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The Role of Free-Streaming Neutrinos and Standard Model Particles in Primordial Gravitational Waves Background

Abstract

In this review, I mentioned the nature of gravitational waves (GW) as a perturbation that propagates through the space-time. Then I defined important thermodynamical properties of the Early Universe and I underlined how the changes of the d.o.f. (g_*) of Standard Model (SM) particles can affect the evolution of those and of the scale factor $a(t)$. This in particular reflects itself in the evolution of primordial GW energy density spectrum, where it is possible to see different features given by several changes of g_*. Then I summarized the effects of free-streaming neutrinos on cosmological GW, both at first order and at second order in perturbation theory. The first order approach shows how free-streaming neutrinos do act on GW spectrum, damping it up to 35.5% between $kc \simeq 10^{-16}$ and 2×10^{-10}. The second order results instead show how free-streaming neutrinos can be an actual source of second order GWs, whose spectrum is also damped by first-order neutrinos as it should be. Finally, I briefly reported how GW can produce temperature and polarisation anisotropies in the CMB spectrum.

1 Introduction

In the middle of the past century a first "Cosmological Standard Model" was proposed. The so called "Hot Big-Bang" model explained the expansion of the Universe from its initial conditions to the Universe we observe nowadays, the light-element abundances and the presence of the Cosmic Microwave Background (CMB) with a temperature of $T = 2.7$ K. If this model shed light on several topics, other issues remained unsolved: for instance, the horizon problem. In the 1980s the inflation model was proposed. This explained the accelerated expansion of the Universe as a consequence of the displacement of a scalar field (the "inflaton") from the minimum of its potential $V(\phi)$. The inflation solved the problem of the horizon producing at the same time the nearly flat Universe we observe (thus solving even the flatness problem). Moreover it showed that this flat Universe must have a spectrum very similar to the one which is required to explain the origin of the large scale structures as galaxies or clusters of galaxies. Inflation has been corroborated by several tests concerning the spectrum of primordial perturbations and the temperature fluctuations of CMB. Therefore, it results natural to continue to test inflation model through new and more precise ways. A very important target for possible experiments concerns the primordial-gravitational-waves (PGW) background. In fact, inflation predicts the existence of a nearly scale-invariant stochastic gravitational-waves background (GWB) and that the square of the amplitude of these waves should be proportional to the energy density $V(\phi)$ during inflation. The important point of these predictions is that we can confirm or deny them by observing the temperature fluctuations and polarisation of the CMB. When we observe the polarisation, we describe it as a 2×2 symmetric tensor, which can be decomposed into a curl-free and a curl component. Even if primordial perturbations produced scalar perturbations to the spacetime metric, GWs are tensor metric perturbations and they can produce a curl. The amplitude of this "curl-signal" depends upon the amplitude of the PGWs background and hence it is dependent on the energy density $V(\phi)$. The detection of this curl component in the polarisation of the CMB would be a signal for inflation, and it has become a widely sought feature in CMB experiments [1]. Gravitational waves are produced by scalar density perturbations at second order in perturbation theory [2]. Here I report a summary of how the changes of relativistic degrees of freedom which happened during phase transitions of the universe (as, for instance, the quark gluon plasma to hadron gas phase transition, or the electron-positron annihilation) could affect the Hubble rate and thus produce features in the energy density spectrum of the PGW background. Then I underline the role of free streaming neutrinos in doing so, for energy scales lower than neutrino decoupling ($\sim$ 2MeV, [3]) and how they impact on the evolution of the PGW background. At second order in perturbation theory, it turns out that cosmic neutrinos are also a very important source for the second order GW background during the radiation dominated epoch, when neutrinos represented a strong contribution to the total energy density of the Universe. These new source terms arise because, as neutrinos are collisionless highly relativistic particles, the dispersion of their velocity acts as an effective extra source for the second order tensor perturbations (i.e. GW) [4]. In the end I briefly discuss how the presence of GW affects the CMB power spectrum [5]. It will be shown an analytic account of the features present in the tensor energy density spectrum, as the neutrinos driven damping and the several suppressions due to phase transitions.

2 Gravitational Waves spectrum

Gravitational waves represent a perturbation of the metric which describes the considered spacetime. In principle, considering a flat spacetime with no curvature it is possible to define the tensor perturbation, e.g. the tensor which describes GWs in that spacetime, as: $h_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$. By imposing the so called TT-Gauge to the perturbation $h_{\mu\nu}$ (traceless and transverse GW), one obtains $h_{00} = h_{0i} = 0$, and since the perturbation has to be traceless ($h^i{}_i = 0$), the two possible solutions for this 4×4 tensor are given by the two polarisations $\lambda = +, \times$:

$$h_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

For a tensor perturbation of an isotropic, uniform and flat spacetime, one has to follow the same procedure by substituting the usual Minkowski metric with the FLRW one: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ (in natural units). By using the relation between the cosmological time t and the so called "conformal time" ($dt = d\tau/a(t)$), and applying the same tensor perturbations as before, one finally gets:

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j]. \tag{2}$$

By requiring the perturbed metric $g_{\mu\nu} = a^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu})$ to satisfy the Einstein equations at the first order in perturbations, it is possible to write the tensor perturbations in its plane wave components in Fourier space:

$$h_{ij}(\tau, \vec{x}) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} h_\lambda e^{i\vec{k}\cdot\vec{x}} \epsilon^{\lambda}_{ij}, \tag{3}$$

where $|\vec{k}| = k$ is the comoving wave number. Once one imposes the perturbed Einstein equations to be satisfied, on finds that, at linear order

$$-\frac{1}{2} h_{ij;\nu}{}^{;\nu} = 8\pi G \Pi_{ij}, \tag{4}$$

with ";" indicating the covariant derivative.
Here Π_{ij} represents the anisotropic part of the stress-energy tensor obtained by considering the spatial part of the perturbed energy-momentum tensor $T_{ij} = pg_{ij} + a^2\Pi_{ij}$. The anisotropic part is often considered to be equal to zero when it is referred to perfect fluids, which in principle (i.e. without perturbations) are defined as isotropic. However, when neutrinos leave their last scattering surface, starting to free stream, they affect the GWs amplitude by anisotropic stresses [6,7]. As the components of tensors are scalars, we have that, for fixed i, j, $h_{ij;\mu} = h_{ij,\mu}$, where "," represents the ordinary derivative. The left part of the perturbed Einstein equation becomes, by expliciting the covariant derivative,

$$\begin{aligned} h_{ij;\nu}{}^{;\nu} &= (h_{ij;\nu}{}^{,\nu} - \Gamma^{\nu\alpha}_{\nu} h_{ij}{}^{;\alpha}) \\ &= g^{\bar{\mu}\nu}(h_{ij,\mu\nu} - \Gamma^{\alpha}_{\mu\nu} h_{ij,\alpha}) \\ &= -\ddot{h}_{ij} + \left(\frac{\nabla^2}{a^2}\right) h_{ij} - \left(\frac{3\dot{a}}{a}\right) \dot{h}_{ij}, \end{aligned} \tag{5}$$

where $g^{\bar{\mu}\nu} = \left(\bar{g_{\mu\nu}}\right)^{-1}$ is the inverse of the unperturbed metric and $\Gamma^0_{0\nu} = \Gamma^0_{\mu 0} = 0$, $\Gamma^0_{ij} = \delta_{ij}\dot{a}a$. Since the laplacian ∇^2 becomes $-k^2$ in the Fourier space, we get:

$$h''_{\lambda,k} + \frac{2a'}{a} h'_{\lambda,k} + k^2 h_{\lambda,k} = 16\pi G a^2 \Pi_{\lambda,k} \tag{6}$$

where the comma does not indicate the canonical derivative anymore and the conformal time derivative has been used $' \equiv \frac{\partial}{\partial\tau}$ $(\dot{h}_{\lambda,k} = a^{-1}h'_{\lambda,k}$, $\ddot{h}_{\lambda,k} = a^{-2}h''_{\lambda,k} - a^{-3}a'h'_{\lambda,k})$. The wave amplitude has been defined as $h_{ij,\lambda}(\tau, \vec{x}) = h_{\lambda,k}$. This is the massless Klein-Gordon equation for a plane wave in an expanding space with a source term. This implies that each polarization solution behaves as a massless, minimally coupled, real scalar field, with a normalization factor of $\sqrt{16\pi G}$. After the perturbations left the horizon, $k \ll aH$ and thus one gets:

$$\frac{h''_{\lambda,k}}{h'_{\lambda,k}} \approx -\frac{2a'}{a} \tag{7}$$

By putting $h'_{ij} = y$ it follows:

$$\begin{cases} h'_{\lambda,k} = y \\ \frac{y'}{y} = -\frac{2a'}{a} \end{cases}$$

which gives $y = a^{-2}$, and hence

$$h_{\lambda,k} = A + B \int^{\tau} \frac{d\tau'}{a^2(\tau')}. \tag{8}$$

Ignoring the second term, which represents a decaying mode [8], one gets a constant $h_{\lambda,k}$ outside the horizon. Thus, one can write

$$h_{\lambda,k} = h_{\lambda,k}^{prim} \mathcal{T}(\tau, k), \tag{9}$$

where $h_{\lambda,k}^{prim}$ is the primordial GW mode that left the horizon during the inflation and $\mathcal{T}(\tau, k)$ is the transfer function which describes the subhorizon evolution of GW modes after they entered the horizon. It is now possible to define the power spectrum of GWs as the Fourier transform of the autocorrelation function of the GWs amplitude:

$$\langle h_{ij}(\tau, \vec{x}) h^{ij}(\tau, \vec{x})) \rangle = \int \frac{dk}{k} \triangle_h^2(\tau, k),$$

and thus

$$\triangle_h^2(\tau, k) = \frac{k^3}{\pi^2} \sum_{\lambda} \langle |h_{\lambda,k(\tau)}|^2 \rangle. \tag{10}$$

