CONVEX HULLS

Chan's optimal output sensitive algorithms for convex hulls

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December 15, 2016

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- → **Algorithms for convex hulls**

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- → Optimal 2D algorithm

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BASIC NOTIONS

Crash course on convex hulls

A *convex set* S is a set in which, $\forall\, x, y \in S$, the segment $xy \subseteq S$.

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$\rightarrow$ the *minimal convex set* containing P.

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Given a set of points P in d dimensions, the **Convex Hull** $\mathbf{CH(P)}$ of P is:

→ the *minimal convex set* containing P.

→ the union of *all convex combinations* of points in P, i.e. the points $CH(P)$ are s.t.

$$\sum_{i=1}^{|P|} w_i \cdot x_i,\ \forall x_i \in P,\ \forall w_i :\ w_i \geq 0 \textit{ and } \sum_{i=1}^{|P|} w_i = 1$$

Output sensitive algorithm

The complexity of an *output-sensitive* algorithm is a function of both the *input size* and the *output size*.

ALGORITHMS FOR CONVEX HULLS

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- At each step we scan all the $\boldsymbol{n}$ points.
- How many steps? $\boldsymbol{h}$, size of the hull.
- **Complexity:** $\boldsymbol{O(nh)}$

One step, visually

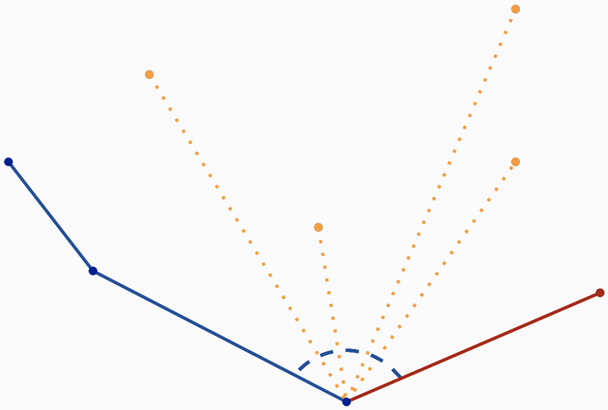

Figure: *A step of the Jarvis March. The red line is the next segment that will be added to the hull.*

Jarvis's march

Algorithm 1: Jarvis March

Input: a list S of bidimensional points.
Output: the convex hull of the set, sorted counterclockwise.

```
hull =[]
x_0 = the leftmost point.
hull.push(x_0)
Loop hull.last() != hull.first()
    candidate = S.first()
    foreach p in S do
        if p != hull.last() and p is on the right of the segment
          "hull.last(), candidate" then
            candidate = p
    if candidate != hull.first then hull.push(candidate) else break
return hull
```

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- **Overall complexity:** $\boldsymbol{O(n \log n)}$

Graham's scan

Algorithm 2: Graham Scan

Input: a list S of bidimensional points.
Output: the convex hull of the set, sorted counterclockwise.

```
hull =[]
x_0 = the leftmost point.
Put x_0 as the first element of S.
Sort the remaining points in counter-clockwise order, with
  respect to x_0.
Add the first 3 points in S to the hull.
forall the remaining points in S do
    while hull.second_to_last(), hull.last(), p form a right turn do
        hull.pop()
    hull.push(p)
return hull
```

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Can we get to $O(n \log h)$?

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    - Pick the tangent point p that maximizes $\angle h_{curr-1} h_{curr} p$.

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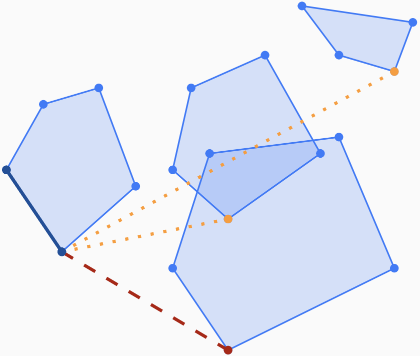

Figure: *A step of Chan's algorithm. In blue, the existing hull, in orange, the tangents, in red, the new edge that will be added.*

Chan's 2D algorithm, part 2

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- → Iteration cost: $\boldsymbol{O(n \log H) = O(n2^i)}$
- → $O\left(\sum_{i=1}^{\lceil \log \log h \rceil} n2^i\right) = O(n2^{\lceil \log \log h \rceil + 1}) = \boldsymbol{O(n \log h)}$

Chan's 2D pseudo-code

Algorithm 3: ChanHullStep, a step of Chan's algorithm

Input: a list S of bidimensional points, the parameters m, H
Output: the convex hull of the set, sorted counterclockwise, or an empty list, if H is $< h$

