$I^2(\phi)\tilde{F}F$ pseudo-scalar Inflation's correlation functions

Alberto Salvarese

1 Introduction

In order to summarize the complete procedure one has to follow to compute trispectra for a certain Inflation's model, we briefly present here the calculations of some correlation functions for the $I^2(\phi)\tilde{F}^{\mu\nu}F_{\mu\nu}$ pseudo-scalar model. We start studying it by introducing its action [1,2]

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + I^2(\phi)\left(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\gamma}{4}\tilde{F}^{\mu\nu}F_{\mu\nu}\right)\right], \tag{1}$$

where γ is a constant parameter, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength of A^μ and $\tilde{F}^{\mu\nu} = \frac{\epsilon^{\mu\nu\alpha\beta}}{2\sqrt{-g}}F_{\alpha\beta}$ is its dual tensor. While all the coupling terms affect the inflaton's dynamics, only the last one can produce signatures of parity violation, as it is a pseudo-scalar quantity [2]. Since the deep study of this model is not the primary goal of this work, we report here only the most important steps that are needed to compute the correlation functions we want. We perturb the field, allowing for the presence of fluctuations, getting $\phi(\tau,\vec{x}) = \phi_0(\tau) + \delta\phi(\tau,\vec{x})$ and hence $I(\phi) = I_0(\phi) + \delta I(\tau,\vec{x})$. It is shown that the second order term in the field fluctuations is actually subdominant with respect to the other ones, and thus we can neglect it in our calculations. Another useful assumption is to put $I_0(\phi) \propto a^n(\tau)$ during Inflation, and that it becomes a constant when Inflation ends.We also introducing now the electromagnetic convention $\vec{E} = -\frac{I_0(\tau)}{a^2}\vec{A}'$, $\vec{B} = \frac{I_0(\tau)}{a^2}\vec{\nabla}\times\vec{A}$ and put ourselves in the Lorentz gauge, where $A_0 = \vec{\nabla}\cdot\vec{A} = 0$.

As we wish to reproduce some parity violation, we need to set $\gamma \neq 0$. But as we stated in the first chapter, a non vanishing vev for a vector field is not symmetric under rotations leading to an anisotropic expansion of the Universe. However, for $\gamma = 0$, it was shawn that the energy density of the vector field must be much smaller than the inflaton's one, for the model to be consistent with the CMB results. The situation gets even more delicate with $\gamma \neq 0$, and hence we can neglect the geometric contributions than can arise from the vector field and spoil the FLRW metric [3]. By introducing now the canonical vector field $\vec{V} = I_0(\tau)\vec{A}$ and expanding it at first order, we see that the gauge field's vev must satisfies $\vec{V}_0'' - \frac{I_0''}{I}\vec{V}_0 = 0$, which leads to a vanishing and a constant vev for $\vec{B}_0$ and $\vec{E}_0$ respectively. Specifically, the latter depends upon the dependence of I_0 with respect to the scale factor. For $n = -2$, $\vec{E}_0(\tau) = \vec{E}^{vev}$. We skip now the construction of the mode-functions of our fields, whose procedure can be found in [1,2]. The results that are important for our purposes are

$$\delta E_+(\tau,k) = -\frac{e^{\pi\xi}}{\xi^{3/2}}\frac{3H^2}{2\sqrt{\pi k^3}} = \frac{3}{k\tau}\delta B_+(\tau,k) \tag{2}$$

and

$$\langle\delta E_i(\vec{k_1},\tau_1)\delta E_j(\vec{k_2},\tau_2)\rangle \approx (2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right)\frac{9H^4}{4\pi}\frac{e^{2\pi\xi}}{\xi^3}k_1^{-3}\epsilon_i^{(+)}(\hat{k_1})\epsilon_j^{(+)}(\hat{k_2}). \tag{3}$$

We reported only the one-point correlation function between two electric fields since, as the previous equation shows, on superhorizon scales the magnetic components are much smaller than the other ones. In this model, the power spectra of curvature perturbations depend upon the corrections given by the interaction with the vector field [4]. For example, the direction dependence of the vector field's vev is straightly shown in the curvature power spectra, as we will see. We set ourselves in the spatially flat gauge, where the only contributions to the spatial part of the metric are given by gravitational waves $\delta g_{ij} = a^2 h_{ij}$. We use the scalar curvature perturbations defined as $\zeta = -H\frac{\delta\rho}{\dot\rho} \approx -\frac{H}{\dot\phi}\delta\phi$. In principle, the vector contributions enter in perturbing the g_{00} and g_{0i} components of the metric, but since the subsequent couplings are suppressed, with respect to the topological $\delta\phi\delta A^2$-couplings, it is possible to set $\delta g_{00} = \delta g_{0i} = 0$ [1, 2, 4]. Then, since we are interested in studying the power spectra given by the gauge field-curvature interactions, we can just look at the interaction Hamiltonian that carries these information. In other words, by writing $\langle\zeta\zeta\rangle \approx \langle\zeta\zeta\rangle_0 + \langle\zeta\zeta\rangle_1$, we focus ourselves on the correlators at first order in the gauge field. By treating the vector field perturbatively, we expand the different terms in the I^2-coupling searching for contributions up to $\delta\phi\delta A^2$. We get

