Scalar-tensor-tensor bispectrum

Alberto Salvarese

From the famous paper of J. Maldacena [1], the action for a single field slow-roll inflation expressed in terms of the so called shifted field ζ^c reads

$$S = \int d^4x \frac{1}{2} \epsilon H a^5 \dot{\gamma_{ij}} \dot{\gamma_{ij}} \partial^{-2} \dot{\zeta}^c + ..., \tag{1}$$

where

$$\zeta = \zeta^c + f(\zeta^c), \quad f(\zeta^c) = -\frac{1}{32}\gamma_{ij}\gamma^{ij} + \frac{1}{16}\partial^{-2}(\gamma_{ij}\partial^2\gamma^{ij}) + ... \tag{2}$$

Here, the dots indicate terms that are higher order in the slow roll approximation. In order to evaluate the two-point function between a scalar and two tensors, we thus have to take into account this shift. Specifically, we have to evaluate two different terms:

$$\langle \zeta(\vec{x},\eta)\gamma_{ij}(\vec{y},\eta)\gamma_{ij}(\vec{z},\eta)\rangle = \underbrace{\langle \zeta^c(\vec{x},\eta)\gamma_{ij}(\vec{y},\eta)\gamma_{ij}(\vec{z},\eta)\rangle}_{(1)} + \underbrace{\langle f(\zeta^c(\vec{x},\eta))\gamma_{ij}(\vec{y},\eta)\gamma_{ij}(\vec{z},\eta)\rangle}_{(2)}. \tag{3}$$

In Fourier space we get

$$(1) = \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i\vec{x}\vec{k_1}} e^{i\vec{y}\vec{k_2}} e^{i\vec{z}\vec{k_3}} \sum_{ss'} \epsilon^s_{ij}\epsilon^{s'}_{ij} \langle \zeta^c_{k_1}\gamma^s_{k_2}\gamma^{s'}_{k_3}\rangle \tag{4}$$

We want to calculate this correlation function in Fourier space, and in order to do that we need to expand each field in its Fourier modes. Doing so we obtain

$$\zeta(\vec{x},\eta) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \zeta_{\vec{p}}(\eta), \quad \gamma_{ij}(\vec{x},\eta) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \sum_s \epsilon^s_{ij}(\vec{p})\gamma^s_{\vec{p}}(\eta). \tag{5}$$

Before continuing, we need to find the mode functions for both the scalar and the tensor fields. We would like to resemble the usual harmonic oscillator's mode function, for which we already know the solution. To do so we exploit the action of a canonically normalized scalar field in de Sitter space, thereby imposing it to describe our fields. In order to find the perturbations' equation of motion at first order, we need the action perturbed up to second order in the specific perturbation. The kinetic part of the action of a free scalar field f is

$$S = \frac{1}{2}\int d^3x d\eta a^2 \left[f'^2 - (\partial_l f)^2\right], \tag{6}$$

where $' = \frac{\partial}{\partial\eta}$. For our curvature perturbations, we have instead

$$S = \frac{1}{2}\int d^3x d\eta a^2 2\epsilon \left[\zeta'^2 - (\partial_l \zeta)^2\right], \tag{7}$$

Thus, to easily describe ζ as f, and hence quantize it with the usual mode functions for such a field, we consider $\zeta = \frac{f}{\sqrt{2\epsilon}}$. For the tensor field one more step is required. The second order action for the gravity component is given by

$$S = \frac{1}{16\pi G}\int d^3x d\eta a^2 \frac{1}{4}\left[\gamma'_{ij}\gamma'_{ij} - \partial_l \gamma_{ij}\partial_l \gamma_{ij}\right]. \tag{8}$$

By opening now the two fields in Fourier space, we notice that each tensor field contributes with two scalar fields. By remembering the property $\epsilon_{ij}(\vec{k})^s \epsilon_{ij}(\vec{k})^{s'} = 2\delta^{ss'}$, we find the following action for each real, massless, scalar field

$$S^{(s)} = \frac{1}{16\pi G}\int d^3x d\eta a^2 \frac{1}{2}\left[(\gamma^{s}_{\vec{k}}{}')^2 - (\partial_i \gamma^s_{\vec{k}})^2\right] \tag{9}$$

