Finding Time-dependent Shortest Paths over Large Graphs

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DSSC - Algorithmic Design exam

Piano della presentazione

Introduction

1. TSDP (time-dependent shortest path problem): to find the optimal path (with minimum travel time) from a source to a destination, over a time-dependent graph.
2. We concentrate on finding the *least total travel time* (LTT) from source node v_s to a dest. node v_e with starting time $t \in [t_s, t_e]$, chosen from the user. Such a query is called an LTT *query*, denoted as LTT(v_s, v_e).
3. More focused in on a specific class of graphs, called FIFO time-dependent graphs.

Problem Definition

Definition

A *time-dependent graph* is defined as $G_T(V, E, W)$ (or G_T for short):

1. $V = \{v_i\}$ is a set of nodes.
2. $E \subseteq V \times V$ is a set of edges.
3. W is a set of positive valued functions.

For every edge $(v_i, v_j) \in E$, there is a function $\omega_{i,j}(t)$, where t is a time variable in a domain T: It specifies how much time it takes to travel from v_i to v_j, if departing v_i at time t.

Problem Definition

Definition

Given a time-dependent graph $G_T(V, E, W)$ and a LTT(v_s, ve, T), where $v_s, v_e \in V$ and $T \in \tau$ is a starting-time interval, the *time-dependent shortest path* (TDSP) problem is to minimize LTT:

$$g_{p^*}(t^*) - t^* = \min_{p,\omega(\cdot),t} g_p(t) - t$$

where p^* is the path $v_s - v_e$, $\omega^*(v_i)$ is the waiting time in v_i and t^* is the best starting time, with which results in the minimum travel time $g_p(t) - t$.

Existing Solutions

three different types of algorithm:

1. Discrete-time algorithm.
2. *Bellman-Ford* based algorithm.
3. A* algorithm.

observation

Main challenge: edge delays are different function of departure times, the $v_s - v_e$ path with the least total travel time changes in a complicated manner as the starting time changes

Discrete-time algorithm

1. Find approximate LTT by globally discretizing time interval into time points.
2. Given a graph $G_T(V, E, W)$, we discretize the starting-time interval $T = [t_s, t_e]$ into k points and contructs a static graph $G'_T(V', E', W')$ by making k copies of each node and each edge.
3. For each edge (v'_i, v'_j), $w'_{i,j}$ is equal to the value of $w_{i,j}(t)$ on a fixed time point.
4. static-single-source shortest path problem on $G'_T(V', E', W')$.

observation

Increasing k deteriorates the efficiency of discrete-time approaches, since G'_t is k times larger than G_t.

Bellman-Ford Based Algorithm

Definition

Let we give the following definitions:

1. $g_l(t)$ the *earliest arrival time* function at node v_l, from source v_s, for starting time t.
2. $h_{k,l}(t)$ the *earliest arrival time* at v_l, from source v_s via edge (v_k, v_l), for starting time t.

how does the algorithm work?

1. It updates $g_l(t)$ and $h_{k,l}(t)$ until they converge to the correct values.
2. it returns the best starting time t^* and the optimal path p^*.
3. Time complexity is $O(|V| \cdot |E| \cdot \alpha(T))$ where $\alpha(T)$ is the time required in a function operation in T.

Bellman-Ford Based Algorithm

Pseudo-code

```
1: for all $v_l \in V$ do $g_l \leftarrow \infty$ for $t \in T$;
2: for all $(v_k, v_l) \in E$ do $h_{k,l} \leftarrow \infty$ for $t \in T$;
3: $g_s(t) \leftarrow t$ for $t \in T$;
4: repeat
5:     for all $(v_k, v_l) \in E$ do $h_{k,l} \leftarrow g_k(t) + w_{k,l}(g_k(t))$;
6:     for all $v_l \in V$ do $g_l(t) \leftarrow \min_{v_k \in N(v_l)} h_{k,l}(t)$;
7: until all functions $g_l(t)$ are unchanged;
8: return $(t* \leftarrow \arg\min_{t \in T} g_e(t) - t, p*)$
```

A* Algorithm

Main idea: maintain a priority queue of all paths to be expanded.

