Algebraic Geometry I

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Roughly speaking, the goal of algebraic geometry is to study systems of polynomial equations $F_1(X) = \ldots = F_n(X) = 0$ for polynomials $F_i \in K[X_1, \ldots, X_m]$ over a field (or ring) K. The set of solutions of this system is a geometric object, which we try to understand using algebraic methods, for example considering the ideal $I = (F_1, \ldots, F_n)$ in $K[X_1, \ldots, X_m]$ or the quotient $K[X_1, \ldots, X_m]/I$. Lecture 1 Oct 13, 2025

There is a very strong relation between these objects in the case that $K = \overline{K}$ is algebraically closed (e.g. $\mathbb{C}$). If K is not algebraically closed, or some generic ring, things get more complicated: For example, there are many equations over $\mathbb{R}$ with no solutions, like $x^2+y^2+1 = 0$, which behave differently when considered over $\mathbb{C}$. The wish to still study these equations geometrically leads to the idea of spectra (the set of all prime ideals of a ring), and later the theory of sheaves and schemes.

1 Algebraic Sets and Affine Varieties

Let K be an algebraically closed field.

Definition 1.1. For $n \in \mathbb{N}$ define *affine n-space* over K as

$$\mathbb{A}^n := \mathbb{A}^n_K := K^n.$$

Definition 1.2. Let $I \subset K[x_1, \ldots, x_n]$ be a subset. The associated *(affine) algebraic set* is

$$V(I) := \{x \in \mathbb{A}^n_K \mid f(x) = 0 \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n$ is called *algebraic* if $X = V(I)$ for some $I \subset K[x_1, \ldots, x_n]$.

Remark 1.3. By definition $V(I) = V(\langle I \rangle) = V(f_1, \ldots, f_m)$ where $\langle I \rangle = (f_1, \ldots, f_m)$ is finitely generated because $K[x_1, \ldots, x_n]$ is Noetherian. Therefore, $X \subseteq \mathbb{A}^n$ is algebraic if and only if $X = V(I)$ for some ideal I if and only if $X = V(f_1, \ldots, f_m)$ for a finite number of polynomials f_i.

Example 1.4. The following sets are algebraic:

- A parabola $\{(x, x^2) \mid x \in K\} = V(y - x^2)$
- $\emptyset = V(K[x_1, \ldots, x_n])$
- $\mathbb{A}^n = V(0)$
- Points: $\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$

Lemma 1.5. *Let $I, J \lhd K[x_1, \ldots, x_n]$ be ideals. Then*

(a) If $I \subseteq J$, then $V(I) \supseteq V(J)$.

(b) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$

(c) For any family $(I_t)_{t \in T}$ of ideals, $\bigcap_t V(I_t) = V(\bigcup_t I_t) = V(\sum_t I_t)$

Proof. (a) is clear.

For (b), part (a) yields $V(I \cap J) \subseteq V(IJ)$ and $V(I), V(J) \subseteq V(I \cap J)$, so it remains to show $V(IJ) \subseteq V(I) \cup V(J)$. Let $a \in V(IJ)$. Assume $a \notin V(I)$, i.e. there is $f \in I$ such that $f(a) \neq 0$. Let $g \in J$. Then $fg \in IJ$, so $0 = (fg)(a) = f(a)g(a)$. Since $f(a) \neq 0$, we conclude $g(a) = 0$.

The first equation of (c) is tautological, the second one is remark 1.3, □

Definition 1.6. The *Zariski topology* on $\mathbb{A}^n$ is the topology whose closed subsets are exactly the algebraic sets. That is, $U \subseteq \mathbb{A}^n$ is open iff its complement is algebraic.

Remark 1.7. This is indeed a topology by example 1.4 and lemma 1.5. Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set $X \subseteq \mathbb{A}^n$, which is also called the Zariski topology.

Recall from general topology that a topological space $X \neq \emptyset$ is called irreducible if $X \neq X_1 \cup X_2$ with $X_i \subsetneq X$ closed. $\emptyset$ is not considered irreducible.

For example, $V(xy) = V(x) \cup V(y)$ (the union of the coordinate axes in $\mathbb{A}^2$) is not irreducible, while a parabola $V(y - x^2)$ is irreducible (we will see how to check this later).

Definition 1.8. An *affine algebraic variety* is an irreducible closed subset of $\mathbb{A}^n$.