Therefore, by considering $M_{pl}^2 = 1$, we get $\gamma = \sqrt{16\pi G} f = \sqrt{2} f$, where $u^f_k(\eta) = \frac{iH}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}$. Finally, we can quantize the two fields as

$$\zeta_{\vec{p}}(\eta) = u^\zeta_p(\eta) a_{\vec{p}} + u^{\zeta *}_{\ p}(\eta) a^\dagger_{-\vec{p}} \qquad \gamma^s_{\vec{p}}(\eta) = u^\gamma_p(\eta) a^s_{\vec{p}} + u^{\gamma *}_{\ p}(\eta) a^{s\,\dagger}_{\ -\vec{p}} \tag{10}$$

with

$$u^\zeta_p(\eta) = \frac{iH}{\sqrt{4\epsilon p^3}}(1+ip\eta)e^{-ip\eta}, \quad u^\gamma_p(\eta) = \frac{iH}{\sqrt{p^3}}(1+ip\eta)e^{-ip\eta}[a_{\vec{p}} \tag{11}$$

and where the usual commutation relations hold

$$[a_{\vec{p}}, a^\dagger_{-\vec{q}}] = \delta^{(3)}(\vec{p}+\vec{q})(2\pi)^3, \quad [a^r_{\vec{p}}, a^{s\,\dagger}_{\ -\vec{q}}] = \delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}(2\pi)^3 \tag{12}$$

The former is referred to the scalar field, whereas the latter to the tensor one. All the other scalar-scalar and tensor-tensor commutators vanish. To evaluate the first term of Eq. 3, we proceed using the in-in formalism [2–4]:

$$\langle \zeta^c_{k_1}\gamma^s_{k_2}\gamma^{s'}_{k_3}\rangle = -i\int d^3w \int_{-\infty}^{\eta\to 0} d\eta' a \int \frac{d^3p d^3q d^3k}{(2\pi)^9} e^{i\vec{w}(\vec{p}+\vec{q}+\vec{k})} \sum_{r,r'} \epsilon^r_{lm}(\vec{p})\epsilon^{r'}_{lm}(\vec{q}) \frac{\epsilon H a^5}{2}\frac{1}{k^2 a^3}\langle T[\zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}, \gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}]\rangle, \tag{13}$$

where $\partial^{-2}\zeta'^c_{\vec{k}} = -\frac{1}{k^2}\zeta'^c_{\vec{k}}$, $H^I_{int} = -\int d^3w \mathcal{L}_{int}$ and $\eta \to 0$ since we are looking for the bispectrum well outside the horizon. By expanding now the time-ordered product inside the integral through the Wick theorem [3,4] we have

$$\begin{aligned}\langle T[\zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}, \gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}]\rangle &= \langle T\left(\zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}\right)\rangle - c.c. \\ &= \langle \zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}\rangle + \langle \zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}\rangle - c.c.\end{aligned} \tag{14}$$

A contraction between two fields is nothing but vacuum expectation value of the time-ordered product of the two fields. As reported in Chen et al. 2010 [3], in an inflationary background there is not a simple Feynman propagator taking care of the time-ordering. Therefore, we can evaluate the contractions forgetting about the time-ordering and leave this complication to the final integration in time. By remembering that $\eta \to 0$ and decomposing each Fourier mode in its ladder operators, one has

$$
\begin{aligned}
\langle 0|\gamma^s_{\vec{p}}(\eta)\gamma'^r_{\vec{q}}(\eta')|0\rangle &= \langle\left[\gamma^{s+}_{\vec{p}}(\eta), \gamma'^{r-}_{\vec{q}}(\eta')\right]\rangle \\
&= (16\pi G)(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}\left(\frac{iH}{\sqrt{2p^3}}(1+ip\eta)e^{-ip\eta}\right)\left(\frac{-iH}{\sqrt{2q^3}}(1-iq\eta')e^{+iq\eta'}\right)' \\
&= (16\pi G)(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}\frac{H^2}{2\sqrt{p^3q^3}}q^2\eta' e^{+iq\eta'},
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\langle 0|\zeta^c_{\vec{p}}(\eta)\zeta^{c\prime}_{\vec{q}}(\eta')|0\rangle &= \langle\left[\zeta^{c+}_{\vec{p}}(\eta), \zeta^{c\prime-}_{\vec{q}}(\eta')\right]\rangle \\
&= (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\left(\frac{iH}{\sqrt{4\epsilon p^3}}(1+ip\eta)e^{-ip\eta}\right)\left(\frac{-iH}{\sqrt{4\epsilon q^3}}(1-iq\eta')e^{+iq\eta'}\right)' \\
&= (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\frac{H^2}{4\epsilon\sqrt{p^3q^3}}q^2\eta' e^{-iq\eta'}.
\end{aligned}
\tag{16}
$$

