ZONOIDS

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April 19, 2017

A super short summary:

→ Some interesting facts

→ Zonoids & K-Sets

→ Zonoids & K-Levels

→ Applications

DEFINITIONS & PROPERTIES

Definition

You might recall...

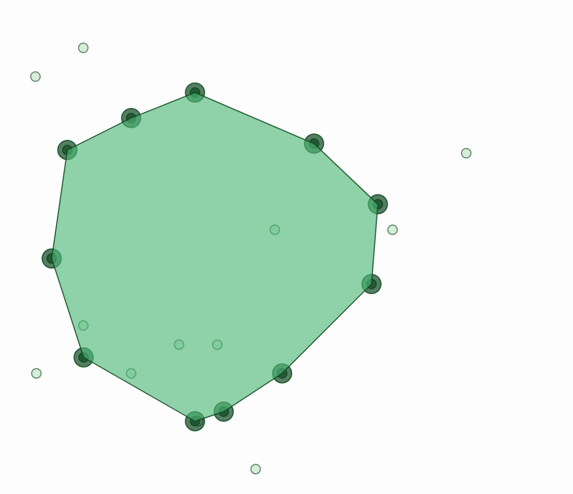
Given a set of points
$S = \{p_1,\ p_2,\ \dots,\ p_n\} \subset \mathbb{R}^d$

→ **Convex Hull:**
$CH(S) = \{\sum_{i=1}^{n} \lambda_i p_i :\ 0 \leq \lambda_i \leq 1, \sum_{i=1}^{n} \lambda_i = 1\}$

Let's impose a constraint on λ_i, i.e.
$\lambda_i \leq \frac{1}{k}\ \forall k \in [1, n]$

→ $Z_k(S) = \{\sum_{i=1}^{n} \lambda_i p_i :\ 0 \leq \lambda_i \leq \frac{1}{k}, \sum_{i=1}^{n} \lambda_i = 1\}$

What does it look like?



A place-holder zonoid, for $k = 3$. Look at the demo instead!

Cool facts!

- For $k = 1$, we get the convex hull.
- For $k = n$, we find the mean point of S (as $\lambda_i = \frac{1}{n}\ \forall i$).
- $\forall k,\ Z_k(S)$ is a **convex** polygon.
- If $k_1 > k_2$, then $Z_{k_1} \subseteq Z_{k_2}$.

Zonoid depth

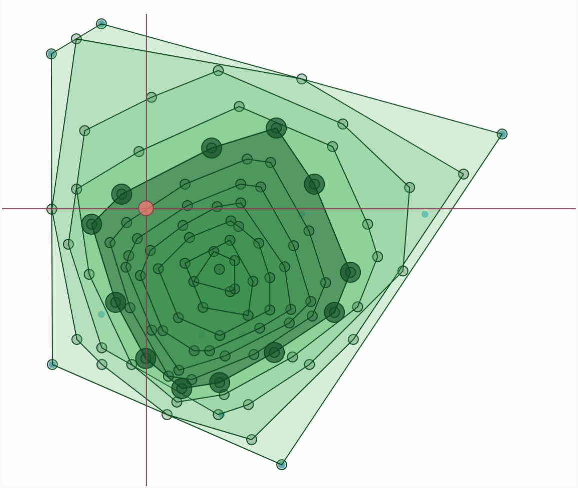
The **zonoid depth** of x with respect to S is the **max** value of k s.t. $x \in Z_k(S)$.

$$D(x, S) = \begin{cases} max\{k \mid x \in Z_k(S)\}, & if\ x \in CH(S) \\ 0, & otherwise \end{cases}$$

→ **Affine invariant:**
$D(Ax + Ab, AS + Ab) = D(x, S),\ \forall A : \mathbb{R}^{d\times d}, b \in \mathbb{R}^d$

→ **Zero at infinity**: $\lim_{\|x\|\to\infty} D(x, S) = 0$.

What does it look like?



Zonoid depth $= 5$. Look at the demo instead!

K-SETS & ZONOIDS

K-Sets, definition

- → How can we draw a zonoid?
- → Exploit **K-Sets**!
- → **K-Set**: subset of S of size k that can be separated from the other points in S with a straight line.

