[Brief Review*]

Interior Product, Lie derivative and Wilson Line in the *KBc* Subsector of Open String Field Theory

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KBc sector is a subspace of bosonic open string field theory (SFT) expanded by

$$K = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} T(z), \quad B = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} b(z), \quad c = c(0), \tag{1}$$

where T is the energy-momentum tensor, b is the anti-ghost field, and c is the ghost field. Here operators are defined in sliver frame z, which is defined through the map $z = (2/\pi)\arctan w$ with the coordinate w used in radial quantization. The following relations called KBc algebra are satisfied by K, B, c and BRST operator Q_{B}:

$$[K, B] = 0, \quad \{B, c\} = 1, \quad B^2 = 0, \quad c^2 = 0, \tag{2}$$

$$Q_{\mathrm{B}} K = 0, \quad Q_{\mathrm{B}} B = K, \quad Q_{\mathrm{B}} c = cKc. \tag{3}$$

Since any CFT has KBc sector, any solution in KBc sector is universal. Other Analytic solutions are studied by combining KBc sector with some matter operators, where solutions are not universal in general. The review paper [1] is useful to learn KBc sector and classical solutions including tachyon vacuum.

We established the notion of the manifold in KBc sector in the paper. A top-down approach is taken in this review, while a heuristic approach is in the paper. Let $\xi = (\xi^1, \xi^2)$ a two-component function, $\xi : \mathbb{R}_{\geq 0} \to \mathbb{C}^2$.[1)] Then it is easily confirmed that the new triad $(K(\xi), B(\xi), c(\xi))$ defined as follows again forms KBc algebra:

$$K(\xi) := e^{\xi^1(K)} K, \quad B(\xi) := e^{\xi^1(K)} B, \quad c(\xi) = e^{-i\xi^2(K)} c e^{-\xi^1(K)} B c e^{i\xi^2(K)}. \tag{4}$$

It is known that the eigen value of K runs over non-negative real number. We define KBc manifold $\mathcal{K}$ as follows:

- The triad $(K(\xi), B(\xi), c(\xi))$ is a point of $\mathcal{K}$.
- The coordinate is the function ξ.

Next, we define the interior product $\mathcal{I}_X$, where $X = (X^1, X^2)$ is two-component function as the coordinate ξ is. It is known that the action of bosonic open SFT (called Witten's action) has similar structure to the Chern-Simons (CS) action. Especially, the ghost number in bosonic open SFT corresponds to the form number in CS theory. Therefore, we assume that $\mathcal{I}_X$ lower the ghost number by one. In addition, KBc algebra should not be broken by

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1) $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} | x \geqq 0\}$

the operation of $\mathcal{I}_X$ at each point of $\mathcal{K}$.[2)] Under the two conditions,[3)] the general form of the operation of $\mathcal{I}_X$ at ξ is given by[4)]

$$\mathcal{I}_X K(\xi) = iB(\xi)X^1(K), \quad \mathcal{I}_X B(\xi) = 0, \quad \mathcal{I}_X c(\xi) = \left\{ \frac{X^2(K)}{K(\xi)} B(\xi), c(\xi) \right\}. \tag{5}$$

We further impose on the interior product Leibniz-rule

$$\mathcal{I}_X(PQ) = (I_X P)Q + (-1)^{|P|} P(\mathcal{I}_X Q), \tag{6}$$

for the operation of $\mathcal{I}_X$ to be defined for any quantities in KBc sector. Here $|P|$ is 0 (1) when P is Grassmann-even (odd).

By analogy with the ordinary manifold ($L_V = \{\mathrm{d}, i_X\}$), we define the Lie derivative as

$$\pounds_X = -i\{Q_\mathrm{B}, \mathcal{I}_X\}. \tag{7}$$

We have used a known fact that BRST operator corresponds to the exterior derivative in the similarities described above. The concrete forms of the operation of $\pounds_X$ at ξ are given by

$$\begin{aligned} &\pounds_X K(\xi) = K(\xi)X^1(K), \quad \pounds_X B(\xi) = B(\xi)X^1(K), \\ &\pounds_X c(\xi) = -c(\xi)X^1(K)B(\xi)c(\xi) - i[X^2(K), c(\xi)]. \end{aligned} \tag{8}$$

From the definition and (6), $\pounds_X$ follows Liebniz-rule

$$\pounds_X(PQ) = (\pounds_X P)Q + P(\pounds_X Q). \tag{9}$$

The ordinary properties and formulas satisfied by the ordinary interior product, exterior product and Lie derivative still hold here, by replacing the exterior derivative with BRST operator. For example, $\{I_X, I_Y\} = 0$, $[Q_B, \pounds_X] = 0$ and $[\pounds_X, \pounds_Y] = \pounds_{[X,Y]}$, where $[X, Y]$ is a Lie bracket properly defined in the oroginal paper. As other benefits, the followings are introduced in the paper.

- The KBc triad at $\xi + \delta\xi$ can be obtained by acting $\pounds_{\delta\xi}$ on the triad at ξ for any infinitesimal $\delta\xi$. Then any two points (in other words, triads) on a continuous curve on $\mathcal{K}$ are related by the continuous operation of Lie derivative.
- If Ψ is a solution described only by KBc sector, then $\Psi(\xi)$, which is obtained by the replacement of (K, B, c) with $(K(\xi), B(\xi), c(\xi))$, is also a solution.
- Wilson line is also defined by analogy with CS theory. Although some similar properties to the ordinary one still hold, the construction appears to be incomplete.

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2) For example, $\mathcal{I}_X(\{B, c\}) = \mathcal{I}_X(1)$ should hold for the relation $\{B, c\} = 1$ not to be broken by introducing $\mathcal{I}_X$ to KBc sector.

3) We imposed more conditions in the original paper, for example nilpotency, but now it was turned out for these to be sufficient to determine the forms of (5).

4) Here $X^{1,2}$ can depend on the original K not through $K(\xi)$, and the factor $1/K(\xi)$ accompanying X^2 is just for convenience to simplify the expression of the Lie derivative introduced in the next paragraph.

Possible future works are as follows.

- The physical interpretation of KBc manifold.
- Improving the Wilson line.
- The extension to the remaining sector of bosonic open SFT.
- The case of super SFT.

References

[1] Y. Okawa, *Analytic methods in open string field theory*, *Prog. Theor. Phys.* **128** (2012) 1001.