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The ReesAlgebra package in Macaulay2

amended title to match package name

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ABSTRACT: This note introduces Rees algebras and some of their uses, with illustrations from version 2.2 of the Macaulay2 package ReesAlgebra.m2.

INTRODUCTION. A central construction in modern commutative algebra starts from an ideal I in a commutative ring R, and produces the *Rees algebra*

$$\mathcal{R}(I) := R \oplus I \oplus I^2 \oplus I^3 \oplus \cdots \cong R[It] \subset R[t],$$

where $R[t]$ denotes the polynomial algebra in one variable t over R. For basics on Rees algebras, see [Vasconcelos 1994] and [Huneke and Swanson 2006].

From the point of view of algebraic geometry, the Rees algebra $\mathcal{R}(I)$ is a homogeneous coordinate ring for the graph of a rational map whose total space is the blowup of Spec R along the scheme defined by I. (In fact, the "Rees algebra" is sometimes called the "blowup algebra".)

Rees algebras were first studied in the algebraic context by David Rees, in the now-famous paper [Rees 1958]. Actually, Rees mainly studied the ring $R[It, t^{-1}]$, now also called the *extended Rees algebra* of I.

removed duplicated "by"

Mike Stillman and I, if memory serves, wrote a Rees algebra script for Macaulay classic. It was augmented, and made into the package ReesAlgebra.m2 around 2002, to study a generalization of Rees algebras to modules described in [Eisenbud et al. 2003]. Subsequently Amelia Taylor, Sorin Popescu, the present author, and, at the Macaulay2 Workgroup in July 2017, Ilir Dema, Whitney Liske, and Zhangchi Chen contributed routines for computing many of the invariants of an ideal or module defined in terms of Rees algebras. These routines comprise the package's primary utility, since Rees algebras of modules other than ideals are comparatively little studied.

your memory appears to be correct, according to Mdocs.tar's docs/man/ap3.tex on http://www.math.columbia.edu/ bayer – should I remove this clause?

We first describe the construction and an example from [Eisenbud et al. 2003]. Then we list some of the functionality the package now has and illustrate it with a theorem of Morey and Ulrich. Finally we give examples of how Rees algebras

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appear in the Fulton–MacPherson intersection theory and in the resolution of singularities.

1. THE REES ALGEBRA OF A MODULE. There are several possible ways of extending the Rees algebra construction from ideals to modules. For simplicity we will henceforward only consider finitely generated modules over Noetherian rings. Huneke and Ulrich and I argued in [Eisenbud et al. 2003] that the most natural way to extend the definition is to think of $R[It]$ as the image of the map of symmetric algebras $\mathrm{Sym}(\phi) : \mathrm{Sym}_R(I) \to \mathrm{Sym}_R(R) = R[t]$, and to generalize it to the case of an arbitrary finitely generated module M by setting

$$\mathcal{R}(M) = \operatorname{image} \mathrm{Sym}(\phi),$$

where ϕ is a *versal* map from M to a free module. Such a versal map may be computed as the composition of the diagonal embedding

$$M \to \oplus_{i=1}^{m} M,$$

with the map

$$\oplus_{i=1}^{m} \phi_i : \oplus_{i=1}^{m} M \to R^m,$$

where $\phi_1, \ldots, \phi_m$ generate $\mathrm{Hom}_R(M, R)$.

