ZONOIDS

Alberto Parravicini
Université libre de Bruxelles

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A super short summary:

→ **Some interesting facts**

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→ **Zonoids & K-Sets**

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→ Applications

DEFINITIONS & PROPERTIES

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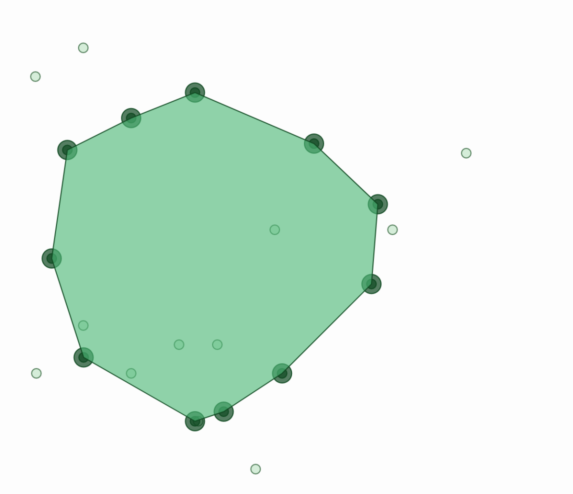
→ $Z_k(S) = \{\sum_{i=1}^{n} \lambda_i p_i : \ 0 \leq \lambda_i \leq \frac{1}{k}, \sum_{i=1}^{n} \lambda_i = 1\}$

What does it look like?



A place-holder zonoid, for $k = 3$. Look at the demo instead!

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→ $\forall k,\ Z_k(S)$ is a **convex** polygon.

→ If $k_1 > k_2$, then $Z_{k_1} \subseteq Z_{k_2}$.

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→ **Affine invariant:**
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→ **Zero at infinity:** $\lim_{\|x\| \to \infty} D(x, S) = 0$.

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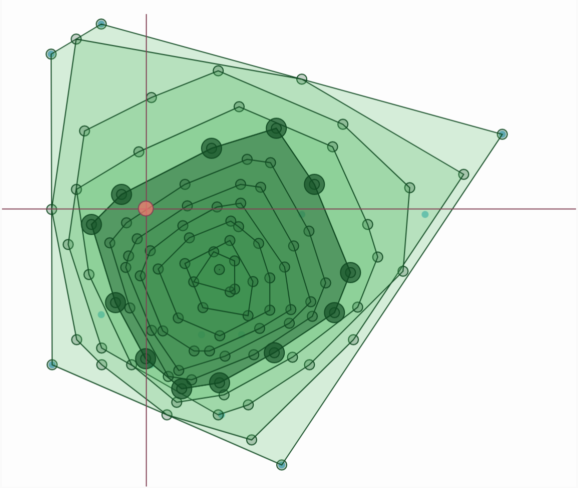

Zonoid depth $= 5$. Look at the demo instead!

K-SETS & ZONOIDS

K-Sets, definition

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→ How can we draw a zonoid?

→ Exploit **K-Sets**!

→ **K-Set**: subset of S of size k that can be separated from the other points in S with a straight line.

K-Sets & Zonoids

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→ Equal to taking the mean of a certain k-set!

→ Repeat for all k-sets and find all the vertices.

→ Formally, the k-zonoid vertex x in a direction p is:
$\arg\max_x \{p \cdot x \mid x \in Z_k(S)\} = \left(\sum_{i=1}^{k} \frac{1}{k} S_i^p\right)$

K-LEVELS & ZONOIDS

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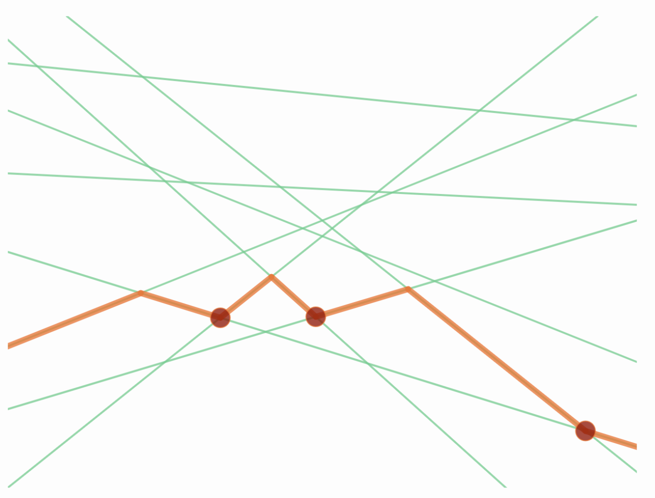
K-Levels

