$\langle\gamma\gamma\zeta\rangle Bispectrum$

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From Maldacena's paper about cosmological correlation functions, the third order action for one graviton and two scalar fields reads [1]

$$S = \int d^4x \frac{\dot{\phi}^2}{2\dot{\rho}^2} e^{\rho} \gamma_{ij} \partial^i \zeta \partial^j \zeta. \tag{1}$$

Therefore, since we need the interaction Hamiltonian to use to in-in formalism [2–4], we have

$$\begin{aligned} H_{int}(t) &= -\int d^3\vec{x} \frac{\dot{\phi}^2}{2\dot{\rho}^2} e^{\rho} \gamma_{ij} \partial^i \zeta \partial^j \zeta \\ &= -\int d^3x \frac{\dot{\phi}^2}{2\dot{\rho}^2} e^{\rho} \int \frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi)^9} \sum_{r=\pm} \epsilon^r_{ij}(\hat{p}_1) i^2 p_2^i p_3^j \gamma^r_{\vec{p_1}}(t) \zeta_{\vec{p_2}}(t) \zeta_{\vec{p_3}}(t) e^{i\vec{x}(\vec{p_1}+\vec{p_2}+\vec{p_3})} \\ &= \epsilon a(t) \int \frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi)^6} \delta^{(3)}(\vec{p_1}+\vec{p_2}+\vec{p_3}) \sum_r \epsilon^r_{ij}(\hat{p}_1) p_2^i p_3^j \gamma^r_{\vec{p_1}}(t) \zeta_{\vec{p_2}}(t) \zeta_{\vec{p_3}}(t) \end{aligned} \tag{2}$$

where we considered $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2 M_{Pl}^2} = \frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}$ and expanded each field in its Fourier modes:

$$\zeta(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \zeta_{\vec{k}}(t) e^{i\vec{x}\cdot\vec{k}}, \qquad \gamma_{ij}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}(\hat{k}) \gamma^s_{\vec{k}}(t) e^{i\vec{x}\cdot\vec{k}}. \tag{3}$$

Thus, we simply have

$$\begin{aligned} \langle \gamma_{ij}\zeta\zeta \rangle &= i\int_{-\infty}^{t} dt' \langle T\left[H_{int}(t'), \gamma_{ij}(\vec{x},t)\zeta(\vec{y},t)\zeta(\vec{z},t)\right]\rangle \\ &= i\int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i(\vec{k_1}\cdot\vec{x}+\vec{k_2}\cdot\vec{y}+\vec{k_3}\cdot\vec{z})} \sum_{s=\pm} \epsilon^s_{ij}(\hat{k_1}) \int_{-\infty}^{t} dt' \langle T\left[H_{int}, \gamma^s_{\vec{k_1}}(t)\zeta_{\vec{k_2}}(t)\zeta_{\vec{k_3}}(t)\right]\rangle \\ &= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i(\vec{k_1}\cdot\vec{x}+\vec{k_2}\cdot\vec{y}+\vec{k_3}\cdot\vec{z})} \sum_{s=\pm} \epsilon^s_{ij}(\hat{k_1}) \langle \gamma^s_{\vec{k_1}} \zeta_{\vec{k_2}} \zeta_{\vec{k_3}} \rangle \end{aligned} \tag{4}$$

where, as usual, T denotes the time ordered product. The wave functions for our graviton and density perturbations are respectively

$$\zeta_{\vec{k}} = u_k^\zeta(t) a_{\vec{k}} + u_k^{\zeta *}(t) a^\dagger_{-\vec{k}} \quad \text{and} \quad \gamma^s_{\vec{k}} = u_k^\gamma(t) a^s_{\vec{k}} + u_k^{\gamma *}(t) a^{s\dagger}_{-\vec{k}}. \tag{5}$$

