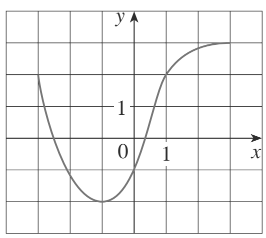
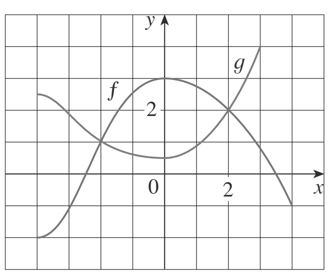
1.1 Exercises

1. The graph of a function f is given.
(a) State the value of $f(-1)$.
(b) Estimate the value of $f(2)$.
(c) For what values of x is $f(x) = 2$?
(d) Estimate the values of x such that $f(x) = 0$.
(e) State the domain and range of f.
(f) On what interval is f increasing?



2. The graphs of f and g are given.
(a) State the values of $f(-4)$ and $g(3)$.
(b) For what values of x is $f(x) = g(x)$?
(c) Estimate the solution of the equation $f(x) = -1$.
(d) On what interval is f decreasing?
(e) State the domain and range of f.
(f) State the domain and range of g.

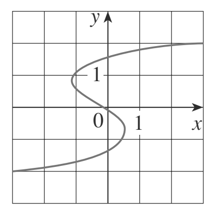


3. Figures 1, 11, and 12 were recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use them to estimate the ranges of the vertical, north-south, and east-west ground acceleration functions at USC during the Northridge earthquake.

4. In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

5–8 Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

5.



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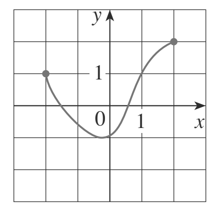

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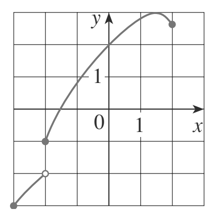

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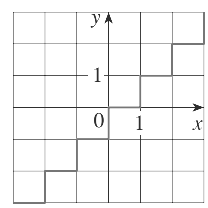
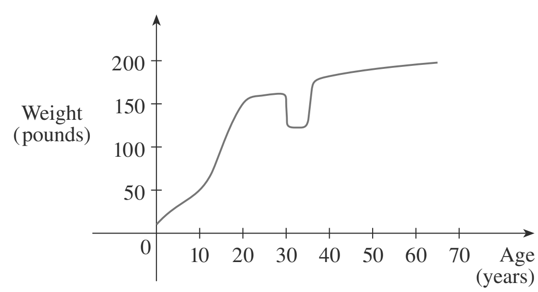
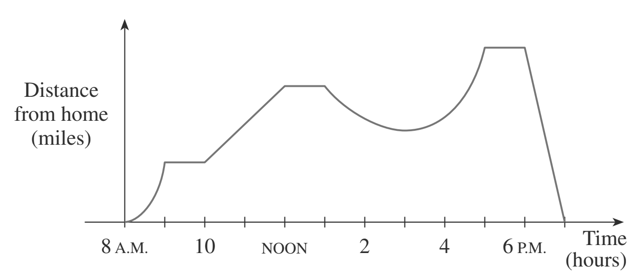


9. The graph shown gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?



10. The graph shown gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



11. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

12. Sketch a rough graph of the number of hours of daylight as a function of the time of year.

13. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

14. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

15. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

16. An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.
(a) Sketch a possible graph of $x(t)$.
(b) Sketch a possible graph of $y(t)$.
(c) Sketch a possible graph of the ground speed.
(d) Sketch a possible graph of the vertical velocity.

17. The number N (in thousands) of cellular phone subscribers in Malaysia is shown in the table. (Midyear estimates are given.)

t	1991	1993	1995	1997
N	132	304	873	2461

(a) Use the data to sketch a rough graph of N as a function of t.
(b) Use your graph to estimate the number of cell-phone subscribers in Malaysia at midyear in 1994 and 1996.

18. Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 P.M. in Dallas on June 2, 2001. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	73	73	70	69	72	81	88	91

(a) Use the readings to sketch a rough graph of T as a function of t.
(b) Use your graph to estimate the temperature at 11:00 A.M.

19. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a+1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a+h)$.

20. A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

21–22 Find $f(2+h)$, $f(x+h)$, and $\dfrac{f(x+h) - f(x)}{h}$, where $h \neq 0$.

21. $f(x) = x - x^2$

22. $f(x) = \dfrac{x}{x+1}$

23–27 Find the domain of the function.

23. $f(x) = \dfrac{x}{3x - 1}$

24. $f(x) = \dfrac{5x + 4}{x^2 + 3x + 2}$

25. $f(t) = \sqrt{t} + \sqrt[3]{t}$

26. $g(u) = \sqrt{u} + \sqrt{4 - u}$

27. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

28. Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

29–40 Find the domain and sketch the graph of the function.

29. $f(x) = 5$

30. $F(x) = \frac{1}{2}(x + 3)$

31. $f(t) = t^2 - 6t$

32. $H(t) = \dfrac{4 - t^2}{2 - t}$

33. $g(x) = \sqrt{x - 5}$

34. $F(x) = |2x + 1|$

35. $G(x) = \dfrac{3x + |x|}{x}$

36. $g(x) = \dfrac{|x|}{x^2}$

37. $f(x) = \begin{cases} x & \text{if } x \leqslant 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

38. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geqslant -1 \end{cases}$

39. $f(x) = \begin{cases} x + 2 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

40. $f(x) = \begin{cases} -1 & \text{if } x \leqslant -1 \\ 3x + 2 & \text{if } |x| < 1 \\ 7 - 2x & \text{if } x \geqslant 1 \end{cases}$

41–46 Find an expression for the function whose graph is the given curve.

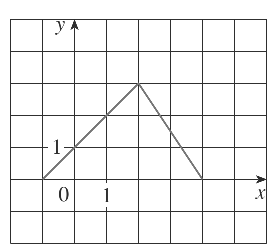
41. The line segment joining the points $(-2, 1)$ and $(4, -6)$

42. The line segment joining the points $(-3, -2)$ and $(6, 3)$

43. The bottom half of the parabola $x + (y - 1)^2 = 0$

44. The top half of the circle $(x - 1)^2 + y^2 = 1$

45.



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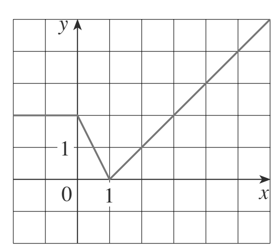


47–51 Find a formula for the described function and state its domain.

47. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

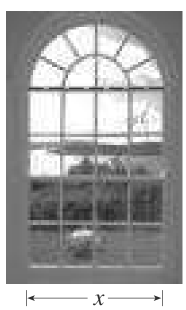
48. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

49. Express the area of an equilateral triangle as a function of the length of a side.

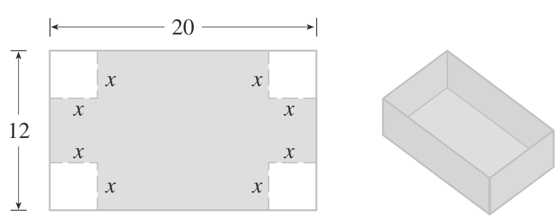
50. Express the surface area of a cube as a function of its volume.

51. An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

52. A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.



53. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



54. A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C (in dollars) of a ride as a function of the distance x traveled (in miles) for $0 < x < 2$, and sketch the graph of this function.

55. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

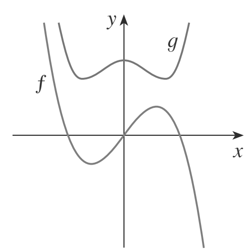
(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

56. The functions in Example 10 and Exercises 54 and 55(a) are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

57–58 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

57.



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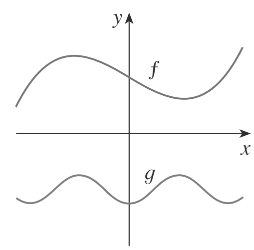


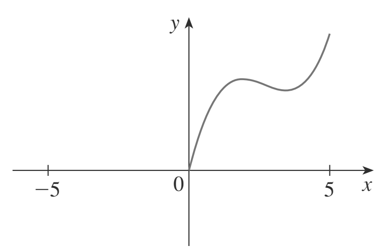
59. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?

(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

60. A function f has domain $[-5, 5]$ and a portion of its graph is shown.

(a) Complete the graph of f if it is known that f is even.

(b) Complete the graph of f if it is known that f is odd.



61–66 Determine whether f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

61. $f(x) = x^{-2}$

62. $f(x) = x^{-3}$

63. $f(x) = x^2 + x$

64. $f(x) = x^4 - 4x^2$

65. $f(x) = x^3 - x$

66. $f(x) = 3x^3 + 2x^2 + 1$

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. They will be studied in Section 1.6. Figure 21 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

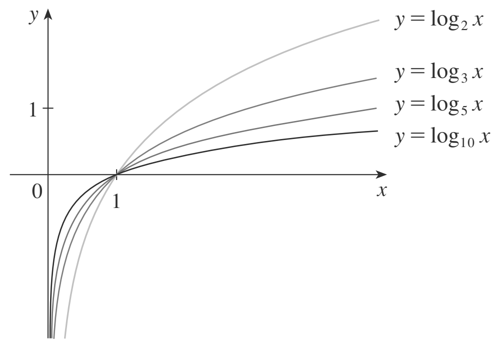


FIGURE 21

Transcendental Functions

These are functions that are not algebraic. The set of transcendental functions includes the trigonometric, inverse trigonometric, exponential, and logarithmic functions, but it also includes a vast number of other functions that have never been named. In Chapter 11 we will study transcendental functions that are defined as sums of infinite series.

EXAMPLE 5 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

1.2 Exercises

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = \sqrt[5]{x}$ (b) $g(x) = \sqrt{1 - x^2}$

(c) $h(x) = x^9 + x^4$ (d) $r(x) = \dfrac{x^2 + 1}{x^3 + x}$

(e) $s(x) = \tan 2x$ (f) $t(x) = \log_{10} x$

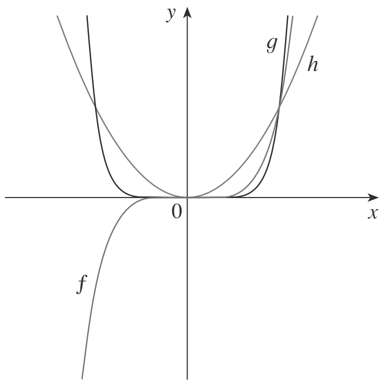
2. (a) $y = \dfrac{x - 6}{x + 6}$ (b) $y = x + \dfrac{x^2}{\sqrt{x - 1}}$

(c) $y = 10^x$ (d) $y = x^{10}$

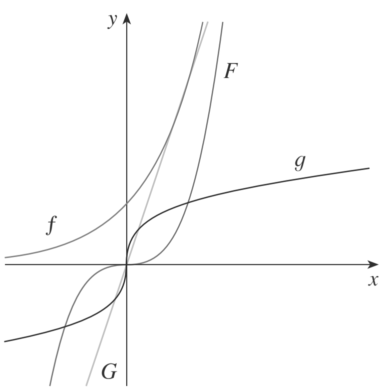
(e) $y = 2t^6 + t^4 - \pi$ (f) $y = \cos\theta + \sin\theta$

3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

3. (a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



4. (a) $y = 3x$ (b) $y = 3^x$
(c) $y = x^3$ (d) $y = \sqrt[3]{x}$



5. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
(b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
(c) Which function belongs to both families?

6. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.

7. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.

8. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the flea market, then the number y of spaces he can rent is given by the equation $y = 200 - 4x$.
(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

9. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
(a) Sketch a graph of this function.
(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

10. Jason leaves Detroit at 2:00 P.M. and drives at a constant speed west along I-96. He passes Ann Arbor, 40 mi from Detroit, at 2:50 P.M.
(a) Express the distance traveled in terms of the time elapsed.
(b) Draw the graph of the equation in part (a).
(c) What is the slope of this line? What does it represent?

11. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F.
(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.
(b) What is the slope of the graph? What does it represent?
(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

12. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y-intercept of the graph and what does it represent?

13. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.
(a) Express the water pressure as a function of the depth below the ocean surface.
(b) At what depth is the pressure 100 lb/in^2?

14. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship gives a suitable model.
(b) Use part (a) to predict the cost of driving 1500 miles per month.
(c) Draw the graph of the linear function. What does the slope represent?
(d) What does the y-intercept represent?
(e) Why does a linear function give a suitable model in this situation?

(b) $$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$$

For $\sqrt{x}$ to be defined we must have $x \geqslant 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geqslant 0$, that is, $\sqrt{x} \leqslant 2$, or $x \leqslant 4$. Thus, we have $0 \leqslant x \leqslant 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

If $0 \leqslant a \leqslant b$, then $a^2 \leqslant b^2$.

(c) $$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $$(g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$$

This expression is defined when $2 - x \geqslant 0$, that is, $x \leqslant 2$, and $2 - \sqrt{2 - x} \geqslant 0$. This latter inequality is equivalent to $\sqrt{2 - x} \leqslant 2$, or $2 - x \leqslant 4$, that is, $x \geqslant -2$. Thus, $-2 \leqslant x \leqslant 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 9 Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION
$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 3)) \\ &= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}\end{aligned}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to decompose a complicated function into simpler ones, as in the following example.

EXAMPLE 10 Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then

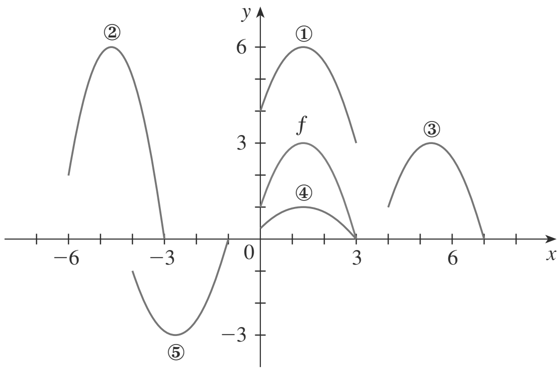
$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) \\ &= [\cos(x + 9)]^2 = F(x)\end{aligned}$$

1.3 Exercises

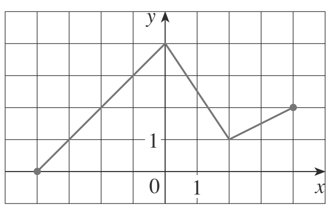
1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.
   (a) Shift 3 units upward.
   (b) Shift 3 units downward.
   (c) Shift 3 units to the right.
   (d) Shift 3 units to the left.
   (e) Reflect about the x-axis.
   (f) Reflect about the y-axis.
   (g) Stretch vertically by a factor of 3.
   (h) Shrink vertically by a factor of 3.

2. Explain how the following graphs are obtained from the graph of $y = f(x)$.
(a) $y = 5f(x)$ (b) $y = f(x - 5)$
(c) $y = -f(x)$ (d) $y = -5f(x)$
(e) $y = f(5x)$ (f) $y = 5f(x) - 3$

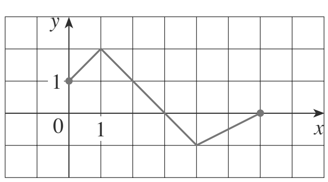
3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.
(a) $y = f(x - 4)$ (b) $y = f(x) + 3$
(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$
(e) $y = 2f(x + 6)$



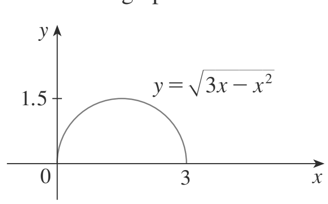
4. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x + 4)$ (b) $y = f(x) + 4$
(c) $y = 2f(x)$ (d) $y = -\frac{1}{2}f(x) + 3$



5. The graph of f is given. Use it to graph the following functions.
(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$
(c) $y = f(-x)$ (d) $y = -f(-x)$



6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



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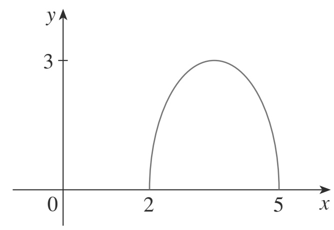

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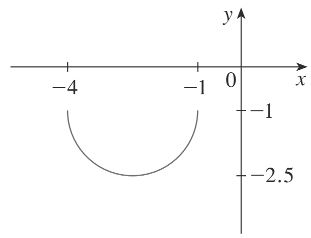


8. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2 \sin x$.
(b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

9–24 Graph the function, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

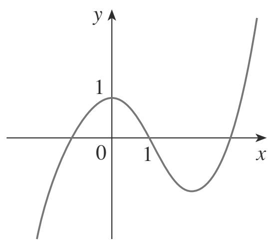
9. $y = -x^3$
10. $y = 1 - x^2$
11. $y = (x + 1)^2$
12. $y = x^2 - 4x + 3$
13. $y = 1 + 2 \cos x$
14. $y = 4 \sin 3x$
15. $y = \sin(x/2)$
16. $y = \dfrac{1}{x - 4}$
17. $y = \sqrt{x + 3}$
18. $y = (x + 2)^4 + 3$
19. $y = \frac{1}{2}(x^2 + 8x)$
20. $y = 1 + \sqrt[3]{x - 1}$
21. $y = \dfrac{2}{x + 1}$
22. $y = \dfrac{1}{4}\tan\left(x - \dfrac{\pi}{4}\right)$
23. $y = |\sin x|$
24. $y = |x^2 - 2x|$

25. The city of New Orleans is located at latitude 30°N. Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. Use the fact that on March 31 the Sun rises at 5:51 A.M. and sets at 6:18 P.M. in New Orleans to check the accuracy of your model.

26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

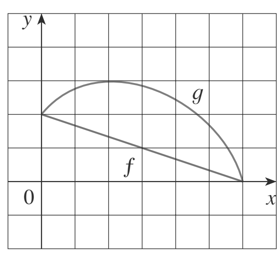
27. (a) How is the graph of $y = f(|x|)$ related to the graph of f?
(b) Sketch the graph of $y = \sin|x|$.
(c) Sketch the graph of $y = \sqrt{|x|}$.

28. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



29–30 Use graphical addition to sketch the graph of $f + g$.

29.



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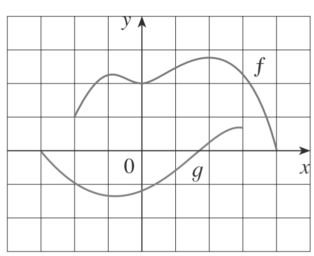


31–32 Find $f + g$, $f - g$, fg, and f/g and state their domains.

31. $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$

32. $f(x) = \sqrt{1 + x}$, $g(x) = \sqrt{1 - x}$

33–34 Use the graphs of f and g and the method of graphical addition to sketch the graph of $f + g$.

33. $f(x) = x$, $g(x) = 1/x$

34. $f(x) = x^3$, $g(x) = -x^2$

35–40 Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

35. $f(x) = 2x^2 - x$, $g(x) = 3x + 2$

36. $f(x) = 1 - x^3$, $g(x) = 1/x$

37. $f(x) = \sin x$, $g(x) = 1 - \sqrt{x}$

38. $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$

39. $f(x) = x + \dfrac{1}{x}$, $g(x) = \dfrac{x + 1}{x + 2}$

40. $f(x) = \sqrt{2x + 3}$, $g(x) = x^2 + 1$

41–44 Find $f \circ g \circ h$.

41. $f(x) = x + 1$, $g(x) = 2x$, $h(x) = x - 1$

42. $f(x) = 2x - 1$, $g(x) = x^2$, $h(x) = 1 - x$

43. $f(x) = \sqrt{x - 1}$, $g(x) = x^2 + 2$, $h(x) = x + 3$

44. $f(x) = \dfrac{2}{x + 1}$, $g(x) = \cos x$, $h(x) = \sqrt{x + 3}$

45–50 Express the function in the form $f \circ g$.

45. $F(x) = (x^2 + 1)^{10}$

46. $F(x) = \sin(\sqrt{x})$

47. $G(x) = \dfrac{x^2}{x^2 + 4}$

48. $G(x) = \dfrac{1}{x + 3}$

49. $u(t) = \sqrt{\cos t}$

50. $u(t) = \dfrac{\tan t}{1 + \tan t}$

51–53 Express the function in the form $f \circ g \circ h$.

51. $H(x) = 1 - 3^{x^2}$

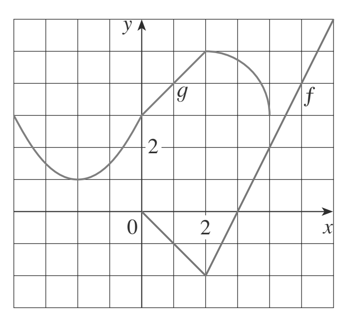
52. $H(x) = \sqrt[3]{\sqrt{x} - 1}$

53. $H(x) = \sec^4(\sqrt{x})$

54. Use the table to evaluate each expression.
(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
(d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

55. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



1.5 Exercises

1. (a) Write an equation that defines the exponential function with base $a > 0$.
(b) What is the domain of this function?
(c) If $a \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

4. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$

5. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 14 and, if necessary, the transformations of Section 1.3.

