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Chapter 1

Quantum circuits

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Operations on a qubit must preserve the norm of the qubit, i.e., given an operation O on a single qubit and two qubits $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\psi'\rangle = O|\psi\rangle = a'|0\rangle + b'|1\rangle$, the normalization conditions

$$a^2 + b^2 = a'^2 + b'^2 = 1 \tag{1.1}$$

must hold. For this reason, operators on single qubits are 2x2 unitary matrices.

The most common single qubit operations are represented by the Pauli matrices. The Pauli matrices are shown below.

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ; Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.2}$$

Three other matrices that are commonly used in quantum computing are the Hadamard (H), $\pi/8$ (T) and phase (S) gates. These are shown below.

$$H \equiv \frac{X+Z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} ; T \equiv \begin{bmatrix} 1 & 0 \\ 0 & exp(i\pi/4) \end{bmatrix} ; S \equiv T^2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{1.3}$$

In Appendix A, we have shown how to exponentiate matrices. Using these results, we now introduce three additional unitary matrices known as *rotation matrices* corresponding to the Pauli matrices. These are shown below.

$$R_x(\theta) \equiv \exp(-i\theta X/2) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{1.4}$$

$$R_y(\theta) \equiv \exp(-i\theta Y/2) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{1.5}$$

$$R_z(\theta) \equiv \exp(-i\theta Z/2) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) Z = \begin{bmatrix} \exp(-i\theta/2) & 0 \\ 0 & \exp(i\theta/2) \end{bmatrix} \tag{1.6}$$

In general, the rotation by θ about an axis defined by the real unit vector $\hat{n} = (n_x, n_y, n_z)$ is applied using the following matrix.

$$R_{\hat{n}}(\theta) \equiv \exp\left(-i\theta\frac{n_xX + n_yY + n_zZ}{2}\right) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(n_xX + n_yY + n_zZ) \tag{1.7}$$

Without proof, we present a very useful way of representing unitary operator matrices below. Any unitary operator U can be represented by a matrix which is a product of rotations in the y and z axes plus a global phase. Interested readers are

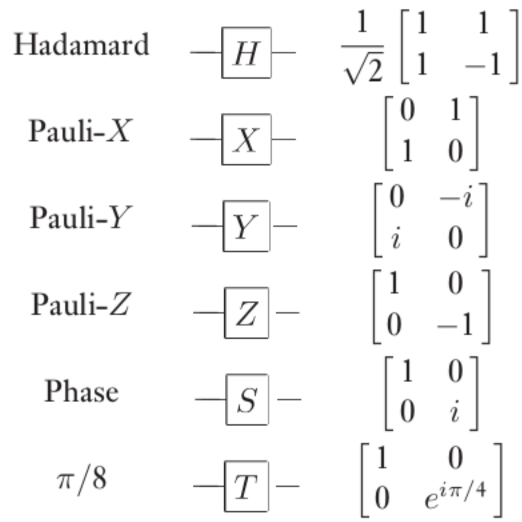

Figure 1.1: Single-qubit gates and their circuit notations

pointed to reference [6] for a detailed proof.

$$U = \exp\left(i\alpha\right) R_z(\beta) R_y(\gamma) R_z(\delta) \tag{1.8}$$

Yet another representation of a unitary operator matrix which follows from above is shown here without proof. Interested readers are pointed to reference [6] for a detailed proof.

$$U = \exp\left(i\alpha\right) AXBXC \tag{1.9}$$

In the above representation, A, B and C are unitary themselves and $ABC = I$. These representations will be important as we define controlled operations using multiple qubits in the next section. One additional set of identities that we need to keep in mind is the following. The proofs for these are straightforward substitutions and are not shown here.

$$HXH = Z \ ; \ HYH = -Y \ ; \ HZH = X \tag{1.10}$$

Figure 1.1 summarizes all the single qubit gates and shows their circuit notations.

