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Epidemics on critical random graphs with heavy-tailed degree distribution

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ABSTRACT

We study the susceptible–infected–recovered (SIR) epidemic on a random graph chosen uniformly over all graphs with certain critical, heavy-tailed degree distributions. We prove process level scaling limits for the number of individuals infected on day h on the largest connected components of the graph. The scaling limits contain non-negative jumps corresponding to some vertices of large degree. These weak convergence techniques allow us to describe the height profile of the α-stable continuum random graph (Goldschmidt et al., 2022; Conchon-Kerjan and Goldschmidt, 2023), extending results known in the Brownian case (Miermont and Sen, 2022). We also prove model-independent results that can be used on other critical random graph models.

1. Introduction

Consider the following simple susceptible–infected–recovered (SIR) model of disease spread in discrete time. On day 0, a single individual becomes infected with a disease. On day 1, that single infected individual comes into contact with some random number (possibly zero) of non-infected individuals and transmits the disease. After transmitting the disease to others, this initial infected individual is cured and can never catch the disease again. On subsequent days each infected individual does the same thing: they come into contact with some non-infected individuals, transmit the disease but then are cured. The study of how the disease spreads over time naturally gives rise to a graph [1] constructed in a breadth-first order, see Fig. 1 for an example of a small outbreak and Fig. 2 for an example of a larger outbreak. The individuals are represented by vertices, and an edge between two vertices represents that a vertex closer to the source transmitted the disease to the other. Knowing the graph and the source tells us more information than the number of individuals infected on a particular day, it tells us the history of how the disease spread from individual to individual.

The number of people eventually infected in the course of the outbreak then corresponds to the number of vertices (or size) of a connected component in the graph and, more importantly for our work, the number of people infected on day $h = 0, 1, \cdots$ is just the number of vertices at distance h from a root vertex corresponding to the initially infected individual. Let $Z_n(h)$ represent the number of people infected on day $h \geq 0$ when the total population is of size n. The process $Z_n(h)$ is just the *height profile* of the component containing the initially infected individual, where the height of a vertex v in a rooted graph is simply its graph distance to the root. We are interested in describing $n \to \infty$ scaling limits of $Z_n(h)$ for the macroscopic outbreaks for certain critical random graphs which exhibit a "super-spreader" phenomena — that is they possess vertices with large degree.

A classical probabilistic model in this area is the so-called Reed–Frost model, where each individual comes into contact with every non-infected individual independently with probability p. It is not hard to see that the corresponding graph is the Erdős-Rényi random graph $G(n, p)$ where each edge is independently added with probability p. This object is well-studied, and we know that in

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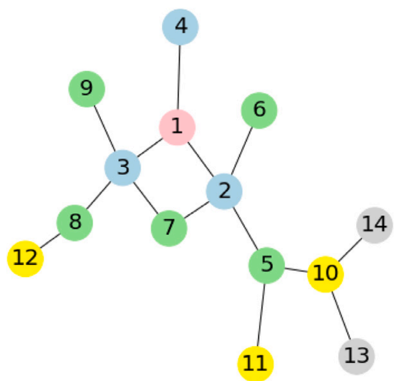

Fig. 1. A small outbreak. Here, on day 0 the vertex labeled 1 is infected. The vertex 1 transmits the disease to vertices 2, 3 and 4 (in blue) who become the infected population on day 1. The vertices infected on day 1 will infect the green vertices (5 through 9) who are infected on day 2. This continues with the yellow vertices becoming infected on day 3, and the grey vertices on day 4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

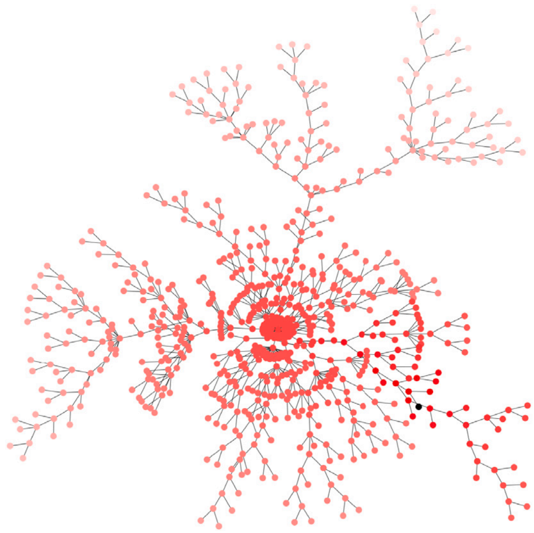

Fig. 2. A simulation of the largest outbreak on a configuration model with heavy-tailed degree distribution with $\alpha = 3/2$. This component has 735 vertices, while the entire graph has 70,000. The black node is the first vertex to be infected, and then darker shades indicate that the corresponding vertex infected earlier in the outbreak. Most of the vertices have small degree (≤ 3); however, there are some vertices with large degree. The large red blob in the middle of the image comes from a vertex of relatively large degree, i.e. a super-spreader. We can also see that there is another super-spreader depicted just below that red blob. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the critical window $p = p(n) = n^{-1} + \lambda n^{-4/3}$ the size of the macroscopic outbreaks are of order $n^{2/3}$ [2]. Within this critical window each vertex has approximately Poisson(1) many neighbors, so in particular the degree distribution of the near-critical Erdős-Rényi random graph has light tails. In turn, the height-profile $(Z_n(h); h \geq 0)$ of the largest component has a scaling limit and that limit is a continuous process [3]. We stress that this is not because we are looking only at an epidemic started from a single individual. The same can be said if we infect $O(n^{1/3})$ individuals on day 0 [4].

To capture some super-spreading phenomena we turn to a random graph model which can contain vertices of large degree, the configuration model. The configuration model is a random graph with a prescribed degree distribution chosen randomly over all such graphs. See Chapter 7 of [5] for an introduction to this model. We will focus on the case where the prescribed degree distribution has heavy tails, because otherwise this model falls within the same universality class as the critical Erdős-Rényi random graph $G(n, n^{-1} + \lambda n^{-4/3})$ [6,7] under some additional technical assumptions. In turn, up to some scaling factors, the behavior of the processes $Z_n(h)$ on largest components (which correspond to the largest possible outbreaks) will be asymptotically the same as those in the Erdős-Rényi random graph. In the asymptotic regime we study the scaling limits of the height profiles $Z_n(h)$ which possess positive jumps. These positive jumps come from the presence of the super-spreading individuals.

We also restrict our focus to critical regimes. One reason is a general principle that what happens at a phase transition is often interesting. Another is that while there are some important results on the structure of the largest components of the critical heavy-tailed configuration model [7,8], there is not much information on the structure of the disease outbreaks. In this vein, there are results in the literature on the behavior of the largest outbreak when initially only a single individual is infected. While studying a model similar to ours where edges are kept with probability $p \in [0,1]$ but are otherwise deleted, the authors of [9] show that there is a parameter R_0 such that if $R_0 \le 1$ then only outbreaks of size $o(n)$ as $n \to \infty$ can occur whereas if $R_0 > 1$ there is a positive probability that an outbreak of size $O(n)$ occurs as $n \to \infty$. See also [10–12]. A continuous time analog of that model was studied in [13] and there the authors show that there is a similar phase transition between outbreaks of size $o(n)$ and outbreaks which are of size $O(n)$ with positive probability. Those authors also describe some of the large n behavior of $Z_n(t)$ (the number of individuals infected at a continuous time $t \ge 0$) conditionally on having an outbreak of size $O(n)$, but they do not provide information for what happens at the phase transition. We hope to fill in this gap in the literature.

1.1. Weak convergence results

Let us discuss a little more formally the configuration model. Before doing so, we recall that a multi-graph can have multiple edges and self-loops while a simple graph does not contain multiple edges nor self-loops. In terms of our approach to studying epidemics, self-loops and multiple edges do not make any physical sense because, for example, infected individuals cannot reinfect themselves.

Given $\mathbf{d}^n = (d_1, \cdots, d_n)$ a finite sequence of strictly positive integers $d_j \ge 1$, the configuration model $M(\mathbf{d}^n)$ is the random multi-graph chosen randomly over all multi-graphs G on the vertex set $[n] := \{1, \cdots, n\}$ where the degree (counted with multiplicity) of vertex j is $\deg(j) = d_j$. In order to construct such a multi-graph we need $\sum_{j=1}^n d_j$ to be even, and two algorithms for its construction will be discussed in Section 5.1. We say that any such graph G has degree sequence $\mathbf{d}^n$.

A priori it may not be possible to construct a simple graph with degree sequence $\mathbf{d}^n$ because, for example, a single vertex may have degree $d_i > \sum_{j \ne i} d_j$. However, if there is a simple graph with degree sequence $\mathbf{d}^n$, then conditionally on the event $\{M(\mathbf{d}^n) \text{ is simple}\}$ the graph is uniformly distributed over all simple graphs with degree sequence $\mathbf{d}^n$ [5, Proposition 7.15]. Moreover for the asymptotic regime we study it makes no difference [7] whether or not we examine simple graphs or multi-graphs so we will just say "graph".

One aspect of randomness for the configuration model comes from taking the graph to be randomly constructed over all graphs with a fixed deterministic degree sequence. Another comes from taking the degree sequence itself to be random, say, with a common distribution ν on $\{1, 2, \cdots\}$. We then generate the graph conditionally given this degree distribution. That is we generate $M(\mathbf{d}^n)$ where the d_j are i.i.d. with common law ν. We may have to replace d_n with $d_n + 1$ to obtain the proper parity; however, this does not affect the analysis [7]. To distinguish between these two situations we will write $M_n(\nu)$ instead of $M(\mathbf{d}^n)$.

We focus on the degree distributions studied by Joseph [8] and Conchon-Kerjan and Goldschmidt [7] which satisfy

$$\lim_{k \to \infty} k^{(\alpha+2)} \nu(k) = c \in (0, \infty), \qquad \mathbb{E}[d_1] = \delta \in (1, 2), \qquad \mathbb{E}[d_1^2] = 2\mathbb{E}[d_1], \tag{1}$$

for some $\alpha \in (1, 2)$, for some c and some δ. The third statement involving the second moment and mean implies that we are examining the random graph at criticality [10–12] and that there is no giant component, i.e. there is no single component which contains a positive proportion of the total number of vertices. Instead, there are macroscopic components which are of order $\Theta(n^{\frac{\alpha}{\alpha+1}})$.

In order to obtain scaling limits for a height profile representing the number of people infected on day h, we would either need to look at the case where a significant number of individuals are infected on day zero, or focus on the largest possible outbreaks. We focus on the latter situation and hence decompose the graph $M_n(\nu)$ into its connected components $G_n^1, G_n^2, \cdots$ where they are indexed so

$$\#G_n^1 \ge \#G_n^2 \ge \cdots.$$

In order to know how a disease spreads through G_n^i, we need to know its source. We will start the spread from a single vertex ρ_n^i chosen with probability proportional to its degree, and we will say that the component G_n^i is *rooted* (or *pointed*) at the vertex ρ_n^i.

The selection of ρ_n^i is size-biased by its degree and is not a uniform sample, but this is for good reason. In terms of how a disease spreads through a community, vertices with higher degree have more neighbors from whom they can catch the disease and so we should expect these vertices to be infected earlier in the outbreak. This has been observed in a survey of how influenza (seasonal or the H1N1 variant) spread through Harvard in 2009 [14]. Researchers surveyed two sets of students twice-weekly to see when they developed flu-like symptoms. One set was a random sample of all students and the other was a sample of friends nominated by this original set. The set of friends was sample of the students biased by their number of friends at Harvard and not a uniform sample. Sometimes called "the friendship paradox", this is just the observation that the average number of friends of friends is always greater than the average number of friends [15]. In the study of influenza, the set of friends showed flu-like symptoms earlier than the uniform random sample. See also [16].

Of course for each n, there are only a finite number K_n, say, of connected components which correspond to each of the outbreaks. To simplify the presentation we set G_n^i for $i > K_n$ as the graph on a single vertex with no edges and rooted at its only vertex.

The ith largest outbreak is then described by the process $Z_{n,i} = (Z_{n,i}(h) : h = 0, 1, \cdots)$ defined by

$$Z_{n,i}(h) = \#\{v \in G_n^i : \operatorname{dist}(\rho_n^i, v) = h\}, \tag{2}$$

where $\operatorname{dist}(-,-)$ is the graph distance on G_n^i. In terms of the graph, $Z_{n,i}$ is the height profile of the component G_n^i. Our first result is the joint convergence of the rescaled processes $Z_{n,i}$ to a time-change of some excursion processes $\tilde{e}_i = (\tilde{e}_i(t); t \ge 0)$. The processes $\tilde{e}_i$, for $i \ge 1$, are the excursions above past minima of a certain stochastic process $\tilde{X}$ obtained by an exponential tilting of a spectrally positive α-stable process. See Section 3.1 for more information on these processes.

Theorem 1. *Fix some $\alpha \in (1,2)$, and some distribution ν satisfying (1). In the product Skorohod topology on $\mathbb{D}(\mathbb{R}_+)^\infty$, the following convergence holds*

$$\left(\left(n^{-\frac{1}{\alpha+1}} Z_{n,i}(\lfloor n^{\frac{\alpha-1}{\alpha+1}} t\rfloor); t \geq 0\right); i \geq 1\right) \overset{d}{\Longrightarrow} ((Z_i(t); t \geq 0); i \geq 1),$$

where, for all $i \geq 1$, Z_i is the unique càdlàg solution to

$$Z_i(t) = \tilde{e}_i \circ C_i(t), \qquad C_i(t) = \int_0^t Z_i(s)\,ds, \qquad \inf\{t > 0 : C_i(t) > 0\} = 0,$$

where $\tilde{e}_i = (\tilde{e}_i(t) : t \geq 0)$ are defined in Eq. (12) and depend on the value α, c and δ in (1).

1.2. A single macroscopic outbreak and the α-stable graph

More has been said about the graph components G_n^i in the literature. Joseph [8] has argued that the size of the component G_n^i scaled down by $n^{\frac{\alpha}{\alpha+1}}$ converges to a random variable ζ_i for each $i \geq 1$, which in fact can be seen to be $\zeta_i = \inf\{t > 0 : \tilde{e}_i(t) = 0\}$. Conchon-Kerjan and Goldschmidt [7] generalize Joseph's results and show that the graph G_n^i itself has a scaling limit which is a random rooted compact measured metric space $\mathcal{M}_i = (\mathcal{M}_i, \mathrm{d}_i, \rho_i, \mu_i)$. Here d_i is a metric on $\mathcal{M}_i$, $\rho_i \in \mathcal{M}_i$ is a specified element and μ_i is a finite Borel measure on $\mathcal{M}_i$.

This means not only does the height profile (the number of infected people on day h) converge by Theorem 1, but there is some limiting continuum structure of the components G_n^i which is represented by these continuum spaces $(\mathcal{M}_i; i \geq 1)$. The standard construction of the i^{th} continuum limit $\mathcal{M}_i$, first obtained in the critical Erdős-Rényi case in [17,18], is by taking a continuum random tree $\mathcal{T}_i$ (built directly from the excursion $\tilde{e}_i$) and quotienting by an equivalence relation such that $u_j \sim v_j$ for a finite collection of pairs $(u_1, v_1), \cdots, (u_m, v_m) \in \mathcal{T}_i \times \mathcal{T}_i$. The continuum metric space $\mathcal{M}_i$ is the quotient $\mathcal{T}_i / \sim$. The map $\mathcal{T}_i \twoheadrightarrow \mathcal{M}_i$ is not an isometry, and, in particular, it is non-trivial to argue convergence of the height profiles similar to Theorem 1 directly from the results of [7].

The processes $\tilde{e}_i = (\tilde{e}_i(t); t \geq 0)$ in Theorem 1 and the continuum measured metric space $\mathcal{M}_i$ are of a random length and mass, respectively. That is

$$\tilde{e}_i(t) > 0 \qquad \text{if and only if} \quad t \in (0, \zeta_i)$$

for some random ζ_i and, moreover, one can construct both $\tilde{e}_i$ and $\mathcal{M}_i$ on the same probability space so that $\zeta_i = \mu_i(\mathcal{M}_i)$. This does complicate the analysis somewhat; however, conditionally given the values $(\zeta_i; i \geq 1)$ the excursions $\tilde{e}_i$ (resp. the spaces $\mathcal{M}_i$) are independent and are described by a scaling of an excursion (resp. metric measure space) of unit length (resp. unit mass) [7]. Therefore in order to understand the scaling limit $Z_1 = (Z_1(t); t \geq 0)$ of a single macroscopic outbreak $Z_{n,1}$, we can study the structure of the process $\tilde{e}_1$ conditioned on $\zeta_1 = 1$.

To do this we let $\mathbf{e} = (\mathbf{e}(t); t \in [0,1])$ denote a standard excursion [19] of a spectrally positive α-stable Lévy process $X = (X(t); t \geq 0)$. To simplify our proofs, we will work with a situation where the Laplace transform of X satisfies

$$\mathbb{E}\left[\exp(-\lambda X(t))\right] = \exp\left(A\lambda^\alpha t\right), \qquad \forall \lambda, t \geq 0, \quad \text{where} \quad A = \frac{c\Gamma(2-\alpha)}{\delta\alpha(\alpha-1)}, \tag{3}$$

for c, δ defined in (1). this excursion depends on the value A; however, the results also hold for any value of A by using scaling properties of Lévy processes and their associated height processes.