7. $y = 4^x - 3$

8. $y = 4^{x-3}$

9. $y = -2^{-x}$

10. $y = 1 + 2e^x$

11. $y = 3 - e^x$

12. $y = 2 + 5(1 - e^{-x})$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward
(b) shifting 2 units to the right
(c) reflecting about the x-axis
(d) reflecting about the y-axis
(e) reflecting about the x-axis and then about the y-axis

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$
(b) reflecting about the line $x = 2$

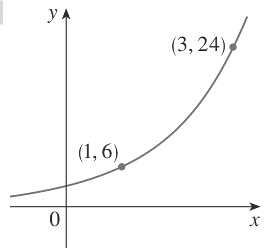
15–16 Find the domain of each function.

15. (a) $f(x) = \dfrac{1}{1 + e^x}$ (b) $f(x) = \dfrac{1}{1 - e^x}$

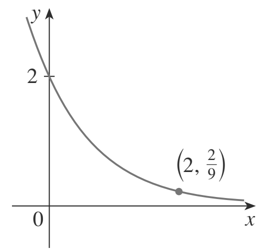
16. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

17–18 Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



18.



19. If $f(x) = 5^x$, show that

$$\frac{f(x + h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

20. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?
I. One million dollars at the end of the month.
II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day.

21. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

22. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

23. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?

24. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

25. Under ideal conditions a certain bacteria population is known to double every three hours. Suppose that there are initially 100 bacteria.
(a) What is the size of the population after 15 hours?
(b) What is the size of the population after t hours?

EXAMPLE 14 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad \text{(since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2\text{)}$$

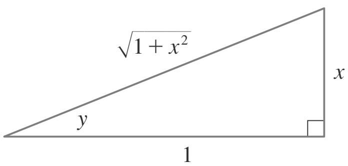


FIGURE 24

Thus $$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$

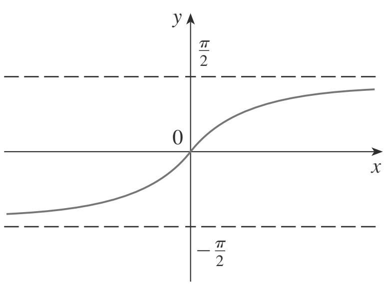


FIGURE 25
$y = \tan^{-1}x = \arctan x$

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.

We know that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

11
$$y = \csc^{-1}x \ (|x| \geqslant 1) \iff \csc y = x \text{ and } y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$
$$y = \sec^{-1}x \ (|x| \geqslant 1) \iff \sec y = x \text{ and } y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x \text{ and } y \in (0, \pi)$$

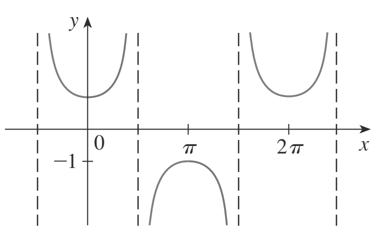


FIGURE 26
$y = \sec x$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 26 that both this choice and the one in (11) will work.]

1.6 Exercises

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

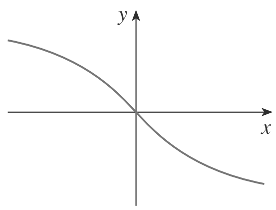
3–14 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

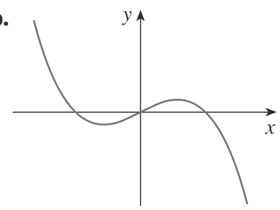
3.

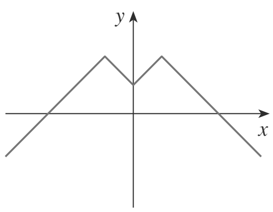
x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

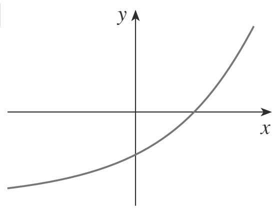
4.

x	1	2	3	4	5	6
$f(x)$	1	2	4	8	16	32

5. 

6. 

7. 

8. 

9. $f(x) = \frac{1}{2}(x + 5)$

10. $f(x) = 1 + 4x - x^2$

11. $g(x) = |x|$

12. $g(x) = \sqrt{x}$

13. $f(t)$ is the height of a football t seconds after kickoff.

14. $f(t)$ is your height at age t.

15–16 Use a graph to decide whether f is one-to-one.

15. $f(x) = x^3 - x$

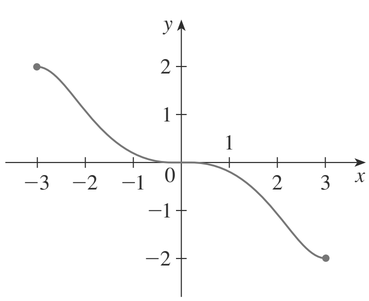
16. $f(x) = x^3 + x$

17. If f is a one-to-one function such that $f(2) = 9$, what is $f^{-1}(9)$?

18. Let $f(x) = 3 + x^2 + \tan(\pi x/2)$, where $-1 < x < 1$.
(a) Find $f^{-1}(3)$.
(b) Find $f(f^{-1}(5))$.

19. If $g(x) = 3 + x + e^x$, find $g^{-1}(4)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) State the domain and range of f^{-1}.
(c) Estimate the value of $f^{-1}(1)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–28 Find a formula for the inverse of the function.

23. $f(x) = \sqrt{10 - 3x}$

24. $f(x) = \dfrac{4x - 1}{2x + 3}$

25. $f(x) = e^{x^3}$

26. $y = 2x^3 + 3$

27. $y = \ln(x + 3)$

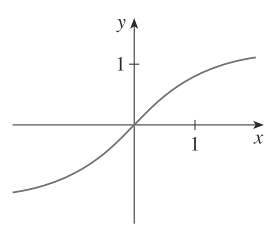
28. $y = \dfrac{1 + e^x}{1 - e^x}$

29–30 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about the line.

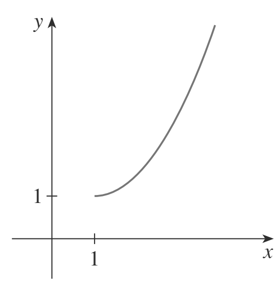
29. $f(x) = 1 - 2/x^2, \quad x > 0$

30. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

31. Use the given graph of f to sketch the graph of f^{-1}.



32. Use the given graph of f to sketch the graphs of f^{-1} and $1/f$.



33. (a) How is the logarithmic function $y = \log_a x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

34. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

35–38 Find the exact value of each expression.

35. (a) $\log_2 64$ (b) $\log_6 \frac{1}{36}$

36. (a) $\log_8 2$ (b) $\ln e^{\sqrt{2}}$

37. (a) $\log_{10} 1.25 + \log_{10} 80$
(b) $\log_5 10 + \log_5 20 - 3\log_5 2$

38. (a) $2^{(\log_2 3 + \log_2 5)}$ (b) $e^{3\ln 2}$

39–41 Express the given quantity as a single logarithm.

39. $2\ln 4 - \ln 2$ **40.** $\ln x + a\ln y - b\ln z$

41. $\ln(1 + x^2) + \frac{1}{2}\ln x - \ln\sin x$

42. Use Formula 10 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} 10$ (b) $\log_2 8.4$

43–44 Use Formula 10 to graph the given functions on a common screen. How are these graphs related?

43. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

44. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

45. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

46. Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?

47–48 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 12 and 13 and, if necessary, the transformations of Section 1.3.

47. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

48. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

49–52 Solve each equation for x.

49. (a) $2\ln x = 1$ (b) $e^{-x} = 5$

50. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

51. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

52. (a) $\ln(\ln x) = 1$ (b) $e^{ax} = Ce^{bx}$, where $a \neq b$

53–54 Solve each inequality for x.

53. (a) $e^x < 10$ (b) $\ln x > -1$

54. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

55–56 Find (a) the domain of f and (b) f^{-1} and its domain.

55. $f(x) = \sqrt{3 - e^{2x}}$ **56.** $f(x) = \ln(2 + \ln x)$

CAS **57.** Graph the function $f(x) = \sqrt{x^3 + x^2 + x + 1}$ and explain why it is one-to-one. Then use a computer algebra system to find an explicit expression for $f^{-1}(x)$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

CAS **58.** (a) If $g(x) = x^6 + x^4$, $x \geqslant 0$, use a computer algebra system to find an expression for $g^{-1}(x)$.
(b) Use the expression in part (a) to graph $y = g(x)$, $y = x$, and $y = g^{-1}(x)$ on the same screen.

59. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$. (See Exercise 25 in Section 1.5.)
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

60. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

61. Starting with the graph of $y = \ln x$, find the equation of the graph that results from
(a) shifting 3 units upward
(b) shifting 3 units to the left
(c) reflecting about the x-axis
(d) reflecting about the y-axis
(e) reflecting about the line $y = x$
(f) reflecting about the x-axis and then about the line $y = x$
(g) reflecting about the y-axis and then about the line $y = x$
(h) shifting 3 units to the left and then reflecting about the line $y = x$

62. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

63–68 Find the exact value of each expression.

63. (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1}(-1)$

64. (a) $\arctan(-1)$ (b) $\csc^{-1} 2$

65. (a) $\tan^{-1}\sqrt{3}$ (b) $\arcsin(-1/\sqrt{2})$

66. (a) $\sec^{-1}\sqrt{2}$ (b) $\arcsin 1$

67. (a) $\sin(\sin^{-1} 0.7)$ (b) $\tan^{-1}\left(\tan \dfrac{4\pi}{3}\right)$

68. (a) $\sec(\arctan 2)$ (b) $\cos\left(2 \sin^{-1}\left(\frac{5}{13}\right)\right)$

69. Prove that $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$.

70–72 Simplify the expression.

70. $\tan(\sin^{-1}x)$

71. $\sin(\tan^{-1}x)$

72. $\sin(2 \cos^{-1}x)$

73–74 Graph the given functions on the same screen. How are these graphs related?

73. $y = \sin x, \ -\pi/2 \leqslant x \leqslant \pi/2; \quad y = \sin^{-1}x; \quad y = x$

74. $y = \tan x, \ -\pi/2 < x < \pi/2; \quad y = \tan^{-1}x; \quad y = x$

75. Find the domain and range of the function

$$g(x) = \sin^{-1}(3x + 1)$$

76. (a) Graph the function $f(x) = \sin(\sin^{-1}x)$ and explain the appearance of the graph.
(b) Graph the function $g(x) = \sin^{-1}(\sin x)$. How do you explain the appearance of this graph?

