A body of mass m moves in a two-dimensional plane (x,y) with initial velocity (v_{x0},v_{y0}) subject to gravity along the y axis ($-mg\hat{y}$) and to friction, modelled as $-\beta\overrightarrow{v}$. The equation for the velocity is

$$m\dot{v}_x = -\beta v_x$$
$$m\dot{v}_y = -mg - \beta v_y$$

The analytical solution of the problem is

$$v_x(t) = v_{x0}e^{-\beta t/m}$$
$$v_y(t) = v_{y0}e^{-\beta t/m} - \frac{mg}{\beta}(1 - e^{-\beta t/m})$$

A numerical solution can be estimated with the iterative 2nd-order Runge-Kutta method for 1st order differential equations $\frac{dy}{dx} = f(x, y)$ with $y(0) = y_0$; given a value x, the solution y(x) can be estimated iteratively as :

$$K_1 = h \cdot f(x_n, y_n)$$
$$K_2 = h \cdot f(x_n + h/2, y_n + K_1 \cdot h/2)$$
$$y_{n+1} = y_n + K_2$$

where ***h*** is the step size for the iterative method and must be one of the parameters to be passed to the method. In this case, ***y*** is the velocity and ***x*** is the time t.

Implement two functions **`analytical()`** and **`rungekutta()`** with proper arguments to estimate $v_x(t)$ and $v_y(t)$.

Assuming the initial values $v_{x0} = v_{y0} = 10\,\mathrm{m}/s$, m = 10 kg, and $\beta = 0.1\,\mathrm{N}s/m$, plot $v_x(t)$ and $v_y(t)$ as a function of time t. In each plot, show both the analytical and the Runge-Kutta solutions with different line type and/or colours with proper legend.

Compute the difference between the analytical and numerical trajectories (residual) as a function of time and make a 1D histogram of the residuals.

You can customise the calculation by asking the user for initial values of the velocity, mass, and β.

Try using numpy arrays and comprehensions instead of C-style loops and structures.