CONVEX HULLS

Chan's optimal output sensitive algorithms for convex hulls

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What's on the menu?

→ Basic notions

→ Algorithms for convex hulls

→ Optimal 2D algorithm

→ Optimal 3D algorithm

BASIC NOTIONS

Crash course on convex hulls

A *convex set* S is a set in which, $\forall\, x, y \in S$, the segment $xy \subseteq S$.

Given a set of points P in d dimensions, the **Convex Hull $\mathbf{CH(P)}$** of P is:

→ the *minimal convex set* containing P.

→ the union of *all convex combinations* of points in P, i.e. the points $CH(P)$ are s.t.

$$\sum_{i=1}^{|P|} w_i \cdot x_i,\ \forall x_i \in P,\ \forall w_i :\ w_i \geq 0 \textit{ and } \sum_{i=1}^{|P|} w_i = 1$$

Output sensitive algorithm

The complexity of an *output-sensitive* algorithm is a function of both the *input size* and the *output size*.

ALGORITHMS FOR CONVEX HULLS

Jarvis's march

- **Core idea:** given an edge $\boldsymbol{pq}$ of the convex hull, the next edge $\boldsymbol{qr}$ to be added will maximize the angle $\angle\boldsymbol{pqr}$

- At each step we scan all the $\boldsymbol{n}$ points.
- How many steps? $\boldsymbol{h}$, size of the hull.
- **Complexity:** $\boldsymbol{O(nh)}$

One step, visually

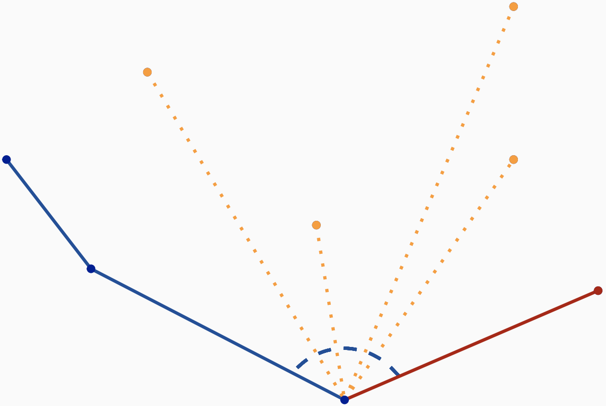


Figure: *A step of the Jarvis March. The red line is the next segment that will be added to the hull.*

Jarvis's march

Algorithm 1: Jarvis March

Input: a list *S* of bidimensional points.
Output: the convex hull of the set, sorted counterclockwise.

```
hull =[]
x_0 = the leftmost point.
hull.push(x_0)
Loop hull.last() != hull.first()
    candidate = S.first()
    foreach p in S do
        if p != hull.last() and p is on the right of the segment
          "hull.last(), candidate" then
            candidate = p
    if candidate != hull.first then hull.push(candidate) else break
return hull
```

Graham's scan

- Sort the points in clockwise order around the leftmost point - **Cost:** $O(n \log n)$
- Keep the current hull in a stack $H = \{h_0, \ldots, h_{curr}\}$.
- Inspect each point p in order - **Cost:** $O(n)$
  - While $h_{curr-1}h_{curr}p$ is a right turn, pop h_{curr}.
  - Push p to H.
- **Overall complexity:** $O(n \log n)$

Graham's scan

Algorithm 2: Graham Scan

Input: a list *S* of bidimensional points.
Output: the convex hull of the set, sorted counterclockwise.

```
hull =[]
x_0 = the leftmost point.
Put x_0 as the first element of S.
Sort the remaining points in counter-clockwise order, with
  respect to x_0.
Add the first 3 points in S to the hull.
forall the remaining points in S do
    while hull.second_to_last(), hull.last(), p form a right turn do
        hull.pop()
    hull.push(p)
return hull
```

Can we do better?

So far:

- → **Jarvis's march:** $\boldsymbol{O(nh)}$
- → **Graham's scan:** $\boldsymbol{O(n \log n)}$

→ How do we compare their complexity?

→ We would like to do even better!

Can we get to $\boldsymbol{O(n \log h)}$?

CHAN'S 2D ALGORITHM

Chan's 2D algorithm

- **Idea:** Some points will never be in the hull! Discard them to speed up *Jarvis's march*.
- Combine *Jarvis's march* and *Graham's scan*.

