[Overview*] last updated: July 4, 2022

Holographic Holes and Differential Entropy [1]

Matthew Headrick, Robert C. Myers, Jason Wien

Hole-ography is a method to reconstruct a curve on $t=$ const and its length in pure AdS_3 from the differential entropy, which is a quantity defined by a family of the entanglement entropy of CFT_2 [2] (see my overview). Here t is the time of the static coordinate. In [1], they showed that the hole-ography can be also used for more generic bulk geometries subject to some assumptions, with curves not restricted to be on $t=$ const.

A spacelike geodesic was characterized by an θ-interval, $[\theta-\alpha,\theta+\alpha]$, in [2]. However, if we want to describe curves not limited to be on $t=$ const, θ-intervals are not useful. Thus we might better to express a geodesic by its boundary endpoints: $\gamma_L(\lambda),\gamma_R(\lambda)$. Here λ is a parameter labeling a family of the endpoint pairs to specify a spatial geodesic, and the family creates a spacelike curve in the bulk, as its envelop. We call the bulk curve $c_\gamma=c_\gamma^M(\lambda)$, where M expresses the bulk coordinates and the geodesic specified by $\gamma_L(\lambda)$ and $\gamma_R(\lambda)$ is tangent at $c_\gamma^M(\lambda)$.

The differential entropy is defined as[1)]

$$E=\int \mathrm{d}\lambda\ \left.\frac{\partial S(\gamma_L(\lambda),\gamma_R(\lambda'))}{\partial\lambda'}\right|_{\lambda'=\lambda}, \tag{1}$$

where $S(\gamma_L,\gamma_R)$ is the length of the geodesic from γ_L to γ_R, which is dual to the boundary entanglement entropy, or more precisely the entwinement [3]. The following statement was shown by them: *given that the pair $(\gamma_L(\lambda),\gamma_R(\lambda))$ is periodic and smooth, then the length of c_γ is equal to E.*[2)]

Let us follow the proof. Length S in (1) is expressed as

$$S(\gamma_L(\lambda),\gamma_R(\lambda))=\int_{s_L}^{s_R}\mathrm{d}s\,\sqrt{(\dot{x}(s;\lambda),\dot{x}(s;\lambda))}\qquad (x(s_{L,R})=\gamma_{L,R}(\lambda)), \tag{2}$$

where $(\ ,\ )$ is the inner product defined by the bulk metric, $x(s,\lambda)$ is the geodesic from $\gamma_L(\lambda)$ to $\gamma_R(\lambda)$, and $\dot{}:=\partial/\partial s$. Since S is of the ordinary action form,[3)] we can use the knowledge of analytical mechanics. The integrand in (1) is rewritten as

$$\begin{aligned}\left.\frac{\partial S(\gamma_L(\lambda),\gamma_R(\lambda'))}{\partial\lambda'}\right|_{\lambda'=\lambda}&=\gamma_R^{\mu}{}'(\lambda)\frac{\partial S}{\partial\gamma_R^{\mu}}(\gamma_L(\lambda),\gamma_R(\lambda))\\&=\gamma_R^{M}{}'(\lambda)\frac{\partial S}{\partial\gamma_R^{M}}(\gamma_L(\lambda),\gamma_R(\lambda))\\&=\frac{\partial x^M(s_R;\lambda)}{\partial\lambda}p_M(x(s_R;\lambda),\dot{x}(s_R;\lambda)),\end{aligned} \tag{3}$$

* Written by Daichi Takeda (takedai.gauge@gmail.com)

1) This definition is equivalent to that of [2] when geodesics are on $t=$ const, as explained in [1].

2) Great attention should be paid to that the periodicity is assumed. For example, BTZ black hole unfortunately breaks this assumption.

3) The reparameterization-invariance of s and that it consists of up to the first derivative of the position.

where $p_M(x(s),\dot{x}(s))$ is the conjugate momentum of off-shell position $x^M(s)$. (If those are accompanied by "$;\lambda$", then they take the on-shell value defined above.) In the last equality, we have used the knowledge that the derivative of the on-shell action with the final position is the conjugate momentum at the position.

On the other hand, the length of c_γ, which we call L, is given as

$$L = \oint \mathrm{d}\lambda \sqrt{(c'_\gamma(\lambda), c'_\gamma(\lambda))}. \tag{4}$$

This is of the off-shell version of (2). It is easy to show that equation

$$\sqrt{(\dot{y}(s),\dot{y}(s))} = \dot{y}^M(s) p_M(y(s),\dot{y}(s)) \tag{5}$$

holds in general (even in off-shell),[4] and hence,

$$L = \oint \mathrm{d}\lambda\, c_\gamma^M{}'(\lambda) p_M(c_\gamma(\lambda), c'_\gamma(\lambda)). \tag{6}$$

As c_γ is the envelop of $\{\dot{x}(s;\lambda)\}$, at the tangent point $s = s_c$, we have

$$x(s_c;\lambda) = c_\gamma(\lambda), \qquad \exists\alpha(\lambda) > 0,\ \alpha(\lambda)\dot{x}(s_c;\lambda) = c'_\gamma(\lambda). \tag{7}$$

Note that s_c can be chosen to be independent of λ by reparameterizing s. Then we obtain

$$\begin{aligned} L &= \oint \mathrm{d}\lambda\, \frac{\partial x^M(s_c;\lambda)}{\partial\lambda} p_M(x(s_c;\lambda), \alpha(\lambda)\dot{x}(s_c;\lambda)) \\ &= \oint \mathrm{d}\lambda\, \frac{\partial x^M(s_c;\lambda)}{\partial\lambda} p_M(x(s_c;\lambda), \dot{x}(s_c;\lambda)), \end{aligned} \tag{8}$$

where in the last equality, we have used the property of the momentum that $p(y(s),\alpha\dot{y}(s)) = p(y(s),\dot{y}(s))$ which follows from (5).

Therefore, the remaining task to accomplish is to show that

$$E - L = \oint \mathrm{d}\lambda\ \frac{\partial x^M(s;\lambda)}{\partial\lambda} p_M(x(s;\lambda),\dot{x}(s;\lambda))\bigg|_{x_c}^{s_R} \tag{9}$$

must vanish. Using the relation between the on-shell action and the momenta of endpoints again, we can rewrite the above as

$$E - L = \oint \mathrm{d}\lambda\, \frac{\partial}{\partial\lambda} S(x(s_c;\lambda), x(s_R;\lambda)). \tag{10}$$

Since $x(s;\lambda)$ is also periodic about λ by the assumption, we see this vanish.

[4] This generally follows from the reparameterization-invariance, but can also be shown directly of course.

References

[1] M. Headrick, R.C. Myers and J. Wien, *Holographic Holes and Differential Entropy*, *JHEP* **10** (2014) 149 [1408.4770].

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[3] V. Balasubramanian, B.D. Chowdhury, B. Czech and J. de Boer, *Entwinement and the emergence of spacetime*, *JHEP* **01** (2015) 048 [1406.5859].