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Entanglement Entropy and Quantum Field Theory [1]

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It is well-known that the entanglement entropy in CFT_2 can be calculated by applying *replica trick*. They introduced two local fields, which is later called *branch-point twist fields* [2], so as to compute the Rényi entropy. As the consequence, the trace of the n-th power of the reduced density matrix is expressed by the two-point correlation function of the two twisted fields, and hence the entanglement entropy is derived from the n-derivative of it. Since now that our concern is the two-point correlator, the entanglement entropy of a compactified CFT_2 can also be calculated by applying the conformal transformation to the correlator. I also referred review paper [3] to write the following.

Let ϕ be the fundamental field in the theory, then the partition function is written as follows:

$$Z = \int \mathcal{D}\phi \, \exp\left[-\int_{\mathbb{C}} \mathrm{d}^2 z \, \mathcal{L}[\phi]\right]. \tag{1}$$

Here the complex coordinate z is decomposed as $z = \sigma + i\tau$ with τ being the imaginary time. We now consider the vacuum entanglement entropy of region

$$A = \{z \in \mathbb{C} \,|\, 0 \leq \sigma \leq u, \ \tau = 0\} \tag{2}$$

at the time slice $\tau = 0$. In this case, we have

$$\mathrm{Tr}\rho_A^n = \frac{Z_{\mathcal{R}_n}}{Z^n}, \tag{3}$$

$$Z_{\mathcal{R}_n} = \int_{\mathcal{R}_n} \mathcal{D}\phi \, \exp\left[-\int_{\mathcal{R}_n} \mathrm{d}^2 w \, \mathcal{L}[\phi]\right], \tag{4}$$

where $\mathcal{R}_n$ is the Riemann surface constructed by joining n Riemann sheets at A as the usual way, and w is its coordinate. Since the Lagrangian density is local, (4) can be re-written as

$$Z_{\mathcal{R}_n} = \int_{\mathrm{B.C.}} \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_n \, \exp\left[-\int_{\mathbb{C}} \mathrm{d}^2 z \, (\mathcal{L}[\phi_1] + \cdots + \mathcal{L}[\phi_n])\right], \tag{5}$$

where B.C. denotes the boundary condition at A on each sheet:

$$\mathrm{B.C.} : \phi_i(\sigma, +0) = \phi_{i+1}(\sigma, -0) \quad (\sigma \in A) \quad \text{and} \quad \phi_i(\sigma, +0) = \phi_i(\sigma, -0) \quad (\sigma \notin A). \tag{6}$$

Note that $i+n$-th and i-th sheet are identical on $\mathcal{R}_n$.

The partition function (5) is now a path integral of n fields $\phi_i \, (i = 1, \cdots, n)$, each of which are defined on the same complex plane $\mathbb{C}$ with (6). We may expect that $Z_{\mathcal{R}_n}$ would be expressed by the operator insertions at $z = 0$ and $z = u$ on $\mathbb{C}$, instead of using (6).

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The operators are branch-point twist operators, which we write as $\mathcal{T}_n(0), \bar{\mathcal{T}}_n(u)$ here. Those operators are formally defined through

$$\langle \mathcal{T}_n(0)\bar{\mathcal{T}}_n(u)\rangle_{\mathbb{C},\mathcal{L}^{(n)}} \propto \int_{\mathrm{B.C.}} \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_n \, \exp\left[-\int_{\mathbb{C}} \mathrm{d}^2 z\, \mathcal{L}^{(n)}[\phi_1, \cdots, \phi_n]\right], \tag{7}$$

where

$$\mathcal{L}^{(n)}[\phi_1, \cdots, \phi_n](z) = \mathcal{L}[\phi_1](z) + \cdots + \mathcal{L}[\phi_n](z). \tag{8}$$

The correlator of the l.h.s. in (7) is defined on the theory of ϕ_i's on $\mathbb{C}$ without B.C. Twist operators $\mathcal{T}, \bar{\mathcal{T}}$ are in principle some composite operators of ϕ_i's. Assuming that $\mathcal{T}, \bar{\mathcal{T}}$ are primary[1)] having weight d_n, we obtain

$$\langle \mathcal{T}_n(0)\bar{\mathcal{T}}_n(u)\rangle_{\mathbb{C},\mathcal{L}^{(n)}} \propto \frac{1}{u^{2d_n}}. \tag{9}$$

To calculate the entropy, we need the vacuum expectation value $\langle T(w)\rangle_{\mathcal{R}_n,\mathcal{L}}$, where $T(w)$ is the energy momentum tensor of theory $(\mathcal{R}_n, \mathcal{L})$. Through map

$$z = \left(\frac{w}{w-u}\right)^{1/n}, \tag{10}$$

the Riemann surface $\mathcal{R}_n$ is mapped to $\mathbb{R}$. The energy momentum tensor $T(\zeta)$ of $(\mathbb{R}, \mathcal{L})$ is related with $T(w)$ as

$$T(w) = \left(\frac{\partial \zeta}{\partial w}\right)^2 T(z) + \frac{c}{12}\{\zeta, w\}, \tag{11}$$

where the Schwarzian derivative is

$$\{\zeta, w\} = \frac{\zeta'''\zeta' - (3/2)(\zeta'')^2}{(\zeta')^2}, \tag{12}$$

and c is the central charge. Taking the vacuum expectation value of (11), we obtain

$$\langle T(w)\rangle_{\mathcal{R}_n,\mathcal{L}} = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)\frac{u^2}{w^2(w-u)^2}, \tag{13}$$

by using $\langle T(\zeta)\rangle_{\mathbb{C},\mathcal{L}} = 0$ which follows from the translational invariance.