1.4 Multiple qubits and controlled operations

So far, we have talked about operations on single qubits. We now discuss a class of families on multiple qubits known as *controlled operations.* In particular, we will discuss two-qubit controlled operations. These discussions can intuitively be generalized for larger numbers of qubits.

The simplest controlled operation is the controlled version of the classical NOT gate, known as the CNOT gate. It operates on two qubits known as the *control qubit* and the *target qubit.* Its operation is described as follows: if the control qubit is set (to $|1\rangle$), then the target qubit is inverted (from $|0\rangle$ to $|1\rangle$ and vice versa). The shorthand notation for the CNOT gate is given below.

$$\text{CNOT: } |control\rangle|target\rangle \rightarrow |control\rangle|control \oplus target\rangle \tag{1.11}$$

The matrix representation for the CNOT operator can be determined as follows from the above shorthand notation.

$$\text{CNOT: } |0\rangle|0\rangle \rightarrow |0\rangle|0\rangle \tag{1.12}$$

$$\text{CNOT: } |0\rangle|1\rangle \rightarrow |0\rangle|1\rangle \tag{1.13}$$

$$\text{CNOT: } |1\rangle|0\rangle \rightarrow |1\rangle|1\rangle \tag{1.14}$$

$$\text{CNOT: } |1\rangle|1\rangle \rightarrow |1\rangle|0\rangle \tag{1.15}$$

In the computational basis $\{|0\rangle, |1\rangle\}$, the CNOT gate acts on two qubits and is therefore 4x4. The columns of the CNOT matrix are the outputs of $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$ and $|1\rangle|1\rangle$ respectively. Hence,

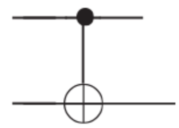

Figure 1.2: Circuit notation for a CNOT gate

$$\text{CNOT} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.16}$$

The circuit notation for the CNOT operator is shown in figure 1.2. In general, a controlled-U operator can be represented by a matrix by noting how it affects combinations of input qubits. The shorthand notation is shown below.

$$\text{controlled-U: } |control\rangle|target\rangle \rightarrow |control\rangle U^{control}|target\rangle \tag{1.17}$$

This general notation leads to a general form for the matrix of a controlled-U operator, which is shown below.

$$\text{controlled-U} \equiv \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \tag{1.18}$$

Note that the above matrix is 4x4 because both I and U are 2x2. Also note that the CNOT operator has the same matrix representation as a controlled-X operator. Notationally,

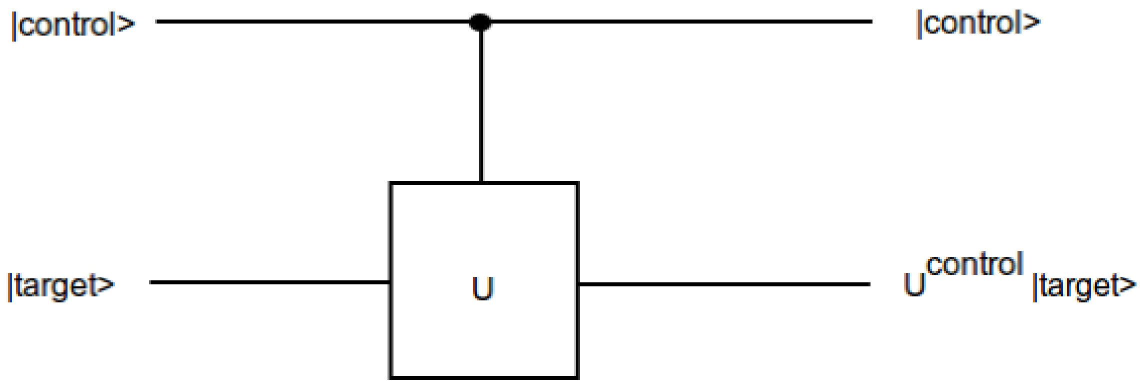

Figure 1.3: A controlled-U operator. Note the control and target qubits

$$\text{CNOT} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \tag{1.19}$$

The commonly used circuit notation for a controlled-U operation is shown in figure 1.3.