By substituting Eq. 9 to the amplitude of the GW mode, it is possible to show that using the amplitude of GWs from de Sitter inflation one obtains ([8])

$$\triangle_{h,prim}^2(\tau, k) = \frac{16}{\pi} \left(\frac{H_{inf}}{m_{Pl}} \right)^2, \tag{11}$$

with H_{inf} the Hubble constant during inflation.
The energy density of GWs is given by the 00-component of the stress-energy tensor, that by definition reads

$$\rho_h(\tau) = \frac{\langle h'_{ij}(\tau, \vec{x}) h^{ij\prime}(\tau, \vec{x}) \rangle}{32\pi G a^2(\tau)}. \tag{12}$$

For the derivation of the spectral energy density one can proceed as reported in Appendix 1. The relative spectral energy density is given by:

$$\Omega_h(\tau, k) = \frac{\triangle_{h,prim}^2}{12 H^2(\tau) a^2 \rho_c(\tau)} [T'(\tau, k)]^2. \tag{13}$$

By neglecting the dark energy contribution, as it is important only at low frequency, it is possible to solve this equation numerically for different values of k. The result is reported in Fig.1 for values of

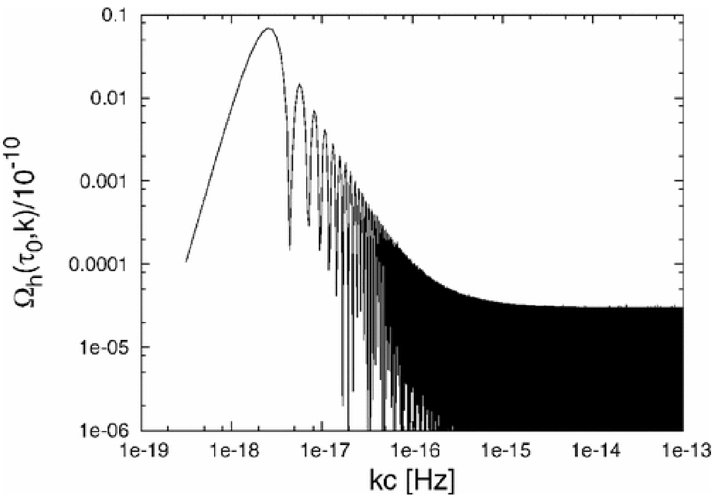

Figure 1: The primordial GW spectrum is plotted as a function of the comoving wavenumber k. It is possible to see the scale-invariance of the spectrum at large wavenumbers [8].

matter and radiation density (Ω_m and Ω_r) given by $\Omega_m = 1 - \Omega_r$, $\Omega_r h^2 = 4.15 \times 10^{-5}$ and $h = 0.7$ ($h = H_0/100\text{km}^{-1}\text{sec Mpc}$, $H_0 \approx 72\text{km sec}^{-1}\text{Mpc}^{-1}$ is the Hubble rate) [9]. The plot shows the evolution of the relative spectral energy density of GWs. The energy density evolves in the same way of radiation inside the horizon ($k \gg aH$), as it has been already pointed out. Massless particles have a time-independent energy density during the first times of radiation era, until the rate of reciprocal scatterings was greater than the expansion rate. When the scattering rate became lower than the expansion one, the energy density of massless particles started to decrease as $a(\tau)^{-1}$. Therefore, $\Omega_h(\tau, k)$ remains independent of time during the radiation era, while it decreases as $\Omega_h(\tau, k) \propto a^{-1}$ once matter era sets in. This straightforwardly implies that GW modes that enter the horizon during the matter era later will decay less. The modes that entered the horizon later are represented by the low frequency ones (where frequency is given by $f_0 = kc/2\pi$), and thus one expects to see the relative spectral energy density to rise up towards lower wavenumbers, which is exactly what happens in the diagram. As reported in [8], $\Omega_h(\tau, k < k_{eq}) \propto k^{-2}$. Instead, the wavenumbers $k \gtrsim 10^{-15}$ Hz refer to modes that entered the horizon earlier, during the radiation dominated era (i.e. $k > k_{eq}$). This caused all these modes to suffer the same amount of redshift from the radiation-matter transition era, and therefore the shape of $\Omega_h(\tau, k)$ remains scale-invariant for all the considered modes. It is possible to see this by looking at the "tail" of the diagram, which is perfectly flat from $kc \sim 10^{-15}$ Hz on.

3 Relativistic Degrees of Freedom and their effect on the spectral energy density of GWs

In order to study the primordial universe and how its components interact with each other, one needs to define the number density of particles, which is given by

$$n(T, \mu) = \frac{g}{(2\pi)^3} \int f(\vec{q}, T, \mu) d^3 q, \tag{14}$$

where $\vec{q}$ are the momenta of particles and g their helicity states. In particular g refers to a single particle species (one has $g = 2$ for photons and other massless vector bosons, $g = 3$ for massive vector bosons and $g = 2$ for spinors). Therefore, f results to be a probability density in momentum space, since we are considering an isotropic and homogeneous Universe, which is already "mediated" over positions. In an analogous way, it is possible to define the energy density and pressure respectively as:

$$\rho(T,\mu) = \frac{g}{(2\pi)^3}\int E(\vec{q})f(\vec{q},T,\mu)d^3q, \tag{15}$$

(with $E = \sqrt{q^2+m^2}$) and:

$$P(T,\mu) = \frac{g}{(2\pi)^3}\int \frac{q^2}{3E(\vec{q})}f(\vec{q},T,\mu)d^3q. \tag{16}$$

In each of these definitions, T is the temperature and μ the chemical potential, while in the last one, the factor $1/3$ shows up because of the isotropy of the space. It is possible to show that:

$$f(\vec{q},T,\mu) = \left[e^{\frac{E}{T}} \mp 1\right]^{-1}, \tag{17}$$

where $+$ is for bosons (Bose-Einstein statistics), $-$ for fermions (Fermi-Dirac statistics) and the chemical potential has been neglected (this is due to the fact that the Universe is globally neutral and that baryons are far fewer than photons). By considering now the ultra-relativistic and the non-relativistic limits ($T \gg m$ and $T \ll m$ respectively) we get

$$n(T) = \begin{cases} g\frac{\zeta(3)}{\pi^2}T^3 & \text{BE} \\ \frac{3}{4}g\frac{\zeta(3)}{\pi^2}T^3 & \text{FD} \end{cases} \tag{18}$$

$$\rho(T) = \begin{cases} g\frac{\pi^2}{30}T^4 & \text{BE} \\ \frac{7}{8}g\frac{\pi^2}{30}T^4 & \text{FD} \end{cases} \tag{19}$$

for $T \gg m$, and

$$n(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}, \tag{20}$$

$$\rho(T) = mn(T), \tag{21}$$

for $T \ll m$, regardless the considered type of particle. At this point it is easy to note that $\rho_{rel} \gg \rho_{nonrel}$, because of the exponential factor of $n_{nonrel}(T)$ (if $m \gg T$, then $e^{-m/T}$ rapidly decreases, and so does ρ_{nonrel}). In a situation in which there are different types of particles interacting with each other, the energy density of the relativistic ones overcomes the non relativistic contribution. One thus would have [8]:

$$\rho = g_*(T)\frac{\pi^2}{30}T^4, \tag{22}$$

where

$$g_* = \sum_{i=BE} g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{j=FD} g_i\left(\frac{T_j}{T}\right)^4, \tag{23}$$

defines the so called "effective relativistic degrees of freedom". It is now possible to verify that the entropy density can be written as (Appendix 2)

$$s(T) = \frac{2\pi^2}{45} g_{*s} T^3, \tag{24}$$

and since $S(T) = s(T)a(T)^3$ =constant [10], the entropy per unit comoving volume then reads

$$S(T) = \frac{2\pi^2}{45} g_{*s} T^3 a^3. \tag{25}$$

The energy density and pressure in the primorial Universe are respectively given by

$$\rho(T) = \frac{\pi^2}{30} g_*(T) T^4 \qquad P(T) = \frac{1}{3}\rho(T), \tag{26}$$

where g_{*s} and g_* have been defined as the number of relativistic species contributing to entropy and radiation energy density. Eq. 25 and 26 imply that energy density of the Universe during the radiation era should have evolved as

$$\rho \propto g_* g_{*s}^{-4/3} a^{-4}. \tag{27}$$

This shows a very important result, because if g_{*s} and g_* were not constant, it would mean that ρ could deviate from the usual known relation $\rho \propto a^{-4}$. Somehow, the evolution of energy density during the radiation era is sensitive to the relativistic species the Universe had at a given time. This result naturally extends itself to gravitational waves, as the wave equation contains (a'/a). One has indeed

$$\frac{a'(\tau)}{a^2} = H_0 \sqrt{\left(\frac{g_*}{g_{*0}}\right)\left(\frac{g_{*s}}{g_{*s0}}\right)^{-4/3} \Omega_r \left(\frac{a}{a_0}\right)^{-4} + \Omega_m \left(\frac{a}{a0}\right)^{-3}}. \tag{28}$$

