

# Nuts and Bolts for Creating Space [1]

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They introduced the method to reconstruct a time slice surface of  $\text{AdS}_3$  from the differential entropy of  $\text{CFT}_2$  [2] (see [my overview](#)). Specifically they found the method to detect the bulk points and to measure the distances between two points on the time slice.

## The detection of points

Let  $\alpha(\theta)$  be a boundary function introduced in [2], which is also seen in <sup>1)</sup> Any bulk closed curve  $R = R(\phi)$  corresponds to a certain boundary function  $\alpha(\theta)$ . In the limit of the bulk curve shrinking, the curve becomes a point.

They used this shrinking limit to find the way to detect bulk points from the boundary. First, Gauss-Bonnet theorem states that the following relation holds in negatively curved spaces:

$$\oint_{\partial A} d\tau \sqrt{h} K = 2\pi - \int_A dA R \geq 2\pi. \quad (1)$$

Here  $\partial A$  means the closed curve  $R = R(\phi)$ ,  $A$  is the area enclosed by the curve,  $d\tau \sqrt{h}$  is the line element along the curve,  $K$  is the extrinsic curvature, and  $dA$  is the areal element. In the shrinking limit, we obtain

$$\lim_{A \rightarrow \text{point}} \oint_{\partial A} d\tau \sqrt{h} K = 2\pi. \quad (2)$$

In addition, the integrand of l.h.s. in (1) can be expressed by  $\alpha(\theta)$  as

$$d\tau \sqrt{h} K = \frac{d\theta \sqrt{1 - \alpha'(\theta)^2}}{\sin \alpha(\theta)}. \quad (3)$$

From these facts, the boundary function  $\alpha(\theta)$  corresponds to a certain bulk point if and only if it extremizes the functional

$$I[\alpha] = \int_0^{2\pi} d\theta \frac{\sqrt{1 - \alpha'(\theta)^2}}{\sin \alpha(\theta)}, \quad (4)$$

of which the Euler-Lagrange equation is a second order differential equation. Therefore,  $\alpha(\theta)$  has two constants, which we regard as a bulk point on the time slice. Noting the relation

$$\frac{1}{\sin^2 \alpha} = -\frac{2G}{L} \frac{d^2 S(\alpha)}{d\alpha^2}, \quad (5)$$

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<sup>1)</sup> The notation adopted here is checked in my overview.

the action (4) can be rewritten as

$$I[\alpha] = \int_0^{2\pi} d\theta \sqrt{-\frac{d^2 S(\alpha)}{d\alpha^2} \Big|_{\alpha=\alpha(\theta)}} (1 - \alpha'(\theta)^2) . \quad (6)$$

They conjectured that this action also holds for other cases having the rotational symmetry. In other words, the collection of functions  $\alpha(\theta)$  extremizing (6) is the copy of the bulk points. The elements of the collection are called “point function” here.<sup>2)</sup> A counterexample is discussed in [3] (see [my overview](#)).

### The measurement of distances

Next, let us see the holographic distances introduced by them, which is defined in the set of point functions  $\mathcal{P}$ . Now the differential entropy  $E[\alpha]$  is used to define the distance.

Let us define  $\gamma_{AB}(\theta) := \min\{\alpha_A(\theta), \alpha_B(\theta)\}$  for  $\alpha_A, \alpha_B \in \mathcal{P}$  ( $\gamma$  is defined as a point-wise minimum among  $\alpha_A$  and  $\alpha_B$ ). The functional  $d(A, B)$  defined as follows is the holographic distance between  $A$  and  $B$  measured in the units of  $4G$ :

$$d(A, B) := \frac{1}{2} E[\gamma]. \quad (7)$$

It is shown that this functional satisfies the axioms of a distance function.<sup>3)</sup>

In the remaining, we see why  $d(A, B)$  can be regarded as the distance on the time slice. Let us consider that a boundary function  $\alpha(\theta)$  has a point  $\theta_k$  at which  $\alpha'(\theta)$  jumps. From the corollary shown in [2], we have

$$\begin{aligned} E[\alpha] &= \frac{1}{2} \int_0^{\theta_k-0} d\theta \frac{dS(\alpha)}{d\alpha} \Big|_{\alpha=\alpha(\theta)} + \frac{1}{2} \int_{\theta_k+0}^{2\pi} d\theta \frac{dS(\alpha)}{d\alpha} \Big|_{\alpha=\alpha(\theta)} \\ &= \frac{(\text{left length}) + (\text{right length}) + (\text{geodesic length in } \phi(\theta_k-0) \leq \phi \leq \phi(\theta_k+0))}{4G}. \end{aligned} \quad (8)$$

This means that the bulk curve  $(R, \phi) = (R(\theta), \phi(\theta))$  corresponding to  $\alpha(\theta)$  has a jump at  $\theta_k$ , and the following holds with the curve compensated by the geodesic:

$$E[\alpha] = \frac{\text{length}}{4G}. \quad (9)$$

The relation is introduced in [2]. (see ).

Let us consider  $\gamma(\theta) = \min\{\alpha(\theta), \beta(\theta)\}$  with  $\alpha$  and  $\beta$  smooth. Generally,  $\gamma$  has points at which  $\gamma'(\theta)$  jumps, and these points correspond to the points  $\gamma$  changes the value from  $\alpha$  to  $\beta$  or vice versa. Therefore, for each  $\theta_0$  at which  $\gamma$  jumps, there must exist a geodesic which smoothly connects  $(R_\alpha(\theta_0), \phi_\alpha(\theta_0))$  and  $(R_\beta(\theta_0), \phi_\beta(\theta_0))$ , where  $(R_\alpha(\theta), \phi_\alpha(\theta))$  is a curve corresponding to  $\alpha$  and  $(R_\beta(\theta), \phi_\beta(\theta))$  to  $\beta$ . This means that there is a convex curve which covers  $(R_\alpha(\theta), \phi_\alpha(\theta))$  and  $(R_\beta(\theta), \phi_\beta(\theta))$  (see Figure 6 in [1]). In the shrinking limit of  $(R_\alpha(\theta), \phi_\alpha(\theta)) \rightarrow A$  and  $(R_\beta(\theta), \phi_\beta(\theta)) \rightarrow B$ , the convex curve becomes the identical two geodesic going from  $A$  to  $B$ , by which we can verify (7).

The extension to conical defect  $\text{AdS}_3$  and BTZ are also discussed in the paper.

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<sup>2)</sup>There is no specific terminology yet.

<sup>3)</sup> Note that this is the distance defined on a time slice, so the distance is always positive.

## References

- [1] B. Czech and L. Lamprou, *Holographic definition of points and distances*, *Phys. Rev. D* **90** (2014) 106005 [1409.4473].
- [2] V. Balasubramanian, B.D. Chowdhury, B. Czech, J. de Boer and M.P. Heller, *Bulk curves from boundary data in holography*, *Phys. Rev. D* **89** (2014) 086004 [1310.4204].
- [3] P. Burda, R. Gregory and A. Jain, *Holographic reconstruction of bubble spacetimes*, *Phys. Rev. D* **99** (2019) 026003 [1804.05202].