1 Count the number of possibilities to put k balls into n boxes

1.1 Balls distinguishable, boxes distinguishalbe

1.1.1 No restriction

Assume the balls and boxes are numbered. Then a possible distribution of balls corresponds to a function $f : [[1, k]] \to [[1, n]]$ - alternatively a k-list - where $f(i)$ is the number of the box where ball number i is put.

So e.g. the list $(1, 3, 4, 1, 3, 3, 1)$ $(k = 7,\ n = 4)$ corresponds to:

box 1	box 2	box 3	box 4
1,4,7	.	2,5,6	3

Solution: n^k.

1.1.2 At least one ball into each box

This means that the function $f : [[1, k]] \to [[1, n]]$ (as above) is surjective. Hence:

Condition: $k \geq n$. Solution: $n!S(k, n)$ (with the Stirling numbers of the second kind).

1.1.3 At most one ball into each box

This means that the function $f : [[1, k]] \to [[1, n]]$ (as above) is injective. Hence:

Condition: $k \leq n$. Solution: $n(n - 1) \cdots (n - k + 1)$.

1.2 Balls distinguishable, boxes not distinguishalbe

1.2.1 No restriction

Choose a j-partition of $[[1, k]]$ to fill j boxes. E.g. for $k = 10$, $n = 5$:

Partition: $\{1, 4, 5, 6\}, \{2, 3, 10\}, \{8\}, \{7, 9\}$

2 3 10	8	1 4 5 6	.	7 9

Solution: $\sum_{j=1}^{n} S(k, j)$

1.2.2 At least one ball into each box

So there is no empty box allowed. We need a n-partition of $[[1, k]]$.

Condition: $k \geq n$. Solution: $S(k, n)$

1.2.3 At most one ball into each box

If $k \leq n$ then put each ball into one box. Solution: 1.

1.3 Balls not distinguishable, boxes distinguishalbe

1.3.1 No restriction

The distribution may be represented as a k-multiset from the n-set of boxes: If box i appears j-times it gets j balls. E.g. for $k = 10$ and $n = 4$:

Multiset: $\{1, 1, 1, 1, 2, 3, 3, 3, 4, 4\}$

box 1	box 2	box 3	box 4
oooo	o	ooo	oo

Note: The picture suggests a direct interpretation of the solution: There are $n+k-1$ places (balls or box-delimiters) and we choose k to take balls. In particular two adjacent delimiters indicate an empty box.

Solution: $\binom{n+k-1}{k}$.

1.3.2 At least one ball into each box

Align all balls and choose from the $k-1$ spaces between the balls $n-1$ to take a box delimiter. This problem is also related to the problem of number-partitions: In how many ways can we represent k as

$$k = \sum_{i=1}^{n} m_i, \quad m_i \in \mathbb{N}$$

if the order matters. Here m_i is the number of balls which go into box i.

Condition: $k \geq n$. Solution: $\binom{k-1}{n-1}$

1.3.3 At most one ball into each box

Choose a subset of the boxes to take 1 ball.

Condition: $k \leq n$. Solution: $\binom{n}{k}$.

1.4 Balls not distinguishable, boxes not distinguishalbe

1.4.1 No restriction

For this problem we don't have a closed form solution. It is related to the number representation:

$$k = \sum_{i=1}^{n} m_i, \quad m_i \in \mathbb{N} \cup \{0\}$$

where the order doesn't matter.

1.4.2 At least one ball into each box

As above except that $m_i \in \mathbb{N}$, condition: $k \geq n$

1.4.3 At most one ball into each box

There is just 1 solution if $k \leq n$.

$\boxed{\text{BallBox}}$