Recall from equation 1.9 that we can represent unitary operators by elementary operators. Therefore, for a controlled-U gate, we have the following: when the control is disabled, the identity matrix is applied to the target qubit. Otherwise, $U = \exp(i\alpha) AXBXC$ is applied to the target qubit. Also recalling the constraints on A, B and C such that $ABC = I$, this implies that the control qubit affects the X operators, turning them into CNOT gates.

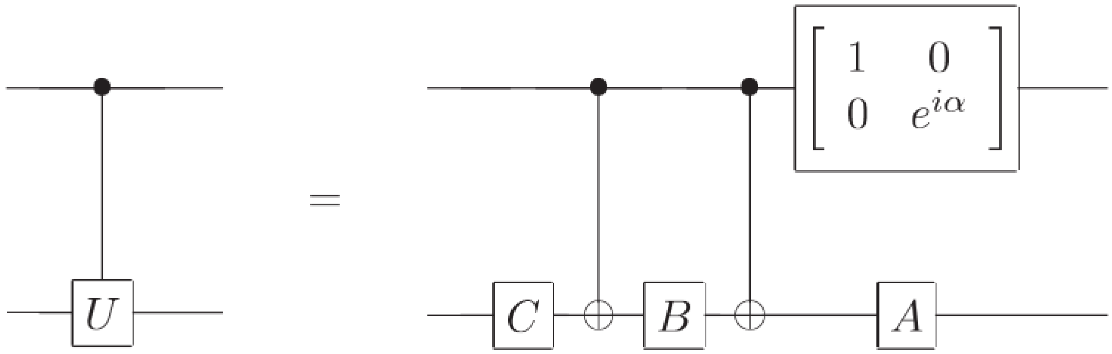

Figure 1.4: A controlled-U gate decomposed into elementary single-qubit gates

Also note the following.

$$\exp(i\alpha): \ |0\rangle|0\rangle \rightarrow |0\rangle|0\rangle \tag{1.20}$$

$$\exp(i\alpha): \ |0\rangle|1\rangle \rightarrow |0\rangle|1\rangle \tag{1.21}$$

$$\exp(i\alpha): \ |1\rangle|0\rangle \rightarrow |1\rangle\exp(i\alpha)(|0\rangle) = \exp(i\alpha)(|1\rangle)|0\rangle \tag{1.22}$$

$$\exp(i\alpha): \ |1\rangle|1\rangle \rightarrow |1\rangle\exp(i\alpha)(|1\rangle) = \exp(i\alpha)(|1\rangle)|1\rangle \tag{1.23}$$

From the above equations, we observe that the effect of the global phase can be applied to the control or target qubits. Since it only applies when the control qubit is set, we will apply it there, giving the circuit diagram in figure 1.4.

The above discussion easily applies when there are multiple control and target qubits. Given a unitary operator U applied on n control qubits $\{x_1, x_2, \ldots, x_n\}$ and k target qubits $\{y_1, y_2, \ldots, y_k\}$, we have the following shorthand notation.

$$\text{controlled-U: } |x_1\rangle|x_2\rangle \ldots |x_{n-1}\rangle|x_n\rangle|y_1\rangle|y_2\rangle \ldots |y_{k-1}\rangle|y_k\rangle \rightarrow \tag{1.24}$$

$$|x_1\rangle|x_2\rangle \ldots |x_{n-1}\rangle|x_n\rangle U^{x_1 x_2 \ldots x_{n-1} x_n}|y_1\rangle|y_2\rangle \ldots |y_{k-1}\rangle|y_k\rangle \tag{1.25}$$

This shorthand notation is shown in circuit notation in figure 1.5.