We also recall from above that $\mathcal{M}_i$ are obtained by gluing together a finite collection of pairs of points in a continuum tree. We let $\mathscr{G}^{(\alpha,k)} = (\mathscr{G}^{(\alpha,k)}, d, \rho, \mu)$ denote the graph $\mathcal{M}_1$ conditioned on $\mu_1(\mathcal{M}_1) = 1$ and having surplus k. For more details see Section 3.4. For now it suffices to say that it will be constructed from an excursion $\mathbf{e}^{(k)} = (\mathbf{e}^{(k)} : t \in [0,1])$ defined by the polynomial tilting

$$\mathbb{E}\left[f(\mathbf{e}^{(k)}(t); t \in [0,1])\right] \propto \mathbb{E}\left[\left(\int_0^1 \mathbf{e}(t)\,dt\right)^k f(\mathbf{e}(t); t \in [0,1])\right]. \tag{4}$$

Recall that Theorem 1 tells us the scaling limit of the height profile of the random graph G_n^i, which in turn provides information of the mass measure on the connected component G_n^i. The following theorem provides us with information about mass measure μ on the continuum metric space $\mathscr{G}^{(\alpha,k)}$ and the proceeding corollary provides more distributional identities relating the metric space $\mathscr{G}^{(\alpha,k)}$ back to the excursion $\tilde{\mathbf{e}}^{(k)}$.

Theorem 2. *Fix $k \geq 0$, and $\alpha \in (1,2)$. Let $\mathscr{G}^{(\alpha,k)}$ be the α-stable continuum random graph, constructed from a spectrally positive Lévy process with Laplace exponent (3) and rooted at a point $\rho \in \mathscr{G}^{(\alpha,k)}$.*

(i) Let $B(x,t)$ be the closed ball of radius t centered at x. The process $\mathbf{c} = (\mathbf{c}(t); t \geq 0)$ defined by $\mathbf{c}(t) = \mu(B(\rho,t))$ is absolutely continuous and

$$\mathbf{c}(t) = \int_0^t \mathbf{z}(s)\,ds$$

for a càdlàg process $\mathbf{z} = (\mathbf{z}(t); t \geq 0)$.

(ii) The process $(\mathbf{z}(t); t \geq 0) \stackrel{d}{=} (z(t); t \geq 0)$ *where* z *is the unique càdlàg solution to*

$$z(t) = \mathbf{e}^{(k)}\left(\int_0^t z(s)\,ds\right), \qquad \inf\left\{t > 0 : \int_0^t z(s)\,ds > 0\right\} = 0.$$

We remark that the claim in Theorem 2(i) follows from the construction in [7] and well-known properties of the mass measure of Lévy trees [20]. Thus, the new contribution is the distributional identity in Theorem 2(ii). In particular, this result allows us to obtain new results about the structure of the metric space $\mathscr{G}^{(\alpha,k)}$, which are summarized in the following corollary.

Corollary 3.

(i) The radius of the graph $\mathscr{G}^{(\alpha,k)}$ *satisfies*

$$\sup_{v\in\mathscr{G}^{(\alpha,k)}} d(\rho, v) \stackrel{d}{=} \int_0^1 \frac{1}{\mathbf{e}^{(k)}(s)}\,ds.$$

(ii) The width of the graph $\mathscr{G}^{(\alpha,k)}$ *satisfies*

$$\sup_{t\geq 0} \mathbf{z}(t) \stackrel{d}{=} \sup_{t\in[0,1]} \mathbf{e}^{(k)}(t).$$

(iii) Let $V \in \mathscr{G}^{(\alpha,k)}$ *be distributed according to the mass measure* μ*, and let* U *denote a uniform random variable on* $(0,1)$*. Then*

$$d(\rho, V) \stackrel{d}{=} \int_0^U \frac{1}{\mathbf{e}^{(k)}(s)}\,ds.$$

(iv) More generally, for any $n \geq 1$*, let* $V_1, \cdots, V_n$ *denote random points distributed according to* μ *on* $\mathscr{G}^{(\alpha,k)}$*. Let* $R_{(1)} \leq R_{(2)} \leq \cdots R_{(n)}$ *denote the order statistics of* $d(\rho, V_1), \cdots, d(\rho, V_n)$*. Let* $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$ *denote the order statistics for an i.i.d. sample of* n *uniform random variables. Then*

$$\left(R_{(1)}, \cdots, R_{(n)}\right) \stackrel{d}{=} \left(\int_0^{U_{(1)}} \frac{1}{\mathbf{e}^{(k)}(s)}\,ds, \cdots, \int_0^{U_{(n)}} \frac{1}{\mathbf{e}^{(k)}(s)}\,ds\right).$$

After discussing the integral equation involved in the statement of part 2 of Theorem 2, which is called the Lamperti transform in the literature, we will show how Corollary 3 follows from Theorem 2 in Section 3.2.

1.3. Relation to other works and proof structure

Epidemics on random graphs are important for many areas in the applied sciences, see [21–24] and references therein for a non-exhaustive collection of such works. One difficulty in describing the behavior of an epidemic on a graph as the graph grows large comes from analyzing the influence that the specific degree distribution has on the local structure of the graph.

One approach to overcoming this issue is by using a *mean-field* approach [21]. A typical approximation is in continuous time where each infected vertex v is infected for an exponential time, and infects each neighbor independently and at an exponential rate. On homogeneous networks the behavior of $\mathbb{E}[Z(t)]$, the number of people infected at time $t \in \mathbb{R}_+$, is modeled by the ordinary differential equation

$$\frac{dz(t)}{dt} = \lambda\mu z(t)(1 - z(t)) \qquad \text{where} \quad z(t) = \frac{1}{n}\mathbb{E}[Z(t)].$$

A more careful analysis can be done on heterogeneous networks, where one can track the proportion of vertices with degree k infected at a certain time.

A more detailed approach to studying the time-evolution of the number of individuals infected with a disease when the community structure is modeled by a heterogeneous networks was taken by Volz in [22], subsequently studied in [25]. In these works the population of size n is broken into 3 compartments – the susceptible, the infected and the recovered – and individuals in one compartment are moved to another compartment (i.e. an infected individual recovers or a susceptible individual is infected) at certain exponential rates. The global changes in the proportional size of the outbreak is described, to first order, by just the size of the respective compartments and the degree distribution. Here the limiting structure is described by a system of deterministic ordinary differential equations, which depend on the degree distribution. Deterministic limiting equations were also obtained in [26] under a second moment condition. Some results in [26] hold only once the number of susceptible individuals is sufficiently small, giving a deterministic approximation for the epidemic after some random time.

Our approach is different and based on the so-called Lamperti transformation appearing in the random tree and branching process literature. The Lamperti transformation gives a path-by-path bijection between continuous state branching processes and a certain class of Lévy processes. It was stated (without proof) by Lamperti in [27], but was proved later by Silverstein [28]. See also [29].

The transformation can be described as follows. Instead of looking at the total number of people infected on day h, we look at the number of individuals that person v_j infects, when v_j is the jth individual who contracts the disease. In terms of the graph, this can be done by doing a breadth-first labeling of the connected component containing the initially infected individual. Call χ_j the number of infected individuals that vertex v_j infects χ_j, and let X be the (breadth-first) walk

$$X(k) = \sum_{j=1}^{k} (\chi_j - 1).$$

It was this walk on the Erdős-Rényi random graph that Aldous used in [2] to describe the scaling limits of the component sizes of $G(n,p)$ in the critical window, and an analogous walk was used by Joseph [8] for the configuration model.

The number of people infected on day h, $Z(h)$, solves the difference equation

$$Z(h) = Z(0) + X\left(C(h-1)\right), \qquad C(h) = \sum_{j=0}^{h} Z(j),$$

for each h. See the Introduction of [30] for a proof of this equality. As far as the author is aware, the first instance of this identity can be found in [31] with a slightly more complicated formulation. The authors of [30,32] studied the scaled convergence of solutions of the above equation (with the addition of an immigration term) to its continuum analog

$$Z(t) = X\left(\int_0^t Z(s)\,ds\right).$$

Unfortunately, there is not a unique solution to this integral equation when $X(0) = 0$ and so proving weak convergence is quite difficult. For certain models of random trees one can prove a weak convergence result [31,33,34], and this also works for the Erdős-Rényi random graph when $X(0) > 0$ [4].

We overcome the uniqueness problem by arguing that processes $(n^{\frac{-1}{\alpha+1}} Z_{n,i}(\lfloor n^{\frac{\alpha-1}{\alpha+1}} t\rfloor); t \ge 0)$ are tight, and we further show that each subsequential weak limit must be of a particular form. This approach to overcoming the uniqueness problem was used in [34] to study trees with a certain degree distribution, as opposed to graphs with a given degree distribution in the present situation. While, at first, these two discrete models may seem related, the proofs are quite different. In [34], the authors use a combinatorial transformation of the tree to show that that subsequential limits must be of a particular form. We, instead, show that this follows automatically once we know the underlying graph converges to a measured metric space.

1.4. Organization

The rest of the article is organized as follows. In Section 2 we discuss a general result about the convergence of height profiles of discrete metric measure spaces when those spaces converge in the Gromov–Hausdorff–Prohorov topology. We assume some familiarity with convergence of metric measure spaces in that section.

Section 3 is a fairly detailed section containing preliminaries. Section 3.1 provides an overview of spectrally positive Lévy processes, height processes, their excursions, and the processes under an exponential tilting of the measure. Section 3.2 provides a description of the Lamperti transform, recalling an important uniqueness result from [34] (Proposition 8). Since Corollary 3 follows from Proposition 8, we prove that there. In Section 3.3, we provide some background information on the Gromov–Hausdorff–Prohorov topology and establish two lemmas that are used in the sequel. Finally, in Section 3.4 we describe the general constructions of continuum random trees and continuum random graphs. Readers familiar with any of these topics can skip these sections and come back to them only when needed.

In Section 4 we provide proofs of the abstract convergence theorems found in Section 2.

Section 5 provides some background on the configuration and contain the proofs of Theorems 1 and 2. In Section 5.1 we a detailed description of the exploration of the configuration model used by Joseph [8] and Conchon-Kerjan and Goldschmidt [7] to describe the component sizes and structure of the critical configuration model with random degree distribution. The proof of Theorem 1 is contained in Section 5.2 and the proof of Theorem 2 is contained in Section 5.3.

2. General weak convergence results

2.1. General weak convergence approach

Let us now discuss the general set up for our weak convergence arguments. In the introduction we discussed the number of infected individuals in an epidemic as a function in time, which can be realized as the height profile of a connected component of a random graph. Explicitly those graphs were viewed as a metric space, but we implicitly equipped them with the counting measure. We phrase our results in terms of more general measures on random graphs, which will likely be useful the study of some inhomogeneous models appearing in [35–37]. The epidemiological interpretation of considering non-uniform measures is not immediately clear; however, we could think of the unequal mass of vertices as measuring the size of a clique in a community which was reduced to a single vertex.

A major assumption of these results is the convergence of graphs as measured metric spaces. We delay a more detailed discussion of this topic until Section 3.3. For now it suffices to say that we can equip the space $\mathfrak{X}$ of (equivalent classes) of pointed measured metric spaces with additional boundedness assumptions with a metric which turns $\mathfrak{X}$ into a Polish space. This metric is called the Gromov–Hausdorff–Prokhorov metric, and we will denote it by d_{GHP}.

We will denote a generic element of $\mathfrak{X}$ as $\mathcal{M} = (\mathcal{M}, \rho, \mathrm{d}, \mu)$ where $(\mathcal{M}, \mathrm{d})$ is a metric space such that bounded sets have compact closure, $\rho \in \mathcal{M}$ is a specified point and μ is a Borel measure on $\mathcal{M}$ such that bounded sets have finite mass. For each $\alpha > 0, \beta \ge 0$ define the scaling operation by

$$\mathrm{scale}(\alpha,\beta)\mathcal{M} := (\mathcal{M}, \rho, \alpha\cdot \mathrm{d}, \beta\cdot\mu).$$

Now let G denote a connected graph on, say, n vertices, with $\rho \in G$ a specified vertex and equipped with a finite measure $\mathfrak{m}$ such that each vertex has strictly positive mass. We view $(G, \rho, \mathfrak{m})$ as a rooted measured metric space. As we did implicitly before, we explore the graph in a breadth-first manner. The precise way in which this is done can vary depending on the graph model, but we assume that the vertices are labeled by $v_1, \cdots, v_n$ such that if $i < j$ then $d(\rho, v_i) \le d(\rho, v_j)$. This trivially implies that $\rho = v_1$. This labeling can be viewed as an indexing of each individual who gets infected, so that if person B got infected after person A, then person A has a smaller index than person B.

We now discuss an underlying tree structure and breadth-first walk for the graph, which draws inspiration from the breadth-first tree and walk in [35]. The tree is constructed by looking at which vertices v_i infects in the graph G. More formally, we will say that vertex v_j is the child of v_i if $\{v_i, v_j\}$ is an edge in G, but $\{v_l, v_j\}$ is *not* an edge for all $l < i$. This implies $i < j$. In most models with a breadth-first exploration, v_j will be a child of v_i if vertex v_j is discovered while exploring the vertices attached to v_i.

We define the breadth-first walk X_G^{BF} as

$$X_G^{\mathrm{BF}}(\tau(k)) = X_G^{\mathrm{BF}}(\tau(k-1)) - \mathfrak{m}(v_k) + \sum_{u \text{ child of } v_k} \mathfrak{m}(u), \qquad \tau(k) = \sum_{j \le k} \mathfrak{m}(v_j), \tag{5}$$

and with $X_G^{\mathrm{BF}}(0) = 0$. How the process X_G^{BF} behaves on the intervals $(\tau(k-1), \tau(k))$ will play no important role. The breadth-first walk used by Aldous and Limic in their classification of the multiplicative coalescence [35] satisfies Eq. (5). Later an analogous walk satisfying (5) was used in [31] to describe the inhomogeneous continuum random tree and extend Jeulin's identity [38]. When $\mathfrak{m}$ is a uniform measure on G, this walk will be the breadth-first Łukasiewicz path [39].

Importantly for us, the walk X_G^{BF} encodes the masses and tree structure of $v_1, \cdots, v_n$. However there is no clear functional amenable to scaling limits which allows us to reconstruct the genealogical structure from this breadth-first walk.

We now define the height profile of G by

$$Z_G(h) = \mathfrak{m}\{v \in G : d(\rho, v) = h\}.$$

It will be useful to define its cumulative sum as well:

$$C_G(h) = \sum_{j=0}^{h} Z_G(j) = \mathfrak{m}\{v \in G : d(\rho, v) \le h\}.$$

As observed in [31, equations (13-14)], $Z_G(h)$ solves the following difference equation:

$$Z_G(h+1) = Z_G(0) + X_G^{\mathrm{BF}} \circ C_G(h). \tag{6}$$

To describe what happens in the $n \to \infty$ limit, let $(G_n; n \ge 1)$ be a sequence of connected random graphs on a finite number of vertices, rooted at a point $\rho_n \in G_n$ and equipped with a mass measure $\mathfrak{m}_n$. We write X_n^{BF} for the breadth-first walk $X_{G_n}^{\mathrm{BF}}$. We prove the following in Section 4.

Theorem 4. *Suppose that $\gamma_n := \mathfrak{m}_n(G_n) \to \infty$ a.s. and there exists a sequence $(\alpha_n; n \ge 1)$ such that $\alpha_n \to \infty$. In addition assume that:*

1. *In the Skorohod space $\mathbb{D}([0,1], \mathbb{R})$, the following weak convergence holds*
$$\left(\frac{\alpha_n}{\gamma_n} X_n^{\mathrm{BF}}(\gamma_n t); t \in [0,1]\right) \overset{d}{\Longrightarrow} (X(t); t \in [0,1]),$$
where X is a process such that almost surely $X(0) = X(1) = 0$, $X(t) > 0$ for all $t \in (0,1)$ and $X(t) - X(t-) \ge 0$ for all t;
2. *There exists a random pointed measured metric space $\mathcal{M} = (\mathcal{M}, \rho, \mathrm{d}, \mu)$ which is locally compact and has a boundedly finite measure such that*
$$\mathrm{scale}(\alpha_n^{-1}, \gamma_n^{-1}) G_n \overset{d}{\Longrightarrow} \mathcal{M},$$
weakly in the Gromov–Hausdorff–Prokhorov topology.
3. *For each $\varepsilon > 0$, $\mu(B(\rho, \varepsilon) \setminus \{\rho\}) > 0$.*
4. *$\frac{\alpha_n}{\gamma_n} \sup_{v \in G_n} \mathfrak{m}_n(v) \to 0$ as $n \to \infty$ in probability.*

Then

(i) There is the joint convergence in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^2$ of:
$$\left(\left(\frac{\alpha_n}{\gamma_n} Z_n(\lfloor \alpha_n t \rfloor); t \ge 0\right), \left(\frac{1}{\gamma_n} C_n(\lfloor \alpha_n t \rfloor); t \ge 0\right)\right) \overset{d}{\Longrightarrow} ((Z(t); t \ge 0), (C(t); t \ge 0))$$
where Z and C are the unique càdlàg solution to
$$Z(t) = X \circ C(t), \qquad C(t) = \int_0^t Z(s)\, ds, \qquad \inf\{t : C(t) > 0\} = 0; \tag{7}$$

(ii) The measure μ on $\mathcal{M}$ satisfies
$$(\mu(B(\rho, t)); t \ge 0) \overset{d}{=} (C(t); t \ge 0)$$

Let us make some important remarks on the assumptions in Theorem 4. Assumption (1) is the convergence of the breadth-first walk, which is required in order to have a description of the limiting process Z as described above, barring some stochastic analysis tools that can be used in particular cases [40]. Assumptions (2) and (3) are how we overcome any possible uniqueness problems that were identified in [34] (see Proposition 8 below). Particularly, assumption (3) allows for the classification of the limit C satisfying $\inf\{t : C(t) > 0\} = 0$. Lastly, assumption (4) is so that the term $Z_n(0) \overset{d}{\Longrightarrow} 0$ as $n \to \infty$. Without this assumption we are left to deal with a simpler situation in which we can use the known weak convergence results in [30].

We remark that we can also apply Theorem 4 to a sequence of random graphs $(G_n)_{n\ge1}$ which are not close to being trees. For example, if we obtain $\tilde{G}_n$ from G_n by adding edges between every pair of vertices at the same height, i.e. for each h we add the edge $\{u, v\}$ for each $u, v \in \{x : \mathrm{d}(x, \rho_n) = h\}$, then $\tilde{G}_n$ has the same breadth-first walk as G_n. The new space $\tilde{G}_n$ together with its breadth-first walk would easily satisfy hypothesis (1) and (4). It is also easy to see that there is a limiting metric space $\tilde{\mathcal{M}}$ such that hypotheses (2) and (3) hold for the sequence $\tilde{G}_n$ and, moreover, $\tilde{\mathcal{M}}$ will be a line segment whose mass profile is given by the limiting process Z of Theorem 4. In this example, the graphs $\tilde{G}_n$ can have surplus tending to infinity but the limiting metric space $\tilde{\mathcal{M}}$ would still be a continuum random tree.