- $\rightarrow$ The **k-level** of a set of lines L is the the set of points on one line of L and strictly **above** $k-1$ lines.
- $\rightarrow$ **Reflex vertices**: points on the k-level found at the intersection of 2 lines, and above $k-2$ lines.
- $\rightarrow$ **Point-line duality**: the dual of $p=(p_1,p_2)$ is the line $p* = \{(x,y) : y = p_1 x - p_2\}$. The dual of a line $l = \{(x,y) : y = ax+b\}$ is the point $l^* = (a,-b)$.

K-Level, visually



k-level of a set of lines, for $k = 3$

K-Levels & Zonoids, 1

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- → **Reflex vertices** are above or equal k lines.
- → In the primal plane, a reflex vertex becomes a line above k points.
- → These k points are a k-set in the primal plane!

K-Levels & Zonoids, 2

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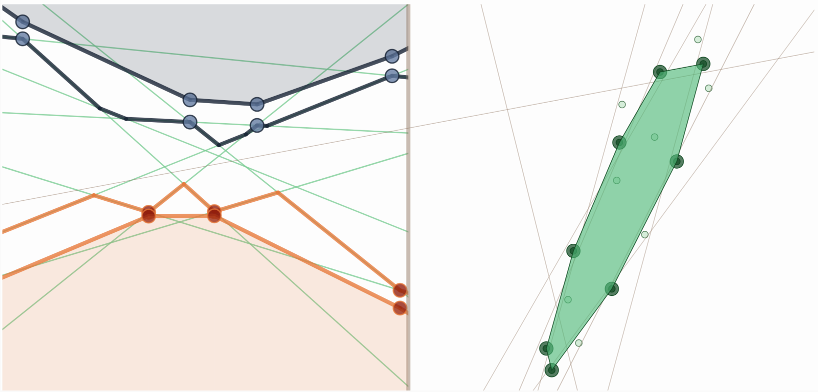
- → It will be a line of the zonoid boundary, in the primal plane!
- → Repeat for the $n - k$-level, with an upward ray.

K-Level & Zonoids, visually



3-zonoid computed from the k-levels.

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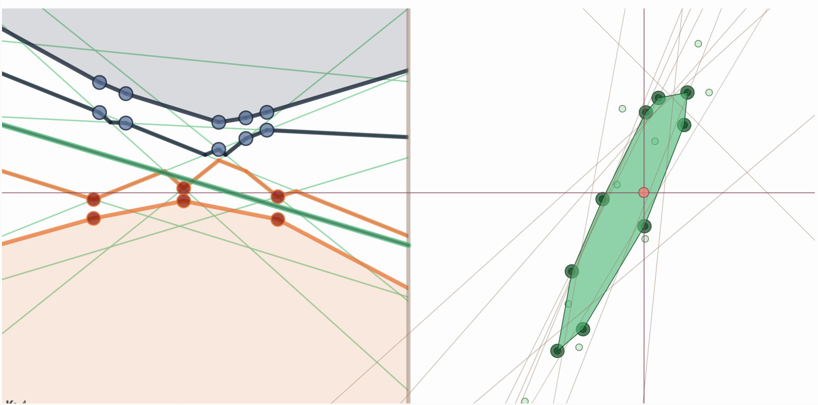
  It works as *duality* is **order-preserving**:

  If a primal point is outside, it is above (below) 2 edges of the zonoid. In the dual, the dual line will be above the duals of those 2 edges, i.e. points on the 2 envelopes.

K-Level & Zonoids, visually



The point is inside the zonoid, the dual line doesn’t intersect the envelopes

THANK YOU!

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Beamer theme: *Presento*, by Ratul Saha. *The research system in Germany*, by Hazem Alsaied