University of Glasgow | School of Computing Science

Introduction to Data Science and Systems

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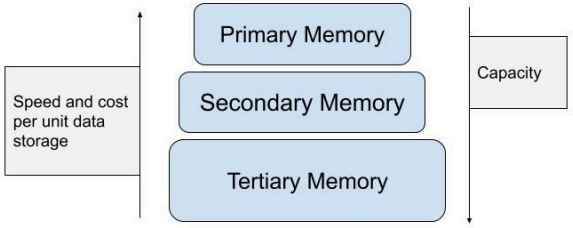
Lecture 9 - Indexing

Recap of Physical Storage

- 3-level hierarchy for data systems
  - Data must first be loaded into primary from secondary
  - Secondary storage access time becomes the bottleneck



- Physical representation of secondary storage
  - Records are stored in blocks
  - *File* is a set of *blocks*, and each block a set of *records*

Recap of Physical Storage

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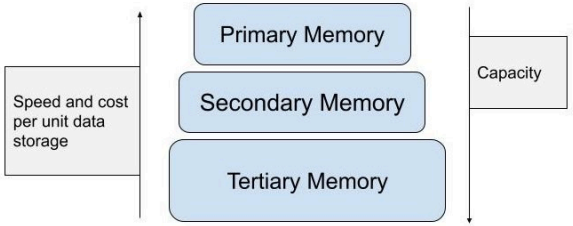

- Physical representation of secondary stor
  - Records are stored in blocks
  - *File* is a set of *blocks*, and each block a set of *records*
- We measure I/O cost for:
  - Retrieving/Inserting/deleting/updating a record x from memory to the disk (update cost)
  - **Cost Function**: # of block accesses (read/write) to search/insert/delete/update record x

Recap of Physical Storage

- File Organizations
  - Heap (random order) files:
    - Suitable when typical access is a file scan retrieving all records.
  - Sequential Files:
    - Best if records must be retrieved in some order, or only a `range' of records is needed.
  - Hash Files
    - Find records in constant time by hash functions.
    - Cannot efficiently scan sequentially or for range queries

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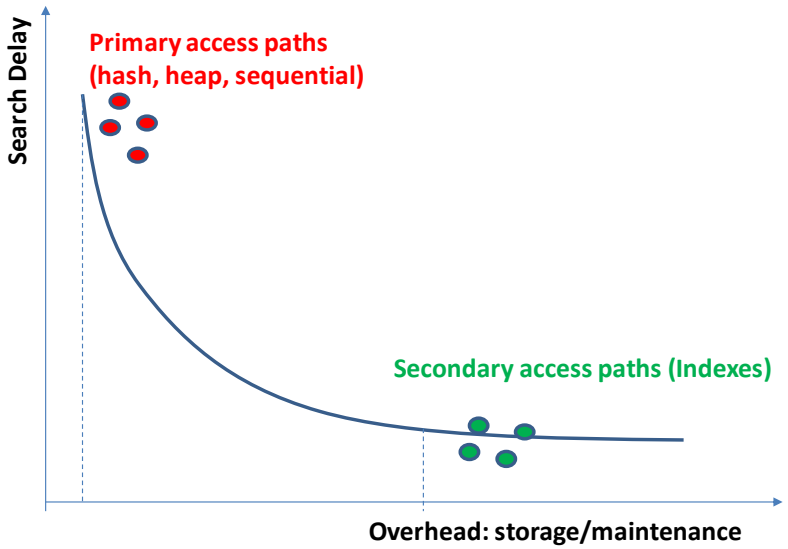
  - Hash Files
    - Find records in constant time by hash functions.
    - Cannot efficiently scan sequentially or for range queries
  - Indexes:
    - Data structures to organize records via trees.
    - Like Sequential files, they speed up searches for a subset of records, based on values in certain ("search key") fields
    - Updates are much faster than in sequential files.

Objectives

- Physical Design
  - Objective: given a specific file type provide a primary access path based on a specific field; e.g., search via SSN
- Index Design
  - Objective: given any file type provide secondary access paths using more than one fields; e.g., search via SSN, Salary, Name, etc.
- Cost: Additional meta-data file on the disk (index)
- Benefit: Significantly accelerate the search process avoiding Linear Scans



Principles & Indexes: 1-d space

- Key Principles:
  - #1: An index is created over predefined (index) field(s)
  - #2: An index is a file separate to that whose data it indexes
  - #3: All index entries are unique and sorted w.r.t. the index field

- Key Hypotheses:
  - #1: Index file occupies less blocks than the data file
    - Fact: index entries are much smaller than data file records: {*index-key*, *block-pointer*}
  - #2: Searching through the index is faster than through the data file
    - Fact: By definition, index is an ordered file; can use binary- and/or tree-based methods to find a record given its key

Principles & Indexes: 1-d space

- Major index types:
  - Primary Index: index field is an ordering-key field of a sequential file
    - e.g., SSN; relation is sorted w.r.t. SSN
  - Clustering Index: index field is an ordering non-key field of a seq. file
    - e.g., Age; relation is sorted w.r.t. Age
  - Secondary index: index field is a non-ordering field over an ordered or non-ordered file…
    - e.g., key-field *unique passport number*
    - e.g., non-key-field *salary*
- Indexes can be:
  - Dense: an index entry for every record in the file
  - Sparse: index entries only for some of records

Primary Index

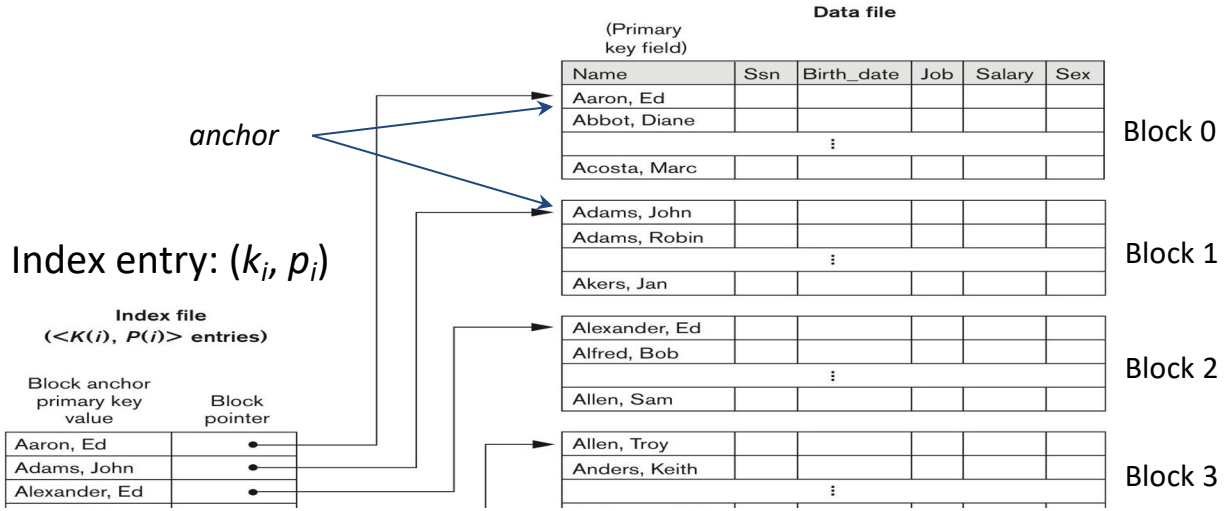
- An ordered file over an ordering key field of a sequential data-file:
  - fixed-length index entries := pair $\{k_i, p_i\}$
  - k_i is a value of the index field
  - p_i is a pointer to the block containing the record with key k_i

Recall: **unique** value for every record; used to **physically order records** in the file

- Sparseness: one index-entry per data block
  - i^{th} index entry $\{k_i, p_i\}$ refers to the i^{th} data block
  - Anchor of block i: The first data-record in the block -- i.e., the one with the index field value k_i

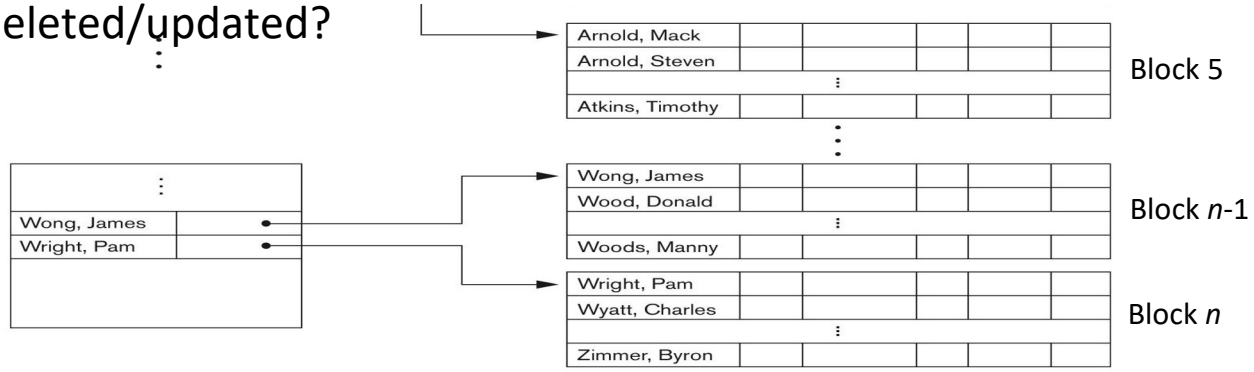
- Theorem: Create a primary index iff a block can accommodate at least two index entries
  - Proof: cost over the index < cost over linear search

Primary Index



Q1: What happens if the anchor record is deleted/updated?

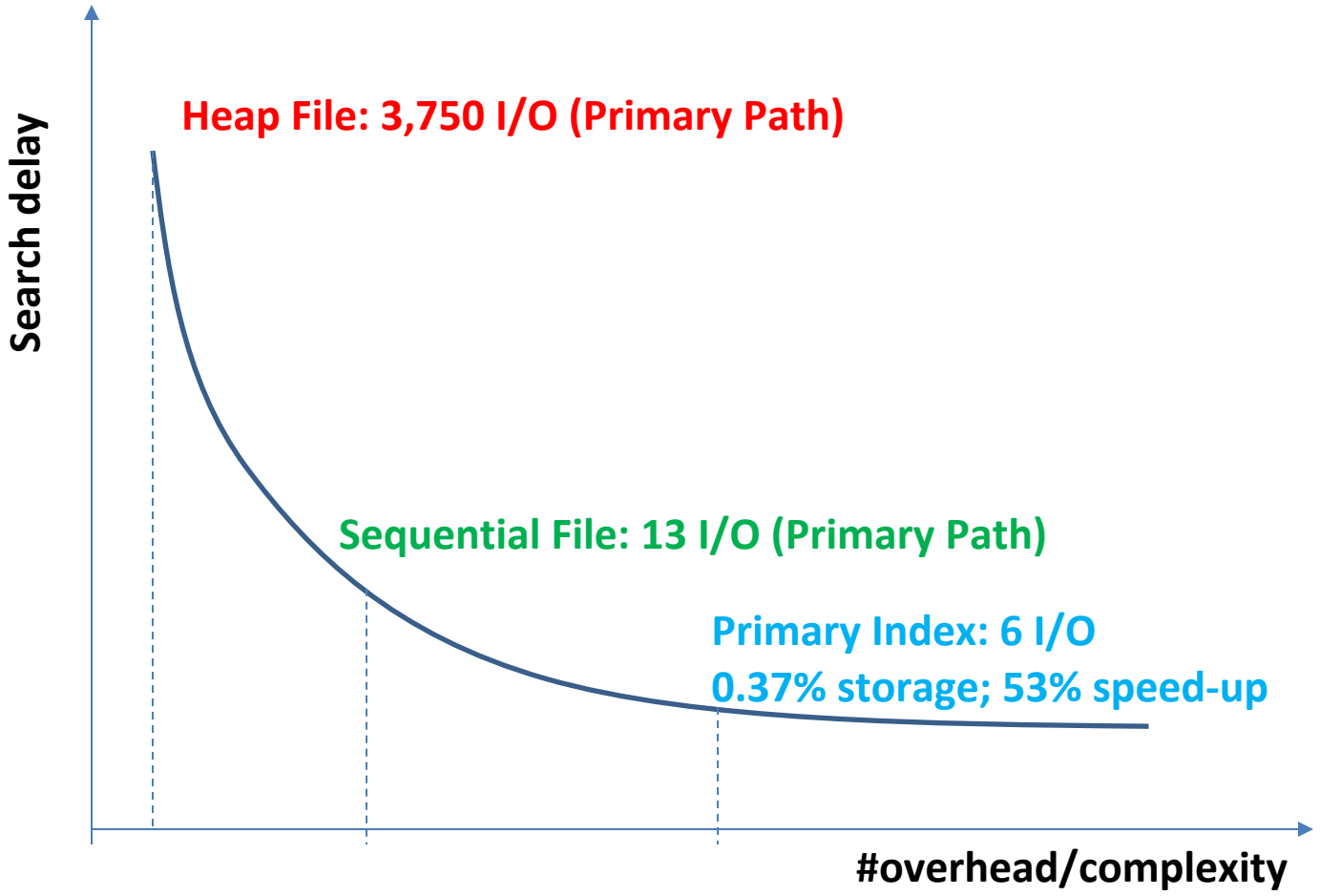
Q2: What happens if a non-anchor record is deleted/updated?