At a first approximation, the interactions between all the involved particles can be assumed to be weak enough to treat them as ideal gases. In the picture of an ideal gas of temperature T, a species of a mass $m_i = x_i T$ would contribute to g_* andg_{*s} as follows:

$$g_{*,i}(T) = g_i \frac{15}{\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{e^u \pm 1} u^2 du \tag{29}$$

$$g_{*s,i}(T) = g_i \frac{15}{\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{e^u \pm 1} \left(u^2 - \frac{x_i^2}{4}\right) du \tag{30}$$

where the signs + and − are respectively referred to bosons and fermions, and g_i is the number of helicity states of particle and antiparticle. The integral variable is defined as $u = E/T$, with E the energy of the particle. The effective number of relativist d.o.f. is finally obtained by the temperature-weighted sum of all the particles species:

$$g_*(T) = \sum_i g_{\star,i}(T) \left(\frac{T_i}{T}\right)^4, \qquad g_{*s}(T) = \sum_i g_{*s,i}(T) \left(\frac{T_i}{T}\right)^3 \tag{31}$$

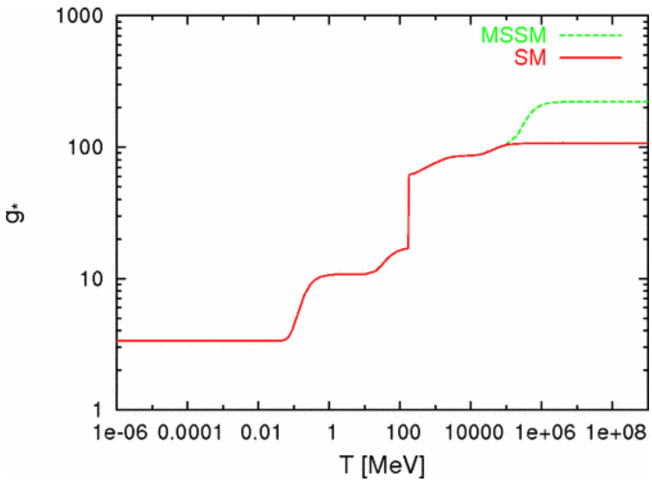

Figure 2: The evolution of g_* from higher temperatures, above supersymmetry breaking (very early universe), to lower ones [8].

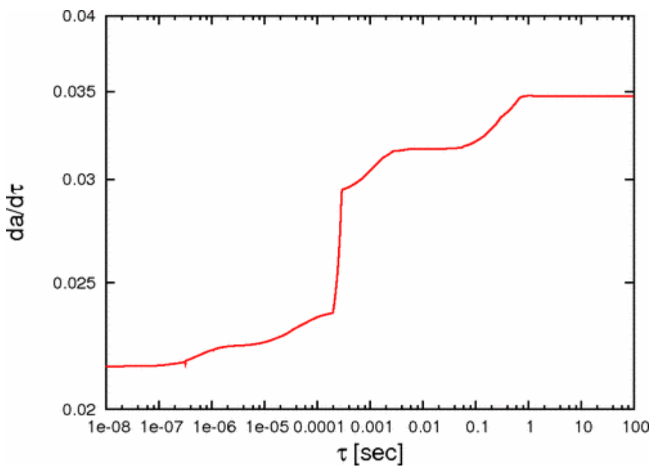

Figure 3: The evolution of a' expressed as function of the conformal time τ. If the relativistic degrees of freedom were constant, the relation $\rho \propto g_* g_{*s}^{-4/3} a^{-4}$ would reduce to the usua one $\rho \propto a^{-4}$ and a' would be constant [8].

In Fig. 2 it is possible to see the evolution of the relativistic degrees of freedom associated to the energy density. Here all the particles in the Standard Model (SM) of particle physics have been taken into account, and the green dashed line concerns the possible existence of superpartners in the minimal extension of supersymmetric standard model (MSSM). Fig. 3 shows instead the evolution of a' as a consequence of the evolution of the relativistic degrees of freedom. If both the d.o.f. were constant, it would be that even for a', which shows instead a series of bumps [8]. However, the interactions between particles actually change the ideal gas results, and the Eqs. 29, 30 and 31 cannot be used to calculate the actual effective relativistic degrees of freedom [8]. From $S(T)$ and $\rho(T)$ it is possible to perform a numerical extrapolation of g_* and g_{*s}. The role that particles interactions have in modifying the evolution of g_* has been pointed out by [3, 11, 12].
It has been already pointed out how the expansion of the Universe could affect the relative energy density spectrum $\Omega_h(\tau, k)$. Inside the horizon, the energy density of GWs evolves ideed like the photons one and while it remains constant in time during the radiation epoch, it decreases as a^{-1} during the matter dominated era. From this it follows that modes that entered the horizon later would be less decreased in energy density (Fig. 1). Now, since gravitons (the hypothetical quanta

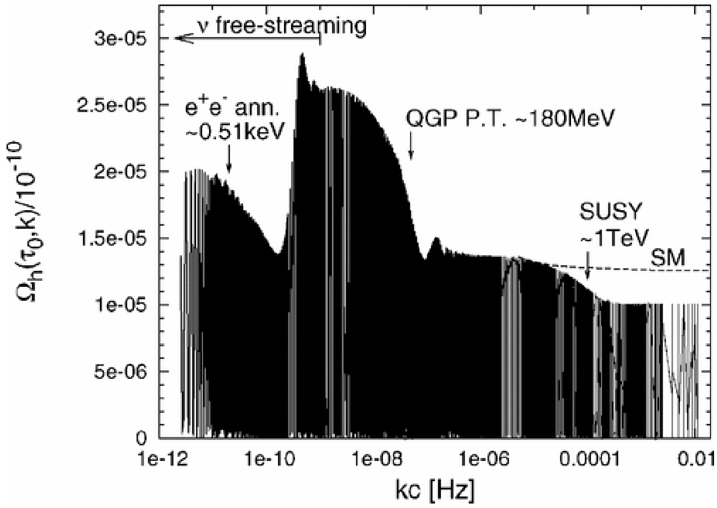

Figure 4: The image shows different features of $\Omega_h(\tau_0, k)$ between $kc = 10^{-12}$ and 10^{-3} Hz. Electron-positron annihilation ($\sim$ 0.51 MeV), QGP to hadron gas ($\sim$ 180 MeV) and the possible breaking of Supersymmetry ($\sim$ 1 TeV) are reported. It is also possible to note the amplification of GW spectrum at about $kc \sim 10^{-9}$Hz due to free streaming neutrinos that entered the horizon at $k\tau_{\nu dec} = 5.0$ [8].

of gravity) are not in thermal equilibrium with other particles [8], while the energy density of the universe (ρ_{cr}) during the radiation epoch is influenced by both density and entropy d.o.f., the energy density of gravitational waves always evolves as $\tilde{\rho}_h(\tau, k) \propto a^{-4}$ inside the horizon, regardless of g_*, $g_{*,s}$. It is indeed the difference in the evolution of $\tilde{\rho}_h$ and ρ_{cr} (Appendix 1 Eq. 1.72) that modifies the scale invariant spectrum, which has been shown for $k > k_{eq}$. After entering the horizon at time $\tau_{hc} < \tau_{eq}$ and temperature $T = T_{hc}$, the amplitude of a GW mode with wavenumber k is suppressed by the cosmological redshift. Thus the relative spectral density now would be

$$\Omega_h(\tau_0, k > k_{eq}) = \Omega_h(\tau_{hc}, k > k_{ue})\Omega_{r0} \left[\frac{g_{*s}(T_{hc})}{g_{*s0}}\right]^{-4/3} \left[\frac{g_*(T_{hc})}{g_{*0}}\right], \tag{32}$$

where Ω_r represents the relative energy density of radiation and 0 indicates the different quantities nowadays. It is now possible to see clearly how the change of the relativistic degrees of freedom could affect not only the energy density of the universe, Eq. 27, but also $\Omega_h(\tau_0, k)$. For a wavenumber k, it would correspond a horizon-crossing epoch τ_{hc}. Then the amount by which the relative spectral energy density of that mode would be suppressed depends on g_* and g_{*s} at τ_{hc}. Since, as one can see in Fig. 2, the relativistic d.o.f. are larger at higher temperatures, the mode which enters the horizon earlier will be suppressed more. It is possible to use an approximation for $T \geq 0.1$MeV, and considering g_* and g_{*s} to be very similar (however, for $T \lesssim 0.1$MeV, $g_* = 3.3626$ and $g_{*s} = 3.9091$). Thus, the suppression factor is expected to be $\sim (g_*/g_{*0})^{-1/3}$. Then, as it has been already pointed out, after the equivalence the relativistic d.o.f. do not change and so do the modes that enter the horizon during matter era. The effect of evolutions of the two relativistic d.o.f. that have been considered so far is quite prominent. During the electron-positron annihilation epoch, at about $T \sim 0.51$ MeV, i.e. $kc \sim 2 \times 10^{-11}$Hz, a big change in g_* occured and thus it is possible to note a suppression of the spectral energy density of GW of about 20%, as Fig. 4 shows. At the quark gluon plasma (QGP) to hadron gas transition, another change in the d.o.f. occured with a resultant 30% suppression factor of the GW energy density spectrum, as well as at the hypothetical supersymmetry breaking, $T \sim 1$ TeV ($\sim 1 \times 10^{-4}$ Hz), correspondingly to that the spectrum would be suppressed by at least $\sim 20\%$ [8].