As the reader may guess, this formulation will not be helpful for the proof of Theorem 1 nor in the study of any of the macroscopic outbreaks for random graphs. Instead, the above theorem works only with a single macroscopic component. In order to prove Theorem 1 we must develop a joint convergence result where each of the macroscopic components of a graph converges to some limiting graph structure. This is something that appears quite often in the literature on continuum random graphs, dating back to the celebrated result of Addario-Berry, Broutin and Goldschmidt [18]. We now suppose that we have a sequence of graphs $(G_n; n \ge 1)$ on a finite number of vertices with a measure $\mathfrak{m}_n$. For each n we denote the connected components of G_n as $(G_n^i; i \ge 1)$, ordered so that

$$\mathfrak{m}_n\left(G_n^1\right) \ge \mathfrak{m}_n\left(G_n^2\right) \ge \cdots .$$

Again, for convenience we will say that G_n^i is a graph on a single vertex where the vertex has mass 0 for all $i > K_n$, where we remind the reader K_n is the number of connected components. We view each of the components as a measured metric space with graph distance, and we select a vertex ρ_n^i from each component to start the breadth-first walks. Here we write $X_{n,i}^{\mathrm{BF}}$ for the breadth-first walk on G_n^i which, by assumption, satisfies Eq. (5) with the obvious notation changes. Additionally we extend it by constancy to be a function on all of $\mathbb{R}_+$:

$$X_{n,i}^{\mathrm{BF}}(t) = X_{n,i}^{\mathrm{BF}}(\mathfrak{m}_n(G_n^i)) \qquad \forall t \ge \mathfrak{m}_n(G_n^i).$$

Let $Z_{n,i} = (Z_{n,i}(h))$ be the height profile of the ith component G_n^i. They solve an equation analogous to (6) with the obvious notation change.

We prove the following

Theorem 5. *Suppose there exists two sequences $\alpha_n \to \infty$ and $\gamma_n \to \infty$ such that*

1. *In the product Skorohod space $\mathbb{D}^\infty$ the following weak convergence holds:*
$$\left(\left(\frac{\alpha_n}{\gamma_n} X_{n,i}^{\mathrm{BF}}(\gamma_n t); t \ge 0\right); i \ge 1\right) \overset{d}{\Longrightarrow} \left(\left(X_i(t); t \ge 0\right); i \ge 1\right)$$
where almost surely, X_i does not possess negative jumps and there exists a $\zeta_i > 0$ such that $X(t) > 0$ if and only if $t \in (0, \zeta_i)$.
2. *There exists a sequence of pointed measured metric spaces $\mathcal{M}_i = (\mathcal{M}_i, \rho_i, d_i, \mu_i)$ which is locally compact and has a boundedly finite measure such that*
$$\left(\mathrm{scale}(\alpha_n^{-1}, \gamma_n^{-1}) G_n^i; i \ge 1\right) \overset{d}{\Longrightarrow} (\mathcal{M}_i; i \ge 1)$$
weakly in the product Gromov–Hausdorff–Prokhorov topology.
3. *Suppose that $\mu_i(B(\rho_i, \varepsilon) \setminus \{\rho_i\}) > 0$ for all $\varepsilon > 0$ and all $i \ge 1$.*
4. *$\frac{\alpha_n}{\gamma_n} \sup_{v \in G_n} \mathfrak{m}_n(v) \to 0$ as $n \to \infty$ in probability.*

Then

(i) In the product Skorohod topology
$$\left(\left(\frac{\alpha_n}{\gamma_n} Z_{n,i}(\lfloor \alpha_n t \rfloor), \frac{1}{\gamma_n} C_{n,i}(\lfloor \alpha_n t \rfloor); t \ge 0\right); i \ge 1\right) \overset{d}{\Longrightarrow} \left(\left(Z_i(t), C_i(t); t \ge 0\right); i \ge 1\right),$$
where (Z_i, C_i) is the unique càdlàg solution to
$$Z_i(t) = X_i \circ C_i(t), \qquad C_i(t) = \int_0^t Z_i(s)\, ds, \qquad \inf\{t : C_i(t) > 0\} = 0.$$

(ii) For each $i \ge 1$,
$$\left(\mu_i(B(\rho_i, t)); t \ge 0\right) \overset{d}{=} \left(C_i(t); t \ge 0\right).$$

2.2. Compactness corollaries

Let us begin with the first corollary, which follows from Theorem 4 and a result in [34] (see Proposition 8 below).

Corollary 6. *If the hypotheses of Theorem 4 are met, then*

$$\int_{0+} \frac{1}{X(s)}\,ds < \infty \qquad a.s.$$

The above corollary avoids a hypothesis in Theorem 1 in [34], but this comes at the expense of assuming convergence in the Gromov–Hausdorff–Prokhorov topology of an underlying metric space, which is a difficult hypothesis to verify. It is assumed in [34] that the limiting process X satisfies $\int^{1-} \frac{1}{X(s)}\,ds < \infty$. The contrapositive of Corollary 6 is interesting, because it gives a necessary condition for convergence in the Gromov–Hausdorff–Prokhorov topology.

For certain models of random trees and random graphs, determining compactness of the candidates for limiting metric space is difficult. This has been a particular problem for the inhomogeneous continuum random trees introduced by Aldous, Camarri and Pitman [41,42]. These trees are characterized by a parameter $\theta = (\theta_0, \theta_1, \cdots)$ and in [31], the authors showed that boundedness of the continuum random tree is equivalent to the almost sure finiteness of an integral $\int_0^1 \frac{1}{X(s)}\,ds$. A question was posed in [31] to develop useful criteria for compactness of the ICRT and determine if boundedness implied compactness. This problem was open for 16 years, but appears to be solved very recently in [43].

It is in this vein that we state the next corollary. It is a more abstract version of Corollary 3(i) above, and follows from Theorem 4(ii) and Proposition 8.

Corollary 7. *Let $\mathcal{M}$ and X be as in Theorem 4, assume the hypotheses of Theorem 4 are met, and let $\mathrm{spt}(\mu) \subset \mathcal{M}$ denote the topological support of the measure μ. Then*

$$\sup_{v\in \mathrm{spt}(\mu)} d(\rho, v) \overset{d}{=} \int_0^1 \frac{1}{X(s)}\,ds.$$

3. Preliminaries

3.1. Lévy processes, height processes, excursions

In this section we recall the construction of Ψ-height processes and their excursions. For more in depth discussion on the height processes and their excursions see the works of Le Gall, Le Jan and Duquesne in [44–46]. For information about spectrally positive Lévy processes, see Bertoin's monograph [47].

Let $X = (X(t); t \ge 0)$ denote a spectrally positive, i.e. no negative jumps, Lévy process, and let $-\Psi$ denote its Laplace transform:

$$\mathbb{E}\left[\exp\{-\lambda X(t)\}\right] = \exp(t\Psi(\lambda)) \qquad \forall t \ge 0.$$

In order to discuss Ψ-height processes, we restrict our attention to the situation where Ψ is of the form

$$\Psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r 1_{[r<1]})\,\pi(dr),$$

where $\alpha \ge 0$, $\beta \ge 0$, $(r \wedge r^2)\,\pi(dr)$ is a finite measure along with

$$\beta > 0 \qquad \text{or} \qquad \int_{(0,1)} r\pi(dr) = \infty.$$

The last assumption occurs if and only if the paths of X have infinite variation almost surely.

The Ψ-height process $H = (H(t); t \ge 0)$ is a way to give a measure (in a local time sense) to the set

$$\{s \in [0,t] : X(s) = \inf_{s\le r\le t} X(r)\}. \tag{8}$$

Slightly more formally, under the additional assumption that

$$\int_1^\infty \frac{1}{\Psi(\lambda)}\,d\lambda = \infty,$$

there exists a continuous process $H = (H(t); t \ge 0)$ such that for all $t \ge 0$ then

$$H(t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t 1_{[X(s) - \inf_{r\in[s,t]} X(r) \le \varepsilon]}\,ds, \tag{9}$$

where the limit is in probability. See [46] and [44, Section 1.2] for more details. In the case where $\beta > 0$, the process H can be seen [44, equation (1.7)] to satisfy

$$H(t) = \frac{1}{\beta}\,\mathrm{Leb}\left\{\inf_{s\le r\le t} X(r) : s \in [0,t]\right\}.$$

In particular, when X is a standard Brownian motion then

$$H(t) = 2\left(X(t) - \inf_{s\le t} X(s)\right)$$

is twice a reflected Brownian motion.

One can also do this same procedure to the excursions of X. That is, if $I(t) = \inf_{s\le t} X(s)$ is the running infimum of X then the process $-I$ acts as a (Markovian) local time at level 0 for the reflected process $X - I$ [47, Chapter IV]. Moreover, by looking at $T(y) = \inf\{t \ge 0 : I(t) < -y\}$, we can talk about the excursions of $X - I$ between times $T(y-)$ and $T(y)$. As in [44, Section 1.1.2], it is possible to define the height process H for the excursions of X above its running infimum. The associated excursion measure will be denoted by N. To avoid confusion, we will write (e, h) for (X, H) under the excursion measure N.

3.1.1. Stable processes and tilting

We now restrict our attention to the stable case where

$$\Psi(\lambda) = A\lambda^\alpha, \qquad \alpha \in (1, 2]. \tag{10}$$

The process X satisfies the scaling [47]

$$(X(t); t \ge 0) \overset{d}{=} \left(k^{-1/\alpha} X(kt); t \ge 0\right), \qquad \forall k > 0.$$

Similarly, the height process H satisfies the scaling

$$(H(t); t \ge 0) \overset{d}{=} \left(k^{(1-\alpha)/\alpha} H(kt); t \ge 0\right), \qquad \forall k > 0,$$

which can be derived from (9).

Remark 3.1. By scaling the Lévy process X, the constant A in (10) can be taken to equal 1 and this is typically done in the literature. We will not do this when proving Theorems 1 or 2 in order to simplify the presentation. By using scaling properties for both X and H, it is possible to prove that the results in Theorem 2 and Corollary 3 continue to hold when $A = 1$.

As originally observed by Aldous [2], one can encode the size of components of a random graph by a certain walk which possesses a scaling limit of the form $X(t)$ where X is a stochastic process. Aldous first proved this [2] within the critical window of Erdős-Rényi random graph where X is a Brownian motion with quadratic drift. Joseph [8] showed that one can encode the sizes of connected components of the configuration model by a certain walk whose scaling limit X process with independent increments that possesses positive jumps.

For the α-stable case ($\alpha \in (1, 2)$), Conchon-Kerjan and Goldschmidt [7] described the process in [8] via an exponential tilting of a Lévy process. That is they examine an α-stable process X and its associated height process H of the form and define $\tilde{X}$ and $\tilde{H}$ by

$$\mathbb{E}\left[F(\tilde{X}, \tilde{H}; [0, t])\right] = \mathbb{E}\left[\exp\left(-\frac{1}{\delta}\int_0^t s\, dX(s) - A\frac{t^{\alpha+1}}{(\alpha+1)\delta^\alpha}\right) F(X, H; [0, t])\right] \tag{11}$$

where A is as in (10) and F is a function on the paths of upto time t. The A in our notation is $\frac{C_\alpha}{\delta}$ in the notation of [7].

The excursions of the process $\tilde{X}$ before time $t > 0$ can be described via the absolute continuity relationship in (11) and the excursions of X prior to time t. What is very useful for us is that all the excursions of

$$\tilde{R}(t) = \tilde{X}(t) - \tilde{I}(t)$$

above zero can be ordered by decreasing length [7, Lemma 3.5]. That is the lengths of the excursion intervals, $(\zeta_i; i \ge 1)$, can be indexed such that $\zeta_1 \ge \zeta_2 \ge \cdots \ge 0$. To each value ζ_i corresponds an excursion interval (g_i, d_i) of length $d_i - g_i = \zeta_i$ such that $\tilde{R}(g_i) = \tilde{R}(d_i) = 0$ and $\tilde{R}(t) > 0$ for all $t \in (g_i, d_i)$. We define the excursion $\tilde{e}_i = (e_i(t); t \ge 0)$ by

$$\tilde{e}_i(t) = \tilde{R}((g_i + t) \wedge d_i), \qquad t \ge 0. \tag{12}$$

These are the excursions which appear in Theorem 1.

We also let $\tilde{h}_i = (\tilde{h}_i(t); t \ge 0)$ be the excursion of $\tilde{H}$ which straddles (g_i, d_i) defined by

$$\tilde{h}_i(t) = \tilde{H}\left((g_i + t) \wedge d_i\right).$$

3.1.2. Normalized excursions and tilting

We now recall Chaumont's path construction of a normalized excursion of a spectrally positive α-stable Lévy process X. See [19] or [47, Chapter VIII] for more details on this. This allows for a simple description when conditioning the excursion measure $N(\cdot|\zeta = x)$, for a fixed constant (deterministic) $x > 0$ and ζ is the duration of the excursion. These results also hold in the Brownian case $\alpha = 2$, and we refer to Chapter XII of [48] for that treatment.

Define $\hat{g}_1$ and $\hat{d}_1$ by

$$\hat{g}_1 = \sup\{s \le 1 : X(s) = I(s)\}, \qquad \hat{d}_1 = \inf\{s > 1 : X(s) = I(s)\},$$

and define

$$\mathbf{e}(t) = \frac{1}{(\hat{d}_1 - \hat{g}_1)^{1/\alpha}} \left(X(\hat{g}_1 + (\hat{d}_1 - \hat{g}_1)t) - X(\hat{g}_1) \right), \qquad t \in [0,1]. \tag{13}$$

The normalized excursion $\mathbf{e} = (\mathbf{e}(t); t \in [0,1])$ has duration $\zeta = 1$, and its law is $N(\cdot|\zeta = 1)$. We obtain the law $N(\cdot|\zeta = x)$ by scaling. Namely, set

$$\mathbf{e}_x(t) = x^{1/\alpha}\mathbf{e}(x^{-1}t), \qquad t \in [0,x],$$

and then $\mathbf{e}_x = (\mathbf{e}_x(t); t \in [0,x])$ has law $N(\cdot|\zeta = x)$.

This can also be done under the conditioning on the lifetime of the excursion of the height process H. See [49] or [50] for more information. We denote $\mathbf{h} = (\mathbf{h}(t); t \in [0,1])$ as the height process under the measure $N(\cdot|\zeta = 1)$ and (by the scaling as the height process) we write

$$\mathbf{h}_x(t) = x^{(\alpha-1)/\alpha}\mathbf{h}(x^{-1}t), \qquad t \in [0,x].$$

The normalized excursions of $\tilde{X}$ and $\tilde{H}$ are trickier to handle because the process $\tilde{X}$ does not have stationary increments. However, there is a relatively simple way of describing these in terms of an exponential tilting of the excursions $\mathbf{e}$ and $\mathbf{h}$ similar to Aldous' description in [2] in the Brownian case. We define the tilted processes denoted by, $\tilde{\mathbf{e}}_x^{(\delta)}$ and $\tilde{\mathbf{h}}_x^{(\delta)}$, by

$$\mathbb{E}\left[F(\tilde{\mathbf{e}}_x^{(\delta)}, \tilde{\mathbf{h}}_x^{(\delta)})\right] = \frac{\mathbb{E}[\exp(\frac{1}{\delta}\int_0^x \mathbf{e}_x(t)\,dt)F(\mathbf{e}_x, \mathbf{h}_x)]}{\mathbb{E}[\exp(\frac{1}{\delta}\int_0^x \mathbf{e}_x(t)\,dt)]} \tag{14}$$

When $x = 1$ or $\delta = 1$ we omit it from notation. The excursions $\tilde{\mathbf{e}}_x^{(\delta)}$ and $\tilde{\mathbf{h}}_x^{(\delta)}$ are shown in [7] to be the excursions $(\tilde{e}_i, \tilde{h}_i)$ conditioned on their duration being exactly x.

Remark 3.2. To clear up any confusion between $\tilde{\mathbf{e}}^{(\delta)}$ defined in (14) and $\mathbf{e}^{(k)}$ defined in (4), we note that we use the tilde ˜ to denote tilting associated with an exponential tilting of an excursion. We do not include a tilde when discussing the polynomial tilting in (4).

3.2. Lamperti transform

The Lamperti transform relates continuous state branching processes and Lévy processes via a time-change. This relationship dates back to a path-by-path relationship observed by Lamperti [27], although only proved later by Silverstein [28]. More recently the authors of [30] gave a path-by-path transformation between certain pairs of Lévy processes and continuous state branching processes with immigration. The bijective relationship was known before the path-by-path connection as well, see [51]. For more information on this transformation see [29] for a description in the continuum, see [30,32] for scaling limits related to continuous state branching processes and their generalizations affine processes, and see [34] for a scaling limits involving a similar situation of non-uniqueness of the limiting equation.

We will focus on the transform applied to excursions. Given a non-decreasing function $c : \mathbb{R}_+ \to \mathbb{R}$ denote its right-hand derivative by D_+c, provided it exists. If it exists then

$$D_+c(t) = \lim_{\varepsilon \downarrow 0} \frac{c(t+\varepsilon) - c(t)}{\varepsilon}.$$

We now define the Lamperti transform and the Lamperti pair.

Definition 3.1. Given a càdlàg function $f \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ let

$$\imath(t) = \int_0^t \frac{1}{f(s)}\,ds.$$

Define the right-continuous inverse of $\imath$, denoted by c^0, by

$$c^0(t) = \inf\{s \ge 0 : \imath(s) > t\},$$

with the convention $\inf \emptyset = \inf\{t > 0 : f(t) = 0\}$. The *Lamperti transform* of f is the function $h^0 = f \circ c^0$ and we call the pair (h^0, c^0) the *Lamperti pair* associated to f.