Hence, by taking all the contractions we finally get

$$
\begin{aligned}
\langle T[\zeta^c_{\vec{k_1}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}, \gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}]\rangle = (16\pi G)^2(2\pi)^9\delta^{(3)}(\vec{k_1}+\vec{k})\frac{H^6}{16\epsilon\sqrt{k_1^3k_2^3k_3^3p^3q^3k^3}}p^2q^2k^2\eta'^3 e^{i\eta'(p+q+k)} \\
\left(\delta^{(3)}(\vec{k_2}+\vec{p})\delta^{(3)}(\vec{k_3}+\vec{q})\delta^{rs}\delta^{r's'} + \delta^{(3)}(\vec{k_2}+\vec{q})\delta^{(3)}(\vec{k_3}+\vec{p})\delta r' s\delta^{rs'}\right) - c.c.
\end{aligned}
\tag{17}
$$

Now, by putting it inside the integral, we note that the different deltas act in the same way on the momentum-dependent factors. In fact, the permutation $k_2 \leftrightarrow k_3$ would just produce a switch between two momenta, which can be rearranged in a single order since all the remain terms commute. We thus obtain

$$
\begin{aligned}
\langle\zeta^c_{k_1}\gamma^s_{k_2}\gamma^{s'}_{k_3}\rangle &= -i\frac{H^7}{4\prod_i(k_i^3)}\epsilon_{lm}(-\vec{k_2})^s\epsilon_{lm}(-\vec{k_3})^{s'}(2\pi)^3k_2^2k_3^2\int\frac{d^3w}{(2\pi)^3}e^{i\vec{w}(\vec{k_1}+\vec{k_2}+\vec{k_3})}\left(\int_{-\infty}^{\eta\to 0}d\eta' a^3 e^{+i\eta' k_t}\eta'^3 - c.c\right). \\
&= -i\frac{H^7}{4\prod_i(k_i^3)}\epsilon^s_{lm}\epsilon^{s'}_{lm}(2\pi)^3k_2^2k_3^2\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\left(\int_{-\infty}^{\eta\to 0}d\eta'\left(-\frac{1}{H^3\eta'^3}\right)e^{+i\eta' k_t}\eta'^3 - c.c.\right) \\
&= i\frac{H^4}{4\prod_i(k_i^3)}\epsilon^s_{lm}\epsilon^{s'}_{lm}(2\pi)^3k_2^2k_3^2\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\left(\int_{-\infty}^{\eta\to 0}d\eta' e^{+i\eta' k_t} - c.c.\right)
\end{aligned}
\tag{18}
$$

where $k_t = \sum_i k_i$. By applying now a Wick rotation $\eta' \to -i\eta''$ we have a simple integral of an exponential factor multiplied by $-i$, which simplifies to 1 with the i in front. Finally, summing also the complex conjugate that gives us a factor $\times 2$, we obtain

$$\langle \zeta^c_{\vec{k_1}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k_1} + \vec{k_2} + \vec{k_3}) \frac{H^4}{\prod_i (2k_i^3)} 4 \frac{k_2^2 k_3^2}{k_t} \epsilon^s_{lm} \epsilon^{s'}_{lm} \tag{19}$$

For the second term we have to express the shifting function in Fourier space, just as the other tensor fields. Doing that we obtain

$$(2) = \int \frac{d^3p d^3q d^3k_2 d^3k_3}{(2\pi)^{12}} e^{i\vec{x}(\vec{p}+\vec{q})} e^{i\vec{y}\vec{k_2}} e^{i\vec{z}\vec{k_3}} \sum_{ss'} \epsilon^s_{ij} \epsilon^{s'}_{ij} \sum_{rr'} \epsilon^r_{lm} \epsilon^{r'}_{lm} \left(-\frac{1}{32} \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle + \frac{1}{16} \frac{q^2}{|\vec{p}+\vec{q}|^2} \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle \right) \tag{20}$$

Here the factor $\frac{q^2}{|\vec{p}+\vec{q}|^2}$ shows up from the term $\partial^{-2}(\gamma_{ij}\partial^2\gamma_{ij})$ in the shifted action. To evaluate now the correlation functions inside the parenthesis, we must use the Wick theorem again. However, in order to obtain only connected diagrams, we avoid contractions between $\vec{x}$ and itself. So that the only contractions will be

$$\langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle = \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle + \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle, \tag{21}$$

where

$$\langle \gamma^r_{\vec{p}}(\eta) \gamma^s_{\vec{k_2}}(\eta) \rangle = (16\pi G)(2\pi)^3 \delta^{(3)}(\vec{p}+\vec{k_2}) \delta^{rs} \left(\frac{iH}{\sqrt{2p^3}} (1+ip\eta) e^{-ip\eta} \right) \left(\frac{-iH}{\sqrt{2k_2^3}} (1-ik_2\eta) e^{ik_2\eta} \right). \tag{22}$$

