Exercise 1

See official solution.

Exercise 2

(1) By assumption, the inverse A^{-1} exists. Consider the automorphism of $K[X_0, \ldots, X_3]$ induced by $X \mapsto A^{-1}X$, $X = (X_0, X_1, X_2, X_3)^t$. This induces an automorphism of $\mathbb{P}^3$ that maps $V^p(f)$ to $V^p(X_0^2 + \ldots + X_3^2)$.

(2) Check that $X_0 \mapsto X_0 + iX_1$, $X_1 \mapsto X_0 - iX_1$, $X_2 \mapsto X_2 + iX_3$, $X_3 \mapsto X_2 - iX_3$ works.

(3) By (1) and (2), it suffices to check $V^p(X_0X_1 - X_2X_3) \cong \mathbb{P}^1 \times \mathbb{P}^1$. But this is exactly the image of the Segre embedding, so the result follows by definition/from the lecture.

Exercise 3

φ is given as $\varphi = (f_1, \ldots, f_n)$ for polynomials $f_i \in K[x]$. Let d be the largest degree of the f_i, and consider
$$\widetilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^n, \quad [x : y] \mapsto [y^d, y^d f_1(\tfrac{x}{y}) : \ldots : y^d f_n(\tfrac{x}{y})].$$
We have $\widetilde{\varphi}([x : 1]) = [1 : \varphi(x)]$ and $\widetilde{\varphi}([1 : 0])$ does not vanish, since if, say, f_1 has degree d, then $(y^d f_1(\tfrac{x}{y}))(1,0)$ equals the leading coefficient of f_1. So $\widetilde{\varphi}$ is well-defined. By the lecture, $\widetilde{\varphi}$ is closed, in particular, $\varphi(\mathbb{A}^1) = \widetilde{\varphi}(U_1)$ is closed in $\mathbb{P}^n$, thus also in $\mathbb{A}^n$.

Exercise 4

(1) $\operatorname{im}\varphi$ contains $D(x)$, which is nonempty open, hence dense. Hence if $\operatorname{im}\varphi$ were closed, we'd have $\operatorname{im}\varphi = \mathbb{A}^2$. But $(0,1) \notin \operatorname{im}\varphi$.

(2) One constructs the "map" $[x : y : z] \mapsto [z^2 : xz : xy]$, which is not well-defined on $[1 : 0 : 0]$.