1 Review

CONCEPT CHECK

1. (a) What is a function? What are its domain and range?
(b) What is the graph of a function?
(c) How can you tell whether a given curve is the graph of a function?

2. Discuss four ways of representing a function. Illustrate your discussion with examples.

3. (a) What is an even function? How can you tell if a function is even by looking at its graph?
(b) What is an odd function? How can you tell if a function is odd by looking at its graph?

4. What is an increasing function?

5. What is a mathematical model?

6. Give an example of each type of function.
(a) Linear function (b) Power function
(c) Exponential function (d) Quadratic function
(e) Polynomial of degree 5 (f) Rational function

7. Sketch by hand, on the same axes, the graphs of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$
(c) $h(x) = x^3$ (d) $j(x) = x^4$

8. Draw, by hand, a rough sketch of the graph of each function.
(a) $y = \sin x$ (b) $y = \tan x$
(c) $y = e^x$ (d) $y = \ln x$
(e) $y = 1/x$ (f) $y = |x|$
(g) $y = \sqrt{x}$ (h) $y = \tan^{-1}x$

9. Suppose that f has domain A and g has domain B.
(a) What is the domain of $f + g$?
(b) What is the domain of fg?
(c) What is the domain of f/g?

10. How is the composite function $f \circ g$ defined? What is its domain?

11. Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
(a) Shift 2 units upward.
(b) Shift 2 units downward.
(c) Shift 2 units to the right.
(d) Shift 2 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 2.
(h) Shrink vertically by a factor of 2.
(i) Stretch horizontally by a factor of 2.
(j) Shrink horizontally by a factor of 2.

12. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
(b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?

13. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
(b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
(c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range?

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

3. Explain the meaning of each of the following.

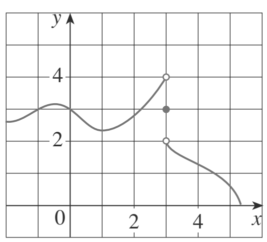
(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

4. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 3^-} f(x)$

(c) $\lim_{x \to 3^+} f(x)$ (d) $\lim_{x \to 3} f(x)$

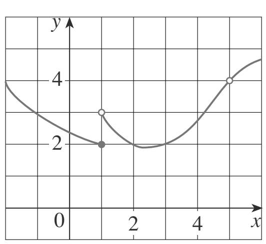
(e) $f(3)$



5. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$ (c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 5} f(x)$ (e) $f(5)$



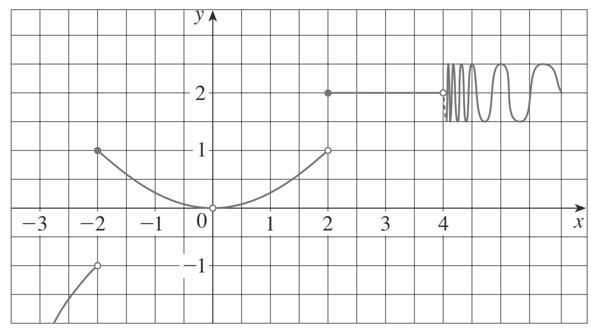
6. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -2^-} g(x)$ (b) $\lim_{x \to -2^+} g(x)$ (c) $\lim_{x \to -2} g(x)$

(d) $g(-2)$ (e) $\lim_{x \to 2^-} g(x)$ (f) $\lim_{x \to 2^+} g(x)$

(g) $\lim_{x \to 2} g(x)$ (h) $g(2)$ (i) $\lim_{x \to 4^+} g(x)$

(j) $\lim_{x \to 4^-} g(x)$ (k) $g(0)$ (l) $\lim_{x \to 0} g(x)$

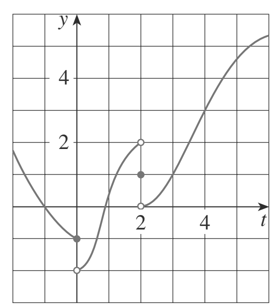


7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$

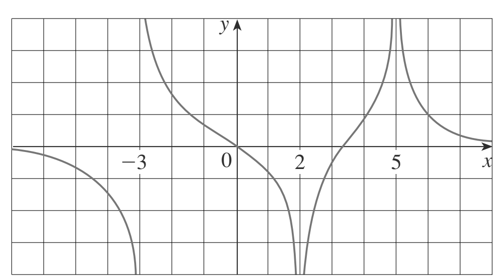


8. For the function R whose graph is shown, state the following.

(a) $\lim_{x \to 2} R(x)$ (b) $\lim_{x \to 5} R(x)$

(c) $\lim_{x \to -3^-} R(x)$ (d) $\lim_{x \to -3^+} R(x)$

(e) The equations of the vertical asymptotes.

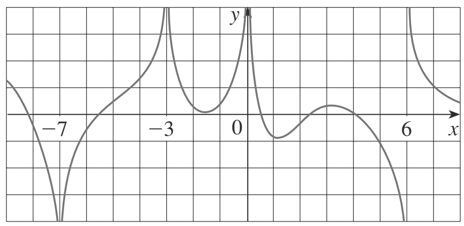


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x\to -7} f(x)$ (b) $\lim_{x\to -3} f(x)$ (c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 6^-} f(x)$ (e) $\lim_{x\to 6^+} f(x)$

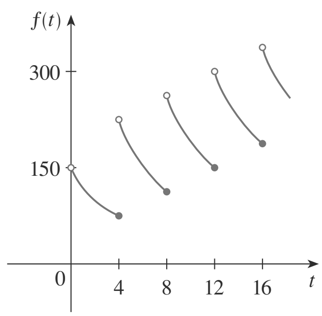
(f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. (Later we will be able to compute the dosage and time interval to ensure that the concentration of the drug does not reach a harmful level.) Find

$$\lim_{t\to 12^-} f(t) \qquad \text{and} \qquad \lim_{t\to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11. Use the graph of the function $f(x) = 1/(1 + e^{1/x})$ to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

12. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ (x-1)^2 & \text{if } x \geqslant 1 \end{cases}$$

13–14 Sketch the graph of an example of a function f that satisfies all of the given conditions.

13. $\lim_{x\to 3^+} f(x) = 4, \quad \lim_{x\to 3^-} f(x) = 2, \quad \lim_{x\to -2} f(x) = 2,$
$f(3) = 3, \quad f(-2) = 1$

14. $\lim_{x\to 0^-} f(x) = 1, \quad \lim_{x\to 0^+} f(x) = -1, \quad \lim_{x\to 2^-} f(x) = 0$
$\lim_{x\to 2^+} f(x) = 1, \quad f(2) = 1, \quad f(0)$ is undefined

15–18 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

15. $\displaystyle\lim_{x\to 2} \frac{x^2 - 2x}{x^2 - x - 2}$, $x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$ $1.9, 1.95, 1.99, 1.995, 1.999$

16. $\displaystyle\lim_{x\to -1} \frac{x^2 - 2x}{x^2 - x - 2}$, $x = 0, -0.5, -0.9, -0.95, -0.99,$ $-0.999, -2, -1.5, -1.1, -1.01, -1.001$

17. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$, $x = \pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01$

18. $\lim_{x\to 0^+} x\ln(x + x^2)$, $x = 1, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001$

19–22 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

19. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+4} - 2}{x}$

20. $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\tan 5x}$

21. $\displaystyle\lim_{x\to 1} \frac{x^6 - 1}{x^{10} - 1}$

22. $\displaystyle\lim_{x\to 0} \frac{9^x - 5^x}{x}$

23–30 Determine the infinite limit.

23. $\displaystyle\lim_{x\to 5^+} \frac{6}{x - 5}$

24. $\displaystyle\lim_{x\to 5^-} \frac{6}{x - 5}$

25. $\displaystyle\lim_{x\to 1} \frac{2 - x}{(x-1)^2}$

26. $\displaystyle\lim_{x\to 0} \frac{x - 1}{x^2(x+2)}$

27. $\displaystyle\lim_{x\to -2^+} \frac{x - 1}{x^2(x+2)}$

28. $\lim_{x\to \pi^-} \csc x$

29. $\lim_{x\to (-\pi/2)^-} \sec x$

30. $\lim_{x\to 5^+} \ln(x - 5)$

31. Determine $\displaystyle\lim_{x\to 1^-} \frac{1}{x^3 - 1}$ and $\displaystyle\lim_{x\to 1^+} \frac{1}{x^3 - 1}$

(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,

(b) by reasoning as in Example 9, and

(c) from a graph of f.

32. (a) Find the vertical asymptotes of the function

$$y = \frac{x}{x^2 - x - 2}$$

(b) Confirm your answer to part (a) by graphing the function.

33. (a) Estimate the value of the limit $\lim_{x\to 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?

(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

34. The slope of the tangent line to the graph of the exponential function $y = 2^x$ at the point $(0, 1)$ is $\lim_{x\to 0} (2^x - 1)/x$. Estimate the slope to three decimal places.

EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we *cannot* use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

Watch an animation of a similar limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Exists

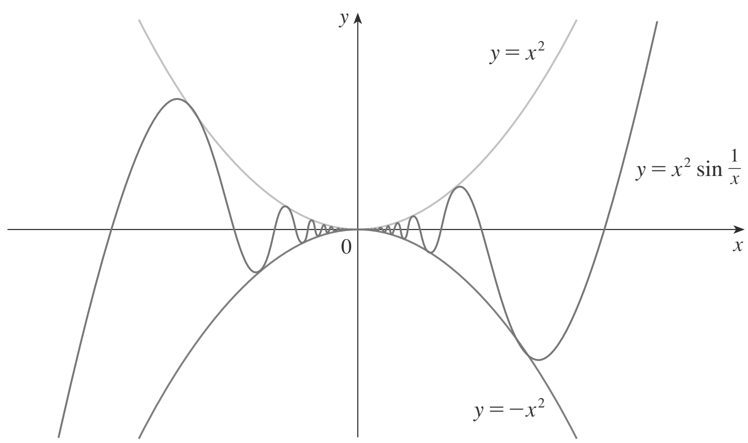


FIGURE 8

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

2.3 Exercises

1. Given that

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

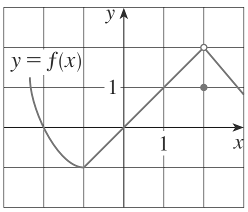
(a) $\lim_{x \to a} [f(x) + h(x)]$ (b) $\lim_{x \to a} [f(x)]^2$

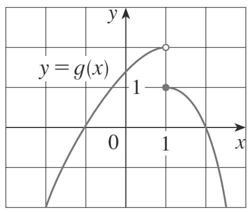
(c) $\lim_{x \to a} \sqrt[3]{h(x)}$ (d) $\lim_{x \to a} \dfrac{1}{f(x)}$

(e) $\lim_{x \to a} \dfrac{f(x)}{h(x)}$ (f) $\lim_{x \to a} \dfrac{g(x)}{f(x)}$

(g) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ (h) $\lim_{x \to a} \dfrac{2f(x)}{h(x) - f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x\to 2}[f(x)+g(x)]$ (b) $\lim_{x\to 1}[f(x)+g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ (d) $\lim_{x\to -1}\frac{f(x)}{g(x)}$

(e) $\lim_{x\to 2}x^3 f(x)$ (f) $\lim_{x\to 1}\sqrt{3+f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to -2}(3x^4+2x^2-x+1)$ **4.** $\lim_{x\to 2}\frac{2x^2+1}{x^2+6x-4}$

5. $\lim_{x\to 3}(x^2-4)(x^3+5x-1)$ **6.** $\lim_{t\to -1}(t^2+1)^3(t+3)^5$

7. $\lim_{x\to 1}\left(\frac{1+3x}{1+4x^2+3x^4}\right)^3$ **8.** $\lim_{u\to -2}\sqrt{u^4+3u+6}$

9. $\lim_{x\to 4^-}\sqrt{16-x^2}$

10. (a) What is wrong with the following equation?

$$\frac{x^2+x-6}{x-2}=x+3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}(x+3)$$

is correct.

