Exercise 1

Suppose $p, q, r \in \mathbb{P}^2$ are not colinear, and assume $I(\{p, q, r\}) = (f, g)$ is generated by two polynomials. Then by Bézout

$$3 = \deg\{p, q, r\} = \deg(f, g) = \deg f \deg g,$$

so wlog $\deg f = 1$. But then $p, q, r \in V^p(f)$, which is a line.

Exercise 2

Let $\mathbb{P}^n = \bigcup_i U_i$ be the standard open cover and let Y be an irreducible component of $X \cap V(F)$. Let $U \in \{U_0, \dots, U_n\}$ be such that $Y \cap U \neq \emptyset$. Then $V^p(F) \cap U$ is nonempty open, hence dense in $V^p(F)$, so taking closures of $Y \cap U \subseteq V^p(F) \cap U$ would imply $\overline{X \cap U} \subseteq X \subseteq V^p(F)$, contradicting the assumption.

Then $X \cap V(F) \cap U = (X \cap U) \cap (V^p(F) \cap U)$. Under the identification $U \cong \mathbb{A}^n$, $X \cap U$ is an affine algebraic set, and we have $V^p(F) = V(F^i)$, so by the affine case (applicable by the previous paragraph) every irreducible component of $X \cap V(F) \cap U$ has dimension $\dim(X \cap U) - 1 = \dim X - 1$. Every irreducible component X' of $X \cap V(F) \cap U_0$ is contained in some irreducible component(s) X_i of $X \cap V(F)$, thus $\dim X - 1 = \dim X' \leq \dim X_i \leq \dim X \cap V(F) = \dim X - 1$, so $\dim X_i = \dim X - 1$ as well. Therefore, every irreducible component that intersects U has dimension $\dim X - 1$. In particular, $\dim Y = \dim X - 1$.

Exercise 3

(1) Let $H \subseteq \mathbb{P}^n$ be a hyperplane not containing X. Then by Bézout $|X \cap H| = \deg(X \cap H) = \deg X \cdot \deg H = 1 \cdot 1 = 1$. Contrapositively, if a hyperplane H contains more than one point of X, then $X \subseteq H$.

Now we proceed by induction. For $n = 1$ the claim is clear, since $\mathbb{P}^1$ is the only subvariety of $\mathbb{P}^1$ of dimension 1. So now let $n > 1$ and $X \subseteq \mathbb{P}^n$ be a curve of degree n, and let $p \neq q \in X$. Then there exists a hyperplane H containing p, q (equivalently, for any two lines in $\mathbb{A}^n$ with $n > 2$ there exists a hyperplane containing them, which is clear from basis completion), so by the previous observation, $X \subseteq H \cong \mathbb{P}^{n-1}$. By induction, we are done.

(2) Under the canonical inclusion $\mathbb{P}^n \subseteq \mathbb{P}^{2n}$, neither the degree nor whether X is linear changes, so we work in $\mathbb{P}^{2n}$ instead. Now we can proceed as in (1): Let H_1 be a hyperplane not containing X, and let $L_1 = X \cap H_1$. Then by exercise 2, L_1 is pure-dimensional of dimension $\dim X - 1$ and of degree 1 by Bézout, so a linear subspace. Let H_2 be another hyperplane with $L_1 \not\subseteq H_2$, then by the same arguments, $L_2 = X \cap H_2$ is a linear subspace of dimension $\dim X - 1$. Now there exists a hyperplane H containing both L_1 and L_2: In the affine cones, $\langle C(L_1), C(L_2) \rangle$ has dimension at most $2 \dim X < 2n + 1 = \dim C(\mathbb{P}^{2n})$. By Bézout it follows that $X \subseteq H$. But then also $X \subseteq \mathbb{P}^n \cap H \cong \mathbb{P}^{n-1}$. An easy induction on $n \geq \dim X$ finished the proof.

Exercise 4

Let S be the intersection point of l_1 and l_2. If $S \in \{A, B, C, D, E, F\}$, say $S = A$, then $P = Q = D$, so of course P, Q and R are colinear. Now consider the more interesting case $S \notin \{A, B, C, D, E, F\}$.

Consider the reducible cubics $X_1 = AE + BF + CD$ and $X_2 = AF + BD + CE$. Let $I(X_i) = (f_i)$ for $i = 1, 2$. Since $f_i(S) \neq 0$ (otherwise $\deg((l_1 \cup l_2) \cap X_i) > 6$ contradicting Bézout's Theorem), we can find $\alpha, \beta \in K$ with $(\alpha f_1 + \beta f_2)(S) = 0$. Observe that $\alpha f_1 + \beta f_2 \neq 0$, since otherwise $V(f_1) = V(f_2)$.

Let $Y = V^p(\alpha f_1 + \beta f_2)$. Then $\deg Y = \deg(\alpha f_1 + \beta f_2) = 3$. Now $l_1 \cap Y$ contains S, A, B, C, contradicting Bézout's Theorem unless l_1 and Y share an irreducible component. Since l_1 is irreducible, $l_1 \subseteq Y$. By the exact same argument, $l_2 \subseteq Y$. Since Y is pure-dimensional of dimension 1, comparing degrees we find $Y = l_1 \cup l_2 \cup L$ for some pure-dimensional L with $\deg L = 1$, i.e. L is a line. Since $P, Q, R \in Y \setminus (l_1 \cup l_2)$, we find $P, Q, R \in L$, i.e. P, Q, R are colinear.