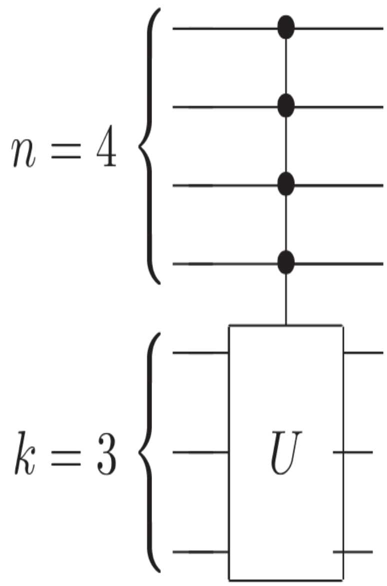

Figure 1.5: A multiple-control multiple-target controlled-U operator with 4 control and 3 target qubits

1.5 Summary

In this chapter, we have introduced the two-state quantum system used in quantum computation known as the qubit. We have indicated its wavefunction and used the Bloch sphere as a means of visualizing qubits. The Bloch sphere is particularly important when we consider single qubit gates. It helps us visualize the effects of the unitary operators that we normally encounter in matrix form.

We have also discussed the fundamental single-qubit gates – the Pauli gates, their derived rotation operators, the Hadamard, phase and $\pi/8$ gates.

After introducing single-qubit gates, we showed how they can be controlled using additional qubits to form multiple-qubit gates. We showed how to derive the matrices corresponding to these operators and derived general controlled operations in terms of elementary single-qubit gates. The method of control was shown to be applicable even when the number of control and target qubits increases.

For additional reading, including topics such as universal gates and operator approximations, the interested reader is referred to chapter 4 of [6].

Chapter 2

Quantum Mechanical Analysis of the Harmonic Oscillator

2.1 Motivation

This chapter will provide the necessary background for understanding optical implementations of quantum gates. We introduce relevant operators such as annihilators and creators based on a quantum mechanical description of the harmonic oscillator. The material in this chapter is based on analysis of the harmonic oscillator provided in [2] and [3]. Additional material concerned with annihilators and creators can be found in [5].

2.2 Creation and Annihilation Operators

In quantum mechanics, physical observables are represented by Hermitian operators that have real eigenvalues. We represent operators with boldface symbols throughout the chapter. The Hamiltonian of a one-dimensional harmonic oscillator is given below.

$$\boldsymbol{H} = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}K\boldsymbol{x}^2 \tag{2.1}$$

In addition, the commutator relationship $[\boldsymbol{x}, \boldsymbol{p}]$ relates the two physical observables in the above equation. The first term in the Hamiltonian is the kinetic energy, and the second is the potential energy. Thus, the Hamiltonian is the total energy of the system. The normalized position and momentum operators are

$$\boldsymbol{X} = \sqrt{\frac{K}{\hbar\omega_0}}\boldsymbol{x} \tag{2.2}$$

$$\boldsymbol{P} = \frac{\boldsymbol{p}}{\sqrt{m\hbar\omega_0}} \tag{2.3}$$

where $\omega_0 = \sqrt{\frac{K}{m}}$. The Hamiltonian expressed in terms of these normalized operators becomes

$$\boldsymbol{H} = \frac{\hbar\omega_0}{2}\left(\boldsymbol{P}^2 + \boldsymbol{X}^2\right) \tag{2.4}$$

with a new commutator relationship $[\boldsymbol{X}, \boldsymbol{P}] = j$. We introduce the non-Hermitian annihilator $\boldsymbol{a}$ and its adjoint $\boldsymbol{a}^\dagger$ as

$$\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(\boldsymbol{X} + j\boldsymbol{P}\right) \tag{2.5}$$

$$\boldsymbol{a}^\dagger = \frac{1}{\sqrt{2}}\left(\boldsymbol{X} - j\boldsymbol{P}\right) \tag{2.6}$$

with a commutator $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$. We are also interested in the anti-commutator $\{\boldsymbol{a}, \boldsymbol{a}^\dagger\}$ of the annihilator and its adjoint. Note that

$$\boldsymbol{a}\boldsymbol{a}^\dagger = \frac{1}{2}\left(\boldsymbol{X}^2 + \boldsymbol{P}^2 + 1\right) \tag{2.7}$$

$$\boldsymbol{a}^\dagger\boldsymbol{a} = \frac{1}{2}\left(\boldsymbol{X}^2 + \boldsymbol{P}^2 - 1\right) \tag{2.8}$$