- Algorithm (**Chan 2D**):
  - Split the n points in groups of size m ($\lceil n/m \rceil$ groups).
  - Compute the hull H_i of each group
    Cost: $\boldsymbol{O((n/m) \cdot (m \log m)) = O(n \log m)}$
  - Until the hull is complete, repeat:
    - Given the current hull $H_{tot} = \{h_0, \ldots, h_{curr}\}$, find for each H_i the point right tangent to h_{curr} (binary search, $\boldsymbol{O(\log m) \lceil n/m \rceil}$).
    - Pick the tangent point p that maximizes $\angle h_{curr-1} h_{curr} p$.

One step, visually

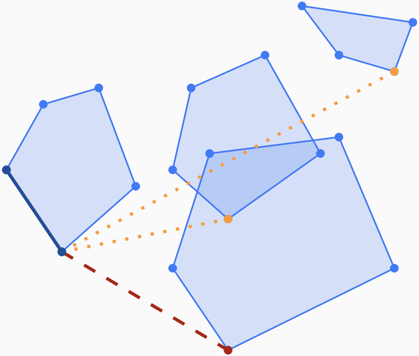


Figure: *A step of Chan's algorithm. In blue, the existing hull, in orange, the tangents, in red, the new edge that will be added.*

Chan's 2D algorithm, part 2

Complexity: $\boldsymbol{O(n \log m + (h(n/m)\log m))}$

What about m? Let's pretend the size h of the final hull is know.

With $m = h$, we get complexity $O(n \log h + (h(n/h)\log h)) = \boldsymbol{O(n \log h)}$

But h is not known!

- → Reiterate the algorithm, with $m = 2^{2^i}$
- → Iteration cost: $\boldsymbol{O(n \log H) = O(n2^i)}$
- → $O\left(\sum_{i=1}^{\lceil \log \log h \rceil} n2^i\right) = O(n2^{\lceil \log \log h \rceil + 1}) = \boldsymbol{O(n \log h)}$

Chan's 2D pseudo-code

Algorithm 3: ChanHullStep, a step of Chan's algorithm

Input: a list S of bidimensional points, the parameters m, H
Output: the convex hull of the set, sorted counterclockwise, or an empty list, if H is $< h$

Partition S into subsets $S_1, \ldots, S_{\lceil n/m \rceil}$.
for $i = 1, \ldots, \lceil n/m \rceil$ **do**
  Compute the convex hull of S_i by using Graham Scan, store the output in a counter-clockwise sorted list.
$p_0 = (0, -\infty)$
$p_1 =$ the leftmost point of S.
for $k = 1, \ldots, H$ **do**
  for $i = 1, \ldots, \lceil n/m \rceil$ **do**
    Compute the points $q_i \in S$ that maximizes $\angle p_{k-1} p_k q_i$, with $q_i \neq p_k$, by performing binary search on the vertices of the partial hull S_i.
  $p_{k+1} =$ the point $q \in \{q_1, \ldots, q_{\lceil n/m \rceil}\}$.
  if $p_{k+1} = p_t$ **then** return $\{p_1, \ldots, p_k\}$
return *incomplete*

Chan's 2D pseudo-code, reprise

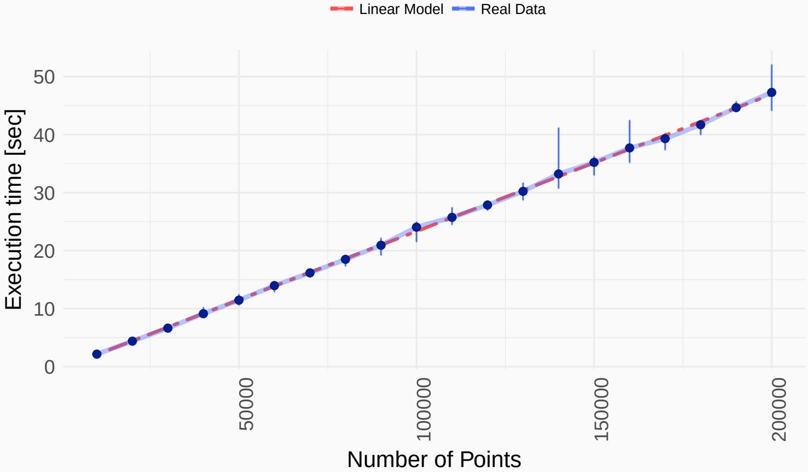
Algorithm 4: Chan's algorithm

Input: a list S of bidimensional points
Output: the convex hull of the set
for $i = 1, 2, \ldots$ **do**
  L = ChanHullStep(S, m, H), where $m = H = min\{|S|, 2^{2^i}\}$
  if $L \neq$ *incomplete* **then** return L