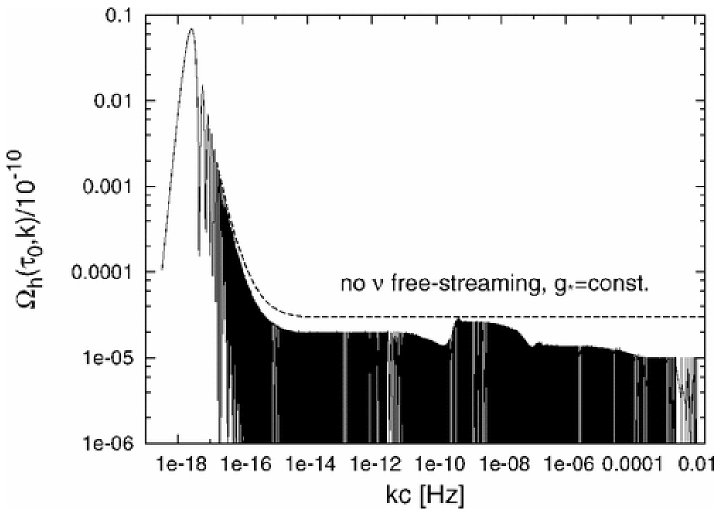

Figure 5: Relative spectral energy density at present as a function of the wavenumber and modified by the presence of free streaming neutrinos. The non-modified spectrum is shown with a dashed line [8].

4 The effects of Free-Streaming Neutrinos

Another important effect which strongly affects the spectral energy of GWs background is given by the anisotropic stress present in the right hand side of Eq. 6. In fact, up to now we've only considered the modifications given by the amplitude h of GWs. However, free streaming neutrinos that decouple from thermal equilibrium ($T \lesssim 2$MeV) contribute to the anisotropic stress damping of the amplitude of GWs [1, 7]. In an imperfect fluid of viscosity η one has indeed $\Pi_{ij} = -\eta \dot{h_{ij}}$ [13]. Nevertheless h_{ij} becomes time-independent as the wavelength of the mode leaves the horizon, and it remains so until horizon re-entry. Since all modes that are interesting from a cosmological point of view are far outside the horizon at the temperature $\sim 10^{10}$ K, where neutrinos are leaving their last scattering surface, h_{ij} can be anisotropically affected only when neutrinos are freely streaming. Later on it will be shown that the right-hand side of Eq 6 can be rewritten as

$$16Ga^2\pi\Pi_{\lambda,k} = -24f_\nu(\tau)\left[\frac{a'(\tau)}{a(\tau)}\right]^2\int_{\tau_{\nu dec}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}\right] h'_k(\tau'), \tag{33}$$

from which it is possible to see that the anisotropic stress is proportional to the fraction of the total energy density of neutrinos. Calculations show that neutrino stress damps the relative spectral energy density of GW by 35.5% for frequencies between $\simeq 10^{-16}$ and $\simeq 2 \times 10^{-10}$Hz. For frequencies below, the universe is dominated by matter and hence the damping effect is much less significant, since the energy density of neutrinos becomes very small after the radiation-matter equality [8]. Fig. 5 reports the result of numerical integration of $\Omega_h(\tau_0, k)$ modified by free streaming neutrinos.

In order to see how these results have been obtained, it is useful to change the point of view and considering the one of free streaming neutrinos. These are considered as massless particles here and hence their line element will be $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$. Moreover, for the purposes of this review, it can be assumed that a gas of relativistic neutrinos could be treated by classical kinetic theory. In this scenario, the Einstein-Boltzmann equation can be written as an integro differential equation that shows how h_{ij} suffers the presence of free streaming neutrinos ([7]).
After neutrinos decoupled, by supposing an instantaneous decoupling ($T \sim 2$ MeV), the free streaming neutrino gas satisfies the so called Vlasov equation (i.e. the collisionless Boltzmann equation):

$$\frac{dF(x,P)}{dt} = 0, \tag{34}$$

where: $F(x,P)$ is the distribution function of neutrinos, $P^\mu = \frac{dx^\mu}{d\lambda}$ is the 4-momentum of a neutrino, and since, considering massless neutrinos, $g_{\mu\nu}P^\mu P^\nu = 0$, we get only three independent components of the 4-momentum, i.e. $P^0 = \sqrt{g_{ij}P^i P^j}$. These three components can be written, in terms of the directional cosines $\gamma^i = \cos\gamma^i = \frac{\vec{v}\cdot\vec{e^i}}{|\vec{v}|}$, as

$$P^i = \pm\frac{\gamma^i P^0}{a}\left(\frac{1}{2}h_{jk}\gamma^j\gamma^k\right). \tag{35}$$

Therefore, since $P^i = P^\mu(P^0, \gamma^i)$:

$$\frac{dF(t,x^i,\gamma^i,P^0)}{dt} = \frac{\partial F}{\partial t} + \frac{dx^i}{dt}\frac{\partial F}{\partial x^i} + \frac{dP^0}{dt}\frac{\partial F}{\partial P^0} + \frac{d\gamma^i}{dt}\frac{\partial F}{\partial \gamma^i} = 0. \tag{36}$$

Now, $\frac{\partial F}{\partial \gamma^i}$ is a the first order term in perturbation theory, as the unperturbed distribution of neutrinos cannot depend upon the direction, and since $\dot{\gamma^i} = -\frac{1}{2a}h_{jk,i}\gamma^j\gamma^k$, the last term of the equation can be neglected at first order. By considering the second term we get:

$$\frac{dx^i}{dt}\frac{\partial F}{\partial x^i} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt}\frac{\partial F}{\partial x^i} = \frac{P^i}{P^0}\frac{\partial F}{\partial x^i},$$

where the $\partial F/\partial x^i$ is of the first order. The third term concerns how the energy of neutrinos changes in time while they propagate through the space-time. In order to obtain its evolution one can consider the geodesic equation of neutrinos;

$$\frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\nu\rho}P^\nu P^\rho \tag{37}$$

where $\Gamma^\mu_{\nu\rho} = \frac{g^{\mu\tau}}{2}(g_{\tau\nu,\rho} + g_{\tau\rho,\nu} - g_{\nu\rho,\tau})$ are the Chrystoffel symbols. By considering the perturbed metric given by Eq. 2, one has:

$$\begin{aligned}
\Gamma^0_{00} &= 0 \\
\Gamma^0_{0i} &= 0 \\
\Gamma^0_{ij} &= \frac{1}{2}\left[2a\dot{a}(\delta_{ij} + h_{ij}) + a^2 h_{ij,0}\right],
\end{aligned} \tag{38}$$

where it has been considered $\partial_0 g_{\mu\nu} = \partial_t g_{\mu\nu}$. Now, by replacing P^i with Eq. 35 and considering only first order terms in perturbation theory, the geodesic equation for the time component of P^μ finally reads

$$\frac{dP^0}{d\lambda} = \frac{dt}{d\lambda}\frac{dP^0}{dt} = -\frac{\dot{a}}{a}(P^0)^2 - \frac{1}{2}\frac{\partial h_{ij}}{\partial t}\gamma^i\gamma^j(P^0)^2, \tag{39}$$

thus (since $dt/d\lambda = P^0$):

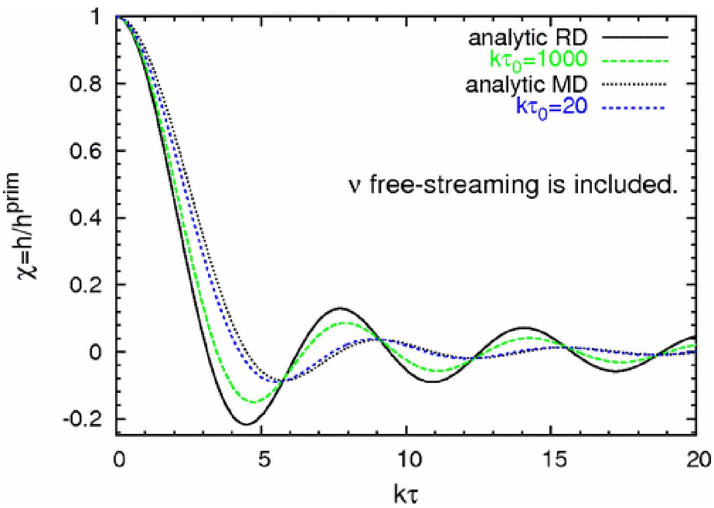

Figure 6: The figure shows how the presence of free streaming neutrinos damps the amplitude of primordial GWs. The solid black line is an analytical solutions of tensor perturbations, which does not include the effect of neutrinos. The dashed and short-dashed lines instead are numerical solutions which do include neutrinos' effects respectively for higher and lower frequency modes [8].

$$\frac{1}{P^0}\frac{dP^0}{dt} = -\frac{\dot{a}}{a} - \frac{1}{2}\frac{\partial h_{ij}}{\partial t}\gamma^i\gamma^j \tag{40}$$

