Exercise 1

The prime ideals of A/I and $A/\sqrt{I}$ correspond bijectively, and inclusion-preservingly, to ideals of A containing I and $\sqrt{I}$, respectively. Hence it is enough to show that for a prime ideal $P \subseteq A$ one has $I \subseteq P \iff \sqrt{I} \subseteq P$.

Since $I \subseteq \sqrt{I}$, the direction "$\Longrightarrow$" is trivial. Conversely, assume $I \subseteq P$ and let $x \in \sqrt{I}$, i.e. $x^m \in I$ for some $m > 0$. Then $x^m \in P$, and since P is prime, it follows that $x \in P$, as desired.

Exercise 2

(1) Moving to the affine cone, we may equivalently ask whether there exists a hyperplane through 0 not containing any of a given set of lines $l_1, \dots, l_r \subseteq \mathbb{A}^{n+1}$. Let V_i be the set of hyperplanes containing l_i. Consider the map

$$\{\text{linear hyperplanes}\} \to \mathbb{P}^n, \quad V(a_0 X_0 + \ldots + a_n X_n) \mapsto [a_0 : \ldots : a_n].$$

This is clearly a bijection, and if $0 \neq v_i \in l_i$ is some point, V_i is identified with a proper closed subspace of $\mathbb{P}^n$. In particular, $\bigcup_i V_i$ is a finite union of proper closed subspaces, hence cannot equal the whole space because it has smaller dimension. Therefore, there exists a hyperplane in the complement, which does the job.

(2) Pick points $a_i \in X_i$. By (1), there exists a linear homogeneous polynomial F s.t. $V(F) \cap \{a_1, \dots, a_n\} = \emptyset$. But then $a_i \in X_i \setminus V(F)$, so $X_i \not\subseteq V(F)$ for any i.

Exercise 3

Consider

$$\varphi : K[\{Z_{ij}\}_{0 \le i \le n, 0 \le j \le m}] \to K[X_0, \dots, X_n, Y_1, \dots, Y_m], \qquad Z_{ij} \mapsto X_i Y_j.$$

Then $\ker \varphi = I(\Sigma_{n,m})$, since $f(\sigma(x,y)) = f(\{x_i y_j\}) = \varphi(f)(x,y)$. Hence, to compute $\chi_{\Sigma_{n,m}}(d)$, we have to count the homogeneous polynomials of degree d in $K[\{Z_{ij}\}_{i,j}]/I(\Sigma_{n,m})$. By the isomorphism theorem, these correspond exactly to the homogeneous polynomials of degree $2d$ in $\operatorname{im} \varphi$, that is, bihomogeneous polynomials of degree (d,d). But there are exactly $\binom{n+d}{d}\binom{m+d}{d}$ such polynomials, so

$$\chi_{\Sigma_{n,m}}(d) = \binom{n+d}{d}\binom{m+d}{d} = \frac{(n+d)!(m+d)!}{n!m!d!d!} = \frac{1}{n!m!} d^{n+m} + O(d^{n+m-1}),$$

hence $\deg(\Sigma_{n+m}) = \frac{(n+m)!}{n!m!} = \binom{n+m}{n}$. (In other words, $\chi_{\mathbb{P}^n \times \mathbb{P}^m} = \chi_{\mathbb{P}^n} \chi_{\mathbb{P}^m}$)

Exercise 4

As in exercise 3, one finds $I(v_d(\mathbb{P}^n)) = \ker \varphi$, where $\varphi : K[Z_0, \dots, Z_N] \to K[X_0, \dots, X_n]$ maps Z_i to the i-th degree-d monomial in the X_j. Hence

$$\chi_{v_d(\mathbb{P}^n)}(x) = \chi_{\mathbb{P}^n}(xd) = \binom{n+xd}{n} = \frac{d^n}{n!} x^n + O(x^{n-1}),$$

and $\deg(v_d(\mathbb{P}^n)) = d^n$.