

# Algebraic Geometry I

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Roughly speaking, the goal of algebraic geometry is to study systems of polynomial equations  $F_1(X) = \dots = F_n(X) = 0$  for polynomials  $F_i \in K[X_1, \dots, X_m]$  over a field (or ring)  $K$ . The set of solutions of this system is a geometric object, which we try to understand using algebraic methods, for example considering the ideal  $I = (F_1, \dots, F_n)$  in  $K[X_1, \dots, X_m]$  or the quotient  $K[X_1, \dots, X_m]/I$ .

There is a very strong relation between these objects in the case that  $K = \overline{K}$  is algebraically closed (e.g.  $\mathbb{C}$ ). If  $K$  is not algebraically closed, or some generic ring, things get more complicated: For example, there are many equations over  $\mathbb{R}$  with no solutions, like  $x^2 + y^2 + 1 = 0$ , which behave differently when considered over  $\mathbb{C}$ . The wish to still study these equations geometrically leads to the idea of spectra (the set of all prime ideals of a ring), and later the theory of sheaves and schemes.

## 1 Algebraic Sets and Affine Varieties

Let  $K$  be an algebraically closed field.

**Definition 1.1.** For  $n \in \mathbb{N}$  define *affine  $n$ -space* over  $K$  as

$$\mathbb{A}^n := \mathbb{A}_K^n := K^n.$$

**Definition 1.2.** Let  $I \subset K[x_1, \dots, x_n]$  be a subset. The associated (*affine*) *algebraic set* is

$$V(I) := \{x \in \mathbb{A}_K^n \mid f(x) = 0 \text{ for all } f \in I\}.$$

A subset  $X \subset \mathbb{A}^n$  is called *algebraic* if  $X = V(I)$  for some  $I \subset K[x_1, \dots, x_n]$ .

**Remark 1.3.** By definition  $V(I) = V(\langle I \rangle) = V(f_1, \dots, f_m)$  where  $\langle I \rangle = (f_1, \dots, f_m)$  is finitely generated because  $K[x_1, \dots, x_n]$  is Noetherian. Therefore,  $X \subseteq \mathbb{A}^n$  is algebraic if and only if  $X = V(I)$  for some ideal  $I$  if and only if  $X = V(f_1, \dots, f_m)$  for a finite number of polynomials  $f_i$ .

**Example 1.4.** The following sets are algebraic:

- A parabola  $\{(x, x^2) \mid x \in K\} = V(y - x^2)$
- $\emptyset = V(K[x_1, \dots, x_n])$
- $\mathbb{A}^n = V(0)$
- Points:  $\{(a_1, \dots, a_n)\} = V(x_1 - a_1, \dots, x_n - a_n)$

**Lemma 1.5.** Let  $I, J \triangleleft K[x_1, \dots, x_n]$  be ideals. Then

- (a) If  $I \subseteq J$ , then  $V(I) \supseteq V(J)$ .
- (b)  $V(I \cap J) = V(IJ) = V(I) \cup V(J)$
- (c) For any family  $(I_t)_{t \in T}$  of ideals,  $\bigcap_t V(I_t) = V(\bigcup_t I_t) = V(\sum_t I_t)$

*Proof.* (a) is clear.

For (b), part (a) yields  $V(I \cap J) \subseteq V(IJ)$  and  $V(I), V(J) \subseteq V(I \cap J)$ , so it remains to show  $V(IJ) \subseteq V(I) \cup V(J)$ . Let  $a \in V(IJ)$ . Assume  $a \notin V(I)$ , i.e. there is  $f \in I$  such that  $f(a) \neq 0$ . Let  $g \in J$ . Then  $fg \in IJ$ , so  $0 = (fg)(a) = f(a)g(a)$ . Since  $f(a) \neq 0$ , we conclude  $g(a) = 0$ .

The first equation of (c) is tautological, the second one is remark 1.3, □

**Definition 1.6.** The *Zariski topology* on  $\mathbb{A}^n$  is the topology whose closed subsets are exactly the algebraic sets. That is,  $U \subseteq \mathbb{A}^n$  is open iff its complement is algebraic.

**Remark 1.7.** This is indeed a topology by example 1.4 and lemma 1.5. Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set  $X \subseteq \mathbb{A}^n$ , which is also called the Zariski topology.

Recall from general topology that a topological space  $X \neq \emptyset$  is called irreducible if  $X \neq X_1 \cup X_2$  with  $X_i \subsetneq X$  closed.  $\emptyset$  is not considered irreducible.

For example,  $V(xy) = V(x) \cup V(y)$  (the union of the coordinate axes in  $\mathbb{A}^2$ ) is not irreducible, while a parabola  $V(y - x^2)$  is irreducible (we will see how to check this later).

**Definition 1.8.** An *affine algebraic variety* is an irreducible closed subset of  $\mathbb{A}^n$ .

**Definition 1.9.** Let  $X \subseteq \mathbb{A}^n$  be an arbitrary set. We define the *vanishing ideal* of  $X$  as

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

**Lemma 1.10.** Let  $X \subseteq \mathbb{A}^n$  and  $S \subseteq K[x_1, \dots, x_n]$ . Then

- (a)  $X \subseteq V(I(X))$  and  $S \subseteq I(V(S))$ .
- (b)  $V(I(X)) = \overline{X}$  is the closure of  $X$  (w.r.t. the Zariski topology).

*Proof.* (a) is clear, (b) is left as an exercise. □

**Proposition 1.11.** An affine algebraic set  $X \subseteq \mathbb{A}^n$  is a variety if and only if  $I(X)$  is a prime ideal.

*Proof.* Let  $X$  be a variety and let  $fg \in I(X)$  for  $f, g \in K[x_1, \dots, x_n]$ . We have  $X \subseteq V(fg) \stackrel{1.5}{=} V(f) \cup V(g)$ . Hence we can write  $X = (X \cap V(f)) \cup (X \cap V(g))$  as the union of two closed subsets. By irreducibility, wlog we have  $X = X \cap V(f)$ , i.e.  $X \subseteq V(f)$ , which is equivalent to  $f \in I(X)$ .

Conversely, suppose that  $X = A \cup B$  is not irreducible. Choose points  $a \in A \setminus B$  and  $b \in B \setminus A$ . By Lemma 1.10 and since  $A, B$  are closed, we get  $V(I(A)) = A$  and  $V(I(B)) = B$ . Hence there exist  $f \in I(A)$  and  $g \in I(B)$  with  $f(b) \neq 0$  and  $g(a) \neq 0$ . Thus  $fg \in I(X)$ , but both  $f, g \notin I(X)$  □

**Remark 1.12.** If  $X = V(I)$  is an affine variety, this does not necessarily imply that  $I$  is prime: Consider  $V((x^2)) \subseteq \mathbb{A}^1$ :  $V((x^2)) = \{0\}$  is irreducible, but  $(x^2)$  is not prime.

Note that  $\mathbb{A}^n$  is irreducible since  $K$  is infinite. However, this is no longer true if one considers finite fields, since then  $\mathbb{A}^n$  is the union of its finitely many points. For example,  $I(\mathbb{A}_{\mathbb{F}_p}^1) = (X^p - X)$  is not prime.

