

# A hole-ographic spacetime [1]

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In pure AdS<sub>3</sub>/CFT<sub>2</sub> correspondence, they proposed a boundary quantity called differential entropy, which measures the circumference of a bulk curve.

The static coordinate of AdS<sub>3</sub> is

$$ds^2 = - \left(1 + \frac{R^2}{L^2}\right) dT^2 + \left(1 + \frac{R^2}{L^2}\right)^{-1} dR^2 + R^2 d\phi^2. \quad (1)$$

The vacuum entanglement entropy for any interval with its length  $2\alpha$ , is computed as

$$S(\alpha) = \frac{c}{3} \ln \left( \frac{2L}{\mu} \sin \alpha \right). \quad (2)$$

In the paper, they introduced *differential entropy* as the modified version of the entanglement entropy, in which the UV regulator  $\mu$  is removed.

**Definition** Let  $\alpha(\theta)$  be a smooth function of  $\theta$  ( $\alpha(\theta + 2\pi) = \alpha(\theta)$ ) defined on a boundary time slice  $T = 0$ . The differential entropy, which is a functional of  $\alpha(\theta)$ , is defined as

$$E[\alpha] := \frac{1}{2} \int_0^{2\pi} d\theta \left. \frac{dS(\alpha)}{d\alpha} \right|_{\alpha=\alpha(\theta)}. \quad (3)$$

In this holographic formula, function  $\alpha$  corresponds a certain closed bulk curve on  $T = 0$ . This is understood as follows. There is a unique spacelike geodesic  $\gamma_\alpha(\theta)$  anchored at  $\theta \pm \alpha(\theta)$  on the boundary. If we move  $\theta$ ,  $\gamma_\alpha(\theta)$  sweeps out the slice  $T = 0$ , leaving a hole on the slice. The hole is the one we are considering and each  $\gamma_\alpha(\theta)$  is tangent on the hole.

In the remaining of this note, we overview the derivation of the formula. First, let us see how  $R(\phi)$  is related to  $\alpha(\theta)$ . For each point  $\phi$  on  $R = R(\phi)$ , there is a unique geodesic on  $T = 0$  which is tangent to the curve at  $\phi$ . Each of such spacelike geodesics parametrized by  $\phi$  reaches the boundary points  $\theta = \theta(\phi) \pm \alpha(\phi)$ :

$$\alpha(\phi) = \arctan \left[ \frac{L}{R} \sqrt{1 + \frac{L^2}{R^2 + L^2} \left( \frac{d \ln R(\phi)}{d\phi} \right)^2} \right], \quad (4)$$

$$\theta(\phi) = \phi - \arctan \left[ \frac{L^2}{R^2 + L^2} \frac{d \ln R(\phi)}{d\phi} \right]. \quad (5)$$

These two equations relate the curve  $R = R(\phi)$  to the interval function  $\alpha(\theta)$ . These formula are inverted to be

$$R(\theta) = L \cot \alpha(\theta) \sqrt{\frac{1 + \alpha'(\theta)^2 \tan^2 \alpha(\theta)}{1 - \alpha'(\theta)^2}}, \quad (6)$$

$$\phi(\theta) = \theta - \arctan[\alpha'(\theta) \tan \alpha(\theta)]. \quad (7)$$

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Next, let us see the formula

$$E[\alpha] = \frac{\text{circumference of } (R, \phi) = (R(\theta), \phi(\theta))}{4G}. \quad (8)$$

To see this, we rewrite the right hand side by the bulk terms as

$$\text{r.h.s.} = \int_0^{2\pi} \frac{d\phi}{4G} \sqrt{\left(1 + \frac{R^2}{L^2}\right)^{-1} \left(\frac{dR}{d\phi}\right)^2 + R^2}. \quad (9)$$

Note that we should regard  $\phi$  and  $R(\phi)$  in this expression as  $\phi(\theta)$  and  $R(\theta)$  given by (6) and (7). Comparing each integrand of both hand sides in (8), we get

$$\text{l.h.s.} \rightarrow \frac{d\theta}{2} \frac{dS(\alpha)}{d\alpha} \Big|_{\alpha=\alpha(\theta)}, \quad (10)$$

$$\text{r.h.s.} \rightarrow \frac{d\phi}{4G} \sqrt{\left(1 + \frac{R^2}{L^2}\right)^{-1} \left(\frac{dR}{d\phi}\right)^2 + R^2}. \quad (11)$$

Subtracting the integrand of the l.h.s. from the r.h.s., we find the result is  $df$ , with  $f$  defined as

$$f(\theta) := \left[ \frac{L}{8G} \ln \left( \frac{\sin(\alpha(\theta) + \phi(\theta) - \theta)}{\sin(\alpha(\theta) - \phi(\theta) + \theta)} \right) \right]. \quad (12)$$

Therefore the formula (8) is shown, since  $df$  is an exact form. It is also shown that  $f(\theta)$  is equal to the proper length along the spacelike geodesic on  $T = 0$  between two points  $\phi$  and  $\theta(\phi)$  of  $R = R(\phi)$ . Then we get a corollary: let  $L$  be a proper length of the bulk curve  $\{(R, \phi) = (R(\theta), \phi(\theta)) | \theta_i \leq \theta \leq \theta_f\}$ , then

$$\frac{L}{4G} = \frac{1}{2} \int_{\theta_i}^{\theta_f} d\theta \frac{dS(\alpha)}{d\alpha} \Big|_{\alpha=\alpha(\theta)} + f(\theta_f) - f(\theta_i). \quad (13)$$

A heuristic approach is taken in their paper, and a brief but more concrete review appears in [2].

## References

- [1] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [1310.4204].
- [2] B. Czech and L. Lamprou, *Holographic definition of points and distances*, *Phys. Rev. D* **90** (2014) 106005 [1409.4473].