Partition S into subsets $S_1, \ldots, S_{\lceil n/m \rceil}$.
for $i = 1, \ldots, \lceil n/m \rceil$ **do**
    Compute the convex hull of S_i by using Graham Scan, store the output in a counter-clockwise sorted list.
$p_0 = (0, -\infty)$
$p_1 =$ the leftmost point of S.
for $k = 1, \ldots, H$ **do**
    for $i = 1, \ldots, \lceil n/m \rceil$ **do**
        Compute the points $q_i \in S$ that maximizes $\angle p_{k-1} p_k q_i$, with $q_i \neq p_k$, by performing binary search on the vertices of the partial hull S_i.
    $p_{k+1} =$ the point $q \in \{q_1, \ldots, q_{\lceil n/m \rceil}\}$.
    if $p_{k+1} = p_t$ **then** return $\{p_1, \ldots, p_k\}$
return *incomplete*

Chan's 2D pseudo-code, reprise

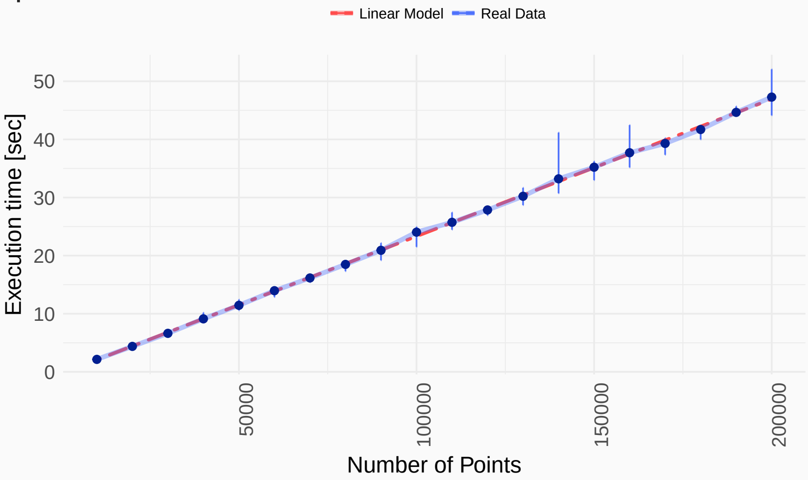
Algorithm 4: Chan's algorithm

Input: a list *S* of bidimensional points
Output: the convex hull of the set
for $i = 1, 2, \ldots$ **do**
  L = ChanHullStep(S, m, H), where $m = H = min\{|S|, 2^{2^i}\}$
  if $L \neq incomplete$ **then** return L

Chan's 2D - Empirical analysis

Is a real implementation ***O*(*n log h*)**?

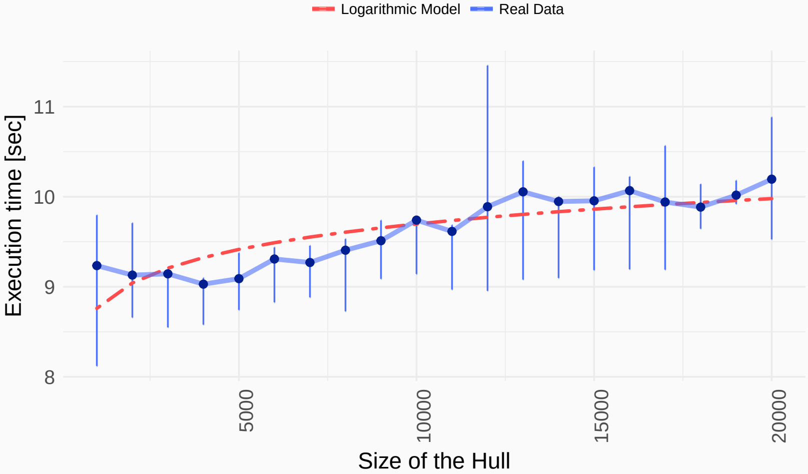
Test 1: fixed hull size (1000), increasing number of points.



Chan's 2D - Empirical analysis

Is a real implementation ***O(n log h)***?

Test 2: fixed number of points (40000), increasing hull size.



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  - Finding tangents in 2D $\longrightarrow$ **Supporting planes in 3D with Dobkin-Kirkpatrick hierarchy** - $O(\log n)$.

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The overall complexity is still $\boldsymbol{O(n \log h)}$.

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  - Merge the partial hulls.

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  - The DK hierarchy has $\boldsymbol{O(\log m)}$ height.

THANK YOU!

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Beamer theme: *Presento*, by Ratul Saha. *The research system in Germany*, by Hazem Alsaied