Though this is not immediate, the Rees algebra of an ideal in a Noetherian ring, in this sense, is the same as the Rees algebra in the classical sense, and in most cases one can take any embedding of the module into a free module in the definition:

Theorem 1.1 [Eisenbud et al. 2003, Theorems 0.2 and 1.4]. *Let R be a Noetherian ring and let M be a finitely generated R-module. Let $\phi : M \to G$ be a versal map of M to a free module. Suppose that ϕ is an inclusion, and let $\psi : M \to G'$ be any inclusion of M into a free module G'. If R is torsion-free over $\mathbb{Z}$ or R is unmixed and generically Gorenstein or M is free locally at each associated prime of R, or $G' = R$, then the image of* $\mathrm{Sym}(\phi)$ *and the image of* $\mathrm{Sym}(\psi)$ *are naturally isomorphic.* added link, expanded “Thms”

Nevertheless some examples do violate the conclusion of Theorem 1.1. Here is one from [Eisenbud et al. 2003] in characteristic 5 (any finite characteristic would work similarly).

```
i1 : p = 5;
i2 : R = ZZ/p[x,y,z]/(ideal(x^p,y^p)+(ideal(x,y,z))^(p+1));
i3 : M = module ideal(z);
```

It is easy to check that $M \cong R^1/(x, y, z)^p$. We write $\iota : M \to R^1$ for the embedding as an ideal and ψ for the embedding $M \to R^2$ sending z to the vector (x, y).

```
i4 : iota = map(R^1,M,matrix{{z}});
i5 : psi = map(R^2,M,matrix{{x},{y}});
```

Finally, we choose a versal embedding $M \to R^3$. It sends z to the vector (x, y, z):

```
i6 : phi = versalEmbedding(M);
```

We now compute the kernels of the three maps on symmetric algebras:

```
i7 : Iiota = symmetricKernel iota;
i8 : Ipsi = symmetricKernel psi;
i9 : Iphi = symmetricKernel phi;
```

and check that the ones corresponding to ϕ and ι are equal, whereas the ones corresponding to ψ and ϕ are not — they differ in degree p.

```
i10 : Iiota == Iphi
o10 = true
i11 : Ipsi == Iphi
o11 = false
i12 : numcols basis(p,Iphi)
o12 = 3
i13 : numcols basis(p,Ipsi)
o13 = 1
```

2. THE REES ALGEBRA AND ITS RELATIONS. The central routine, `reesIdeal` (with synonym: `reesAlgebraIdeal`), computes an ideal defining the Rees algebra $\mathcal{R}(M)$ as a quotient of a polynomial ring over R from a free presentation of M. From the Rees ideal we immediately get `reesAlgebra` M. In the case when M is an ideal in R we also compute the important `associatedGradedRing` $M = \mathcal{R}(M)/M$ (and the more geometric sounding but identical `normalCone` M). If I is a (homogeneous) ideal primary to the maximal ideal of a standard graded ring R we compute the Hilbert–Samuel multiplicity of I with the routine `multiplicity`. multiplicity I → the routine multiplicity

We now describe the basic computation. Suppose that M has a set of generators represented by a map from a free module, and a free presentation added "and" is this okay?

$$F \xrightarrow{\alpha} M \to 0,$$

and suppose $F = R^n$. The symmetric algebra of F over R is then a polynomial ring $\mathrm{Sym}_R(F) = R[t_1, \dots, t_n]$ on n new indeterminates $t_1, \dots, t_n$. By the universal property of the symmetric algebra there is a canonical surjection $\mathrm{Sym}_R(F) \to \mathrm{Sym}_R(M)$, so we may compute the Rees algebra of M as a quotient of $\mathrm{Sym}_R(F)$. removed "the "

The expression

$$I = \texttt{reesIdeal}\ M$$

first uses `versalEmbedding` M to compute a versal map from M to a free module $\beta : M \to G$. The expression `symmetricKernel` $\alpha \circ \beta$ then constructs the map of symmetric algebras $\beta \circ \alpha : \mathrm{Sym}_R(F) \to \mathrm{Sym}_R(G)$ and uses the built-in Macaulay2 routine to compute the kernel

$$I = \texttt{reesIdeal}\ M = \ker \mathrm{Sym}(\beta \circ \alpha) : \mathrm{Sym}_R(F) \to \mathrm{Sym}_R(G).$$

There is a different way of computing the Rees algebra that is often much more efficient. It begins by constructing the symmetric algebra of M, and uses the observation that the construction of the Rees algebra commutes with localization. See [Eisenbud 1995, Appendix 2] for the necessary facts about symmetric algebras.

