Algebraic Geometry I

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Contents

Roughly speaking, the goal of algebraic geometry is to study systems of polynomial equations $F_1(X) = \ldots = F_n(X) = 0$ for polynomials $F_i \in K[X_1, \ldots, X_m]$ over a field (or ring) K. The set of solutions of this system is a geometric object, which we try to understand using algebraic methods, for example considering the ideal $I = (F_1, \ldots, F_n)$ in $K[X_1, \ldots, X_m]$ or the quotient $K[X_1, \ldots, X_m]/I$. Lecture 1 Oct 13, 2025

There is a very strong relation between these objects in the case that $K = \overline{K}$ is algebraically closed (e.g. $\mathbb{C}$). If K is not algebraically closed, or some generic ring, things get more complicated: For example, there are many equations over $\mathbb{R}$ with no solutions, like $x^2+y^2+1 = 0$, which behave differently when considered over $\mathbb{C}$. The wish to still study these equations geometrically leads to the idea of spectra (the set of all prime ideals of a ring), and later the theory of sheaves and schemes.

1 Algebraic Sets and Affine Varieties

Let K be an algebraically closed field.

Definition 1.1. For $n \in \mathbb{N}$ define *affine n-space* over K as

$$\mathbb{A}^n := \mathbb{A}^n_K := K^n.$$

Definition 1.2. Let $I \subset K[x_1, \ldots, x_n]$ be a subset. The associated *(affine) algebraic set* is

$$V(I) := \{x \in \mathbb{A}^n_K \mid f(x) = 0 \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n$ is called *algebraic* if $X = V(I)$ for some $I \subset K[x_1, \ldots, x_n]$.

Remark 1.3. By definition $V(I) = V(\langle I \rangle) = V(f_1, \ldots, f_m)$ where $\langle I \rangle = (f_1, \ldots, f_m)$ is finitely generated because $K[x_1, \ldots, x_n]$ is Noetherian. Therefore, $X \subseteq \mathbb{A}^n$ is algebraic if and only if $X = V(I)$ for some ideal I if and only if $X = V(f_1, \ldots, f_m)$ for a finite number of polynomials f_i.

Example 1.4. The following sets are algebraic:

- A parabola $\{(x, x^2) \mid x \in K\} = V(y - x^2)$
- $\emptyset = V(K[x_1, \ldots, x_n])$
- $\mathbb{A}^n = V(0)$
- Points: $\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$

Lemma 1.5. *Let $I, J \triangleleft K[x_1, \ldots, x_n]$ be ideals. Then*

(a) If $I \subseteq J$, then $V(I) \supseteq V(J)$.

(b) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$

(c) For any family $(I_t)_{t \in T}$ of ideals, $\bigcap_t V(I_t) = V(\bigcup_t I_t) = V(\sum_t I_t)$

Proof. (a) is clear.

For (b), part (a) yields $V(I \cap J) \subseteq V(IJ)$ and $V(I), V(J) \subseteq V(I \cap J)$, so it remains to show $V(IJ) \subseteq V(I) \cup V(J)$. Let $a \in V(IJ)$. Assume $a \notin V(I)$, i.e. there is $f \in I$ such that $f(a) \neq 0$. Let $g \in J$. Then $fg \in IJ$, so $0 = (fg)(a) = f(a)g(a)$. Since $f(a) \neq 0$, we conclude $g(a) = 0$.

The first equation of (c) is tautological, the second one is remark 1.3, □

Definition 1.6. The *Zariski topology* on $\mathbb{A}^n$ is the topology whose closed subsets are exactly the algebraic sets. That is, $U \subseteq \mathbb{A}^n$ is open iff its complement is algebraic.

Remark 1.7. This is indeed a topology by example 1.4 and lemma 1.5. Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set $X \subseteq \mathbb{A}^n$, which is also called the Zariski topology.

Recall from general topology that a topological space $X \neq \emptyset$ is called irreducible if $X \neq X_1 \cup X_2$ with $X_i \subsetneq X$ closed. $\emptyset$ is not considered irreducible.

For example, $V(xy) = V(x) \cup V(y)$ (the union of the coordinate axes in $\mathbb{A}^2$) is not irreducible, while a parabola $V(y - x^2)$ is irreducible (we will see how to check this later).