We use the following result from commutative algebra without proof:

**Theorem 1.13** (Hilbert Nullstellensatz). Let  $J \triangleleft K[x_1, \dots, x_n]$ . Then

- (a)  $V(J) = \emptyset$  if and only if  $J = K[x_1, \dots, x_n]$ .
- (b)  $I(V(J)) = \sqrt{J} = \{f \in K[x_1, \dots, x_n] \mid f^n \in J \text{ for some } n\}$
- (c) If  $J$  is a maximal ideal, then  $J = (x_1 - a_1, \dots, x_n - a_n)$  for some  $a_i \in K$ .

**Corollary 1.14.** *There are inclusion-reversing bijections*

$$\begin{aligned} \{\text{affine algebraic sets } X \subseteq \mathbb{A}^n\} &\xrightleftharpoons[V]{I} \{\text{radical ideals in } K[x_1, \dots, x_n]\} \\ \{\text{affine algebraic varieties } X \subseteq \mathbb{A}^n\} &\xrightleftharpoons[V]{I} \{\text{prime ideals in } K[x_1, \dots, x_n]\} \\ \{\text{points } a \in \mathbb{A}^n\} &\xrightleftharpoons[V]{I} \{\text{maximal ideals in } K[x_1, \dots, x_n]\} \end{aligned}$$

*Proof.* Clear from 1.13, 1.10 and 1.11.  $\square$

**Example 1.15.** Let  $f$  be irreducible in  $K[x_1, \dots, x_n]$ . Then  $V(f)$  is an affine variety. Varieties of this form are called hypersurfaces in  $\mathbb{A}^n$  (curves for  $n = 2$ , surfaces for  $n = 3$ ).

**Remark 1.16.** If  $X \subseteq \mathbb{A}^n$  is a variety, by proposition 1.11  $I(X)$  is prime, and  $K[x_1, \dots, x_n]/I$  is an integral domain. We can consider its fraction field  $\text{Frac}(K[x_1, \dots, x_n]/I)$ .

**Theorem 1.17.** *Any affine algebraic set can be uniquely written as a finite union of affine varieties.*

For the proof, we need some preparations.

**Definition 1.18.** A topological space  $X$  is called *Noetherian* if any chain of descending closed subsets  $X \supseteq X_1 \supseteq X_2 \supseteq \dots$  becomes stationary, i.e. there exists  $n$  s.t.  $X_m = X_n$  for all  $m > n$ .

**Lemma 1.19.** *Affine space  $\mathbb{A}^n$  is Noetherian.*

*Proof.* Let  $\mathbb{A}^n \supseteq X_1 \supseteq X_2 \supseteq \dots$  be a chain of closed subsets. Applying  $I(-)$  yields an ascending chain  $(0) \subseteq I(X_1) \subseteq I(X_2) \subseteq \dots$  of ideals in  $K[x_1, \dots, x_n]$ . This is a Noetherian ring, so there is some  $m$  such that  $I(X_n) = I(X_{n+1})$  for all  $n \geq m$ . By corollary 1.14(a),  $I$  is injective on closed subsets, so we are done.  $\square$

More generally,

**Corollary 1.20.** *Any affine algebraic space  $X \subseteq \mathbb{A}^n$  is Noetherian.*

*Proof.* Any chain in  $X$  is also a chain in  $\mathbb{A}^n$ .  $\square$

**Proposition 1.21.** *Let  $X \neq \emptyset$  be a Noetherian topological space.*

- (a) *Then  $X$  can be written as a finite union of irreducible closed subspaces.*
- (b) *Moreover, if we assume that  $X_i \not\subseteq X_j$  for  $i \neq j$ , then the above decomposition is unique up to permutation. In this case, the  $X_i$  are called irreducible components of  $X$ .*

*Proof.* Assume that (a) fails for  $X$ . Consider  $S = \{Y \subseteq X \mid Y \text{ closed, cannot be written as a finite union of irreducible closed subsets}\}$ . Since  $X$  is Noetherian,  $S$  must have some minimal element  $Y$  w.r.t. inclusion.  $Y$  is not irreducible, so we can write  $Y = Y_1 \cup Y_2$  with  $Y_{1,2}$  proper closed subspaces. By minimality,  $Y_1$  and  $Y_2$  can be written as finite unions of irreducible closed subsets, thus so can  $Y$ , contradicting  $Y \in S$ .

To check uniqueness, assume we have two decompositions  $X = X_1 \cup \dots \cup X_r = X'_1 \cup \dots \cup X'_s$  as in (b). Then  $X'_1 = \bigcup_i (X_i \cap X'_1)$ . Since  $X'_1$  is irreducible, wlog  $X'_1 \subseteq X_1$ . By the same argument,  $X_1 \subseteq X'_i$  for some  $i$ . If  $i \neq 1$ , then  $X'_1 \subseteq X'_i$ , contradicting our assumption. Hence  $i = 1$  and  $X_1 = X'_1$ . Proceed inductively with  $X \setminus X_1 = X_2 \cup \dots \cup X_r = X'_2 \cup \dots \cup X'_s$ .  $\square$

Combining 1.20 and 1.21 yields theorem 1.17.

**Remark 1.22.** The proof strategy for (a) can be summarized as follows: Let  $X$  be a Noetherian space and  $P$  a property of closed subsets. To show that  $P$  holds for all subsets of  $X$  (thus in particular for  $X$ ), it suffices to show that for all  $Y \subseteq X$  closed, if  $P$  holds for all proper closed subsets of  $Y$ , then it also holds for  $Y$ . This is called *Noether induction* (a special case of well-founded induction).

**Example 1.23.** Let  $f \in K[x_1, \dots, x_n]$ . This is a factorial ring, so we may write  $f = g_1^{k_1} \cdots g_r^{k_r}$  with  $g_i$  irreducible and pairwise different. Then

$$V(f) = V(g_1^{k_1}) \cup \cdots \cup V(g_r^{k_r}) = V(g_1) \cup \cdots \cup V(g_r)$$

is the decomposition of  $V(f)$  into irreducible subsets:  $V(g_i)$  is irreducible by proposition 1.11, since  $I(V(g_i)) = (g_i)$  is prime.

In general, finding this composition for  $V(f_1, \dots, f_r)$  is not easy.

**Example 1.24.** What is the Zariski topology on  $\mathbb{A}^1$ ? By definition, a closed/algebraic set is of the form  $V(I)$  for some ideal  $I \subseteq K[x]$ . Since  $K[x]$  is a PID,  $I = (f)$  for some  $f = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \in K[x]$ . If  $f$  is not constant, we see as in example 1.23 that

$$X = V(f) = \bigcup_i V(x - a_i) = \{a_1, \dots, a_r\}.$$

Hence the closed sets are exactly  $V(0) = \mathbb{A}^1$ ,  $V(1) = \emptyset$ , and finite unions of points. In other words, the Zariski topology coincides with the cofinite topology on  $\mathbb{A}^1$ . The affine varieties on  $\mathbb{A}^1$  are therefore either  $\mathbb{A}^1$  itself or a single point.

We also see that any two non-empty open subsets have nontrivial intersection, so  $\mathbb{A}^1$  with the Zariski topology is not Hausdorff.

**Definition 1.25.** Let  $X$  be a nonempty topological space. We define the dimension of  $X$  as the supremum of all  $n \in \mathbb{N}$  such that there is a chain of irreducible subspaces  $\emptyset \neq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \subseteq X$ .