1. Let p_k be a path from v_s to v_k (there can be multiple paths of this type in the queue).
2. Each path is associated with a function $f_{p_k}(t) = g_{p_k}(t) + d_{k,e} - t$.
3. In each iteration we pick the path which $\min_t\{f_{p_i}(t)\}$ is minimum and we extend it with one more edge (v_i, v_j): New path is added in priority queue and the old one is deleted.
4. In worst case,all possible paths are enumerated and time/space complexity is exponential with respect the size of G_T.

Notation

First of all, we introduce some important notations:

1. $G_T(V, E, W)$: time-dependent graph (G_T).
2. $w_{i,j}(t)$: edge-delay function for $(v_i, v_j) \in E$.
3. v_s, v_e, T: source,destination and starting-time interval.
4. p^*: optimal path from v_s to v_e.
5. t^*: optimal starting time.
6. $\omega^*(v_i)$: optimal waiting time at node v_i.
7. $g_i(t)$: $v_s - v_i$ earliest arrival-time function.
8. $g_p(t)$: arrival-time function (along path p).
9. $\alpha(T)$: time/space required to maintain a function or to manipulate a function operation over time interval T

FIFO time-dependent graph

Introduction

We focus in answering LTT queries in an FIFO (*First-in* and *Firt-out*) time-dependent graph G_T,where no waiting time is needed in optimal solution.

Definition

Time-dependent graph G_T is a FIFO graph,iff every edge (v_i, v_j) has FIFO property. An edge (v_i, v_j) has FIFO property, iff $w_{i,j}(t_{t_0}) \leq t_\Delta + w_{i,j}(t_0 + t_\Delta)$ for $t_\Delta \geq 0$.

Teorema

For a given LTT query on a FIFO time-dependent graph G_T, there exists an optimal path $p*$ along which the optimal waiting time is 0 for every v_i on $p*$.

New Dijkstra Based Algorithm

Organization of the algorithm

Answering LTT query can be done in two decoupled steps:

1. *time-refinement*: for every node $v_i \in V$ to compute the earliest arrival time $g_i(t)$,departing from v_s at any starting time $t \in T$.
2. *path-selection*: we select from first step one of the paths from v_s to v_e,which matches the optimal travel time $g_e(t*) - t*$.

These two steps are grouped in the so called **TWO-STEP-LTT**.

Two-step-LTT

```
TWO-STEP-LTT($G_T(V, E, W), v_s, v_e, T$):
1: $\{g_i(t)\} \leftarrow$ timeRefinement($G_T, v_s, v_e, T$);
2: if !($g_e(t) = \infty$ for the entire $[t_s, t_e]$)then
3 :     t* $\leftarrow \arg\min_{t \in T}\{g_e(t) - t\}$;
4:      $p* \leftarrow$ pathSelection($G_T, g_i(t), v_s, v_e, t*$);
5:      return($t*, p*$);
6: else return 0;
```

Time-refinement

How does the algorithm works?

The first step is dominating factor in terms of computational cost. We compute $g_i(t)$ for every node $v_i \in V$, through this recursive equation:

$$g_i(t) = \min_{v_j \in N(v_i), \omega(v_j)} (g_j(t) + \omega(v_j)) + w_{i,j}(g_j(t) + \omega(v_j))$$

where $N(v_i) = \{v_j | (v_i, v_j) \in E\}$

Definition

We say that function $g_i(t)$ is *well-refined* in a starting-time subinterval I_i,if it specifies the earliest arrival time at v_i from v_s for any starting time $t \in I_i$.

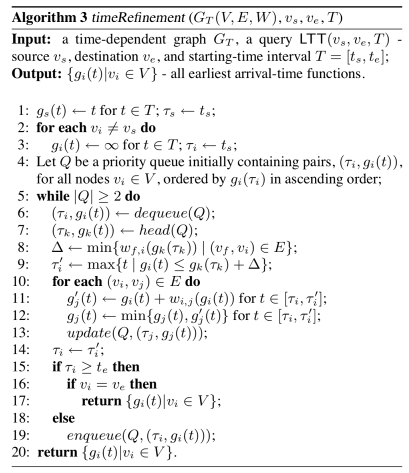
Time-refinement

How does the algorithm works?