Suppose that M has a free presentation,

$$G \xrightarrow{\alpha} F \xrightarrow{\epsilon} M \to 0.$$

The right exactness of the symmetric algebra functor implies that the symmetric algebra of M is the quotient of $\mathrm{Sym}_R(F)$ by an ideal I_0 that is generated by the entries of the matrix

$$(t_1 \ \cdots \ t_n) \circ \phi,$$

(where we have identified ϕ with $\mathrm{Sym}_R(F) \otimes_R \phi$). Thus I_0 is generated by polynomials that are linear in the variables t_i (and because M is the degree 1 part of $\mathcal{R}(M)$, these are the only linear forms in the t_i in the Rees ideal).

If $f \in R$ is an element such that $M[f^{-1}]$ is free on generators $g_1, \ldots, g_n$, it follows that after inverting f, the Rees algebra of M becomes a polynomial ring over $R[f^{-1}]$ on indeterminates corresponding to the g_i:

$$\mathcal{R}(M)[f^{-1}] = \mathrm{Sym}_R(M[f^{-1}]) = R[G_1, \ldots, G_n].$$

Now suppose in addition that f is a non-zerodivisor in R. In the diagram

$$\begin{tikzcd}
\mathrm{Sym}_R(F) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Sym}_R(M) \arrow[r, "\beta"] \arrow[d] & \mathrm{Sym}_R(G) \arrow[d] \\
\mathrm{Sym}_R(F)[f^{-1}] \arrow[r, "\alpha"] & \mathrm{Sym}_R(M)[f^{-1}] \arrow[r, "\beta"] & \mathrm{Sym}_R(G)[f^{-1}]
\end{tikzcd}$$

the two outer vertical maps are inclusions, and it follows that the Rees ideal, which is the kernel of the map $\mathcal{R}(F) = \mathrm{Sym}_R(F) \to \mathcal{R}(M)$, is equal to the intersection of $\mathcal{R}(F)$ with the kernel of

$$\mathrm{Sym}_R(F)[f^{-1}] \xrightarrow{\beta} \mathrm{Sym}_R(G)[f^{-1}].$$

This intersection may be computed as $I_0 : f^{\infty}$. The command

$$\texttt{reesIdeal}(I, f)$$

computes the Rees ideal in this way.

More generally, we say that a module N is *of linear type* if the Rees ideal of M is equal to the ideal of the symmetric algebra of M; for example, any complete intersection ideal is of linear type, and the condition can be tested by the command

$$\texttt{isLinearType}\ M.$$

The procedure above really requires only that f be a non-zerodivisor in R and that $M[f^{-1}]$ be of linear type over $R[f^{-1}]$.

3. Reductions and the special fiber. A *reduction* J of an ideal I is a subideal $J \subset I$ over which I is *integrally dependent*. In concrete terms this means that there is some integer r such that $JI^r = I^{r+1}$, and the minimal r with this property is called the reduction number. The property of being a reduction is tested by `isReduction` I, and `reductionNumber` I computes the reduction number. it→the reduction number

Now suppose that $\mathfrak{m}$ is a maximal ideal containing I. The special fiber ring is by definition $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. It is a standard graded algebra over the field $k := R/\mathfrak{m}$, a quotient of $\mathrm{Sym}_R(F)/\mathfrak{m} = k[t_1, \dots, t_n]$ where, as before, F is a free module of rank n with a surjection to M. The defining ideal of the special fiber ring, and the ring itself, are computed using `specialFiberIdeal` I and `specialFiberRing` I. removed "by" to avoid overfull line