Hopefully the choice of denoting the Lamperti pair by (h^0, c^0) will be clear after the statement of the next proposition, which we recall from [34] while introducing a trivial scaling argument and fixing a typo:

Proposition 8 (*[34, Proposition 2]*). *Let $f \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ with non-negative jumps. Assume that $f(t) = 0$ if and only if $t \in \{0\} \cup [\zeta, \infty)$ for some $\zeta \in (0,\infty)$. Let (h^0, c^0) denote the Lamperti pair associated to f. Then, solutions to*

$$c(0) = 0, \qquad D_+c = f \circ c \tag{15}$$

can be characterized as follows:

(i) If $\int_{0+} \frac{1}{f(s)}\,ds = \infty$ *then* $h = c = 0$ *is the unique solution to* (15).

(ii) If $\int_{0+} \frac{1}{f(s)}\,ds < \infty$ *then* c^0 *is not identically zero,* $D_+c^0 = h^0$*, and* c^0 *solves* (15)*. Furthermore, solutions to* (15) *are a one-parameter family* $(c^\lambda; \lambda \in [0,\infty])$ *given by*

$$c^\lambda(t) = c^0\left((t-\lambda)_+\right), \qquad (x)_+ := x \vee 0.$$

In addition,

(a) If $\int^{\zeta-} \frac{1}{f(s)}\,ds = \infty$ *then* c^0 *is strictly increasing with* $\lim_{t\to\infty} c^0(t) = \zeta$.

(b) If $\int^{\zeta-} \frac{1}{f(s)}\,ds < \infty$ *then* c^0 *is strictly increasing until reaching* ζ *at* $\iota(\zeta) = \int_0^\zeta \frac{1}{f(s)}\,ds$.

The above proposition states that all the solutions to (15) are determined by time-shifts of the Lamperti pair associated with f, or is identically zero. As we will see in the sequel, a major part of the proof of Theorem 4 is showing that every subsequential (weak) limit of the $\tilde{C}_n$ is of the form C^0 and not a time-shift, C^Λ, of C^0 for some random Λ.

With Proposition 8 recalled, we can prove Corollary 3 from Theorem 2.

Proof of Corollary 3. Let us apply Proposition 8 with $f(t) = \mathbf{e}^{(k)}(t)$. Theorem 2(ii) then implies that $\mathbf{c}(t)$ is the corresponding process $c^0(t)$ defined in Proposition 8(ii).

Now observe that

$$\sup_{v\in \mathrm{spt}(\mu)} \mathrm{d}(\rho, v) \overset{d}{=} \int_0^1 \frac{1}{\mathbf{e}^{(k)}(s)}\,ds$$

follows from Theorem 2 by an application of Proposition 8(ii)(b). To replace the support of the measure μ with the graph $\mathscr{G}^{(\alpha,k)}$ we observe that

$$\mathrm{spt}(\mu) = \mathscr{G}^{(\alpha,k)}.$$

Indeed, this holds for Lévy trees [20,49] and will hence holds for quotients of Lévy trees. This establishes conclusion (i).

Conclusion (ii) trivially follows from Theorem 2 and the observation that $\mathbf{c}$ increases continuously from 0 to 1 as t ranges from 0 to ∞. We restrict the rest our proof to conclusion (iii), the argument of which will imply part (iv) with minor modifications.

We recall the well-known fact that if X is a real random variable taking values in (a,b) with cumulative distribution function F which is strictly increasing on (a,b), then $X \sim F^{-1}(U)$ where U is a standard uniform random variable and $F^{-1}(y) = \inf\{t : F(t) > y\}$ is the right-continuous inverse. Typically this is stated with the left-continuous inverse of F; however, when F is strictly increasing these two inverses agree on (a,b).

Now, conditionally given $\mathscr{G}^{(\alpha,k)}$, Theorem 2 implies that

$$\mathbb{P}(\mathrm{d}(\rho, V) \le t | \mathscr{G}^{(\alpha,k)}) = \mu(B(\rho,t)) = \mathbf{c}(t).$$

Thus,

$$\mathrm{d}(\rho, V) \overset{d}{=} \inf\{t : \mathbf{c}(t) > U\}, \qquad U \sim \mathrm{Unif}(0,1).$$

However, the process $\mathbf{c}$ is equal in distribution to $c(t) = \int_0^t z(s)\,ds$ where z is as in Theorem 2(ii). It is easy to see by definition of c^0 (Definition 3.1), that

$$\mathbf{c}(t) = \inf\left\{u : \int_0^u \frac{1}{\mathbf{e}^{(k)}(s)}\,ds > t\right\} \wedge 1.$$

The result now follows by properties of right-inverses of strictly increasing functions

$$\mathrm{d}(\rho, V) \overset{d}{=} \inf\{t : \mathbf{c}(t) > U\} = \int_0^U \frac{1}{\mathbf{e}^{(k)}(s)}\,ds.$$

The proof of conclusion (iv) is a trivial generalization involving order statistics. □

3.3. Convergence of metric spaces

In this section we discuss how to topologize the collection of pointed measured metric spaces with some additional compactness assumptions. We start with a definition:

Definition 3.2. A collection $\mathcal{M} = (\mathcal{M}, \rho, \mathrm{d}, \mu)$ is a *pointed measured metric* (PMM) space if $(\mathcal{M}, \mathrm{d})$ is a metric space, μ is a Borel measure on $\mathcal{M}$ and $\rho \in \mathcal{M}$ is a distinguished point. We say that $\mathcal{M}$ is *boundedly compact* if bounded sets are pre-compact and we say that μ is a *boundedly finite* measure if bounded sets have finite mass. We say that $\mathcal{M}$ and $\mathcal{M}'$ are *isomorphic* if there exists a bijective isometry $f : \mathcal{M} \to \mathcal{M}'$ such that $f(\rho) = f(\rho')$ and $f_\#\mu = \mu'$ and $f_\#^{-1}\mu' = \mu$. We denote the collection of all (isomorphism classes of) boundedly compact PMM spaces equipped with boundedly finite measures by $\mathfrak{X}$. Let $\mathfrak{X}_c \subset \mathfrak{X}$ consist of all compact elements of $\mathfrak{X}$.

We leave a more detailed accounting of the metric space structure of $\mathfrak{X}$ to the texts [52,53]. We do recall some useful properties which will be used in the sequel.

Theorem 9 (*[52,53]*)**.**

1. *There exists a metric d_{GHP} on the space $\mathfrak{X}$ which makes $(\mathfrak{X}, d_{\mathrm{GHP}})$ a Polish space.*
2. *There exists a metric d^c_{GHP} on the space $\mathfrak{X}_c$ which makes $(\mathfrak{X}_c, d^c_{\mathrm{GHP}})$ a Polish space.*

We now prove the following simple lemma, which we cannot find in the existing literature. This will be used in the proof of Theorem 4. We denote by $B(y,r)$, the closed ball of radius r centered at y in the appropriate metric space.

Lemma 10. *Let $\mathcal{M}_n = (\mathcal{M}_n, \rho_n, \mathrm{d}_n, \mu_n)$, $\mathcal{M} = (\mathcal{M}, \rho, \mathrm{d}, \mu)$ be random elements of $\mathfrak{X}$ such that*

$$\mathcal{M}_n \overset{d}{\Longrightarrow} \mathcal{M}, \qquad \text{as random elements of } (\mathfrak{X}, d_{\mathrm{GHP}}),$$

and μ on $\mathcal{M}$ is almost surely not the zero measure. Then for all but countably many $r \in (0,\infty)$:

$$\mu_n(B(\rho_n, r)) \overset{d}{\Longrightarrow} \mu(B(\rho, r)), \qquad \text{as random real numbers.}$$

Both convergences above can be replaced with almost sure convergence as well.

We prove this by first appealing to a deterministic lemma.

Lemma 11. *Let $\mathcal{M}_n \to \mathcal{M}$ in d_{GHP} and suppose that the measure μ on $\mathcal{M}$ is not the zero measure. Let r be a radius such that*

$$\mu\{x \in \mathcal{M} : \mathrm{d}(\rho, x) = r\} = 0.$$

Then

$$\mu_n(B(\rho_n, r)) \to \mu(B(\rho, r)).$$

Proof. By Theorem 3.16 in [53], it suffices to consider the compact case where $\mathcal{M}_n, \mathcal{M} \in \mathfrak{X}_c$ and $\mathcal{M}_n \to \mathcal{M}$ with respect to the d^c_{GHP} metric and that $r = \sup_{x\in\mathcal{M}} \mathrm{d}(\rho, x)$.

We recall that, for metric spaces X and Y, a function $f : X \to Y$ is an ε-isometry if f is measurable and

$$\sup\{|d(x_1,x_2) - d(f(x_1), f(x_2))| : x_1, x_2 \in X\} \le \varepsilon$$

and for all $y \in Y$ there exists some $x \in X$ such that $d(y, f(x)) < \varepsilon$.

By Theorem 3.18 in [53], there exists a sequence $\varepsilon_n \to 0$ and a sequence of functions $f^n : \mathcal{M}_n \to \mathcal{M}$ such that f^n is an ε_n-isometry and such that

$$f^n_{\#}\mu_n \to \mu,$$

with respect to the weak-* topology of measures on $\mathcal{M}$, that is convergence of the integrals against compactly supported continuous functions. However, $1_{\mathcal{M}}$ is continuous and compactly supported since $\mathcal{M}$ is compact. So the following convergence holds in because of convergence in the weak-* topology:

$$\begin{aligned}\mu_n(\mathcal{M}_n) &= \int_{\mathcal{M}_n} 1_{\mathcal{M}_n}\, d\mu_n = \int_{\mathcal{M}_n} 1_{\mathcal{M}} \circ f^n(x)\, \mu_n(dx)\\ &= \int_{\mathcal{M}} 1_{\mathcal{M}}(x) (f^n_{\#}\mu_n)(dx) \to \int_{\mathcal{M}} 1_{\mathcal{M}}\, d\mu = \mu(\mathcal{M}).\end{aligned}$$

Therefore, there is no loss in generality in assuming that the measures μ_n and μ are probability measures, since we can just rescale the measures by their (non-zero) total mass. Since weak-* convergence of probability measures on a compact space is simply weak convergence of probability measures, the desired convergence holds by Portmanteau. □

Proof of Lemma 10. By Lemma 11, we have shown that the map $\Phi_r : \mathfrak{X} \to \mathbb{R}$ by

$$\Phi_r(\mathcal{M}) = \mu(B(\rho, r))$$

is continuous at each $\mathcal{M}$ such that $\mu(\{x \in \mathcal{M} : \mathrm{d}(\rho,x) = r\}) = 0$. But $r \mapsto \Phi_r(\mathcal{M})$ is a non-decreasing càdlàg process with $\Phi_r(\mathcal{M}) - \Phi_{r-}(\mathcal{M}) = \mu(\{x \in \mathcal{M} : \mathrm{d}(\rho, x) = r\})$. Hence $\{r : \mathbb{P}(\Phi_r(\mathcal{M}) \neq \Phi_{r-}(\mathcal{M}) > 0) > 0\}$ is countable (Section 13 of [54]) and so

$$\{r : \mathbb{P}[\mu(\{x \in \mathcal{M} : \mathrm{d}(\rho, x) = r\}) > 0] > 0\} \quad \text{is countable.} \quad \square$$

3.4. Continuum random trees and continuum random graphs

In this section we briefly recall the definition of continuum random trees and continuum random graphs. This will not be a full description of what these metric spaces are, but will be enough to define the metric spaces we use in the sequel. For a more abstract account of these metric spaces see Section 2.2 of [55], for example.

We briefly describe a real tree encoded by a continuous function h, see [39] and references therein for more information. Let $h : [0,x] \to [0,\infty)$ be a continuous function such that $h(0) = h(x) = 0$ and $h(t) > 0$ for all $t \in (0,x)$. We can define a pseudo-distance d_h on $[0,x]$ by

$$d_h(s,t) = h(s) + h(t) - 2 \inf_{s\wedge t \le r \le s\vee t} h(r).$$

We then define an equivalence relation by $s \sim_h t$ if $d_h(s,t) = 0$. The random tree $\mathscr{T}_h$ is defined as the quotient space

$$\mathscr{T}_h = [0,x]/\sim_h .$$

and let $q_h : [0,x] \to \mathscr{T}_h$ denote the canonical quotient map. The topological space $\mathscr{T}_h$ can be made into a PMM by setting the specified point as $\rho := q_h(0) = q_h(x)$, the distance as $d(q_h(s), q_h(t)) = d_h(s,t)$ which is a well-defined metric and the measure as $\mu = (q_h)_\# \mathrm{Leb}\,|_{[0,x]}$. We call $\mathscr{T}_h$ the tree encoded by the function h, and call h the height function (or process) of $\mathscr{T}_h$.

The spaces $\mathscr{T}_h$ are tree-like in the sense that given any two elements $a, b \in \mathscr{T}_h$, there exists a unique isometry $f_{a,b} : [0, d(a,b)] \to \mathscr{T}_h$ such that $f_{a,b}(0) = a$ and $f_{a,b}(d(a,b)) = b$ and every continuous injection $f : [0,1] \to \mathscr{T}_h$ such that $f(0) = a$ and $f(b) = 1$ is a reparametrization of $f_{a,b}$.

Let us now describe how to add *shortcuts* to the tree $\mathscr{T}_h$ in order to form a graph-like metric space. We fix a càdlàg function $g : [0,x] \to [0,\infty)$ such that $g(0) = g(x) = 0$ and g does not jump downwards, i.e. $g(t) - g(t-) \ge 0$ for all t. We also suppose that we have a finite set $Q = \{(t_j, y_j) : j = 1, \cdots, s\}$ of points in $[0,x] \times \mathbb{R}_+$ such that $y_j \le g(t_j)$. For each of these values t_j, we define the value $\tilde{t}_j$ by

$$\tilde{t}_j = \inf\{u \ge t_j : g(u) = y_j\}.$$

The infimum above is taken over a non-empty set because g does not jump downwards. These times t_j and $\tilde{t}_j$ will come to represent the points in the tree that are glued together.

Let us now go back into the continuum tree $\mathscr{T}_h$. Define the vertices $u_j = q_h(t_j)$ and $v_j = q_h(\tilde{t}_j)$ where q_h is the canonical quotient map. We define a new equivalence relation $\sim$ on $\mathscr{T}_h$ which depends on both g and Q by setting $u_j \sim v_j$ for each $j = 1, 2, \cdots, s$. We define the set

$$\mathscr{G}(h,g,Q) = \mathscr{T}_h/\sim .$$

It is not difficult to turn $\mathscr{G}(h,g,Q)$ into a PMM where the distance between u_j and v_j is zero.

In the description of the construction of the graph $\mathscr{G}(h,g,Q)$ it is easier to consider only the case where Q consists of points (t,y) such that $0 \le y \le g(t)$ and $t \in [0,x]$. We can equally as well consider the situation where Q is a discrete set in $\mathbb{R}_+ \times \mathbb{R}_+$ with finitely many elements in any compact set, and define

$$\mathscr{G}(h,g,Q) = \mathscr{G}(h,g,g\cap Q), \quad \text{where} \quad g \cap Q = \{(t,y) \in Q : y \le g(t)\}.$$

Let us now describe the graphs $\mathscr{G}^{(\alpha,k)}$ that we mentioned in the introduction. See [7,56] for more information on these graphs. We first define the α-stable graph $\mathscr{G}^{(\alpha)}$ where we let the surplus be a random non-negative integer. The graphs $\mathscr{G}^{(\alpha)}$ are the graphs $\mathscr{G}(\tilde{\mathbf{h}}_1^{(1)}, \tilde{\mathbf{e}}_1^{(1)}, \mathscr{P})$ for a Poisson random measure $\mathscr{P}$ on $\mathbb{R}_+^2$ with Lebesgue intensity. The Poisson point process $\mathscr{P}$ has only a finite number of points (t,y) such that $0 \le y \le \tilde{\mathbf{e}}_1^{(\delta)}(t)$, and this is the surplus of the random graph $\mathscr{G}^{(\alpha)}$. The graph $\mathscr{G}^{(\alpha,k)}$ is just the graph $\mathscr{G}^{(\alpha)}$ conditioned on having fixed surplus k. This conditioning on the number of points of $\mathscr{P}$ which lie under the curve $\tilde{\mathbf{e}}_1^{(1)}$ changes the exponential tilting appearing in (14) to the polynomial tilting appearing in (4). See the discussion around equation (3) and the proof of Theorem 1.2 on page 24 of [7].

For more information on random trees and graphs see [39,57–59] for the Brownian CRT, see [44,49,50] for the stable-trees, see [6,18] for the Brownian random graph and [7,56] for the stable graph.

4. Proofs of weak convergence results

We now turn our attention to proving the abstract weak convergence results: Theorems 4 and 5. To simplify the notation in the proof of Theorem 4, we write $X_n(\cdot) = Z_n(0) + X_n^{\mathrm{BF}}(\cdot)$. By assumption (4) in Theorem 4, $\frac{\alpha_n}{\gamma_n} Z_n(0) \to 0$ in probability and so assumption (1) in Theorem 4 holds with X_n replacing X_n^{BF} by Slutsky's theorem. Moreover, changing (6) to match this notation, the process Z_n solves

$$Z_n(h+1) = X_n \circ C_n(h), \qquad C_n(h) = \sum_{j=0}^{h} Z_n(j), \qquad C_n(-1) = 0.$$

We define the rescalings:

$$\tilde{Z}_n(t) = \frac{\alpha_n}{\gamma_n} Z_n(\lfloor \alpha_n t \rfloor), \qquad \tilde{C}_n(t) = \frac{1}{\gamma_n} C_n(\lfloor \alpha_n t \rfloor), \qquad \tilde{X}_n(t) = \frac{\alpha_n}{\gamma_n} X_n(\gamma_n t).$$

We begin by proving the tightness of $\tilde{X}_n$ and $\tilde{C}_n$.