This equation describes how the energy of a neutrino, which is propagating through a perturbed space-time (i.e. a FLRW space-time with a GW background at first order in perturbation theory), changes in time. It is possible to see that the amount of energy, as well as its evolution, depends upon two term: the first one is just the redshift term accounting for the isotropic expansion of the universe, whereas the second one describes the variation of GWs' amplitude in time, and thus it connects the energy of free streaming neutrinos to the amplitude of GWs. Neutrinos gain energy if $\frac{\partial h_{ij}}{\partial t} < 0$ and lose energy if $\frac{\partial h_{ij}}{\partial t} > 0$. It is indeed this flow of energy between free streaming neutrinos and GWs that causes a collisionless damping and amplification of GWs' amplitude [8]. In Fig. 6 it is possible to note that while, entering the horizon earlier, the higher k-modes (blue short dashed line) do suffer the damping of free streaming neutrinos, the lower k-modes (green dashed line) enter the horizon much later, therefore their numerical solutions are much closer to the analytical one, which does not include free-streaming neutrinos' effects. By combining now Eqs. 36, 4 and 40, the collisionless Boltzmann equation at first order in perturbation reads:

$$\left(\frac{dF}{dt}\right)_{first order} = \frac{\partial \delta F}{\partial t} + \frac{\gamma^i}{a}\frac{\partial \delta F}{\partial x^i} - P^0\frac{\partial \delta F}{\partial P^0}\frac{\dot{a}}{a} - P^0\frac{\partial \bar{F}}{\partial P^0}\frac{1}{2} \times \frac{\partial h_{ij}}{\partial t}\gamma^i\gamma^j = 0 \tag{41}$$

where $F = \bar{F} + \delta F(t, x^i, \gamma^i, P^0)$ and δF is a tensor-type perturbation of the neutrinos' distribution function. As it can be seen, the zeroth order of the equation only accounts for the cosmological redshift $P^0 \propto a^{-1}$. The first order of the Vlasov equation written in the Fourier space reads

$$\frac{\partial f_k}{\partial t} - \frac{\dot{a}}{a}P^0\frac{\partial f_k}{\partial P^0} + \frac{ik\mu}{a}f_k = P^0\frac{\partial \bar{F}}{\partial P^0}\frac{1}{2}\frac{\partial h_k}{\partial t} \tag{42}$$

where it has been considered the generalization of Eq. 3 for a general spatial geometry which describes the universe, i.e.

$$h_{ij}(t,\vec{x}) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} h_{\lambda,k}(t) Q^{\lambda}_{ij}(\vec{x}), \tag{43}$$

$$\delta F = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} f_{\lambda,k}(t, P^0, \mu)\gamma^i\gamma^j Q^{\lambda}_{ij}(\vec{x}), \tag{44}$$

where $Q^{\lambda}_{ij}(\vec{x})$ are the symmetric, traceless and divergenceless solutions of the tensorial Helmholtz equation: $Q^{\lambda}_{ij|a}(\vec{x}) + k^2 Q^{\lambda}_{ij}(\vec{x}) = 0$, $\partial_l Q^{\lambda}_{ij} = ik_l Q^{\lambda}_{ij}$ ($Q^{\lambda}_{ij} = Q^{\lambda}_{ji}$, $\gamma^{ij} Q^{\lambda}_{ij} = Q^{\lambda}_{ii} = 0$), with $\gamma^{ij} = a^2 \bar{g}^{ij}$ and "$|$" representing the covariant derivative for the metric γ^{ij}. In Eq. 3 it has been used $Q^{\lambda}_{ij}(\vec{x})$ as a simple wave equation, since it has been described a perturbation in a flat geometry (i.e. $\Gamma^{\mu}_{\nu\rho}|_{\delta_{ij}} = 0$, thus $\nabla_{\mu} = \partial_{\mu}$ and therefore $Q^{\lambda}_{ij}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} e_{ij}$ solves the equation). It is possible to notice that from the presence of $\frac{\dot{a}}{a} P^0 \frac{\partial f_k}{\partial P^0}$ on the left-hand side of Eq. 42, this equation is not solvable. Therefore, in order to make it solvable another step is necessary. By defining the comoving momentum as $q^{\mu} \equiv aP^{\mu}$ and considering F as a function of comoving energy $q = q^0 = aP^0$ and conformal time τ, the third term of Eq. 36 can be rewritten as $(\frac{dq}{d\tau}\frac{\partial F}{\partial q})_{linearorder} = -\frac{1}{2} q h'_{ij} \gamma^i \gamma^j \frac{\partial \bar{F}}{\partial q}$. Thus, the linearized Vlasov equation now reads

$$\frac{\partial f_k}{\partial \tau} + ik\mu f_k = q \frac{\partial \bar{F}}{\partial q} \frac{1}{2} \frac{h_k}{\partial \tau}, \tag{45}$$

with $f_k = f_k(\tau, q, \mu)$, whose solution becomes

$$f_k(\tau, q, \mu) = e^{-i\mu k(\tau - \tau_{\nu dec})} f_k(\tau_{\nu dec}, q, \mu) + \frac{q}{2} \frac{\partial \bar{F}}{\partial q} \int_{\tau_{\nu dec}}^{\tau} d\tau' h'_k(\tau') e^{-i\mu k(\tau-\tau')}, \tag{46}$$

where $h'_k(\tau)$ represents the derivative of $h_k(\tau)$ with respect to the conformal time. Before decoupling, the neutrino distribution function does not have any tensor perturbations, thus $f_k(\tau_{\nu dec}, q, \mu) = 0$. The right hand side of linearly perturbed Einstein's equations (Eq. 4) contains anisotropic stress due to free streaming neutrinos:

$$\delta T^{(\nu)}_{ij} = a^2 \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} \Pi_{\lambda,k} Q^{\lambda}_{ij}(\vec{x}), \tag{47}$$

where $T^{(\nu)}_{ij}$ represents the stress-energy tensor of neutrinos and $\delta T^{(\nu)}_{ij}$ is just its perturbation. Since $T^{(\nu)}_{ij} = \frac{1}{\sqrt{-det(g_{\mu\nu})}} \int \frac{d^3q}{q^0} q_i q_j F(q)$, this perturbation can be simply expressed as

$$\delta T^{(\nu)}_{ij} = a^{-4} \int \frac{d^3q}{q^0} [\bar{q}_i \bar{q}_j \delta F + (\delta q_i \bar{q}_j + \bar{q}_{ij})\bar{F}]. \tag{48}$$

One can now neglect the last two terms of the perturbation, since they are not at linear order, and by substituting it into Eq. 47, and observing that $q_i = aq^i = aq\gamma^i$, it is possible to obtain:

$$\Pi_{\lambda,k} Q^{\lambda}_{ij}(\vec{x}) = a^{-4} \int \frac{d^3q}{q^0} q^2 \gamma^i \gamma^j \gamma^l \gamma^m f_{\lambda,k} Q^{\lambda}_{lm}(\vec{x}), \tag{49}$$

By inserting the solution given by Eq. 46 and using $\int d\Omega_q \gamma^i \gamma^j \gamma^l \gamma^m e^{-i\hat{\gamma}\cdot\hat{k}u} Q^\lambda_{ij}(\vec{x}) = \frac{1}{8}(\delta^{il}\delta^{jm} + \delta^{im}\delta^{jl})\Omega_q e^{-i\mu u}(1-2\mu^2+\mu^4 0)Q^\lambda_{ij}$, one gets

$$\Pi_k = \frac{1}{4a^4}\int d^3 q q(1-2\mu^2+\mu^4) f_k. \tag{50}$$

Using the identity $\frac{1}{16}\int_{-1}^{+1} d\mu(1-2\mu^2+\mu^4)e^{i\mu u} = \frac{j_2(u)}{u}$, where $\frac{j_2(u)}{u}$ is even and such that $\int_{-\infty}^{\infty}\frac{j_2(u)}{u} = \frac{\pi}{8}$ and $\lim_{u\to 0}\frac{j_2(u)}{u} = \frac{1}{15}$, and integrating by parts, it is possible to obtain

$$\Pi_k = -4\bar{\rho_\nu}(\tau)\int_{\tau_{\nu dec}}^{\tau} d\tau' \left(\frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}\right) h'_k(\tau'). \tag{51}$$

Here $\bar{\rho_\nu}(\tau) = a^{-4}\int d^3 q q \bar{F}(q)$ represents the unperturbed neutrino energy density.
Therefore, the linearly perturbed Einstein-Vlasov equation for the GWs amplitude accounting for free streaming neutrinos' damping takes the form of an integro-differential equation [7,8]:

$$h''_k(\tau) + \left[\frac{2a'(\tau)}{a(\tau)}\right] h'K(\tau) + k^2 h_k(\tau) = -24 f_\nu(\tau)\left[\frac{a'(\tau)}{a(\tau)}\right]\int_{\tau_{\nu dec}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}\right] h'_k(\tau'). \tag{52}$$