**Example 1.26.** By example 1.24, a maximal chain of affine varieties in  $\mathbb{A}^1$  is  $\{0\} \subsetneq \mathbb{A}^1$ , hence  $\dim \mathbb{A}^1 = 1$ .

**Definition 1.27.** Let  $R$  be a (commutative) ring. The Krull dimension of  $R$  is the supremum over all  $l$  such that there is a chain of prime ideals  $\mathfrak{p}_l \subsetneq \mathfrak{p}_{l-1} \subsetneq \cdots \subsetneq \mathfrak{p}_0 \subsetneq R$ .

Recall from corollary 1.14 that there is an inclusion-reversing correspondence between prime ideals of  $K[x_1, \dots, x_n]$  and affine algebraic varieties in  $\mathbb{A}^n$ . Fixing some variety  $X$ , it follows that subvarieties correspond bijectively to prime ideals that contain  $I(X)$ , i.e. prime ideals of  $K[x_1, \dots, x_n]/I(X)$ . Hence

**Proposition 1.28.** If  $X$  is an affine algebraic variety, then  $\dim X = \dim K[x_1, \dots, x_n]/I(X)$ .

## 2 Morphisms of Affine Varieties

### 2.1 Regular Morphisms

**Definition 2.1.** Let  $X \subseteq \mathbb{A}_K^n$  be an algebraic set. A function  $f : X \rightarrow K$  is *regular* if there is a polynomial  $F \in K[x_1, \dots, x_n]$  such that  $f = F|_X$ , i.e.  $f(x) = F(x)$  for all  $x \in X$ . Write  $A(X)$  for the set of regular functions on  $X$ .

**Remark 2.2.**  $A(X)$  is a ring (and even a  $K$ -algebra) in a natural way, with addition and multiplication defined pointwise. Moreover, there is a homomorphism of  $K$ -algebras

$$K[x_1, \dots, x_n] \twoheadrightarrow A(X), \quad F \mapsto F|_X.$$

The kernel of this morphism is exactly  $I(X)$ , so that  $A(X) \cong K[x_1, \dots, x_n]/I(X)$  canonically.

**Remark 2.3.** By corollary 1.14,  $A(X)$  is always reduced,  $A(X)$  is integral iff  $X$  is a variety, and  $A(X)$  is a field iff  $X$  is a point (in which case  $A(X) \cong K$ ).

**Definition 2.4.** Let  $X \subseteq \mathbb{A}^n$ ,  $Y \subseteq \mathbb{A}^m$  be affine algebraic sets. A map  $\varphi : X \rightarrow Y$  is called *regular* if  $\varphi = (f_1, \dots, f_m)$  for some regular  $f_1, \dots, f_m \in A(X)$ . A regular map  $\varphi$  is an isomorphism if it has an inverse which is also regular.

**Example 2.5.** (i)  $f : \mathbb{A}^1 \rightarrow V(y - x^2) \subseteq \mathbb{A}^2$ ,  $t \mapsto (t, t^2)$  is a regular map. It has inverse  $(x, y) \mapsto x$ , which is also regular, hence  $\mathbb{A}^1 \cong V(y - x^2)$ .

(ii)  $\varphi : \mathbb{A}^1 \rightarrow V(y^2 - x^3) \subseteq \mathbb{A}^2$ ,  $t \mapsto (t^2, t^3)$  is regular and bijective as well, but its inverse  $(x, y) \mapsto \frac{y}{x}$  is not regular, so  $\varphi$  is not an isomorphism.

**Proposition 2.6.** Let  $X \subseteq \mathbb{A}^n$  and  $Y \subseteq \mathbb{A}^m$  be algebraic sets, and let  $\varphi : X \rightarrow Y$  be a regular map. Then  $\varphi$  is continuous (w.r.t. the Zariski topology on  $X$  and  $Y$ ).

*Proof.* Let  $\varphi = (f_1, \dots, f_m)$  and  $J = \langle F_1, \dots, F_k \rangle \subseteq K[x_1, \dots, x_m]$  with  $V(J) \subseteq Y$ . Then

$$\varphi^{-1}(V(J)) = \varphi^{-1}(V(F_1, \dots, F_k)) = \{x \in X \mid F_j(f_1(x), \dots, f_m(x)) = 0, j = 1, \dots, k\}$$

Now  $F_j(f_1(x), \dots, f_m(x))$  is a composition of polynomials, hence a polynomial, call it  $\tilde{F}_j$ . We conclude  $\varphi^{-1}(V(J)) = X \cap V(\tilde{F}_1, \dots, \tilde{F}_k)$  as desired.  $\square$

**Remark 2.7.** The converse is false. For example, one easily concludes from example 1.24 that every bijective map  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$  is continuous, but there are way more bijections than polynomials (say because polynomials are defined by their values on any infinite subset). On the other hand, if  $K$  is finite (loosing algebraic closedness), then every function  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$  is regular.

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**Remark 2.8.** Let  $X$  be an algebraic set, and let  $f : X \rightarrow \mathbb{A}^1$ . Then  $f$  is a regular map if and only if  $f$  is a regular function. Note that the composition of regular maps is regular, since compositions of polynomials are polynomials.

**Definition 2.9.** Let  $X, Y$  be algebraic sets and  $F : X \rightarrow Y$  be regular. Then we set  $F^* : A(Y) \rightarrow A(X)$ ,  $g \mapsto g \circ F$ . This is well-defined by remark 2.8, and  $F^*$  clearly preserves addition and multiplication, so it is a morphism of  $K$ -algebras.

**Remark 2.10.** Let  $F = (f_1, \dots, f_m) : X \rightarrow Y$ ,  $f_i \in K[x_1, \dots, x_n]$ , then  $F^*$  is given by the  $K$ -algebra homomorphism  $A(Y) \cong K[y_1, \dots, y_m]/I(Y) \rightarrow K[x_1, \dots, x_n]/I(X) \cong A(X)$  (see remark 2.2) defined by  $y_i \mapsto f_i$ . Hence  $F(x) = (F_1^*(y_1), \dots, F_m^*(y_m))$ .

**Theorem 2.11.** (i) *There is a bijection  $\text{Mor}(X, Y) \rightarrow \text{Hom}_{K\text{-Alg}}(A(Y), A(X))$  given by  $F \mapsto F^*$ .*

(ii) *If  $F : X \rightarrow Y$  and  $H : Y \rightarrow Z$  are regular, then  $(H \circ F)^* = F^* \circ H^*$ . Further,  $\text{id}_X^* = \text{id}_{A(X)}$ .*

(iii) *Let  $F : X \rightarrow Y$  be regular. Then  $F$  is an isomorphism of affine sets if and only if  $F^*$  is an isomorphism of  $K$ -algebras*

*Proof.* Injectivity in (i) follows from remark 2.10. For surjectivity, let  $\varphi : A(Y) \rightarrow A(X)$  be a  $K$ -algebra homomorphism and define  $F : X \rightarrow Y$  by  $F = (\varphi(y_1), \dots, \varphi(y_m))$ . We need to check that this is well-defined, i.e. that the image of  $F$  lies in  $Y$ . Then it is clear that  $F$  is regular and that  $F^* = \varphi$ , again by remark 2.10.

So let  $g \in I(Y)$ , we need to show  $g \circ F = 0$ . But this is exactly the statement  $\varphi([g]) = \varphi(0) = 0$ .