$$\delta\left(I^2(\phi)\right) = 2I_0(\phi)\frac{\partial I_0(\phi)}{\partial\phi}\delta\phi = -2I_0\frac{I_0'}{a}\left(-\frac{\delta\phi}{\dot\phi}\right) \approx -2\frac{I_0 I_0'}{aH}\zeta, \tag{4}$$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} =& - a^{-2}\partial_\tau A^i\partial_\tau A_i a^{-2}2 + \left(\partial_i A_j - \partial_j A_i\right)\left(\partial_k A_l - \partial_l A_k\right)a^{-4}\delta^{ik}\delta^{jl} \\ =& - 2\frac{|\vec E|^2}{I_0^2} + 2\frac{|\vec B|^2}{I_0^2} \approx -2E_i^{(0)}\delta E^i + 2B_i^{(0)}\delta B^i - \delta E_i\delta E^i + \delta B_i\delta B^i, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{\gamma}{8}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu} =& \frac{\gamma}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\partial_\alpha A_\beta = \frac{\gamma}{2}\left[-2\left(-\frac{A_i'}{a^2}\right)\left(\frac{\vec\nabla\times\vec A}{a^2}\right)^i\frac{a^3}{I_0^2}I_0^2\right] \\ &\approx -\frac{\gamma a^3}{I_0^2}\left[\delta E_i B^{(0)i} + E_i^{(0)}\delta B^i + \delta E_i\delta B^i\right]. \end{aligned} \tag{6}$$

Thus, we can separate the so-perturbed action in the $A\delta\phi\delta A$ and $\delta\phi\delta A^2$ contributions

$$S^{(\zeta\delta A)} = \int d\tau d^3x\left(-\frac{2a^3}{H}\right)\frac{I_0'}{I_0}\zeta\left[\left(E_i^{(0)}\delta E^i - B_i^{(0)}\delta B^i\right) - \gamma\left(E_i^{(0)}\delta B^i + B_i^{(0)}\delta E^i\right)\right], \tag{7}$$

$$S^{(\zeta\delta A^2)} = \int d\tau d^3x\left(-\frac{2a^3}{H}\right)\frac{I_0'}{I_0}\zeta\left[\frac{1}{2}\left(\delta E_i\delta E^i - \delta B_i\delta B^i\right) - \gamma\delta E_i\delta B^i\right] \tag{8}$$

.

For studying the perturbed power spectra for gravitational modes, one has to follow the same procedure perturbing $\sqrt{-\delta g} \approx \sqrt{-g} + \delta g_{\mu\nu}\sqrt{-g}g^{\mu\nu}$ getting

$$\begin{aligned} S^{(h\delta A^2)} =& \int d\tau d^3x a I_0^2\delta g_{\mu\nu}\sqrt{-g}g^{\mu\nu}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \\ =& \int d\tau d^3x a^4 I_0^2 a^2 h_{ij}g^{ij}\left[-\frac{1}{2I_0^2}\left(2B_i^{(0)}\delta B^i + 2E_i^{(0)}\delta E^i + \delta B_i\delta B^i + \delta E_i\delta E^i\right)\right]. \end{aligned} \tag{9}$$

By focusing now only to scalar correlation functions, we can neglect the magnetic modes, as we saw they contribute much less than the electric ones. Hence, we can find the two interaction Hamiltonian that follow from

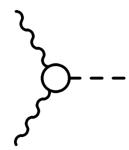
Eqs. 7, 8. By simply expanding each field in the Fourier space and substituting them inside the corresponding action, knowing that $S = \int dt d^3x \mathcal{L} = -\int d\tau a \mathcal{H}$, we get

$$H^{(\zeta\delta A)} \approx -\frac{4E_i^{vev}}{H^4\tau^4}\int \frac{d^3p}{(2\pi)^3}\delta E^i(\vec{p},\tau)\zeta_{-\vec{p}}(\tau) \tag{10}$$

and

$$H^{\zeta\delta A^2} \approx -\frac{2\eta^{ij}}{H^4\tau^4}\int \frac{d^3pd^3p'}{(2\pi)^6}\delta E_i(\vec{p},\tau)\delta E_j(\vec{p},\tau)\zeta_{-\vec{p}'}(\tau)\zeta_{-\vec{p}-\vec{p}'}(\tau). \tag{11}$$

We used also the relation $\frac{I_0'}{I_0} = \frac{\left[(-H\tau)^2\right]'}{(-H\tau)} = 2\frac{H}{H\tau}$. It is now useful to diagrammatically represent these Hamiltonians. These read

$H^{(\zeta\delta A)} =$  and $H^{(\zeta\delta A^2)} =$ 

As, in the computation of the bispectra, we will disassemble them in several power-spectra computations, and since in a single graph we will encounter two or more different vertices (at different times), it is useful to compute firstly the 0-th order scalar power spectrum at different times. For the scalar field we have

$$\begin{aligned}
\langle\left[\zeta_{k_1}(\tau_1),\zeta_{\vec{k}_2}(\tau_2)\right]\rangle =&\langle 0|\left\{u_{k_1}(\tau_1)u^*_{k_2}(\tau_2)\left[a(\vec{k_1}),a^\dagger(-\vec{k_2})\right] + u^*_{k_1}(\tau_1)u_{k_2}(\tau_2)\left[a^\dagger(-\vec{k_1}),a(\vec{k_2}))\right]\right\}|0\rangle \\
=&u_{k_1}(\tau_1)u^*_{k_2}(\tau_2)\langle\left[a(\vec{k_1}),a^\dagger(-\vec{k_2})\right]\rangle - u^*_{k_1}(\tau_1)u_{k_2}(\tau_2)\langle\left[a(\vec{k_2}),a^\dagger(-\vec{k_1})\right]\rangle \\
=&(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right)\frac{H^2}{4\epsilon k_1^3}\left((1+ik_1\tau_1)(1-ik_1\tau_2)e^{-ik_1(\tau_1+\tau_2)} - c.c.\right).
\end{aligned} \tag{12}$$