Definition 1.8. An *affine algebraic variety* is an irreducible closed subset of $\mathbb{A}^n$.

Definition 1.9. Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. We define the *vanishing ideal* of X as

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

Lemma 1.10. *Let $X \subseteq \mathbb{A}^n$ and $S \subseteq K[x_1, \dots, x_n]$. Then*

(a) $X \subseteq V(I(X))$ and $S \subseteq I(V(S))$.

(b) $V(I(X)) = \overline{X}$ is the closure of X (w.r.t. the Zariski topology).

Proof. (a) is clear, (b) is left as an exercise. □

Proposition 1.11. *An affine algebraic set $X \subseteq \mathbb{A}^n$ is a variety if and only if $I(X)$ is a prime ideal.*

Proof. Let X be a variety and let $fg \in I(X)$ for $f, g \in K[x_1, \dots, x_n]$. We have $X \subseteq V(fg) \overset{1.5}{=} V(f) \cup V(g)$. Hence we can write $X = (X \cap V(f)) \cup (X \cap V(g))$ as the union of two closed subsets. By irreducibility, wlog we have $X = X \cap V(f)$, i.e. $X \subseteq V(f)$, which is equivalent to $f \in I(X)$.

Conversely, suppose that $X = A \cup B$ is not irreducible. Choose points $a \in A \setminus B$ and $b \in B \setminus A$. By Lemma 1.10 and since A, B are closed, we get $V(I(A)) = A$ and $V(I(B)) = B$. Hence there exist $f \in I(A)$ and $g \in I(B)$ with $f(b) \neq 0$ and $g(a) \neq 0$. Thus $fg \in I(X)$, but both $f, g \notin I(X)$ □

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Remark 1.12. If $X = V(I)$ is an affine variety, this does not necessarily imply that I is prime: Consider $V((x^2)) \subseteq \mathbb{A}^1$: $V((x^2)) = \{0\}$ is irreducible, but (x^2) is not prime.

Note that $\mathbb{A}^n$ is irreducible since K is infinite. However, this is no longer true if one considers finite fields, since then $\mathbb{A}^n$ is the union of its finitely many points. For example, $I(A^1_{\mathbb{F}_p}) = (X^p - X)$ is not prime.

We use the following result from commutative algebra without proof:

Theorem 1.13 (Hilbert Nullstellensatz). *Let $J \triangleleft K[x_1, \dots, x_n]$. Then*

(a) $V(J) = \emptyset$ if and only if $J = K[x_1, \dots, x_n]$.

(b) $I(V(J)) = \sqrt{J} = \{f \in K[x_1, \dots, x_n] \mid f^n \in J \text{ for some } n\}$

(c) If J is a maximal ideal, then $J = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in K$.

Corollary 1.14. *There are inclusion-reversing bijections*

$$\{\textit{affine algebraic sets } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{radical ideals in } K[x_1, \dots, x_n]\}$$

$$\{\textit{affine algebraic varieties } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{prime ideals in } K[x_1, \dots, x_n]\}$$

$$\{\textit{points } a \in \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{maximal ideals in } K[x_1, \dots, x_n]\}$$

Proof. Clear from 1.13, 1.10 and 1.11. □

Example 1.15. Let f be irreducible in $K[x_1, \dots, x_n]$. Then $V(f)$ is an affine variety. Varieties of this form are called hypersurfaces in $\mathbb{A}^n$ (curves for $n = 2$, surfaces for $n = 3$).

Remark 1.16. If $X \subseteq \mathbb{A}^n$ is a variety, by proposition 1.11 $I(X)$ is prime, and $K[x_1, \dots, x_n]/I$ is an integral domain. We can consider its fraction field $\mathrm{Frac}(K[x_1, \dots, x_n]/I)$.

Theorem 1.17. *Any affine algebraic set can be uniquely written as a finite union of affine varieties.*

For the proof, we need some preparations.

Definition 1.18. A topological space X is called *Noetherian* if any chain of descending closed subsets $X \supseteq X_1 \supseteq X_2 \supseteq \dots$ becomes stationary, i.e. there exists n s.t. $X_m = X_n$ for all $m > n$.