Hence, $\{\boldsymbol{a}, \boldsymbol{a}^\dagger\} = \boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a} = \boldsymbol{P}^2 + \boldsymbol{X}^2$. The Hamiltonian can now be expressed in terms of this anti-commutator.

$$\boldsymbol{H} = \frac{\hbar\omega_0}{2}\left(\boldsymbol{P}^2 + \boldsymbol{X}^2\right) = \frac{\hbar\omega_0}{2}\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}^\dagger\right) \tag{2.9}$$

$$= \hbar\omega_0\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{1}{2}\right) \tag{2.10}$$

We introduce an additional Hermitian operator known as the number operator $\boldsymbol{N}$ such that

$$\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a} \tag{2.11}$$

that counts the number of energy quanta excited in the harmonic oscillator. The eigenvectors $|n\rangle$ of $\boldsymbol{N}$ are known as *Fock* states and the corresponding eigenvalues are denoted as N_n such that

$$\boldsymbol{N}|n\rangle = N_n|n\rangle \tag{2.12}$$

Since $\boldsymbol{N}$ is Hermitian, it has orthonormal eigenvectors and real eigenvalues. In other words, for two eigenvectors $|n\rangle, |m\rangle$ of $\boldsymbol{N}$,

$$\langle m|n\rangle = \delta_{mn} \tag{2.13}$$

The annihilation operator and its adjoint have special effects on the Fock states [3].

$$\boldsymbol{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{2.14}$$

$$\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.15}$$

We define $\boldsymbol{a}|0\rangle = 0$ for the ground state of the harmonic oscillator. Note that the application of $\boldsymbol{a}$ leads to one less quantum while that of $\boldsymbol{a}^\dagger$ leads to an additional quantum. For this reason, the two operators are usually called the annihilation and creation operators respectively. Starting from the ground state, we can reach a Fock state $|n\rangle$ by repeated application of the creation operator followed by normalization.

$$|n\rangle = \frac{\boldsymbol{a}^{\dagger}|n-1\rangle}{\sqrt{n}} = \frac{\left(\boldsymbol{a}^{\dagger}\right)^{2}|n-2\rangle}{\sqrt{n}\sqrt{n-1}} = \dots \tag{2.16}$$

$$= \frac{1}{\sqrt{(n)!}}\left(\boldsymbol{a}^{\dagger}\right)^{n}|0\rangle \tag{2.17}$$

Since the ground state has energy $\frac{\hbar\omega_0}{2}$ with each additional quantum contributing the same energy, the eigenvalues of the Hamiltonian of the harmonic oscillator are related to the Hamiltonian as

$$\boldsymbol{H}|n\rangle = E_n|n\rangle = \hbar\omega_0\left(n + \frac{1}{2}\right) \tag{2.18}$$

where E_n is the energy eigenvalue corresponding to the Fock state $|n\rangle$.

2.3 Matrix Elements of the Creation and Annihilation Operators

In this section, we will investigate the elements of the matrices that represent the annihilation and creation operators. From (2.5), we have

$$\boldsymbol{X} = \frac{1}{\sqrt{2}}\left(a^{\dagger} + a\right) \tag{2.19}$$

$$\boldsymbol{P} = \frac{j}{\sqrt{2}}\left(a^{\dagger} - a\right) \tag{2.20}$$

For Fock states $|m\rangle$ and $|n\rangle$, we now derive a series of relations that reveal the matrix elements of the annihilation and creation operators. Note that for an operator O,

$\langle m|O|n\rangle$ is the element O_{mn} of the matrix representation of O.

