Demonstration of the insolubility of the Solitaire Mancala

Luca Stornaiuolo – DOSE 2015, group 5

Hypothesis:

3.1 Solitaire Mancala Rules

SR001: Initial Setup – Mancala board. The game shall be played using a standard Mancala board consisting of two rows, where each row has 6 holes and 1 store. The board is defined as:

<table>
<tr><td rowspan="2">store01</td><td>hole01</td><td>hole02</td><td>hole03</td><td>hole04</td><td>hole05</td><td>hole06</td><td rowspan="2">store02</td></tr>
<tr><td>hole07</td><td>hole08</td><td>hole09</td><td>hole10</td><td>hole11</td><td>hole12</td></tr>
</table>

SR002: Initial Setup – initial configuration. The game shall be initiated with 48 stones randomly split into the 12 holes (hole01, hole02, … hole12; see SR001), leaving each hole with at least 1 stone (all 48 stones shall all be placed in the board, leaving the stores empty).

SR003: Selection of hole before first round. The player shall select a hole (in any row, thus any of the 12 holes). SR004: A round. When a hole is selected, all stones in the selected hole are distributed either clockwise or anti-clockwise (the user can choose), as follows:

SR004(A): Dropping clockwise. One stone is placed in each hole starting with the hole next to the selected one, in clockwise direction:

- If the number of stones remaining to be distributed is more than 1 after dropping in hole06 or hole07, then store02 or store01 respectively is skipped, and the next stone is dropped in hole12 or hole01 respectively. The round is over when there are no more stones to distribute. If the game is not over (see SR005 and SR006), a new round starts as 4 described in SR004, where the selected hole will be the one where the last stone of this round was dropped.
- If the number of stones remaining to be distributed is 1 after dropping in hole06, this stone is dropped in store02. The round is over. If the game is not over (see SR005 and SR006), a new round starts as described in SR004, where the selected hole will be hole06.
- If the number of stones remaining to be distributed is 1 after dropping in hole07, this stone is dropped in store01. The round is over. If the game is not over (see SR005 and SR006), a new round starts as described in SR004, where the selected hole will be hole07.

SR004(B): Dropping anti-clockwise. One stone is placed in each hole starting with the hole next to the selected one, in anti-clockwise direction:

- If the number of stones remaining to be distributed is more than 1 after dropping in hole01 or hole12, the store01 or store02 respectively is skipped, and the next stone is dropped in hole07 or hole06 respectively. The round is over when there are no more stones to distribute. If the game is not over (see SR005 and SR006), a new round starts as described in SR004, where the selected hole will be the one where the last stone of this round was dropped.

- If the number of stones remaining to be distributed is 1 after dropping in hole01, this stone is dropped in store01. The round is over. If the game is not over (see SR005 and SR006), a new round starts as described in SR004, where the selected hole will be hole01.
- If the number of stones remaining to be distributed is 1 after dropping in hole12, this stone is dropped in store02. The round is over. If the game is not over (see SR005 and SR006), a new round starts as described in SR004, where the selected hole will be hole12.

SR005: Game over – player loses. When the last stone distributed in a round is placed in an empty hole, the player loses and the game is over.

SR006: Game over – player wins. The player wins the game if no stone remains in any of the 12 holes.

Thesis:

The player never wins.

The condition described in paragraph SR006 is unreachable using the provided game rules.

<table>
<tr><td rowspan="2">store01</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td rowspan="2">store02</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

Definitions:

s_i: a valid game configuration (see SR001, SR002 and SR003);

I: the set of initial configurations (see SR002);

G: the set of configurations corresponding to a game-losing configuration (see SR005);

x: the number of stones used in the game | $0 \leq x \leq 48$;

$S(s_i)$: set of valid successors of the game configuration s_i (a successor of s_i is a valid game configuration reachable from s_i using the game rules (see SR004(A) and SR004(B));

$P(s_i)$: set of valid predecessors of the game configuration s_i (a predecessor of s_i is a valid game configuration from which it is possible to reach s_i using the game rules (see SR004(A) and SR004(B));

s_i → SR004(A): Dropping clockwise. → s_j: s_j is reachable from s_i using the game rules (see SR004(A));

s_i → SR004(B): Dropping anti-clockwise. → s_j: s_j is reachable from s_i using the game rules (see SR004(B));

red hole: is the selected hole.

e.g.

store01	x	x	x	x	x	x	store02
	x	x	x	x	x	x	

Demonstration:

D1:

- if the selected hole is not equal to hole01, hole06, hole07, hole12 and the number of stones in the selected hole is equal to 1 then the configuration is game-losing.