The fraction of the total energy density in neutrinos is now simply

$$f_\nu \equiv \frac{\rho_{\bar{\nu}(\tau)}}{\rho(\bar{\tau})} = \frac{\Omega_\nu (a_0/a)^4}{\Omega_M (a_0/a)^3 + (\Omega_\gamma + \Omega_\nu)(a_0/a)^4} = \frac{f_\nu(0)}{1 + a(\tau)/a_{EQ}} \tag{53}$$

where $f_\nu(0) = \frac{\Omega_\nu}{\Omega_\gamma + \Omega_\nu} = 0.40523$ [8]. Since during the matter dominated era $f_\nu \to 0$, the anisotropic stress-energy tensor Π_k tends to zero, which states that the damping effect due to free streaming neutrinos is unimportant during the matter era. The solution of Eq. 52 can be expressed in the general form $h_\lambda(u) \equiv h_\lambda(0)\chi(u)$ [7].Therefore, Eq. 52 can be rewritten in terms of this new variable $\chi(u)$

$$\chi''(u) + \left[\frac{2a'(u)}{a}\right]\chi'(u) + \chi(u) = -24 f_\nu(u)\left[\frac{a'(u)}{a}\right]^2 \int_{u_{\nu dec}}^{u} dU \left[\frac{j_2(u-U)}{(u-U)^2}\right]\chi'(U), \tag{54}$$

where $u = h\tau$ and the derivatives are taken with respect to u. Since the amplitude of perturbations becomes rapidly independent upon time after horizon exit [7], it is conserved until the mode re-enters the horizon, thus $h_\lambda(0) = h_{\lambda,\vec{k}}(\tau_{end})$. Moreover, even the anisotropic damping due to neutrinos disappears on super-horizon scales, as neutrinos can affect the metric perturbation only inside the horizon [8]. The initial condition are $\chi(0) = 1$, $\chi'(0) = 0$ ($h_{ij}(0) = h_{ij}(0)$, $h'_{ij}(0) = h'_{ij}(0)$). The numerical solutions of this equation are the ones reported in Fig. 6.
In order to estimate the effect of damping, let's consider the radiation era after neutrinos became free streaming, i.e. after neutrinos decoupling. During the radiation era $a'(u)/a = 1/u$ and, in the absence of the neutrinos' contribution (thus setting $\Pi_k = 0$), Eq. 54 becomes a Bessel's equation of order zero, hence its solution is given by $\chi(u) = j_0(u)$. In the presence of neutrino free streaming, the solution shows an asymptotic tendency [7] (for $u \gg 1$)

$$\chi \to A\frac{\sin u + \delta}{u}, \tag{55}$$

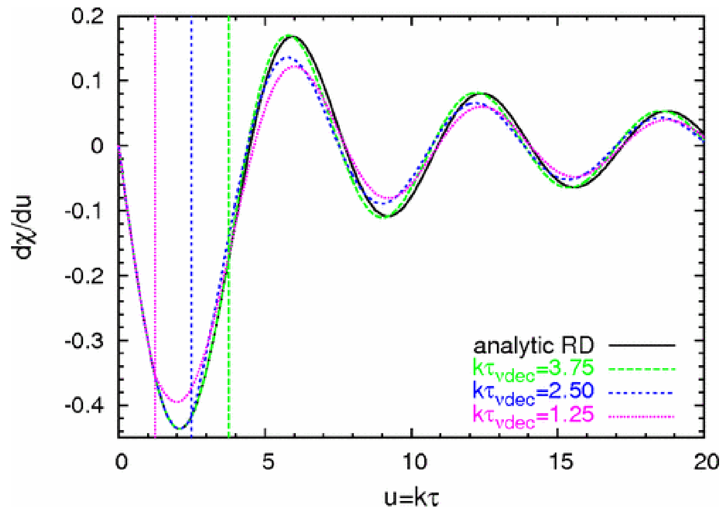

Figure 7: The image shows the derivative of the modes that entered the horizon at different times: $k\tau_{\nu dec} = 1.25$, 2.50 and 3.75. The solid line refers to the analytical solution $\chi'(u) = -j_1(u)$ which does not take into account the presence of free streaming neutrinos. The dotted, short-dashed and dashed lines reports numerical solutions that consider the effect of neutrinos with different modes. The vertical lines show the correspondent decoupling times for each solution [8].

where $A = 0.80313$ and $\delta = 0$ as it has been obtained numerically by Y. Watanabe and E. Komatsu (2006) [8]. This asymptotic form provides the initial conditions for the later period, when the matter energy density becomes greater than the radiation's one (i.e. matter dominated era). The effect of neutrino damping at these times is still only to reduce the tensor amplitude by the factor A. Hence, the suppression factor $A^2 = 0.64502$ applies to the gravitational waves' spectrum that entered the horizon after neutrino decoupling but well before the radiation-matter equivalence [8]. In Eq. 40 we saw that, depending upon the time derivative of the neutrino energy, the amplitude of the mode can be damped or amplified. By integrating the amplitude of gravitational waves over the time, one obtains that free streaming neutrinos almost always contribute to the GWs' spectrum damping the modes. This can be seen in Fig. 5, where the spectrum of GWs is affected by an overall damping due to neutrinos.
For wavelengths that entered the horizon right after the neutrino decoupling and before the equivalence era, all quadratic effects of the tensor modes on the cosmic microwave background (as the tensor contribution to the temperature multipole coefficients, for instance) are dumped by the 35.5% by free streaming neutrinos.

Let's now consider the higher k-modes: $k\tau_{\nu dec} \sim 1$, or $k \sim 10^{-10} - 10^{-9}$ Hz. The condition $k\tau_{\nu dec} = 1$ corresponds to k-modes that entered the horizon just at the neutrino decoupling time, while by requiring $k\tau_{\nu dec} > 1$ are indicated the modes that entered the horizon earlier. In Fig. 7, 8 numerical solutions for $\chi'(u)$ are reported at different k-modes. It is possible to see that for $k\tau_{\nu dec} = 1.25$, 2.50 neutrino decoupling took place at the first depression of $\chi(u)$, where it is negative. Therefore, the modes are damped releasing energy to free streaming neutrinos. Indeed if $\frac{\chi(u)}{\partial u} < 0$, with $u = k\tau$, by using $h_\lambda(u) \equiv h_\lambda(0)\chi(u)$ one gets $\frac{\partial h_\lambda \chi(u)}{\partial u} = h_\lambda \frac{\partial \chi(u)}{\partial u} < 0$ where $\frac{\partial h_\lambda(u)}{\partial u}$ corresponds to the Fourier transform of the last term in Eq. 40. Since it is negative, the time derivative of neutrino energy gets enhanced. For higher k-modes ($k\tau_{\nu dec} = 3.75$), neutrinos decouple at higher value of $\chi'(u)$, which is closer to zero, giving an almost vanishing amplifying effect. At $k\tau_{\nu dec} = 5.0$ the decoupling corresponds to the first peak of $\chi'(u)$, and hence the amplitude of GWs is amplified by neutrinos. This feature can be seen in Fig. 4.

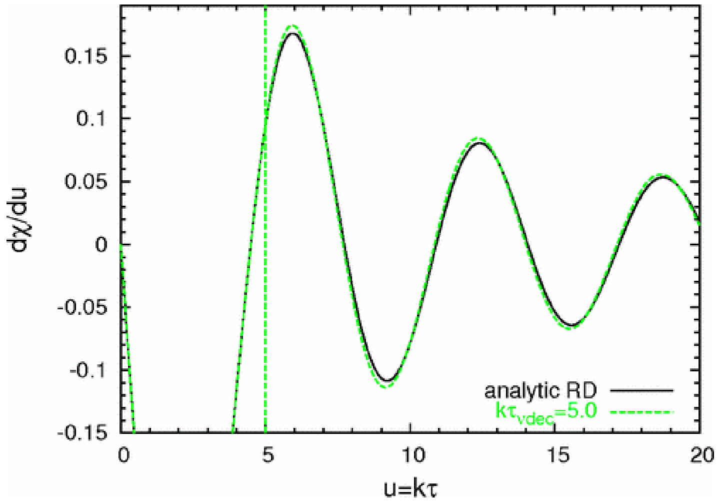

Figure 8: The image shows the derivative of a mode that entered the horizon at $k\tau_{\nu dec} = 5.0$. The solid line refers to the analytical solution $\chi'(u) = -j_1(u)$ which does not take into account the presence of free streaming neutrinos. The dashed lines reports the numerical solution that considers the effect of neutrinos with $k\tau_{nudec} = 5.0$. The vertical line shows the correspondent decoupling time [8].

5 Second Order Perturbations

A further step in the determination of how free-streaming neutrinos can affect the primordial GW background would be considering the second order in perturbation theory. Here it is reported a very brief discussion of this topic; the complete analysis is given by [4]. This could be done by considering the solution of the second order Boltzmann equation for neutrinos

$$\frac{\partial F_\nu}{\partial \tau} + \frac{\partial F_\nu}{\partial x^i} \cdot \frac{dx^i}{d\tau} + \frac{\partial F_\nu}{\partial E}\frac{dE}{d\tau} + \frac{\partial F_\nu}{\partial n^i} \cdot \frac{dn^i}{d\tau} = 0, \tag{56}$$

where here $F_\nu = \bar{F}_\nu(E,\tau) + F_\nu^{(1)}(E,\tau,x^i,n^i) + \frac{F_\nu^{(2)}}{2}(E,\tau,x^i,n^i)$. By translating it into the Fourier space one can find the following expression for the solution of the Boltzmann equation:

$$F_{\vec{k}}^{(2)}(\tau) = \int_{\tau_{dec}}^{\tau} d\tau' e^{ik\mu(\tau'-\tau)}[G_k(\tau') - T_k(\tau')] \tag{57}$$

where $G_k(\tau')$ accounts for the second-order scalar and tensor perturbations and T_k is the "pure" tensor contribution. Following the analogue steps underlined above, it is then possible to compute the second-order neutrino energy-momentum tensor, which reads

$$(\delta T_j^{i(\nu)})_{\vec{k}}^{(2)} = 2a^{-3}\int \frac{d^3q}{(2\pi)^3} q n^i n_j \left[\int_{\tau_{dec}}^{\tau} d\tau' e^{ik\mu(\tau'-\tau)} \times [G_k(\tau') - T_k](\tau')\right]. \tag{58}$$