$$\langle m|\boldsymbol{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle = \sqrt{n}\delta_{m,n-1} \tag{2.21}$$

$$\langle m|\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle = \sqrt{n+1}\delta_{m,n+1} \tag{2.22}$$

$$\langle m|\boldsymbol{a}^\dagger\boldsymbol{a}|n\rangle = n\langle m|n\rangle = n\delta_{m,n} \tag{2.23}$$

$$\langle m|\boldsymbol{a}\boldsymbol{a}^\dagger|n\rangle = (n+1)\langle m|n\rangle = (n+1)\,\delta_{m,n} \tag{2.24}$$

$$\langle m|\boldsymbol{X}|n\rangle = \langle m|\frac{1}{\sqrt{2}}\left(a^\dagger + a\right)|n\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right) \tag{2.25}$$

$$\langle m|\boldsymbol{P}|n\rangle = \langle m|\frac{j}{\sqrt{2}}\left(a^\dagger - a\right)|n\rangle = \frac{j}{\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right) \tag{2.26}$$

$$\langle m|\boldsymbol{a}^2|n\rangle = \sqrt{n}\langle m|\boldsymbol{a}|n-1\rangle = \sqrt{n(n-1)}\langle m|n-2\rangle = \sqrt{n(n-1)}\delta_{m,n-2} \tag{2.27}$$

$$\langle m|\left(\boldsymbol{a}^\dagger\right)^2|n\rangle = \sqrt{n+1}\langle m|\boldsymbol{a}^\dagger|n+1\rangle = \sqrt{(n+1)(n+2)}\langle m|n+2\rangle \tag{2.28}$$

$$= \sqrt{(n+1)(n+2)}\delta_{m,n+2} \tag{2.29}$$

$$\langle m|\boldsymbol{X}^2|n\rangle = \langle m|\frac{1}{2}\left(\left(\boldsymbol{a}^\dagger\right)^2 + \boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^2\right)|n\rangle \tag{2.30}$$

$$= \frac{1}{2}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} + n\delta_{m,n} + (n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2}\right) \tag{2.31}$$

$$= \frac{1}{2}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} + (2n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2}\right) \tag{2.32}$$

$$\langle m|\boldsymbol{P}^2|n\rangle = \langle m| -\frac{1}{2}\left(\left(\boldsymbol{a}^\dagger\right)^2 - \boldsymbol{a}^\dagger\boldsymbol{a} - \boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}^2\right)|n\rangle \tag{2.33}$$

$$= -\frac{1}{2}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} - n\delta_{m,n} - (n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2}\right) \tag{2.34}$$

$$= -\frac{1}{2}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} - (2n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2}\right) \tag{2.35}$$

$$= \frac{1}{2}\left(-\sqrt{(n+1)(n+2)}\delta_{m,n+2} + (2n+1)\delta_{m,n} - \sqrt{n(n-1)}\delta_{m,n-2}\right) \tag{2.36}$$

$$\tag{2.37}$$

Chapter 3

Optical Implementations

3.1 What makes a good quantum computer?

3.2 Optical devices for quantum computing

3.3 Quantum gates using optical devices

3.3.1 Single qubit gates

3.3.2 Multiple qubit gates

3.4 Drawbacks of the optical scheme

Appendix A

Exponentiating Matrices

In this appendix, we prove the following.

$$\exp{(iAx)} = cos(x)I + isin(x)A \tag{A.1}$$

for a real number x and matrix A such that $A^2 = I$ and $A^0 = I$. Recall the power series expansion for the exponential $\exp{(x)}$ for all x.

$$\exp{(x)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{A.2}$$

We now rewrite this power series expansion for $\exp{(x)}$ after replacing x by iAx.

$$\begin{aligned}
\exp{(iAx)} &= \sum_{n=0}^{\infty} \frac{(iAx)^n}{n!} && \text{(A.3)}\\
&= \frac{I}{0!} + \frac{iAx}{1!} + \frac{(iAx)^2}{2!} + \frac{(iAx)^3}{3!} + \dots && \text{(A.4)}\\
&= \frac{A^2}{0!} + \frac{iAx}{1!} + \frac{(iAx)^2}{2!} + \frac{(iAx)^3}{3!} + \dots && \text{(A.5)}
\end{aligned}$$

