Branch & Bound

Alberto Parravicini
Université libre de Bruxelles

June 1, 2017

A brief introduction...

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→ **Branch**: find the optimal solution by exploring smaller sub-problems.

→ **Bound**: use bounds on the objective function to implicitly enumerate all possible solutions.

Branching

Observation: if we partition the feasible region X into $\mathbf{X} = \mathbf{X_1} \cup \ldots \cup \mathbf{X_k}$ and let $\mathbf{z_i} = \mathbf{max}\{\mathbf{c}(\mathbf{x}) : \mathbf{x} \in \mathbf{X_i}\}$, then:

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We can't explore all possible branches though, they are too many!

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Lower bounds are feasible solutions, while *upper* bounds can be found by solving a relaxation.
In **ILP**, the linear relaxation will provide an upper bound $\mathbf{x^*_{LP}}$.

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- **Infeasibility**: if $\mathbf{X_i} = \emptyset$.
- **Bounding**: if the **upper** bound $\bar{z}_i$ is $\leq$ than some other **lower** bound $\underline{z}_j$, or of the best feasible solution found so far.

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- **Best Bound**, find the upper bounds of the children and choose the best (slow!).

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$X_1 = \mathbf{x} : \mathbf{x_i} = \mathbf{0}, \mathbf{i} = \mathbf{0}, \ldots, \mathbf{r}$

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- Branch only when adding more cuts stops being effective.
- Computational tradeoff, as cuts are valid only for a node and its children. Very effective in practice, however.

THANK YOU!