In-Class Example

- EMPLOYEE:
  - r = 300,000 fixed-length records of size R = 100 bytes each; block size B = 4,096 bytes; SSN size = 9 bytes; pointer size = 6 bytes
- Task: Expected cost of SELECT * FROM EMPLOYEE WHERE SSN = 'k'
  - Blocking factor: bfr = floor(B/R) = 40 records per block
  - File: b = ceil(r/bfr) = **7,500 blocks**
  - Linear Search over the File: b/2 = **3,750 block accesses**
  - Binary Search/ordered by key SSN: ceil($\log_2$(b)) = **13 block accesses**
- Create Primary Index on SSN: CREATE INDEX ON EMPLOYEE(SSN)
  - Index Entry: {SSN, Pointer}; size: V = 9 bytes for SSN and P = 6 bytes for Pointer
  - Index Blocking Factor: ibfr = floor(B/(P+V)) = 273 entries/block
  - Primary Index requires 7,500 entries: one per block (b = 7500)
  - Index blocks: ib = ceil(7,500/273) = **28 blocks**
  - Storage overhead: **28 blocks**; 0.37% additional storage
  - Performance gain: Binary Search on Index: ceil($\log_2$(ib)) = 5 block accesses + one block access to load the data block pointed by index = **6 block accesses**; **53% (Binary)** and **99.84% (Linear) speed up**

Trade off: Overhead vs Speed



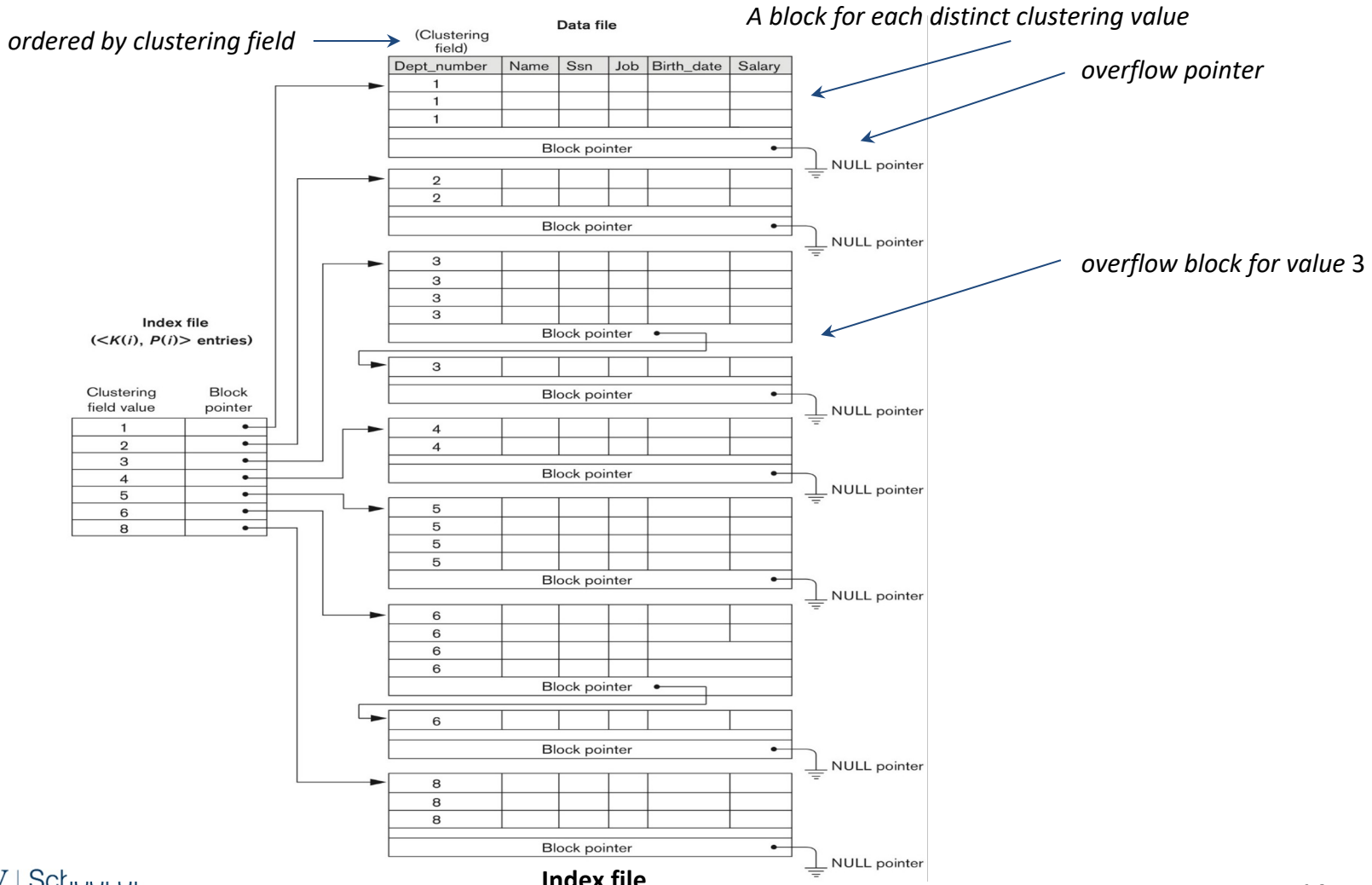
Clustering Index

- Challenge: Index a sequential file on a non-key ordering field
  - e.g., create an index on EMPLOYEE ordered by DNO (dept. no)

Recall: used to **physically order records** in the file; **non unique** values

- Idea: The file is a *set of clusters*, one per distinct value of the above field (a.k.a. the *clustering* field)
  - **index-entry** := {clustering-value, block-pointer}
    - One *index-entry* per *distinct clustering value*
  - Block pointer points to the *first* block of *a cluster*
  - Is the clustering index sparse or dense?

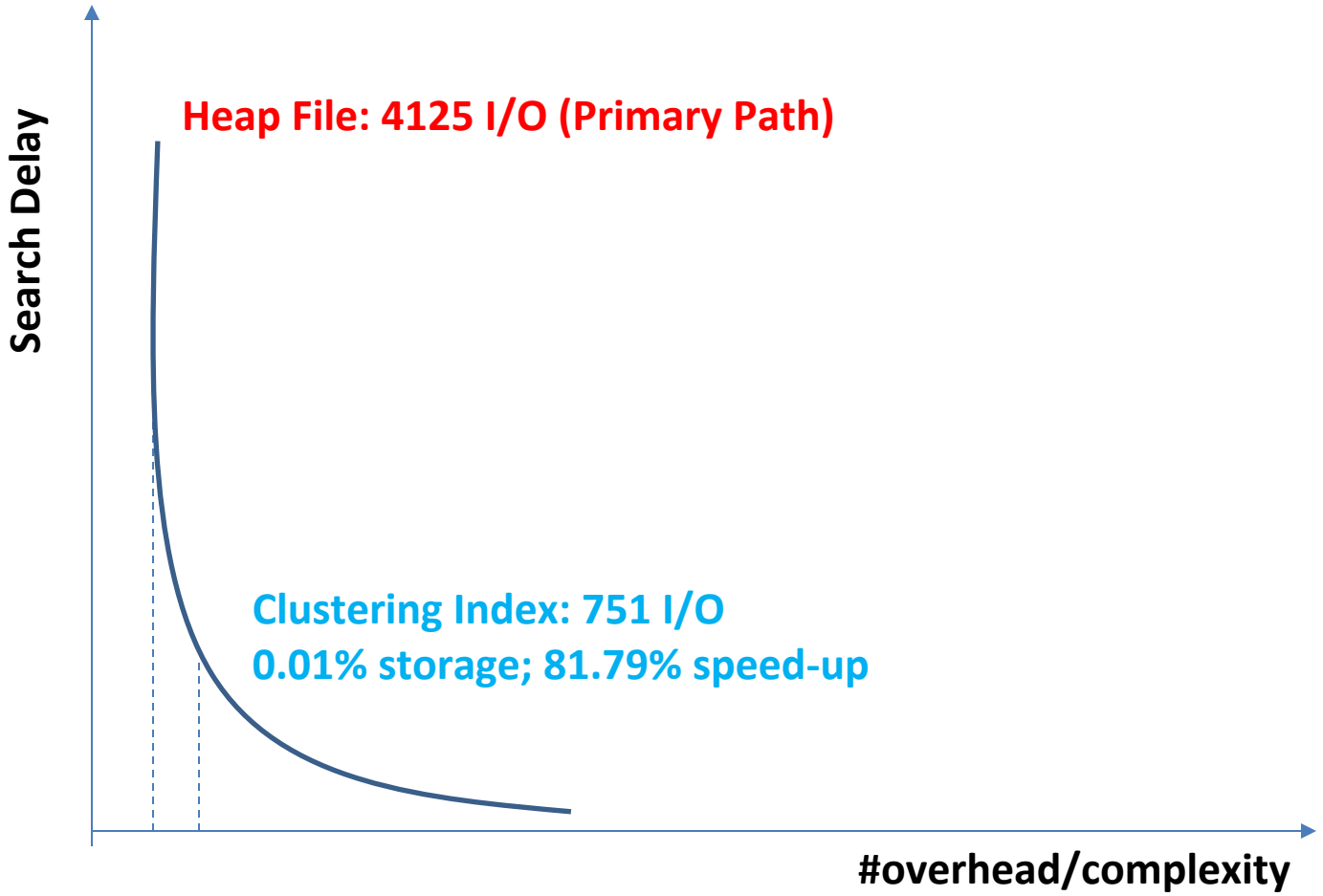
Clustering Index



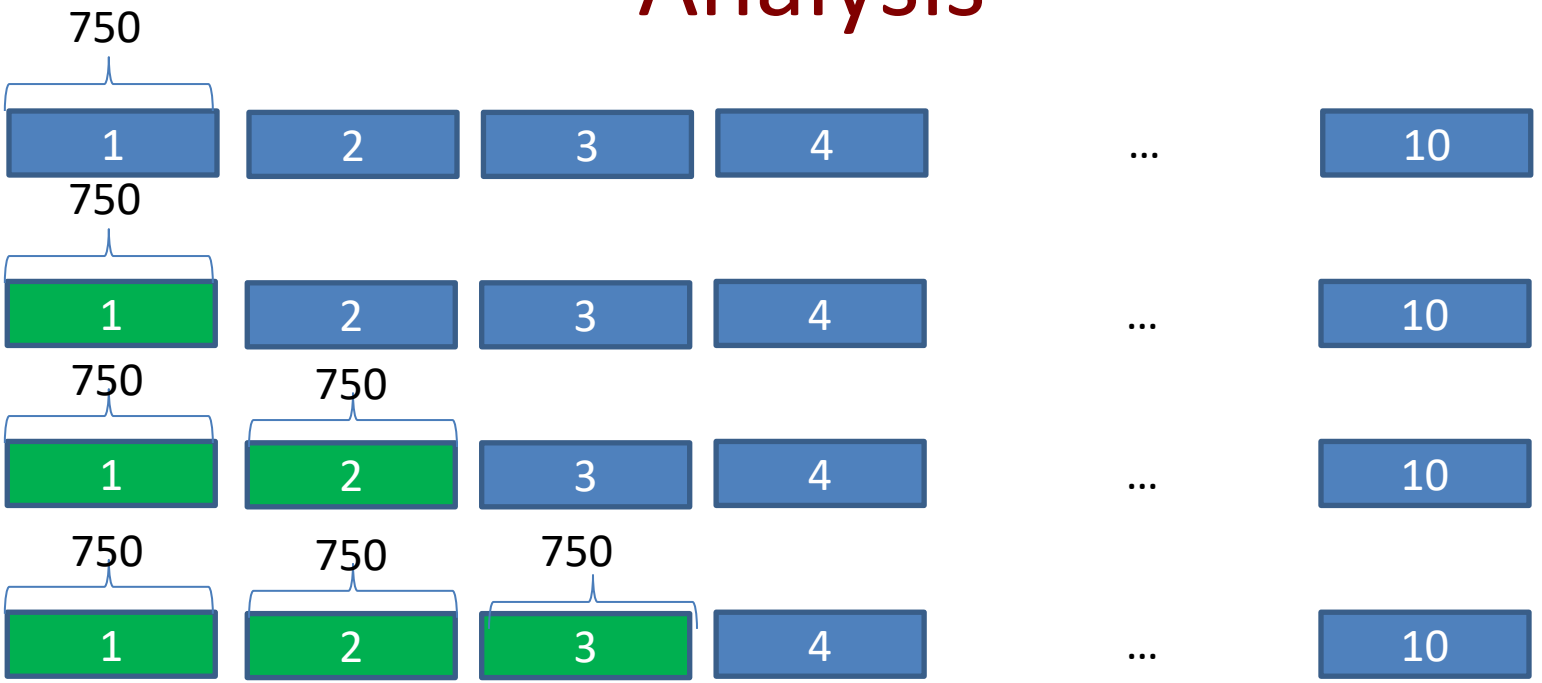
Analysis

- EMPLOYEE: r = 300,000 fixed-length records of size R = 100 bytes each; block size B = 4,096 bytes; DNO size = 9 bytes; P = 6 bytes (pointer); ordered by DNO
- **Assume:** 10 departments; DNO values are uniformly distributed over the tuples
- Compute the expected cost of SELECT * FROM EMPLOYEE WHERE DNO = 'd'
- Compare with naïve linear search over the data file
  - Data File: b = ceil(r/bfr) = **7,500 blocks**
  - Index Entry: {DNO, Pointer}; size: V = 9 bytes for DNO and P = 6 bytes for a Pointer
  - Index Blocking Factor: ibfr = floor(B/(P+V)) = **273 entries/block**
  - Clustering Index requires **10 index entries**: one per cluster!
  - Index blocks: ib = ceil(10/273) = **1 block**
  - Storage overhead: 1 block ☺ (0.01% additional storage)
  - Performance gain:
    - Search on Index: 1 block access
    - 7500/10 = 750 block accesses to load the data blocks belonging to a cluster
    - Total: 1 + 750 = **751 block accesses (81.79%)**
    - Linear Search on File: **4,125 block accesses! (not simply 7,500/2 = 3,750)**

Trade off: Overhead vs Speed



Analysis



1/10(750) + 1/10(750+750) + 1/10(3·750) + … + 1/10(10·750) = 4125

$$\sum_{k=1}^{n}\left(\frac{b}{n}\right)\left(\frac{1}{n}\right)k = \left(\frac{b}{n^2}\right)\left(\frac{n(n+1)}{2}\right) = \frac{b(n+1)}{2n}$$

n = number of clusters
b = number of blocks

$$\log_2(m) + \frac{b}{n}$$

Clustering index: *m* blocks

Clustering Index

- Fact: If $n \rightarrow \infty$, i.e., infinite number of distinct values, then the linear search over an ordering non-key field is b/2:

$$lim_{n \rightarrow \infty} \frac{b(n+1)}{2n} = \left(\frac{b}{2}\right)$$

- **Conclusion**: meaningless to sort a file w.r.t. a non-key field when *n* is large (compared to the number of tuples)

- Create a clustering Index of m blocks iff:

$$m < 2^{\frac{b(n+1)}{2n}}$$

where m is the number of index blocks

- Proof: cost over the index < cost over linear search

Range queries over a clustering index

- Query index for block of lower bound value:
  - Expected cost $\log_2(m)$ block accesses
- Fetch data for lower bound cluster value:
  - Expected cost (b/n) block accesses
- Assuming L additional clusters will be fetched in total:
  - Expected cost L * (b/n) block accesses
- Total: **$\log_2(m) + (L + 1)*(b/n)$** block accesses
- Linear search: b block accesses
- Benefit iff:

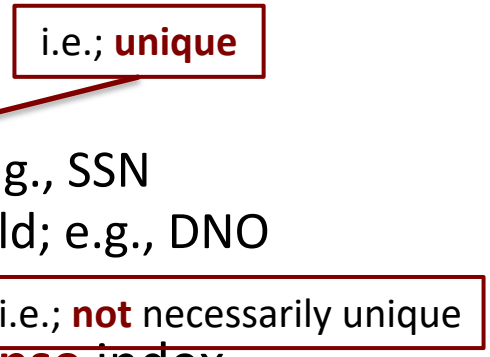
$$\log_2(m) + (L + 1) * (b/n) < b \iff \mathbf{m < 2^{b(n-(L+1))/n}}$$