11–30 Evaluate the limit, if it exists.

11. $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$ **12.** $\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4}$

13. $\lim_{x\to 2}\frac{x^2-x+6}{x-2}$ **14.** $\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4}$

15. $\lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3}$ **16.** $\lim_{x\to -1}\frac{x^2-4x}{x^2-3x-4}$

17. $\lim_{h\to 0}\frac{(4+h)^2-16}{h}$ **18.** $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

19. $\lim_{h\to 0}\frac{(1+h)^4-1}{h}$ **20.** $\lim_{h\to 0}\frac{(2+h)^3-8}{h}$

21. $\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}$ **22.** $\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$

23. $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$ **24.** $\lim_{x\to 2}\frac{x^4-16}{x-2}$

25. $\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}$ **26.** $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

27. $\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}$ **28.** $\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}$

29. $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$ **30.** $\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

31. (a) Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}$$

by graphing the function $f(x)=x/\left(\sqrt{1+3x}-1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

32. (a) Use a graph of

$$f(x)=\frac{\sqrt{3+x}-\sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0}f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

33. Use the Squeeze Theorem to show that $\lim_{x\to 0}x^2\cos 20\pi x=0$. Illustrate by graphing the functions $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$, and $h(x)=x^2$ on the same screen.

34. Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

35. If $1\leqslant f(x)\leqslant x^2+2x+2$ for all x, find $\lim_{x\to -1}f(x)$.

36. If $3x\leqslant f(x)\leqslant x^3+2$ for $0\leqslant x\leqslant 2$, evaluate $\lim_{x\to 1}f(x)$.

37. Prove that $\lim_{x\to 0}x^4\cos\frac{2}{x}=0$.

38. Prove that $\lim_{x\to 0^+}\sqrt{x}\,e^{\sin(\pi/x)}=0$.

39–44 Find the limit, if it exists. If the limit does not exist, explain why.

39. $\lim_{x\to -4}|x+4|$ **40.** $\lim_{x\to -4^-}\frac{|x+4|}{x+4}$

41. $\lim_{x \to 2} \dfrac{|x - 2|}{x - 2}$

42. $\lim_{x \to 1.5} \dfrac{2x^2 - 3x}{|2x - 3|}$

43. $\lim_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

44. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

45. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.
(i) $\lim_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \to 0^-} \operatorname{sgn} x$
(iii) $\lim_{x \to 0} \operatorname{sgn} x$ (iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

46. Let

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \leqslant 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

(a) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.
(b) Does $\lim_{x \to 2} f(x)$ exist?
(c) Sketch the graph of f.

47. Let $F(x) = \dfrac{x^2 - 1}{|x - 1|}$.
(a) Find
(i) $\lim_{x \to 1^+} F(x)$ (ii) $\lim_{x \to 1^-} F(x)$
(b) Does $\lim_{x \to 1} F(x)$ exist?
(c) Sketch the graph of F.

48. Let

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leqslant 2 \\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.
(i) $\lim_{x \to 0^+} h(x)$ (ii) $\lim_{x \to 0} h(x)$ (iii) $\lim_{x \to 1} h(x)$
(iv) $\lim_{x \to 2^-} h(x)$ (v) $\lim_{x \to 2^+} h(x)$ (vi) $\lim_{x \to 2} h(x)$
(b) Sketch the graph of h.

49. (a) If the symbol $[\![\;]\!]$ denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$
(b) If n is an integer, evaluate
(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$
(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

50. Let $f(x) = x - [\![x]\!]$.
(a) Sketch the graph of f.
(b) If n is an integer, evaluate
(i) $\lim_{x \to n^-} f(x)$ (ii) $\lim_{x \to n^+} f(x)$
(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

51. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

52. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

53. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

54. If r is a rational function, use Exercise 53 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

55. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

56. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

57. Show by means of an example that $\lim_{x \to a} [f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

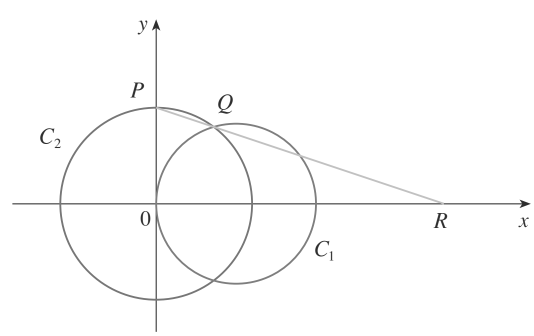
58. Evaluate $\lim_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

59. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

60. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



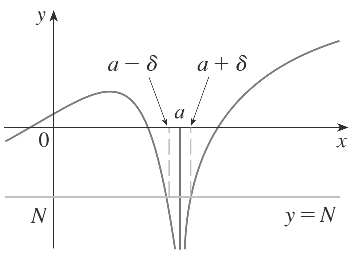


FIGURE 11

Similarly, the following is a precise version of Definition 5 in Section 2.2. It is illustrated by Figure 11.

> **7 Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then
>
> $$\lim_{x \to a} f(x) = -\infty$$
>
> means that for every negative number N there is a positive number δ such that
>
> $$f(x) < N \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

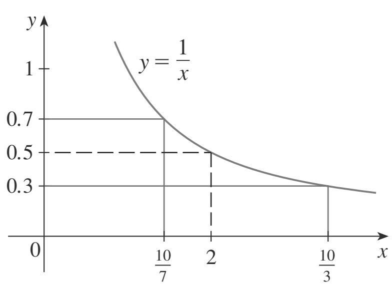
2.4 Exercises

1. How close to 2 do we have to take x so that $5x + 3$ is within a distance of (a) 0.1 and (b) 0.01 from 13?

2. How close to 5 do we have to take x so that $6x - 1$ is within a distance of (a) 0.01, (b) 0.001, and (c) 0.0001 from 29?

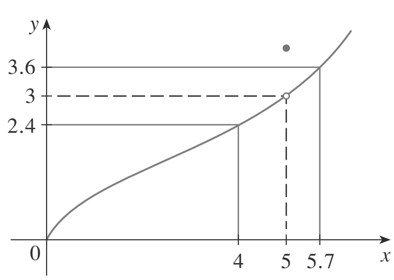
3. Use the given graph of $f(x) = 1/x$ to find a number δ such that

$$\left| \frac{1}{x} - 0.5 \right| < 0.2 \qquad \text{whenever} \qquad |x - 2| < \delta$$



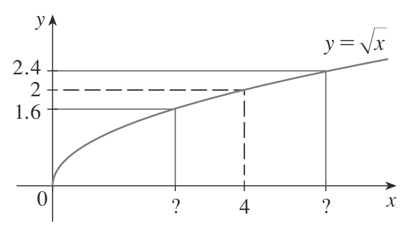
4. Use the given graph of f to find a number δ such that

$$|f(x) - 3| < 0.6 \qquad \text{whenever} \qquad 0 < |x - 5| < \delta$$



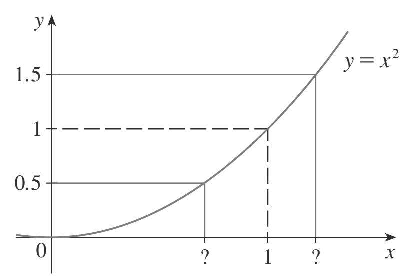
5. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$|\sqrt{x} - 2| < 0.4 \qquad \text{whenever} \qquad |x - 4| < \delta$$



6. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$|x^2 - 1| < \tfrac{1}{2} \qquad \text{whenever} \qquad |x - 1| < \delta$$



7. Use a graph to find a number δ such that

$$|\sqrt{4x + 1} - 3| < 0.5 \qquad \text{whenever} \qquad |x - 2| < \delta$$

8. Use a graph to find a number δ such that

$$|\sin x - \tfrac{1}{2}| < 0.1 \qquad \text{whenever} \qquad \left| x - \frac{\pi}{6} \right| < \delta$$

9. For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 1$ and $\varepsilon = 0.1$.

10. For the limit

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

11. Use a graph to find a number δ such that

$$\frac{x}{(x^2+1)(x-1)^2} > 100 \qquad \text{whenever} \qquad 0 < |x-1| < \delta$$

12. For the limit

$$\lim_{x\to 0}\cot^2 x = \infty$$

illustrate Definition 6 by finding values of δ that correspond to (a) $M = 100$ and (b) $M = 1000$.

13. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of $\pm 5\text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

14. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.
(a) How much power is needed to maintain the temperature at 200°C?
(b) If the temperature is allowed to vary from 200°C by up to ± 1°C, what range of wattage is allowed for the input power?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

15–18 Prove the statement using the ε, δ definition of limit and illustrate with a diagram like Figure 9.

15. $\lim_{x\to 1}(2x+3) = 5$

16. $\lim_{x\to -2}\left(\frac{1}{2}x+3\right) = 2$

17. $\lim_{x\to -3}(1-4x) = 13$

18. $\lim_{x\to 4}(7-3x) = -5$

19–32 Prove the statement using the ε, δ definition of limit.