Definition 1.9. Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. We define the *vanishing ideal* of X as

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

Lemma 1.10. *Let $X \subseteq \mathbb{A}^n$ and $S \subseteq K[x_1, \dots, x_n]$. Then*

(a) $X \subseteq V(I(X))$ and $S \subseteq I(V(S))$.

(b) $V(I(X)) = \overline{X}$ is the closure of X (w.r.t. the Zariski topology).

Proof. (a) is clear, (b) is left as an exercise. □

Proposition 1.11. *An affine algebraic set $X \subseteq \mathbb{A}^n$ is a variety if and only if $I(X)$ is a prime ideal.*

Proof. Let X be a variety and let $fg \in I(X)$ for $f, g \in K[x_1, \dots, x_n]$. We have $X \subseteq V(fg) \overset{1.5}{=} V(f) \cup V(g)$. Hence we can write $X = (X \cap V(f)) \cup (X \cap V(g))$ as the union of two closed subsets. By irreducibility, wlog we have $X = X \cap V(f)$, i.e. $X \subseteq V(f)$, which is equivalent to $f \in I(X)$.

Conversely, suppose that $X = A \cup B$ is not irreducible. Choose points $a \in A \setminus B$ and $b \in B \setminus A$. By Lemma 1.10 and since A, B are closed, we get $V(I(A)) = A$ and $V(I(B)) = B$. Hence there exist $f \in I(A)$ and $g \in I(B)$ with $f(b) \neq 0$ and $g(a) \neq 0$. Thus $fg \in I(X)$, but both $f, g \notin I(X)$ □

Remark 1.12. If $X = V(I)$ is an affine variety, this does not necessarily imply that I is prime: Consider $V((x^2)) \subseteq \mathbb{A}^1$: $V((x^2)) = \{0\}$ is irreducible, but (x^2) is not prime. Lecture 2 Oct 15, 2025

Note that $\mathbb{A}^n$ is irreducible since K is infinite. However, this is no longer true if one considers finite fields, since then $\mathbb{A}^n$ is the union of its finitely many points. For example, $I(A^1_{\mathbb{F}_p}) = (X^p - X)$ is not prime.

We use the following result from commutative algebra without proof:

Theorem 1.13 (Hilbert Nullstellensatz). *Let $J \lhd K[x_1, \dots, x_n]$. Then*

(a) $V(J) = \emptyset$ if and only if $J = K[x_1, \dots, x_n]$.

(b) $I(V(J)) = \sqrt{J} = \{f \in K[x_1, \dots, x_n] \mid f^n \in J$ for some $n\}$

(c) If J is a maximal ideal, then $J = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in K$.

Corollary 1.14. *There are inclusion-reversing bijections*

$$\{\textit{affine algebraic sets } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{radical ideals in } K[x_1, \ldots, x_n]\}$$

$$\{\textit{affine algebraic varieties } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{prime ideals in } K[x_1, \ldots, x_n]\}$$

$$\{\textit{points } a \in \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{maximal ideals in } K[x_1, \ldots, x_n]\}$$

Proof. Clear from 1.13, 1.10 and 1.11. □

Example 1.15. Let f be irreducible in $K[x_1, \ldots, x_n]$. Then $V(f)$ is an affine variety. Varieties of this form are called hypersurfaces in $\mathbb{A}^n$ (curves for $n = 2$, surfaces for $n = 3$).

Remark 1.16. If $X \subseteq \mathbb{A}^n$ is a variety, by proposition 1.11 $I(X)$ is prime, and $K[x_1, \ldots, x_n]/I$ is an integral domain. We can consider its fraction field $\mathrm{Frac}(K[x_1, \ldots, x_n]/I)$.

Theorem 1.17. *Any affine algebraic set can be uniquely written as a finite union of affine varieties.*

For the proof, we need some preparations.

Definition 1.18. A topological space X is called *Noetherian* if any chain of descending closed subsets $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$ becomes stationary, i.e. there exists n s.t. $X_m = X_n$ for all $m > n$.

Lemma 1.19. *Affine space $\mathbb{A}^n$ is Noetherian.*

Proof. Let $\mathbb{A}^n \supseteq X_1 \supseteq X_2 \supseteq \ldots$ be a chain of closed subsets. Applying $I(-)$ yields an ascending chain $(0) \subseteq I(X_1) \subseteq I(X_2) \subseteq \ldots$ of ideals in $K[x_1, \ldots, x_n]$. This is a Noetherian ring, so there is some m such that $I(X_n) = I(X_{n+1})$ for all $n \geq m$. By corollary 1.14(a), I is injective on closed subsets, so we are done. □

More generally,

Corollary 1.20. *Any affine algebraic space $X \subseteq \mathbb{A}^n$ is Noetherian.*

Proof. Any chain in X is also a chain in $\mathbb{A}^n$. □

Proposition 1.21. *Let $X \neq \emptyset$ be a Noetherian topological space.*

(a) Then X can be written as a finite union of irreducible closed subspaces.