The dimension of the special fiber ring is called the analytic spread of I, usually denoted

$$\ell(I) = \texttt{analyticSpread}\ I.$$

Northcott and Rees [1954] proved that if k is infinite then there always exist reductions generated by $\ell(I)$ elements, and this is the minimum possible number; these are called minimal reductions. The smallest possible reduction number for I with respect to a minimal reduction is by definition `reductionNumber` I. (This is always achieved by any ideal generated by $\ell(I)$ sufficiently general scalar linear combinations of the generators of I; but note that when I is homogeneous but has generators of different degrees such linear combinations are sometimes necessarily inhomogeneous.) reductionnumber → reductionNumber

An interesting special case occurs when R is a graded ring over $k = R_0$ and the generators $g_1, \dots, g_n$ of I are all homogeneous of the same degree. In this case the special fiber ring is easily seen to be equal to the subring $k[g_1, \dots, g_n]$ (usually *not* a polynomial ring) generated by the elements g_i.

For example, if I is the ideal of $p \times p$ minors of a $p \times (p+q)$ matrix, then the special fiber ring is equal to the homogeneous coordinate ring $\mathbb{G}$ of the Grassmannian of p-planes in $p+q$ space. It follows that $\ell(I) = \dim \mathbb{G} = pq + 1$, and the reduction number of I is $(p-1)(q-1)$.

4. Finding elements of the Rees ideal. Let M be an R-module and let $\phi : R^s \to R^m$ be its presentation matrix. We identify $\mathrm{Sym}_R(R^m)$ with the polynomial ring $R[t_1, \dots, t_m]$. By the universality of the symmetric algebra construction, the symmetric algebra of I has the form

$$\mathrm{Sym}_R(I) = R[t_1, \dots, t_m]/(T\phi),$$

where we have written T for the vector $(t_1\ \dots\ t_m) \in R[t_1, \dots, t_m]^m$, whose entries correspond to the generators of I, and written $(T\phi)$ for the ideal generated by the entries of the product

$$(t_1\ \cdots\ t_m)\phi.$$

If $J := (x_1, \dots, x_n) \subset R$ is an ideal containing I, and we write

$$X = (x_1 \ \cdots \ x_n) \in R[t_1, \dots, t_m]^n,$$

then there is a matrix ψ defined over $R[t_1, \dots, t_m]$, called the Jacobian dual of ϕ with respect to X, such that $T\phi = X\psi$. (The matrix ψ is generally not unique; Macaulay2 computes it using Gröbner division with remainder.)

If I, J each contain a non-zerodivisor then J will have grade ≥ 1 on the Rees algebra $\mathcal{R}(I)$. Since $(T\phi)$ is contained in the defining ideal of the Rees algebra, the vector X is annihilated by the matrix ψ when regarded over the Rees algebra, and the relation $X\psi \equiv 0$ in $\mathcal{R}(I)$ implies that the $m \times m$ minors of ψ are in the Rees ideal of I.

In very favorable circumstances, one may even have the equality

$$\texttt{reesIdeal}\ I == \texttt{ideal}(T\phi) + \texttt{minors}(m, \psi).$$

changed math from texttt font, as seems consistent with previous equations; please check

We illustrate with a theorem of Morey and Ulrich. Recall that an ideal I is said to satisfy the condition G_ℓ if the number of generators of the localized ideal I_P is $\leq \operatorname{codim} P$ for every prime ideal P of codimension $< \ell$; equivalently, if I has presentation matrix ϕ as above,

$$\operatorname{codim} I_{m-p}(\phi) > p$$

for $1 \leq p < \ell$.