19. $\lim_{x\to 3}\frac{x}{5} = \frac{3}{5}$

20. $\lim_{x\to 6}\left(\frac{x}{4}+3\right) = \frac{9}{2}$

21. $\lim_{x\to -5}\left(4-\frac{3x}{5}\right) = 7$

22. $\lim_{x\to 3}\frac{x^2+x-12}{x-3} = 7$

23. $\lim_{x\to a} x = a$

24. $\lim_{x\to a} c = c$

25. $\lim_{x\to 0} x^2 = 0$

26. $\lim_{x\to 0} x^3 = 0$

27. $\lim_{x\to 0} |x| = 0$

28. $\lim_{x\to 9^-}\sqrt[4]{9-x} = 0$

29. $\lim_{x\to 2}(x^2-4x+5) = 1$

30. $\lim_{x\to 3}(x^2+x-4) = 8$

31. $\lim_{x\to -2}(x^2-1) = 3$

32. $\lim_{x\to 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.

34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ is $\delta = \sqrt{9+\varepsilon} - 3$.

CAS **35.** (a) For the limit $\lim_{x\to 1}(x^3+x+1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.
(b) By using a computer algebra system to solve the cubic equation $x^3+x+1 = 3+\varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.
(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$.

37. Prove that $\lim_{x\to a}\sqrt{x} = \sqrt{a}$ if $a > 0$.

$$\left[\textit{Hint}\text{: Use } \left|\sqrt{x}-\sqrt{a}\right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}.\right]$$

38. If H is the Heaviside function defined in Example 6 in Section 2.2, prove, using Definition 2, that $\lim_{t\to 0} H(t)$ does not exist. [*Hint*: Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1 in Section 2.3.

41. How close to -3 do we have to take x so that

$$\frac{1}{(x+3)^4} > 10{,}000$$

42. Prove, using Definition 6, that $\lim_{x\to -3}\frac{1}{(x+3)^4} = \infty$.

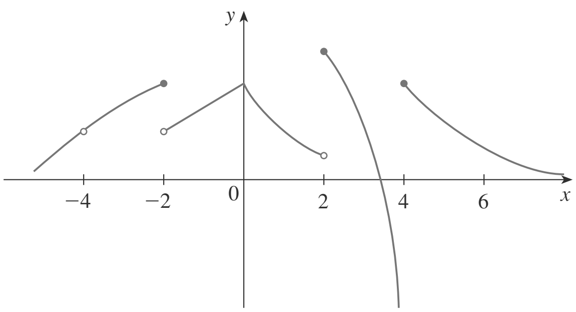
43. Prove that $\lim_{x\to 0^+}\ln x = -\infty$.

2.5 Exercises

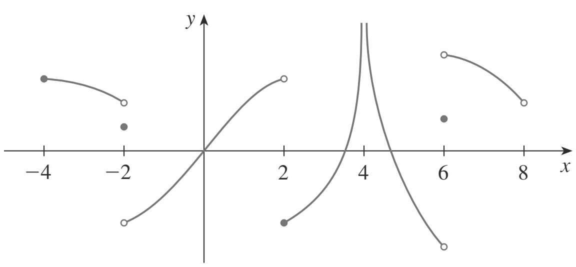
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of g, state the intervals on which g is continuous.



5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
(a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.
(b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

9. If f and g are continuous functions with $f(3) = 5$ and $\lim_{x\to 3}[2f(x) - g(x)] = 4$, find $g(3)$.

10–12 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

10. $f(x) = x^2 + \sqrt{7 - x}, \quad a = 4$

11. $f(x) = (x + 2x^3)^4, \quad a = -1$

12. $g(x) = \dfrac{x + 1}{2x^2 - 1}, \quad a = 4$

13–14 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

13. $f(x) = \dfrac{2x + 3}{x - 2}, \quad (2, \infty)$

14. $g(x) = 2\sqrt{3 - x}, \quad (-\infty, 3]$

15–20 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

15. $f(x) = \ln|x - 2| \qquad a = 2$

16. $f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \qquad a = 1$

17. $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geqslant 0 \end{cases} \qquad a = 0$

18. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \qquad a = 1$

19. $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases} \qquad a = -3$

20. $f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geqslant 1 \end{cases} \qquad a = 1$

21–28 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$

22. $G(x) = \sqrt[3]{x}(1 + x^3)$

23. $R(x) = x^2 + \sqrt{2x - 1}$

24. $h(x) = \dfrac{\sin x}{x + 1}$

25. $f(x) = e^x \sin 5x$

26. $F(x) = \sin^{-1}(x^2 - 1)$

27. $G(t) = \ln(t^4 - 1)$

28. $H(x) = \cos\left(e^{\sqrt{x}}\right)$

29–30 Locate the discontinuities of the function and illustrate by graphing.

29. $y = \dfrac{1}{1 + e^{1/x}}$

30. $y = \ln(\tan^2 x)$

31–34 Use continuity to evaluate the limit.

31. $\displaystyle\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$

32. $\displaystyle\lim_{x \to \pi} \sin(x + \sin x)$

33. $\displaystyle\lim_{x \to 1} e^{x^2 - x}$

34. $\displaystyle\lim_{x \to 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$

35–36 Show that f is continuous on $(-\infty, \infty)$.

35. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geqslant 1 \end{cases}$

36. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

37–39 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

37. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leqslant 0 \\ 2 - x & \text{if } 0 < x \leqslant 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

38. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x - 3} & \text{if } x \geqslant 3 \end{cases}$

39. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

40. The gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\[2ex] \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

41. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leqslant 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

42. Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geqslant 4 \end{cases}$$

43. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous on $\mathbb{R}$.

(a) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}, \quad a = -2$

(b) $f(x) = \dfrac{x - 7}{|x - 7|}, \quad a = 7$

(c) $f(x) = \dfrac{x^3 + 64}{x + 4}, \quad a = -4$

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}, \quad a = 9$

44. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

45. If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

46. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$. (This proves the existence of the number $\sqrt{2}$.)

47–50 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

47. $x^4 + x - 3 = 0, \quad (1, 2)$

48. $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

49. $\cos x = x, \quad (0, 1)$

50. $\ln x = e^{-x}, \quad (1, 2)$

51–52 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

51. $e^x = 2 - x$

52. $x^5 - x^2 + 2x + 3 = 0$

53–54 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

53. $x^5 - x^2 - 4 = 0$

54. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.

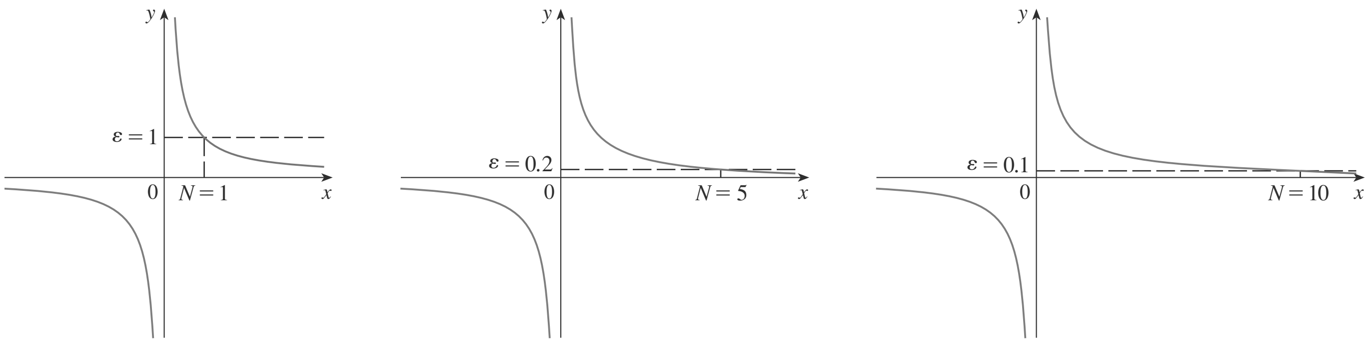


FIGURE 18

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

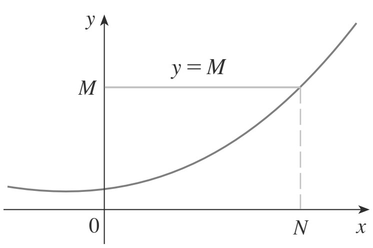


FIGURE 19
$\lim_{x \to \infty} f(x) = \infty$

> **9 Definition** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x \to \infty} f(x) = \infty$$
>
> means that for every positive number M there is a corresponding positive number N such that
>
> $$f(x) > M \qquad \text{whenever} \qquad x > N$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 66.)

2.6 Exercises

1. Explain in your own words the meaning of each of the following.

(a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.

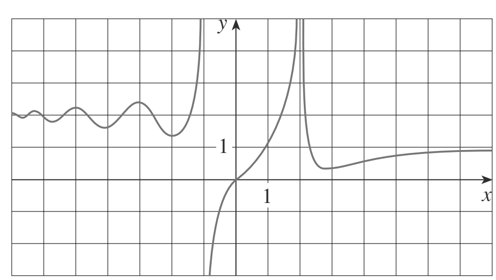
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.

(a) $\lim_{x \to 2} f(x)$ (b) $\lim_{x \to -1^-} f(x)$ (c) $\lim_{x \to -1^+} f(x)$

(d) $\lim_{x \to \infty} f(x)$ (e) $\lim_{x \to -\infty} f(x)$

(f) The equations of the asymptotes

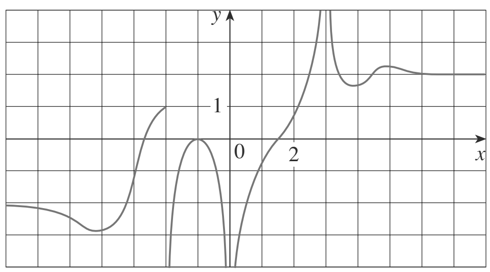


4. For the function g whose graph is given, state the following.

(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$

(c) $\lim_{x \to 3} g(x)$ (d) $\lim_{x \to 0} g(x)$

(e) $\lim_{x \to -2^+} g(x)$ (f) The equations of the asymptotes



5–8 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(0) = 0, \quad f(1) = 1, \quad \lim_{x \to \infty} f(x) = 0, \quad f$ is odd

6. $\lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = 1,$
$\lim_{x \to -\infty} f(x) = 1$

7. $\lim_{x \to 2} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = 0,$
$\lim_{x \to 0^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = -\infty$

8. $\lim_{x \to -2} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = 3, \quad \lim_{x \to \infty} f(x) = -3$

9. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

10. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

11–12 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

11. $\lim_{x \to \infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$

12. $\lim_{x \to \infty} \sqrt{\dfrac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

13–34 Find the limit.