Secondary Index

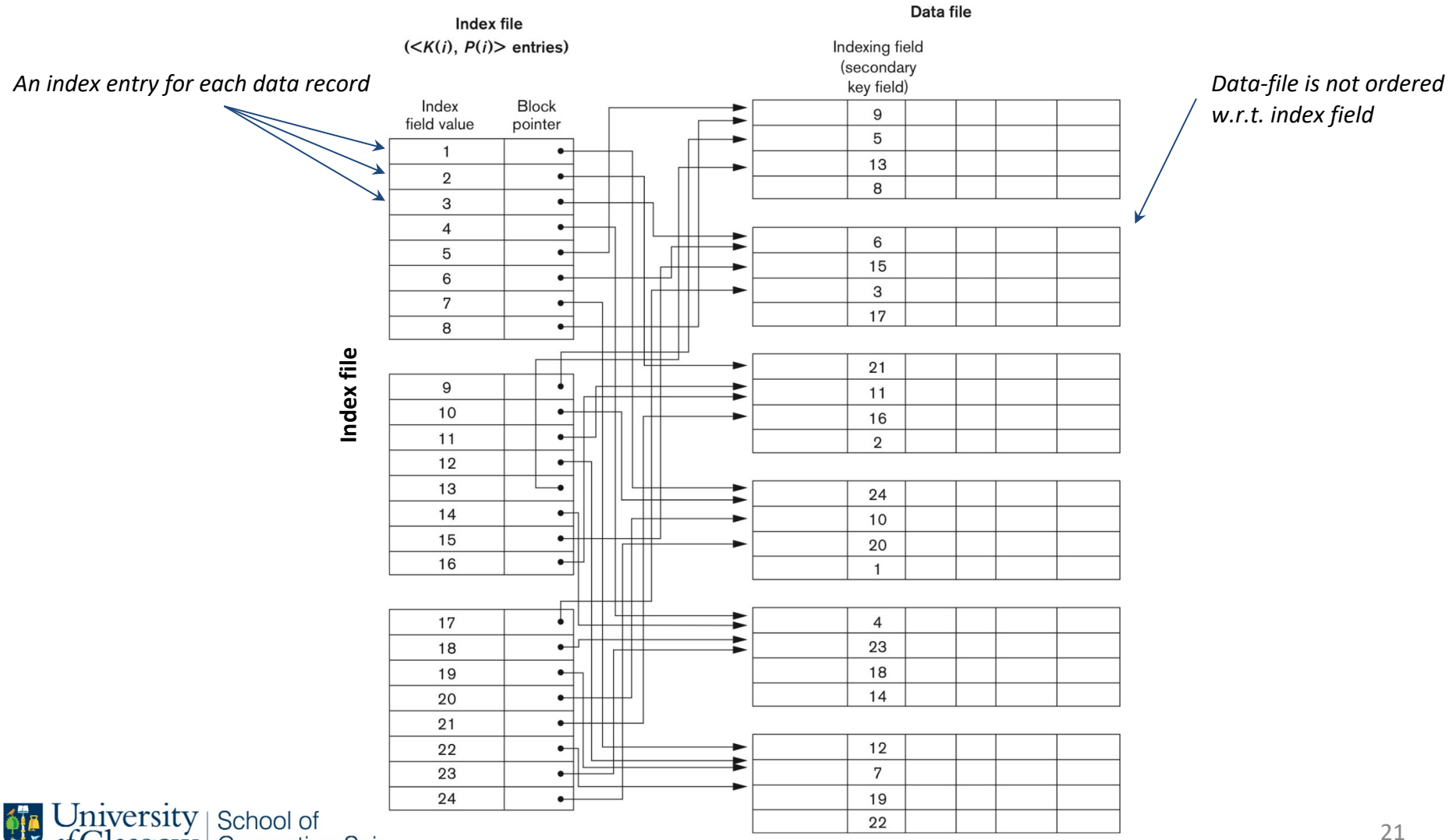
- Challenge: Index a file on a non-ordering field
  - The file might be unordered, hashed, or ordered but not according to the indexing field

- Cases:
  - [S1] Secondary Index on a non-ordering, key field; e.g., SSN
  - [S2] Secondary Index on a non-ordering, non-key field; e.g., DNO

- Type [S1]: One index entry per data record --> a **dense** index
- Why?
  - The file is not ordered on the indexing field --> we cannot use anchor records



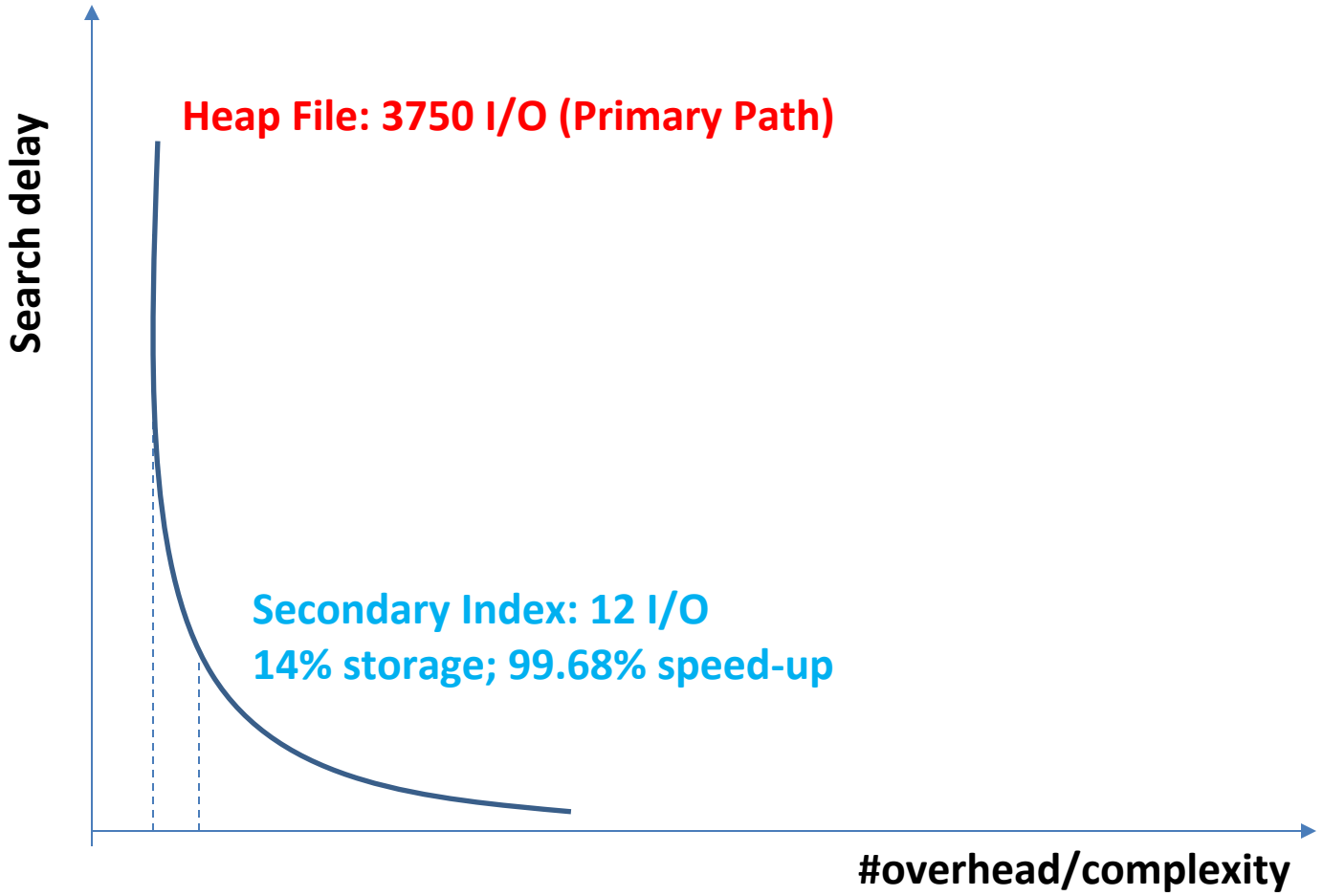
[S1] Secondary Index (non-ordering; key)



In-Class Activity

- Task: Secondary Index on a non-ordering, key attribute: ID
- File: r = 300,000 fixed-length records of size R = 100 bytes each; block size B = 4096 bytes; ID is V = 9 bytes, pointer P = 6 bytes
- Compute cost of SELECT * FROM EMPLOYEE WHERE ID = 'x'
  - Blocking factor bfr = floor(B/R) = 40 records per block
  - File blocks: b = ceil(r/bfr) = **7,500** data blocks
  - Linear Search on File: b/2 = **3,750 block accesses**
  - **We cannot do binary search since it is non-ordering**
  - Blocking factor of index: ibfr = floor(B/(V+P)) = **273 entries/block**
  - Secondary Index is dense: n = 300,000 index entries
  - Index File blocks ib = ceil(n/ibfr) = 1,099 blocks (14% overhead)
  - Cost: Binary Search on Index: ceil($\log_2$(ib)) = 11 block accesses
  - … plus one more block access to load the block pointed by the index
  - Total: 11 + 1 = **12 block accesses** (99.68% speed-up)
- We can do better by splitting the data space into more than two subspaces

Trade off: Overhead vs Speed



Secondary Index

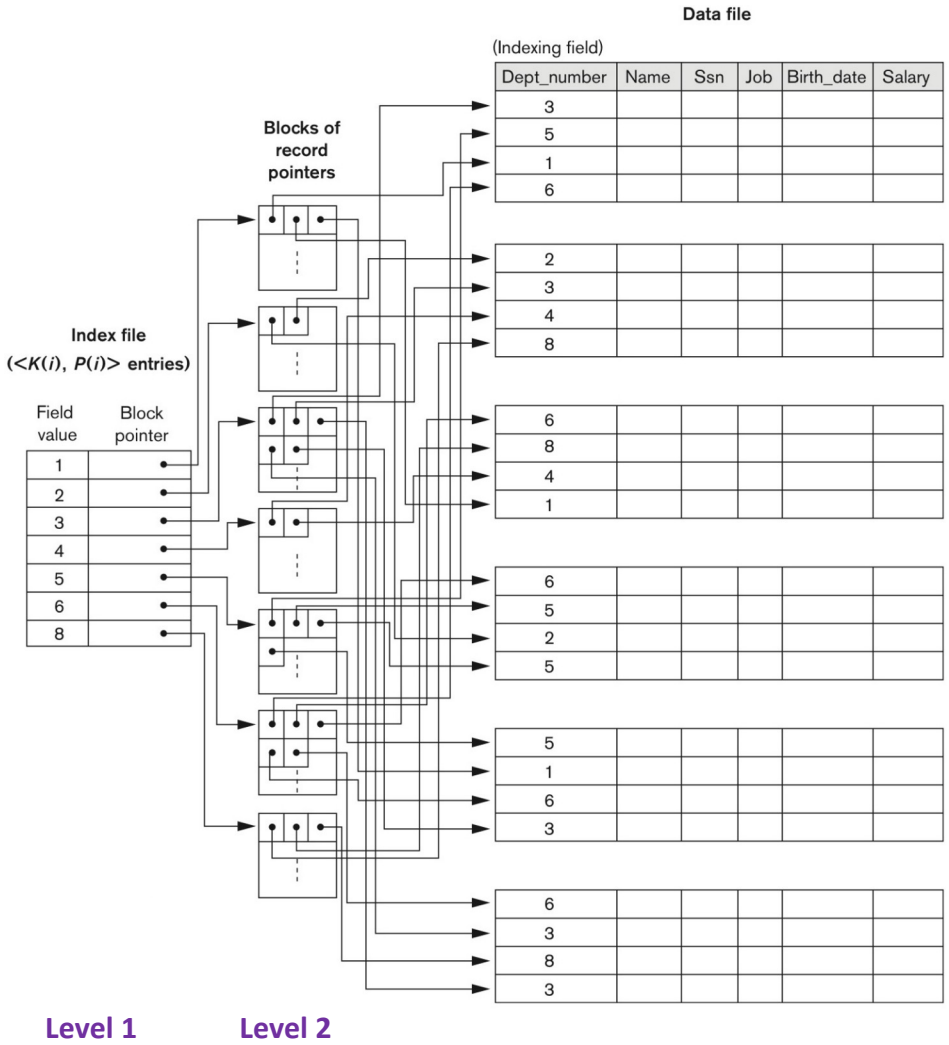
- Type [S2]: Indexing field is not a key & the file is not ordered w.r.t. indexing field
- Idea 1: cluster the block addresses of the records having the same index value
- Idea 2: assign an index entry per cluster (of block addresses)
- **index-entry** := {cluster-value, cluster-pointer}

Remember this trick; applicable in many cases

cluster pointers *block pointers* *records*

- A cluster-pointer points to:
  - (Level 1) a block of {block-pointers} of a cluster
  - (Level 2) a block-pointer points to the data-block that has records with this distinct index value

[S2] Secondary Index (non-ordering; non-key)



Note:

- Index is *sparse*
- One entry *per* cluster
- Level 1 is a *set* of blocks
- Each Level-1 block contains records of block pointers

Search for ID = 3:

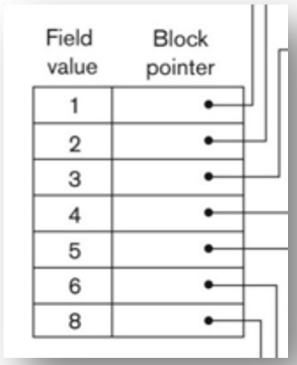
1. Binary search in Level-1 (1)
2. Direct access to Level-2 (1)
3. Load *all* the corresponding data-blocks (4)

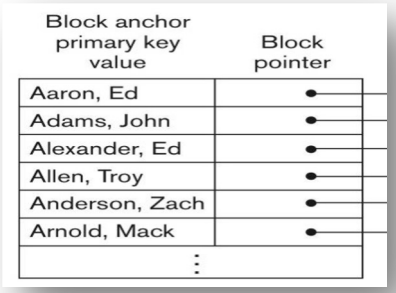
Total:

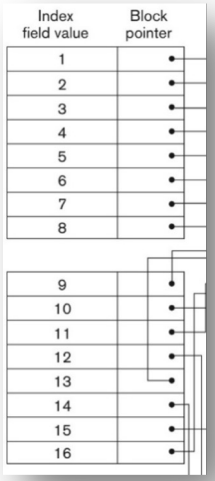
1 + 1 + 4 = 6 block accesses

Multilevel Index

- Observation: in all index files it holds true that:
  - They are ordered on the indexing field
  - The indexing field has unique (distinct) values
  - Each index entry is of fixed length