For (ii),  $\text{id}_X^* = \text{id}_{A(X)}$  is clear, and for  $f \in A(Z)$  one has

$$(H \circ F)^*(f) = f \circ H \circ F = H^*(f) \circ F = (F^* \circ H^*)(f),$$

so  $(H \circ F)^* = F^* \circ H^*$ . Then (iii) follows from (i) and (ii).  $\square$

**Example 2.12.** Looking again at the maps from example 2.5, we see that  $f : \mathbb{A}^1 \rightarrow V(y-x^2), t \mapsto (t, t^2)$  is an isomorphism, because  $f^* : K[x, y]/(y-x^2) \rightarrow K[t], x \mapsto t, y \mapsto t^2$  clearly is. On the other hand, let  $\varphi : \mathbb{A}^1 \rightarrow V(y^2-x^3), t \mapsto (t^2, t^3)$ . We saw that this is a bijective regular map and gave intuitive reasoning for why this map isn't an isomorphism. But now we can prove it: We have

$$f^* : K[x, y]/(y^2-x^3) \rightarrow K[t], \quad x \mapsto t^2, y \mapsto t^3$$

is not surjective, for the image does not contain  $t$ .

**Remark 2.13.** In categorical terms, theorem 2.11 says that

$$\begin{array}{c} \left\{ \begin{array}{l} \text{algebraic sets} \\ \text{regular maps} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{finitely generated reduced } K\text{-algebras} \\ K\text{-algebra homomorphisms} \end{array} \right\} \\ X \mapsto A(X) \\ F \mapsto F^* \end{array}$$

is a contravariant functor, and even an equivalence of categories: For essential surjectivity, note that every finitely generated  $K$ -algebra can be written as a quotient  $K[x_1, \dots, x_n]/I$  by choosing generators. Then consider  $X = V(I)$ .

**Proposition 2.14.** *Let  $X, Y$  be algebraic sets, and let  $f : X \rightarrow Y$  be a regular map. Then*

(i)  *$f^* : A(Y) \rightarrow A(X)$  is surjective if and only if  $\overline{f(X)} = Y$ , i.e. if the image of  $f$  is dense in  $Y$ .*

(ii)  *$f^*$  is injective if and only if  $f(X) \subseteq Y$  is closed and  $f : X \rightarrow f(X)$  is an isomorphism.*

*Proof.* Exercise.  $\square$

## 2.2 Rational Maps of Varieties

Let  $X \subseteq \mathbb{A}^n$  be an affine algebraic variety. Then  $I(X)$  is prime, so  $A(X) \cong K[x_1, \dots, x_n]/I(X)$  is an integral domain. Hence we can define its field of fractions  $K(X) := \text{Frac } A(X)$ .

**Definition 2.15.** An element  $\varphi \in K(X)$  is called regular at  $x \in X$  if there exist  $f, g \in A(X)$  with  $\varphi = \frac{f}{g}$  and  $g(x) \neq 0$ .

**Example 2.16.** Let  $X = V(x^2 - yz) \subseteq \mathbb{A}^3$  and  $x = (0, 0, 1)$ . Consider  $\varphi = \frac{y}{x} \in K(X)$ . Even though it may look like  $\varphi$  might not be regular at  $x$ , one can note that  $\frac{y}{x} = \frac{x}{z}$  in  $K(X)$ , so actually  $\varphi(x)$  can be defined and  $\varphi$  is regular at  $x$ .

**Proposition 2.17.** Let  $\varphi \in K(X)$ . Then  $\varphi$  is regular at every  $x \in X$  if and only if  $\varphi \in A(X)$ .

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**Remark 2.18.** If  $X \subseteq \mathbb{A}^n$  is an affine algebraic variety, the closed sets are exactly of the form  $V_X(I) := \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$  for ideals  $I \subseteq A(X)$ , and  $V_X$  is still an inclusion-reversing bijection between radical ideals and closed subsets, compare exercises.

*Proof.* Assume  $\varphi \in K(X)$  is regular at every point  $x \in X$ . Consider  $I := \{f \in A(X) \mid f\varphi \in A(X)\}$ . Then the claim is equivalent to  $I = A(X)$ , hence to  $V_X(I) = \emptyset$  by remark 2.18. Assume there exists  $x \in V_X(I)$ . Since  $\varphi$  is regular at  $x$ , we can write  $\varphi = \frac{g}{h}$  with  $g, h \in A(X)$  and  $h(x) \neq 0$ . Hence  $h \in I$ , and  $h(x) = 0$  by choice of  $x$ , contradiction.  $\square$

**Definition 2.19.** Let  $X \subseteq \mathbb{A}^n$  be an affine variety and  $U \subseteq X$  be open. Denote  $\mathcal{O}_X(U) := \{\varphi \in K(X) \mid \varphi \text{ regular at all } x \in U\}$ . For  $\varphi \in K(X)$ , its domain is  $\text{dom}(\varphi) := \{a \in X \mid \varphi \text{ is regular at } a\}$ . In other words,  $\mathcal{O}_X(U) = \{\varphi \in K(X) \mid U \subseteq \text{dom}(\varphi)\}$ .

By proposition 2.17,  $\mathcal{O}_X(X) = A(X)$ .

**Example.** (i)  $\varphi = \frac{y}{x}$  on  $X = V(y - x^2)$  is regular, since  $\varphi = x$ . Hence  $\text{dom}(\varphi) = X$ .

(ii)  $\varphi = \frac{y}{x}$  on  $X = V(y^2 - x^3)$  has  $\text{dom}(\varphi) = X \setminus \{(0, 0)\}$ .

**Proposition 2.20.** Let  $\varphi \in K(X)$ . Then  $\text{dom}(\varphi)$  is an open non-empty set in  $X$ .

*Proof.* Define  $I := \{f \in A(X) \mid f\varphi \in A(X)\}$ . As before, we have  $\varphi$  is regular at  $x$  if and only if  $x \notin V_X(I)$ , so  $\text{dom } \varphi = X \setminus V_X(I)$  is open.  $\square$

**Remark 2.21.** Let  $X$  be an irreducible topological space. Then

- (i) Every non-empty open subset  $U \subseteq X$  is dense in  $X$ .
- (ii) If  $U_1, U_2 \subseteq X$  are open and non-empty, then  $U_1 \cap U_2 \neq \emptyset$ .

Hence, if  $X$  is an affine variety and  $f \in A(X)$  evaluates to zero on some non-empty open, then already  $f = 0$ .

**Remark 2.22.** Let  $U \subseteq X$  be a non-empty open. Any regular  $\varphi \in \mathcal{O}_X(U) \subseteq K(X)$  defines a set-theoretical function  $\varphi : U \rightarrow K$ , by sending  $a \in U$  to  $\frac{f(a)}{g(a)}$ , where  $\varphi = \frac{f}{g}$  with  $f, g \in A(X)$  and  $g(a) \neq 0$ . This is well-defined, for if  $\varphi = \frac{f_1}{g_1}$  with  $g_1(a) \neq 0$ , then  $f g_1 - f_1 g = 0$  in  $A(X)$ .

Conversely, let  $\varphi : U \rightarrow K$  be a (set-theoretical) function. Then  $\varphi$  defines a regular function on  $U$  if for every  $a \in U$  there is an open neighbourhood  $a \in V \subseteq U$  such that  $\varphi(b) = \frac{f(b)}{g(b)}$  for all  $b \in V$ , where  $f, g \in K[x_1, \dots, x_n]$  and  $g(b) \neq 0$  for all  $b \in B$ .