13. $\lim_{x \to \infty} \dfrac{1}{2x + 3}$

14. $\lim_{x \to \infty} \dfrac{3x + 5}{x - 4}$

15. $\lim_{x \to -\infty} \dfrac{1 - x - x^2}{2x^2 - 7}$

16. $\lim_{y \to \infty} \dfrac{2 - 3y^2}{5y^2 + 4y}$

17. $\lim_{x \to \infty} \dfrac{x^3 + 5x}{2x^3 - x^2 + 4}$

18. $\lim_{t \to -\infty} \dfrac{t^2 + 2}{t^3 + t^2 - 1}$

19. $\lim_{u \to \infty} \dfrac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)}$

20. $\lim_{x \to \infty} \dfrac{x + 2}{\sqrt{9x^2 + 1}}$

21. $\lim_{x \to \infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

22. $\lim_{x \to -\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

23. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$

24. $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2x}\right)$

25. $\lim_{x \to \infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

26. $\lim_{x \to \infty} \cos x$

27. $\lim_{x \to \infty} \sqrt{x}$

28. $\lim_{x \to -\infty} \sqrt[3]{x}$

29. $\lim_{x \to \infty} \left(x - \sqrt{x}\right)$

30. $\lim_{x \to \infty} \dfrac{x^3 - 2x + 3}{5 - 2x^2}$

31. $\lim_{x \to -\infty} (x^4 + x^5)$

32. $\lim_{x \to \infty} \tan^{-1}(x^2 - x^4)$

33. $\lim_{x \to \infty} \dfrac{x + x^3 + x^5}{1 - x^2 + x^4}$

34. $\lim_{x \to (\pi/2)^+} e^{\tan x}$

35. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

36. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

37–42 Find the horizontal and vertical asymptotes of each curve. Check your work by graphing the curve and estimating the asymptotes.

37. $y = \dfrac{x}{x + 4}$

38. $y = \dfrac{x^2 + 4}{x^2 - 1}$

39. $y = \dfrac{x^3}{x^2 + 3x - 10}$

40. $y = \dfrac{x^3 + 1}{x^3 + x}$

41. $h(x) = \dfrac{x}{\sqrt[4]{x^4 + 1}}$

42. $F(x) = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 2}}$

the slope of the tangent line is

$$\frac{|BC|}{|AC|} = \frac{10.3}{5.5} \approx 1.9$$

IIII Another method is to average the slopes of two secant lines. See Example 2 in Section 2.1.

Therefore, the instantaneous rate of change of temperature with respect to time at noon is about 1.9°C/h.

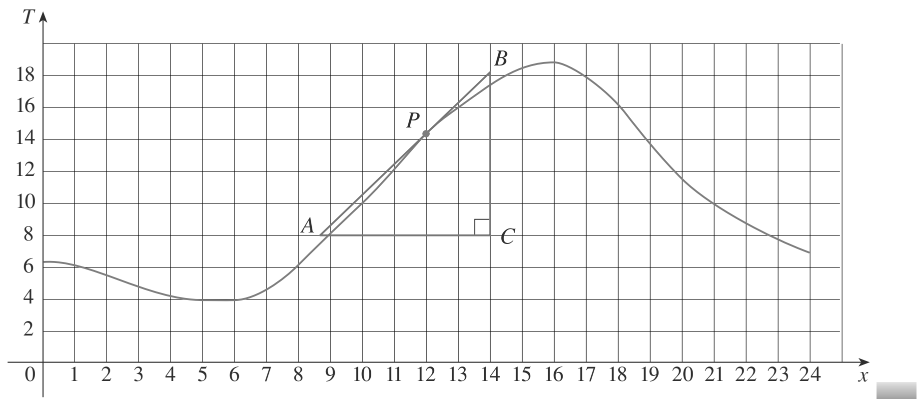


FIGURE 8

The velocity of a particle is the rate of change of displacement with respect to time. Physicists are interested in other rates of change as well—for instance, the rate of change of work with respect to time (which is called *power*). Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A steel manufacturer is interested in the rate of change of the cost of producing x tons of steel per day with respect to x (called the *marginal cost*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.3.

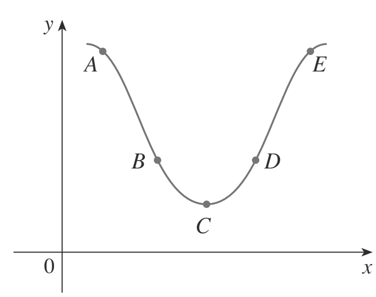
All these rates of change can be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.7 Exercises

1. A curve has equation $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P.

2. Suppose an object moves with position function $s = f(t)$.
(a) Write an expression for the average velocity of the object in the time interval from $t = a$ to $t = a + h$.
(b) Write an expression for the instantaneous velocity at time $t = a$.

3. Consider the slope of the given curve at each of the five points shown. List these five slopes in decreasing order and explain your reasoning.



4. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?

5. (a) Find the slope of the tangent line to the parabola $y = x^2 + 2x$ at the point $(-3, 3)$
   (i) using Definition 1
   (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(-3, 3)$ until the parabola and the tangent line are indistinguishable.

6. (a) Find the slope of the tangent line to the curve $y = x^3$ at the point $(-1, -1)$
   (i) using Definition 1
   (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(-1, -1)$ until the curve and the line appear to coincide.

7–10 Find an equation of the tangent line to the curve at the given point.

7. $y = 1 + 2x - x^3$, $(1, 2)$

8. $y = \sqrt{2x + 1}$, $(4, 3)$

9. $y = (x - 1)/(x - 2)$, $(3, 2)$

10. $y = 2x/(x + 1)^2$, $(0, 0)$

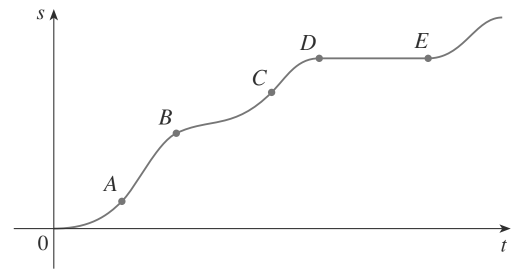
11. (a) Find the slope of the tangent to the curve $y = 2/(x + 3)$ at the point where $x = a$.
(b) Find the slopes of the tangent lines at the points whose x-coordinates are (i) -1, (ii) 0, and (iii) 1.

12. (a) Find the slope of the tangent to the parabola $y = 1 + x + x^2$ at the point where $x = a$.
(b) Find the slopes of the tangent lines at the points whose x-coordinates are (i) -1, (ii) $-\frac{1}{2}$, and (iii) 1.
(c) Graph the curve and the three tangents on a common screen.

13. (a) Find the slope of the tangent to the curve $y = x^3 - 4x + 1$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, -2)$ and $(2, 1)$.
(c) Graph the curve and both tangents on a common screen.

14. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 1)$ and $\left(4, \frac{1}{2}\right)$.
(c) Graph the curve and both tangents on a common screen.

15. The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.
(a) What was the initial velocity of the car?
(b) Was the car going faster at B or at C?
(c) Was the car slowing down or speeding up at A, B, and C?
(d) What happened between D and E?



16. Valerie is driving along a highway. Sketch the graph of the position function of her car if she drives in the following manner: At time $t = 0$, the car is at mile marker 15 and is traveling at a constant speed of 55 mi/h. She travels at this speed for exactly an hour. Then the car slows gradually over a 2-minute period as Valerie comes to a stop for dinner. Dinner lasts 26 min; then she restarts the car, gradually speeding up to 65 mi/h over a 2-minute period. She drives at a constant 65 mi/h for two hours and then over a 3-minute period gradually slows to a complete stop.

17. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

18. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.
(a) Find the velocity of the arrow after one second.
(b) Find the velocity of the arrow when $t = a$.
(c) When will the arrow hit the moon?
(d) With what velocity will the arrow hit the moon?

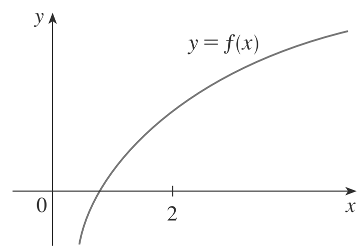
19. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

20. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
(a) Find the average velocity over each time interval:
   (i) $[3, 4]$ (ii) $[3.5, 4]$
   (iii) $[4, 5]$ (iv) $[4, 4.5]$
(b) Find the instantaneous velocity when $t = 4$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

21. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

2.8 Exercises

1. On the given graph of f, mark lengths that represent $f(2)$, $f(2+h)$, $f(2+h)-f(2)$, and h. (Choose $h>0$.) What line has slope $\dfrac{f(2+h)-f(2)}{h}$?

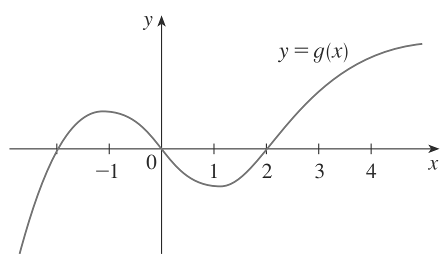


2. For the function f whose graph is shown in Exercise 1, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad f'(2) \qquad f(3)-f(2) \qquad \tfrac{1}{2}[f(4)-f(2)]$$

3. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



4. If the tangent line to $y=f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

5. Sketch the graph of a function f for which $f(0)=0$, $f'(0)=3$, $f'(1)=0$, and $f'(2)=-1$.

6. Sketch the graph of a function g for which $g(0)=0$, $g'(0)=3$, $g'(1)=0$, and $g'(2)=1$.

7. If $f(x)=3x^2-5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $y=3x^2-5x$ at the point $(2, 2)$.

8. If $g(x)=1-x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y=1-x^3$ at the point $(0, 1)$.

9. (a) If $F(x)=x^3-5x+1$, find $F'(1)$ and use it to find an equation of the tangent line to the curve $y=x^3-5x+1$ at the point $(1, -3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

10. (a) If $G(x)=x/(1+2x)$, find $G'(a)$ and use it to find an equation of the tangent line to the curve $y=x/(1+2x)$ at the point $\left(-\frac{1}{4}, -\frac{1}{2}\right)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

11. Let $f(x)=3^x$. Estimate the value of $f'(1)$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By zooming in on the graph of $y=3^x$ and estimating the slope.

12. Let $g(x)=\tan x$. Estimate the value of $g'(\pi/4)$ in two ways:
(a) By using Definition 2 and taking successively smaller values of h.
(b) By zooming in on the graph of $y=\tan x$ and estimating the slope.

13–18 Find $f'(a)$.