By expanding now the exponentials up to third order in $-k_i\tau_i \ll 1$ (i.e. when the modes are well outside the horizon), all the real term cancel between themselves, and we are left with

$$\langle\left[\zeta_{k_1}(\tau_1),\zeta_{\vec{k}_2}(\tau_2)\right]\rangle = -(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{iH^2}{6\epsilon}(\tau_1^3-\tau_2^3) \tag{13}$$

For the tensorial modes the same procedure holds, but the mode functions, as we saw, are given by $u_k^{h_{ij}}(\tau) = u_k^{\zeta}(\tau)\sqrt{2\epsilon}$. We are now ready to calculate the power spectra

Power spectrum

At tree-level the only two-points contribution that can arise from the scalar-vector interaction is given by

$\langle\zeta_{\vec{k_1}}(\tau)\zeta_{\vec{k_2}}(\tau)\rangle_1 =$ 

By means of the in-in formalism, we thus have

$$\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 = -\int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \left\langle \left[\left[\prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau), H^{(\zeta\delta A)}(\tau_1) \right], H^{(\zeta\delta A)}(\tau_2) \right] \right\rangle . \tag{14}$$

As we are interested in the result in the $-k\tau \ll 1$ limit, the mode functions behave as classical fields and evolve independently between each other. Thanks to this fact, and to Wick's theorem, the expectation value can be divided in two expectation values [2, 5]. One then obtains

$$\begin{aligned}
\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 =& -\left(\frac{4}{H^2}\right)^2 E_i^{vev} E_j^{vev} \int^{\tau} \frac{d\tau_1}{\tau_1^4} \int^{\tau_1} \frac{d\tau_2}{\tau_2^4} \\
& \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \langle \delta E^i(\vec{p_1}, \tau_1) \delta E^j(\vec{p_2}, \tau_2) \rangle \left\langle \prod_l \zeta_{\vec{k_l}}(\tau) \zeta_{-\vec{p_1}}(\tau_1) \zeta_{-\vec{p_2}}(\tau_2) \right\rangle \\
=& -\left(\frac{4}{H^2}\right)^2 E_i^{vev} E_j^{vev} \int^{\tau} \frac{d\tau_1}{\tau_1^4} \int^{\tau_1} \frac{d\tau_2}{\tau_2^4} \\
& \int \frac{d_1^3 d^3p_2}{(2\pi)^6} \epsilon^{(+)i}(\hat{p_1}) \epsilon^{(+)j}(\hat{p_2}) \left(-\frac{e^{\pi\xi}}{\xi^{3/2}}\right)^2 \left(\frac{3H^2}{2\sqrt{\pi}}\right)^2 \frac{(2\pi)^3 \delta^{(3)}(\vec{p_1} + \vec{p_2})}{(p_1 p_2)^{3/2}} \times \\
& \times (2\pi)^6 \left[\left(-\frac{iH^2}{6\epsilon}\right)^2 (\tau^3 - \tau_1^3)(\tau^3 - \tau_2^3) \delta^{(3)}(\vec{k_1} - \vec{p_1}) \delta^{(3)}(\vec{k_2} - \vec{p_2}) + (\vec{k_1} \leftrightarrow \vec{k_2}) \right] \\
=& (2\pi)^3 \delta^{(3)}\left(\vec{k_1} + \vec{k_2}\right) \left(\frac{4}{H^4}\right)^2 \frac{E_i^{vev} E_j^{vev}}{k_1^3} \int^{\tau} \frac{d\tau_1}{\tau_1^4} \int^{\tau_1} \frac{d\tau_2}{\tau_2^4} (\tau^3 - \tau_1^3)(\tau^3 - \tau_2^3) \\
& \left[\epsilon^{(+)i}(\hat{k_1}) \epsilon^{(+)j}(\hat{k_2}) + \epsilon^{(+)i}(\hat{k_2}) \epsilon^{(+)j}(\hat{k_1}) \right] \frac{e^{2\pi\xi}}{\xi^3} \frac{H^8}{16\pi\epsilon^2}
\end{aligned} \tag{15}$$

where we simply substitute the corresponding contractions' forms to the various expectation values. In the integration over time, only the long-wavelength contributions survive, as the small-wavelength cancel each other out due to their rapid oscillation [5]. This means that the dominant terms are given by considering the time interval $\left[\tau_{in} = -\frac{1}{k_{in}} \ll 1, \tau \to 0\right]$. The time integrals then reduce to

