[Overview*] last updated: May 19, 2022

Entwinement and the emergence of spacetime [1]

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In the bulk reconstruction program, it is known that there is a bulk region called entanglement shadow, which is a region minimal surfaces cannot enter. This is problematic because Ryu-Takayanagi formula, which is known as a key formula in reconstructing spacetimes, cannot be used to construct the metric in an entanglement shadow. In the paper, they proposed another notion called *entwinement* as a boundary quantity to go deeper into the bulk. Recalling that the radial coordinate r in the bulk is interpreted as the energy scale of the boundary, we need a boundary quantity characterizing smaller energy scale to know the inside of an entanglement shadow. The entwinement reflects the entanglement entropy between "internal" degrees of freedom, while the ordinary spatial entanglement entropy does not.

The setup

As an example, they considered the AdS_3 spacetime with a conical defect:

$$\mathrm{d}s^2 = -\left(\frac{1}{n^2}+\frac{r^2}{L^2}\right)\mathrm{d}t^2 + \left(\frac{1}{n^2}+\frac{r^2}{L^2}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \qquad n \in \mathbb{N}. \tag{1}$$

Here θ and $\theta+2\pi$ are identified. Since this geometry can be obtained by the identification $\theta \sim \theta + 2\pi/n$ in the pure AdS_3, it is denoted as $\mathrm{AdS}_3/\mathbb{Z}_n$. The $\mathbb{Z}_n$-identification can be regarded as a discrete gauge symmetry, hence the corresponding boundary theory is in a $\mathbb{Z}_n$-symmetric state. Thus, the boundary theory has the internal degrees of freedom associated with this $\mathbb{Z}_n$-symmetry.

The geometry $\mathrm{AdS}_3/\mathbb{Z}_n$ is known to be dual to a state in the D1-D5 CFT.[1)] This state is identified by a twist field, which makes $N = pn$ fields $X^1, \cdots, X^N$ subject to the condition that under a 2π-rotation, they transform as follows ($k = 0, 2, \cdots p-1$):

$$(X^{kn+1}, \cdots, X^{(k+1)n-1}, X^{(k+1)n}) \to (X^{kn+2}, \cdots, X^{(k+1)n}, X^{kn+1}). \tag{2}$$

This reflects the global $\mathbb{Z}_n$-symmetry that corresponds to the $\mathbb{Z}_n$-gauge symmetry of the bulk.

The above theory can be embedded to a larger theory defined on n times longer circle: $\theta \sim \theta + 2n\pi$. The theory consists of p single-valued fields $\tilde{X}^k$, each of which is obtained by glueing $X^{kn+1}, \cdots X^{k(n+1)}$ in order. In this covering CFT, the original CFT is a sector which is symmetric under 2π times integer rotation.

The dual geometry of the covering CFT is still (1), except that θ runs over n times larger region: $\theta \sim \theta + 2n\pi$. The conical singularity has already been removed, and (1) is completely equivalent to the pure AdS_3, meaning that the $\mathbb{Z}_n$-gauge is ungauged now. In other words, the original bulk theory can be realized by gauging this covering geometry with the $\mathbb{Z}_n$-identification. Thus, any quantity that descends to the original bulk theory must be $\mathbb{Z}_n$-symmetric in the larger bulk.

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1) see the references on p.8 in [1].

Note that in the covering CFT, the central charge is $\tilde{c} = c/n$, where c is that of the original CFT, because the degrees of freedom is now $p = N/n$ instead of N. According to this, the gravitational constants of the two bulk are related as $\tilde{G} = nG$.

The entanglement shadow

Here we refere figure 2 in [1]. There are n geodesics that connects given two boundary points in $\mathrm{AdS}_3/\mathbb{Z}_n$. According to the Ryu-Takayanagi formula, the shortest one gives the entanglement entropy of the region between the two points. It is easily confirmed by geodesic computations that the minimum geodesic cannot enter the region

$$r < L\cot(\pi/2n)/n =: r_{\mathrm{c}}, \tag{3}$$

no matter how long is the interval between the two boundary points. This is the entanglement shadow of (1), which cannot be described by the entanglement. However, the other $n-1$ geodesics do enter the entanglement shadow.

