First order N,Ni

Alberto Salvarese

In order to compute the expression for the lapse and shift functions up to first order, we need their two constraint equations. Specifically, we need to find their equations of motion. These are simply given by solving the following Euler-Lagrange equations respectively for N and N_i

$$\frac{\partial \mathcal{L}}{\partial N} - \nabla_i \frac{\partial \mathcal{L}}{\partial \partial_i N} = 0 \qquad \frac{\partial \mathcal{L}}{\partial N_j} - \nabla_i \frac{\partial \mathcal{L}}{\partial \partial_i N_j} = 0, \tag{1}$$

where ∇_i denotes the spatial part of the covariant derivative. From these two equations we find the so called Hamiltonian and momentum constraints

$$\hat{R} - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V = 0 \tag{2}$$

$$\hat{\nabla}_j[N^{-1}(E_i^j - \delta_i^j E)] = N^{-1}(\dot{\phi} - N^j\partial_j\phi)\partial_i\phi. \tag{3}$$

Now, the actual fluctuations of fields and metric are considered to be very small with respect to the background. Therefore, in order to study the scalar perturbations we can expand the action up to the desired order in the scalar fluctuations. Since then the ADM formalism does not fix a gauge, we can choose one that simplifies our calculations. A general parametrization which is often used is

$$N = 1 + 2\Phi(t, \vec{x}) \qquad N^i = \delta^{ij}\partial_j B \qquad h_{ij} = e^{2\rho(t) + 2\zeta(t,\vec{x})}\hat{h_{ij}} \tag{4}$$

where $\hat{h_{ij}}$ encodes all the tensor perturbations of the metric. The particular gauge we choose is the so called spatially flat gauge or uniform curvature gauge, in which the spatial fluctuations of the inflaton field are considered to be zero. Thus, we are left with

$$\hat{R} - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 = 0, \tag{5}$$

$$\hat{\nabla}_j[N^{-1}(E_i^j - \delta_i^j E)] = 0. \tag{6}$$

We note that no conditions have been expressed for the potential. The only thing we require is for it to satisfy the slow-roll conditions, which implies $V = 3\dot{\rho}^2 - \frac{1}{2}\dot{\phi}^2$. To proceed, we must compute each term of the above constraints up to first order in scalar fluctuations. We start with the Hamiltonian constraint. The first step to do so is to calculate the Chrystoffel symbol, for which we have.

$$\Gamma^k_{ij} = \frac{1}{2}h^{kl}(h_{il,j} + h_{jl,i} - h_{ij,l}) = \partial_j\zeta\delta_i^k + \partial_i\zeta\delta_j^k - \partial^k\zeta\delta_{ij} \tag{7}$$

Hence, for the spatial Ricci curvature we obtain

$$\begin{aligned}\hat{R} &= h^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}\right) \\ &= e^{-2\rho-2\zeta}\delta^{ij}\Big[2\partial_i\partial_j\zeta - \partial_k\partial^k\zeta\delta_{ij} - 3\partial_i\partial_j\zeta\left(\partial_j\zeta\delta^l_i + \partial_i\zeta\delta^l_j - \partial^l\zeta\delta_{ij}\right)3\partial_l\zeta \\ &\quad - \left(\partial_k\zeta\delta^l_i + \partial_i\zeta\delta^l_k - \partial_l\zeta\delta_{ik}\right)\left(\partial_j\zeta\delta^k_l + \partial_l\zeta\delta^k_j - \partial^k\zeta\delta_{jl}\right)\Big] \\ &= -4e^{-2\rho}\partial_k\partial^k\zeta + \mathcal{O}(\zeta^2)\end{aligned} \tag{8}$$

where the Taylor expansion at first order $e^{-2\zeta} \approx 1 - 2\zeta$ was used. In the same way, by observing that $N^k = \delta^{kl}\partial_l B \sim \mathcal{O}(\zeta)$, for the extrinsic curvature E_{ij} we have

$$\begin{aligned}E_{ij} &= \frac{1}{2}\left[\dot{h^{ij}} - h_{ik}\partial_j N^k - h_{jk}\partial_i N^k - N^k\partial_k h_{ij}\right] \\ &= \frac{1}{2}e^{2\rho+2\zeta}\left[2(\dot{\rho}+\dot{\zeta})\delta_{ij} - \delta_{ik}\partial_j N^k - \delta_{jk}\partial_i N^k - N^k\delta_{ij}\partial_k\zeta\right] \\ &= \frac{1}{2}e^{2\rho}\left[(\dot{\rho}+\dot{\rho}\zeta+\dot{\zeta})\delta_{ij} - \partial_i\partial_j B\right] + \mathcal{O}(\zeta^2).\end{aligned} \tag{9}$$