$$\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 = (2\pi)^3 \delta^{(3)}(\vec{k_1} + \vec{k_2}) \left(\frac{4}{H^4}\right)^2 \frac{H^8 e^{2\pi\xi}}{16\pi\epsilon^2\xi^3} \sum_{s=+,-} \frac{\epsilon^{(s)i}(\hat{k_1}) \epsilon^{(s)j*}(\hat{k_1}) E_i^{vev} E_j^{vev}}{k_1^3} \int_{-k_1^{-1}}^{0} \frac{d\tau_1}{\tau_1} \int_{-k_2^{-1}}^{\tau_1} \frac{d\tau_2}{\tau_2} \tag{16}$$

where we used the property $\epsilon_i^s(-\hat{k}) = \epsilon_j^{s*}(\hat{k})$, with $s = +, -$. We note that if we had the same upper bound for the τ_2-integral, we would resemble the exact definition of the number of e-folds N_{k_2} calculated from k-mode's horizon exit to the end of inflation, just as done by the τ_1-integration. There is however a well known result that relates nested time integrals to a simpler time integral in which the boundaries are the same for all the integrated variables. It is well known in fact that $\int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \cdots \int^{\tau_{n-1}} d\tau_n f(\tau_1, ..., \tau_n) = \frac{1}{n!} \int^{\tau} f(\tau_1, ...\tau_n)$. Therefore, from the whole integration in time, we get a factor $\frac{1}{2} N_{k_1}^2$ as the Dirac's delta forces the two momenta to be the same in modulus. For the directional part we have

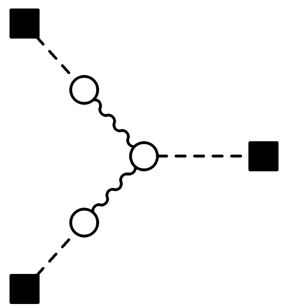
$$\sum_{s=+,-} \epsilon^{(s)i}(\hat{k_1}) \epsilon^{(s)*j}(\hat{k_1}) E_i^0 E_j^0 \delta^{(3)}(\vec{k_1} + \vec{k_2}) = \left[1 - \left(\hat{k_1} \cdot \hat{E}^{vev}\right)^2 \right] (E^{vev})^2 \delta^{(3)}(\vec{k_1} + \vec{k_2}) \tag{17}$$

where we applied the well known equality $\sum_s \epsilon_i^s(\hat{k})\epsilon_j^{s*}(\hat{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}$[1]
In the end it thus follows

$$\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 = (2\pi)^3 \frac{E_{vev}^2}{2\pi\epsilon^2 M_{Pl}^4} \frac{e^{2\pi\xi}}{\xi^3} \frac{N_{k_1}^2}{k_1^3} \left[1 - \left(\hat{k_1} \cdot \hat{E}^{vev} \right)^2 \right] \delta^{(3)} \left(\vec{k_1} + \vec{k_2} \right) \tag{18}$$

Bispectrum

Analogously, we want now to calculate the 1-st order perturbation of the three-point scalar correlation function. The tree-level contribution to this bispectrum is given by

$\left\langle \prod_{l=1}^3 \zeta_{\vec{k_l}}(\tau) \right\rangle_1 =$ 

where we have that each vertex identifies a different interaction-time. Then, we write our bispectrum as

$$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 = \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} + \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{121} + \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{112} \tag{19}$$

with $\left\langle \prod_{l=1}^3 \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{abc} = i \int^\tau d\tau_a \int^{\tau_a} d\tau_b \int^{\tau_b} d\tau_c \left\langle \left[\left[\left[\prod_{l=1}^3 \zeta_{\vec{k_l}}(\tau), H(\tau_a) \right], H(\tau_b) \right], H(\tau_c) \right] \right\rangle$. For each vertex we will use the corresponding Hamiltonian calculated above. We have to use here both types of Hamiltonian, since we introduced a vertex that connects two internal vector fields. By following the same line of reasoning exposed above, for each piece of Eq. 19 we can decompose the whole expectation value in two separate ones, respectively for the scalar and vector part. Considering as an example $\left\langle \prod_{l=1}^3 \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211}$, we have

$$\begin{aligned} \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} = &- i \int^\tau d\tau_1 \int^{\tau_1} d\tau_2 \int^{\tau_2} d\tau_3 \int \frac{\prod_{n=1}^4 d^3p_n}{(2\pi)^{12}} \frac{2}{H^4\tau_1^4} \langle \delta E_i(\vec{p_1}, \tau_1) \delta E^j(\vec{p_2}, \tau_1) \delta E^k(\vec{p_3}, \tau_2) \delta E^l(\vec{p_4}, \tau_3) \rangle \times \\ &\times \frac{E_k^{vev} E_l^{vev}}{H^8 \tau_2^4 \tau_3^4} 16 \left[\langle \zeta_{\vec{k_1}}(\tau) \zeta_{-\vec{p_1}-\vec{p_2}}(\tau_!) \rangle \langle \zeta_{\vec{k_2}}(\tau) \zeta_{-\vec{p_3}}(\tau_2) \rangle \langle \zeta_{\vec{k_3}}(\tau) \zeta_{-\vec{p_4}}(\tau_3) \rangle + 5 \text{ perms in } \vec{k_l} \right] \end{aligned} \tag{20}$$

where have already dissembled the scalar fields' expectation value in its contractions, by mean of Wick's theorem. Just for simplicity, we redefine $\frac{32 E_k^{vev} E_l^{vev}}{(H2\pi)^{12}} = B_{kl}$. Then, from Eq. 13, it follows