(b) Moreover, if we assume that $X_i \not\subseteq X_j$ for $i \neq j$, then the above decomposition is unique up to permutation. In this case, the X_i are called irreducible components of X.

Proof. Assume that (a) fails for X. Consider $S = \{Y \subseteq X \mid Y$ closed, cannot be written as a finite union of irreducible closed subsets$\}$. Since X is Noetherian, S must have some minimal element Y w.r.t. inclusion. Y is not irreducible, so we can write $Y = Y_1 \cup Y_2$ with $Y_{1,2}$ proper closed subspaces. By minimality, Y_1 and Y_2 can be written as finite unions of irreducible closed subsets, thus so can Y, contradicting $Y \in S$.

To check uniqueness, assume we have two decompositions $X = X_1 \cup \ldots X_r = X_1' \cup \ldots X_s'$ as in (b). Then $X_1' = \bigcup_i (X_i \cap X_1')$. Since X_1' is irreducible, wlog $X_1' \subseteq X_1$. By the same argument, $X_1 \subseteq X_i'$ for some i. If $i \neq 1$, then $X_1' \subseteq X_i'$, contradicting our assumption. Hence $i = 1$ and $X_1 = X_1'$. Proceed inductively with $\overline{X \setminus X_1} = X_2 \cup \ldots \cup X_r = X_2' \cup \ldots \cup X_s'$. □

Combining 1.20 and 1.21 yields theorem 1.17.

Remark 1.22. The proof strategy for (a) can be summarized as follows: Let X be a Noetherian space and P a property of closed subsets. To show that P holds for all subsets of X (thus in particular for X), it suffices to show that for all $Y \subseteq X$ closed, if P holds for all proper closed subsets of Y, then it also holds for Y. This is called *Noether induction* (a special case of well-founded induction).

Example 1.23. Let $f \in K[x_1, \ldots, x_n]$. This is a factorial ring, so we may write $f = g_1^{k_1} \cdots g_r^{k_r}$ with g_i irreducible and pairwise different. Then Lecture 3 Oct 22 , 2025

$$V(f) = V(g_1^{k_1}) \cup \cdots V(g_r^{k_r}) = V(g_1) \cup \cdots \cup V(g_r)$$

is the decomposition of $V(f)$ into irreducible subsets: $V(g_i)$ is irreducible by proposition 1.11, since $I(V(g_i)) = (g_i)$ is prime.

In general, finding this composition for $V(f_1, \ldots, f_r)$ is not easy.

Example 1.24. What is the Zariski topology on $\mathbb{A}^1$? By definition, a closed/algebraic set is of the form $V(I)$ for some ideal $I \subseteq K[x]$. Since $K[x]$ is a PID, $I = (f)$ for some $f = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \in K[x]$. If f is not constant, we see as in example 1.23 that

$$X = V(f) = \bigcup_i V(x - a_i) = \{a_1, \ldots, a_r\}.$$

Hence the closed sets are exactly $V(0) = \mathbb{A}^1, V(1) = \emptyset$, and finite unions of points. In other words, the Zariski topology coincides with the cofinite topology on $\mathbb{A}^1$. The affine varieties on $\mathbb{A}^1$ are therefore either $\mathbb{A}^1$ itself or a single point.

We also see that any two non-empty open subsets have nontrivial intersection, so $\mathbb{A}^1$ with the Zariski topology is not Hausdorff.

Definition 1.25. Let X be a nonempty topological space. We define the dimension of X as the supremum of all $n \in \mathbb{N}$ such that there is a chain of irreducible subspaces $\emptyset \neq Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n \subseteq X$

Example 1.26. By example 1.24, a maximal chain of affine varieties in $\mathbb{A}^1$ is $\{0\} \subsetneq \mathbb{A}^1$, hence $\dim \mathbb{A}^1 = 1$.

Definition 1.27. Let R be a (commutative) ring. The Krull dimension of R is the supremum over all l such that there is a chain of prime ideals $\mathfrak{p}_l \subsetneq \mathfrak{p}_{l-1} \subsetneq \ldots \subsetneq \mathfrak{p}_0 \subsetneq R$.