Then, by considering $E^i_i = E_{ij}h^{ji}$ it is easy to show that

$$E^2 = \left(E^i_i\right)^2 = 9\dot{\rho}^2 + 18\dot{\rho}\dot{\zeta} - 6\dot{\rho}(\partial_k\partial^k B) + \mathcal{O}(\zeta^2). \tag{10}$$

whence it follows that

$$E_{ij}E^{ij} - E^2 = -6\dot{\rho}^2 - 12\dot{\rho}\dot{\zeta} + 4\dot{\rho}(\partial^2 B) + \mathcal{O}(\zeta^2). \tag{11}$$

Now we have all the ingredients needed to compute the Hamiltonian constraints. We just have to sum up all the terms and to truncate them at first order in scalar fluctuations. We get

$$-\partial_k\partial^k\left[e^{-2\rho}\zeta + \dot{\rho}B\right] - 3\dot{\rho}\left[2\Phi\dot{\rho} - \dot{\zeta}\right] + \Phi\dot{\phi}^2 = 0. \tag{12}$$

For the momentum constraint we have to evaluate only E^j_i, since the other element we need has already been computed above. By using again the metric to raise indices we have

$$E^j_i = E_{il}h^{lj} = (\dot{\rho}+\dot{\zeta})\delta^j_i - \partial_i\partial^j B + \mathcal{O}(\zeta^2). \tag{13}$$

Then, by considering $N^{-1} \approx 1 - 2\Phi(t,\vec{x})$, we obtain

$$\begin{aligned}&\hat{\nabla}_j\left[(1-2\Phi)\left((\dot{\rho}+\dot{\zeta})\delta^j_i - \partial_i\partial^j B - \delta^j_i(3\dot{\rho}+3\dot{\zeta} - \partial_l\partial^l B)\right)\right] = 0 \\ &\hat{\nabla}_j\left[-2(\dot{\rho}+\dot{\zeta}-2\dot{\rho}\Phi)\delta^j_i - \partial_i\partial^j B + \delta^j_i\partial_l\partial^l B\right] = 0\end{aligned} \tag{14}$$

As we saw above, the Crystoffel's symbols are of order one in scalar fluctuations. Hence, when we apply the covariant derivative on the left hand side, they can be forgotten. In fact, they act only on the tensors present in the equation, which are already of order one in scalar perturbations. By focusing for a moment only on these tensor terms we have $-\partial_j\partial_i\partial^j B + \Gamma^j_{jl}\partial_i\partial^l B - \Gamma^l_{ji}\partial_l\partial^j B + \partial_i\partial_l\partial^l B + \Gamma^k_{il}\partial_k\partial^l B - \Gamma^l_{ik}\partial_l\partial^k B \approx -\partial_i\partial_j\partial^j B + \partial_i\partial_l\partial^l B = 0$. Therefore our momentum constraint turns out to be simply

$$2\partial_i(2\dot{\rho}\Phi - \dot{\zeta}) = 0, \tag{15}$$

whence we can easily obtain the expression for Φ at first order in scalar perturbations:

$$\Phi^{(1)} = \frac{\dot{\zeta}}{\dot{\rho}}. \tag{16}$$

Putting now this form of N inside Eq.12 leads to

$$\begin{aligned}
&\partial_k\partial^k \left[e^{-2\rho}\zeta + \dot{\rho}B\right] - \dot{\phi}^2\frac{1}{2}\frac{\dot{\zeta}}{\dot{\rho}} = 0 \\
&\partial_k\partial^k B = -\frac{e^{-2\rho}}{\dot{\rho}}\partial_k\partial^k\zeta + \frac{\dot{\phi}^2}{2}\frac{\dot{\zeta}}{\dot{\rho}}.
\end{aligned} \tag{17}$$

At the very end, we can rewrite the lapse and shift functions at first order in scalar perturbations as

$$N^{(1)} = 1 + \frac{\dot{\zeta}}{\dot{\rho}}, \quad N_i^{(1)} = \partial_i B, \quad B = -\frac{e^{-2\rho}}{\dot{\rho}}\zeta + \frac{\dot{\phi}^2}{2\dot{\rho}}\partial^{-2}\dot{\zeta}. \tag{18}$$