s_1

store01	x	1	x	x	x	x	store02
	x	x	x	x	x	x	

s_2

store01	x	x	1	x	x	x	store02
	x	x	x	x	x	x	

s_3

store01	x	x	x	1	x	x	store02
	x	x	x	x	x	x	

s_4

store01	x	x	x	x	1	x	store02
	x	x	x	x	x	x	

s_5

store01	x	x	x	x	x	x	store02
	x	1	x	x	x	x	

s_6

store01	x	x	x	x	x	x	store02
	x	x	1	x	x	x	

s_7

store01	x	x	x	x	x	x	store02
	x	x	x	1	x	x	

s_8

store01	x	x	x	x	x	x	store02
	x	x	x	x	1	x	

$s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in G$ (see SR004(A), SR004(B) and SR005)

D2:

- If s_i is a game-losing configuration then it has no successors.

$S(s_1), S(s_2), S(s_3), S(s_4), S(s_5), S(s_6), S(s_7), S(s_8) \in \emptyset$ (see SR005)

D3:

- If s_i is a game-losing configuration then it is not a valid predecessor (see predecessor definition and D2).

$s_i \in G \Rightarrow \nexists\, s_j \mid s_i \in P(s_j)$

D4:

- To win the game one must reach the game-winning configuration s_9 (see SR006).

s_9

<table>
<tr><td rowspan="2">store01</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td rowspan="2">store02</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

D5:

- List of the valid predecessors of game-winning configuration s_9 (see the definition of predecessor).

- **P(s_9):**

s_{10}

<table>
<tr><td rowspan="2">store01</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td rowspan="2">store02</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

s_{10} → SR004(B): Dropping anti-clockwise. → s_9

s_{11}

<table>
<tr><td rowspan="2">store01</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td rowspan="2">store02</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

s_{11} → SR004(A): Dropping clockwise. → s_9

s_{12}

<table>
<tr><td rowspan="2">store01</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td rowspan="2">store02</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

s_{12} → SR004(A): Dropping clockwise. → s_9

s_{13}

<table>
<tr><td rowspan="2">store01</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td rowspan="2">store02</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td></tr>
</table>

s_{13} → SR004(B): Dropping anti-clockwise. → s_9

Author's note:

As s_{10}, s_{11}, s_{12} and s_{13} are clearly symmetrical, only the proof for s_{10} will be reported in details, without loss of generality; similar demonstrations could be done to prove that configurations s_{11}, s_{12}, s_{13} are unreachable from any valid starting configuration.

- **P(s_{10}):**

s_{14}

store01	0	**2**	0	0	0	0	store02
	0	0	0	0	0	0	

s_{14} → SR004(B): Dropping anti-clockwise. → s_{10}

s_{15}

store01	0	**1**	0	0	0	0	store02
	0	0	0	0	0	0	

s_{15} → SR004(B): Dropping anti-clockwise. → s_{10}

s_{15} ϵ G (see D1) => s_{15} is not a valid predecessor of s_{10} (see D3)

- **P(s_{14}):**

s_{16}

store01	**1**	1	0	0	0	0	store02
	0	0	0	0	0	0	

s_{16} → SR004(A): Dropping clockwise. → s_{14}

s_{17}

store01	0	1	**1**	0	0	0	store02
	0	0	0	0	0	0	

s_{17} → SR004(B): Dropping anti-clockwise. → s_{14}

s_{17} ϵ G (see D1) => s_{17} is not a valid predecessor of s_{14} (see D3)

- **P(s_{16}):**

s_{18}

store01	0	0	2	0	0	0	store02
	0	0	0	0	0	0	

s_{18} → SR004(B): Dropping anti-clockwise. → s_{16}

If s_{16} is reached from s_{18} then it becomes a game-losing configuration (see SR004(B)), but $s_{14} \in S(s_{16})$ so s_{18} is not a valid predecessor of s_{16} (see D2).

s_{19}

store01	0	0	3	0	0	0	store02
	0	0	0	0	0	0	

s_{19} → SR004(B): Dropping anti-clockwise. → s_{16}

- **P(s_{19}):**

s_{20}

store01	0	1	2	0	0	0	store02
	0	0	0	0	0	0	

s_{20} → SR004(A): Dropping clockwise. → s_{19}

$s_{20} \in G$ (see D1) => s_{20} is not a valid predecessor of s_{19} (see D3)

