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Integral Geometry and Holography [1]

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Hole-ography, which originates in [2], is known as a method to reconstruct the bulk geometry in AdS/CFT correspondence. In ref. [1], they reinterpreted hole-ographic reconstruction formulas by a mathematic tool called *kinematic space*, which is the space of oriented geodesics. By that, the geometric information of a time slice in AdS_3 can be considered to be encoded in the kinematic space, which is intrinsically a property of the boundary theory.

As a warm-up, let us start with Euclidean $\mathbb{R}^2$. All geodesics, lines, in this space are written in the form $x\cos\theta + y\sin\theta - p = 0$.[1)] The kinematic space is the space of all oriented geodesics, so we have to define the orientation of each line. If we continuously change the line from (θ, p) to $(\theta+\pi, -p)$, we see that the line comes back to the original one, with the formal endpoints at infinity exchanged. Therefore, we regard $(\theta+\pi, p)$ as the inversely oriented version of (θ, p). Then the kinematic space is characterized by the pair $(\theta, p) \in \mathbb{R}^2$ with the identification $\theta \sim \theta + 2\pi$.

We can measure the length of an arbitrary curve γ by calculating a volume form defined on the kinematic space $\mathcal{K}$. Crofton formula gives it as

$$\text{length of } \gamma = \frac{1}{4}\int_{\mathcal{K}} \omega\, n_\gamma(\theta, p), \qquad \omega = \mathrm{d}\theta \wedge \mathrm{d}p, \tag{1}$$

where $n_\gamma(\theta, p)$ is the number of times the line (θ, p) intersects with γ. In the integral, p runs over $\mathcal{R}$ and θ on $[0, 2\pi]$.

They applied the above story to the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$. We consider a time slice Σ on static asymptotically AdS_3 spacetime M in global patch, and a curve γ on Σ; the metric on ∂M is $\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2\mathrm{d}\theta^2$. Let $[u, v]$ be an interval of θ on $\partial\Sigma = \Sigma \cap \partial M$, and $S(u, v)$ be the length of the bulk geodesic connecting u and v. Note that interval $[u, v]$ corresponds to the identical two geodesics having different orientations. Their conjecture is that, if we replace ω in (1) with

$$\omega = \frac{\partial^2 S(u, v)}{\partial u \partial v}\mathrm{d}u \wedge \mathrm{d}v, \tag{2}$$

then eq.(1) holds on Σ.

The formula is correct at least for any convex closed curve γ. To show this, we define $\ell(u)(>0)$ such that the geodesic corresponding to $[u, u+\ell(u)]$ is tangent to γ. Then, we see $n_\gamma(u, v) = 0$ for $v < u + \ell(u)$ and $n_\gamma(u, v) = 2$ for $v > u + \ell(u)$, and hence[2)]

$$\text{length of } \gamma = \frac{2\cdot 2}{4}\int_0^{2\pi}\mathrm{d}u \int_{u+\ell(u)}^{u+\pi}\mathrm{d}v\, \frac{\partial^2 S}{\partial u \partial v} = -\int_0^{2\pi}\mathrm{d}u\, \left.\frac{\partial S(u, v)}{\partial u}\right|_{v=u+\ell(u)}. \tag{3}$$

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1) Note that $|p|$ is the distance from the origin to the line.

2) The other factor 2 comes from the two types of the orientation.

Here we have used $(\partial_u S)|_{v=u\pm\pi} = 0$, which holds because the geodesic length is maximum when the interval length is π.[3)] Eq.(3) is exactly the differential entropy formula (my overview) under Ryu-Takayanagi formula, and gives the right length of γ according to [3].

From the strong subadditivity, we can show that $\partial_u \partial_v S$ is always positive. The strong subadditivity is an inequality given by

$$S(AB) + S(BC) - S(B) - S(ABC) \geq 0. \tag{4}$$

If we choose A, B and C as

$$A = [u - \mathrm{d}u], \qquad B = [u, v], \qquad C = [v, v + \mathrm{d}v] \tag{5}$$

with $\mathrm{d}u, \mathrm{d}v > 0$, then we have

$$S(u - \mathrm{d}u, v) + S(u, v + \mathrm{d}v) - S(u, v) - S(u - \mathrm{d}u, v + \mathrm{d}v) = \frac{\partial^2 S(u, v)}{\partial u \partial v} \mathrm{d}u \mathrm{d}v \geq 0. \tag{6}$$

Next, let us discuss how points on Σ are interpreted in $\mathcal{K}$. As considered in [4] (my overview), using shrinking limit of closed curves is useful. If a closed curve γ shrinks up to a point A, then for each u, only one v makes geodesic (u, v) intersect with $\gamma = A$. Let $v_A(u)$ denote the critical v, and the curve $v = v_A(u)$ on $\mathcal{K}$ called "point-curve." Ref. [4] have already given a conjecture to define point-curves from the boundary entanglement entropy, and in ref. [1], the extension of it under the assumption that Σ is Riemannian.

Finally, the distance between any two points on Σ is also given in terms of $\mathcal{K}$. Let $v_A(u)$ and $v_B(u)$ be point-curves on $\mathcal{K}$, and γ_{AB} be the geodesic from A to B. As depicted in fig.10 in [1], if v satisfies

$$\min\{v_A(u), v_B(u)\} \leq v \leq \max\{v_A(u), v_B(u)\} \tag{7}$$

for fixed u, then geodesic (u, v) intersects with γ_{AB} once, and does not intersect otherwise.[4)] Thus, from (1), we conclude,

$$\text{geodesic length between } A \text{ and } B = \frac{2}{4} \int_0^{2\pi} \mathrm{d}u \int_{\text{eq.(7)}} \mathrm{d}v \, \frac{\partial^2 S(u, v)}{\partial u \partial v}. \tag{8}$$

References

[1] B. Czech, L. Lamprou, S. McCandlish and J. Sully, *Integral Geometry and Holography*, *JHEP* **10** (2015) 175 [1505.05515].

[2] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [1310.4204].

[3)] Comment: this statement is not true when the bulk has a hole like a black hole, thus the conjecture must be modified for holographic theories having such bulks.

[4)] If we choose γ_{AB} as the other curves connecting A and B, there is a curve which intersects with γ_{AB} twice.

[3] M. Headrick, R.C. Myers and J. Wien, *Holographic Holes and Differential Entropy*, *JHEP* **10** (2014) 149 [1408.4770].

[4] B. Czech and L. Lamprou, *Holographic definition of points and distances*, *Phys. Rev. D* **90** (2014) 106005 [1409.4473].