Here, in order to simplify things, we can use the trick of describing each field (two scalars coming from the two density perturbation fields, and two more scalars given by the two polarizations of the graviton) as a canonically normalized scalar field. This is the same procedure we followed to calculate the $\langle\gamma\gamma\zeta\rangle$ bispectrum. In such a way, our mode functions read

$$u_k^\zeta(\eta) = \frac{iH}{\sqrt{4\epsilon k^3}}(1+ik\eta)e^{-ik\eta}, \quad u_k^\gamma(\eta) = \frac{iH}{\sqrt{k^3}}(1+ik\eta)e^{-ik\eta}, \tag{6}$$

where η is the conformal time $ad\eta = dt$, being $a(t)$ the scale factor. Just to clarify, we note that we obtain this mode functions by implicitly considering the Bunch-Davis vacuum state. The usual commutation relations are valid:

$$[a_{\vec{p}}, a^\dagger_{-\vec{q}}] = \delta^{(3)}(\vec{p}+\vec{q})(2\pi)^3, \quad [a^r_{\vec{p}}, a^{s\dagger}_{-\vec{q}}] = \delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}(2\pi)^3. \tag{7}$$

Since by using the in-in formalism we can neglect the time ordering, and leaving this issue to the final integration [4]. Also, we want to calculate our bispectrum when it is well outside the horizon. This corresponds to send $\eta \to 0$. Thus, one has

$$\gamma^r_{\vec{k_1}}(\eta')\gamma^s_{\vec{k_2}}(\eta) = (2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\delta^{rs}\frac{H^2}{\sqrt{k_1^3 k_2^3}}(1+ik_1\eta')e^{-ik_1\eta}. \tag{8}$$

for a connection between two gravitons, and

$$\zeta_{\vec{k_1}}(\eta')\zeta_{\vec{k_2}}(\eta) = (2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{H^2}{4\epsilon\sqrt{k_1^3 k_2^3}}\left(1+ik_1\eta'\right)e^{-ik_1\eta'} \tag{9}$$

for a scalar-scalar one.

Hence, by means of Wick's theorem, we find

$$\begin{aligned}
\langle T\left[\gamma_{\vec{p_1}}\zeta_{\vec{p_2}}\zeta_{\vec{p_3}}, \gamma_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\right]\rangle &= \left(\gamma^r_{\vec{p_1}}\zeta_{\vec{p_2}}\zeta_{\vec{p_3}}\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}} + \gamma^r_{\vec{p_1}}\zeta_{\vec{p_2}}\zeta_{\vec{p_3}}\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}} - c.c.\right) \\
&= (2\pi)^9\delta^{(3)}\left(\vec{k_1}+\vec{p_1}\right)\delta^{rs}\frac{H^6}{M_{Pl}^2 16\epsilon^2}\frac{1}{\sqrt{\prod_i k_i^3 p_i^3}}(1+ip_1\eta')(1+ip_2\eta')(1+ip_3\eta')\times \\
&\quad \times e^{-i\eta'(p_1+p_2+p_3)}\left(\delta^{(3)}(\vec{p_2}+\vec{k_2})\delta^{(3)}(\vec{p_3}+\vec{k_3}) + \delta^{(3)}(\vec{p_2}+\vec{k_3})\delta^{(3)}(\vec{p_3}+\vec{k_2})\right) - c.c.
\end{aligned} \tag{10}$$

where we considered $8\pi G = M_{Pl}^{-2}$. Now, by considering the conformal time rather than the cosmic one, and approximating $a(\eta) \approx \frac{1}{\eta H}$, we obtain

