MATH 214 - Intermediate Calculus III

Review Formulas

In the table below, $u = u(x), v = v(x)$, $f = f(x)$ and $g = g(x)$ represent four differentiable functions of x, $n \in \mathcal{R}$ and $a > 0$.

$$y = u \pm v \quad \Rightarrow \quad y' = u' \pm v'$$
$$y = uv \quad \Rightarrow \quad y' = u'v + uv'$$
$$y = \frac{u}{v} \quad \Rightarrow \quad y' = \frac{u'v - uv'}{v^2} \qquad (v \neq 0)$$
$$y = (f \circ g)(x))' = (f(g(x))' \quad \Rightarrow \quad y' = f'(g(x)) \cdot g'(x)$$

$$y = x^n \Rightarrow y' = nx^{n-1} \qquad y = u^n \Rightarrow y' = nu^{n-1}u'$$
$$y = \sqrt{x} \Rightarrow y' = \frac{1}{2\sqrt{x}} \qquad y = \sqrt{u} \Rightarrow y' = \frac{u'}{2\sqrt{u}}$$
$$y = \frac{1}{x} \Rightarrow y' = \frac{-1}{x^2} \qquad y = \frac{1}{u} \Rightarrow y' = \frac{-u'}{u^2}$$

$$y = \sin(x) \Rightarrow y' = \cos(x) \qquad y = \sin(u) \Rightarrow y' = \cos(u)u'$$
$$y = \cos(x) \Rightarrow y' = -\sin(x) \qquad y = \cos(u) \Rightarrow y' = -\sin(u)u'$$
$$y = \tan(x) \Rightarrow y' = \sec^2(x) \qquad y = \tan(u) \Rightarrow y' = \sec^2(u)u'$$
$$y = \cot(x) \Rightarrow y' = -\csc^2(x) \qquad y = \cot(u) \Rightarrow y' = -\csc^2(u)u'$$
$$y = \sec(x) \Rightarrow y' = \sec(x)\tan(x) \qquad y = \sec(u) \Rightarrow y' = \sec(u)\tan(u)u'$$
$$y = \csc(x) \Rightarrow y' = -\csc(x)\cot(x) \qquad y = \csc(u) \Rightarrow y' = -\csc(u)\cot(u)u'$$

$$y = e^x \Rightarrow y' = e^x \qquad y = e^u \Rightarrow y' = e^u u'$$
$$y = a^x \Rightarrow y' = a^x \ln(a) \qquad y = a^u \Rightarrow y' = a^u u' \ln(a)$$

$$y = \ln(x) \Rightarrow y' = \frac{1}{x} \qquad y = \ln(u) \Rightarrow y' = \frac{u'}{u}$$

$$y = \log_a(x) \Rightarrow y' = \frac{1}{x\ln(a)} \qquad y = \log_a(u) \Rightarrow y' = \frac{u'}{u\ln(a)}$$

$$y = \sin^{-1}(x) \Rightarrow y' = \frac{1}{\sqrt{1-x^2}} \qquad y = \sin^{-1}(u) \Rightarrow y' = \frac{u'}{\sqrt{1-u^2}}$$
$$y = \cos^{-1}(x) \Rightarrow y' = \frac{-1}{\sqrt{1-x^2}} \qquad y = \cos^{-1}(u) \Rightarrow y' = \frac{-u'}{\sqrt{1-u^2}}$$
$$y = \tan^{-1}(x) \Rightarrow y' = \frac{1}{1+x^2} \qquad y = \tan^{-1}(u) \Rightarrow y' = \frac{u'}{1+u^2}$$
$$y = \cot^{-1}(x) \Rightarrow y' = \frac{-1}{1+x^2} \qquad y = \cot^{-1}(u) \Rightarrow y' = \frac{-u'}{1+u^2}$$
$$y = \sec^{-1}(x) \Rightarrow y' = \frac{1}{x\sqrt{x^2-1}} \qquad y = \sec^{-1}(u) \Rightarrow y' = \frac{u'}{u\sqrt{u^2-1}}$$
$$y = \csc^{-1}(x) \Rightarrow y' = \frac{-1}{x\sqrt{x^2-1}} \qquad y = \csc^{-1}(u) \Rightarrow y' = \frac{-u'}{u\sqrt{u^2-1}}$$

In the formulae below, $u = u(x)$ is a differentiable function, $n \neq -1$ and $a > 0$, and c represents an arbitrary constant.

$$\begin{aligned} \int u^n du &= \tfrac{u^{n+1}}{n+1} + c, & \int \tfrac{du}{u} &= \ln|u| + c \\ \int e^u du &= e^u + c, & \int a^u du &= \tfrac{a^u}{\ln a} + c \end{aligned}$$

