Exercise 1

Suppose otherwise. Then $V^p(F) \subseteq \mathbb{P}^2 \setminus V^p(G)$. By proposition 5.31, the latter is affine, hence by remark 5.30, $V^p(F)$ is a point, contradiction.

Exercise 2

(1) By sheet 8, exercise 1, such a morphism is given as $[F(X,Y,Z) : G(X,Y,Z)]$. If they are not both constant, then by exercise 1, they have a common zero, contradicting the well-definedness of the morphism.

(2) Otherwise, the projections would induce non-constant morphisms $\mathbb{P}^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, contradicting (1).

Exercise 3

(1) $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, hence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Conversely, $\overline{A \cup B}$ is a closed set containing A, so $\overline{A} \subseteq \overline{A \cup B}$, and similarly for B. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

(2) Assume that A is irreducible, and write $\overline{A} = X_1 \cup X_2$ with X_1, X_2 closed in $\overline{A}$. Then $A = (A \cap X_1) \cup (A \cap X_2)$, so wlog $A = A \cap X_1$. It follows $\overline{A} = \overline{A} \cap \overline{X}_1 = X_1$.

Conversely, assume that $\overline{A}$ is irreducible, and write $A = X_1 \cup X_2$ with X_1, X_2 closed in A. Then, taking closures in $\overline{A}$, we have $\overline{A} = \overline{X}_1 \cup \overline{X}_2$, so $\overline{A} = \overline{X}_1$, say, hence $A = X_1$.

(3) Let $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subsetneq Y$ be a chain of irreducible closed subsets. Its closure is also a chain of irreducible closed subsets contained in X, so $n \leq \dim X$. Since this holds for all such chains, $\dim Y \leq \dim X$.

(4) Let $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subseteq Y$ be a maximal chain of irreducible closed subsets of Y. Since Y is closed, all the Y_i are closed in X as well. If $Y \neq X$, then $Y_0 \subsetneq \ldots \subsetneq Y_n \subsetneq X$ would be a larger chain, contradicting $X = Y$.

Exercise 4

(1) From 3(3) we immediately get $\sup_i \dim U_i \leq \dim X$. Conversely, let $X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X$ be a chain of irreducible closed subsets. Let $x \in X_0$, and wlog $x \in U_0$. We claim that $X_0 \cap U_0 \subsetneq X_1 \cap U_0 \subsetneq \ldots \subsetneq X_n \cap U_0 \subseteq U_0$ is a chain of irreducible closed subsets in U_0, which proves the reverse inequality. Closedness is clear.

Assume $X_i \cap U_0 = A \cup B$ with A, B closed in $X_i \cap U_0$. Then $\overline{A} \cup \overline{B} \cup (X_i \setminus U) = X_i$, hence by irreducibility $X_i = \overline{A}$, say. Then $X_i \cap U_0 = \overline{A} \cap U_0 = A$, since A is closed in U_0. Hence $X_i \cap U_0$ is irreducible.

Finally, the inclusions are proper: If $X_i \cap U_0 = X_{i+1} \cap U_0$. Otherwise, $X_{i+1} = X_i \cup (X_{i+1} \setminus U_0)$ and $x \notin X_{i+1} \setminus U_0$, so by irreducibility $X_{i+1} = X_i$, contradiction.

(2) $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ with $\dim U_i = \dim A^n = n$.