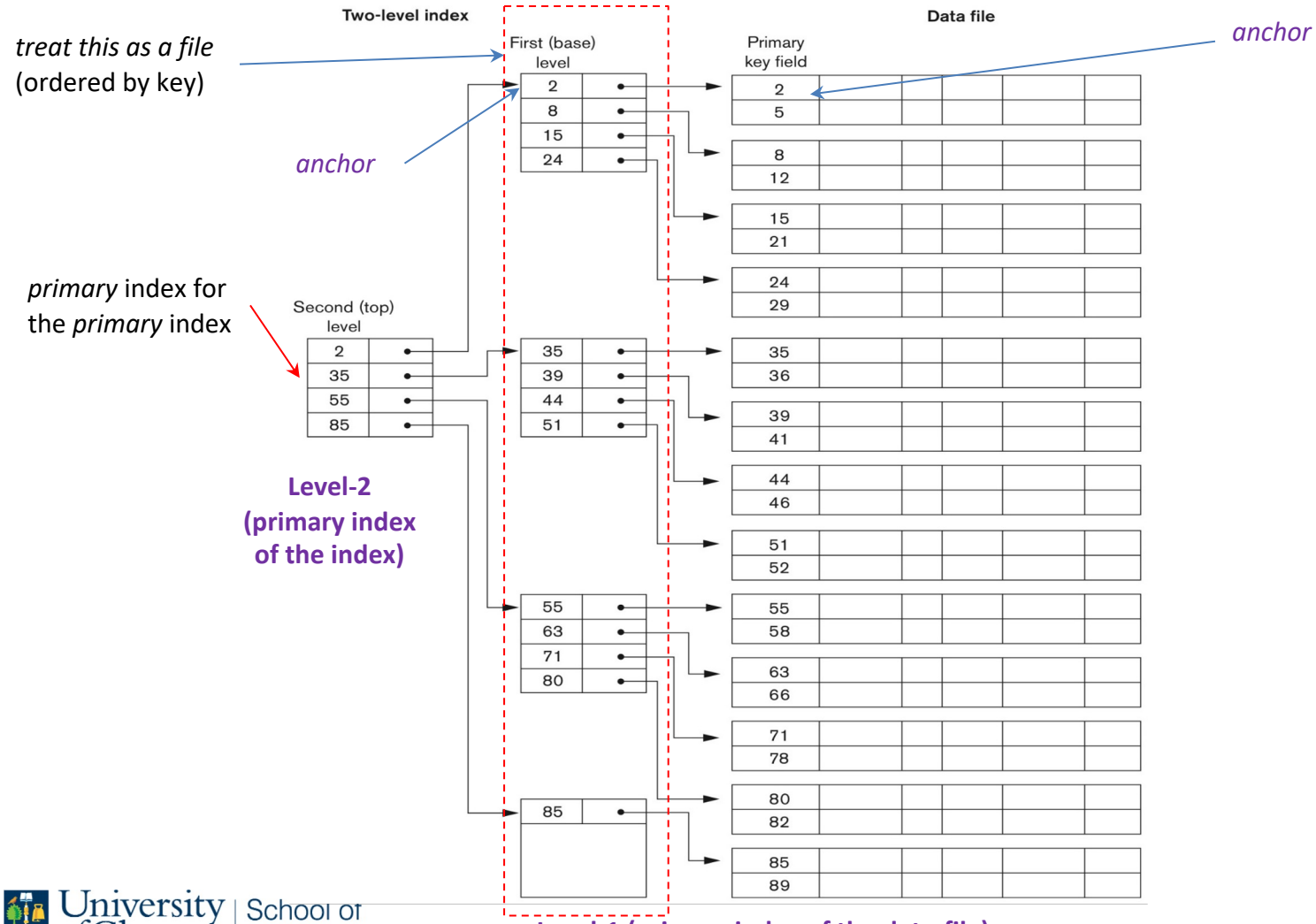
- Conclusion: we can build a primary index over any index file, since it is an ordered file w.r.t. a key field (*index of an index*)

Multilevel Index

- To index the index…
  - The original index file is referred to as the base or Level-1 index
  - The additional index is referred to as Level-2 index (index of an index)
  - …
  - if we repeat this to level > 2 we obtain… Level-t index, i.e., index of an index of an index …

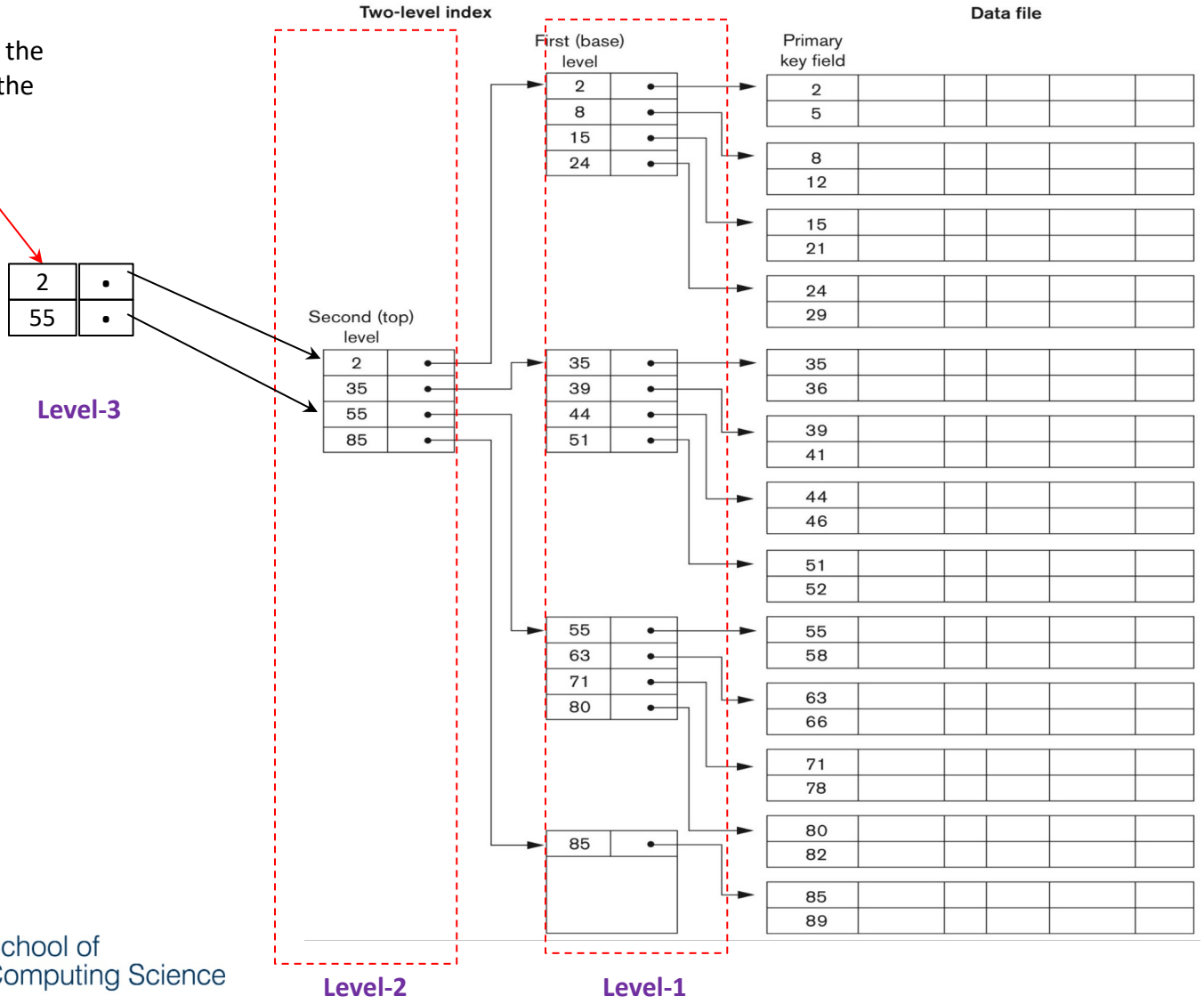
- Challenge: Identify the best level t of a multi-level index to expedite the search process trading off speed-up with overhead

2-level Index



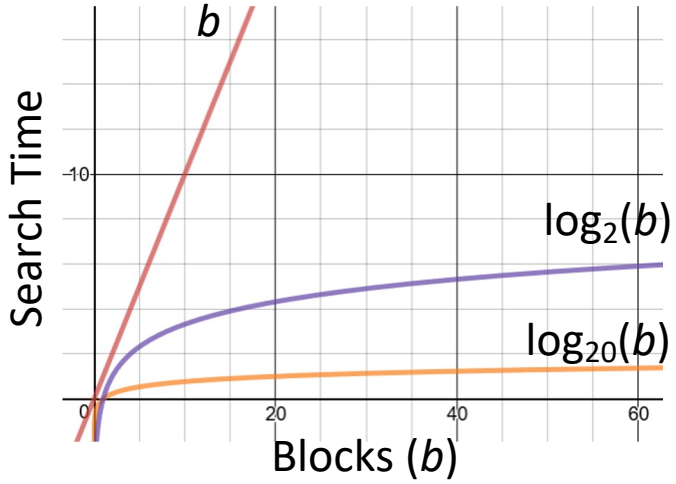
3-level Index

primary index for the
primary index of the
primary index



Multilevel Index: Reasoning

- **History**: Logarithms by John Napier (1550-1617) here in Scotland (Uni St Andrews)
- **Idea**: A logarithm with base m > 2 splits the search space into m sub-spaces
- **Target**: searching steps: $t = \log_m(b) < \ldots < \log_2(b)$ iff m > 2 (logarithm with base m)
- Theorem: Given a Level-1 Index with m being its blocking factor, then we need a multi-level index of Level $t = \log_m(b)$
  - Proof: soon…
- Note: m is also known as the fan-out



… and now the proof

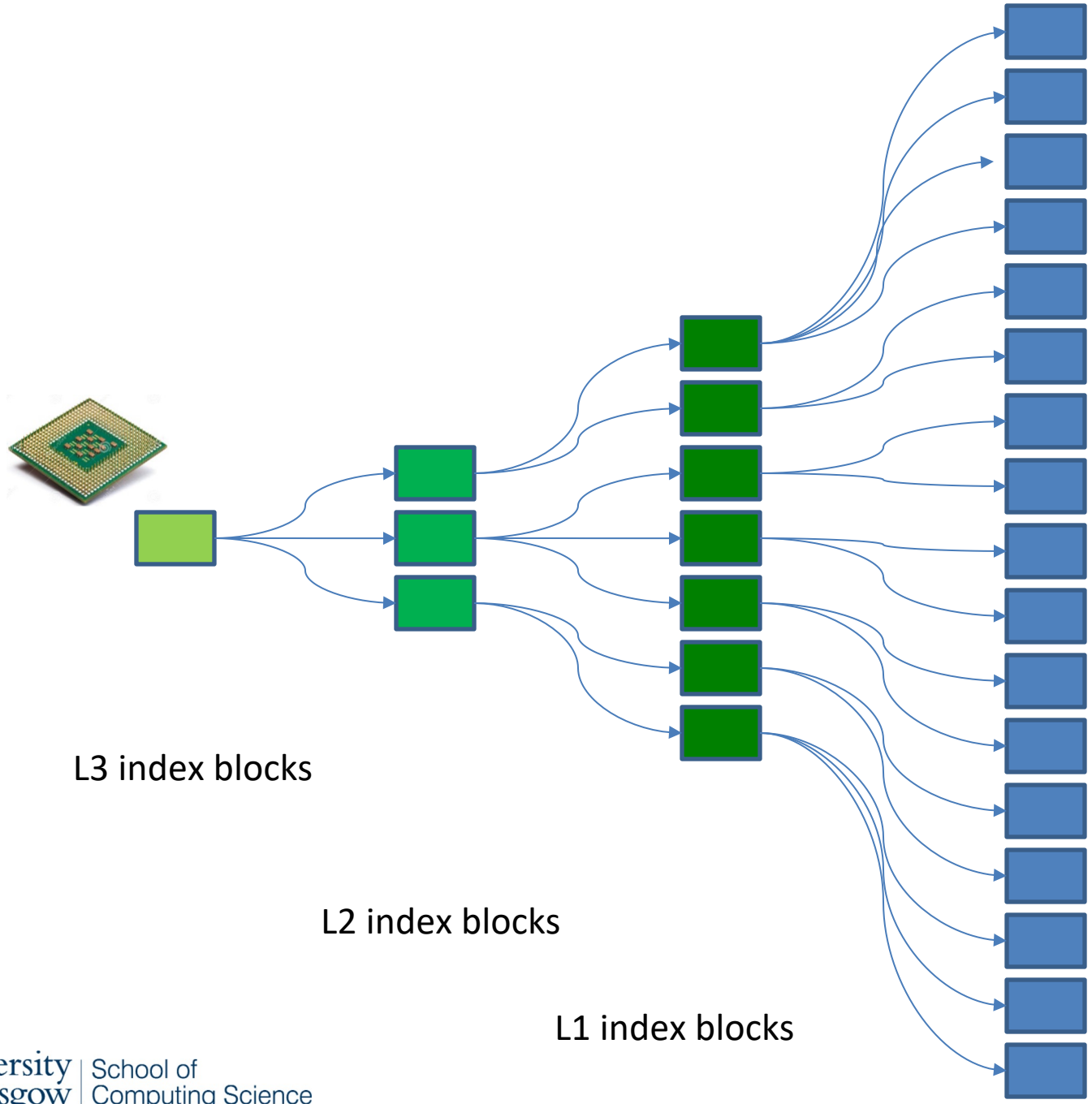
- Block size B bytes; File with r records; data-record has size s.
  - Blocking factor for the data-file is f = floor(B/s) records/block
  - Data file is b = ceil(r / f) data-blocks.