On the other hand, (5) and (7) means that $\langle T(w)\rangle_{\mathcal{R}_n,\mathcal{L}}$ can be computed as a correlator with the twist operators. In concrete, we have

$$\langle T(z)\rangle_{\mathcal{R}_n,\mathcal{L}} = \frac{\langle \mathcal{T}_n(0)\bar{\mathcal{T}}_n(u)T_j(z)\rangle_{\mathbb{C},\mathcal{L}^{(n)}}}{\langle \mathcal{T}_n(0)\bar{\mathcal{T}}_n(u)\rangle_{\mathbb{C},\mathcal{L}^{(n)}}} \tag{14}$$

[1)] I could not understand whether or not twist operators do exist and are primary, but it seems reasonable to think it true according to [2].

for z describing j-th sheet.[2] Here $T(w)$ corresponds to $T_j(w)$ which is defined by $\mathcal{L}[\phi_j]$ in theory $(\mathbb{C}, \mathcal{L}^{(n)})$.

Since the energy momentum tensor of the whole theory of $(\mathbb{C}, \mathcal{L}^{(n)})$, $T^{(n)}$, is given by the summation of T_j,[3] we get

$$\frac{\langle \mathcal{T}_n(0)\bar{\mathcal{T}}_n(u)T^{(n)}(z)\rangle_{\mathbb{C},\mathcal{L}^{(n)}}}{\langle \mathcal{T}_n(0)\bar{\mathcal{T}}_n(u)\rangle_{\mathbb{C},\mathcal{L}^{(n)}}} = \frac{nc}{24}\left(1-\frac{1}{n^2}\right)\frac{u^2}{w^2(w-u)^2}. \tag{15}$$

From the usual formula[4]

$$\begin{aligned}&\langle \mathcal{T}_n(a)\bar{\mathcal{T}}_n(b)T^{(n)}(z)\rangle_{\mathbb{C},\mathcal{L}^{(n)}} = \\ &\quad\left(\frac{1}{z-a}\frac{\partial}{\partial a}+\frac{d_n}{(w-a)^2}+\frac{1}{z-b}\frac{\partial}{\partial b}+\frac{d_n}{(w-b)^2}\right)\langle \mathcal{T}_n(a)\bar{\mathcal{T}}_n(b)\rangle_{\mathbb{C},\mathcal{L}^{(n)}}\end{aligned} \tag{16}$$

with (9), the weight now can be identified:

$$d_n = \frac{c}{12}\left(n-\frac{1}{n}\right). \tag{17}$$

Thus, combining (3), (9) and (17), we conclude

$$\mathrm{Tr}\rho_A^n = c_n\left(\frac{u}{a}\right)^{-c(n-1/n)/6}, \tag{18}$$

where c_n and a are constants independent of u, in particular we have $c_1 = 1$ by $\mathrm{Tr}\rho_A = 1$.

The entanglement entropy of A is

$$S_A = \lim_{n\to 1}\frac{1}{1-n}\mathrm{Tr}\rho_A^n = \frac{c}{3}\ln\left(\frac{u}{a}\right) - c_1', \tag{19}$$

where c_n' denotes n-derivative of c_n (c_1' is non-universal constant).

If the CFT of interest is on the cylinder obtained by compactifying σ-direction with length L, we use the map $\xi = (L/2\pi)\ln(-iz)$ to compute the entropy. Applying the conformal transformation to $\langle \mathcal{T}_n(a)\bar{\mathcal{T}}_n(b)\rangle_{\mathbb{C},\mathcal{L}^{(n)}}$ and putting $a/b = \exp(2\pi i v/L)$, we obtain

$$\mathrm{Tr}\rho_{A'}^n = c_n\left(\frac{L}{a\pi}\sin\frac{v\pi}{L}\right)^{-c(n-1/n)/6}, \qquad \text{i.e.} \qquad S_{A'} = \frac{c}{3}\log\left(\frac{L}{a\pi}\sin\frac{v\pi}{L}\right) - c_1'. \tag{20}$$

The entropy of thermalized CFT_2 can also be computed by following the same process. In this case, we compactify the τ-direction.

[2] I used w to describe the coordinate of $\mathcal{R}_n$, but now used z to describe the coordinate of $\mathbb{C}$. Note that $\mathcal{R}_n$ can be described by the pair (j, z).

[3] Note that $T^{(n)}$ is defined by $\mathcal{L}^{(n)}$ of (8).

[4] Since theory $(\mathbb{C}, \mathcal{L}^{(n)}$ must be rotational invariant, the

References

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[2] J.L. Cardy, O.A. Castro-Alvaredo and B. Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, *J. Statist. Phys.* **130** (2008) 129 [`0706.3384`].

[3] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, *J. Phys. A* **42** (2009) 504005 [`0905.4013`].