[1] We expect $\sum_{s=+,-} \epsilon^{(s)i}(\hat{k}_1)\epsilon^{(s)*j}(\hat{k_1})$ to be equal to a certain tensor D_{ij}. Now, we exploit two properties of polarization vectors to identify D_{ij}. Since $\epsilon_i^s(\hat{k})\epsilon_i^r(\hat{k}) = \delta^{rs}$, then we have $\epsilon^{(r)i}(\hat{k}) \sum_{s=+,-} \epsilon^{(s)i}(\hat{k}_1)\epsilon^{(s)*j}(\hat{k_1}) = \epsilon^{(}r)_j(\hat{k})$, and the analog would be true if we multiplied by $\epsilon^{(r)j}$. Therefore, D_{ij} should have a form which reduces to a Kronecker delta when contracted with one polarization vector. The general form is therefore $D_{ij} = \delta_{ij} + c_{ij}$, with $c_{ij}\epsilon^{(r)i}(\hat{k}) = 0$. Another property of polarization vectors is that $\hat{k}_i \epsilon^{(r)i}(\hat{k}) = 0$. Hence, in the end we can write the equality used above

$$
\begin{aligned}
\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} =& - iB_{kl}(2\pi)^9 \left(-\frac{iH^2}{6\epsilon}\right)^3 \int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \int^{\tau_2} d\tau_3 \prod_{m=1}^{3} \frac{(\tau^3-\tau_m^3)}{\tau_m^4} \\
& \int d^3p_1 \delta^{(3)}(\vec{k_1}-\vec{p_1}-\vec{p_2}) \langle \delta E_i(\vec{p_1},\tau_1)\delta E^i(\vec{k_1}-\vec{p_1},\tau_1)\delta E^k(\vec{k_2},\tau_2)\delta E^l(\vec{k_3},\tau_3)\rangle \\
& + 2\text{perms in } \vec{k_l}
\end{aligned}
\tag{21}
$$

We focus now on the expectation value between vector fields. Again, owing to Wick's theorem, and by looking only to connected diagrams, we have

$$
\begin{aligned}
\langle \delta E_i \delta E^i \delta E^k \delta E^l\rangle =& \delta E_i \delta E^i \delta E^k \delta E^l + \delta E_i \delta E^i \delta E^k \delta E^l \\
=& (2\pi)^6 \epsilon^{(+)i}(\hat{p_1})\epsilon^{(+)k}(\hat{k_2})\epsilon_i^{(+)}(k_1 \hat{-} p_1)\epsilon^{(+)l}(\hat{k_3}) \frac{e^{4\pi\xi}}{\xi^6}\left(\frac{3H^2}{2\sqrt{\pi}}\right)^4 \frac{1}{k_2^3 k_3^3} \\
& \left(\delta^{(3)}(\vec{p_1}+\vec{k_2})\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_3}) + \delta^{(3)}(\vec{p_1}+\vec{k_3})\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_2})\right)
\end{aligned}
\tag{22}
$$

Integrating out the different Dirac's deltas, we end up with two equal contributions. Namely, the last line of Eq. 21 reads

$$
\int d^3p_1 d^3p_2 \langle \dots \rangle = \frac{e^{4\pi\xi}}{\xi^6}\left(\frac{3H^2}{2\sqrt{\pi}}\right)^4 \frac{2}{k_2^3 k_3^3} \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\epsilon^{(-)i}(\hat{k_2})\epsilon^{(+)k}(\hat{k_2})\epsilon_i^{(-)}(\hat{k_3})\epsilon^{(+)l}(\hat{k_3}) \tag{23}
$$

This shows explicitly a symmetry of the system. That is, if we exchange the two external legs connected to the $H^{\zeta\delta A}$-vertex, we obtain the same result. Once the external field connected through $H^{\zeta\delta A}$ is fixed, gives us the same factor. Therefore, since we have only three momenta, the total number of independent diagram is three and a factor $\times 2$ shows up. Hence, at the end we have

$$
\begin{aligned}
\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} =& (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{3H^2 E_k^{vev} E_l^{vev}}{2\pi^2\epsilon^3} \frac{e^{4\pi\xi}}{\xi^6} \frac{1}{k_2^3 k_3^3} \epsilon^{(-)i}(\hat{k_2})\epsilon^{(+)k}(\hat{k_2})\epsilon_i^{(-)}(\hat{k_3})\epsilon^{(+)l}(\hat{k_3}) \\
& \int^{\tau} d\tau_1 \frac{\tau^3-\tau_1^3}{\tau_1^4} \int^{\tau_1} d\tau_2 \frac{\tau^3-\tau_2^3}{\tau_2^4} \int^{\tau_2} d\tau_3 \frac{\tau^3-\tau_3^3}{\tau_3^4} \\
& + 2\text{perms in } \vec{k_l}
\end{aligned}
\tag{24}
$$