These assignments  $(\varphi \in \mathcal{O}_X(U)) \mapsto (\varphi : U \rightarrow K)$  and  $(\varphi : U \rightarrow K) \mapsto [\frac{f}{g}]$  are clearly well-defined and mutually inverse, so this is an equivalent view on regular functions on  $U$ .

One sees easily that the composition of regular maps is again regular.

**Remark 2.23.** Let  $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$  be two regular functions, and let  $U \subseteq V$  be nonempty open. If  $\varphi_1|_U = \varphi_2|_U$  then  $\varphi_1 = \varphi_2$ .

**Definition 2.24.** (i) A *quasi-affine variety* is an open subset of an affine algebraic variety.

(ii) A regular map between quasi-affine varieties  $U \subseteq \mathbb{A}^n, V \subseteq \mathbb{A}^m$  is a map  $\varphi : U \rightarrow V$  given by  $\varphi = (\varphi_1, \dots, \varphi_m)$  with  $\varphi_i$  regular on  $U$ .  $\varphi$  is an isomorphism if there is a regular inverse.

**Remark 2.25.** For affine varieties, by remark 2.13 all information on regular maps  $f : X \rightarrow Y$  could be obtained from their induced coordinate maps  $f^* : A(Y) \rightarrow A(X)$ . This is no longer true for quasi-affine varieties: for example,  $\mathbb{A}^2 \setminus 0 \hookrightarrow \mathbb{A}^2$  induces an isomorphism of coordinate rings

**Definition 2.26.** Let  $X$  be an affine variety and  $f \in A(X)$ . Then  $D(f) := X \setminus V_X(f)$  is called the *distinguished open subset* of  $f$  in  $X$ .

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**Remark 2.27.** Since  $D(f) \cap D(g) = D(fg)$ , finite intersections of distinguished opens are again distinguished open. Any open  $U \subseteq X$  is a finite union of distinguished open subsets. Indeed,  $U = X \setminus V_X(f_1, \dots, f_n) = \bigcup_i D(f_i)$ .

**Proposition 2.28.** Let  $X$  be an affine variety and  $0 \neq f \in A(X)$ . Then  $\mathcal{O}_X(D(f)) = A(X)_f = \{\frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N}\} \subseteq K(X)$ . In particular, on a distinguished open subset a regular function is always globally the quotient of two elements from  $A(X)$ .

*Proof.*  $\supseteq$  is clear. So let  $\varphi \in \mathcal{O}_X(D(f))$  and consider

$$I = \{h \in A(X) \mid h\varphi \in A(X)\} \subseteq A(X).$$

This is an ideal which clearly satisfies  $V_X(I) \cap D(f) = \emptyset$ . Hence  $V_X(I) \subseteq V_X(f)$ , and by the Nullstellensatz 1.13 we see that  $f \in \sqrt{I}$ , i.e.  $f^n \in I$  for some  $n$ .  $\square$

**Example 2.29.** Consider  $D(x) = \mathbb{A}^1 \setminus 0 \rightarrow V(xy-1) \subseteq \mathbb{A}^2, x \mapsto (x, \frac{1}{x})$ . This is an isomorphism (with inverse  $(x, y) \mapsto x$  between the quasi-affine  $\mathbb{A}^1 \setminus 0$  and the affine variety  $V(xy-1)$ ). Note that this is not true in general: not every quasi-affine variety is isomorphic to an affine variety. For example,  $\mathbb{A}^2 \setminus 0$  isn't isomorphic to any affine variety. However, we have

**Proposition 2.30.** Let  $X$  be an affine variety and  $f \in A(X)$ . Then  $D(f)$  is isomorphic to an affine variety  $Y$  with  $A(Y) \cong A(X)_f$ .

*Proof.* Set

$$Y := \{(x, t) \in X \times \mathbb{A}^1 \mid tf(x) = 1\} \subseteq X \times \mathbb{A}^1 \subseteq \mathbb{A}^{n+1}.$$

Then as in example 2.29,  $D(f) \rightarrow Y, x \mapsto (x, \frac{1}{f(x)})$  is an isomorphism with inverse  $(x, y) \mapsto x$ , so  $D(f) \cong Y$  and  $A(Y) \cong \mathcal{O}_X(D(f)) \cong A(X)_f$ .  $\square$

We have seen that for  $X$  an algebraic set and  $f \in A(X)$  regular,  $V_X(f)$  is closed in  $X$ . The same is true for quasi-affine varieties:

**Lemma 2.31.** Let  $X$  be an affine variety and  $U \subseteq X$  open. Let  $\varphi \in \mathcal{O}_X(U)$ . Then  $V_U(\varphi) := V(\varphi) \cap U = \{x \in U \mid \varphi(x) = 0\}$  is closed in  $U$ .

*Proof.* Let  $a \in U$ . Then there exists an open neighbourhood  $a \in U_a \subseteq U$  and  $f, g \in A(X)$  such that  $\varphi = \frac{f_a}{g_a}$  on  $U_a$ . Then

$$U_a \setminus V(\varphi) = \{x \in U_a \mid \varphi(x) \neq 0\} = \{x \in U_a \mid f_a(x) \neq 0\} = U_a \setminus V(f_a)$$

is open in  $X$ , hence  $U \setminus V(\varphi) = \bigcup_a U_a \setminus V(\varphi)$  is open.  $\square$

**Proposition 2.32.** *Let  $X$  be a quasi-affine variety and  $U \subseteq X$  be open. Let  $\varphi, \psi$  be two regular functions on  $X$  such that  $\varphi|_U = \psi|_U$ . Then  $\varphi = \psi$  on  $X$ .*

*Proof.*  $V_X(\varphi - \psi)$  contains the open, hence dense by 2.21, set  $U$ .  $\square$

**Proposition 2.33.** *Let  $X, Y$  be algebraic sets and  $U \subseteq X$  be open. Then any regular map  $\varphi : U \rightarrow Y$  is continuous (w.r.t. the Zariski topology). In particular,  $\varphi \in \mathcal{O}_X(U)$  is a continuous map  $U \rightarrow \mathbb{A}^1$ .*

*Proof.* Let  $\varphi = (\varphi_1, \dots, \varphi_m)$  and let  $Z = V_Y(g_1, \dots, g_m) \subseteq Y$  be a closed subset. Then  $\varphi^{-1}(Z) = \{x \in U \mid g_i(\varphi_1(x), \dots, \varphi_m(x)) = 0 \text{ for all } i\}$ , which is closed by lemma 2.31.  $\square$

Let  $\varphi : U \rightarrow V$  be regular. For any regular map  $f \in \mathcal{O}(V)$ , the composition  $f \circ \varphi \in \mathcal{O}(U)$  is well-defined, hence we get as before a  $K$ -algebra homomorphism

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$$\varphi^* : \mathcal{O}(V) \rightarrow \mathcal{O}(U), \quad f \mapsto f \circ \varphi.$$

The assignment  $U \mapsto \mathcal{O}(U)$ ,  $\varphi \mapsto \varphi^*$  is a contravariant functor as before, but no longer an equivalence of categories, see exercises.