Proposition 12. *Under the assumptions of Theorem 4, and the above notation, the sequence* $((\tilde{C}_n, \tilde{X}_n); n \ge 1)$ *is tight in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^2$. *Moreover, any subsequential limit of* $(\tilde{C}_n, \tilde{X}_n; n \ge 1)$, *say* (C, X), *must satisfy*

$$C(t) = \int_0^t X \circ C(s)\, ds.$$

Proof. We alter the proof of Proposition 7 in [34]. That proof involves a linear interpolation of C_n instead, which makes their proof slightly simpler. The differences are easily overcome using compactness results in Billingsley's monograph [54] (Theorem 16.8).

Because tightness of marginals implies tightness of the pair of random elements, in order to show the tightness claimed, it suffices to show that $(\tilde{C}_n; n \ge 1)$ is tight, since we assume that $\tilde{X}_n$ converges weakly and is therefore tight. Towards this end, observe that $\tilde{C}_n$ is uniformly bounded:

$$0 \le \tilde{C}_n(t) \le \frac{1}{\gamma_n} \mathfrak{m}_n(G_n) = 1. \tag{16}$$

We now set $t > s$. We have

$$\begin{aligned}
\tilde{C}_n(t) - \tilde{C}_n(s) &= \frac{1}{\gamma_n}(C_n(\lfloor \alpha_n t \rfloor) - C_n(\lfloor \alpha_n s \rfloor)) \\
&= \frac{1}{\gamma_n} \sum_{h=\lfloor \alpha_n s \rfloor + 1}^{\lfloor \alpha_n t \rfloor} Z_n(h) \\
&= \frac{1}{\gamma_n} \int_{\alpha_n s + 1}^{\alpha_n t + 1} Z_n(\lfloor u \rfloor)\, du \\
&= \frac{1}{\gamma_n} \int_{\alpha_n s + 1}^{\alpha_n t + 1} X_n \circ C_n(\lfloor u \rfloor - 1)\, du \\
&= \frac{\alpha_n}{\gamma_n} \int_{s + \alpha_n^{-1}}^{t + \alpha_n^{-1}} X_n \circ C_n(\lfloor \alpha_n u \rfloor - 1)\, du \\
&\le (t-s) \|\tilde{X}_n\|,
\end{aligned}$$

where

$$\|f\| = \sup_{t \in [0,1]} |f(t)|.$$

Define the functions

$$w(f; I) := \sup_{s,t \in I} |f(t) - f(s)|, \qquad f \in \mathbb{D}(\mathbb{R}_+), I \subset \mathbb{R},$$

and, for $\delta > 0$,

$$w'_N(f; \delta) := \inf_{\{t_i\}} \max_{1 \le i \le v} w(f; [t_{i-1}, t_i)), \qquad N = 1, 2, \cdots$$

where the infimum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_v = N$ such that $t_i - t_{i-1} > \delta$ for $1 \le i < v$.

From the above string of inequalities, for any integer $N > 0$ and any $\delta > 0$, we have

$$w'_N(\tilde{C}_n; \delta) \le 2\delta \|\tilde{X}_n\|, \qquad \forall n \ge 1,$$

Fix $\varepsilon > 0$ and an integer N. Applying Theorem 13.2 in [54] gives

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}\left(w'_N(\tilde{C}_n; \delta) \ge \varepsilon\right) \le \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}\left(\|\tilde{X}_n\| \ge \frac{\varepsilon}{2\delta}\right) = 0.$$

Hence, by Theorem 16.8 in [54], the process $\tilde{C}_n$ is tight.

The statement about the form of the subsequential weak limits follows as in the proof of Proposition 7 in [34] with little alteration, so we only sketch the proof. Since we have established that $(\tilde{C}_n, \tilde{X}_n)$ are tight, we can pass to a subsequence (which we index still by n) such that $(\tilde{C}_n, \tilde{X}_n) \stackrel{d}{\Longrightarrow} (C, X)$. By using Skorohod's representation theorem, we can work on a probability space where this convergence holds almost surely. Since X has only positive jumps by assumption

$$X_- \circ C(t) \le \liminf_n \tilde{X}_n \circ \tilde{C}_n(t) \le \limsup_n \tilde{X}_n \circ \tilde{C}_n \le X \circ C(t), \qquad \forall t.$$

Hence Fatou's lemma together with the above analysis of $\tilde{C}_n(t) - \tilde{C}_n(s)$ implies that

$$\int_s^t X_- \circ C(r)\, dr \le C(t) - C(s) \le \int_s^t X \circ C(r)\, dr.$$

Such solutions must satisfy $C(t) = \int_0^t X \circ C(r)\, dr$. $\square$

4.1. Proofs of Theorems 4 and 5

We now move to describe the possible subsequential limits in Proposition 12.

Proposition 13. *Let (Z^0, C^0) be the Lamperti pair of X. Then, under the assumptions of Theorem 4, the following weak convergence holds*

$$(\tilde{C}_n, \tilde{X}_n) \Longrightarrow (C^0, X),$$

in the product Skorokhod space $\mathbb{D} \times \mathbb{D}$.

Moreover, since C^0 is not identically zero X must satisfy

$$\int_{0+} \frac{1}{X(s)}\, ds < \infty.$$

Proof. By Proposition 12, the sequence $(\tilde{C}_n, \tilde{X}_n)$ is tight. Moreover, by Proposition 12, the subsequential limits $(\tilde{C}_{n_\ell}, \tilde{X}_{n_\ell}) \overset{d}{\Longrightarrow} (C, X)$ where $C(t) = \int_0^t X \circ C(s)\, ds$ and so solves (15). Observe that the limit C must be of the form $C(t) = C^0((t-\Lambda)_+)$ for some (random) $\Lambda := \inf\{t : C(t) > 0\} \in [0, \infty]$ where C^0 is the Lamperti transform of the function X. Indeed, on the event $\int_{0+} \frac{1}{X(s)} = +\infty$ the process $C^0 \equiv 0$ is the unique solution by Proposition 8(i), but on the event $\int_{0+} \frac{1}{X(s)} < \infty$ we have used that any solution to (15) must be of the form C^Λ by Proposition 8(ii). By a standard subsequence argument, it suffices to show that $\Lambda = 0$ almost surely and that $\mathbb{P}(C^0 \equiv 0) = 0$. The second conclusion follows from $\mathbb{P}(C^0 \equiv 0) = 0$ by contraposition of Proposition 8.

By the Skorokhod representation theorem and by possible taking a further subsequence, we can assume that we are working on a probability space such that both

$$\operatorname{scale}(\alpha_{n_\ell}^{-1}, \gamma_{n_\ell}^{-1}) G_{n_\ell} \to \mathcal{M}, \qquad \text{and} \qquad (\tilde{C}_{n_\ell}, \tilde{X}_{n_\ell}) \to (C, X)$$

occur almost surely in their respective topologies: the first convergence is with respect to the pointed Gromov–Hausdorff–Prokhorov topology and the second convergence is with respect to the product topology on $\mathbb{D} \times \mathbb{D}$. We write $\tilde{G}_{n_\ell} = (\tilde{G}_{n_\ell}, \rho_{n_\ell}, \tilde{d}, \tilde{\mathfrak{m}}_{n_\ell})$ for $\operatorname{scale}(\alpha_{n_\ell}^{-1}, \gamma_{n_\ell}^{-1}) G_n^i$. By Lemma 10, we have for all but countably many $t > 0$,

$$\begin{aligned} \tilde{C}_{n_\ell}(t) &= \frac{1}{\gamma_{n_\ell}} \mathfrak{m}_{n_\ell}\{v \in G_{n_\ell} : d(v, \rho_{n_\ell}) \le \alpha_{n_\ell} t\} \\ &= \tilde{\mathfrak{m}}_{n_\ell}\left(\{v \in \tilde{G}_{n_\ell} : \tilde{d}(v, \rho_{n_\ell}) \le t\}\right) \longrightarrow \mu(\{x \in \mathcal{M} : d(\rho, x) \le t\}). \end{aligned} \tag{17}$$

Similarly, by the convergence of $\tilde{C}_{n_\ell}$ in $\mathbb{D}$ we have for all but countably many $t > 0$

$$\tilde{C}_{n_\ell}(t) \to C(t).$$

By a standard diagonalization argument there exists a sequence $t_m \downarrow 0$ such that

$$C(t_m) = \mu\left(B(\rho, t_m)\right) > 0.$$

The inequality follows from Assumption (3) in Theorem 4. Hence $\mathbb{P}(C^0 \equiv 0) = 0$ and

$$\Lambda = \inf\{t : C(t) > 0\} = 0$$

establishing the result. $\square$

We now finish the proof of Theorem 4.

Proof of Theorem 4. Note that the second conclusion of Proposition 13 implies that $\int_{0+} \frac{1}{X(s)} < \infty$ a.s. In particular, when applying Proposition 8 with $f = X$, we are almost surely in the case (ii).

The second conclusion of Theorem 4 follows from Eq. (17) in the proof of Proposition 13, and so we finish the proof of conclusion (i).

By Propositions 12 and 13, and the Skorohod representation theorem, we can assume that we are working on a probability space such that

$$(\tilde{C}_n, \tilde{X}_n) \longrightarrow (C^0, X) \qquad \text{a.s.}$$

Then, by a result of Wu [60, Theorem 1.2] which extends a result of Whitt [61], the following convergence holds in $\mathbb{D}$:

$$\tilde{Z}_n = \tilde{X}_n \circ \tilde{C}_n \longrightarrow X \circ C^0 \qquad \text{a.s.}$$

Using Proposition 8(ii), we observe that

$$Z^0 := X \circ C^0 = D_+ C^0.$$

That is (Z^0, C^0) is the Lamperti pair associated with X and in $\mathbb{D}^2$

$$(\tilde{Z}_n, \tilde{C}_n) \overset{d}{\Longrightarrow} (Z^0, C^0). \quad \square$$

The proof of Theorem 4 can be easily extended to the joint convergence of the of finitely many graphs G_n^i of random masses $\mathfrak{m}_n(G_n^i)$ as $n \to \infty$. Since the graph G_n^i are ordered by decreasing mass, the excursion lengths also decrease: $\zeta_1 \geq \zeta_2 \geq \cdots$. The only part that changes is that (16) is replaced with an analogous tightness bound on

$$\max_{j \leq N} \frac{1}{\gamma_n} \mathfrak{m}_n(G_n^j) = \frac{1}{\gamma_n} \mathfrak{m}_n(G_n^1)$$

This will yield a proof of Theorem 5. The details are omitted.

5. The configuration model

In this section we focus on the applications to the configuration model when one specifies a critical degree distribution ν in the domain of attraction of a stable law. We will focus on the case $\alpha \in (1, 2)$, although the Brownian case $\alpha = 2$ can be obtained by these methods. The results can easily be altered to cover the case $\alpha = 2$ as well, by instead considering the case where ν is critical and has finite third moment omitting the cases $\nu(2) = 1$ and $\nu(0) > 0$.

We will be using the results of Joseph [8] and Conchon-Kerjan and Goldschmidt [7] on scaling limits related to the configuration model. The latter reference provides a metric space scaling limit for the components of the graph, which allows us to utilize the proof technique of Theorem 5. See also Riordan's work [62] and also [63].

5.1. *Preliminaries: The configuration model and convergence*

Let us describe briefly the configuration model, some of the associated walks on the graphs, and their scaling limits. For a more detailed account of the configuration model, see Chapter 7 of [5].

The multigraph $M(\mathbf{d}^n)$ is a random graph on vertex set $(i; i \in [n])$ where the vertex i has degree (counted with multiplicity) d_i. We can construct this graph by viewing the vertices i as hubs with d_i *half-edges* jutting out from the vertex i. We then pair half-edges uniformly at random to create a multigraph. Below we describe two different algorithms for how to construct the multigraph and describe associated walks. It is also convenient to assume that the d_i half-edges connected to a vertex i are ordered, so that we can talk about the "least" half-edge. We remark that this random construction described above is taken from a deterministic sequence of degrees $\mathbf{d}^n$, later on we will take the vertex degrees to be random.

Following Joseph [8], we partition the $\sum_{j=1}^n d_j$ half-edges into three disjoint subsets: the set $\mathcal{S}$ of sleeping half-edges, the set $\mathcal{A}$ of active half-edges, and the set $\mathcal{D}$ of dead half-edges. We call the set $\mathcal{S} \cup \mathcal{A}$ the collection of alive half-edges. Initially all half-edges are sleeping.

5.1.1. *Breadth-first construction*

We construct a graph $M^{\mathrm{BF}}(\mathbf{d}^n)$ (we initially include a BF to specify the construction) as follows:

To initialize at step 1, we pick a sleeping half-edge uniformly at random. Label the corresponding vertex as v_1 and declare all of the half-edges attached to v_1 as active.

Suppose that we have just finished step j. There are three possibilities: (1) $\mathcal{A} \neq \emptyset$, (2) $\mathcal{A} = \emptyset$ and $\mathcal{S} \neq \emptyset$, (3) all half-edges are dead.

In case 1, we proceed as follows:

1. Let i be the **smallest** integer k such that there exists an active half-edge attached to v_k.
2. Pick the least half-edge l from all active half-edges attached to v_i.
3. Kill l, that is, remove it from $\mathcal{A}$ and add it to $\mathcal{D}$.
4. Choose a half-edge r uniformly at random from all living half-edges and pair it with l, that is, add an edge between the vertex v_i (which is attached to l) and the corresponding vertex connected to r.
5. If r is sleeping, then we have discovered a new vertex. Label this new vertex v_{m+1} where we have discovered the vertices $v_1, \cdots, v_m$ up to this point. Declare that all the half-edges of v_{m+1} are active.
6. Kill r.

In case 2, we have finished exploring a connected component of $M^{\mathrm{BF}}(\mathbf{d}^n)$. We proceed by picking a sleeping half-edge uniformly at random. We then label the corresponding vertex v_{m+1} if we have discovered vertices $v_1, \cdots, v_m$ up to this point, and we declare all the half-edges connected to v_{m+1} as active.

In case 3, we have explored the entire graph and we are done (see Fig. 3). The above is the breadth-first construction of the multigraph $M^{\mathrm{BF}}(\mathbf{d}^n)$. In the sequel, we denote the ordering of the vertices in the exploration/construction above by $v_1^{\mathrm{BF}}, \cdots, v_n^{\mathrm{BF}}$.

Remark 5.1. While the above algorithm gives a breadth-first construction of the graph $M(\mathbf{d}^n)$, it can also be slightly altered to give an exploration of a fixed realization of $M(\mathbf{d}^n)$. Indeed, if all the half-edges are paired, then in step (4) while exploring the half-edge l, a half-edge is already paired with l and so we do not have to sample one uniformly.

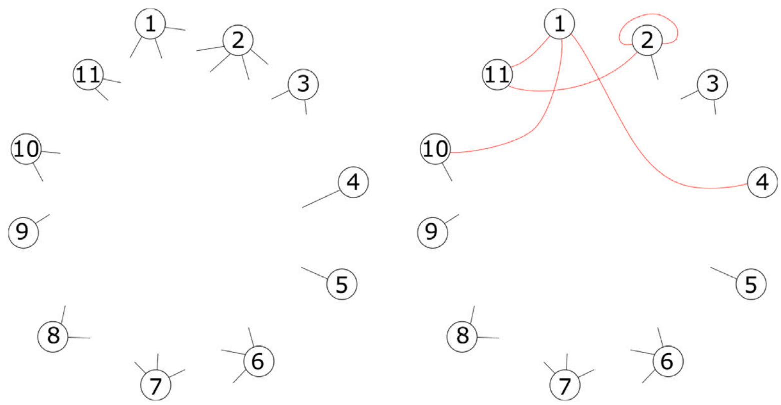

Fig. 3. Left: the initial collection of 11 vertices with half-edges appearing from the center. Right: the structure of the breadth-first constructed graph after initially selecting a half-edge connected to vertex $v_1^{\mathrm{BF}} = 11$. The edges were added in this order $\{11,1\},\{11,2\},\{1,4\},\{1,10\},\{2,2\}$. Consequently, $v_2^{\mathrm{BF}} = 1$, $v_3^{\mathrm{BF}} = 2$, $v_4^{\mathrm{BF}} = 4$, $v_5^{\mathrm{BF}} = 10$. The next half-edge to be explored is the remaining half-edge jutting out from vertex 2.

In case 1 above, it is possible that we match two half-edges l and r where r is already active. We call the corresponding edge in the multigraph $M^{\mathrm{BF}}(\mathbf{d}^n)$ a *breadth-first (bf) backedge*.

We let $F^{\mathrm{BF}}(\mathbf{d}^n)$ denote the forest constructed from the multigraph $M^{\mathrm{BF}}(\mathbf{d}^n)$ obtained by splitting all bf backedges into two half-edges and adding two leaves to each of these half-edges. More formally, if the multigraph $M^{\mathrm{BF}}(\mathbf{d}^n)$ has a bf backedge between vertices v_l, v_r. Remove that edge from the multigraph and add two vertices v'_l and v'_r and add an edge between both pairs (v_l, v'_l) and (v_r, v'_r). Continue this until all bf backedges are removed and replaced.

Remark 5.2. This algorithm can also be used to mark where the new leaves occur within a breadth-first exploration of the forest $F^{\mathrm{BF}}(\mathbf{d}^n)$. When we first find the backedge between half-edges l and r in $M^{\mathrm{BF}}(\mathbf{d}^n)$ we replace it with two new leaves. Then as we are exploring the half-edge l, we find a new leaf and do not “see” the half-edge r. This means we do not kill that half-edge in step (6). This means we will eventually choose half-edge r in step (2) and find a second new leaf for this bf backedge. We then label all the vertices in $F^{\mathrm{BF}}(\mathbf{d}^n)$ in a breadth-first manner by $u_1^{\mathrm{BF}}, \cdots, u_p^{\mathrm{BF}}$ for some $p \ge n$. See Fig. 4 for an example of how this is done for the breadth-first construction.

