

# Holographic Holes and Differential Entropy [1]

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Hole-ography is a method to reconstruct a curve on  $t = \text{const}$  and its length in pure  $\text{AdS}_3$  from the differential entropy, which is a quantity defined by a family of the entanglement entropy of  $\text{CFT}_2$  [2] (see [my overview](#)). Here  $t$  is the time of the static coordinate. In [1], they showed that the hole-ography can be also used for more generic bulk geometries subject to some assumptions, with curves not restricted to be on  $t = \text{const}$ .

A spacelike geodesic was characterized by an  $\theta$ -interval,  $[\theta - \alpha, \theta + \alpha]$ , in [2]. However, if we want to describe curves not limited to be on  $t = \text{const}$ ,  $\theta$ -intervals are not useful. Thus we might better to express a geodesic by its boundary endpoints:  $\gamma_L(\lambda), \gamma_R(\lambda)$ . Here  $\lambda$  is a parameter labeling a family of the endpoint pairs to specify a spatial geodesic, and the family creates a spacelike curve in the bulk, as its envelop. We call the bulk curve  $c_\gamma = c_\gamma^M(\lambda)$ , where  $M$  expresses the bulk coordinates and the geodesic specified by  $\gamma_L(\lambda)$  and  $\gamma_R(\lambda)$  is tangent at  $c_\gamma^M(\lambda)$ .

The differential entropy is defined as<sup>1)</sup>

$$E = \int d\lambda \left. \frac{\partial S(\gamma_L(\lambda), \gamma_R(\lambda'))}{\partial \lambda'} \right|_{\lambda'=\lambda}, \quad (1)$$

where  $S(\gamma_L, \gamma_R)$  is the length of the geodesic from  $\gamma_L$  to  $\gamma_R$ , which is dual to the boundary entanglement entropy, or more precisely the entwinement [3]. The following statement was shown by them: *given that the pair  $(\gamma_L(\lambda), \gamma_R(\lambda))$  is periodic and smooth, then the length of  $c_\gamma$  is equal to  $E$ .*<sup>2)</sup>

Let us follow the proof. Length  $S$  in (1) is expressed as

$$S(\gamma_L(\lambda), \gamma_R(\lambda)) = \int_{s_L}^{s_R} ds \sqrt{(\dot{x}(s; \lambda), \dot{x}(s; \lambda))} \quad (x(s_{L,R}) = \gamma_{L,R}(\lambda)), \quad (2)$$

where  $(\ , \ )$  is the inner product defined by the bulk metric,  $x(s, \lambda)$  is the geodesic from  $\gamma_L(\lambda)$  to  $\gamma_R(\lambda)$ , and  $\dot{\ } := \partial/\partial s$ . Since  $S$  is of the ordinary action form,<sup>3)</sup> we can use the knowledge of analytical mechanics. The integrand in (1) is rewritten as

$$\begin{aligned} \left. \frac{\partial S(\gamma_L(\lambda), \gamma_R(\lambda'))}{\partial \lambda'} \right|_{\lambda'=\lambda} &= \gamma_R^{\mu'}(\lambda) \frac{\partial S}{\partial \gamma_R^\mu}(\gamma_L(\lambda), \gamma_R(\lambda)) \\ &= \gamma_R^{M'}(\lambda) \frac{\partial S}{\partial \gamma_R^M}(\gamma_L(\lambda), \gamma_R(\lambda)) \\ &= \frac{\partial x^M(s_R; \lambda)}{\partial \lambda} p_M(x(s_R; \lambda), \dot{x}(s_R; \lambda)), \end{aligned} \quad (3)$$

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<sup>1)</sup> This definition is equivalent to that of [2] when geodesics are on  $t = \text{const}$ , as explained in [1].

<sup>2)</sup> Great attention should be paid to that the periodicity is assumed. For example, BTZ black hole unfortunately breaks this assumption.

<sup>3)</sup> The reparameterization-invariance of  $s$  and that it consists of up to the first derivative of the position.

where  $p_M(x(s), \dot{x}(s))$  is the conjugate momentum of off-shell position  $x^M(s)$ . (If those are accompanied by “;  $\lambda$ ”, then they take the on-shell value defined above.) In the last equality, we have used the knowledge that the derivative of the on-shell action with the final position is the conjugate momentum at the position.

On the other hand, the length of  $c_\gamma$ , which we call  $L$ , is given as

$$L = \oint d\lambda \sqrt{(c'_\gamma(\lambda), c'_\gamma(\lambda))}. \quad (4)$$

This is of the off-shell version of (2). It is easy to show that equation

$$\sqrt{(\dot{y}(s), \dot{y}(s))} = \dot{y}^M(s) p_M(y(s), \dot{y}(s)) \quad (5)$$

holds in general (even in off-shell),<sup>4)</sup> and hence,

$$L = \oint d\lambda c_\gamma^{M'}(\lambda) p_M(c_\gamma(\lambda), c'_\gamma(\lambda)). \quad (6)$$

As  $c_\gamma$  is the envelop of  $\{\dot{x}(s; \lambda)\}$ , at the tangent point  $s = s_c$ , we have

$$x(s_c; \lambda) = c_\gamma(\lambda), \quad \exists \alpha(\lambda) > 0, \quad \alpha(\lambda) \dot{x}(s_c; \lambda) = c'_\gamma(\lambda). \quad (7)$$

Note that  $s_c$  can be chosen to be independent of  $\lambda$  by reparameterizing  $s$ . Then we obtain

$$\begin{aligned} L &= \oint d\lambda \frac{\partial x^M(s_c; \lambda)}{\partial \lambda} p_M(x(s_c; \lambda), \alpha(\lambda) \dot{x}(s_c; \lambda)) \\ &= \oint d\lambda \frac{\partial x^M(s_c; \lambda)}{\partial \lambda} p_M(x(s_c; \lambda), \dot{x}(s_c; \lambda)), \end{aligned} \quad (8)$$

where in the last equality, we have used follows from the property of the momentum that  $p(y(s), \alpha \dot{y}(s)) = p(y(s), \dot{y}(s))$  following from (5).

Therefore, the remaining task to accomplish is to show that

$$E - L = \oint d\lambda \left. \frac{\partial x^M(s; \lambda)}{\partial \lambda} p_M(x(s; \lambda), \dot{x}(s; \lambda)) \right|_{x_c}^{s_R} \quad (9)$$

must vanish. Using the relation between the on-shell action and the momenta of endpoints again, we can rewrite the above as

$$E - L = \oint d\lambda \frac{\partial}{\partial \lambda} S(x(s_c; \lambda), x(s_R; \lambda)). \quad (10)$$

Since  $x(s; \lambda)$  is also periodic about  $\lambda$  by the assumption, we see this vanish.

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<sup>4)</sup> This generally follows from the reparameterization-invariance, but can also be shown directly of course.

## References

- [1] M. Headrick, R.C. Myers and J. Wien, *Holographic Holes and Differential Entropy*, *JHEP* **10** (2014) 149 [[1408.4770](#)].
- [2] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [[1310.4204](#)].
- [3] V. Balasubramanian, B.D. Chowdhury, B. Czech and J. de Boer, *Entwinement and the emergence of spacetime*, *JHEP* **01** (2015) 048 [[1406.5859](#)].