- Level-1 Primary index: each index entry points to a file-block (anchor)
  - Let l be the size of the index entry
  - The Level-1 index has b entries, with blocking factor m = floor(B/l)
  - The Level-1 index has b_1 = ceil(b / m) L1-index-blocks

- Level-2 Primary index: each index entry points to each index-block of Level-1
  - The Level-2 index has b_1 entries, with blocking factor m_2 = floor(B/l)
  - The Level-2 index has b_2 = ceil(b_1 / m) = ceil(b / m^2) L2-index-blocks (1/m^2 less blocks)

…

- Level-t primary index:
  - The t^{th} top level will have only 1 block thus 1 > (b / m^t) or **$t = \log_m(b)$**
  - Split the search space into m sub-spaces ≈ t steps to find the desired block


L3 index blocks
L2 index blocks
L1 index blocks
data file blocks

In-Class Example

- Amazing Gain: one block per level plus one data-block := t+1 block accesses

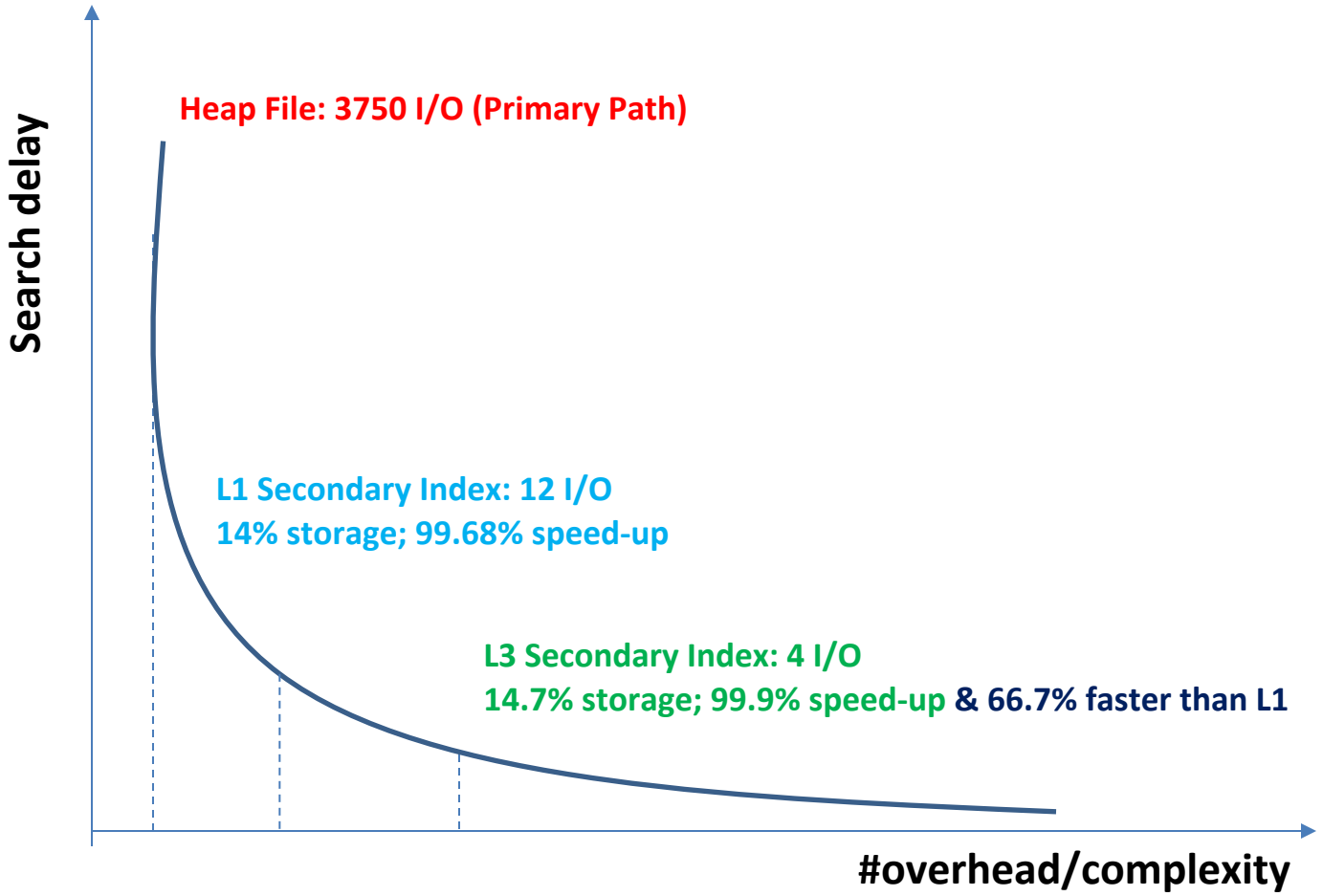
- Build a *non-ordering/key secondary multi-level index* over a file: b = 7,500 blocks and r =300,000 records of size R = 100 bytes each; block size B = 4,096 bytes; ID is V = 9 bytes, pointer P = 6 bytes

- Compute cost of SELECT * FROM EMPLOYEE WHERE ID = 'x'
  - Fan-out (index-blocking factor) m = **273 index entries/block**
  - Level-1: index with $\mathbf{b_1}$ **= 1,099 index-blocks**
  - Level-2: index entries: 1,099 thus, number of blocks $\mathbf{b_2 = ceil(b_1/m)}$ = **5 blocks**
  - Level-3: index entries: 5 thus, number of blocks $\mathbf{b_3 = ceil(b_2/m)}$ = **1 block**

- Structure := L1: Secondary Index (dense); L2 & L3: Primary Indexes (sparse)
- 3-level index search cost: 3 + 1 = **4 block accesses only!**

- Level-1 Secondary Index ceil($\log_2$(1099)) + 1 = 12 blocks accesses
- No Index: 3750 blocks accesses

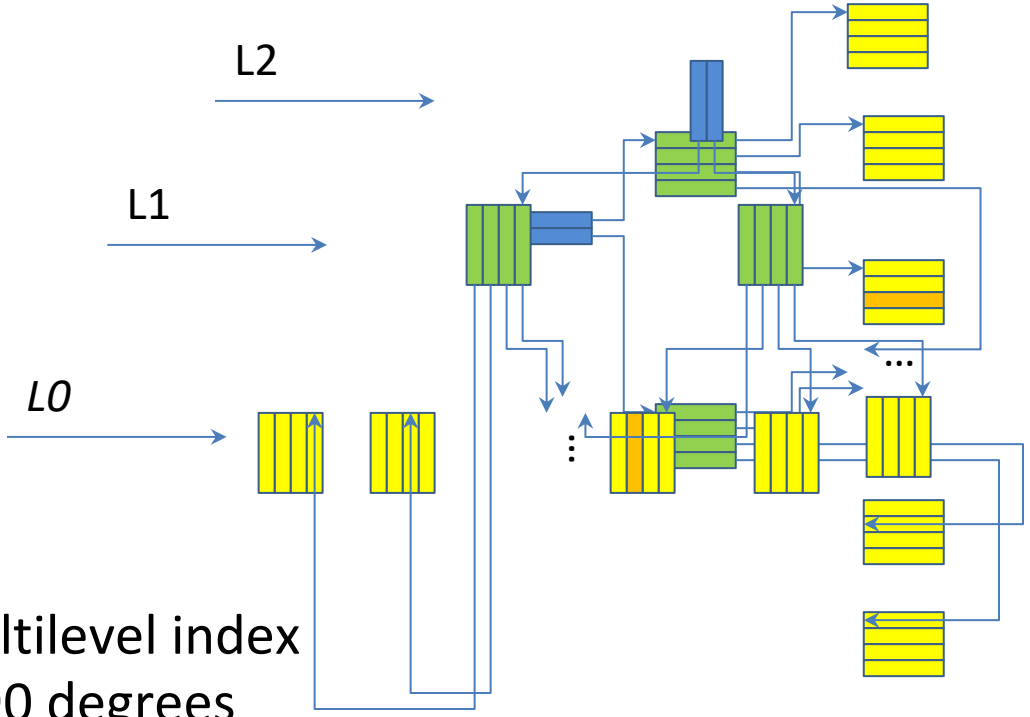
Trade off: Overhead vs Speed



Dynamic Multi-Level Indexes

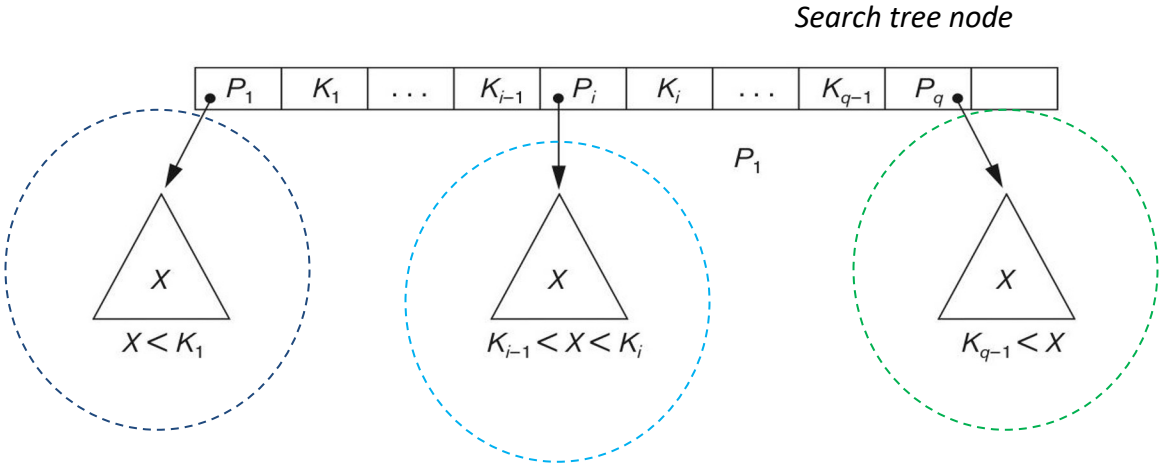
- **Summary so far:** Search for a tuple over a *t*-level index: ***t*+1 block accesses**
- **Challenge:** Insertions, deletions, and updates are costly!
  - **Why?**
  - **Because**: *all* encapsulated indexes are *physically* ordered files → *all* updates should be reflected to *all* levels
- Demand for *dynamic* multi-level indexes
  - I.e., indexes that expand and shrink with insertions/deletions of records

Search Tree



- Observe: 2-level multilevel index
- Turn: clockwise by 90 degrees
- Result: A Tree structure
  - Root is the L2-Index
  - Root's children are blocks of the L1-Index
  - Leaves are actual file data blocks (L0)

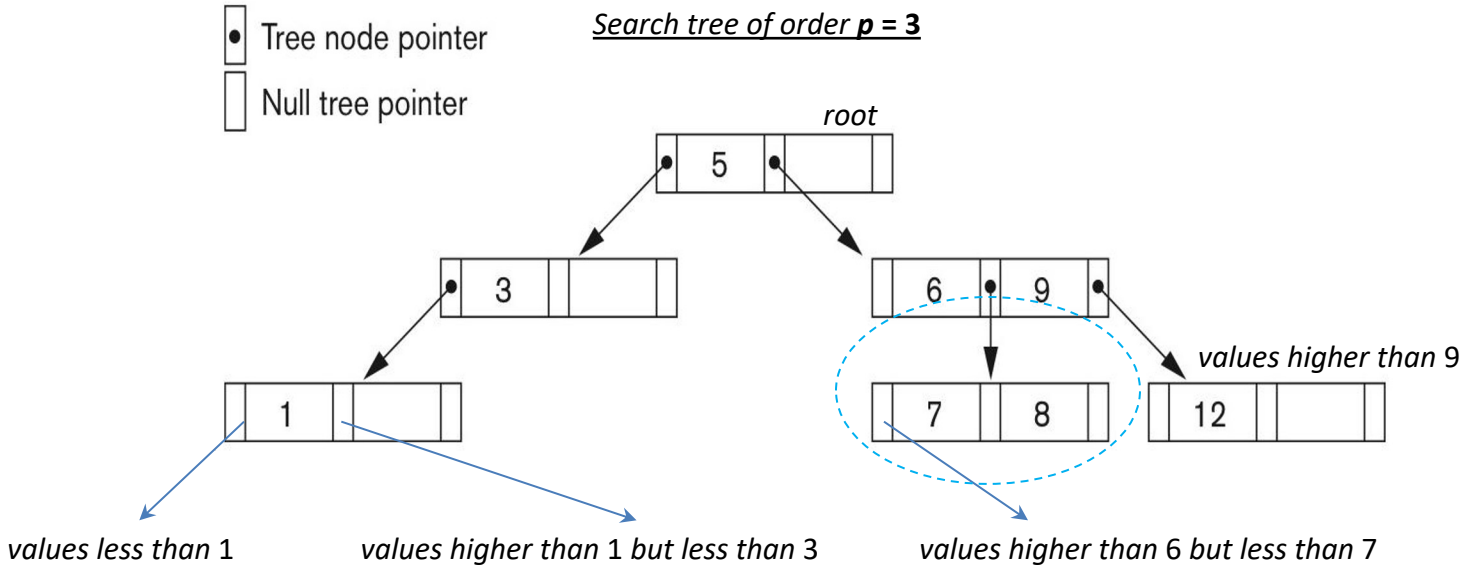
Search Tree over Non-ordering Key



- **Search Tree of order *p* (*splitting factor*) over a *key* field *k* (e.g., SSN)**
  - Each *node* has a set of *q* pairs: {*pointer, key*} where $q \leq p$:

    Tree-Node := {(P_1, **K_1**), (P_2, **K_2**), ..., (P_{q-1}, **K_{q-1}**), P_q}
    - Pointer P_i *either* points to another tree node *or* is NULL
    - Key K_i is a *search key* value: $K_1 < K_2 < K_i < ... < K_{q-1}$
  - For *all* key values **X** in a *subtree* pointed to by P_i :
    - $K_{i-1} <$ **X** $< K_i$, for $1 < i < q$
    - If $i = 1$, $X < K_1$ and if $i = q$, $K_{q-1} < X$

Search Tree over Non-ordering Key



- **Search for key:** *Follow* the tree pointers to and within a tree node
- Search: **13**; **8**; **10**; **0**
  - There is no 13
  - There exists 8
  - There is no 10
  - There is no 0

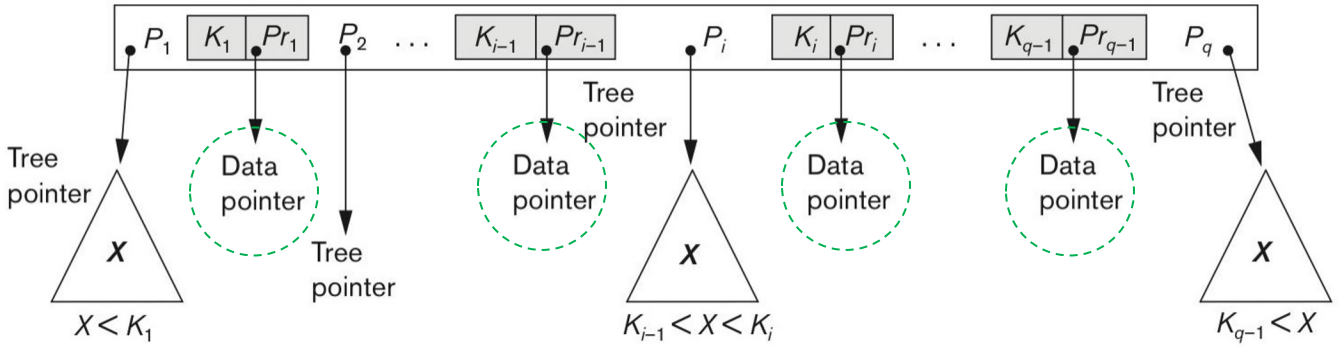
Search Tree: Limitations

- It can become unbalanced
- It does not adjust to the distribution of the keys; i.e., leaf-nodes are at different levels…
  - **Worst case:** a linked-list of nodes instead of a tree structure
  - Larger tree depth t results to higher expected search time O(t)