The contribution of free streaming neutrinos to the evolution of gravitational waves up to second order in perturbation theory is finally given by

$$\Pi^{\nu}_{\vec{k},\lambda} = \bar{\rho}_\nu \int\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta(\vec{k_1}+\vec{k_2}-\vec{k}) \times (k_{2r}k_1\epsilon^{rs}_\lambda)\sum_l \frac{l}{2l+1}\int_{\tau_{dec}}^{\tau} d\tau'(\Phi_{\vec{k}_2}(\tau')+\Psi_{\vec{k}_2}(\tau'))\cdot$$
$$\cdot [B_l - B_l^{(1)}]\int_{\tau_{dec}}^{\tau'} d\tau''(\tau''-\tau')(\Phi'_{\vec{k}_1}(\tau'')+\Psi'_{\vec{k}_1}(\tau''))[A_l - A_l^{(1)}]+ \tag{59}$$
$$-8\bar{\rho}_\nu(\tau)\int_{\tau_{dec}}^{\tau} d\tau' h'_k(\tau') \times \frac{1}{15}[j_0(s)+\frac{10}{7}j_2(s)+\frac{3}{7}j_4(s)],$$

where Ψ and Φ are scalar functions which encode the perturbations of the metric (scalar and tensorial respectively), ϵ^{rs}_λ are the polarisation tensors and A_l, $A_l^{(1)}$, B_l and $B_l^{(1)}$ are functions that come from the angular integration of the new second order terms one can find computing $\Pi_{\vec{k},\lambda}$ and that depend upon the multipole moment l [4]. The last line term is nothing but the collisionless free streaming neutrinos term which one finds in first order perturbation. It corresponds indeed to Eq. 51, where here $s = u - U$ and $\frac{j_2(s)}{s^2} = \frac{1}{15}(j_0(s) + \frac{10}{7}j_2(s) + \frac{3}{7}j_4(s))$ has been used. This shows that even the second order gravitational waves are damped by the presence of free streaming neutrinos [4]. Besides incorporating the first order term, as it should do, this expression also shows new terms that contribute to the source of second-order gravitational waves. These are given by the high dispersion of neutrinos' velocity, which shows up when neutrinos are treated as collisionless ultrarelativistc particles. These new terms, just like the first order one, are non-negligible during the radiation dominated epoch, were the energy density of neutrinos is not negligible with respect to the density of radiation. After the equivalence and during the matter dominated era, the neutrinos' density goes rapidly down and so do the contributions of free streaming neutrinos encoded in $\Pi^{\nu}_{\vec{k},\lambda}$ [4].

6 CMB features

It has been pointed out how primordial gravitational waves' background is sensitive to the free streaming neutrinos, which damp on an overall scale the amplitude of the tensor modes, and to the changes of relativistic degrees of freedom, which cause the scale factor a to change itself. GWs' background however is responsible for some CMB fluctuations, both in temperature and in polarisation. The complete analysis of this was performed by [5, 14]. Gravitational waves cause distorsions with an angular pattern, which depends on the angle between the direction of photons $\hat{n}$ and the wavevector $\hat{k}$ of the GW, and on the azimuthal angle ϕ. A. G. Polnarev recognised the advantage of separating this angular dependence and introduced the so called Polnarev variables $\tilde{\Delta_T}$ and $\tilde{\Delta_P}$ to do so. The complete calculation is illustrated in [15]. The temperature multiple moments due to a single GW of wavenumber k observed at τ_0 are given by

$$\Delta_{Tl} = \sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\tau_0} d\tau(-\dot{h}e^{-k} + g\Psi)\frac{j_1(x)}{x^2}, \tag{60}$$

where $x = k(\tau_0 - \tau)$, $j_1(x)$ is the Bessel's function of order one, g is the visibility function $g(\tau) = \dot{\kappa}e^{-k}$ and Ψ is given by a combination of Polnarev variables [5]. Here $\dot{\kappa}$ represents the differential cross-section for Thomson scattering, so that its integral between τ and τ_0 is the total optical depth within that period of conformal time. The visibility function is defined as the probability distribution of the position of last scattering, and it indicates the probability that a photon last scattered at τ [9].
The second term inside the integral is localised to the last scattering surface (LSS), and it becomes relevant only for l between 200 and 800, whereas at lower and higher l it falls off rapidly, becoming negligible [5]. The first term involves an integral from the LSS to nowadays and it shows that the

temperature power spectrum depends upon the evolution of GWs within all the integrated period (i.e. from the LSS to the present day). It is a form of integrated Sachs-Wolfe (ISW) effect. A gravitational wave alternately stretches and compresses the space-time oscillating, and a photon that travels across that wave can either lose energy if its wavelength is stretched, or gain energy if its wavelength is compressed. Since the GW's amplitude will evolve, the energy of the photon will undergo a net energy change. After entering the horizon, GWs oscillate and decrease their amplitude, and hence a photon will gain energy travelling along the crest of a phase. Photons that travel with a non vanishing angle with respect to the mode will experience further red and blue shifting as they propagate across different phases. The period-averaged amplitude of GW decreases between LSS and nowadays, causing an increasing of the mean energy of the photon. By considering a time in which the amplitude of a GW is essentially negligible, it is possible to see that the energy of the photon is determined by whether it started propagating at LSS in a depression or in a crest of a tensor mode. If a photon started at a crest, then it will have gained energy and hence will appear hotter.
The polarisation anisotropies are instead described by

$$\Delta_{Xl} = \int_0^{\tau_0} d\tau(-g\Psi)P_{Xl}[x]. \tag{61}$$

In this case the source is closely localised to the LSS because of the only presence of the visibility function, and thus it is sensitive to the thermal history and GW evolution at that time. The CMB polarisation is generated by Thomson scattering with an anisotropic intensity distribution inside the unpolarised early radiation's bath. By considering the rest frame of an electron, photons arrive from all directions, with a distance given by the mean free path of photons at recombination epoch. In propagating, these photons suffer the ISW effect briefly explained above, and they arrive at the electron with altered temperatures. The anisotropic temperature distribution is then scattered, producing polarisations. For modes that entered the horizon after last scattering, incident photons experience a little ISW effect and hence a little polarisation is generated. On the contrary, the maximum polarisation is generated by modes with wavelengths very close to the horizon's size at the penultimate scattering [5]. The amplitude of tensor modes decreases fastly after the horizon entry before starting to oscillate with a decreasing amplitude (even if free-streaming neutrinos are not considered, as it has been already pointed out). Modes that enter the horizon just before the penultimate scattering let photons to suffer a ISW effect that samples this slowly decaying regime. Therefore, they cause significant polarisation. All this explanation does not consider the effect of free streaming as the Universe expands, which smooths the final power spectra [5].
One thus expects to have a slow increase in power at the scale of the horizon at the penultimate scattering. After this, there will be a large drop in polarisation, corresponding to the modes that entered the horizon immediately after the last scattering and triggered a much smaller ISW.

7 Conclusion

In this review, the nature of gravitational waves as perturbations of the space-time has been shown and the basic thermodynamical properties of the early Universe have been defined in terms of the relativistic degrees of freedom. Then, it has been pointed out how the change of these d.o.f. could actually affect not only the evolution of the Universe's energy density evolution (Eq. 27), but also the evolution of the scale factor (Eq. 28) and hence of the GW's spectrum. Then even free-streaming neutrinos have been considered, firstly as a damping effect to the GW spectral energy distribution, up to the first order in perturbation theory, and then, by considering the second order expansion, as an effective source of second order gravitational waves during the radiation dominated epoch (Eq. 59). Finally the role of GWs in producing some features in the CMB anisotropies, both in temperature and in polarisation, has been very briefly explained. The study of primordial gravitational waves background plays a fundamental role in the comprehension of our Universe. First of all, for their nature, GWs represent a powerful messenger, both for its distance dependence ($\sim 1/d$) and for the fact that they do not interact with matter as radiation does, propagating a very clean information through the Universe. Primordial gravitational waves were generated by quantum fluctuations of the inflaton field, and hence they are strongly dependent on the model that describes this acceleration process. In particular, it is possible to see that the power spectrum of primordial GWs can be written in a form that accounts for the energy scales at which inflation took place. Therefore, by measuring it one could have hints on the inflation's potential. Another important parameter connected to the GW power spectrum is the so called "tensor to scalar ratio" r which measures the ratio between tensor and scalar perturbations, and that can be very useful to shed light on the Early Universe cosmology. Observations of CMB can give us important constraints on primordial GWs, especially from the B-mode CMB's polarisation. These constraints could be provided by LiteBIRD, CORE and PIXIE among others experiments [16]. The interplay between neutrinos, SM particles and GW can also be a tool for fundamental particle physics and neutrino physics. It is worth to mention for instance that at scales where Supersymmetry is unbroken, g_* should double, and thus it would cause a further suppression of the GW energy density spectrum. One should expect a suppression of $\sim 85\%$ above 10^{-4} Hz in the case of $N = 8$ supersymmetric charges. Therefore, primordial gravitational waves would definitely help us even to constraint the effective number of relativistic degrees of freedom above TeV scales [8], allowing us to better understand a possible way to go beyond the standard model of particle physics.