Let  $X, Y$  be affine algebraic subsets. We know that regular maps  $X \rightarrow Y$  are given by polynomial functions. It may happen that we do not have any "interesting" polynomial maps. For example, over  $K = \mathbb{C}$  consider  $X = \mathbb{A}^1$  and  $Y = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$ . Then the only regular maps  $X \rightarrow Y$  are constant. However, the nontrivial map  $t \mapsto (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$  induces an isomorphism  $\mathbb{A}^1 \setminus \{\pm i\} \rightarrow Y \setminus \{(1, 0)\}$ .

Let  $X$  be an affine algebraic variety. Then  $\varphi \in K(X)$  is a regular function on  $\text{dom } \varphi$ . Moreover, given  $\varphi_1, \dots, \varphi_m \in K(X)$ , we get a regular map on the open set  $\bigcap_i \text{dom } \varphi_i \rightarrow \mathbb{A}^m$ .

**Definition 2.34.** Let  $X$  be an affine algebraic variety and  $Y$  an affine algebraic set. A *rational map*  $\varphi : X \dashrightarrow Y \subseteq \mathbb{A}^m$  is given by  $\varphi = (\varphi_1, \dots, \varphi_m)$  with  $\varphi_i \in K(X)$  such that  $\varphi(x) \in Y$  for every  $x \in \text{dom } \varphi := \bigcap_i \text{dom } \varphi_i$ . A rational map  $\varphi : X \dashrightarrow Y$  is called *dominant* if the image of  $\varphi$  is dense in  $Y$ , i.e. if  $\overline{\varphi(\text{dom } \varphi)} = Y$ .

A rational map  $\varphi : X \dashrightarrow Y$  induces a regular map  $\text{dom } \varphi \rightarrow Y$ . Let  $\varphi : X \dashrightarrow Y$  and  $\psi : Y \dashrightarrow Z$  be rational maps. Then  $\psi$  might not be defined on  $\text{im } \varphi$ . But if  $\varphi$  is dominant, then  $\psi \circ \varphi$  is well-defined on the non-empty open  $\varphi^{-1}(\text{dom } \psi)$ .

**Definition 2.35.** Let  $X$  be an affine algebraic variety and  $Y$  an affine algebraic set. A rational map  $\varphi : X \dashrightarrow Y$  is an equivalence class of pairs  $(U, \varphi_U)$ , where  $U \subseteq X$  is nonempty open,  $\varphi_U : U \rightarrow Y$  is regular, and  $(U, \varphi_U) \sim (V, \varphi_V)$  if and only if  $\varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}$ . The rational map is dominant if for some (and therefore all)  $(U, \varphi_U)$  one has  $\overline{\varphi_U(U)} = Y$ .

**Remark 2.36.** The relation in definition 2.35 is an equivalence relation. Indeed, if  $(U, \varphi_U) \sim (V, \varphi_V) \sim (W, \varphi_W)$ , then  $\varphi_U|_{U \cap W}$  and  $\varphi_W|_{U \cap W}$  are regular maps that agree on the non-empty open  $U \cap V \cap W$ , hence they are equal by proposition 2.32.

The two above definitions are equivalent: If  $\varphi$  is regular in the sense of 2.34, then  $[(\text{dom } \varphi, \varphi)]$  defines a regular map as in 2.35. Conversely, if an equivalence class  $\{(U_i, \varphi_i)\}_i$  is given, then the map  $\bigcup_i U_i, x \mapsto \varphi_i(x)$  for any  $i$  with  $x \in U_i$  is regular, i.e. a rational map as in 2.34. Clearly, the notion of dominance is preserved by these identifications.

One can compose dominant rational maps  $\varphi : X \dashrightarrow Y, \psi : Y \dashrightarrow Z$  by setting

$$[(U, \varphi_U)] \circ [(V, \varphi_V)] := [(\varphi_U^{-1}(V), \psi_V \circ \varphi_U|_{\varphi_U^{-1}(V)})]$$

Write  $\text{Mor}_{\text{rat}}(X, Y)$  for the set of rational morphisms  $X \dashrightarrow Y$ .

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**Definition 2.37.** Let  $\varphi : X \dashrightarrow Y$  be dominant. In the same way as for regular maps, we define

$$\varphi^* : \text{Mor}_{\text{rat}}(Y, \mathbb{A}^1) \rightarrow \text{Mor}_{\text{rat}}(X, \mathbb{A}^1), \quad f \mapsto f \circ \varphi.$$

**Proposition 2.38.** Let  $X$  be an affine algebraic variety. Then  $\text{Mor}_{\text{rat}}(X, \mathbb{A}^1)$  is a field with the operations  $(U, f) * (V, g) := (U \cap V, f|_{U \cap V} + g|_{U \cap V})$  for  $*$   $\in \{+, -, \cdot\}$  and  $(U, f)^{-1} = (U \setminus V(f), \frac{1}{f})$ . Moreover,  $\text{Mor}_{\text{rat}}(X, \mathbb{A}^1) \cong K(X)$  as fields.

*Proof.* It is clear that the given operations are well-defined and make  $\text{Mor}_{\text{rat}}(X, \mathbb{A}^1)$  a field. The equivalence of definitions 2.34 and 2.35 provides a field isomorphism  $K(X) \rightarrow \text{Mor}_{\text{rat}}(X, \mathbb{A}^1)$ ,  $f \mapsto (\text{dom } f, f)$ .  $\square$

**Corollary 2.39.** If  $\varphi : X \dashrightarrow Y$  is a dominant rational map between affine varieties, we get a  $K$ -homomorphism of fields  $\varphi^* : K(Y) \rightarrow K(X)$ ,  $f \mapsto f \circ \varphi$ .

Recall that for regular maps, we had in 2.13 an equivalence between algebraic sets + regular maps, and reduced f.g.  $K$ -algebras +  $K$ -algebra homomorphisms. In the case of rational maps, we get similarly

**Theorem 2.40.**  $\varphi \mapsto \varphi^*$  is a bijection  $\{\varphi \in \text{Mor}_{\text{rat}}(X, Y) \mid \varphi \text{ dominant}\}$  to  $\text{Hom}_K(K(Y), K(X))$ . This assignment is functorial, and induces an equivalence of categories

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{affine algebraic varieties} + \\ \text{dominant rational maps} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{field extensions } L/K \text{ of finite transcendence degree} + \\ K\text{-linear homomorphisms} \end{array} \right\} \\ X & \mapsto & K(X) \\ \varphi & \mapsto & \varphi^* \end{array}$$

*Proof.* To show that  $\varphi \mapsto \varphi^*$  is a bijection, define an inverse by assigning to  $f : K(Y) \rightarrow K(X)$  the morphism  $(f(y_1), \dots, f(y_m))$ . Everything else is clear.  $\square$

**Definition 2.41.** A dominant rational map  $\varphi : X \dashrightarrow Y$  is called a *birational equivalence* (and  $X$  and  $Y$  are called *birational* or *birationally equivalent*) if there exists a rational dominant map  $\psi : Y \dashrightarrow X$  such that  $\varphi \circ \psi = \text{id}_Y$  and  $\psi \circ \varphi = \text{id}_X$  as rational maps.

**Proposition 2.42.** Let  $X, Y$  be affine algebraic varieties. The following statements are equivalent:

- (i)  $X$  and  $Y$  are birational.
- (ii)  $K(X) \cong K(Y)$ .
- (iii) There exist non-empty open subsets  $U \subseteq X$  and  $V \subseteq Y$  such that  $U \cong V$  are isomorphic (in the sense of regular maps).