Noting that even powers of A reduce to identity and $i^2 = -1$, we now rearrange the terms in the above equation as follows.

$$\exp\left(iAx\right) = \frac{A^2}{0!} + \frac{iAx}{1!} + \frac{(iAx)^2}{2!} + \frac{(iAx)^3}{3!} + \ldots \tag{A.6}$$

$$= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) I + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) A \tag{A.7}$$

$$\tag{A.8}$$

The power series expansions for the sine and cosine functions appear in the above equation. We will state the power series expansions for these two functions below.

$$sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) \tag{A.9}$$

$$cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) \tag{A.10}$$

Therefore,

$$\exp\left(iAx\right) = \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) I + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) A \tag{A.11}$$

$$= cos(x)I + isin(x)A \quad \square \tag{A.12}$$

Appendix B

The Baker-Campbell-Hausdorf Formula

The Baker-Campbell-Hausdorf formula for a complex number λ and operators A, G and C_n is given by

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n \tag{B.1}$$

where C_n is defined recursively such that $C_0 = A$ and $C_n = [G, C_{n-1}]$. Several references on Lie Algebras such as [4], [1] and [7] provide interesting proofs of the formula.

Here, we are interested in applying the Baker-Campbell-Hausdorf formula to the unitary matrix which represents the operation of an optical beamsplitter. In particular, for a beamsplitter B of angle θ, the unitary matrix representation is

$$B = \exp\left(\theta(a^\dagger b - ab^\dagger)\right) \tag{B.2}$$

For such a beamsplitter, we will compute $BaB^\dagger$ and $BbB^\dagger$ for annihilation operators a and b and creation operators $a^\dagger$ and $b^\dagger$. Recall from 2.2 that for annihilation operators α and β, $[\alpha, \alpha^\dagger] = [\beta, \beta^\dagger] = 1$ and $[\alpha, \beta] = [\alpha^\dagger, \beta] = [\alpha, \beta^\dagger] = [\alpha^\dagger, \beta^\dagger] = 0$. Setting G

in (B.1) to $a^\dagger b - ab^\dagger$, we have

$$\begin{aligned}
[G,a] &= (a^\dagger b - ab^\dagger)a - a(a^\dagger b - ab^\dagger) \\
&= a^\dagger ba - ab^\dagger a - aa^\dagger b + aab^\dagger \\
&= a^\dagger ba - aa^\dagger b + aab^\dagger - ab^\dagger a \\
&= [a^\dagger b, a] + a\cancelto{0}{[a, b^\dagger]} \\
&= [a^\dagger b, a] \\
&= a^\dagger ba - a^\dagger ab + a^\dagger ab - aa^\dagger b \\
&= a^\dagger \cancelto{0}{[b, a]} - [a, a^\dagger]b \\
&= -b
\end{aligned}$$

Similarly, we can prove that $[G, b] = a$. We summarize these results below.

$$[G,a] = -b \tag{B.3}$$

$$[G,b] = a \tag{B.4}$$

We can now generalize the terms of C_n. Setting $C_0 = a$, $C_1 = [G,a] = -b$, $C_2 = [G, C_1] = -a$, $C_3 = [G, C_2] = b, \ldots$ and generally,

$$C_n = i^n a \text{ for even n}$$

$$C_n = i^{n+1} b \text{ for odd n}$$

Hence,

$$
\begin{aligned}
BaB^\dagger &= e^{\theta G} a e^{-\theta G} && \text{(B.5)}\\
&= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} C_n && \text{(B.6)}\\
&= \sum_{n=even} \frac{(i\theta)^n}{n!} a + i \sum_{n=odd}^{\infty} \frac{(i\theta)^n}{n!} b && \text{(B.7)}\\
&= a\cos\theta + b\sin\theta && \text{(B.8)}
\end{aligned}
$$

Similarly, $BbB^\dagger = -a\sin\theta + b\cos\theta$. $\square$

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