Theorem 4.1 [Morey and Ulrich 1996]. *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect ideal of grade 2 with m generators, let ϕ be the presentation matrix of I, and let ψ be the Jacobian dual matrix. Let $\ell = \ell(I)$ be the analytic spread. Suppose that I satisfies the condition G_ℓ. The following conditions are equivalent*: removed “and”

1. *$\mathcal{R}(I)$ is Cohen–Macaulay and $I_{(m-\ell)}(\phi) = I_1(\phi)^{m-\ell}$.*
2. *$r(I) < \ell$ and $I_{m+1-\ell}\phi = (I_1\phi)^{m+1-\ell}$.*
3. *The ideal of $\mathcal{R}(I)$ is equal to the sum of the ideal of* Sym(I) *with the Jacobian dual minors, $I_m\psi$.*

We can check all these conditions with functions in the package. We start with the presentation matrix ϕ of an $m{=}n{+}1$-generator perfect ideal such that the first row consists of the n variables of the ring, and the rest of the rows are reasonably general (in this case random quadrics): whose→the

```
i2 : setRandomSeed 0
i3 : n=3;
i4 : kk = ZZ/101;
i5 : S = kk[a_0..a_(n-2)];
i6 : phi = transpose map(S^(n-1),S^{-1,(n-1):-2},
             (i,j) -> if j == 0 then a_i else random(2,S));
             3       2
```

```
o6 : Matrix S  <--- S
i7 : I = minors(n-1,phi);
```

This is a perfect codimension 2 ideal, as we see from the Betti table:

```
i8 : betti (F = res I)
            0 1 2
o8 = total: 1 3 2
         0: 1 . .
         1: . . .
         2: . 2 .
         3: . 1 2
```

We compute the analytic spread ℓ and the reduction number r:

```
i12 : ell = analyticSpread I
o12 = 2
i13 : r = reductionNumber(I, minimalReduction I)
o13 = 1
```

Now we can check the condition G_ℓ, first probabilistically:

```
i15 : whichGm I >= ell
o15 = true
```

and now deterministically:

```
i17 : apply(toList(1..ell-1),
                  p-> {p+1, codim minors(n-p, phi)})
o17 = {{2, 2}}
```

We now check the three equivalent conditions of the Morey–Ulrich theorem. Since $\ell = n-1$ in this case, the second parts of conditions (1) and (2) are vacuously satisfied, and since $r < \ell$ the conditions must all be satisfied. We first check that $\mathcal{R}(I)$ is Cohen–Macaulay: part → parts is → are

```
i19 : reesI = reesIdeal I;
o19 : Ideal of S[w , w , w ]
                  0   1   2
i20 : codim reesI
o20 = 2

i21 : betti res reesI
             0 1 2
o21 = total: 1 3 2
          0: 1 . .
          1: . . .
          2: . 2 .
          3: . 1 2
```

Finally, we wish to see that `reesIdeal` I is generated by the ideal of the symmetric algebra together with the Jacobian dual: reesIdeal I → `reesIdeal` I

```
i23 : psi = jacobianDual phi;
                                 2                      2
o23 : Matrix (S[w , w , w ])  <--- (S[w , w , w ])
                 0   1   2              0   1   2
```

We now compute the ideal J of the symmetric algebra; we do this by hand, since the command `symmetricAlgebra` I would return the ideal over a different ring.

rearranged to avoid overfull line

```
i25 : ST = ring psi
i26 : T = vars ST
o26 = | w_0 w_1 w_2 |
i27 : J = ideal(T*promote(phi, ST))
i28 :      betti res J
             0 1 2
o28 = total: 1 2 1
          0: 1 . .
          1: . . .
          2: . 2 .
          3: . . .
          4: . . 1
i29 : J1 = minors(ell, psi)
```

We compute the resolution of $G := J + J1$, to see that the resulting ideal is perfect, which also shows that it is the full ideal of the Rees algebra. We also check directly that it has the same resolution as the computed Rees ideal of I:

```
i30 : betti (G = res trim (J+J1))
             0 1 2
o30 = total: 1 3 2
          0: 1 . .
          1: . . .
          2: . 2 .
          3: . 1 2
i31 : betti res reesIdeal I
             0 1 2
o31 = total: 1 3 2
          0: 1 . .
          1: . . .
          2: . 2 .
          3: . 1 2
```