Again, we can send $\tau \to 0$, and apply the same trick as above to reduce the nested integral in a single one. However, while we have two time integrals that resemble exactly the definition of the number of e-folds, respectively for the modes that exit the horizon at $-\frac{1}{k_2}$ and $-\frac{1}{k_3}$, the integral over τ_1 depends upon the three moment, which are constrained by the Dirac's delta. Since we want to be sure to consider only long-wavelength contributions, and as $k_1 = |\vec{k_2}+\vec{k_3}|$, we require $\tau_1 > \text{Max}\left[-\frac{1}{k_1}, -\frac{1}{k_2}, -\frac{1}{k_3}\right]$. Namely, since a $H^{\zeta\delta A^2}$-vertex connects three momenta, we want to pick the higher time of horizon-crossing and set it to be the lower bound of our final integration in time.

$$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} = (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{H^2 E_{vev}^2}{4\pi^2\epsilon^3} \frac{e^{4\pi\xi}}{\xi^6} \frac{N_{k_3} N_{k_2} \mathrm{Min}\left[N_{k_1}, N_{k_2}, N_{k_3}\right]}{k_2^3 k_3^3} \epsilon^{(-)i}(\hat{k_2}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(-)}(\hat{k_3}) \epsilon^{(+)l}(\hat{k_3}) + 2\text{perms in } \vec{k_l} \tag{25}$$

In principle, we would have to calculate all the other terms of Eq. 19. However, we notice that for each term we would get exactly the same three contributions depicted in the latter equation. Therefore, we can simply skip the whole calculation and multiply the result above by 3. We just have to follow the same reasoning for the nested time integral and relabel its variables in order to end up with an identical result for each term [2]. At the very end, we find[2]

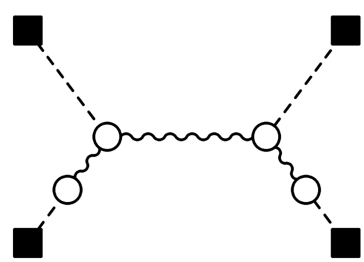
$$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 = (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{3H^2 E_{vev}^2}{4\pi^2\epsilon^3} \frac{e^{4\pi\xi}}{\xi^6} \frac{N_{k_3} N_{k_2} \mathrm{Min}\left[N_{k_1}, N_{k_2}, N_{k_3}\right]}{k_2^3 k_3^3} \mathcal{C}_{\vec{k_2},\vec{k_3}} + 2\text{perms in } \vec{k_l} \tag{26}$$

with $\mathcal{C}_{\vec{k_2},\vec{k_3}} = \hat{E}_k^{vev} \hat{E}_l^{vev} \epsilon^{(-)i}(\hat{k_2}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(-)}(\hat{k_3}) \epsilon^{(+)l}(\hat{k_3})$.
Since we just want to show how this type of calculation is done, we will not go through the procedure to rewrite the directional part in a more compact way. Besides, we are going to do that explicitly in the next chapter. The final equation, in the two opposite limits $\gamma = 0$,$\gamma \neq 0$ is computed in [2].
We note that the two terms arising from a different way to contract the scalar fields can be immediately written as the one above. Namely, by just substituting the old $H^{\zeta\delta A}$- and $H^{\zeta\delta A^2}$-connected fields' momenta with the new ones. While the latter will appear only in $Min[N_{k_1}, N_{k_2}, N_{k_3}]$, the former will define $\mathcal{C}_{\vec{k},\vec{k'}}$, the other two number of e-folds and the $\sim 1/k^3 k'^3$ factor.

Trispectrum

Now, we study the four-points correlation function for the scalar-curvature field. Again, we restrict ourselves to the single tree-level contribution we can draw, which reads

$$\left\langle \prod_{l=1}^{4} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 =$$



Again, the full trispectrum is given by

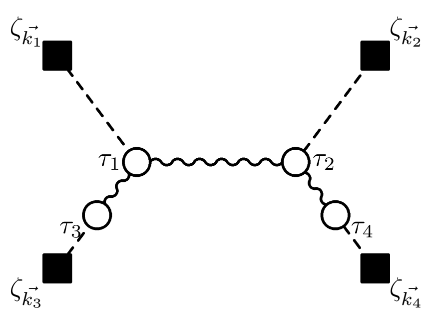
[2] There is a difference with respect to the original calculation done in [2]. Specifically, the authors used a different convention for the Fourier tranforms and hence the mode functions. They do not have a factor $(2\pi)^3$ that shows up in the power-spectra. This is true also for 18

$$\left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1 = \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{1122} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{1212} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{1221} + \\ + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2121} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2112} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2211} \tag{27}$$

In complete analogy with what we have done so far, we proceed in writing explicitly the trispectrum. For the same reason explained above, we can separate the whole expectation value in two simpler ones. Then, we proceed in applying Wick's theorem, thereby simplifying each of them in several power-spectra. We have