- Challenge 1: remove redundant levels, i.e., estimate the best level t
- Challenge 2: ensure balanced tree by minimizing the tree depth t
  - Challenge 2.1: what happens if key values are inserted in a full node? (split)
  - Challenge 2.2: what happens if the last key value in a node is deleted? (merge)
- The maintenance algorithms become very complex → Overhead
- **Merging & Splitting complexity: $O(p^t)$**

B-Tree over Non-ordering Key

- **B Tree of Order p:**
  - **B-Tree-Node** := {P_1, (K_1, Pr_1), P_2, (K_2, Pr_2) ..., P_{q-1}, (K_{q-1}, Pr_{q-1}), P_q}, where $q \leq p$

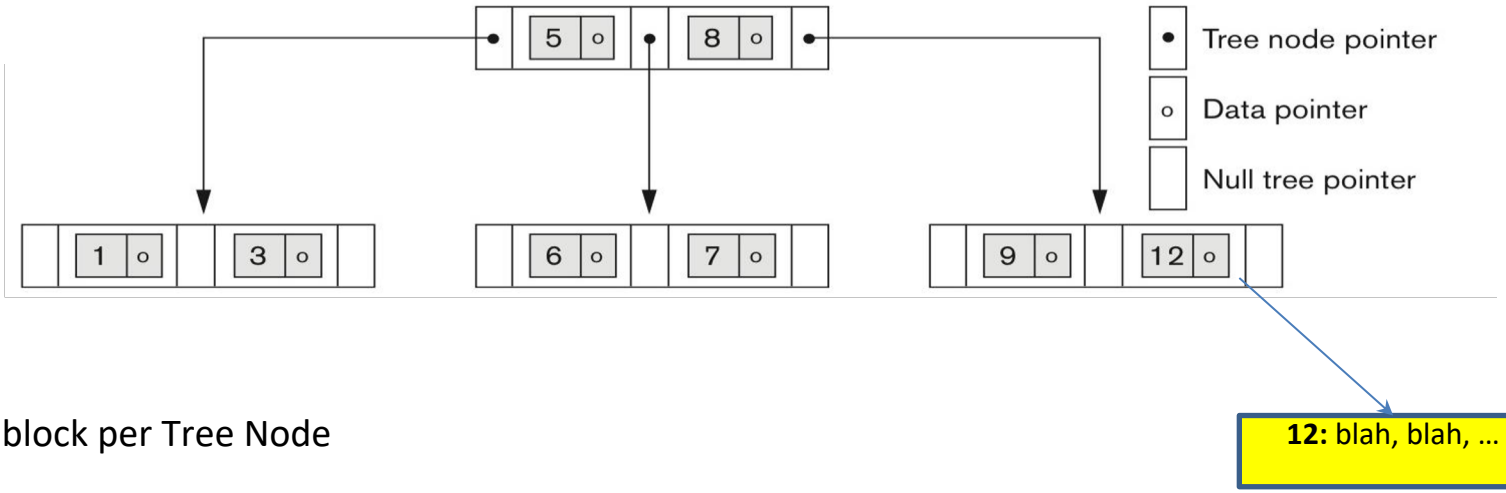


- Pr_i is a data-pointer to the data-block holding the value K_i
- Key order: $K_1 < K_2 < K_i < ... < K_{q-1}$
- For all key values X in a subtree pointed to by tree-pointer Pi:
  - $K_{i-1} < X < K_i$, for $1 < i < q$
  - If $i = 1$, $X < K_1$ and if $i = q$, $K_{q-1} < X$
- Each node with q tree-pointers has at most q-1 values & q-1 data-pointers

B-Tree

Search: Traversing the tree nodes until finding the record
Rationale: Immediate access to the block of the searching key!

B-Tree tree, p = 3 (**balanced tree**)

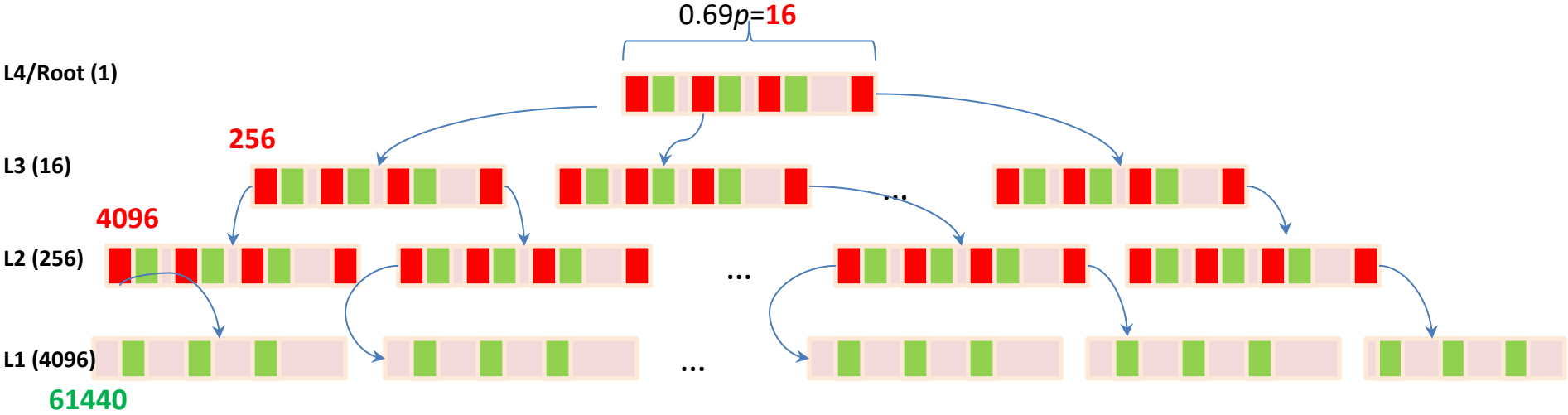


Assume: 1 block per Tree Node

Search 8: 1 index block access, then access the data block *immediately* → **2 block accesses**
Search 5: 1 index block access, plus data block access → **2 block accesses**
Search 7: 2 index block accesses, plus data block access → **3 block accesses**
Search 12: 2 index block accesses, plus data block access → **3 block accesses**
Search 31: 2 index block accesses; *no data block access needed* → **2 block accesses**

In-class Example

- Task 1: Create a 3-level B-Tree index of order p = 23 over a non-ordering / key field
- Assume each B-Tree node is 69% full of information (pointers/keys)
  - On average, each B-Tree node accommodates 0.69p = 16 tree pointers/ 15 key values
  - Average fan-out = 16 per tree node, i.e., split the tree space into 16 sub-trees



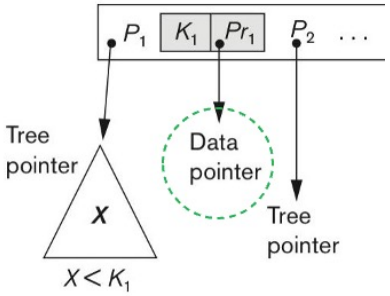
- Level-4/Root: 1 node with 15 keys/data-pointers & 16 pointers to tree nodes
- Level-3: 16 nodes with 16*15=240 keys/data-pointers;16*16 = 256 pointers to nodes
- Level-2: 256 nodes with 256*15=3840 keys/data-pointers; 256*16 = 4096 pointers to nodes
- Level-1: 4096 nodes with 4096*15 = 61440 keys/data-pointers; and null node pointers (leaves)

In-class Example

- How many keys are stored in this tree?
  - Structure:
    - Level-4/Root: 1 node with **15 keys**/data-pointers; 16 pointers to tree nodes;
    - Level-3: 16 nodes with 16*15=**240 keys**/data-pointers; 16*16 = 256 pointers to nodes;
    - Level-2: 256 nodes with 256*15=**3,840 keys**/data-pointers; 256*16 = 4096 pointers to nodes;
    - Level-1: 4096 nodes with 4096*15 = **61,440 keys**/data-pointers; null pointers (leaves);
  - There are: 61,440 + 3,840 + 240 + 15 = **65,535** key entries pointing to data blocks
- What if our file has 65,536 **(65,535+1)** keys to be indexed?
- What if our file has more than 279,840 keys to be indexed?
  - = 22 + 23 * 22 + 23 * 23 * 22 + 23 * 23* 23 * 22

In-class Example

- Assume block B = 512 bytes, data-pointer Pr = 7 bytes, tree-pointer P = 6 bytes, key V = 9 bytes
- How many bytes is the index?
- How many blocks is the index?



  - Storage (used for metadata):
    - Storage for data-pointers = 65,535*7 = 458,745 bytes (460KB)
    - Storage for key entries= 65,535*9 = 589,815 bytes (590KB)
    - Storage for tree-pointers = (4096+256+16)*6 = 4368*6 = 26,208 bytes (27KB)
    - **Total (metadata):** 1,074,768 bytes = **1.07MB index**
  - Storage (complete tree) :
    - Number of nodes: 1 + 16 + 256 + 4096 = 4369 nodes
    - Complete tree node size: 23*6 + 22*7 + 22*9 = 490 bytes
    - **Total:** 4369 * 490 = 2,140,810 bytes = **2.04MB**
  - Blocks:
    - Blocking factor: floor(512/490) = 1 tree-node per block (normally)!
    - 1 node per block, thus **4,369 blocks**!
    - **Actual space on disk:** 4,369 * 512 = 2236928 bytes = **2.13MB**

Maximize fan-out & minimize storage

- Hmmm, a B-Tree stores too much meta-data!
  - Data-pointers to data blocks (addresses, e.g., URI/L average size 647 bytes!)
  - Tree-pointers to tree nodes (structural meta-data)
  - Search key values (data values)

- Objective 1: Be more storage efficient… free up some space from the nodes. Why?
- Objective 2: Maximize the fan-out of a node. Why?

- Recall: The fan-out indicates the number of search-space splits (at every step)
  - Thought: The fan-out is represented through the number of tree-pointers per node
- Note:
  - Maximizing the number of tree-pointers per node maximises fan-out!
  - Data-pointers can be removed from the tree nodes!

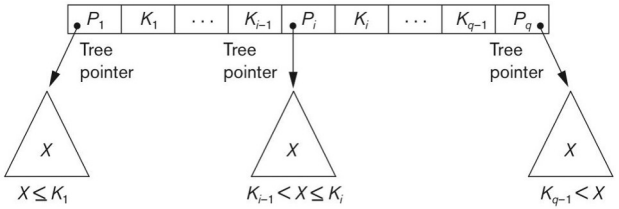
B+-Tree over Non-ordering Key

- Ta-dah: **B+ Tree** (give semantics to the nodes!)
  - Internal Nodes
  - Leaf Nodes
  - Principle: internal nodes have no data pointers to maximize fan-out

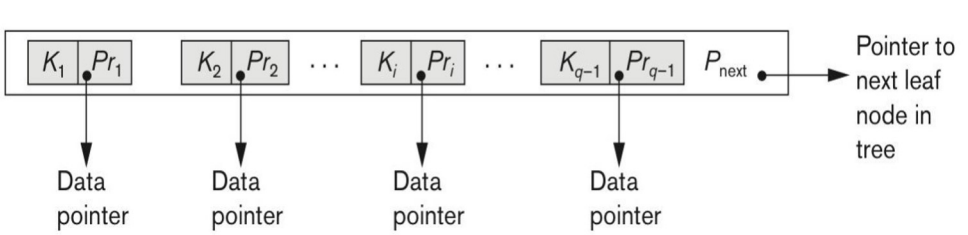
- Key ideas:
  1. Only Leaf Nodes hold the actual data-pointers thus removing them from the Internal Nodes
  2. Leaf Nodes hold all the key values and their corresponding data-pointers
  3. Some key values are replicated in the Internal Nodes to guide the search process
  4. Fusion/combination of ISAM Search Tree with B Tree

B+-Tree Nodes

- **Assume B+ Tree of order p**
- **B+-Internal-Node** := $\{P_1, K_1, P_2, K_2, ..., P_{q-1}, K_{q-1}, P_q\}$, $q \leq p$
  - Order: $K_1 < K_2 < ... < K_{q-1}$
  - For all key values X in a subtree pointed to by tree pointer P_i:
    - $K_{i-1} < X \leq K_i$, for $1 < i < q$
    - If $i = 1$, $X \leq K_1$ and if $i = q$, $K_{q-1} < X$
  - Note: Each internal node with q tree-pointers has at most q-1 key values

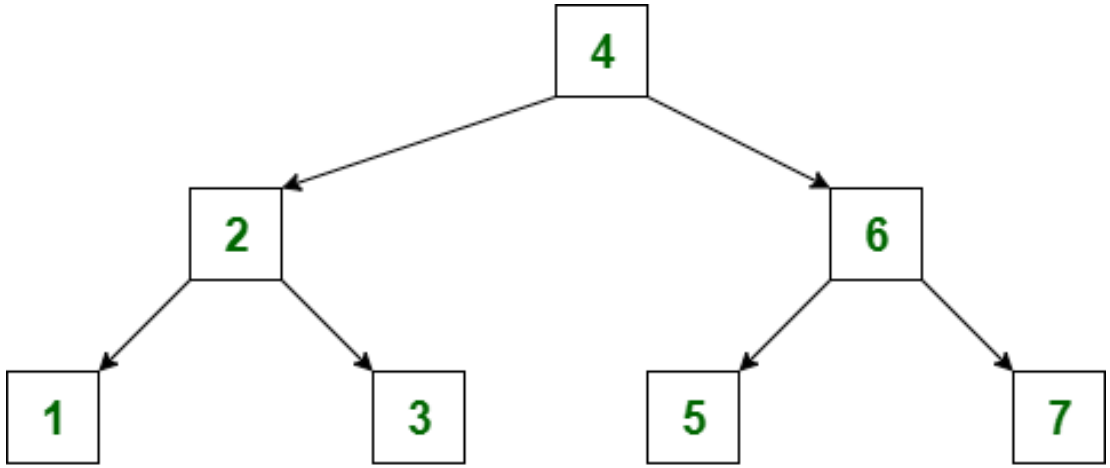


- **B+-Leaf-Node** := $\{(K_1, Pr_1), (K_2, Pr_2) ..., (K_{q-1}, Pr_{q-1}), P_{next}\}$, $q \leq p$
  - Pr_i is a data-pointer to the actual block holding value K_i
  - P_{next} is a tree-pointer to the next leaf node (sibling) → Linked-list of leaf-tree nodes!
  - All leaf nodes are at the same level → tree is balanced