Considering again the limit $\eta \to 0$, we have, by putting $\int D = \int \frac{d^3pd^3qd^3k_2d^3k_3}{(2\pi)^{12}} e^{i\vec{p}\vec{x}} e^{i\vec{q}\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}}$,

$$\begin{aligned}
(2) &= \int D \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2}) \epsilon^{s'}_{ij}(\vec{k_3}) \sum_{r,r'} \epsilon^r_{lm}(\vec{p}) \epsilon^{r'}_{lm}(\vec{q}) \left[-\frac{1}{32} \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle + \frac{1}{16} \frac{q^2}{|\vec{p}+\vec{q}|^2} \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle \right] \\
&= (16\pi G)^2 (2\pi)^6 \int D \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2}) \epsilon^{s'}_{ij}(\vec{k_3}) \sum_{r,r'} \epsilon^r_{lm}(\vec{p}) \epsilon^{r'}_{lm}(\vec{q}) \frac{H^4 (2\pi)^6}{4\sqrt{p^3 q^3 k_2^3 k_3^3}} \left(-\frac{1}{32} + \frac{1}{16} \frac{q^2}{|\vec{p}+\vec{q}|} \right) \times \\
&\quad \times \left(\delta^{rs} \delta^{r's'} \delta^{(3)}(\vec{k_2}+\vec{p}) \delta^{(3)}(\vec{k_3}+\vec{q}) + \delta^{rs'} \delta^{r's} \delta^{(3)}(\vec{k_2}+\vec{q}) \delta^{(3)}(\vec{k_3}+\vec{p}) \right) \\
&= (16\pi G)^2 \int \frac{d^3k_2 d^3k_3}{(2\pi)^{12}} e^{-i(\vec{k_2}+\vec{k_3})\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2}) \epsilon^{s'}_{ij}(\vec{k_3}) \frac{H^4 (2\pi)^6}{4k_2^3 k_3^3} \left(-\frac{2}{32} + \frac{1}{16} \frac{k_2^2 + k_3^2}{|\vec{k_2}+\vec{k_3}|^2} \right) \epsilon^s_{lm}(-\vec{k_2}) \epsilon^{s'}_{lm}(-\vec{k_3})
\end{aligned} \tag{23}$$

Considering now that $(16\pi G)^2 = (2M_P^{-2})^2 = 4$, we obtain

$$(2) = \int \frac{d^3k_2 d^3k_3}{(2\pi)^6} e^{-i(\vec{k_2}+\vec{k_3})\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2}) \epsilon^{s'}_{ij}(\vec{k_3}) \frac{H^4}{16k_2^3 k_3^3} \left(-1 + \frac{k_2^2 + k_3^2}{|\vec{k_2}+\vec{k_3}|^2} \right) \epsilon^s_{lm}(-\vec{k_2}) \epsilon^{s'}_{lm}(-\vec{k_3}). \tag{24}$$

Now, in order to make the expression more symmetric in the momentum space, we can multiply and divide by k_1^3, and integrate over $\vec{k_1}$ under the condition that $\vec{k_1} + \vec{k_2} + \vec{k_3} = 0$

$$
\begin{aligned}
(2) &= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i\vec{k_1}\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij} \epsilon^{s'}_{ij} (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{H^4}{\prod_i (2k_i^3)} \left(-\frac{k_1^3}{2} + \frac{k_1}{2}(k_2^2+k_3^2) \right) \epsilon^s_{lm} \epsilon^{s'}_{lm} \\
&= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i\vec{k_1}\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2}) \epsilon^{s'}_{ij}(\vec{k_3}) \langle f_{\vec{k_1}}(\zeta^c) \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle
\end{aligned}
\tag{25}
$$

In the end, we can rewrite the full three-point function in the momentum space by summing together the two contributions getting [1]

$$
\langle \zeta_{k_1} \gamma^s_{k_2} \gamma^{s'}_{k_3} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{H^4}{\prod_i^3 (2k_i)} \left(-\frac{k_1^3}{2} + \frac{1}{2} k_1 (k_2^2+k_3^2) + 4\frac{k_2^2 k_3^2}{k_t} \right) \epsilon^s_{lm} \epsilon^{s'}_{lm} \tag{26}
$$

References

[1] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP*, vol. 05, p. 013, 2003.

[2] L. V. Keldysh, "Diagram technique for nonequilibrium processes," *Sov. Phys. JETP*, vol. 20, p. 1018, 1965. [Zh. Eksp. Theor. Fiz. **47**, 1515 (1964)].

[3] X. Chen, "Primordial non-gaussianities from inflation models," *Advances in Astronomy*, vol. 2010, p. 1–43, 2010.

[4] S. Weinberg, "Quantum contributions to cosmological correlations," *Phys. Rev. D*, vol. 72, p. 043514, Aug 2005.