In the covering geometry, on the other hand, each of these n geodesics corresponds to a different boundary region; if the interval corresponding to the minimum geodesic is $[\theta_0, \theta_0 + \Delta]$, then the other intervals corresponding to the rest geodesics are given as $[\theta_0, \theta_0 + \Delta + 2m\pi]$. (These regions are originally identical due to the $\mathbb{Z}_n$ gauge symmetry.) Therefore, each of the long geodesics ascends to a minimum geodesic of its corresponding interval, which leads to the notion of the entwinement.

The entwinement

The entwinement is defined on the covering CFT. Let us consider an interval R on the covering CFT and its $\mathbb{Z}_n$ translations $g_m R$, where g_m's are $\mathbb{Z}_n$ generators. The entwinement of the region R is defined as

$$E_R = \sum_{m=1}^{n} S_{g_m R} = \sum_{m=1}^{n} g_m S_R = nS_R, \tag{4}$$

where S_R is the conventional entanglement entropy in the large CFT. Since E_R is $\mathbb{Z}_n$-invariant, it can descends to $\mathrm{AdS}_3/\mathbb{Z}_n$.

If the length of R is $2\alpha\,(0 \leq \alpha \leq n\pi)$, we have

$$S_R = \frac{\tilde{c}}{3}\ln\left(\frac{2L}{\mu}\sin\frac{\alpha}{n}\right), \tag{5}$$

and hence with $c = 3L/(2G)$, we obtain

$$E_R = \frac{c}{3}\ln\left(\frac{2L}{\mu}\sin\frac{\alpha}{n}\right) = \frac{L}{2G}\ln\left(\frac{2L}{\mu}\sin\frac{\alpha}{n}\right) =: E(\alpha). \tag{6}$$

As mentioned above, there are n geodesics for region R, and the lengths of them are given as

$$\ell(\phi + \pi m) = 2GL\ln\left(\frac{2L}{\mu}\sin\frac{\phi + \pi m}{n}\right), \tag{7}$$

where $2\phi\,(\phi \in [0,\pi])$ is the length of R. (The minimum one is given by $m = 0$.) Thus with the identification of $\alpha = \phi + \pi m\,(\alpha \in [0, n\pi])$, we see

$$\ell(\alpha) = \frac{E(\alpha)}{4G}. \tag{8}$$

In the view point of $\mathrm{AdS}_3/\mathbb{Z}_n$, the discussion here says that the entwinement is obtained by the naive continuation of the entanglement for $\alpha \in [0, \pi/2]$ to $\alpha \in [0, n\pi]$, and the entwinement version of the Ryu-Takayanagi formula is the equivalence between entwinements and geodesic lengths.

The entwinement reflects the internal degrees of freedom. For simplicity, let us consider the case $p = 1$. In the covering CFT, (5) measures, for example, the entanglement between X^1 and X^2, the entanglement between internal degrees of freedom associated with $\mathbb{Z}_n$. This is because if we take $[0, \pi]$ for R in the covering space, then R is for X^1 and $\bar{R}$ is for $X^{2,\cdots,n}$, by the definition of $\tilde{X}^1$. Thus, S_R measures how entangled R is with $\bar{R}$, and E_R includes such quantities in a gauge-invariant way.

The bulk reconstruction

If we use the entwinement for the so-called hole-ography [2, 3],[2)] instead of the entanglement itself, entanglement shadows can be reconstructed by following the same way we usually do in the hole-ography.

References

[1] V. Balasubramanian, B.D. Chowdhury, B. Czech and J. de Boer, *Entwinement and the emergence of spacetime*, *JHEP* **01** (2015) 048 [1406.5859].

[2] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [1310.4204].

[3] B. Czech and L. Lamprou, *Holographic definition of points and distances*, *Phys. Rev. D* **90** (2014) 106005 [1409.4473].

2) See also my overview 1 and my overview 2