1 Appendix A: Relative spectral energy density

The Ricci tensor for the perturbed metric can be expanded in orders of h, giving $R_{\mu\nu} = \bar{R_{\mu\nu}} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ... + O(h^n)$, where the i-th term is of the order $O(h^i)$. Since we are considering no massive source of curvature, it is possible to set $R_{\mu\nu} = 0$. This is true also for the linear part of the 1-st order term ($R^{(1)}_{\mu\nu} = R^{(1)linear}_{\mu\nu} + R^{(1)nonlinear}_{\mu\nu}$, where the non linear term represents the non linear correction to the propagation of the perturbation).
By forgetting about the order greater than the 2-nd one, one gets

$$\bar{R_{\mu\nu}} + R^{(1)nonlinear}_{\mu\nu} + R^{(2)}_{\mu\nu} = 0, \tag{1.62}$$

which can be divided into a smooth part and a fluctuating part which varies on smaller scales:

$$\bar{R_{\mu\nu}} + \langle R^{(2)}_{\mu\nu}\rangle = 0, \tag{1.63}$$

$$R^{(1)nonlinear}_{\mu\nu} + R^{(2)}_{\mu\nu} - \langle R^{(2)}_{\mu\nu}\rangle = 0. \tag{1.64}$$

The first equation shows how the stress energy in the GWs creates the background curvature. The vacuum Einstein equation is then

$$\bar{G_{\mu\nu}} = \bar{R_{\mu\nu}} - \frac{1}{2}\bar{R}\bar{g_{\mu\nu}} = 8\pi G T^{(GW)}_{\mu\nu}. \tag{1.65}$$

Thus:

$$T^{(GW)}_{\mu\nu} \equiv -\frac{1}{8\pi G}\left(\langle R^{(2)}_{\mu\nu}\rangle - \frac{1}{2}\bar{g_{\mu\nu}}\langle R^{(2)}\rangle\right); \tag{1.66}$$

this is the definition of the energy-momentum tensor of the GWs and the angle brackets indicate the average over the wavelengths. The importance of this expression is that it tells us how the energy density of GWs affects the expansion law of the background universe [8]. Now since $\langle R^{(2)}\rangle = 0$ [17],

$$\begin{aligned} T^{(GW)}_{\mu\nu} &= \frac{1}{32\pi G}\langle h_{\alpha\beta|\mu} h^{\alpha\beta}_{|\nu}\rangle \\ &= \frac{1}{32\pi G}\langle h_{\alpha\beta,\mu} h^{\alpha\beta}_{,\nu}\rangle + O(h^3), \end{aligned} \tag{1.67}$$

where $|$ indicates the covariant derivative with respect to the background unperturbed metric and therefore, by opening it, we can divide the usual derivative from the perturbation contributions (which are included into $O(h^3)$). The energy density ρ_h is the 00-component of the energy-momentum tensor

$$\rho_h(\tau) \equiv T^{(GW)}_{00} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle, \tag{1.68}$$

as the 0-component of the perturbation tensor is 0 (TT-Gauge). By explicitly showing the presence of the two polarisations one gets

$$\dot{h}_{ij} = \begin{pmatrix} \dot{h}_+ & \dot{h}_\times & 0 \\ \dot{h}_\times & -\dot{h}_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $\dot{h}_{ij}\dot{h}^{ij} = 2(\dot{h}_+^{\,2} + \dot{h}_\times^{\,2})$, and one ends up with

$$\begin{aligned} \rho_h(\tau) &= \frac{2}{32}\langle \dot{h}_+^{\,2} + \dot{h}_\times^{\,2}\rangle = \frac{1}{16\pi G a^2}\langle h'^2_+ + h'^2_\times\rangle = \\ &= \frac{1}{16\pi G a^2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d3k'}{(2\pi)^3}\langle (h'_{+,\vec{k}}h'_{+,\vec{k'}} + h'_{\times,\vec{k}}h'_{\times,\vec{k'}})e^{i(\vec{k}+\vec{k'})\vec{x}}\rangle \end{aligned} \tag{1.69}$$

where the relation between the conformal time and the ordinary time and the Fourier transform of the perturbations have been used. The spatial average over the wavelengths is equivalent to the average in the $\vec{k}$-space: $\langle h'_{\lambda,\vec{k}} h'_{\lambda',\vec{k}'}\rangle = (2\pi)^3 \delta_{\lambda,\lambda'}\delta^{(3)}(\vec{k}+\vec{k}')|h'_{\lambda,\lambda'}|$. From this equality Eq. 1.69 becomes

$$\rho_h(\tau) = \frac{1}{16\pi G a^2}\int \frac{d^3k}{(2\pi)^3}[|h'_{+,k}(\tau)|^2 + |h'_{\times,k}(\tau)|^2]. \tag{1.70}$$

By reasonably supposing primordial GWs to be unpolirized [8], and expressing their amplitude in terms of their original ones and the transfer function $\mathcal{T}(k\tau)$, one obtains:

$$\rho_h(\tau) = \frac{1}{32\pi G a^2}\int d\ln k \triangle^2_{h,prim}[\mathcal{T}'(k\tau)], \tag{1.71}$$

with $\triangle^2_{h,prim} \equiv 4\frac{k^3}{2\pi^2}|h^{prim}_h|^2 = \frac{16}{\pi}\left(\frac{H_{inf}}{m_{Pl}}\right)$, where $|h^{prim}_h|^2$ is the squared amplitude of the GWs outside the horizon, during inflation. Inside the horizon averaging over the periods $[\mathcal{T}'(\bar{k}\tau)]^2$ is proportional to $\tau^{-2} \propto a^{-2}$ during radiation dominated era and to τ^{-4} during the matter one. Therefore well inside the horizon $\rho_h \propto a^{-4}$, consistently with the massless and relativistic nature of the graviton (the same behaviour is followed by photons). By defining the relative spectral density as the normalized energy density per logarithmic scale:

$$\Omega_h(\tau,k) \equiv \frac{\tilde{\rho}_h(\tau,k)}{\rho_{cr}(\tau)}, \quad \tilde{\rho}_h(\tau,k) \equiv \frac{d\rho_h(\tau)}{d\ln k}, \tag{1.72}$$

By inserting these definitions into Eq. 1.71, and recalling the Friedman equation $H^2 = 8\pi G\rho_c/3$ the relative spectral density reads:

$$\Omega_h(\tau,k) = \frac{\triangle^2_{h,prim}}{12H^2(\tau)a^2}[\mathcal{T}'(\tau,k)]^2 \tag{1.73}$$

2 Appendix B: Entropy density

From the thermodynamic we have that the infinitesimal variation of entropy is given by

$$dS = \frac{1}{T}[d(\rho(T)V) + P(T)dV], \tag{2.74}$$

with $\frac{\partial S}{\partial V} = \frac{1}{T}(\rho(T)+P(T))$ and $\frac{\partial S}{\partial T} = \frac{V}{T}\frac{d\rho(T)}{T}$. Since dS is an exact form, it has to be such that $\frac{\partial^2 S}{\partial V\partial T} = \frac{\partial^2 S}{\partial T\partial V}$. Therefore:

$$\frac{\partial}{\partial T}\left[\frac{1}{T}(\rho(T)+P(T))\right] = \frac{\partial}{\partial V}\left(\frac{V}{T}\frac{d\rho(T)}{dT}\right). \tag{2.75}$$

which reads, as ρ and P only depend upon T

$$\frac{dP}{dT} = \frac{1}{T}(\rho(T)+P(T)). \tag{2.76}$$

Now, the third Friedmann equation $\dot{\rho} = -3\frac{\dot{a}}{a}(\rho+P)$ can be written as

$$a^3\dot{P} = \frac{d}{dt}[a^3(\rho+P)]. \tag{2.77}$$

Indeed, $a^3\dot{P} = 3a^2\dot{a}(\rho + P) + a^3(\dot{\rho} + \dot{P})$, which yields to the usual third Friedmann equation. One now has

$$\frac{d}{dt}\left[\frac{a^3}{T}(\rho + P)\right] = \frac{1}{T}\frac{d}{dt}[a^3(\rho + P)] - a^3(\rho + P)\frac{1}{T^2}\frac{dT}{dt} = \frac{1}{T}a^3\dot{P} - a^3(\rho + P)\frac{1}{T^2}\frac{dT}{dt}, \tag{2.78}$$

and by multiplying Eq. 2.76 by a^3, $a^3\frac{dP}{dT} = \frac{a^3}{T}(\rho + P)$. Therefore, it follows

$$\frac{d}{dt}\left[\frac{a^3}{T}(\rho + P)\right] = \frac{1}{T}a^3\dot{P} - a^3(\rho + P)\frac{1}{T^2}\frac{dT}{dt} = \frac{1}{T}a^3\dot{P} - \frac{1}{T}a^3\frac{dP}{dt} = 0. \tag{2.79}$$

By considering again Eq. 2.76, since dS can be written as $dS = d\left[\frac{V}{T}(\rho + P)\right] + \frac{1}{T^2}V(\rho + P)dT - \frac{V}{T}dP$, one gets:

$$dS = d\left[\frac{V}{T}(\rho + P)\right] \quad \Longrightarrow \quad S(V, T) = \frac{V}{T}(\rho(T) + P(T)). \tag{2.80}$$

Considering the volume $V = a^3$, the conserved quantity given by Eq.2.79 states that the entropy of a comoving volume is conserved [10]. Finally, by substituting Eq. 22 and remembering that the equation of state for a gas of relativistic particles is $P = \frac{1}{3}\rho$, the entropy inside a comoving volume is

$$S(T) = \frac{4}{3}g_*\frac{\pi^2}{30}T^3a^3, \tag{2.81}$$

from which we can define the entropy density as $s(T) = \frac{S(T)}{a^3}$.

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