In the formulae below $a \neq 0$.

$$\begin{aligned} \int \sin(ax)dx &= -\frac{\cos(ax)}{a} + c, & \int \cos(ax)dx &= \frac{\sin(ax)}{a} + c \\ \int \tan(ax)dx &= -\frac{\ln|\cos(ax)|}{a} + c, & \int \cot(ax)dx &= \frac{\ln|\sin(ax)|}{a} + c \end{aligned}$$

$$\begin{aligned} \int \sec x dx &= \ln(\sec x + \tan x) + c \\ \int \csc x dx &= \ln(\csc x - \cot x) + c \end{aligned}$$

Integration by Parts

$$\int u dv = uv - \int v du, \qquad \int_a^b u dv = uv|_a^b - \int_a^b v du$$

Trigonometric Integrals

Depending on the integral we have, we use some of the following identities.

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1, & 1 + \tan^2 x &= \sec^2 x, & 1 + \cot^2 x &= \csc^2 x \\ \sin(2x) &= 2\sin x \cos x, & \cos(2x) &= 2\cos^2 x - 1 \\ & & &= 1 - 2\sin^2 x \\ & & &= cos^2 x - \sin^2 x \end{aligned}$$

$$\sin^2 x = \tfrac{1-\cos(2x)}{2}, \qquad \cos^2 x = \tfrac{1+\cos(2x)}{2}$$

$$\begin{aligned} \sin x \cos y &= \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right) \\ \sin x \sin y &= \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right) \\ \cos x \cos y &= \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right) \end{aligned}$$

$$\begin{aligned} \int \frac{du}{a^2 + u^2} &= \frac{1}{a}\tan^{-1}\frac{u}{a} + c \\ \int \frac{du}{\sqrt{a^2 - u^2}} &= \sin^{-1}\frac{u}{a} + c \\ \int \frac{du}{u\sqrt{u^2 - a^2}} &= \frac{1}{a}\sec^{-1}\frac{u}{a} + c \end{aligned}$$

Trigonometric Substitutions

If an integral contains one of the following terms, make the suggested substitution:

$$\begin{aligned} \sqrt{a^2-x^2} &\implies \text{sub in} \quad \mathrm{x = a\sin t} \\ \sqrt{x^2-a^2} &\implies \text{sub in} \quad \mathrm{x = a\sec t} \\ \sqrt{x^2+a^2} &\implies \text{sub in} \quad \mathrm{x = a\tan t} \end{aligned}$$

Partial Fractions

This technique is only used for rational functions of the form $f(x) = \frac{p(x)}{q(x)}$.
If $deg(p(x)) \geq \deg(q(x))$ then we first apply long division.

After we factorizing the denominator $q(x)$, we use a partial fractions decomposition depending on the factors of $q(x)$ *only*.

$q(x)$ contains	*remark*	partial fractions decomposition
$(a_1x+b_1)\cdots(a_nx+b_n)$		$\frac{A_1}{a_1x+b_1}+\cdots+\frac{A_n}{a_nx+b_n}$
$(ax+b)^n$		$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$
$(a_1x^2+b_1x+c_1)\cdots(a_nx^2+b_nx+c_n)$	$\Delta < 0$	$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\cdots+\frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$
$(ax^2+bx+c)^n$	$\Delta < 0$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Improper Integrals

TYPE I

$$\begin{aligned} \int_a^\infty f(x)dx &= \lim_{t\to\infty}\int_a^t f(x)dx, \text{ if f(x) is continuous on } [\mathrm{a},\infty), \\ \int_{-\infty}^a f(x)dx &= \lim_{t\to-\infty}\int_t^a f(x)dx, \text{ if f(x) is continuous on } (-\infty,\mathrm{a}], \\ \int_{-\infty}^\infty f(x)dx &= \lim_{t\to-\infty}\int_t^a f(x)dx + \lim_{t\to\infty}\int_a^t f(x)dx, \quad \text{if f(x) is continuous on } (-\infty,\infty). \end{aligned}$$

TYPE II

$$\begin{aligned} \int_a^b f(x)dx &= \lim_{t\to a^+}\int_t^b f(x)dx, \text{ if f(x) has a vertical asymptote at } \mathrm{x=a}, \\ \int_a^b f(x)dx &= \lim_{t\to b^-}\int_a^t f(x)dx, \text{ if f(x) has a vertical asymptote at } \mathrm{x=b}, \\ \int_a^b f(x)dx &= \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx, \quad \text{if f(x) has a vertical asymptote at } \mathrm{x = c} \in (\mathrm{a},\mathrm{b}). \end{aligned}$$

If we need to check whether an improper integral converges or diverges (without evaluating the integral), we can use the direct or the limit comparison tests.

Faruk Uygul
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