1. We refine $g_i(t)$, incrementally in the given starting-time interval $T = [t_s, t_e]$.
2. $\forall v_i \in V$,let $I_i = [t_s, \tau_i] \subseteq T$ be a starting-time subinterval, where $\tau_i \in T = [t_s, t_e]$.
3. We "incrementally" refine $g_i(t)$ to a largest starting-time subinterval $I'_i{}_i$, in a way that $g_i(t)$ is always well-refined.
4. We go ahead until $g_e(t)$ is well refined in the entire starting-time interval T.

Time-refinement

Pseudo-code



Algorithm 3 *timeRefinement* $(G_T(V,E,W), v_s, v_e, T)$

Input: a time-dependent graph G_T, a query $\mathsf{LTT}(v_s, v_e, T)$ - source v_s, destination v_e, and starting-time interval $T = [t_s, t_e]$;
Output: $\{g_i(t) | v_i \in V\}$ - all earliest arrival-time functions.

```
1: $g_s(t) \leftarrow t$ for $t \in T$; $\tau_s \leftarrow t_s$;
2: for each $v_i \neq v_s$ do
3:     $g_i(t) \leftarrow \infty$ for $t \in T$; $\tau_i \leftarrow t_s$;
4: Let $Q$ be a priority queue initially containing pairs, $(\tau_i, g_i(t))$,
   for all nodes $v_i \in V$, ordered by $g_i(\tau_i)$ in ascending order;
5: while $|Q| \geq 2$ do
6:     $(\tau_i, g_i(t)) \leftarrow dequeue(Q)$;
7:     $(\tau_k, g_k(t)) \leftarrow head(Q)$;
8:     $\Delta \leftarrow \min\{w_{f,i}(g_k(\tau_k)) \mid (v_f, v_i) \in E\}$;
9:     $\tau_i' \leftarrow \max\{t \mid g_i(t) \leq g_k(\tau_k) + \Delta\}$;
10:    for each $(v_i, v_j) \in E$ do
11:        $g_j'(t) \leftarrow g_i(t) + w_{i,j}(g_i(t))$ for $t \in [\tau_i, \tau_i']$;
12:        $g_j(t) \leftarrow \min\{g_j(t), g_j'(t)\}$ for $t \in [\tau_i, \tau_i']$;
13:        $update(Q, (\tau_j, g_j(t)))$;
14:    $\tau_i \leftarrow \tau_i'$;
15:    if $\tau_i \geq t_e$ then
16:        if $v_i = v_e$ then
17:            return $\{g_i(t) | v_i \in V\}$;
18:    else
19:        $enqueue(Q, (\tau_i, g_i(t)))$;
20: return $\{g_i(t) | v_i \in V\}$.
```

More on Time-refinement algorithm

1. We use a priority Queue Q, which initially contains pairs $(\tau_i, g_i(t))$ for all nodes v_i in ascending order of $g_i(\tau_i)$.
2. While loop conducts time-refinement for every node v_i in G_T.
3. Line 6-9 and line 14: we update starting-time interval refinement, by finding the minimum travel time from v_f to fixed v_i (line 8) and by calculating τ' (line 9).
4. line 11-14: we use the well-refined $g_i(t)$ to refine arrival-time functions $g_j(t)$ in starting-time subintervals $[\tau_i, \tau_i']$ of all v_i's outgoing neighbors.

PathSelection

How does the algorithm works?

1. We determine the predecessor of every node in $p*$ backward from v_e to v_s based on $\{g_i(t)$ and $t* \in T$.
2. This algorithm takes five inputs: graph G_t,all the earliest arrival-time functions $\{g_i(t)\}$, the optimal starting time $t* \in T$,source v_s and destination v_e.
3. The predecessor of v_j is determined as v_i,if $g_j(t*) = g_i(t*) + w_{i,j}(g_i(t*))$ for $(v_i, v_j) \in E$.