5. DISTINGUISHED SUBVARIETIES. The key construction in the Fulton–MacPherson definition of the refined intersection product [Fulton 1998, Section 6.1] involves normal cones, and is easy to implement using the tools in this package. The simplest case is the intersection of two subvarieties $X, V \subset Y$. If X and V meet in the *expected dimension*, defined to be $\dim V - \operatorname{codim}_Y X$, and the ambient variety Y is smooth, then one can assign multiplicities m_i to the components W_i of $X \cap V$, and the intersection product has the form $[X][V] = \sum m_i[W_i]$. The astonishing result of the Fulton–MacPherson theory is that if $X \subset Y$ is locally a complete intersection, then, no matter how singular Y and no matter how strange the actual intersection $X \cap V$, the intersection product $X \cdot V$ can be given a meaning as a rational equivalence class of cycles of the expected dimension on X, or even on certain *distinguished* subvarieties Z_i of $X \cap V$. This class comes with a canonical decomposition $\sum_i m_i \alpha_i$, where the m_i are positive integers, and α_i is a cycle of the expected dimension (possibly 0) on $Z_i \subset X \cap V$ (the same Z_i can appear several times, with different multiplicities and cycles). added “a”

In the general case, the subvariety V is replaced by a morphism $f : V \to Y$ from a variety V, and this is the key to the functoriality of the intersection product. The routines in this package work in the general setting, but for simplicity we will stick with the basic case in this description.

We now describe the distinguished subvarieties and their multiplicities. This part of the construction sheafifies, so (as in the package) we work in the affine case. We do not require any hypothesis on X, Y or V.

Let S be a ring (for example, the coordinate ring of Y) and let $I \subset S$ be an ideal (for example, the ideal of X). Write

$$T := \mathrm{gr}_I S = S/I \oplus I/I^2 \oplus \cdots$$

for the associated graded ring of I, and let π be the inclusion of S/I into T as the degree 0 part.

Let $f : S \to R$ be a ring homomorphism (for example, representing the projection $S \to S/(I(V))$). Let $K \subset T$ be the kernel of the induced map $\mathrm{gr}_I S \to \mathrm{gr}_{f(I)R} R$.

Let $P_1, \ldots, P_m$ be the minimal primes over K in $\mathrm{gr}_I R$. We define p_i to be the degree 0 part of P_i; that is, $p_i := P_i \cap S/I$. These are the distinguished prime ideals of S/I, and they clearly contain the kernel of $\overline{f} : S/I \to R/f(I)R$, so in the case where $R = S/J$ they contain $I + J$. Thus, in this case, they represent subvarieties of $X \cap V$.

Let m_i be the multiplicity with which P_i appears in the primary decomposition of K — that is,

$$m_i := \mathrm{length}_{t_{P_i}}\ P_{i\,P_i}/K_{P_i}.$$

Returning to the geometric language, and the case where $X \subset Y$ is locally a complete intersection in a quasiprojective variety, the cycle class α_i in the Chow group of the variety Z_i corresponding to p_i is defined as the Gysin image of the class of the subvariety corresponding to P_i in the projectivized normal bundle of X in Y — a construction not included in this package.

Here are some simple examples in which `distinguished` is used to compute the distinguished varieties of intersections in $\mathbb{A}^n$, via the function `intersectInP`. First, the familiar multiplicity 2 intersection of a conic with a tangent line.

```
i2 : kk = ZZ/101;
i3 : P = kk[x,y];
```

```
i4 : I = ideal"x2-y";J=ideal y;
i6 : intersectInP(I,J)
o6 = {{2, ideal (y, x)}}
```

Slightly more interesting, the following shows what happens when the intersections aren't rational:

```
i7 : I = ideal"x4+y3+1";
i8 : intersectInP(I,J)
                        2                         2
o8 = {{1, ideal (y, x  + 10)}, {1, ideal (y, x  - 10)}}
```