$$\begin{aligned}\left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2211} =& \frac{64E_k^{vev}E_l^{vev}}{H^{16}}\int^{\tau}\frac{d\tau_1}{\tau_1^4}\int^{\tau_1}\frac{d\tau_2}{\tau_2^4}\int^{\tau_2}\frac{d\tau_3}{\tau_3^4}\int^{\tau_3}\frac{d\tau_4}{\tau_4^4}\int\frac{d^3p_1d^3q_1d^3p_2d^3q_2d^3p_3d^3p_4}{(2\pi)^{18}}(2\pi)^{12}\left(\frac{-iH^2}{6\epsilon M_{Pl}^2}\right)^4\\ &\left\langle\delta E^i(\vec{p_1},\tau_1)\delta E_i(\vec{q_1},\tau_1)\delta E^j(\vec{p_2},\tau_2)\delta E_j(\vec{q_2},\tau_2)\delta E^k(\vec{p_3},\tau_3)\delta E^l(\vec{p_4},\tau_4)\right\rangle(\tau^3-\tau_1^3)(\tau^3-\tau_2^3)\\ &(\tau^3-\tau_3^3)(\tau^3-\tau_4^3)\delta^{(3)}\left(\vec{k_1}-\vec{p_1}-\vec{q_1}\right)\delta^{(3)}\left(\vec{k_2}-\vec{p_2}-\vec{q_2}\right)\delta^{(3)}\left(\vec{k_3}-\vec{p_3}\right)\delta^{(3)}\left(\vec{k_4}-\vec{p_4}\right)\\ &+23\text{ perms in }\vec{k_l}\\ =& \frac{64E_k^{vev}E_l^{vev}}{H^{16}}\left(\prod_{i=1}^{4}\int^{\tau_{i-1}}\frac{d\tau_i}{\tau_i^4}(\tau^3-\tau_i^3)\right)\int\frac{d^3p_1d^3p_2d^3}{(2\pi)^6}\left(\frac{-iH^2}{6\epsilon M_{Pl}^2}\right)^4\\ &\left\langle\delta E^i(\vec{p_1},\tau_1)\delta E_i(\vec{k_1}-\vec{p_1},\tau_1)\delta E^j(\vec{p_2},\tau_2)\delta E_j(\vec{k_2}-\vec{p_2},\tau_2)\delta E^k(\vec{k_3},\tau_3)\delta E^l(\vec{k_4},\tau_4)\right\rangle\end{aligned} \tag{28}$$

where we expanded the scalar expectation value by using Eq. 13 and put $\tau_0=\tau$. Since the complete calculation is quite long, we proceed reporting the most important passages. For simplicity, we forget about the 23 permutations for the moment, and depict only the contribution explicitly written above. This corresponds to the diagram



By expanding the expectation value of the vector fields we find, neglecting the non-connected diagrams

$$
\begin{aligned}
\langle \dots \rangle =&(2\pi)^9 \left(\frac{9H^4 e^{2\pi\xi}}{4\pi\xi^3}\right)^3 \frac{\epsilon_i^{(+)}(\hat{p_1})\epsilon^{(+)i}(k_1 \hat{-} p_1)\epsilon_j^{(+)}(\hat{p_2})\epsilon^{(+)j}(k_2 \hat{-} p_2)\epsilon^{(+)k}(\hat{k_3})\epsilon^{(+)l}(\hat{k_4})}{(p_1|\vec{k_1}-\vec{p_1}|p_2|\vec{k_2}-\vec{p_2}|k_3k_4)^{3/2}} \times \\
&\times \Big\{ \delta^{(3)}(\vec{p_1}+\vec{p_2}) \left[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_3})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_4}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_4})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_3})\right] \\
&\delta^{(3)}(\vec{p_1}+\vec{k_2}-\vec{p_2}) \left[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_3})\delta^{(3)}(\vec{p_2}+\vec{k_4}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_4})\delta^{(3)}(\vec{p_2}+\vec{k_3})\right] \\
&\delta^{(3)}(\vec{p_1}+\vec{k_3}) \left[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_2}-\vec{p_2})\delta^{(3)}(\vec{p_2}+\vec{k_4}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{p_2})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_4})\right] \\
&\delta^{(3)}(\vec{p_1}+\vec{k_4}) \left[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_2}-\vec{p_2})\delta^{(3)}(\vec{p_2}+\vec{k_3}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{p_2})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_3})\right] \Big\} \\
=&(2\pi)^9 \delta^{(3)}\left(\sum_{i=1}^4 \vec{k_i}\right) \left(\frac{9H^4 e^{2\pi\xi}}{4\pi\xi^3}\right)^3 4 \frac{\epsilon^{(-)i}(\hat{k_3})\epsilon^{(+)k}(\hat{k_3})\epsilon^{(-)j}(\hat{k_4})\epsilon^{(+)l}(\hat{k_4})}{k_3^3 k_4^3} \times \\
&\times \left[\frac{\epsilon_i^{(+)}(\hat{k_{24}})\epsilon_j^{(-)}(\hat{k_{24}})}{k_{24}^3} + \frac{\epsilon_j^{(+)}(\hat{k_{23}})\epsilon_i^{(-)}(\hat{k_{23}})}{k_{23}^3}\right]
\end{aligned} \tag{29}
$$

From the last line, we see that we we exchange the two external legs connected to the simplest vertex, the result remains the same. This is because all the index are actually dummy and can be relabelled. This reduces the number of independent diagrams to 12, and multiplies the whole result by a factor 2. By calling now the directional part $\mathcal{C}^{k_i,k_j}(k_k)$ we see that for every diagram we have

$$
\mathcal{C}^{k_i,k_j}(k_k) = 4E_k^{\hat{v}ev} E_l^{\hat{v}ev} \frac{\epsilon^{(-)i}(\hat{k_i})\epsilon^{(+)k}(\hat{k_i})\epsilon^{(-)j}(\hat{k_j})\epsilon^{(+)l}(\hat{k_j})}{k_i^3 k_j^3} \left[\frac{\epsilon_i^{(+)}(\hat{k_{ki}})\epsilon_j^{(-)}(\hat{k_{ki}})}{k_{ki}^3} + \frac{\epsilon_j^{(+)}(\hat{k_{kj}})\epsilon_i^{(-)}(\hat{k_{kj}})}{k_{kj}^3}\right] \tag{30}
$$

