[Overview*] last updated: February 25, 2022

Nuts and Bolts for Creating Space [1]

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They introduced the method to reconstruct a time slice surface of AdS_3 from the differential entropy of CFT_2 [2] (see my overview). Specifically they found the method to detect the bulk points and to measure the distances between two points on the time slice.

The detection of points

Let $\alpha(\theta)$ be a boundary function introduced in [2], which is also seen in my overview.[1)] Any bulk closed curve $R = R(\phi)$ corresponds to a certain boundary function $\alpha(\theta)$. In the limit of the bulk curve shrinking, the curve becomes a point.

They used this shrinking limit to find the way to detect bulk points from the boundary. First, Gauss-Bonnet theorem states that the following relation holds in negatively curved spaces:

$$\oint_{\partial A} \mathrm{d}\tau \sqrt{h} K = 2\pi - \int_A \mathrm{d}A\, R \geq 2\pi. \tag{1}$$

Here ∂A means the closed curve $R = R(\phi)$, A is the area enclosed by the curve, $\mathrm{d}\tau\sqrt{h}$ is the line element along the curve, K is the extrinsic curvature, and $\mathrm{d}A$ is the areal element. In the shrinking limit, we obtain

$$\lim_{A \to \text{point}} \oint_{\partial A} \mathrm{d}\tau \sqrt{h} K = 2\pi. \tag{2}$$

In addition, the integrand of l.h.s. in (1) can be expressed by $\alpha(\theta)$ as

$$\mathrm{d}\tau \sqrt{h} K = \frac{\mathrm{d}\theta \sqrt{1 - \alpha'(\theta)^2}}{\sin \alpha(\theta)}. \tag{3}$$

From these facts, the boundary function $\alpha(\theta)$ corresponds to a certain bulk point if and only if it extremizes the functional

$$I[\alpha] = \int_0^{2\pi} \mathrm{d}\theta \frac{\sqrt{1 - \alpha'(\theta)^2}}{\sin \alpha(\theta)}, \tag{4}$$

of which the Euler-Lagrange equation is a second order differential equation. Therefore, $\alpha(\theta)$ has two constants, which we regard as a bulk point on the time slice. Noting the relation

$$\frac{1}{\sin^2 \alpha} = -\frac{2G}{L} \frac{\mathrm{d}^2 S(\alpha)}{\mathrm{d}\alpha^2}, \tag{5}$$

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1) The notation adopted here is checked in my overview.

the action (4) can be rewritten as

$$I[\alpha] = \int_0^{2\pi} \mathrm{d}\theta \sqrt{-\left.\frac{\mathrm{d}^2 S(\alpha)}{\mathrm{d}\alpha^2}\right|_{\alpha=\alpha(\theta)} (1-\alpha'(\theta)^2)} \ . \tag{6}$$

They conjectured that this action also holds for other cases having the rotational symmetry. In other words, the collection of functions $\alpha(\theta)$ extremizing (6) is the copy of the bulk points. The elements of the collection are called "point function" here.[2)] A counterexample is discussed in [3] (see my overview).

The measurement of distances

Next, let us see the holographic distances introduced by them, which is defined in the set of point functions $\mathcal{P}$. Now the differential entropy $E[\alpha]$ is used to define the distance.

Let us define $\gamma_{AB}(\theta) := \min\{\alpha_A(\theta), \alpha_B(\theta)\}$ for $\alpha_A, \alpha_B \in \mathcal{P}$ (γ is defined as a point-wise minimum among α_A and α_B). The functional $d(A, B)$ defined as follows is the holographic distance between A and B measured in the units of $4G$:

$$d(A, B) := \frac{1}{2} E[\gamma]. \tag{7}$$

It is shown that this functional satisfies the axioms of a distance function.[3)]

In the remaining, we see why $d(A, B)$ can be regarded as the distance on the time slice. Let us consider that a boundary function $\alpha(\theta)$ has a point θ_k at which $\alpha'(\theta)$ jumps. From the corollary shown in [2], we have

$$\begin{aligned} E[\alpha] &= \frac{1}{2}\int_0^{\theta_k-0} \mathrm{d}\theta \left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha(\theta)} + \frac{1}{2}\int_{\theta_k+0}^{2\pi} \mathrm{d}\theta \left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha(\theta)} \\ &= \frac{(\text{left length}) + (\text{right length}) + (\text{geodesic length in } \phi(\theta_k-0) \le \phi \le \phi(\theta_k+0))}{4G}. \end{aligned} \tag{8}$$

This means that the bulk curve $(R, \phi) = (R(\theta), \phi(\theta))$ corresponding to $\alpha(\theta)$ has a jump at θ_k, and the following holds with the curve compensated by the geodesic:

$$E[\alpha] = \frac{\text{length}}{4G}. \tag{9}$$

The relation is introduced in [2]. (see my overview).

Let us consider $\gamma(\theta) = \min\{\alpha(\theta), \beta(\theta)\}$ with α and β smooth. Generally, γ has points at which $\gamma'(\theta)$ jumps, and these points correspond to the points γ changes the value from α to β or vice versa. Therefore, for each θ_0 at which γ jumps, there must exist a geodesic which smoothly connects $(R_\alpha(\theta_0), \phi_\alpha(\theta_0))$ and $(R_\beta(\theta_0), \phi_\beta(\theta_0))$, where $(R_\alpha(\theta), \phi_\alpha(\theta))$ is a curve corresponding to α and $(R_\beta(\theta), \phi_\beta(\theta))$ to β. This means that there is a convex curve which covers $(R_\alpha(\theta), \phi_\alpha(\theta))$ and $(R_\beta(\theta), \phi_\beta(\theta))$ (see Figure 6 in [1]). In the shrinking limit of $(R_\alpha(\theta), \phi_\alpha(\theta)) \to A$ and $(R_\beta(\theta), \phi_\beta(\theta)) \to B$, the convex curve becomes the identical two geodesic going from A to B, by which we can verify (7).

The extension to conical defect AdS_3 and BTZ are also discussed in the paper.

2) There is no specific terminology yet.

3) Note that this is the distance defined on a time slice, so the distance is always positive.

References

[1] B. Czech and L. Lamprou, *Holographic definition of points and distances*, *Phys. Rev. D* **90** (2014) 106005 [1409.4473].

[2] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [1310.4204].

[3] P. Burda, R. Gregory and A. Jain, *Holographic reconstruction of bubble spacetimes*, *Phys. Rev. D* **99** (2019) 026003 [1804.05202].