Recall from corollary 1.14 that there is an inclusion-reversing correspondence between prime ideals of $K[x_1, \ldots, x_n]$ and affine algebraic varieties in $\mathbb{A}^n$. Fixing some variety X, it follows that subvarieties correspond bijectively to prime ideals that contain $I(X)$, i.e. prime ideals of $K[x_1, \ldots, x_n]/I(X)$. Hence

Proposition 1.28. *If X is an affine algebraic variety, then* $\dim X = \dim K[x_1, \ldots, x_n]/I(X)$.

2 Morphisms of Affine Varieties

2.1 Regular Morphisms

Definition 2.1. Let $X \subseteq \mathbb{A}^n_K$ be an algebraic set. A function $f : X \to K$ is *regular* if there is a polynomial $F \in K[x_1, \dots, x_n]$ such that $f : F|_X$, i.e. $f(x) = F(x)$ for all $x \in X$. Write $A(X)$ for the set of regular functions on X.

Remark 2.2. $A(X)$ is a ring (and even a K-algebra) in a natural way, with addition and multiplication defined pointwise. Moreover, there is a homomorphism of K-algebras

$$K[x_1, \dots, x_n] \twoheadrightarrow A(X), \qquad F \mapsto F|_X.$$

The kernel of this morphism is exactly $I(X)$, so that $A(X) \cong K[x_1, \dots, x_n]/I(X)$ canonically.

Remark 2.3. By corollary 1.14, $A(X)$ is always reduced, $A(X)$ is integral iff X is a variety, and $A(X)$ is a field iff X is a point (in which case $A(X) \cong K$).

Definition 2.4. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A map $\varphi : X \to Y$ is called *regular* if $\varphi = (f_1, \dots, f_m)$ for some regular $f_1, \dots, f_m \in A(X)$. A regular map φ is an isomorphism if it has in anverse which is also regular.

Example 2.5.
(i) $f : \mathbb{A}^1 \to V(y - x^2) \subseteq \mathbb{A}^2$, $t \mapsto (t, t^2)$ is a regular map. It has inverse $(x, y) \mapsto x$, which is also regular, hence $\mathbb{A}^1 \cong V(y - x^2)$.

(ii) $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subseteq \mathbb{A}^2$, $t \mapsto (t^2, t^3)$ is regular and bijective as well, but its inverse $(x, y) \mapsto \frac{y}{x}$ is not regular, so φ is not an isomorphism.

Proposition 2.6. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets, and let $\varphi : X \to Y$ be a regular map. Then φ is continuous (w.r.t. the Zariski topology on X and Y).*

Proof. Let $\varphi = (f_1, \dots, f_m)$ and $J = \langle F_1, \dots, F_k \rangle \subseteq K[x_1, \dots, x_m]$ with $V(J) \subseteq Y$. Then

$$\varphi^{-1}(V(J)) = \varphi^{-1}(V(F_1, \dots, F_k)) = \{x \in X \mid F_j(f_1(x), \dots, f_m(x)) = 0,\ j = 1, \dots, k\}$$

Now $F_j(f_1(x), \dots, f_m(x))$ is a composition of polynomials, hence a polynomial, call it $\widetilde{F_j}$. We conlcude $\varphi^{-1}(V(J)) = X \cap V(\widetilde{F_1}, \dots, \widetilde{F_k})$ as desired. □

Remark 2.7. The converse is false. For example, one easily concludes from example 1.24 that every bijective map $\mathbb{A}^1 \to \mathbb{A}^1$ is continuous, but there are way more bijections than polynomials (say because polynomials are defined by their values on any infinite subset). On the other hand, if K is finite (loosing algebraic closedness), then every function $\mathbb{A}^1 \to \mathbb{A}^1$ is regular.

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Remark 2.8. Let X be an algebraic set, and let $f : X \to \mathbb{A}^1$. Then f is a regular map if and only if f is a regular function. Note that the composition of regular maps is regular, since compositions of polynomials are polynomials.

Definition 2.9. Let X, Y be algebraic sets and $F : X \to Y$ be regular. Then we set $F^* : A(Y) \to A(X)$, $g \mapsto g \circ F$. This is well-defined by remark 2.8, and F^* clearly preserves addition and multiplication, so it is a morphism of K-algebras.

Remark 2.10. Let $F = (f_1, \dots, f_m) : X \to Y$, $f_i \in K[x_1, \dots, x_n]$, then F^* is given by the K-algebra homomorphism $A(Y) \cong K[y_1, \dots, y_m]/I(Y) \to K[x_1, \dots, x_n]/I(X) \cong A(X)$ (see remark 2.2) defined by $y_i \mapsto f_i$. Hence $F(x) = (F_1^*(y_1), \dots, F_m^*(y_m))$.