*Proof.* (i) $\Leftrightarrow$ (ii) follows from 2.40 and (iii) $\Rightarrow$ (i) from the definition of rational function as regular functions on some open. Now assume (i), i.e. that there exists a birational equivalence  $\varphi = (U, \varphi_U) : X \dashrightarrow Y$  with inverse  $\psi = (V, \psi_V)$ . Then  $\varphi = (U \cap \psi^{-1}(V), \varphi_U|_{U \cap \psi^{-1}(V)})$  and  $\psi = (V \cap \varphi^{-1}(U), \psi_V|_{V \cap \varphi^{-1}(U)})$  are the required isomorphisms  $U \cap \psi^{-1}(V) \cong V \cap \varphi^{-1}(U)$ .  $\square$

**Remark 2.43.** An affine algebraic variety  $X$  is called *rational*, if  $X$  is birational to  $\mathbb{A}^k$  for some  $k$ . Equivalently,  $K(X)/K$  is a purely transcendental field extension. For example, in the exercises we proved that  $S = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$  is rational<sup>1</sup>.

**Theorem 2.44.** Every affine algebraic variety  $X$  is birational to some hypersurface, i.e. a variety  $V(f) \subseteq \mathbb{A}^n$  for some irreducible  $f \in K[x_1, \dots, x_n]$ .

<sup>1</sup>Our proof works in  $\text{char } K \neq 2$ , but otherwise  $\sqrt{(x^2 + y^2 - 1)} = (x + y - 1)$ , so even  $A(S) \cong K[x]$

*Proof.* For simplicity, we only consider the case  $\text{char } K = 0$ . Since  $K(X)/K$  is finitely generated, by basic algebra  $K(X)/K$  factors as a purely transcendental extension followed by a finite one  $K(X)/K(t_1, \dots, t_d)/K$ . Since everything is separable,  $K(X)/K(t_1, \dots, t_d)$  is generated by a primitive element, i.e.  $K(X) = K(t_1, \dots, t_d, \alpha)$  with  $\alpha$  algebraic over  $K(t_1, \dots, t_d)$ . Let wlog  $f \in K[t_1, \dots, t_d]$  be the minimal polynomial of  $\alpha$ . Then  $K(X) \cong \text{Frac } K[t_1, \dots, t_d, s]/(f(s)) \cong K(V(f))$  as desired.  $\square$

**Remark 2.45.** Let  $X \subseteq \mathbb{A}^n$ ,  $Y \subseteq \mathbb{A}^m$  be affine algebraic sets. Then  $X \times Y \subseteq \mathbb{A}^{n+m}$  is also affine algebraic, given by the same equations, now considered in  $K[x_1, \dots, x_n, y_1, \dots, y_m]$ . Furthermore, if  $X, Y$  are irreducible, then so is  $X \times Y$  (Exercise). This is the product in the category of affine algebraic sets (resp. varieties), i.e. for regular maps  $\varphi : Z \rightarrow X$ ,  $\psi : Z \rightarrow Y$ , there exists a unique regular map  $Z \rightarrow X \times Y$ . Therefore  $A(X \times Y) = A(X) \otimes_K A(Y)$ .

### 3 Projective Varieties

**Definition 3.1.** Projective  $n$ -space over  $K$  is given by  $\mathbb{P}_K^n := \mathbb{A}^{n+1} \setminus \{0\} / \sim$ , where  $x \sim y$  if  $x = \lambda y$  for some  $\lambda \in K$ . We notate the equivalence class of  $x$  by  $x = [x_0 : x_1 : \dots : x_n]$ , called the *homogeneous coordinates* of  $x$ .

Note that points in  $\mathbb{P}^n$  correspond to one-dimensional linear subspaces of  $\mathbb{A}^{n+1}$ .

**Remark 3.2.** We would like to define projective algebraic sets as zeroes of polynomials as in the affine case. But this is not well-defined, because evaluation of a polynomial need not respect the equivalence relation of 3.1. For example, let  $f = x_1^2 - x_0 \in K[x_0, x_1]$ . Then  $f(1, 1) = 0$  and  $f(-1, -1) = 2$ , but  $[1 : 1] = [-1 : -1] \in \mathbb{P}_K^1$ .

This problem can be solved by only considering *homogeneous polynomials*. For such a polynomial

$$f = \sum_{k_0 + \dots + k_n = d} a_{k_0, \dots, k_n} x_0^{k_0} \dots x_n^{k_n},$$

we have  $f(\lambda x) = \lambda^d f(x)$ , so  $f(x) = 0$  is well-defined for  $x \in \mathbb{P}^n$ .

**Definition 3.3.** An ideal  $I \subseteq K[x_0, \dots, x_n]$  is called homogeneous if it can be generated by homogeneous polynomials.

**Remark 3.4.** (i) If  $I$  is homogeneous and  $f \in K[x_0, \dots, x_n]$ , write  $f = f_0 + f_1 + \dots + f_d$  with  $f_i$  homogeneous of degree  $i$ . Then  $f \in I$  if and only if  $f_i \in I$  for all  $i$ . (Say  $I = (g_1, \dots, g_n)$  with  $g_i$  homogeneous, write  $f = \sum g_i h_i$ . Then  $f_d = \sum g_i (h_i)_{d - \deg g_i} \in I$ .)  
(ii) If  $I_1, I_2$  are homogeneous ideals, then so are  $I_1 + I_2, I_1 I_2, I_1 \cap I_2, \sqrt{I_1}$ . (For " $\cap$ ", find an arbitrary generating set and then use (i),  $\sqrt{\phantom{x}}$  is exercise.)

**Definition 3.5.** Let  $f_1, \dots, f_k \in K[x_0, \dots, x_n]$  be homogeneous. Then

$$V(f_1, \dots, f_k) := V^p(f_1, \dots, f_k) := \{x \in \mathbb{P}_K^n \mid f_i(x) = 0 \text{ for all } i\}$$

is called a *projective algebraic set*. In the same way, for a homogeneous ideal  $I \subseteq K[x_0, \dots, x_n]$ , set

$$V(I) := V^p(I) := \{x \in \mathbb{P}_K^n \mid f(x) = 0 \text{ for all } f \in I \text{ homogeneous}\}.$$

**Example 3.6.** We have  $V^p(0) = \mathbb{P}^n$ ,  $V^p(1) = \emptyset$ . Further, every point  $x = [x_0 : \dots : x_n]$  forms a projective algebraic set, since  $V^p(a_i x_j - a_j x_i)_{i,j} = \{x\}$ .

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**Remark 3.7.** Just as in 1.5, projective algebraic sets are closed under arbitrary intersections and finite unions.

**Definition 3.8.** The *Zariski topology* on  $\mathbb{P}^n$  is defined as the topology which closed sets the projective algebraic sets. On a projective algebraic set  $X \subseteq \mathbb{P}^n$ , the induced subspace topology is also called the Zariski topology on  $X$ .

**Definition 3.9.** A projective algebraic variety is an irreducible projective algebraic set.

For a subset  $X \subseteq \mathbb{P}^n$  we may set

$$I^p(X) := \{f \in K[x_0, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

$V^p$  and  $I^p$  enjoy many of the same properties as in the affine case. In particular

**Proposition 3.10.** (i) For a subset  $X \subseteq \mathbb{P}^n$ ,  $V^p(I^p(X)) = \overline{X}$ .  
(ii) For a homogeneous ideal  $I \subseteq K[x_0, \dots, x_n]$  with  $(x_0, \dots, x_n) \not\subseteq I$ ,  $I^p(V^p(I)) = \sqrt{I}$ .