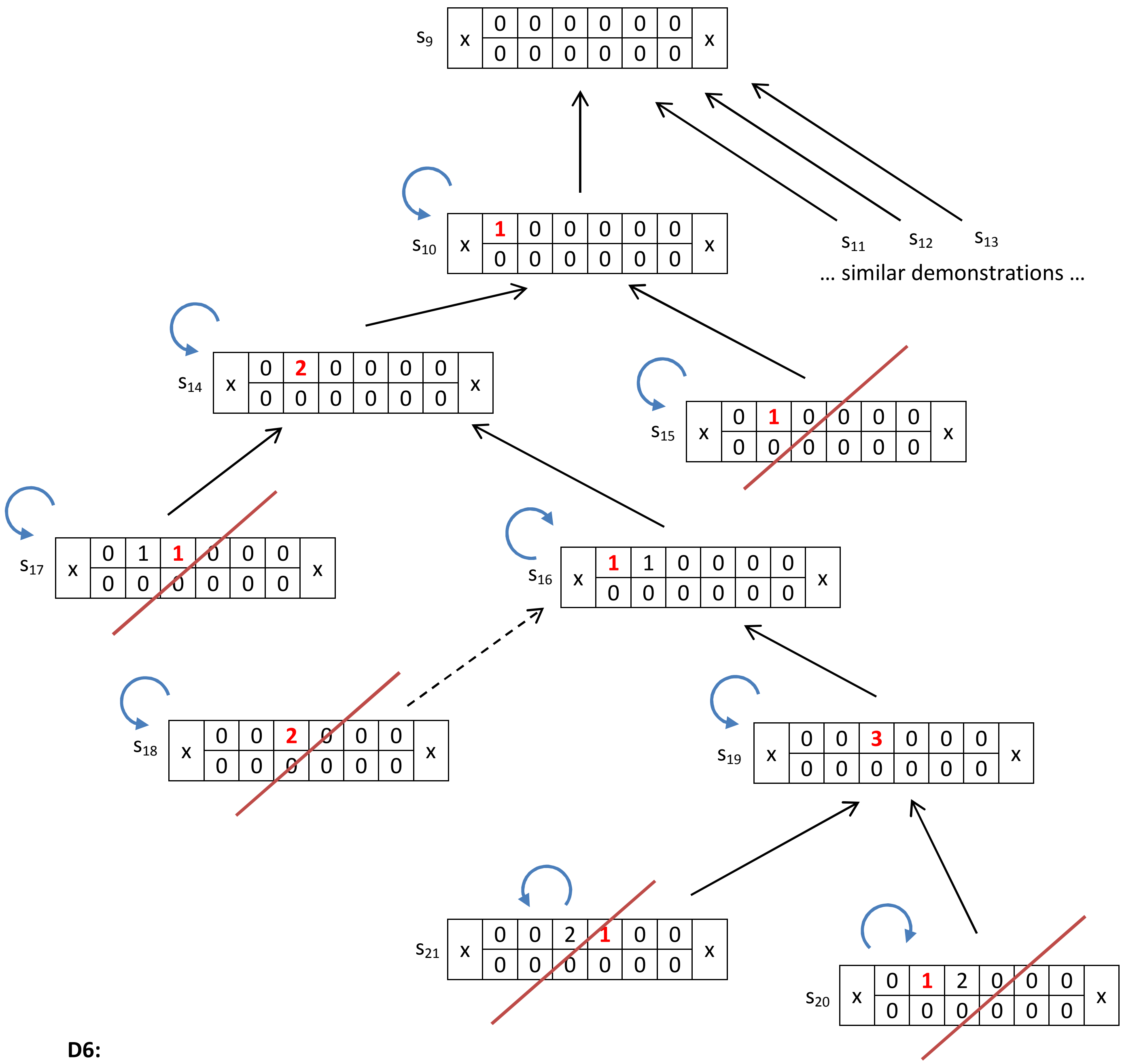
s_{21}

store01	0	0	2	1	0	0	store02
	0	0	0	0	0	0	

s_{21} → SR004(B): Dropping anti-clockwise. → s_{19}

$s_{21} \in G$ (see D1) => s_{21} is not a valid predecessor of s_{19} (see D3)

Configurations reachable from the game-winning configuration, obtained by performing valid moves in reverse order (predecessors of game-winning configuration).



D6:

- valid predecessors of game-winning configuration s_9 are not initial configurations

$\exists$ $hole_i$ | stones of $hole_i = 0$ => $s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21} \notin I$ (see SR002)

D7:

- a state s_i is unreachable using the game rules if

$P(s_i) \in \emptyset$ *or* ($\forall$ $s_j \in P(s_i)$ | $s_j \notin I$ *and* s_j is unreachable using the game rules)

(see SR002, SR003, SR004(A) and SR004(B))

Thesis:

- s_9 is unreachable using the game rules (see D5, D6 and D7)

Q.E.D.