Pathselection

Pseudo-Code

```
PathSelection($G_T(V, E, W)$, $\{g_i(t)\}$, $v_s$, $v_e$, $t*$):
1: $v_j \leftarrow v_e$;
2: $p* \leftarrow \emptyset$;
3: while $v_j \neq v_s$ do
4:     for each $(v_i, v_j) \in E$ do
5:         if $g_i(t*) + w_{i,j}(g_i(t*)) = g_j(t*)$ then
6:             $v_j \leftarrow v_i$; break;
7:         $p* \leftarrow (v_i, v_j) \cdot p*$;
8: return $p*$;
```

A running example

Our query is $\mathrm{LTT}(v_1, v_4, T = [0, 60])$.

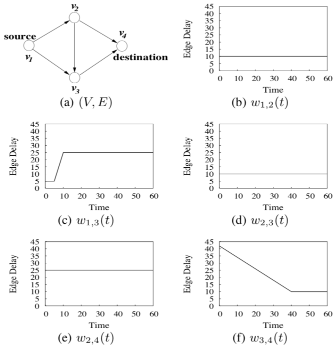
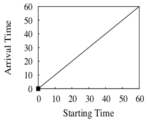


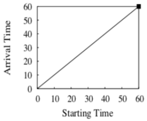
Figura: Our Time-dependent graph, which we use as example

A running example

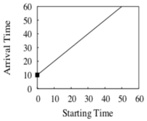
First Iteration:



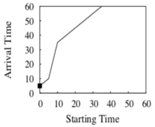
(a) $g_1(t)$



(b) $g_1(t)$

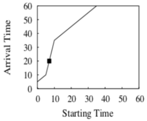


(c) $g_2(t)$

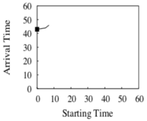


(d) $g_3(t)$

Second Iteration:



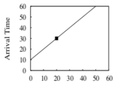
(e) $g_3(t)$



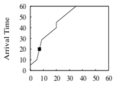
(f) $g_4(t)$

A running example

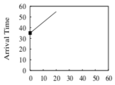
Third iteration:



(g) $g_2(t)$

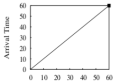


(h) $g_3(t)$

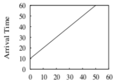


(i) $g_4(t)$

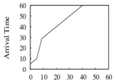
... after 11 iterations we have that all functions are well-refined!



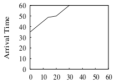
(b) $g_1(t)$



(j) $g_2(t)$



(k) $g_3(t)$



(l) $g_4(t)$

Solution

$t^* = 20$ and $p^* = (v_1, v_2)(v_2, v_3)(v_3, v_4)$.

Time/space complexity of two-Step-LTT

Given a graph G_T with n nodes and m edges in total, consider query LTT(v_s, v_e, T).

Teorema

The time complexity of the timeRefinement algorithm is $O((n \cdot \log n + m)\alpha(T))$.

Teorema

The time complexity of PathSelection is $O(m\alpha(T))$.

Teorema

the time-complexity of Two-step-LTT is $O((n \cdot \log n + m)\alpha(T))$.

Teorema

The space complexity of Two-Step-LTT is $O((n + m)\alpha(T))$.

Solution for Non-FIFO Graphs

How to find an optimal LTT over a non-FIFO graphs?

1. We can transform a non-FIFO graphs G'_T into a FIFO graph G_T,where both V and E remain unchanged.
2. Find optimal path p^* found in G_T can be converted into a optimal path p'^* for G'_T, by inserting some waiting time on each node.

solution for non-FIFO Graphs

IDEA

Foe each $w'_{i,j}(t)$ in the non-FIFO graph, we define $w_{i,j}(t)$ to construct a FIFO graph:

$$w_{i,j} = \Delta_{i,j}(t) + w'_{i,j}(t + \Delta_{i,j}(t))$$

$\Delta_{i,j}(t)$ is the optimal waiting time to traverse edge (v_i, v_j), if arriving at v_i at time t.
Then in the FIFO optimal path p^* we add the term $\Delta_{i,j}(t)$ at node v_i, for $1 \leq i \leq k-1$, where t is the arrival time at node v_i along path p^* in G_T for starting time t^*.