Assignment 1: Brachytherapy

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Exercise 1

$$\min_{t\geq 0} \sum_{i\in I} \max\{0, \alpha_i(L_i - d_i^T t), \beta_i(d_i^T t - U_i)\} \quad (*)$$

Sets: I : set of points in the tumor and surrounding organs
J : set of dwell positions

Decision variables: t_j - dwell time at point $j \in J$
w_i - penalty at calculation point $i \in I$

Parameters: α_i, β_i penalty parameters $(i \in I)$
L_i - lower bound $(i \in I)$
U_i - upper bound $(i \in I)$
d_{ij} - dose rate from the dwell position j to location i $(j \in J, i \in I)$

Then the non-linear minimazation model $(*)$ can be transformed into a linear one:

$$\min_{t,w} \sum_{i\in I} w_i$$

$$\text{s.t.} \quad t_j \geq 0 \quad , j \in J$$

$$w_i \geq 0 \quad , i \in I$$

$$w_i \geq \alpha_i (L_i - d_i^T t) \quad , i \in I$$

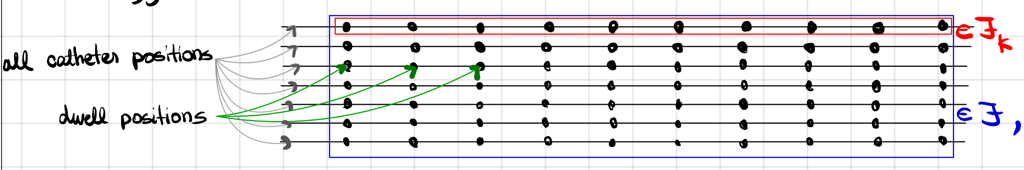
$$w_i \geq \beta_i (d_i^T t - U_i) \quad , i \in I$$

Exercise 2

As the model has to also entail the choice of catheter positions, the sets and the decision variables need to be modified.

Sets: I : set of points in the tumor and surrounding organs $(\{1, \ldots, L\})$
C : set of possible catheter positions $(\{1, \ldots, K\})$
J_k : set of dwell positions for catheter position $k \in C$ $(\{1, \ldots, D\})$ **

** Difference between J_k & J



, J is not 2-dimensional but 1-dimensiona
(vector consisting of the rows)

Decision variables: t_{kj} - dwell time for catheter in position $k \in C$ at dwell position $j \in J_k$
w_i - penalty at calculation point $i \in I$
c_k - catheter in position $k \in C$ (binary variable ***)

*** $c_k = \begin{cases} 1 & \text{if catheter is used in position } k \in C \\ 0 & \text{otherwise} \end{cases}$

Parameters: α_i, β_i } penalty parameters $(i \in I)$

L_i - lower bound $(i \in I)$

U_i - upper bound $(i \in I)$

d_{ijk} - dose rate from the dwell position j to location i (from catheter position k) $(j \in J_k, i \in I, k \in C)$

P - a very large number

Before writing down the minimization model it should be noted that $d_i^T t$ is no longer possible as t isn't a vector of all dwell positions in J. It is now a matrix where rows are the catheter positions $(k \in C)$ and columns are the dwell positions $(j \in J_k)$.

→ Using this information, the new optimization model is:

$$\min_{t,w} \sum_{i \in I} w_i$$

s.t.

(1) $t_{kj} \geq 0 \quad \forall k \in C, \forall j \in J_k$

(2) $w_i \geq 0 \quad \forall i \in I$

(3) $w_i \geq \alpha_i \left(L_i - \sum_{k=1}^{K} \sum_{j=1}^{D} d_{ijk} \cdot t_{kj} \right) \quad \forall i \in I$

(4) $w_i \geq \beta_i \left(\sum_{k=1}^{K} \sum_{j=1}^{D} d_{ijk} \cdot t_{kj} - U_i \right) \quad \forall i \in I$

d_{ijk} t_{kj}

d_{ik} t_k

$L \times K$ $K \times 1$

$L \times 1$

(5) $\sum_{k=1}^{K} c_k \leq N \quad \forall k \in C$

(6) $\sum_{j=1}^{D} t_{kj} \leq P c_k \quad \forall k \in C$

constraint to take entries of row t_k to be 0 if the catheter position k is not used

P is picked to be large enough so that no constraint is imposed for $\sum_{j=1}^{D} t_{kj}$ when catheter position k is picked

Exercise 3 The penalty to one organ should weigh more heavily than penalty to another organ.

Define a parameter vector a which has elements a_i $(i \in I)$. i represents the location in the tumor. Then define the elements in a as follows:

$$\text{for } i \in I: \qquad a_i = \begin{cases} d & \text{if } i \text{ is in the location of a healthy organ} \\ e & \text{otherwise} \end{cases}$$

Here, d & e are constants where $d > e$ as exposure of the dosage should be smaller for healthy organs.

↳ The constraints stay the same but the objective changes:

$$\min_{t,w} \sum_{i \in I} a_i w_i$$

Exercise 4 The doses to the individual calculation points should not differ too much from one another.

