[Overview*] last updated: February 18, 2022

A hole-ographic spacetime [1]

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In pure $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, they proposed a boundary quantity called differential entropy, which measures the circumference of a bulk curve.

The static coordinate of AdS_3 is

$$\mathrm{d}s^2 = -\left(1+\frac{R^2}{L^2}\right)\mathrm{d}T^2 + \left(1+\frac{R^2}{L^2}\right)^{-1}\mathrm{d}R^2 + R^2\mathrm{d}\phi^2. \tag{1}$$

The vacuum entanglement entropy for any interval with its length 2α, is computed as

$$S(\alpha) = \frac{c}{3}\ln\left(\frac{2L}{\mu}\sin\alpha\right). \tag{2}$$

In the paper, they introduced *differential entropy* as the modified version of the entanglement entropy, in which the UV regulator μ is removed.

Definition Let $\alpha(\theta)$ be a smooth function of θ ($\alpha(\theta+2\pi)=\alpha(\theta)$) defined on a boundary time slice $T=0$. The differential entropy, which is a functional of $\alpha(\theta)$, is defined as

$$E[\alpha] := \frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\ \left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha(\theta)}. \tag{3}$$

In this holographic formula, function α corresponds a certain closed bulk curve on $T=0$. This is understood as follows. There is a unique spacelike geodesic $\gamma_\alpha(\theta)$ anchored at $\theta\pm\alpha(\theta)$ on the boundary. If we move θ, $\gamma_\alpha(\theta)$ sweeps out the slice $T=0$, leaving a hole on the slice. The hole is the one we are considering and each $\gamma_\alpha(\theta)$ is tangent on the hole.

In the remaining of this note, we overview the derivation of the formula. First, let us see how $R(\phi)$ is related to $\alpha(\theta)$. For each point ϕ on $R=R(\phi)$, there is a unique geodesic on $T=0$ which is tangent to the curve at ϕ. Each of such spacelike geodesics parametrized by ϕ reaches the boundary points $\theta=\theta(\phi)\pm\alpha(\phi)$:

$$\alpha(\phi) = \arctan\left[\frac{L}{R}\sqrt{1+\frac{L^2}{R^2+L^2}\left(\frac{\mathrm{d}\ln R(\phi)}{\mathrm{d}\phi}\right)^2}\,\right], \tag{4}$$

$$\theta(\phi) = \phi - \arctan\left[\frac{L^2}{R^2+L^2}\frac{\mathrm{d}\ln R(\phi)}{\mathrm{d}\phi}\right]. \tag{5}$$

These two equations relate the curve $R=R(\phi)$ to the interval function $\alpha(\theta)$. These formula are inverted to be

$$R(\theta) = L\cot\alpha(\theta)\sqrt{\frac{1+\alpha'(\theta)^2\tan^2\alpha(\theta)}{1-\alpha'(\theta)^2}}, \tag{6}$$

$$\phi(\theta) = \theta - \arctan[\alpha'(\theta)\tan\alpha(\theta)]. \tag{7}$$

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Next, let us see the formula

$$E[\alpha] = \frac{\text{circumference of } (R,\phi) = (R(\theta),\phi(\theta))}{4G}. \tag{8}$$

To see this, we rewrite the right hand side by the bulk terms as

$$\text{r.h.s.} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{4G}\sqrt{\left(1+\frac{R^2}{L^2}\right)^{-1}\left(\frac{\mathrm{d}R}{\mathrm{d}\phi}\right)^2 + R^2}. \tag{9}$$

Note that we should regard ϕ and $R(\phi)$ in this expression as $\phi(\theta)$ and $R(\theta)$ given by (6) and (7). Comparing each integrand of both hand sides in (8), we get

$$\text{l.h.s.} \to \frac{\mathrm{d}\theta}{2}\left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha(\theta)}, \tag{10}$$

$$\text{r.h.s.} \to \frac{\mathrm{d}\phi}{4G}\sqrt{\left(1+\frac{R^2}{L^2}\right)^{-1}\left(\frac{\mathrm{d}R}{\mathrm{d}\phi}\right)^2 + R^2}. \tag{11}$$

Subtracting the integrand of the l.h.s. from the r.h.s., we find the result is $\mathrm{d}f$, with f defined as

$$f(\theta) := \left[\frac{L}{8G}\ln\left(\frac{\sin(\alpha(\theta)+\phi(\theta)-\theta)}{\sin(\alpha(\theta)-\phi(\theta)+\theta)}\right)\right]. \tag{12}$$

Therefore the formula (8) is shown, since $\mathrm{d}f$ is an exact form. It is also shown that $f(\theta)$ is equal to the proper length along the spacelike geodesic on $T=0$ between two points ϕ and $\theta(\phi)$ of $R=R(\phi)$. Then we get a corollary: let L be a proper length of the bulk curve $\{(R,\phi)=(R(\theta),\phi(\theta))|\theta_i \le \theta \le \theta_f\}$, then

$$\frac{L}{4G} = \frac{1}{2}\int_{\theta_i}^{\theta_f} \mathrm{d}\theta\, \left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha(\theta)} + f(\theta_f) - f(\theta_i). \tag{13}$$

A heuristic approach is taken in their paper, and a brief but more concrete review appears in [2].

References

[1] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [1310.4204].

[2] B. Czech and L. Lamprou, *Holographic definition of points and distances*, *Phys. Rev. D* **90** (2014) 106005 [1409.4473].