(iii) A projective algebraic set  $X$  is a variety if and only if  $I^p(X)$  is a prime ideal.

*Proof.* (i) and (ii) as in the affine case. For (iii), we need the following

**Claim:** A homogeneous ideal  $I \subseteq K[x_0, \dots, x_n]$  is prime if and only if for all homogeneous  $f, g \in L[x_0, \dots, x_n]$  with  $fg \in I$ , one has  $f \in I$  or  $g \in I$ .

Indeed, suppose  $I$  were not prime, and let  $f, g \notin I$  such that  $fg \in I$ . Let  $d_0, e_0$  be maximal w.r.t.  $f_{d_0}, g_{e_0} \notin I$ . Then  $(fg)_{d_0+e_0} = f_{d_0}g_{e_0} + \sum_{i+j=d_0+e_0, i \neq d_0} f_i g_j$ . The left hand side is in  $I$  by remark 3.4, and the sum by the maximality assumption. Hence  $f_{d_0}g_{e_0} \in I$ .  $\square$

**Definition 3.11.** Let  $\pi : \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$  be the canonical projection.

- (i) An algebraic set  $X \subseteq \mathbb{A}^{n+1}$  is called a *cone* if  $0 \in X$  and  $x \in X$  implies  $\lambda x \in X$  for all  $\lambda \in K$ .
- (ii) Given a cone  $X \subseteq \mathbb{A}^{n+1}$ , its *projectivization* is  $\mathbb{P}(X) := \pi(X \setminus \{0\})$ .
- (iii) For a projective algebraic set  $X \subseteq \mathbb{P}^n$ , its *cone* is  $C(X) := \{0\} \cup \pi^{-1}(X)$

Note that  $\mathbb{P}(X)$  and  $C(X)$  are projective resp. affine algebraic sets. Indeed, for a homogeneous ideal  $S \subseteq K[x_0, \dots, x_n]$  we have  $\mathbb{P}(V(S)) = V^p(S)$  and  $C(V^p(S)) = V(S)$ . It remains to show that all cones are of this form, which is

**Proposition 3.12.** Let  $X \subseteq \mathbb{A}^{n+1}$  be a cone. Then  $I(X)$  is a homogeneous ideal.

*Proof.* For  $f = f_0 + \dots + f_d$  and  $x \in X$  we have  $0 = f(\lambda x) = \sum_i \lambda^i f_i(x)$ . As the 0 polynomial function in  $\lambda$ , since  $K$  is infinite we must have  $f_i(x) = 0$  for all  $i$ .  $\square$

The next goal is to prove a projective version of the Nullstellensatz 1.13.

Lecture 11  
Nov 24, 2025

**Definition 3.13.** The (homogeneous maximal) ideal  $I_0 := (x_0, \dots, x_n) \subseteq K[x_0, \dots, x_n]$  is called the *irrelevant* ideal.

Note that  $V^p(I_0) = \emptyset$ , so in general  $I^p(V^p(I)) \neq I$  for radical homogeneous ideals  $I$ . But in some sense this is the only problematic case:

**Proposition 3.14** (Projective Nullstellensatz). For any homogeneous ideal  $J \subseteq K[x_0, \dots, x_n]$  with  $\sqrt{J} \neq I_0$  we have  $I^p(V^p(J)) = \sqrt{J}$ .

*Proof.* The inclusion " $\supseteq$ " is clear. We have

$$\begin{aligned} I^p(V^p(J)) &= \langle f \in K[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in V^p(J) \rangle \\ &= \langle f \in K[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in \overline{V(J)} \setminus \{0\} \rangle \end{aligned}$$

Now  $V(J) \neq \{0\}$ , otherwise  $\sqrt{J} = I(V(J)) = I_0$ , hence  $\overline{V(J)} \setminus \{0\} = V(J)$  (since then either  $V(J) = \emptyset$  or  $V(J)$  contains a line through 0). Then  $I^p(V^p(J))$  is generated by homogeneous polynomials in  $I(V(J)) = \sqrt{J}$ . But  $\sqrt{J}$  is homogeneous itself, so  $I^p(V^p(J)) = \sqrt{J}$  as well.  $\square$

**Corollary 3.15.** (i) If  $I \subseteq K[x_0, \dots, x_n]$  is a homogeneous ideal, then  $V^p(I) = \emptyset$  if and only if  $I_0 \subseteq \sqrt{I}$ , if and only if  $\sqrt{I} = I_0$  or  $I = (1)$ .  
(ii) If  $V^p(J) \neq \emptyset$ , then  $I^p(V^p(J)) = \sqrt{J}$ .  
(iii)  $I^p$  and  $V^p$  define inclusion-reversing bijections

$$\begin{aligned} \{\text{projective algebraic sets in } \mathbb{P}^n\} &\rightleftarrows \{\text{radical hom. ideals } I_0 \neq J \subseteq K[x_0, \dots, x_n]\} \\ \{\text{projective algebraic varieties in } \mathbb{P}^n\} &\rightleftarrows \{\text{prime hom. ideals } I_0 \neq J \subseteq K[x_0, \dots, x_n]\} \\ \{\text{points in } \mathbb{P}^n\} &\rightleftarrows \{\text{maximal hom. ideals } I_0 \neq J \subseteq K[x_0, \dots, x_n]\} \end{aligned}$$

(iv)  $I^p(\mathbb{P}^n) = 0$ , and  $\mathbb{P}^n$  is a variety.

**Remark 3.16.** Let  $U_i := D(x_i) = \{x \in \mathbb{P}^n \mid x_i \neq 0\} = \{x \in \mathbb{P}^n \mid x_i = 1\}$ . Leaving out the  $i$ -th coordinate in the last presentation yields a homeomorphism  $\iota_i : \mathbb{A}^n \rightarrow U_i$  (even an isomorphism of varieties, cf. later).

Therefore  $\mathbb{P}^n = \bigcup_{i=0}^n U_i$  is an open cover of projective space by  $n + 1$  copies of  $\mathbb{A}^n$ .

**Definition 3.17.** (i) For a homogeneous polynomial  $f \in K[x_0, \dots, x_n]$ , its dehomogenization is  $f^i := f(1, x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ . For a homogeneous ideal  $J \subseteq K[x_0, \dots, x_n]$ , write  $J^i = \{f^i \mid f \in J\} \subseteq K[x_1, \dots, x_n]$ . In other words, these are the images of  $f$ , resp.  $J$ , under the natural map  $K[x_0, \dots, x_n] \mapsto K[x_0, \dots, x_n]/(x_0 - 1) \cong K[x_1, \dots, x_n]$ .  
(ii) For  $0 \neq f \in K[x_1, \dots, x_n]$  of  $\deg f = d$ , its homogenization is  $f^h := x_0^d f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \in K[x_0, \dots, x_n]$ . For an ideal  $J \subseteq K[x_1, \dots, x_n]$ , write  $J^h$  for the ideal of  $K[x_0, \dots, x_n]$  generated by  $f^h, f \in J$

For example, if  $f = 1 + X_1 + X_2 + X_1^2$ , then  $f^h = X_0^2 + X_0X_1 + X_0X_2 + X_1^2$ , and  $(f^h)^i = f$ .