Lemma 1.19. *Affine space $\mathbb{A}^n$ is Noetherian.*

Proof. Let $\mathbb{A}^n \supseteq X_1 \supseteq X_2 \supseteq \dots$ be a chain of closed subsets. Applying $I(-)$ yields an ascending chain $(0) \subseteq I(X_1) \subseteq I(X_2) \subseteq \dots$ of ideals in $K[x_1, \dots, x_n]$. This is a Noetherian ring, so there is some m such that $I(X_n) = I(X_{n+1})$ for all $n \geq m$. By corollary 1.14(a), I is injective on closed subsets, so we are done. □

More generally,

Corollary 1.20. *Any affine algebraic space $X \subseteq \mathbb{A}^n$ is Noetherian.*

Proof. Any chain in X is also a chain in $\mathbb{A}^n$. □

Proposition 1.21. *Let $X \neq \emptyset$ be a Noetherian topological space.*

(a) Then X can be written as a finite union of irreducible closed subspaces.

(b) Moreover, if we assume that $X_i \not\subseteq X_j$ for $i \neq j$, then the above decomposition is unique up to permutation. In this case, the X_i are called irreducible components of X.

Proof. Assume that (a) fails for X. Consider $S = \{Y \subseteq X \mid Y$ closed, cannot be written as a finite union of irreducible closed subsets$\}$. Since X is Noetherian, S must have some minimal element Y w.r.t. inclusion. Y is not irreducible, so we can write $Y = Y_1 \cup Y_2$ with $Y_{1,2}$ proper closed subspaces. By minimality, Y_1 and Y_2 can be written as finite unions of irreducible closed subsets, thus so can Y, contradicting $Y \in S$.

To check uniqueness, assume we have two decompositions $X = X_1 \cup \dots X_r = X_1' \cup \dots X_s'$ as in (b). Then $X_1' = \bigcup_i (X_i \cap X_1')$. Since X_1' is irreducible, wlog $X_1' \subseteq X_1$. By the same argument, $X_1 \subseteq X_i'$ for some i. If $i \neq 1$, then $X_1' \subseteq X_i'$, contradicting our assumption. Hence $i = 1$ and $X_1 = X_1'$. Proceed inductively with $\overline{X \setminus X_1} = X_2 \cup \dots \cup X_r = X_2' \cup \dots \cup X_s'$. □

Combining 1.20 and 1.21 yields theorem 1.17.

Remark 1.22. The proof strategy for (a) can be summarized as follows: Let X be a Noetherian space and P a property of closed subsets. To show that P holds for all subsets of X (thus in particular for X), it suffices to show that for all $Y \subseteq X$ closed, if P holds for all proper closed subsets of Y, then it also holds for Y. This is called *Noether induction* (a special case of well-founded induction).

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Example 1.23. Let $f \in K[x_1, \dots, x_n]$. This is a factorial ring, so we may write $f = g_1^{k_1} \cdots g_r^{k_r}$ with g_i irreducible and pairwise different. Then

$$V(f) = V(g_1^{k_1}) \cup \cdots V(g_r^{k_r}) = V(g_1) \cup \cdots \cup V(g_r)$$

is the decomposition of $V(f)$ into irreducible subsets: $V(g_i)$ is irreducible by proposition 1.11, since $I(V(g_i)) = (g_i)$ is prime.

In general, finding this composition for $V(f_1, \dots, f_r)$ is not easy.

Example 1.24. What is the Zariski topology on $\mathbb{A}^1$? By definition, a closed/algebraic set is of the form $V(I)$ for some ideal $I \subseteq K[x]$. Since $K[x]$ is a PID, $I = (f)$ for some $f = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \in K[x]$. If f is not constant, we see as in example 1.23 that

$$X = V(f) = \bigcup_i V(x - a_i) = \{a_1, \dots, a_r\}.$$

Hence the closed sets are exactly $V(0) = \mathbb{A}^1, V(1) = \emptyset$, and finite unions of points. In other words, the Zariski topology coincides with the cofinite topology on $\mathbb{A}^1$. The affine varieties on $\mathbb{A}^1$ are therefore either $\mathbb{A}^1$ itself or a single point.

We also see that any two non-empty open subsets have nontrivial intersection, so $\mathbb{A}^1$ with the Zariski topology is not Hausdorff.