Theorem 2.11. *(i) There is a bijection* $\operatorname{Mor}(X,Y) \to \operatorname{Hom}_{K\text{-}Alg}(A(Y),A(X))$ *given by* $F \mapsto F^*$.

(ii) If $F : X \to Y$ *and* $H : Y \to Z$ *are regular, then* $(H \circ F)^* = F^* \circ H^*$. *Further,* $\mathrm{id}_X^* = \mathrm{id}_{A(X)}$.

(iii) Let $F : X \to Y$ *be regular. Then* F *is an isomorphism of affine sets if and only if* F^* *is an isomorphism of* K*-algebras*

Proof. Injectivity in (i) follows from remark 2.10. For surjectivity, let $\varphi : A(Y) \to A(X)$ be a K-algebra homomorphism and define $F : X \to Y$ by $F = (\varphi(y_1),\dots,\varphi(y_m))$. We need to check that this is well-defined, i.e. that the image of F lies in Y. Then it is clear that F is regular and that $F^* = \varphi$, again by remark 2.10.

So let $g \in I(Y)$, we need to show $g \circ F = 0$. But this is exactly the statement $\varphi([g]) = \varphi(0) = 0$.

For (ii), $\mathrm{id}_X^* = \mathrm{id}_{A(X)}$ is clear, and for $f \in A(Z)$ one has

$$(H \circ F)^*(f) = f \circ H \circ F = H^*(f) \circ F = (F^* \circ H^*)(f),$$

so $(H \circ F)^* = F^* \circ H^*$. Then (iii) follows from (i) and (ii). □

Example 2.12. Looking again at the maps from example 2.5, we see that $f : \mathbb{A}^1 \to V(y-x^2), t \mapsto (t,t^2)$ is an isomorphism, because $f^* : K[x,y]/(y-x^2) \to K[t]$, $x \mapsto t, y \mapsto t^2$ clearly is. On the other hand, let $\varphi : \mathbb{A}^1 \to V(y^2-x^3)$, $t \mapsto (t^2,t^3)$. We saw that this is a bijective regular map and gave intuitive reasoning for why this map isn't an isomorphism. But now we can prove it: We have

$$f^* : K[x,y]/(y^2-x^3) \to K[t], \quad x \mapsto t^2, y \mapsto t^3$$

is not surjective, for the image does not contain t.

Remark 2.13. In categorical terms, theorem 2.11 says that

$$\begin{aligned} \left\{\begin{matrix}\text{algebraic sets}\\ \text{regular maps}\end{matrix}\right\} &\to \left\{\begin{matrix}\text{finitely generated reduced } K\text{-algebras}\\ K\text{-algebra homomorphisms}\end{matrix}\right\} \\ X &\mapsto A(X) \\ F &\mapsto F^* \end{aligned}$$

is a contravariant functor, and even an equivalence of categories: For essential surjectivity, note that every finitely generated K-algebra can be written as a quotient $K[x_1,\dots,x_n]/I$ by choosing generators. Then consider $X = V(I)$.

Proposition 2.14. *Let* X, Y *be algebraic sets, and let* $f : X \to Y$ *be a regular map. Then*

(i) $f^* : A(Y) \to A(X)$ *is surjective if and only if* $\overline{f(X)} = Y$*, i.e. if the image of* f *is dense in* Y.

(ii) f^* *is injective if and only if* $f(X) \subseteq Y$ *is closed and* $f : X \to f(X)$ *is an isomorphism.*

Proof. Exercise. □

2.2 Rational Maps of Varieties

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety. Then $I(X)$ is prime, so $A(X) \cong K[x_1, \ldots, x_n]/I(X)$ is an integral domain. Hence we can define its field of fractions $K(X) := \operatorname{Frac} A(X)$.

Definition 2.15. An element $\varphi \in K(X)$ is called regular at $x \in X$ if there exist $f, g \in A(X)$ with $\varphi = \frac{f}{g}$ and $g(x) \neq 0$.

Example 2.16. Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$ and $x = (0, 0, 1)$. Consider $\varphi = \frac{y}{x} \in K(X)$. Even though it may look like φ might not be regular at x, one can note that $\frac{y}{x} = \frac{x}{z}$ in $K(X)$, so actually $\varphi(x)$ can be defined and φ is regular at x.