The real interest in the construction is in the case of improper intersections. Here are some typical results:

```
i9 : I = ideal"x2y";J=ideal"xy2";
i11 : intersectInP(I,J)
o11 = {{2, ideal x}, {5, ideal (y, x)}, {2, ideal y}}
i12 : intersectInP(I,I)
o12 = {{1, ideal y}, {4, ideal x}, {4, ideal (y, x)}}
```

6. Rees algebras and Desingularization.

We conclude with an example illustrating a general result about projective birational maps of varieties. Recall that a map $B \to X$ of varieties is projective if it is the composition of a closed embedding $B \subset X \times \mathbb{P}^n$ with the projection to X. It is birational if it is generically an isomorphism. The inclusion of a ring into the Rees algebra of an ideal corresponds to a map from proj of the Rees algebra to Spec of the ring, called a blowup, that is such a proper birational transformation, and in fact every proper birational transformation to an affine variety (or more generally to any scheme, if one works with sheaves of ideals) can be realized in this way.

varieites→varieties

Proj → proj
spec → Spec
for consistency with use elsewhere in the paper; is this correct?

The theorem of embedded resolution of singularities (proven by Hironaka in characteristic 0 and conjectured in general) says that, given any subvariety X of a smooth variety Y, there is a finite sequence of blowups

$$B_n \to \cdots \to B_2 \to B_1 \to Y$$

of smooth subvarieties that lie over the singular set of X, and a component of the preimage of X in B_n that is smooth and maps birationally to X. In the case of plane curves, this can be done with a sequence of blowups of closed points. But in fact *any* sequence of blowups of a quasiprojective variety can be replaced with a single blowup [Hartshorne 1977, Theorem II.7.17] of a more complicated ideal. We illustrate this with the desingularization of a tacnode (the union of two smooth curves that meet with a simple tangency).

any→*any*
removed unmatched "("
added "this"

Example 6.1. Blowing-up (x^2, y) in $k[x, y]$ desingularizes the tacnode $x^2 - y^4$ in a single step.

```
i1 : R = ZZ/32003[x,y];
i2 : tacnode = ideal(x^2-y^4);
i3 : mm = ideal(x,y^2);
i4 : B = first flattenRing reesAlgebra mm;
i5 : irrelB = ideal(w_0,w_1);
i6 : proj = map(B,R,{x,y});
i7 : totalTransform = proj tacnode
                4    2
o7 = ideal(- y  + x )
i8 : netList (D = decompose totalTransform)
     +-----------------------+
o8 = |ideal (y, x)           |
     +-----------------------+
     |        2              |
     |ideal (y  + x, w  + w )|
     |                0    1 |
     +-----------------------+
     |        2              |
     |ideal (y  - x, w  - w )|
     |                0    1 |
     +-----------------------+
i9 : exceptional = proj mm
                  2
o9 = ideal (x, y )
i10 : strictTransform = saturate(
                 totalTransform, exceptional);
i11 : netList decompose strictTransform
      +-----------------------+
      |        2              |
o11 = |ideal (y  + x, w  + w )|
      |                0    1 |
      +-----------------------+
      |        2              |
      |ideal (y  - x, w  - w )|
      |                0    1 |
      +-----------------------+
i12 : sing0 = sub(ideal singularLocus strictTransform, B);
i13 : sing = saturate(sing0,irrelB)
o13 = ideal 1
```

The last line asserts that the singular locus of the strict transform is empty; that is, the scheme defined by `strictTransform` is smooth (in this case it is the union of two disjoint smooth curves). removed duplicated "the"s "strictTransform" → `strictTransform`

SUPPLEMENT. The online supplement contains version 2.2 of ReesAlgebra.m2. added this

REFERENCES. [EU], [KU], [U] and [VV] were not referred to and don't appear here. If you'd like to reinstate any of them, please cite it somewhere in the text to make its relevance clear.

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