$$
\begin{aligned}
\langle\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle =& i\epsilon\delta^{rs}\int d\eta' a^2\int\frac{d^3p_1d^3p_2d^3p_3}{(2\pi)^6}(2\pi)^9\delta^{(3)}(\vec{p_1}+\vec{p_2}+\vec{p_3})\frac{H^6}{16\epsilon^2}\delta^{(3)}(\vec{k_1}+\vec{p_1})\frac{p_2^ip_3^j}{\sqrt{\prod_l k_l^3p_l^3}}\sum_r\epsilon^r_{ij}(\hat{p_1})\times\\
&\times\left(1+i\eta'(p_1+p_2+p_3)-\eta'^2(p_1p_2+p_1p_3+p_2p_3)-i\eta'^3p_1p_2p_3\right)e^{-i\eta' K}\times\\
&\times\left(\delta^{(3)}(\vec{p_2}+\vec{k_2})\delta^{(3)}(\vec{p_3}+\vec{k_3})+\delta^{(3)}(\vec{p_2}+\vec{k_3})\delta^{(3)}(\vec{p_3}+\vec{k_2})\right)-c.c.\\
=& i(2\pi)^3\delta^{(3)}\left(\sum_i\vec{k}_i\right)\frac{H^4}{16\epsilon}\epsilon^s_{ij}(\hat{k}_1)\frac{k_2^ik_3^j}{\prod_l k_l^3}\times\\
&\times 2\int d\eta'\frac{1}{\eta'^2}\left(1+i\eta'(_1+k_2+k_3)-\eta'^2(k_1k_2+k_1k_3+k_2k_3)-i\eta'^3k_1k_2k_3\right)e^{-i\eta' K}-c.c.
\end{aligned}
\tag{11}
$$

since when the Dirac's deltas on the second last line are applied, they give us the same term, which is taken into account in the 2 factor in front of the time-integral. We named also K as the sum of the three external momenta k_i. By focusing only on the time integral $I(\eta)$, we have

$$
\begin{aligned}
I =& \int_{-\infty}^{\eta\to 0}d\eta'\left[\frac{e^{-i\eta' K}}{\eta'^2}+\frac{e^{-i\eta' K}}{\eta'}iK-e^{-i\eta' K}(k_1k_2+k_1k_3+k_2k_3)-ik_1k_2k_3e^{-i\eta' K}\eta'-c.c.\right]\\
=& -\frac{1}{\eta'}e^{-i\eta' K}\Big|_{-\infty}^{0}-iK\int_{-\infty}^{0}d\eta'\frac{e^{-i\eta' K}}{\eta'}+iK\int_{-\infty}^{0}d\eta'\frac{e^{-i\eta' K}}{\eta'}-i\sum_{i>j}k_ik_j\frac{1}{K}e^{\eta' K}\Big|_{-\infty}^{0}\\
&-ik_1k_2k_3i^2\left(\frac{1}{K}\eta''e^{K\eta''}\Big|_{-\infty}^{0}-\frac{1}{K}\int_{-\infty}^{0}d\eta''e^{K\eta''}\right)-c.c.
\end{aligned}
\tag{12}
$$

Here, we integrated by part the first term, so that the second piece that we obtain cancels out with its opposite. Also, we Wick-rotated the last two terms. By substituting $\eta'\to i\eta'=\eta''$, we cancel the oscillatory behaviour of the complex exponential when $\eta'\to-\infty$. Finally, by rewriting the first term in terms of sinusoidal functions, we get

$$
I = 2i\left[K-\frac{\sum_{i>j}k_ik_j}{K}-\frac{k_1k_2k_3}{K^2}\right],
\tag{13}
$$

since all the real terms sum themselves to zero due to the complex conjugate. At the very end, we find [1]

$$
\langle\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle = (2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}\right)\frac{2H^4}{\epsilon\prod_i(2k_i^3)}\epsilon^s_{ij}(\hat{k_1})k_2^ik_3^j\left[-K+\frac{\sum_{i>j}k_ik_j}{K}+\frac{k_1k_2k_3}{K^2}\right]
\tag{14}
$$

References

[1] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP*, vol. 05, p. 013, 2003.

[2] L. V. Keldysh, "Diagram technique for nonequilibrium processes," *Sov. Phys. JETP*, vol. 20, p. 1018, 1965. [Zh. Eksp. Theor. Fiz. **47**, 1515 (1964)].

[3] S. Weinberg, "Quantum contributions to cosmological correlations," *Phys. Rev. D*, vol. 72, p. 043514, Aug 2005.

[4] X. Chen, "Primordial non-gaussianities from inflation models," *Advances in Astronomy*, vol. 2010, p. 1–43, 2010.