where k_i,k_j are the momenta connected to the simplest vertex, while k_k is one of the other two. Thanks to this observation, we can now write all the other 11 permutations easily. However, since k_k can be referred either to k_1 or k_2 without having any differences in the final result, we get that by exchanging the two $H^{\zeta\delta A^2}$-external lines we obtain the same form of $\mathcal{C}$. Therefore, the number of independent diagrams further decrease to 6, while another factor $\times 2$ shows up. In the end, we thus have

$$
\left\langle \prod_{l=1}^4 \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{2211} = (2\pi)^3 \delta^{(3)}\left(\sum_{i=1}^4 \vec{k_i}\right) \frac{3^5 H^4 E_{vev}^2}{\pi^3 M_{Pl}^8 \epsilon^4} \frac{e^{6\pi\xi}}{\xi^9} \left(\prod_{i=1}^4 \int^{\tau_{i-1}} \frac{d\tau_i}{\tau_i^4}(\tau^3 - \tau_i^3)\right) \mathcal{C}^{k_2 k_3}(k_2) + 5 \text{ perms in } \vec{k_l} \tag{31}
$$

Now, for the nested time-integrals, we proceed as above. We extend each integral's upper bound to the same value $\tau \to 0$ while dividing by $4! = 24$. Then, we use the same approximation applied in the previous section, which constraints our integral to $\tau_i > -\frac{1}{k_i}$. Just to briefly recap, we neglect the short-wavelength contributions since due to their highly oscillatory behaviour they cancel each other [5]. Therefore, we have

$$
\begin{aligned}
\prod_i \int^\tau d\tau_i \frac{1}{\tau_i} =& \frac{1}{24} \int_{\text{Max}\left[-\frac{1}{k_1},-\frac{1}{k_3},-\frac{1}{k_{24}}\right]}^0 \frac{d\tau_1}{\tau_1} \int_{\text{Max}\left[-\frac{1}{k_2},-\frac{1}{k_4},-\frac{1}{k_{13}}\right]}^0 \frac{d\tau_2}{\tau_2} \int_{-\frac{1}{k_3}}^0 \frac{d\tau_3}{\tau_3} \int_{-\frac{1}{k_3}}^0 \frac{d\tau_4}{\tau_4} \\
=& \frac{1}{24} \text{Min}\left[N_{k_1}, N_{k_3}, N_{k_{24}}\right] \text{Min}\left[N_{k_2}, N_{k_4}, N_{k_{13}}\right] N_{k_3} N_{k_4}
\end{aligned} \tag{32}
$$

In the very end, by calculating all the other terms of Eq. 27 the total trispectrum $t^{\vec{k_1},\vec{k_2}}_{\vec{k_3},\vec{k_4}}$ depicted in the diagram above reads [5][3]

$$\begin{aligned}\mathbf{t}^{\vec{k_1},\vec{k_2}}_{\vec{k_3},\vec{k_4}} =& (2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}+\vec{k_4}\right)\frac{3^5}{4\pi^3}\frac{H^4E^2_{vev}}{M^8_{Pl}\epsilon^8}\frac{e^{6\pi\xi}}{\xi^9}\mathcal{C}^{\vec{k_2},\vec{k_3}}(\vec{k_2}) \\ & \mathrm{Min}\left[N_{k_1},N_{k_3},N_{k_{24}}\right]\mathrm{Min}\left[N_{k_2},N_{k_4},N_{k_{13}}\right]N_{k_3}N_{k_4}\end{aligned} \tag{33}$$

References

[1] N. Barnaby, R. Namba, and M. Peloso, “Observable non-gaussianity from gauge field production in slow roll inflation, and a challenging connection with magnetogenesis,” *Physical Review D*, vol. 85, Jun 2012.

[2] N. Bartolo, S. Matarrese, M. Peloso, and M. Shiraishi, “Parity-violating cmb correlators with non-decaying statistical anisotropy,” *Journal of Cosmology and Astroparticle Physics*, vol. 2015, p. 039–039, Jul 2015.

[3] N. Bartolo, S. Matarrese, M. Peloso, and A. Ricciardone, “Anisotropic power spectrum and bispectrum in the f(phi)f2 mechanism,” *Physical Review D*, vol. 87, Jan 2013.

[4] N. Bartolo, S. Matarrese, M. Peloso, and M. Shiraishi, “Parity-violating and anisotropic correlations in pseudoscalar inflation,” *Journal of Cosmology and Astroparticle Physics*, vol. 2015, p. 027–027, Jan 2015.

[5] M. Shiraishi, “Parity violation in the cmb trispectrum from the scalar sector,” *Physical Review D*, vol. 94, Oct 2016.

[3] The numerical factors between the original result differ from the one reported here. In this review, in fact, a different vector mode function has been used. In [5], a factor $1/2^{15}$ arises from the three vector-vector contractions. Also, for the other reported results, there is a difference in the (2π) proportionality. This is because in several considered papers another convention has been followed for the Fourier tranform, and hence for the form of the mode-functions.