B tree vs B+- tree

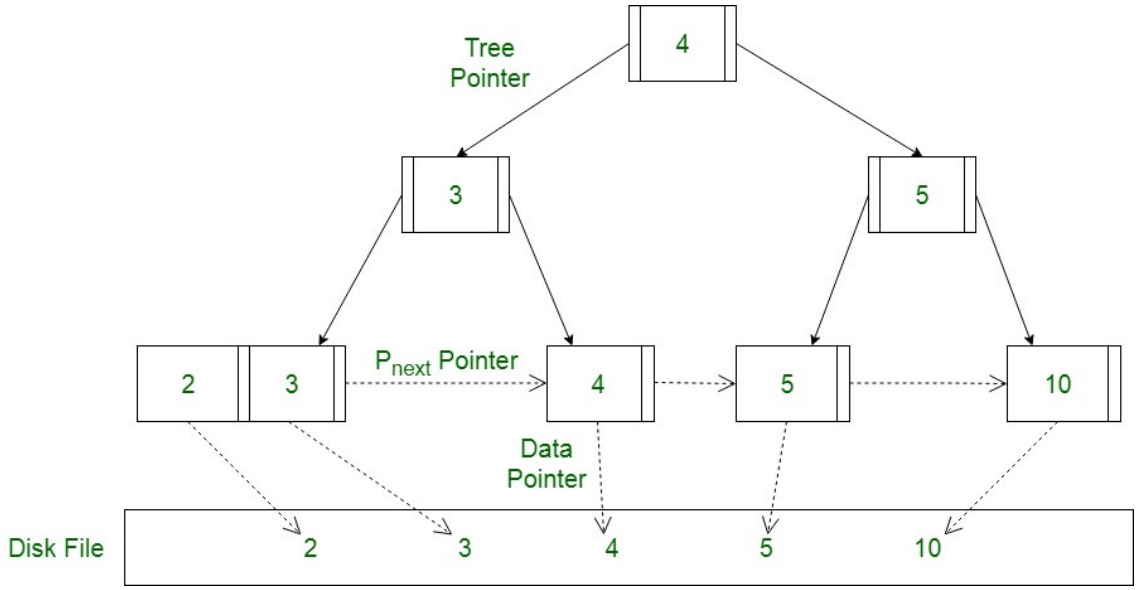
- B Tree:
  - All the leaf nodes of the B-tree must be at the same level.
  - Above the leaf nodes of the B-tree, there should be no empty sub-trees.
  - B- tree's height should lie as low as possible.



B-Tree

B tree vs B+- tree

- B+ Tree:
  - Only leaf nodes have data pointers
  - The leaf nodes are linked to provide ordered access to the records
  - Some of the key values of the leaf nodes also appear in the internal nodes



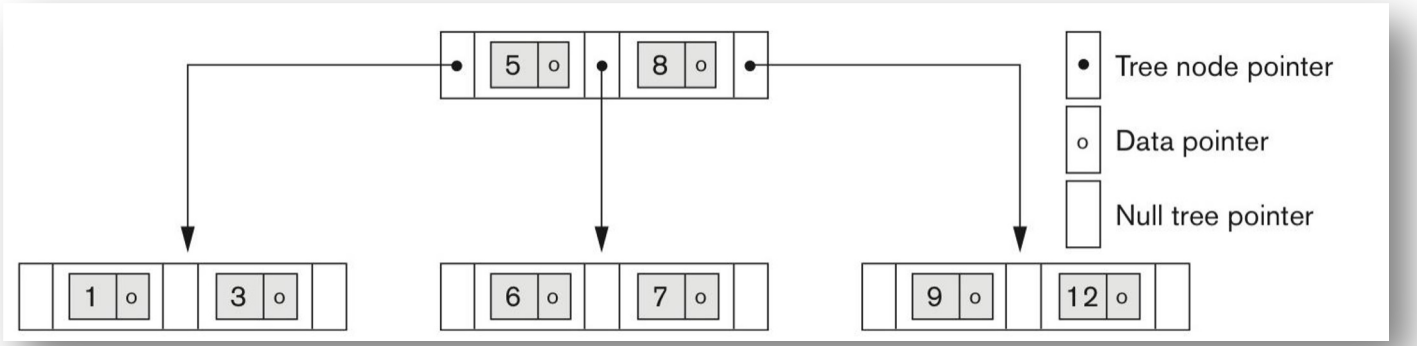
B Tree vs B+-Tree

Observation: No data-pointers in the internal nodes of B+ Tree

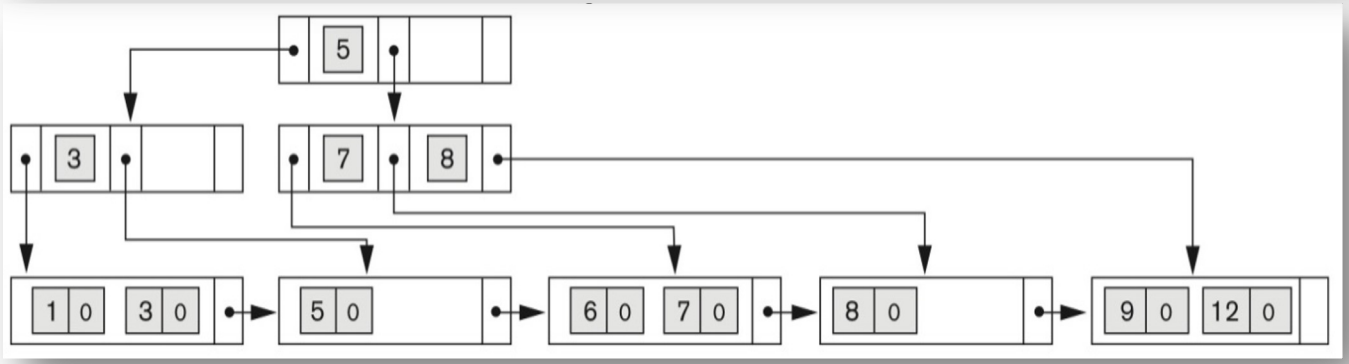
Observation: *values* are distributed *all* over the B Tree while stored *only* in the B+ Tree *leaves*

Target: More space in the nodes, *thus*, more tree-pointers, *thus*, higher fan-out

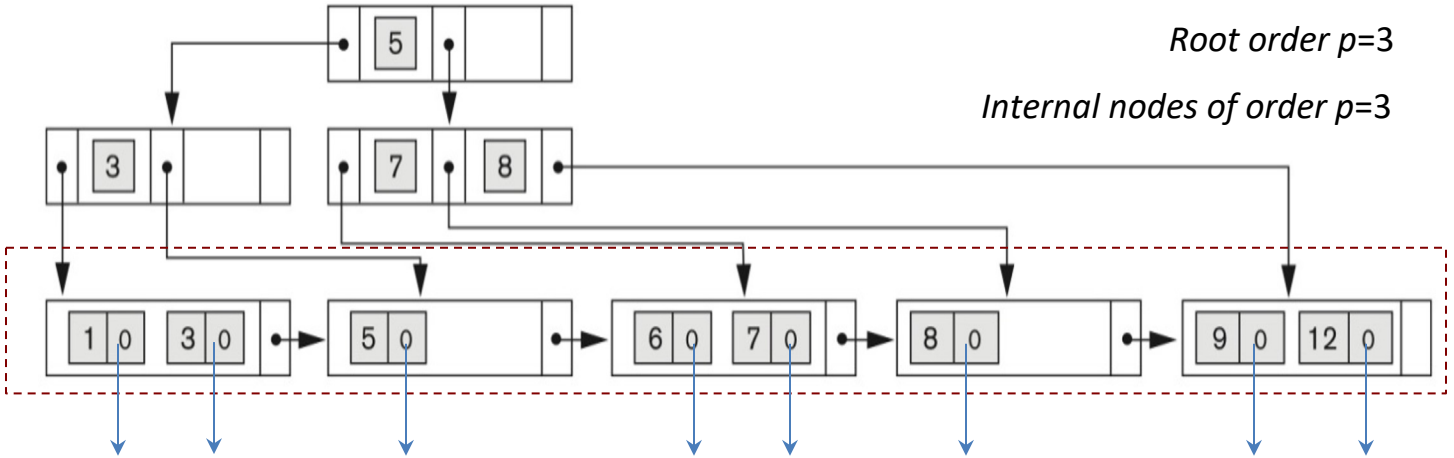
B Tree



B+ Tree



B+-Tree: Example

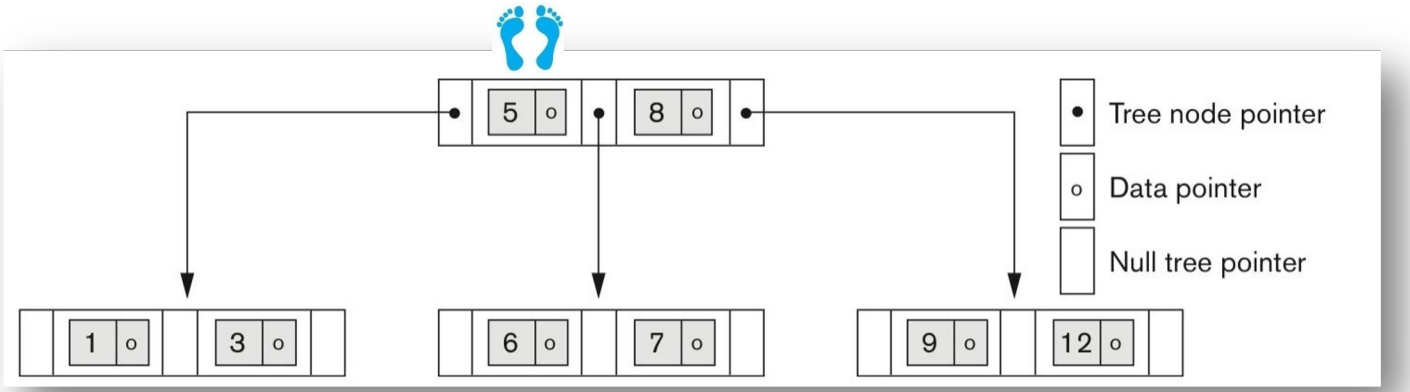


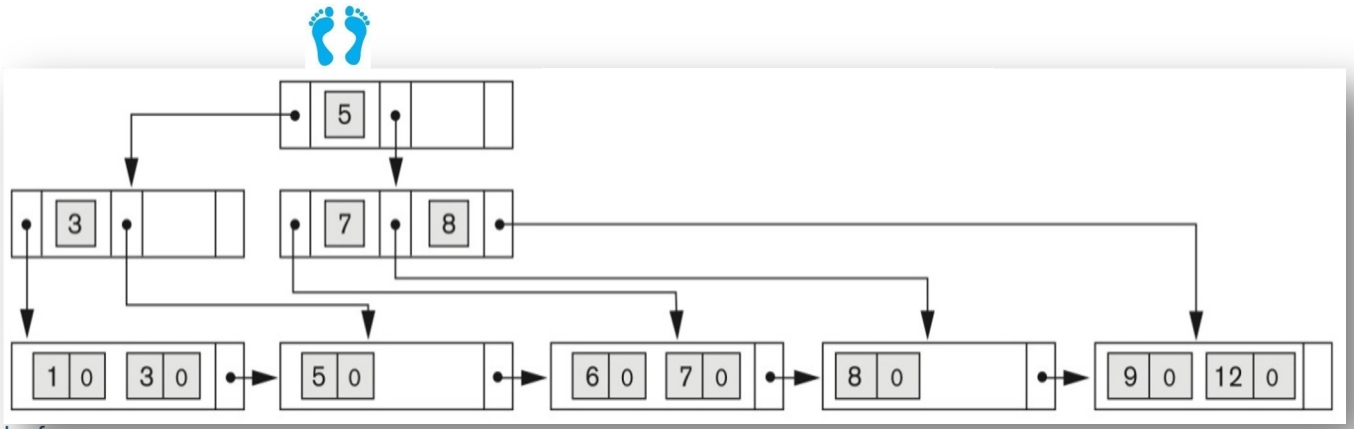
1. Leaf nodes are *linked* & *ordered* w.r.t. key (it is *good*; why?)
2. All *key* values appear at the Leaf nodes! (it is *good*; why?)
3. Leaf nodes contain data-pointers *only* (it is *good*; why?)
4. Leaf nodes are equi-balanced (it is *good*; why?)
5. *Some* selected keys are replicated in the internal nodes (*sparse & dense*)

B & B+-Tree

Range Query:

```
SELECT * FROM R WHERE key > 3 AND key < 10
```





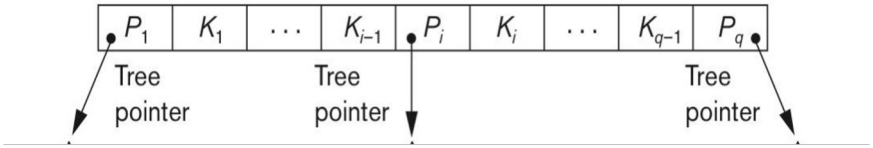
B+-Tree & B-Tree Example

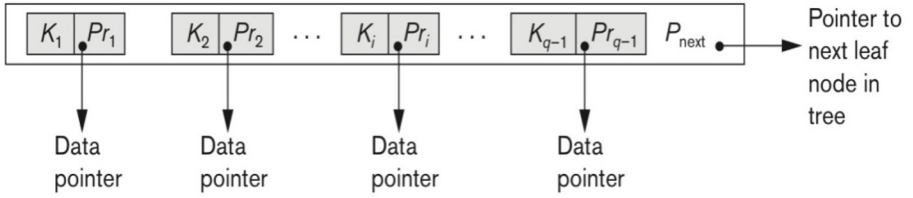
- Hypothesis: Removing the data-pointers from internal nodes results in higher fan-out thus more index-entries thus quicker search process
- Sounds plausible?