5.1.2. Depth-first construction

We construct a graph $M^{\mathrm{DF}}(\mathbf{d}^n)$ (we initially include a DF to specify the construction) as follows:

We initialize at step 1 as before, we pick a sleeping half-edge uniformly at random. Label the corresponding vertex as v_1 and declare all of the half-edges attached to v_1 as active. The only thing that changes on subsequent steps is that in Case 1 we replace part (1) with

(1’) Let i be the **largest** integer k such that there exists an active half-edge attached to v_k.

This above is the depth-first construction of the multigraph $M^{\mathrm{DF}}(\mathbf{d}^n)$. This changes the order in which we find vertices and label them, so we will denote the new ordering and labeling by $v_1^{\mathrm{DF}}, \cdots, v_n^{\mathrm{DF}}$. We analogously construct the depth-first forest, by removing depth-first (df) backedges and replacing them with leaves.

5.1.3. Symmetry between constructions

Recall that the multigraph $M(\mathbf{d}^n)$ is taken to be uniform over all possible pairings of half-edges. The following claim is trivial.

Claim 14. *For any degree sequence $\mathbf{d}^n = (d_1, \cdots, d_n)$ the graphs $M^{\mathrm{BF}}(\mathbf{d}^n)$ and $M^{\mathrm{DF}}(\mathbf{d}^n)$ are equal in distribution. We write both as $M(\mathbf{d}^n)$.*

The symmetry between the constructions allows us to look at two random walks which turn out being equal in distribution. We write $\deg(v)$ for the degree (counted with multiplicity) of the vertex v in a graph, $M(\mathbf{d}^n)$, $F^{\mathrm{DF}}(\mathbf{d}^n)$, etc., which is clear from context. Recall that $(v_j^{\mathrm{BF}}; j \in [n])$ and $(v_j^{\mathrm{DF}}; j \in [n])$ are the vertices in the multigraph $M(\mathbf{d}^n)$ labeled in two distinct ways according the breadth-first exploration or depth-first exploration respectively. Define the following two walks:

$$S_{\mathbf{d}^n}^{\mathrm{BF}}(k) = \sum_{j=1}^{k} (\deg(v_j^{\mathrm{BF}}) - 2), \qquad S_{\mathbf{d}^n}^{\mathrm{DF}}(k) = \sum_{j=1}^{k} (\deg(v_j^{\mathrm{DF}}) - 2). \tag{18}$$

We wish to define analogous walks on the forests $F^{\mathrm{BF}}(\mathbf{d}^n)$ and $F^{\mathrm{DF}}(\mathbf{d}^n)$, to which we remind the reader of Remark 5.2. Hence we have a labeling of all the vertices of the forests as (u_j^{BF}) for the forest $F^{\mathrm{BF}}(\mathbf{d}^n)$ with the breadth-first exploration and (u_j^{DF}) for the forest $F^{\mathrm{DF}}(\mathbf{d}^n)$ with the depth-first exploration.

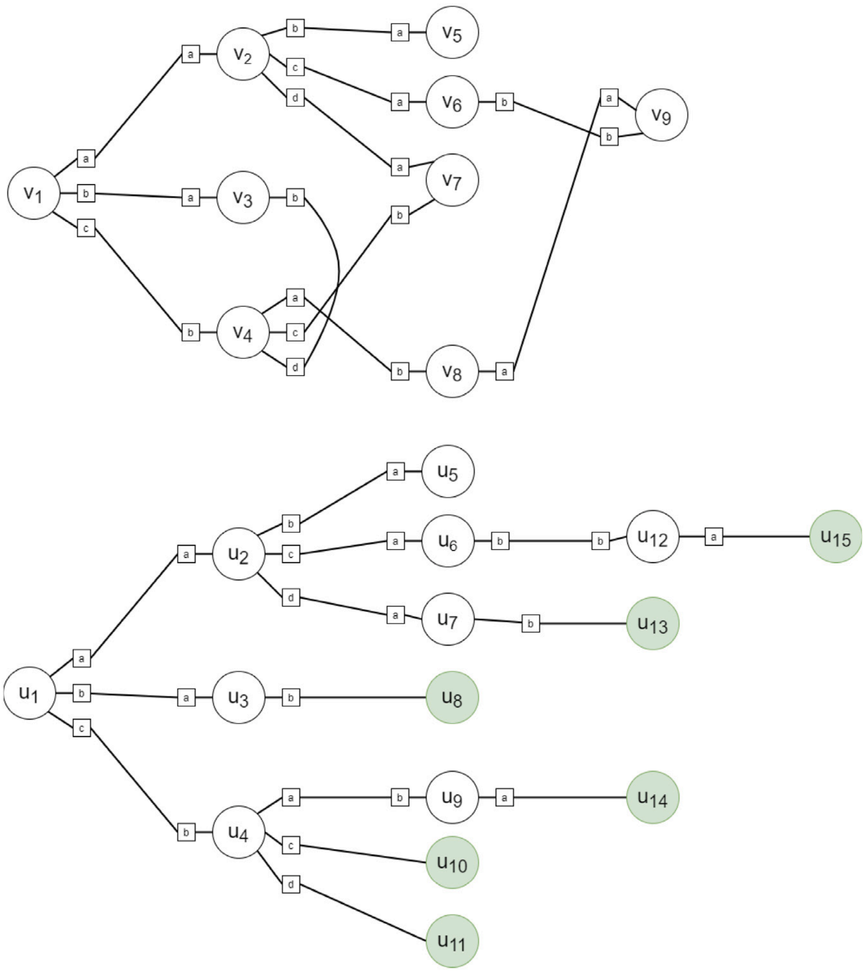

Fig. 4. The first component of $M^{\mathrm{BF}}(\mathbf{d}^n)$ (top) and the corresponding first component of $F^{\mathrm{BF}}(\mathbf{d}^n)$ (bottom). The circles are the vertices, the labeled squares are the ordering of the half-edges connected to each hub with $a < b < c < d$. The three bf backedges in this graph connect half-edge (v_3, b) to (v_4, d), half-edge (v_4, c) to (v_7, b), and half-edge (v_8, a) to (v_9, a). The new leaves are vertices $u_8, u_{10}, u_{11}, u_{13}, u_{14}, u_{15}$ in green and they are ordered according to the ordering of the half-edge to which it is connected. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

For each connected component of the multigraph and hence forest, there is a vertex discovered when the collection of active vertices $\mathcal{A}$ was empty, we call those vertices *roots*. If u is a root in a forest, write $\chi(u) = \deg(u)$, otherwise write $\chi(u) = \deg(u) - 1$. The value of $\chi(u)$ is precisely the number of children that vertex u has in the forest in which it lives. For j sufficiently large, there will be no vertex u_j^{BF} or u_j^{DF}. This will not matter for our scaling limits, but for completeness we define u_j^{BF} and u_j^{DF} as root vertices of components with a single vertex, and therefore $\chi(u_j^{\mathrm{BF}}) = \chi(u_j^{\mathrm{DF}}) = 0$ for sufficiently large j. Define the walks

$$X_{\mathbf{d}^n}^{\mathrm{BF}}(k) = \sum_{j=1}^{k} (\chi(u_j^{\mathrm{BF}}) - 1), \qquad X_{\mathbf{d}^n}^{\mathrm{DF}}(k) = \sum_{j=1}^{k} (\chi(u_j^{\mathrm{DF}}) - 1). \tag{19}$$

As is shown in Section 5.1 of [7], the distribution of the depth first walk $X_{\mathbf{d}^n}^{\mathrm{DF}}$ can be reconstructed from $S_{\mathbf{d}^n}^{\mathrm{DF}}$. This was done when the degree sequence is taken to be random; however it works for a deterministic degree sequence as well. A trivial alteration of that algorithm can be used to construct $X_{\mathbf{d}^n}^{\mathrm{BF}}$ from the walk $S_{\mathbf{d}^n}^{\mathrm{BF}}$ and moreover, we can couple these two constructions to see that backedges have a particular correspondence. We write this as the following lemma.

Lemma 15.

1. *For any degree sequence $\mathbf{d}^n$, the breadth-first walks and the depth-first walks are equal in distribution. That is*

$$(S_{\mathbf{d}^n}^{\mathrm{BF}}(k); k \ge 0) \overset{d}{=} (S_{\mathbf{d}^n}^{\mathrm{DF}}(k); k \ge 0), \qquad \textit{and} \qquad (X_{\mathbf{d}^n}^{\mathrm{BF}}(k); k \ge 0) \overset{d}{=} (X_{\mathbf{d}^n}^{\mathrm{DF}}(k); k \ge 0).$$

2. *There exists a coupling of $F^{\mathrm{DF}}(\mathbf{d}^n)$ and $F^{\mathrm{BF}}(\mathbf{d}^n)$ such that for all $j < i$ a df-backedge appears between u_j^{DF} and u_i^{DF} if and only if a bf-backedge appears between u_j^{BF} and u_i^{BF}.*

In particular, there exists a coupling of X_n^{BF} and X_n^{DF} such that

$$\left(\inf_{i\le k} X_n^{\mathrm{BF}}(i); k\ge 0\right) = \left(\inf_{i\le k} X_n^{\mathrm{DF}}(i); k\ge 0\right).$$

The last part of the above lemma tells us something that will be used several times in the sequel, under this coupling the distribution excursions of $X_{\mathbf{d}^n}^{\mathrm{BF}}$ and $X_{\mathbf{d}^n}^{\mathrm{DF}}$ above the running infimum have the same length.

5.1.4. Random degree distribution and continuum graphs

In this subsection we describe some of what happens when we take the sequence $\mathbf{d}^n = (d_1, \cdots, d_n)$ to be from i.i.d. samples from a distribution ν on $\mathbb{N} = \{1, 2, \cdots\}$. We take $(d_j; j \ge 1)$ to be an i.i.d. sequence with common distribution ν. In order to guarantee that a multigraph with degree sequence $\mathbf{d}^n = (d_1, \cdots, d_n)$ exists, we replace d_n with $d_n + 1$ if the sum has the wrong parity.

Recall from the introduction that we write $M_n(\nu)$ for the random graphs with a random degree distribution. We do the same when referencing the forests, i.e. we will write $F_n^{\mathrm{BF}}(\nu)$ instead of $F^{\mathrm{BF}}(\mathbf{d}^n)$ when the degree sequence is random. We do not emphasize this dependence on ν when describing the random walks, instead we replace subscript $\mathbf{d}^n$ with just n, i.e. we write X_n^{DF} instead of $X_{\mathbf{d}^n}^{\mathrm{DF}}$. We will now restrict our attention to the case where ν is as in (1).

In this setting Joseph [8] gives a scaling limit of a depth-first walk for the multigraph $M_n(\nu)$, which is very slightly different than what we wrote as S_n^{DF}. That work was extended by Conchon-Kerjan and Goldschmidt in [7]. We now recall the scaling limit in the latter reference. Let $\tilde{X} = (\tilde{X}(t); t \ge 0)$ and $\tilde{H} = (\tilde{H}(t); t \ge 0)$ be defined by the change of measure in (11).

Theorem 16 (*Joseph [8], Conchon-Kerjan–Goldschmidt [7]*). *Fix some $\alpha \in (1,2)$. Let ν be a distribution satisfying (1) and write $A = \frac{c\Gamma(2-\alpha)}{\delta\alpha(\alpha-1)}$. Using the notation above, the following joint convergence holds in $\mathbb{D}^2$:*

$$\left(n^{-\frac{1}{\alpha+1}} S_n(\lfloor n^{\frac{\alpha}{\alpha+1}} t\rfloor); t\ge 0\right) \overset{d}{\Longrightarrow} \left(\tilde{X}(t); t\ge 0\right).$$

A similar result is obtained in [64] under a finite third moment condition on the measure ν where the limiting process is a scaled Brownian motion with a parabolic drift.

Crucially for their descriptions of the limiting graphs, the authors of [7] also develop the excursion theory for the process $\tilde{X}$ (which is notated as $\tilde{L}$ in that work). Proposition 3.9 in [7] shows that the excursions of $\tilde{X}, \tilde{H}$, conditioned on their length being exactly x are distributed as $(\tilde{\mathbf{e}}_x^{(\delta)}, \tilde{\mathbf{h}}_x^{(\delta)})$ defined in (14).

We now heuristically describe how the authors of [7] obtain their metric space scaling limit. Let H_n be the height process on the forest $F_n^{\mathrm{DF}}(\nu)$. That is $H_n(k)$ is the distance in $F_n^{\mathrm{DF}}(\nu)$ from vertex u_k^{DF} to the root in its connected component. This process H_n satisfies [39]:

$$H_n(k) = \#\{j \in \{0, \cdots, k-1\} : X_n^{\mathrm{DF}}(j) = \inf_{j\le \ell\le k} X_n^{\mathrm{DF}}(\ell)\}.$$

To examine the components of the graph $M_n(\nu)$, the authors of [7] look at the collection of excursions of the processes X_n^{DF} and H_n. These are defined as follows:

$$\begin{aligned} \widehat{X}_{n,i}^{\mathrm{DF}}(k) &= X_n^{\mathrm{DF}}(\sigma_n(i-1)+k) - X_n^{\mathrm{DF}}(\sigma_n(i-1)), \\ \widehat{H}_{n,i}(k) &= H_n(\sigma_n(i-1)+k), \end{aligned} \qquad k = 0, \cdots, \sigma_n(i) - \sigma_n(i-1)$$

where $\sigma_n(i) = \inf\{j : X_n^{\mathrm{DF}}(j) = -i\}$ is the first hitting time of level $-i$. The process $\widehat{X}_{n,i}^{\mathrm{DF}}$ starts at zero and is non-negative until it hits level -1 at time $k = \sigma_n(i) - \sigma_n(i-1)$. The process $\widehat{H}_{n,i}$ is strictly positive for $k = 1, \ldots, \sigma_n(i) - \sigma_n(i-1) - 1$. We extend both of these by constancy for $k > \sigma_n(i) - \sigma_n(i-1)$. These processes encode the tree structure [39] of the ith connected component of the forest $F_n^{\mathrm{DF}}(\nu)$. By the construction of $M_n^{\mathrm{DF}}(\nu)$, this orders the components of the forest $F_n^{\mathrm{DF}}(\nu)$ in a manner size-biased by the number of edges in the component. There are further only a finite number of indices i such that $\sigma_n(i) - \sigma_n(i-1) \neq 1$ since for sufficiently large i the ith component of $F_n^{\mathrm{DF}}(\nu)$ is simply an isolated vertex.

To study the large components of the graph, we instead reorder the excursions by decreasing lengths with ties broken arbitrarily. Denote this new ordering by omitting the "widehat" notation: $(X_{n,i}^{\mathrm{DF}}; i \ge 1) = \left((X_{n,i}^{\mathrm{DF}}(k); k\ge 0); i \ge 1\right)$ and $(H_{n,i}; i \ge 1) = \left((H_{n,i}(k); k \ge 0); i \ge 1\right)$.

The ith excursion $X_{n,i}^{\mathrm{DF}}$ may not tell us information about the ith largest component of $M_n(\nu)$, G_n^i. This is because the forest $F_n^{\mathrm{DF}}(\nu)$ contains additional vertices which could change the ordering of the components. I.e. if G_n^i had 10 vertices and 0 df backedges and component G_n^{i+1} had 9 vertices and 2 df backedges then the corresponding components in $F_n^{\mathrm{DF}}(\nu)$ will have 10 and 13 vertices respectively. In turn, their indices will appear in the reversed order. We also note that the excursions of the process X_n^{DF} do not identically correspond to the excursions of the process S_n^{DF} discussed previously. While this may cause some problems in the discrete, in the large n limit neither of these problems are relevant as we will shortly explain.

Before turning to the scaling limits, let T_n^1 be the largest connected component, i.e. tree, contained in $F_n^{\mathrm{DF}}(\nu)$. This is encoded by $X_{n,1}^{\mathrm{DF}}$ and $H_{n,1}$. This tree may contain df backedges and these backedges appear in pairs. For concreteness, suppose that there are $m \ge 1$ of these pairs. These can be indexed by $(l_1, r_1), \cdots, (l_m, r_m)$. This means that the l_ith vertex explored in the depth-first

Fig. 5. The excursion $X^{\mathrm{BF}}_{n,i}$ associated with the component shown in Fig. 4 with the marks $\mathscr{P}^{\mathrm{BF}}_{n,i}$ included. The vertices u^{BF}_8 and u^{BF}_{11} are paired, u^{BF}_{10} and u^{BF}_{13} are paired, and u^{BF}_{14} and u^{BF}_{15} are paired. These are represented by the green circles, red triangles and blue squares above.

exploration of the largest component of T^1_n will be paired with the r_ith vertex explored in the corresponding component of $M_n(\nu)$. See Fig. 5 for the analogous pairs for the breadth-first labeling of the component in Fig. 4. We now define $\mathscr{P}^{\mathrm{DF}}_{n,1}$ as the collection of points

$$\mathscr{P}^{\mathrm{DF}}_{n,1} = \left\{ \left(n^{-\frac{\alpha}{\alpha+1}} l_i, n^{-\frac{\alpha}{\alpha+1}} r_i\right) : i = 1, \cdots, m \right\}.$$

When there are no df backedges present, just define the set as the empty set. We call this set the set of *marks*, and we can do the same thing to each of the other connected components as well to get sets $\mathscr{P}^{\mathrm{DF}}_{n,i}$.

An important step in how the authors of [7] proved that the components $G^1_n, G^2_n, \cdots$ have scaling limits was showing scaling limits of the processes $X^{\mathrm{DF}}_{n,i}$ and $H_{n,i}$ and of the set of marks $\mathscr{P}^{\mathrm{DF}}_{n,i}$. The convergence of the sets $\mathscr{P}^{\mathrm{DF}}_{n,i}$ is with respect to the vague topology of its associated counting measure. Namely they prove [7, Proposition 5.16] that

$$\begin{aligned}\left(\left(n^{-\frac{1}{\alpha+1}} X^{\mathrm{DF}}_{n,i}(\lfloor n^{\frac{\alpha}{\alpha+1}} t\rfloor); t \ge 0\right), \left(n^{-\frac{\alpha-1}{\alpha+1}} H_{n,i}(\lfloor n^{\frac{\alpha}{\alpha+1}} t\rfloor); t \ge 0\right), \mathscr{P}^{\mathrm{DF}}_{n,i}; i \ge 1\right)& \\ \overset{d}{\Longrightarrow} \left(\tilde{e}_i, \tilde{h}_i, \mathscr{P}_i; i \ge 1\right),& \end{aligned} \tag{20}$$

for some discrete sets $(\mathscr{P}_i; i \ge 1)$. Here the convergence in the first two coordinates is with respect to the Skorohod topology and the convergence in the third coordinate is with respect to the vague topology of its associated counting measure and then the product topology is taken over the index $i \ge 1$.

