1 Count the number of possibilities to put k balls into n boxes

1.1 Balls distinguishable, boxes distinguishable

1.1.1 No restriction

Assume the balls and boxes are numbered. Then a possible distribution of balls corresponds to a function $f:[[1,k]] \to [[1,n]]$ - alternatively a k-list - where f(i) is the number of the box where ball number i is put.

So e.g. the list (1, 3, 4, 1, 3, 3, 1) (k = 7, n = 4) corresponds to:

box 1	box 2	box 3	box 4
1,4,7		2,5,6	3

Solution: n^k .

1.1.2 At least one ball into each box

This means that the function $f:[[1,k]] \to [[1,n]]$ (as above) is surjective. Hence: Condition: $k \ge n$. Solution: n!S(k,n) (with the Stirling numbers of the second kind).

1.1.3 At most one ball into each box

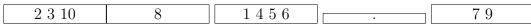
This means that the function $f:[[1,k]] \to [[1,n]]$ (as above) is injective. Hence: Condition: $k \le n$. Solution: $n(n-1)\cdots(n-k+1)$.

1.2 Balls distinguishable, boxes not distinguishable

1.2.1 No restriction

Choose a j-partition of [[1, k]] to fill j boxes. E.g. for k = 10, n = 5:

Partition: $\{1, 4, 5, 6\}, \{2, 3, 10\}, \{8\}, \{7, 9\}$



Solution: $\sum_{j=1}^{n} S(k,j)$

1.2.2 At least one ball into each box

So there is no empty box allowed. We need a n-partition of [[1, k]].

Condition: $k \geq n$. Solution: S(k, n)

1.2.3 At most one ball into each box

If $k \leq n$ then put each ball into one box. Solution: 1.

1.3 Balls not distinguishable, boxes distinguishable

1.3.1 No restriction

The distribution may be represented as a k-multiset from the n-set of boxes: If box i appears j-times it gets j balls. E.g. for k = 10 and n = 4:

Multiset: $\{1, 1, 1, 1, 2, 3, 3, 3, 4, 4\}$

box 1	box 2	box 3	box 4
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Note: The picture suggests a direct interpretation of the solution: There are n+k-1 places (balls or box-delimiters) and we choose k to take balls. In particular two adjacent delimiters indicate an empty box.

Solution: $\binom{n+k-1}{k}$.

1.3.2 At least one ball into each box

Align all balls and choose from the k-1 spaces between the balls n-1 to take a box delimiter. This problem is also related to the problem of number-partitions: In how many ways can we represent k as

$$k = \sum_{i=1}^{n} m_i, \quad m_i \in \mathbb{N}$$

if the order matters. Here m_i is the number of balls which go into box i.

Condition: $k \ge n$. Solution: $\binom{k-1}{n-1}$

1.3.3 At most one ball into each box

Choose a subset of the boxes to take 1 ball.

Condition: $k \leq n$. Solution: $\binom{n}{k}$.

1.4 Balls not distinguishable, boxes not distinguishable

1.4.1 No restriction

For this problem we don't have a closed form solution. It is related to the number representation:

$$k = \sum_{i=1}^{n} m_i, \quad m_i \in \mathbb{N} \cup \{0\}$$

where the order doesn't matter.

1.4.2 At least one ball into each box

As above except that $m_i \in \mathbb{N}$, condition: $k \geq n$

1.4.3 At most one ball into each box

There is just 1 solution if $k \leq n$.

BallBox