Chan's 2D - Empirical analysis

Is a real implementation ***O*(*n log h*)**?

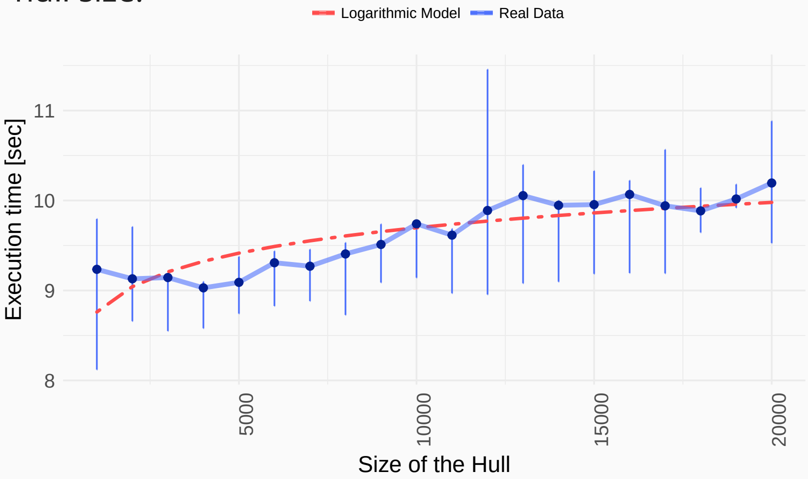
Test 1: fixed hull size (1000), increasing number of points.



Chan's 2D - Empirical analysis

Is a real implementation ***O*(*n log h*)**?

Test 2: fixed number of points (40000), increasing hull size.



Chan's 3D Algorithm

Chan's 3D algorithm

→ **Idea:** The structure of the algorithm doesn't change, just the ingredients.

Given a set S of n points with an hull of size h:

- Jarvis's march $\longrightarrow$ **Chand and Kapur's gift wrapping** - $O(nh)$.
- Graham's scan $\longrightarrow$ **Quickhull 3D** - $O(n \log n)$.
- Finding tangents in 2D $\longrightarrow$ **Supporting planes in 3D with Dobkin-Kirkpatrick hierarchy** - $O(\log n)$.

The overall complexity is still $O(n \log h)$.

Chan's 3D algorithm

- **3D Gift wrapping**
  - Pick an edge e_j of a face f of the partial hull.
  - Find p that maximizes the angle between f and the new face given by e_j and p. Repeat for the 3 edges of f.
  - Use *breadth first visit* to build the entire hull.
- **Quickhull 3D, a divide-and-conquer algorithm for convex hulls**
  - Split the set of points S in S_1 and S_2.
  - Recursively split until the base case, then build the convex hull.
  - Merge the partial hulls.

Chan's 3D algorithm

- To find supporting planes, store each partial hull as a **Dobkin-Kirkpatrick hierarchy**
  - Sequence $P_0, \ldots, P_k$ of increasingly smaller approximations of a polyhedron $P = P_0$.
  - Build P_{i+1} from P_i by picking a maximal set of independent vertices of P_i.
  - Overall, building the hierarchies takes $\boldsymbol{O(n)}$.
- **Finding supporting planes in 3D**
  - **Goal:** given an edge e_j and the DK hierarchy of a partial hull H_i, find the plane passing through e_j and tangent to H_i in p_t.
  - Find p_t in constant time in P_k
  - Step up in the hierarchy: if p_t changes, it will move to a neighbour (constant time check).
  - The DK hierarchy has $\boldsymbol{O(\log m)}$ height.

THANK YOU!

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Beamer theme: *Presento*, by Ratul Saha. *The research system in Germany*, by Hazem Alsaied