- Assume block size B = 512 bytes, key size V = 9 bytes, data-pointer size Pr = 7 bytes, tree-pointer size P = 6 bytes
- Calculate the order p of a B-Tree and a B+ Tree so that each node will fit in one block!
  - Recall: B-Tree nodes and internal B+ Tree nodes can have up to p tree pointers (p-1 key values)
  - B+ Tree internal node:
    - Size: p*P + (p-1)*V
    - To fit it into a block we have: p*P + (p-1)*V ≤ B → p ≤ (B + V) / (P + V)
    - The maximum order is **p = 34** (i.e., 34 tree pointers; 33 key values)
  - B+ Tree leaf node:
    - Size: pL data-pointers, pL key values, one next-tree pointer
    - To fit into a block we have: pL * (Pr + V) + P ≤ B → pL ≤ (B - P) / (Pr + V)
    - Each leaf node can store up to **pL = 31 pairs** of key-value/data-pointers
  - B Tree nodes:
    - Size: p-1 data-pointers, p-1 key values, p tree-pointers
    - To fit into a block we have: p*P + (p-1)*(V + Pr) ≤ B → p ≤ (B + V + Pr) / (P + V + Pr)
    - The maximum order for B Tree is **p = 24 < 34**

In-Class Activity

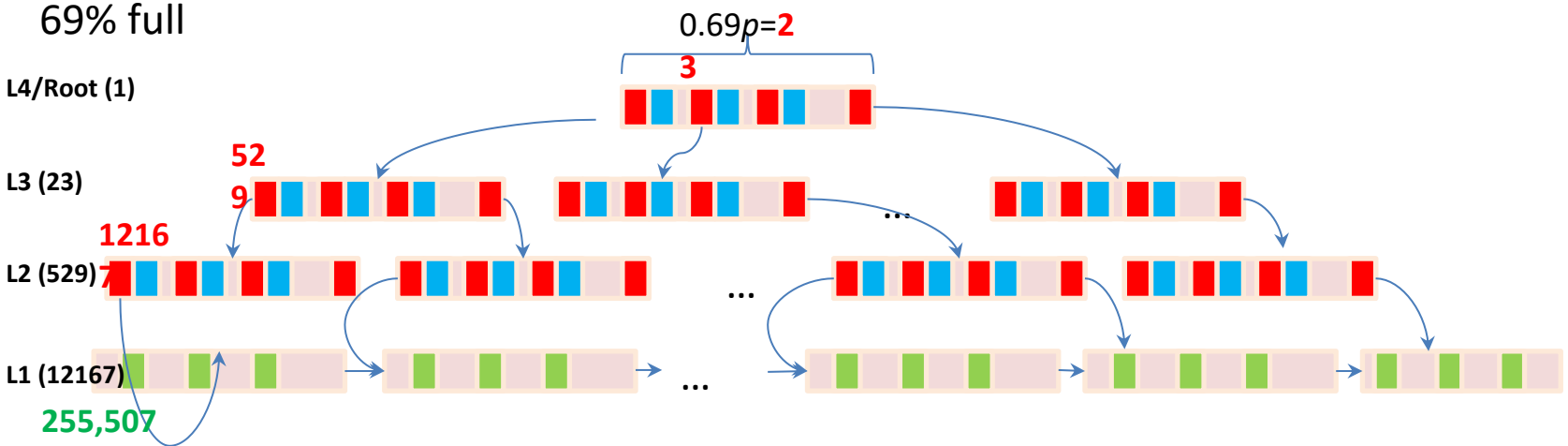
- Consider a 3-level B+ Tree index (i.e., 3 internal levels and one leaf level) of order p = 34 and pL = 31
- Assume that each internal & leaf tree node is 69% full
- **Task 1:** Compare its storage capacity to that of a 3-level B Tree in terms of #keys (earlier example -- 65,535 keys)
- **Task 2:** How many blocks can we store?





In-Class Activity

- 3-level B+ Tree index: 3 internal levels and one leaf level, order p = 34 and pL = 31, 69% full



- **Task 1:** Compare its storage capacity in terms of #keys to that of a 3-level B Tree (65,535 keys)
  - Internal tree-nodes have 0.69p = 23 tree-pointers and 22 keys (on average)
  - Leaf nodes have 0.69pL = 21 data-pointers (on average)
  - Level-4/Root: 1 node with 22 keys and 23 pointers to tree nodes
  - Level-3: 23 nodes with 23*22 = 506 keys and 23*23 = 529 pointers to nodes
  - Level-2: 529 nodes with 529*22 = 11638 keys and 529*23 = 12167 pointers to leaves
  - Level-1/Leaf Level: 12167 nodes with 12167*21 = **255,507 keys/data-pointers**

In-Class Activity

- **Task 2:** How many blocks do we store?
  - Level-4/Root: 1 node with 22 keys and 23 pointers to tree nodes
  - Level-3: 23 nodes with 23*22 = 506 keys and 23*23 = 529 pointers to nodes
  - Level-2: 529 nodes with 529*22 = 11,638 keys and 529*23 = 12,167 pointers to leaves
  - Level-1/Leaf: 12,167 nodes with 12167*21 = **255,507 keys**/data-pointers
  - Number of nodes: 1 + 23 + 529 + 12167 = 12720 nodes > **12,720 blocks!**
  - Compare:
    - B Tree (3-Level) stores 65,535 keys/data-pointers (at a fill factor of 69%)
    - B+ Tree (3-Level) achieves **290%** more capacity over the same file and same fill factor!
    - And, expected search cost is the same! 4 + 1 = 5 block accesses
    - And, if the file needs to store more than 65,535 records (e.g., 65,53**6** records) then we may need to define one more level for the B Tree; B+ Tree level does not change!

Putting it all together

- Assume:
  - Block size B = 512 bytes
  - File r = 100,000 records
  - File blocking factor: bfr = 10 records/block
  - Any pointer = 10 bytes
  - Indexing key = 10 bytes

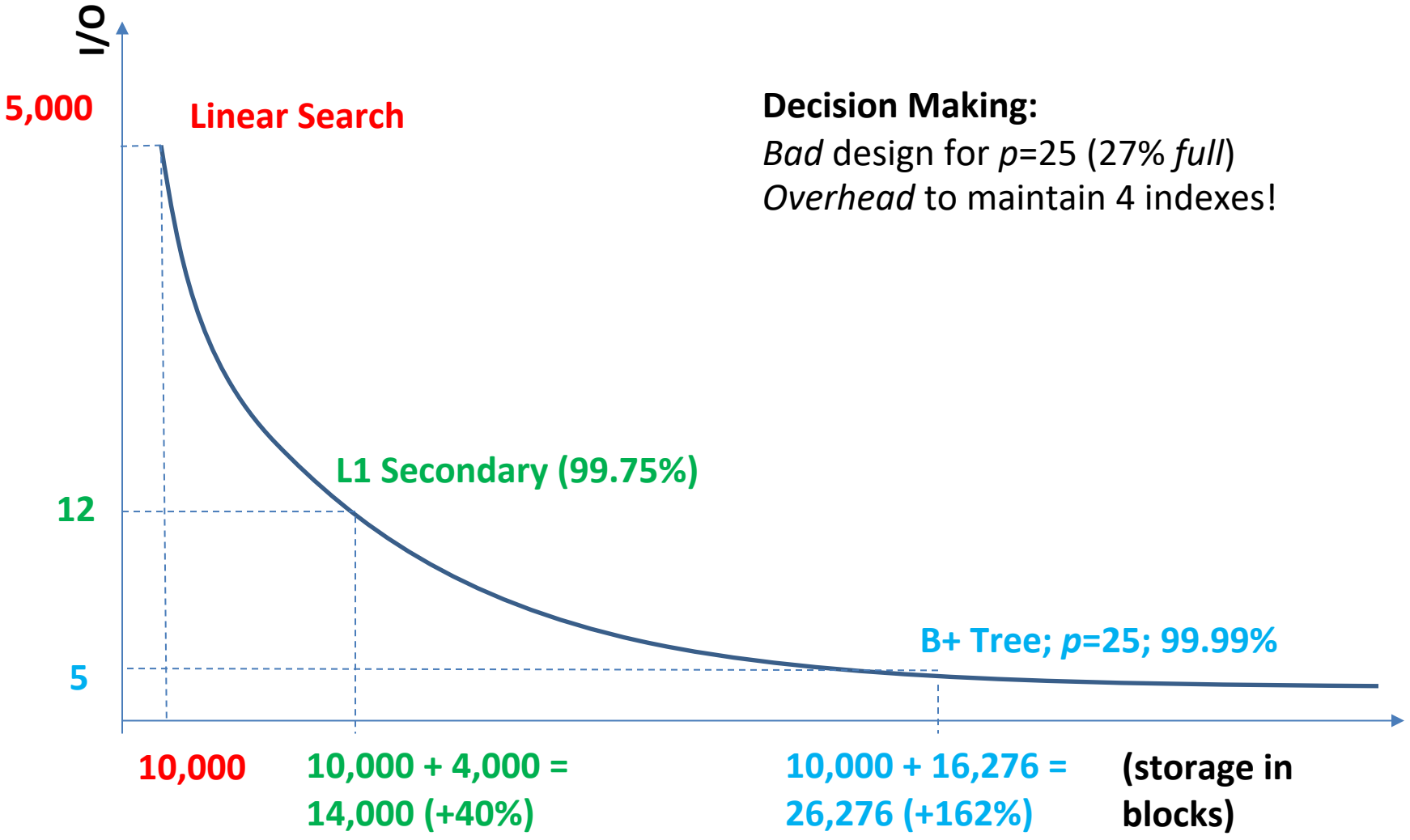
- Consider:
  - L1 Secondary Index on non-ordering key; index blocking factor = m
  - B+ Tree Index of order p = m; 100% full all nodes

- Project onto the Speed-vs-Overhead trade-off plane the performance of:
  - Expected cost of Linear Search on File vs File Storage
  - Expected cost of search on L1 Secondary Index vs Index Storage
  - Expected cost of search on B+ Tree Index vs B+ Tree Storage

Putting it all together

- Block size B = 512 bytes; File r = 100,000 records; blocking factor: bfr = 10 records/block; pointers = 10 bytes; Indexing key = 10 bytes
- File: b = 10,000 blocks → Linear Search on File b/2 = **5,000 block accesses**
- L1 Secondary Index:
  - L1 index-entry = 10 + 10 = 20 bytes
  - L1 blocking factor (fan-out) m = floor(512/20) = 25 entries/block
  - L1 blocks = ceil(r/m) = 4,000 blocks
  - Binary Search on L1 Index: $\log_2(4000)$ = **12 block accesses**
- B+ Index of order p = m = 25 (25 pointers; thus 24 keys)
  - Internal Node size: p*10 + (p-1)*10 = 490 bytes; 1 block per Internal Node
  - Leaf Node size: (p-1)*(10 + 10) + 1*10 = 490 bytes; 1 block per Leaf Node
  - B+ Root: 1 node with 24 keys and 25 pointers to tree nodes;
  - B+ L1: 25 nodes with 25*24 = 600 keys and 25*25 = 625 pointers to nodes;
  - B+ L2: 625 nodes with 625*24 = 15000 keys; 625*25 = 15,625 pointers to leaves;
  - B+ Leaf: 15625 nodes with 15625*24 = 375,000 keys/data-pointers
  - Reasoning: Only 27% (100,000/375,000) of the leaf nodes are occupied! → Redundancy!
  - Total: 15625 + 625 + 25 + 1 = **16,276 nodes/blocks!** (file is only 10,000 blocks)
  - B+ Tree search: 1 + 1 + 1 + 1 + 1 = **5 block accesses**

Trade off: Overhead vs Speed



Decision Making:

Bad design for *p*=25 (27% *full*)

Overhead to maintain 4 indexes!

Putting it all together

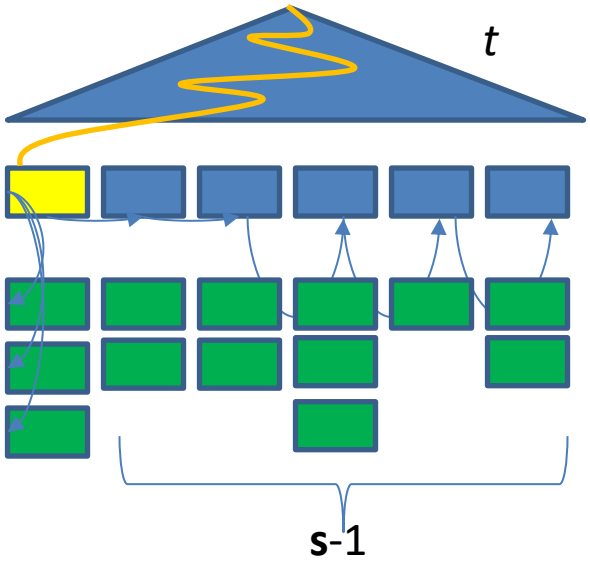
- Consider B+ Tree as Secondary Index of Level t, t > 1, over non-ordering key SSN (Range query)
  - SELECT COUNT(SSN) FROM EMPLOYEE WHERE Salary > £1000

- Context:
  - Salary is a non-unique attribute in the EMPLOYEE
  - B+ Tree occupies s blocks for all the leaf-nodes of order q, q-1 data-pointers/keys + 1 tree-pointer
  - All leaf nodes are 100% full
  - File has b blocks

- Tasks:
  1. Compute the expected cost in block accesses
  2. Compute the maximum level t as a function of b, s, q?
  3. Compute the maximum leaf-node order q?

Putting it all together

- Task 1: Methodology
  - 1. Follow the leftmost pointers to get to the leftmost leaf node
  - 2. There, load the data block for each SSN (visit the data-pointer)
  - 3. Count only the SSN values after checking in-memory: 'Salary > £1000'
  - 3. Follow the sibling leaf node (tree-next pointer)
  - 4. Repeat the same, i.e., GO TO 2
  - 5. Stop at the end, i.e., when tree-next pointer is NULL
  - Sum it up:
    - **t block accesses** to reach the leftmost leaf-node
    - **q-1 block accesses** for each leaf node to load data-blocks ('Salary > £ 1000')
    - Access the next s-1 blocks, i.e., loading **s-1 sibling leaf nodes**
  - Result: t + (q-1) + (s-1) + (s-1)(q-1) = **t + sq block accesses**



- Task 2: Which is the maximum level t as a function of b, s, q
  - B+ Tree Total: t + (q-1) + (s-1) + (s-1)(q-1) = **t + sq block accesses**
  - Naïve solution: just scan the whole file b block accesses
  - Benefit: t + sq < b → t < b – sq → **t_{MAX} = b – sq**
  - **Result:** IF B+ Tree Level > t_{MAX} THEN Linear Scan ELSE Use B+ Tree

- Task 3: Which is the maximum leaf-node order q
  - Since t_{MAX} > 1 then b – sq > 1 or **q_{MAX} = (b-1)/s**
  - **Result:** The blocking factor at the Leaf Level is bounded!

Special Thanks

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