The limiting set $\mathscr{P}_i$ can be described as follows. Let $(\mathcal{Q}_i; i \ge 1)$ denote an i.i.d. collection of Poisson point processes on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\frac{1}{\delta}$ Leb. Only finitely many of these points $(s, y) \in \mathcal{Q}_i$ will satisfy $0 \le y \le \tilde{e}_i(s)$, and index these as $(s^1, y^1), \cdots, (s^m, y^m)$ for some m. The set $\mathscr{P}_i$ is then the collection

$$\mathscr{P}_i = \{(s^p, t^p) : p = 1, \cdots, m\} \qquad t^p = \inf\{u \ge s^p : \tilde{e}_i(u) \le y^p\}.$$

This, in turn, allowed them to show that the ordered sequence of components of $M_n(\nu)$ and $G_n(\nu)$ by conditioning converge after proper rescaling in a product Gromov–Hausdorff–Prokhorov topology to the sequence of continuum random graphs

$$\left(\mathcal{M}_i; i \ge 1\right) = \left(\mathscr{G}(\tilde{h}_i, \tilde{e}_i, \mathcal{Q}_i); i \ge 1\right), \tag{21}$$

where $\mathcal{Q}_i$ were defined above.

Let us summarize these results in a theorem for easy reference.

Theorem 17 (*Conchon-Kerjan–Goldschmidt [7]*). *Let ν be a distribution satisfying (1) and write $A = \frac{c\Gamma(2-\alpha)}{\delta\alpha(\alpha-1)}$. Let $(G^i_n; i \ge 1)$ denote the components of the critical random graph $M_n(\nu)$ ordered by decreasing number of vertices and viewed as pointed measured metric spaces. Fix some $\alpha \in (1, 2)$. The convergence in (20) holds in the product topology (product over the index i). Jointly with (20), the weak convergence*

$$\left(\mathrm{scale}(n^{-\frac{\alpha-1}{\alpha+1}}, n^{-\frac{\alpha}{\alpha+1}}) G^i_n; i \ge 1\right) \overset{d}{\Longrightarrow} (\mathcal{M}_i; i \ge 1) \tag{22}$$

holds with respect to the product Gromov–Hausdorff–Prokhorov topology, where the sequence $(\mathcal{M}_i; i \ge 1)$ is distributed as (21).

Remark 5.3. Theorem 1.1 in [7] does not state the joint convergence between Eqs. (20) and (22); however, the proof of said theorem shows there is joint convergence.

These recalled results from [7] are on the convergence for random graph G^i_n constructed in *depth-first* manner. However, symmetries between depth-first and breadth-first constructions described above in Lemma 15 allow us to have similar results for the analogous *breadth-first* construction. Consequently, if we let $X^{\mathrm{BF}}_{n,i}$ be the breadth-first walk of the ith largest component of $F^{\mathrm{BF}}_n(v)$, or, equivalently stated, it is the ith longest excursion of X^{BF}_n above its running minimum, then

$$\left(\left(n^{-\frac{1}{\alpha+1}}X^{\mathrm{BF}}_{n,i}(\lfloor n^{\frac{\alpha}{\alpha+1}}t\rfloor);t\geq 0\right);i\geq 1\right)\overset{d}{\Longrightarrow}\left(\tilde{e}^*_i;i\geq 1\right), \tag{23}$$

where $\left(\tilde{e}^*_i;i\geq 1\right)\overset{d}{=}\left(\tilde{e}_i;i\geq 1\right)$. We write $\tilde{e}^*_i$ instead of $\tilde{e}_i$ so when we take subsequential weak limits with (20) we can distinguish between the limits of $X^{\mathrm{DF}}_{n,i}$ and $X^{\mathrm{BF}}_{n,i}$. More importantly for our work, there are auxiliary processes described in [7, Section 5.1] from which $\mathscr{P}^{\mathrm{DF}}_{n,k}$ can be constructed. In particular, Lemma 15 extends to the following:

Lemma 18. *There exists a finite set $\mathscr{P}^{\mathrm{BF}}_{n,i}\subset\mathbb{R}^2_+$ of marks corresponding to the ith largest component of $F^{\mathrm{BF}}_n(v)$ which keep track of the bf backedges such that*

$$\left(X^{\mathrm{BF}}_{n,i},\mathscr{P}^{\mathrm{BF}}_{n,i};i\geq 1\right)\overset{d}{=}\left(X^{\mathrm{DF}}_{n,i},\mathscr{P}^{\mathrm{DF}}_{n,i};i\geq 1\right).$$

We now state the following lemma:

Lemma 19. *Couple X^{BF}_n and X^{DF}_n as in Lemma 15 so they have the same excursion intervals. Then, under the assumptions of Theorem 17, any joint subsequential weak limit of (20), (22) and (23) in the product topology satisfies for each fixed $i\geq 1$:*

$$\zeta(\tilde{e}^*_i)=\zeta(\tilde{e}_i)=\mu_i(\mathcal{M}_i), \tag{24}$$

almost surely.

Proof. Under this conditioning, the excursion intervals of X^{DF}_n and X^{BF}_n have the same length. Hence the $\zeta(\tilde{e}^*_i)=\zeta(\tilde{e}_i)$. The equality $\zeta(\tilde{e}_i)=\mu_i(\mathcal{M}_i)$ follows from construction of $\mathcal{M}_i$ as a quotient of a continuum random tree with height function $\tilde{h}_i$ of duration $\zeta(\tilde{e}_i)$. More precisely, with probability 1 the space $\mathcal{M}_i$ is obtained by identifying a finite number of points in a continuum random tree $\mathcal{T}_i$ of total mass $\zeta(\tilde{e}_i)$. Since the mass measure μ_i on $\mathcal{M}_i$ is the pushforward of the mass measure on $\mathcal{T}_i$, we also have $\mu(\mathcal{M}_i)=\zeta(\tilde{e}_i)$. □

We have now gathered most of the required ingredients and background to prove Theorems 1 and 2 using the approach in Theorem 5. The last thing we will verify is that Assumption 3 holds in Theorems 4 and 5. By scaling of the α stable graph [7], in particular Theorem 1.2 and Proposition 3.9 therein, we focus on the case that the total mass equals 1.

Proposition 20. *Fix $\alpha\in(1,2)$. Let $(\tilde{\mathbf{e}}^{(\delta)}_x,\tilde{\mathbf{h}}^{(\delta)}_x)$ be defined as in (14). Let $\mathscr{G}=\mathscr{G}(\tilde{\mathbf{h}}^{(\delta)}_x,\tilde{\mathbf{e}}^{(\delta)}_x,\mathscr{Q})$ denote the continuum random graph where $\mathscr{Q}$ is a Poisson point process with intensity $\frac{1}{\delta}\,\mathrm{Leb}$ for some $\delta>0$. Then, almost surely,*

$$\mu(B(\rho,t)\setminus\{\rho\})>0,\qquad\forall t>0.$$

The same holds for the graphs $\mathcal{M}_i$ appearing in (21).

Proof. The same statement holds for the graphs $\mathcal{M}_i$ by conditioning on their mass $\mu_i(\mathcal{M}_i)$ [7, Theorem 1.2].

We use the tree $\mathscr{T}_{\tilde{\mathbf{h}}^{(\delta)}_x}$ as a measured metric space, and when confusion might arise we will use subscripts to specify whether we are dealing with the tree or the graph.

We observe that from the quotient map

$$q:\mathscr{T}_{\tilde{\mathbf{h}}^{(\delta)}_x}\longrightarrow\mathscr{G}(\tilde{\mathbf{h}}^{(\delta)}_x,\tilde{\mathbf{e}}^{(\delta)}_x,\mathscr{Q})$$

in the construction of the random graph satisfies the following:

$$\mathrm{d}_{\mathscr{G}}(\rho,q(x))\leq\mathrm{d}_{\mathscr{T}_{\tilde{\mathbf{h}}^{(\delta)}_x}}(\rho,x),\qquad\forall x\in\mathscr{T}_{\tilde{\mathbf{h}}^{(\delta)}_x}.$$

Consequently,

$$\mu_{\mathscr{G}}\left(B_{\mathscr{G}}(\rho,t)\right)\geq\mu_{\mathscr{T}_{\tilde{\mathbf{h}}^{(\delta)}_x}}\left(B_{\mathscr{T}_{\tilde{\mathbf{h}}^{(\delta)}_x}}(\rho,t)\right)=\int_0^1 1_{[\tilde{\mathbf{h}}^{(\delta)}_x(s)\in(0,t)]}\,ds.$$

Since the process $\tilde{\mathbf{h}}^{(\delta)}_x$ is continuous, non-negative, almost surely not identically zero, but $\tilde{\mathbf{h}}^{(\delta)}(0)=0$ this integral is strictly positive for all $t>0$. The result follows easily. □

5.2. Proof of Theorem 1

We now prove Theorem 1.

Proof of Theorem 1. Throughout the proof all limits will be as $n\to\infty$, possibly along a subsequence which we denote by n.

The processes $Z_{n,i}$ measure the number of vertices infected on day h, which is simply the number of vertices at distance h from ρ_i in the ith largest connected component G^i_n:

$$Z_{n,i}(h)=\#\{v\in G^i_n\,:\,d(v,\rho_i)=h\}$$

and let process $C_{n,i}=(C_{n,i}(h);h\geq 0)$ denote its running sum $C_{n,i}(h)=\sum_{j=0}^h Z_{n,i}(j)$.

Write $Z^*_{n,i}(h)$ as the discrete Lamperti transform of $X^{\mathrm{BF}}_{n,i}$:

$$Z^*_{n,i}(h+1) = 1 + X^{\mathrm{BF}}_{n,i}\circ C^*_{n,i}(h), \qquad C^*_{n,i}(h) = \sum_{j=0}^{h} Z^*_{n,i}(j). \tag{25}$$

This process $Z^*_{n,i}$ measures the number of vertices at height h in the ith largest component of the forest $F^{\mathrm{BF}}_n(v)$; see the discussion around (6) and more generally [30].

As the forest $F^{\mathrm{BF}}_n(v)$ is obtained by adding at exactly two leaves for each bf backedge discovered in the exploration of $M_n(v)$ we get

$$\sum_{h\geq 0} |Z_{n,i}(h) - Z^*_{n,i}(h)| = 2\cdot \#\{\text{bf backedges in } G^i_n\} =: \kappa_{n,i} \qquad \text{(say)}. \tag{26}$$

It follows from (20) that the sequence of random variables $(\kappa_{n,i})_{n\geq 1} \overset{d}{=} (2\#\mathscr{P}^{\mathrm{DF}}_{n,i})_{n\geq 1}$ is tight for each i.

Unfortunately, a direct application of Theorem 5 is not possible because the (20) only implies the convergence of the trees in the depth-first forest, while we are using a breadth-first forest. However, most of the proof structure still works. To simplify notation, we write

$$\widetilde{Z}_{n,i}(t) = n^{-\frac{1}{\alpha+1}} Z_{n,i}(\lfloor n^{\frac{\alpha-1}{\alpha+1}} t\rfloor), \quad \widetilde{C}_{n,i}(t) = n^{-\frac{\alpha}{\alpha+1}} C_{n,i}(\lfloor n^{\frac{\alpha-1}{\alpha+1}} t\rfloor), \quad \widetilde{X}^{\mathrm{BF}}_{n,i}(t) = n^{-\frac{1}{\alpha+1}} X^{\mathrm{BF}}_{n,i}(\lfloor n^{\frac{\alpha}{\alpha+1}} t\rfloor),$$

and similarly define $\widetilde{C}^*_{n,i}, \widetilde{Z}^*_{n,i}$.

We can apply the reasoning in Proposition 12 to see that for each i,

$$\left((\widetilde{C}^*_{n,i}, \widetilde{X}^{\mathrm{BF}}_{n,i})_{n\geq 1}\right) \qquad \text{is tight in } \mathbb{D}^2$$

and all subsequential limits are of the form $C_i(t) = \int_0^t \tilde{e}^*_i\circ C_i(s)\,ds$ where $\tilde{e}^*_i$ is the weak limit of $\tilde{X}^{\mathrm{BF}}_{n,i}$ appearing in (23). Examining (26), we see that because the sequence $(\kappa_{n,i})_{n\geq 1}$ is tight for each i that

$$\left((\widetilde{C}_{n,i}, \widetilde{X}^{\mathrm{BF}}_{n,i})_{n\geq 1}\right) \qquad \text{is tight in } \mathbb{D}^2 \tag{27}$$

and all subsequential limits must also be of the form $C_i(t) = \int_0^t \tilde{e}^*_i\circ C_i(s)\,ds$.

Note that $C_{n,i}(h)$ is the mass of the closed ball of radius h in the graph G^i_n which converges in the Gromov–Hausdorff–Prohorov metric (Theorem 17) to a metric space $\mathcal{M}_i$ satisfying $\mu_i(B_{\mathcal{M}_i}(\rho,t)\setminus\{\rho\}) > 0$ for all $t > 0$ (Proposition 20). Therefore, as in Proposition 13, we can conclude that all subsequential limits $(C_i, \tilde{e}^*_i)$ of the tight sequence $(\widetilde{C}_{n,i}, X^{\mathrm{BF}}_{n,i})$ must be of the form

$$C_i(t) = \int_0^t \tilde{e}^*_i\circ C_i(s)\,ds, \qquad \text{and} \qquad \inf\{t : C_i(t) > 0\} = 0.$$

Since such solutions are unique (Proposition 8(ii)), we conclude both $(\widetilde{C}_{n,i}(t), \widetilde{X}^{\mathrm{BF}}_{n,i}(t)) \overset{d}{\Longrightarrow} (C_i, \tilde{e}^*_i)$ and $(\widetilde{C}^*_{n,i}(t), \widetilde{X}^{\mathrm{BF}}_{n,i}(t)) \overset{d}{\Longrightarrow} (C_i, \tilde{e}^*_i)$. Moreover, as the weak convergence of $\widetilde{X}^{\mathrm{BF}}_{n,i}$ holds in the product topology ((23)) over the index i and so does the convergence of the metric spaces (Theorem 16), this analysis extends for the joint convergence of $(\widetilde{C}_{n,i}, \widetilde{X}^{\mathrm{BF}}_{n,i})_{i\geq 1}$.

Using [60, Theorem 1.2] as before, $(\widetilde{Z}^*_{n,i})_{i\geq 1} \overset{d}{\Longrightarrow} (Z_i)_{i\geq 1}$ where Z_i are as in the statement of the theorem, i.e. $Z_i(t) = \tilde{e}^*_i\circ C_i(t)$. The tightness of the sequence $\kappa_{n,i}$ and (26) implies the desired convergence. $\square$

Before turning to proof of Theorem 2, we state and prove the following lemma:

Lemma 21. *Couple the depth-first and breadth-first walks as in Lemma 15. Under the assumptions of Theorem 16, and using the notation in (25). There is joint convergence in distribution along a subsequence of the index $n \geq 1$ of the collection*

$$\Bigg(\bigg(\Big(n^{-\frac{1}{\alpha+1}} Z^*_{n,i}(\lfloor n^{\frac{\alpha-1}{\alpha+1}} t\rfloor); t\geq 0\Big), \Big(n^{-\frac{\alpha}{\alpha+1}} C^*_{n,i}(\lfloor n^{\frac{\alpha-1}{\alpha+1}} t\rfloor); t\geq 0\Big),$$
$$\Big(n^{-\frac{1}{\alpha+1}} X^{\mathrm{BF}}_{n,i}(\lfloor n^{\frac{\alpha}{\alpha+1}} t\rfloor); t\geq 0\Big), \mathrm{scale}(n^{-\frac{\alpha-1}{\alpha+1}}, n^{-\frac{\alpha}{\alpha+1}}) G^i_n, \#\mathscr{P}^{\mathrm{BF}}_{n,i}\bigg); i\geq 1\Bigg)_{n\geq 1}$$

towards

$$((Z_i, C_i, \tilde{e}^*_i, \mathcal{M}_i, \mathrm{sur}(\mathcal{M}_i)); i\geq 1),$$

where

1. *$\mathcal{M}_i$ are as in Theorem 17 and (21);*
2. *(Z_i, C_i) is the Lamperti pair associated with the excursion $\tilde{e}^*_i$;*
3. *The process $C_i(t) = \mu_i(B(\rho,t))$ for almost all (and hence all) $t \geq 0$;*
4. *The excursions $(\tilde{e}^*_i; i\geq 1) \overset{d}{=} (\tilde{e}_i; i\geq 1)$ in the construction of $\mathcal{M}_i$, and, in particular, the length of the excursion $\tilde{e}^*_i$ is the mass of the space $\mathcal{M}_i$, i.e.*

$$\zeta(\tilde{e}^*_i) = \mathcal{M}_i;$$

5. *The random variable* $\mathrm{sur}(\mathcal{M}_i)$, *the surplus of the space* $\mathcal{M}_i$ *is*

$$\mathrm{sur}(\mathcal{M}_i) \overset{d}{=} \mathrm{Poisson}\left(\frac{1}{\delta}\int_0^{\zeta(\tilde{e}_i^*)} \tilde{e}_i^*(t)\,dt\right);$$

6. *Lastly, conditionally on the length of excursion* $\zeta_i := \zeta(\tilde{e}_i^*)$ *and the surplus values* $R_i := \mathrm{sur}(\mathcal{M}_i)$, *the graph* $\mathcal{M}_i$ *satisfies*

$$\mathcal{M}_i \overset{d}{=} \mathrm{scale}\left(\zeta_i^{(\alpha-1)/\alpha}, \zeta_i\right) \mathscr{G}^{(\alpha,R_i)}$$

Proof. These are tight random variables in each of the marginals, so joint convergence along a subsequence is standard.

