Exercise 1

(a)⇒(b): Suppose $\varphi : X \to Y$ is a morphism, let $p \in X$. Wlog we may assume $\varphi(p) \notin U_0$. Then $\frac{x_i}{x_0}$ are regular functions on U_0, so by definition $\varphi^*(\frac{x_i}{x_0})$ are regular functions on $U := \varphi^{-1}(U_0)$ (which is a neighbourhood of p). For $x \in U$ we have

$$\varphi = [\varphi^*(x_0) : \ldots : \varphi^*(x_n)] = [1 : \varphi^*(\tfrac{x_1}{x_0}) : \ldots : \varphi^*(\tfrac{x_m}{x_0})].$$

By possibly shrinking U, we may write $\varphi^*(\frac{x_i}{x_0}) = \frac{f_i}{g_i}$ with coprime polynomials $f_i, g_i \in K[x_1, \ldots, x_n]$. Let h be a least common multiple of the g_i, Then we get

$$\varphi(x) = [h : f_1 \frac{h}{g_1} : \ldots : f_m \frac{h}{g_m}]$$

on $U \setminus V(h)$, which is still an open neighbourhood of p since $g_i(p) \neq 0$ for all i.

(b)⇒(a): Since being a morphism is a local property (3.30), we may assume $\varphi = [G_0 : \ldots : G_m]$ globally. Then φ is continuous, and if $\frac{f}{g} : Y \subseteq V \to \mathbb{A}^1$ is a regular function (that is, f, g are homogeneous polynomials of the same degree), then $\varphi^*(\frac{f}{g}) = \frac{f(G_0,\ldots,G_m)}{g(G_0,\ldots,G_m)}$ is clearly regular on $\varphi^{-1}(V)$, as desired.

Exercise 2

(1) Let $v' = av + bw$ and $w' = cv + dw$ be another basis of V. Then $(v', w') = (v, w)T$, where $T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}_2 K$. Hence all 2×2 minors of (v', w') differ from the corresponding minor of (v, w) by $\det(T) \in K^\times$. Thus they define the same point in $\mathbb{P}^5$.

Also, not all minors can vanish, since then v, w would be linearly dependent, hence not span a 2-dimensional subspace.

(2) Let $V \in W_{12}$, say generated by v, w. Let $(v', w') = (v, w)\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}^{-1}$ (where the last matrix exists by assumption). Then $v' = (1, 0, v_3, v_4)^t$ and $w' = (0, 1, w_3, w_4)^t$ are still a basis of V as in (1), and $W_{12} \to \mathbb{A}^4$, $V \mapsto (v_3, v_4, w_3, w_4)$ as constructed above is a bijection. Indeed, the map is clearly injective, and surjective since any vectors of the form v', w' are linearly independent

Furthermore, we can explicitly write down

$$\mathbb{A}^4 \cong W_{12} \xrightarrow{\varphi} U_{12}, \quad (v_3, v_4, w_3, w_4) \mapsto V \mapsto [1 : w_3 : w_4 : -v_3 : -v_4 : v_3 w_4 - w_3 v_4].$$

Since this compositum is clearly injective, so is $\varphi|_{W_{12}}$.

(3) Say $\varphi(V) = \varphi(W)$. Then some coordinate of this point must be nonzero, wlog the first one. But then $V, W \in \varphi^{-1}(U_{12}) = W_{12}$, so by (2) $V = W$. Hence φ is injective.

We claim $\operatorname{im} \varphi = V^p(X_0 X_5 + X_2 X_3 - X_1 X_4) =: Y$. Clearly $\varphi|_{W_{12}}$ maps into Y, and has image $Y \cap U_{12}$. Symmetrically, $\operatorname{im} \varphi|_{W_{ij}} = Y \cap U_{ij}$, and the claim follows.

(4) Let $X = X_{\langle e_1 \rangle}$ and ψ an automorphism of K^4 mapping some nonzero vector v of L to e_1. Then ψ induces a bijection $X_L \to X$, and ψ, ψ^{-1} act by linear maps, hence continuously. Thus X_L and X are isomorphic, and it suffices to show that X is closed. But

$$\varphi(\langle e_1, (a, b, c, d) \rangle) = [b : c : d : 0 : 0 : 0],$$

so $\varphi(X) = V^p(X_3, X_4, X_5) \subseteq Y$ is clearly closed.

(5) As in (4), $\varphi(X_{\langle e_1 \rangle}) = \{[b : c : d : 0 : 0 : 0]\}$, so the space is parametrized by nonzero points on the plane orthogonal to e_1, modulo linear subspaces. The same holds true for any X_L.

(6) As in (4), it suffices to do this for $X = X_{\langle e_1 \rangle}$, where $\varphi(X) = V^p(X_3, X_4, X_5)$, so that $\varphi(X) \to \mathbb{P}^2$, $X \mapsto [X_0 : X_1 : X_2]$ is clearly an isomorphism.