The objective & constraints do not change. However, new constraints and a parameter are added

$$\left| \sum_{j=1}^{D} \sum_{k=1}^{K} d_{ijk} t_{jk} - \underbrace{\frac{1}{L \cdot D \cdot K} \sum^{\forall l} \sum^{\forall j} \sum^{\forall k} d_{ljk} t_{jk}}_{\text{mean}} \right| \leq \varepsilon$$

↳ 2 new constraints

$$(7) \quad \sum_{j=1}^{D} \sum_{k=1}^{K} d_{ijk} t_{jk} - \frac{1}{L \cdot D \cdot K} \sum_{k=1}^{K} \sum_{j=1}^{D} \sum_{l=1}^{L} d_{ljk} t_{jk} \leq \varepsilon \qquad \forall \; i \in I$$

$$(8) \quad \frac{1}{L \cdot D \cdot K} \sum_{k=1}^{K} \sum_{j=1}^{D} \sum_{l=1}^{L} d_{ljk} t_{jk} - \sum_{j=1}^{D} \sum_{k=1}^{K} d_{ijk} t_{jk} \leq \varepsilon \qquad \forall \; i \in I$$

where ε is a parameter (the larger ε is, the more the doses to the individual calculation points can differ)

Exercise 5 Two adjacent catheter positions may not be chosen.

Again objective & constraints do not change. However, a new constraint and a set is added:

A_k - set of all catheter positions adjacent to catheter point k, including the position k itself $(A_k \subseteq C)$

$A = \{A_1, A_2, \dots, A_K\}$

↳ the constraint is

(9) $\sum^{A_k} c_j \leq 1 \quad , \forall A_k \in A$

Exercise 6 The dwell times are multiples of 0.1 seconds.

Replace all t with $0.1 t^*$ where $t^* = 10t$ is an integer $(t^* \in \mathbb{Z})$
This only changes constraints (1), (3), (4), (6) and an additional constraint is added:

Using t

(1) $t_{kj} \geq 0 \quad \forall k \in C, \forall j \in J_k$

(3) $w_i \geq \alpha_i \left(L_i - \sum_{k=1}^{K} \sum_{j=1}^{D} d_{ijk} \cdot t_{kj}\right) \quad \forall i \in I$

(4) $w_i \geq \beta_i \left(\sum_{k=1}^{K} \sum_{j=1}^{D} d_{ijk} \cdot t_{kj} - U_i\right) \quad \forall i \in I$

(6) $\sum_{j=1}^{D} t_{kj} \leq P c_k \quad \forall k \in C$

⟹

Using t^*

(1) $0.1 t^*_{kj} \geq 0 \quad \forall k \in C, \forall j \in J_k$

(3) $w_i \geq \alpha_i \left(L_i - \sum_{k=1}^{K} \sum_{j=1}^{D} d_{ijk} \cdot 0.1 t^*_{kj}\right) \quad \forall i \in I$

(4) $w_i \geq \beta_i \left(\sum_{k=1}^{K} \sum_{j=1}^{D} d_{ijk} \cdot 0.1 t^*_{kj} - U_i\right) \quad \forall i \in I$

(6) $\sum_{j=1}^{D} 0.1 t^*_{kj} \leq P c_k \quad \forall k \in C$

(10) $t^* \in \mathbb{Z}$

Exercise 7

The maximal dose divided by the minimal dose in calculation points in the urethra should be smaller than K^* new parameter

Define a new set : U - set of all calculation points in the urethra

$$\max\{d_{ijk} \cdot t_{jk}\} \leq K^* \min\{d_{ijk} \cdot t_{jk}\} \quad \forall\ i \in U$$

$\hookrightarrow$ Define variable s and r which leads to new constraints:

(11) $s \leq K^* \cdot r$

(12) $s \geq d_{ijk} t_{jk} \qquad \forall\ k \in C, \forall j \in J_k, \forall i \in U$

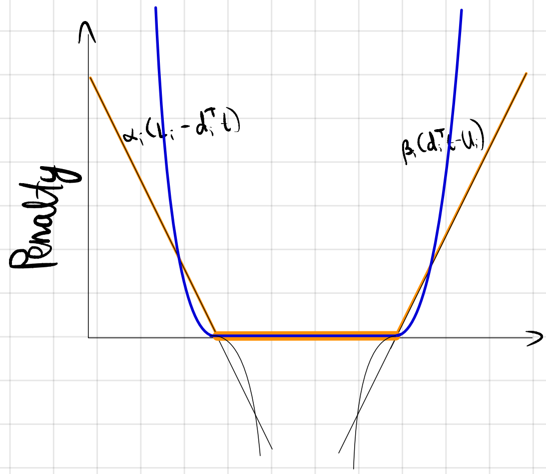
(13) $r \leq d_{ijk} t_{jk} \qquad \forall\ k \in C, \forall j \in J_k, \forall i \in U$

Exercise 8

The penalty increases more than linearly with the deviation from the prescribed dose

The non-linear optimization in exercise 1 had objective:

$$\min_{t \geq 0} \sum_{i \in I} \max\{0, \alpha_i (L_i - d_i^T t), \beta_i (d_i^T t - U_i)\}$$



As can be seen an increase in deviation from the prescribed dose does not make the penalty much larger (it is linear)

To make sure that large deviations suffer from much larger penalty than smaller ones, use cubic functions instead of linear ones for the penalty:

$$\min_{t \geq 0} \sum_{i \in I} \max\{0, (\alpha_i (L_i - d_i^T t))^3, (\beta_i (d_i^T t - U_i))^3\}$$

Larger α_i and/or β_i implies steeper curves