Item 1 follows from the referenced theorem.

Item 2 follows from the proof of Theorem 1 and the identity in distribution $(\tilde{e}_i; i \ge 1) \overset{d}{=} (\tilde{e}_i^*; i \ge 1)$ previously seen.

Item 3 follows from the proof of Theorem 1.

Item 4 is from Lemma 19.

Item 5 follows from Theorem 5.5 and Proposition 5.12 in [7] along with the equality $\#\mathscr{P}_{n,1}^{\mathrm{DF}} = \#\mathscr{P}_{n,i}^{\mathrm{BF}}$.

Item 6 is Theorem 1.2 in [7]. □

5.3. Proof of Theorem 2

Proof of Theorem 2. The big content of this proof is to show that we can condition on the length $\zeta(\tilde{e}_i^*)$ of the excursion $\tilde{e}_i^*$ and the surplus of the graphs $\mathcal{M}_i$ by using Lemma 21 and scaling results for the excursions proved in [7].

By Lemma 21, we know that we can write the height profile of the graph $\mathcal{M}_i$ (which is of random mass) as the process Z_i where (Z_i, C_i) is the Lamperti pair associated with the excursion $\tilde{e}_i^*$. In fact, we know

$$\left(\mu_i(B(\rho_i, v); v \ge 0), \mu_i(\mathcal{M}_i), \mathrm{sur}(\mathcal{M}_i)\right)_{i\ge1} \overset{d}{=} \left((C_i(v); v \ge 0), \zeta(\tilde{e}_i^*), R_i\right)_{i\ge1}$$

where (Z_i, C_i) is the Lamperti pair associated with the excursion $\tilde{e}_i^*$ and $R_i \sim \mathrm{Poisson}\left(\frac{1}{\delta}\int_0^{\zeta(\tilde{e}_i^*)} \tilde{e}_i^*(t)\,dt\right)$.

Conditioning on the values of $\zeta(\tilde{e}_1^*)$ and R_1 gives

$$\left((\mu_1(B(\rho_1, r)); r \ge 0)\middle|\mu_1(\mathcal{M}_1) = 1, \mathrm{sur}(\mathcal{M}_1) = k\right) \overset{d}{=} \left((C_1(r); r \ge 0)\middle|\zeta(\tilde{e}_1^*) = 1, R_1 = k\right). \tag{28}$$

It is shown in the proof of Theorem 1.2 in [7] that for all $i \ge 1$

$$\mathbb{E}\left[g(\tilde{e}_i^*)\middle|R_i = k, \zeta(\tilde{e}_i^*) = 1\right] = \mathbb{E}[g(\mathbf{e}^{(k)})]$$

for all positive functionals g and where $\mathbf{e}^{(k)} = (\mathbf{e}^{(k)}(t); t \in [0,1])$ is defined in (4). This is the displayed equations on page 24 of [7]. Recall from Section 3.2, that the process C_1 is simply a functional of $\tilde{e}_1^*$. Therefore, conditionally on the values of $\zeta(\tilde{e}_1^*)$ and R_1 we have

$$\left((C_1(t); t \ge 0)\middle|\zeta(\tilde{e}_1^*) = 1, R_1 = k\right) \overset{d}{=} \left(\mathbf{c}^{(k)}(t); t \ge 0\right)$$

where $(\mathbf{z}^{(k)}, \mathbf{c}^{(k)})$ is the Lamperti pair associated with the excursion $\mathbf{e}^{(k)}$.

The left-hand side of (28) is easy to condition with part (6) of Lemma 21. Conditionally on the values of $\mu_1(\mathcal{M}_1)$ and $\mathrm{sur}(\mathcal{M}_1)$ (which is precisely the conditioning described above for C_1) the metric spaces $\mathcal{M}_1$ satisfies

$$\left(\mathcal{M}_1\middle|\mu_1(\mathcal{M}_1) = 1, \mathrm{sur}(\mathcal{M}_1) = k\right) \overset{d}{=} \mathscr{G}^{(\alpha,k)}.$$

Hence

$$\left(\mu_{\mathscr{G}^{(\alpha,k)}}(B(\rho, v)); v \ge 0\right) \overset{d}{=} \left(\mathbf{c}^{(k)}(v); v \ge 0\right).$$

An application of Proposition 8 completes the proof. □

6. Discussion

In this work we showed convergence of the height profiles for the macroscopic components of a certain class of critical random graphs. We did this by looking at the height profile of these graphs and we relied on the weak convergence results that exist in the literature on some encoding stochastic processes. We observe that these techniques can likely be extended to other graph models appearing in the literature.

Using the results in the literature on Galton–Watson trees conditioned on having a fixed size [39,49,59] one can recover the Jeulin identity [38] and its α-stable extension due to Miermont [50] from our Theorem 4 as well. The proofs in [38,50] do not rely on weak convergence arguments. For proofs using weak-convergence arguments more in-line with the results of this papers see Kersting's work [33], or the joint work of Angtuncio and Uribe Bravo [34]. See also [31] for a weak convergence result in a slightly weaker topology.

More generally, under certain conditions (see Theorem 2.3.1 in [44]) on the offspring distribution, there is convergence of Galton–Watson forests to continuum forests encoded by spectrally positive Lévy processes. Under these assumptions, one can use a modification of Lemma 4.8 in [65] or Lemma 5.8 in [7], one should be able to prove a Jeulin-type identity for excursions for non-stable Lévy processes and their associated height processes by a simple application of Theorem 5. As far as the author is aware, such results are not present in the literature.

Declaration of competing interest

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References

[1] Andrew Barbour, Denis Mollison, Epidemics and random graphs, Lect. Notes Biomath. 86 (1990) 86–89.
[2] David Aldous, Brownian excursions, critical random graphs and the multiplicative coalescent, Ann. Probab. 25 (2) (1997) 812–854, MR1434128.
[3] Grégory Miermont, Sanchayan Sen, On breadth-first constructions of scaling limits of random graphs and random unicellular maps, Random Structures Algorithms (2022).
[4] David Clancy Jr., A new relationship between erdős-rényi graphs, epidemic models and Brownian motion with parabolic drift, 2020, arXiv e-prints arXiv:2006.06838.
[5] Remco van der Hofstad, Random graphs and complex networks, in: Cambridge Series in Statistical and Probabilistic Mathematics, vol. 1, Cambridge University Press, Cambridge, 2017, p. 43, MR3617364.
[6] Shankar Bhamidi, Nicolas Broutin, Sanchayan Sen, Xuan Wang, Scaling limits of random graph models at criticality: Universality and the basin of attraction of the Erdős-Rényi random graph, 2014, arXiv e-prints arXiv:1411.3417.
[7] Guillaume Conchon–Kerjan, Christina Goldschmidt, The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees, Ann. Probab. 51 (1) (2023) 1–69, MR4515689.
[8] Adrien Joseph, The component sizes of a critical random graph with given degree sequence, Ann. Appl. Probab. 24 (6) (2014) 2560–2594, MR3262511.
[9] Tom Britton, Svante Janson, Anders Martin-Löf, Graphs with specified degree distributions, simple epidemics, and local vaccination strategies, Adv. in Appl. Probab. 39 (4) (2007) 922–948, MR2381582.
[10] Michael Molloy, Bruce Reed, A critical point for random graphs with a given degree sequence, in: Proceedings of the Sixth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science, Random Graphs ’93 (Poznań, 1993), vol. 6, 1995, pp. 161–179, MR1370952.
[11] Michael Molloy, Bruce Reed, The size of the giant component of a random graph with a given degree sequence, Combin. Probab. Comput. 7 (3) (1998) 295–305, MR1664335.
[12] Svante Janson, Malwina J. Luczak, A new approach to the giant component problem, Random Structures Algorithms 34 (2) (2009) 197–216, MR2490288.
[13] Tom Bohman, Michael Picollelli, SIR epidemics on random graphs with a fixed degree sequence, Random Structures Algorithms 41 (2) (2012) 179–214, MR2956054.
[14] Nicholas A. Christakis, James H. Fowler, Social network sensors for early detection of contagious outbreaks, PLOS ONE 5 (9) (2010) 1–8.
[15] Scott L. Feld, Why your friends have more friends than you do, Am. J. Sociol. 96 (6) (1991) 1464–1477.
[16] Manuel Garcia-Herranz, Esteban Moro, Manuel Cebrian, Nicholas A. Christakis, James H. Fowler, Using friends as sensors to detect global-scale contagious outbreaks, PLOS ONE 9 (4) (2014) 1–7.
[17] Louigi Addario-Berry, Nicolas Broutin, Christina Goldschmidt, Critical random graphs: limiting constructions and distributional properties, Electron. J. Probab. 15 (25) (2010) 741–775, MR2650781.
[18] Louigi Addario-Berry, Nicolas Broutin, Christina Goldschmidt, The continuum limit of critical random graphs, Probab. Theory Related Fields 152 (3–4) (2012) 367–406, MR2892951.
[19] Loïc Chaumont, Excursion normalisée, méandre et pont pour les processus de Lévy stables, Bull. Sci. Math. 121 (5) (1997) 377–403, MR1465814.
[20] Thomas Duquesne, Jean-François Le Gall, Probabilistic and fractal aspects of Lévy trees, Probab. Theory Related Fields 131 (4) (2005) 553–603, MR2147221.
[21] Marc Barthélemy, Alain Barrat, Romualdo Pastor-Satorras, Alessandro Vespignani, Dynamical patterns of epidemic outbreaks in complex heterogeneous networks, J. Theoret. Biol. 235 (2) (2005) 275–288, MR2157753.
[22] Erik Volz, SIR dynamics in random networks with heterogeneous connectivity, J. Math. Biol. 56 (3) (2008) 293–310, MR2358436.
[23] Erik Volz, Lauren Ancel Meyers, Susceptible-infected-recovered epidemics in dynamic contact networks, Proceedings of the Royal Society B: Biological Sciences 274 (2007) 2925–2934.
[24] Romualdo Pastor-Satorras, Claudio Castellano, Piet Van Mieghem, Alessandro Vespignani, Epidemic processes in complex networks, Rev. Modern Phys. 87 (3) (2015) 925–979, MR3406040.
[25] Laurent Decreusefond, Jean-Stéphane Dhersin, Pascal Moyal, Viet Chi Tran, Large graph limit for an SIR process in random network with heterogeneous connectivity, Ann. Appl. Probab. 22 (2) (2012) 541–575, MR2953563.
[26] Svante Janson, Malwina Luczak, Peter Windridge, Law of large numbers for the SIR epidemic on a random graph with given degrees, in: Random Structures Algorithms, vol. 45, (4) 2014, pp. 726–763, [Paging previously given as 724–761]. MR3275704.
[27] John Lamperti, Continuous state branching processes, Bull. Amer. Math. Soc. 73 (1967) 382–386, MR208685.
[28] Martin L. Silverstein, A new approach to local times, J. Math. Mech 17 (1967/1968) 1023–1054, MR0226734.
[29] María Emilia Caballero, Amaury Lambert, Gerónimo Uribe Bravo, Proof(s) of the Lamperti representation of continuous-state branching processes, Probab. Surv. 6 (2009) 62–89, MR2592395.

[30] María Emilia Caballero, José Luis Pérez Garmendia, Gerónimo Uribe Bravo, A Lamperti-type representation of continuous-state branching processes with immigration, Ann. Probab. 41 (3A) (2013) 1585–1627, MR3098685.
[31] David Aldous, Grégory Miermont, Jim Pitman, The exploration process of inhomogeneous continuum random trees, and an extension of Jeulin's local time identity, Probab. Theory Related Fields 129 (2) (2004) 182–218.
[32] María Emilia Caballero, José Luis Pérez Garmendia, Gerónimo Uribe Bravo, Affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$ and multiparameter time changes, Ann. Inst. Henri Poincaré Probab. Stat. 53 (3) (2017) 1280–1304, MR3689968.
[33] Götz Kersting, On the Height Profile of a Conditioned Galton–Watson Tree, 2011, arXiv e-prints arXiv:1101.3656.
[34] Osvaldo Angtuncio, Gerónimo Uribe Bravo, On the profile of trees with a given degree sequence, 2020, arXiv e-prints arXiv:2008.12242.
[35] David Aldous, Vlada Limic, The entrance boundary of the multiplicative coalescent, Electron. J. Probab. 3 (3) (1998) 59, MR1491528.
[36] Nicolas Broutin, Thomas Duquesne, Minmin Wang, Limits of multiplicative inhomogeneous random graphs and Lévy trees: The continuum graphs, Ann. Appl. Probab. 32 (4) (2022) 2448–2503, MR4474511.
[37] Nicolas Broutin, Thomas Duquesne, Minmin Wang, Limits of multiplicative inhomogeneous random graphs and Lévy trees: Limit theorems, Probab. Theorey Related Fields 181 (4) (2021) 865–973, MR4344135.
[38] Thierry Jeulin, Marc Yor (Eds.), Grossissements de filtrations: exemples et applications, in: Lecture Notes in Mathematics, vol. 1118, Springer-Verlag, Berlin, 1985, Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982/1983.
[39] Jean-François Le Gall, Random trees and applications, Probab. Surv. 2 (2005) 245–311, MR2203728.
[40] Jim Pitman, The SDE solved by local times of a Brownian excursion or bridge derived from the height profile of a random tree or forest, Ann. Probab. 27 (1) (1999) 261–283, MR1681110.
[41] David Aldous, Jim Pitman, Inhomogeneous continuum random trees and the entrance boundary of the additive coalescent, Probab. Theory Related Fields 118 (4) (2000) 455–482, MR1808372.
[42] Michael Camarri, Jim Pitman, Limit distributions and random trees derived from the birthday problem with unequal probabilities, Electron. J. Probab. 5 (2) (2000) 18, MR1741774.
[43] Arthur Blanc-Renaudie, Compactness and fractal dimensions of inhomogeneous continuum random trees, Probab. Theory Related Fields 185 (3–4) (2023) 961–991, MR4556286.
[44] Thomas Duquesne, Jean-François Le Gall, Random trees, Lévy processes and spatial branching processes, Astérisque (281) (2002) vi+147. MR1954248.
[45] Jean-François Le Gall, Yves Le Jan, Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses, Ann. Probab. 26 (4) (1998) 1407–1432, MR1675019.
[46] Jean-Francois Le Gall, Yves Le Jan, Branching processes in Lévy processes: the exploration process, Ann. Probab. 26 (1) (1998) 213–252, MR1617047.
[47] Jean Bertoin, Lévy processes, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996, MR1406564.
[48] Daniel Revuz, Marc Yor, Continuous martingales and Brownian motion, in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], third ed., vol. 293, Springer-Verlag, Berlin, 1999, MR1725357.
[49] Thomas Duquesne, A limit theorem for the contour process of conditioned Galton-Watson trees, Ann. Probab. 31 (2) (2003) 996–1027, MR1964956.
[50] Grégory Miermont, Self-similar fragmentations derived from the stable tree. I. Splitting at heights, Probab. Theory Related Fields 127 (3) (2003) 423–454, MR2018924.
[51] Kiyoshi Kawazu, Shinzo Watanabe, Branching processes with immigration and related limit theorems, Teor. Verojatnost. i Primenen. 16 (1971) 34–51, MR0290475.
[52] Romain Abraham, Jean-François Delmas, Patrick Hoscheit, A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces, Electron. J. Probab. 18 (14) (2013) 21, MR3035742.
[53] Ali Khezeli, Metrization of the Gromov-Hausdorff (-Prokhorov) topology for boundedly-compact metric spaces, Stochastic Process. Appl. 130 (6) (2020) 3842–3864, MR4092421.
[54] Patrick Billingsley, Convergence of probability measures, in: Wiley Series in Probability and Statistics: Probability and Statistics, second ed., John Wiley & Sons, Inc. New York, 1999, A Wiley-Interscience Publication. MR1700749.
[55] Louigi Addario-Berry, Nicolas Broutin, Christina Goldschmidt, Grégory Miermont, The scaling limit of the minimum spanning tree of the complete graph, Ann. Probab. 45 (5) (2017) 3075–3144, MR3706739.
[56] Christina Goldschmidt, Bénédicte Haas, Delphin Sénizergues, Stable graphs: distributions and line-breaking construction, Ann. H. Lebesgue 5 (2022) 841–904, MR4526241.
[57] David Aldous, The continuum random tree. I, Ann. Probab. 19 (1) (1991) 1–28, MR1085326.
[58] David Aldous, The continuum random tree. II. An overview, in: Stochastic Analysis (Durham, 1990), London Math. Soc. Lecture Note Ser, vol. 167, Cambridge Univ. Press, Cambridge, 1991, pp. 23–70, MR1166406.
[59] David Aldous, The continuum random tree. III, Ann. Probab. 21 (1) (1993) 248–289, MR1207226.
[60] Biao Wu, On the weak convergence of subordinated systems, Statist. Probab. Lett. 78 (18) (2008) 3203–3211, MR2479479.
[61] Ward Whitt, Some useful functions for functional limit theorems, Math. Oper. Res. 5 (1) (1980) 67–85, MR561155.
[62] Oliver Riordan, The phase transition in the configuration model, Combin. Probab. Comput. 21 (1–2) (2012) 265–299, MR2900063.
[63] Shankar Bhamidi, Sanchayan Sen, Geometry of the vacant set left by random walk on random graphs, wright's constants, and critical random graphs with prescribed degrees, Random Structures Algorithms 56 (3) (2020) 676–721, MR4084187.
[64] Shankar Bhamidi, Remco van der Hofstad, Johan S.H. van Leeuwaarden, Scaling limits for critical inhomogeneous random graphs with finite third moments, Electron. J. Probab. 15 (54) (2010) 1682–1703, MR2735378.
[65] James B. Martin, Balázs Ráth, Rigid representations of the multiplicative coalescent with linear deletion, Electron. J. Probab. 22 (83) (2017) 47, MR3718711.