K-Sets & Zonoids

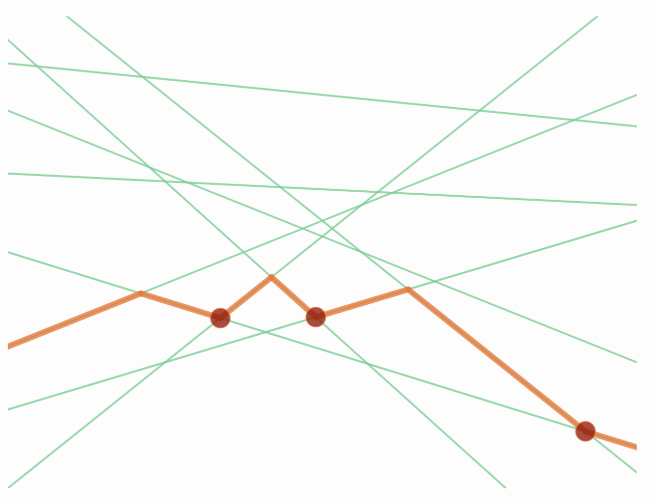
- Obtain a vertex of the *k*-zonoid by giving $\lambda = 1$ to the *k* extreme points in a direction.
- Equal to taking the mean of a certain *k*-set!
- Repeat for all *k*-sets and find all the vertices.
- Formally, the *k*-zonoid vertex *x* in a direction *p* is:
  $\arg\max_x \{p \cdot x \mid x \in Z_k(S)\} = \left(\sum_{i=1}^{k} \frac{1}{k} S_i^p\right)$

K-LEVELS & ZONOIDS

K-Levels

- The **k-level** of a set of lines L is the the set of points on one line of L and strictly **above** $k-1$ lines.
- **Reflex vertices**: points on the k-level found at the intersection of 2 lines, and above $k-2$ lines.
- **Point-line duality**: the dual of $p=(p_1,p_2)$ is the line $p* = \{(x,y) : y = p_1x - p_2\}$. The dual of a line $l = \{(x,y) : y = ax+b\}$ is the point $l^* = (a,-b)$.

K-Level, visually



k-level of a set of lines, for $k = 3$

K-Levels & Zonoids, 1

→ **Reflex vertices** are above or equal k lines.

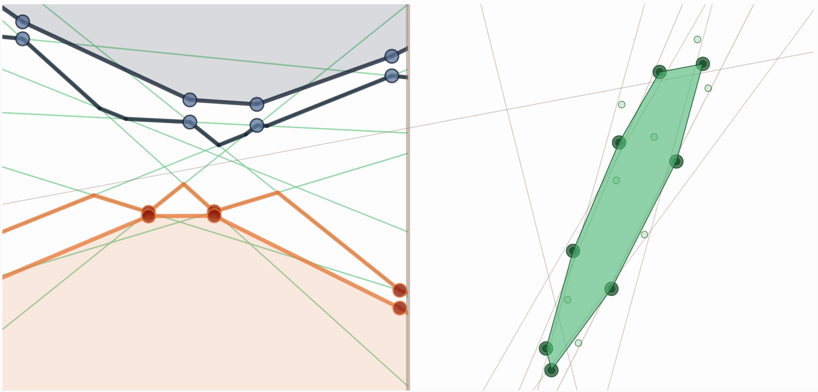
→ In the primal plane, a reflex vertex becomes a line above k points.

→ These k points are a k-set in the primal plane!

K-Levels & Zonoids, 2

- → For each reflex vertex of the k-level, trace a downward vertical ray.
- → Intersect the ray with the lines below the reflex vertex.
- → Compute the mean point of the intersections, find an *envelope*.
- → It will be a line of the zonoid boundary, in the primal plane!
- → Repeat for the $n - k$-level, with an upward ray.

K-Level & Zonoids, visually



3-zonoid computed from the k-levels.

Zoonoid point inclusion

→ How can we check if a point is inside the *k*-zonoid?

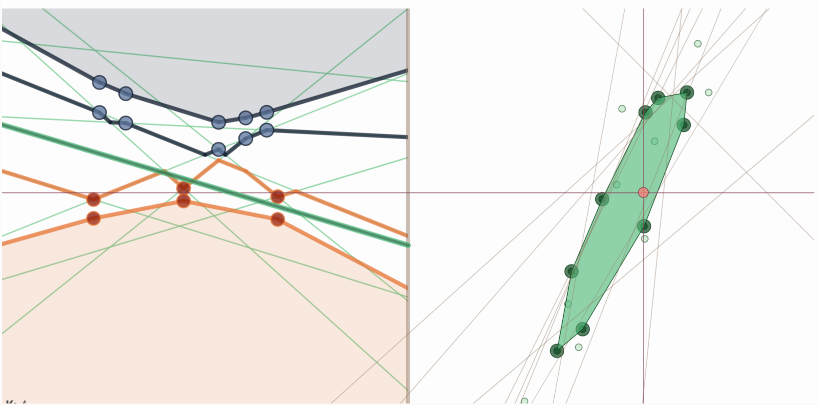
→ Look at the *envelopes* of the zonoid in the dual plane.

→ The point is included if its dual line intersects the envelopes.

It works as *duality* is **order-preserving**:

If a primal point is outside, it is above (below) 2 edges of the zonoid. In the dual, the dual line will be above the duals of those 2 edges, i.e. points on the 2 envelopes.

K-Level & Zonoids, visually



The point is inside the zonoid, the dual line doesn't intersect the envelopes

THANK YOU!

References I

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