13. $f(x)=3-2x+4x^2$

14. $f(t)=t^4-5t$

15. $f(t)=\dfrac{2t+1}{t+3}$

16. $f(x)=\dfrac{x^2+1}{x-2}$

17. $f(x)=\dfrac{1}{\sqrt{x+2}}$

18. $f(x)=\sqrt{3x+1}$

19–24 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

19. $\displaystyle\lim_{h\to 0}\frac{(1+h)^{10}-1}{h}$

20. $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h}$

21. $\displaystyle\lim_{x\to 5}\frac{2^x-32}{x-5}$

22. $\displaystyle\lim_{x\to \pi/4}\frac{\tan x-1}{x-\pi/4}$

23. $\displaystyle\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h}$

24. $\displaystyle\lim_{t\to 1}\frac{t^4+t-2}{t-1}$

25–26 A particle moves along a straight line with equation of motion $s=f(t)$, where s is measured in meters and t in seconds. Find the velocity when $t=2$.

25. $f(t)=t^2-6t-5$

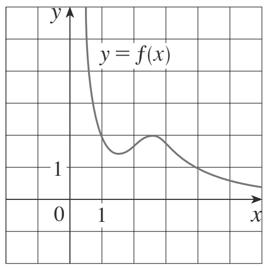
26. $f(t)=2t^3-t+1$

27. The cost of producing x ounces of gold from a new gold mine is $C=f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(800)=17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

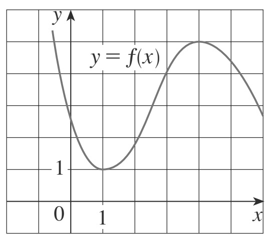
2.9 Exercises

1–3 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

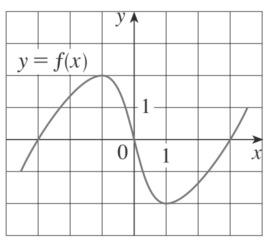
1. (a) $f'(1)$
(b) $f'(2)$
(c) $f'(3)$
(d) $f'(4)$



2. (a) $f'(0)$
(b) $f'(1)$
(c) $f'(2)$
(d) $f'(3)$
(e) $f'(4)$
(f) $f'(5)$

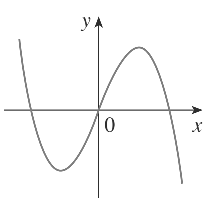


3. (a) $f'(-3)$
(b) $f'(-2)$
(c) $f'(-1)$
(d) $f'(0)$
(e) $f'(1)$
(f) $f'(2)$
(g) $f'(3)$

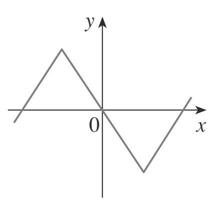


4. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.

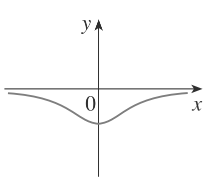
(a)



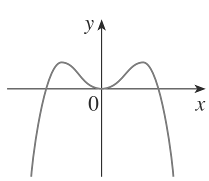
(b)



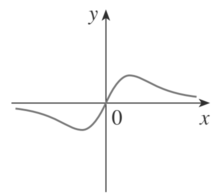
(c)



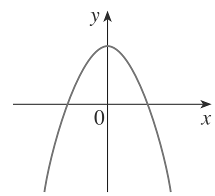
(d)



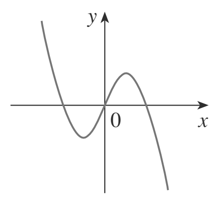
I



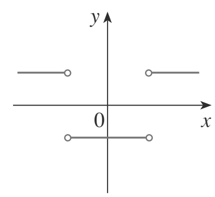
II



III

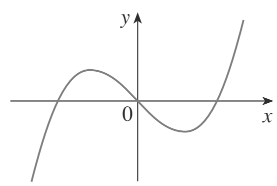


IV



5–13 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

5.



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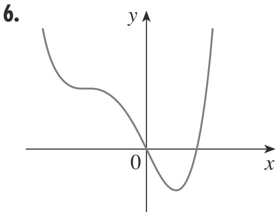

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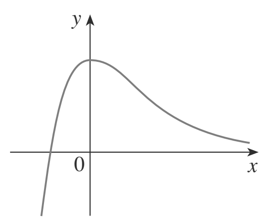

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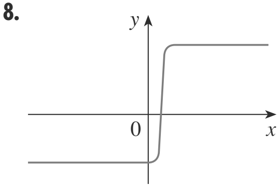
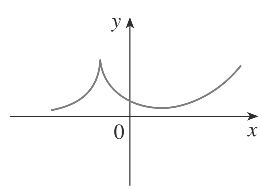
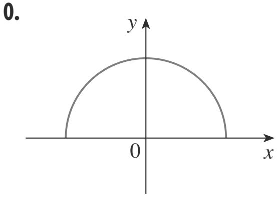


9.



10.



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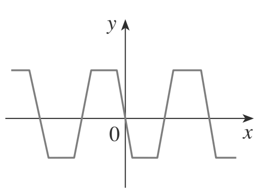

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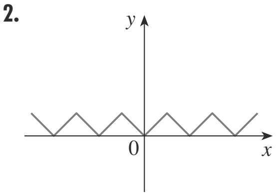
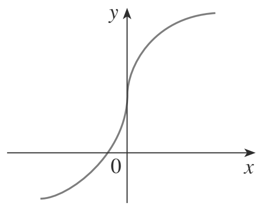
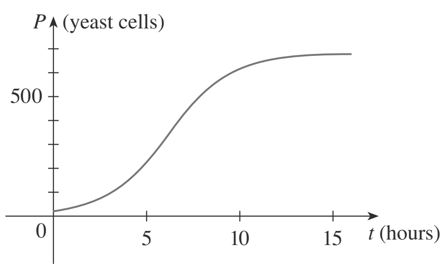


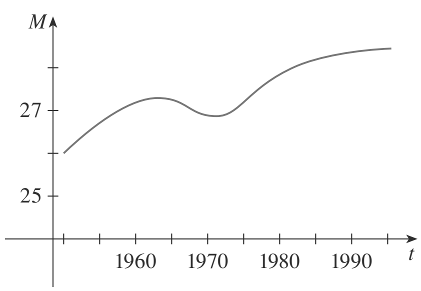
13.



14. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



15. The graph shows how the average age of first marriage of Japanese men varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 5–13. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$

17. $f(x) = e^x$

18. $f(x) = \ln x$

19. Let $f(x) = x^2$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.

(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of a derivative to prove that your guess in part (c) is correct.

20. Let $f(x) = x^3$.

(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Use the values from parts (a) and (b) to graph f'.

(d) Guess a formula for $f'(x)$.

(e) Use the definition of a derivative to prove that your guess in part (d) is correct.

21–31 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21. $f(x) = 37$

22. $f(x) = 12 + 7x$

23. $f(x) = 1 - 3x^2$

24. $f(x) = 5x^2 + 3x - 2$

25. $f(x) = x^3 - 3x + 5$

26. $f(x) = x + \sqrt{x}$

27. $g(x) = \sqrt{1 + 2x}$

28. $f(x) = \dfrac{3 + x}{1 - 3x}$

29. $G(t) = \dfrac{4t}{t + 1}$

30. $g(x) = \dfrac{1}{x^2}$

31. $f(x) = x^4$

32. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.

(b) Use the graph from part (a) to sketch the graph of f'.

(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?

(d) Use a graphing device to graph f' and compare with your sketch in part (b).

33. (a) If $f(x) = x - (2/x)$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

34. (a) If $f(t) = 6/(1 + t^2)$, find $f'(t)$.

(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

35. The unemployment rate $U(t)$ varies with time. The table (from the Bureau of Labor Statistics) gives the percentage of unemployed in the U.S. labor force from 1991 to 2000.

t	$U(t)$	t	$U(t)$
1991	6.8	1996	5.4
1992	7.5	1997	4.9
1993	6.9	1998	4.5
1994	6.1	1999	4.2
1995	5.6	2000	4.0

(a) What is the meaning of $U'(t)$? What are its units?

(b) Construct a table of values for $U'(t)$.

3.1 Exercises

1. (a) How is the number e defined?
   (b) Use a calculator to estimate the values of the limits

$$\lim_{h\to 0}\frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h\to 0}\frac{2.8^h - 1}{h}$$

   correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
   (b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
   (c) Which of the two functions in part (b) grows more rapidly when x is large?

3–32 Differentiate the function.

3. $f(x) = 186.5$
4. $f(x) = \sqrt{30}$
5. $f(x) = 5x - 1$
6. $F(x) = -4x^{10}$
7. $f(x) = x^2 + 3x - 4$
8. $g(x) = 5x^8 - 2x^5 + 6$
9. $f(t) = \frac{1}{4}(t^4 + 8)$
10. $f(t) = \frac{1}{2}t^6 - 3t^4 + t$
11. $y = x^{-2/5}$
12. $y = 5e^x + 3$
13. $V(r) = \frac{4}{3}\pi r^3$
14. $R(t) = 5t^{-3/5}$
15. $Y(t) = 6t^{-9}$
16. $R(x) = \dfrac{\sqrt{10}}{x^7}$
17. $G(x) = \sqrt{x} - 2e^x$
18. $y = \sqrt[3]{x}$
19. $F(x) = \left(\frac{1}{2}x\right)^5$
20. $f(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}}$
21. $g(x) = x^2 + \dfrac{1}{x^2}$
22. $y = \sqrt{x}\,(x - 1)$
23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$
24. $y = \dfrac{x^2 - 2\sqrt{x}}{x}$
25. $y = 4\pi^2$
26. $g(u) = \sqrt{2}u + \sqrt{3u}$
27. $y = ax^2 + bx + c$
28. $y = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2}$
29. $v = t^2 - \dfrac{1}{\sqrt[4]{t^3}}$
30. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$
31. $z = \dfrac{A}{y^{10}} + Be^y$
32. $y = e^{x+1} + 1$

33–36 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

33. $f(x) = e^x - 5x$
34. $f(x) = 3x^5 - 20x^3 + 50x$
35. $f(x) = 3x^{15} - 5x^3 + 3$
36. $f(x) = x + \dfrac{1}{x}$

37–38 Estimate the value of $f'(a)$ by zooming in on the graph of f. Then differentiate f to find the exact value of $f'(a)$ and compare with your estimate.

37. $f(x) = 3x^2 - x^3, \quad a = 1$
38. $f(x) = 1/\sqrt{x}, \quad a = 4$

39–40 Find an equation of the tangent line to the curve at the given point.

39. $y = x^4 + 2e^x, \quad (0, 2)$
40. $y = (1 + 2x)^2, \quad (1, 9)$

41–42 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

41. $y = 3x^2 - x^3, \quad (1, 2)$
42. $y = x\sqrt{x}, \quad (4, 8)$

43. (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
   (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 1 in Section 2.9.)
   (c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

44. (a) Use a graphing calculator or computer to graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
   (b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 1 in Section 2.9.)
   (c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

45. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

46. For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

47. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

48. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

49. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

50. Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.