Definition 1.25. Let X be a nonempty topological space. We define the dimension of X as the supremum of all $n \in \mathbb{N}$ such that there is a chain of irreducible subspaces $\emptyset \neq Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n \subseteq X$

Example 1.26. By example 1.24, a maximal chain of affine varieties in $\mathbb{A}^1$ is $\{0\} \subsetneq \mathbb{A}^1$, hence $\dim \mathbb{A}^1 = 1$.

Definition 1.27. Let R be a (commutative) ring. The Krull dimension of R is the supremum over all l such that there is a chain of prime ideals $\mathfrak{p}_l \subsetneq \mathfrak{p}_{l-1} \subsetneq \ldots \subsetneq \mathfrak{p}_0 \subsetneq R$.

Recall from corollary 1.14 that there is an inclusion-reversing correspondence between prime ideals of $K[x_1, \dots, x_n]$ and affine algebraic varieties in $\mathbb{A}^n$. Fixing some variety X, it follows that subvarieties correspond bijectively to prime ideals that contain $I(X)$, i.e. prime ideals of $K[x_1, \dots, x_n]/I(X)$. Hence

Proposition 1.28. *If X is an affine algebraic variety, then* $\dim X = \dim K[x_1, \dots, x_n]/I(X)$.

2 Morphisms of Affine Varieties

2.1 Regular Morphisms

Definition 2.1. Let $X \subseteq \mathbb{A}^n_K$ be an algebraic set. A function $f : X \to K$ is *regular* if there is a polynomial $F \in K[x_1, \dots, x_n]$ such that $f : F|_X$, i.e. $f(x) = F(x)$ for all $x \in X$. Write $A(X)$ for the set of regular functions on X.

Remark 2.2. $A(X)$ is a ring (and even a K-algebra) in a natural way, with addition and multiplication defined pointwise. Moreover, there is a homomorphism of K-algebras

$$K[x_1, \dots, x_n] \twoheadrightarrow A(X), \qquad F \mapsto F|_X.$$

The kernel of this morphism is exactly $I(X)$, so that $A(X) \cong K[x_1, \dots, x_n]/I(X)$ canonically.

Remark 2.3. By corollary 1.14, $A(X)$ is always reduced, $A(X)$ is integral iff X is a variety, and $A(X)$ is a field iff X is a point (in which case $A(X) \cong K$).

Definition 2.4. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A map $\varphi : X \to Y$ is called *regular* if $\varphi = (f_1, \dots, f_m)$ for some regular $f_1, \dots, f_m \in A(X)$. A regular map φ is an isomorphism if it has in anverse which is also regular.

Example 2.5.
(i) $f : \mathbb{A}^1 \to V(y - x^2) \subseteq \mathbb{A}^2$, $t \mapsto (t, t^2)$ is a regular map. It has inverse $(x, y) \mapsto x$, which is also regular, hence $\mathbb{A}^1 \cong V(y - x^2)$.

(ii) $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subseteq \mathbb{A}^2$, $t \mapsto (t^2, t^3)$ is regular and bijective as well, but its inverse $(x, y) \mapsto \frac{y}{x}$ is not regular, so φ is not an isomorphism.

Proposition 2.6. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets, and let $\varphi : X \to Y$ be a regular map. Then φ is continuous (w.r.t. the Zariski topology on X and Y).*

Proof. Let $\varphi = (f_1, \dots, f_m)$ and $J = \langle F_1, \dots, F_k \rangle \subseteq K[x_1, \dots, x_m]$ with $V(J) \subseteq Y$. Then

$$\varphi^{-1}(V(J)) = \varphi^{-1}(V(F_1, \dots, F_k)) = \{x \in X \mid F_j(f_1(x), \dots, f_m(x)) = 0,\ j = 1, \dots, k\}$$

Now $F_j(f_1(x), \dots, f_m(x))$ is a composition of polynomials, hence a polynomial, call it $\widetilde{F}_j$. We conlcude $\varphi^{-1}(V(J)) = X \cap V(\widetilde{F}_1, \dots, \widetilde{F}_k)$ as desired. □

Remark 2.7. The converse is false. For example, one easily concludes from example 1.24 that every bijective map $\mathbb{A}^1 \to \mathbb{A}^1$ is continuous, but there are way more bijections than polynomials (say because polynomials are defined by their values on any infinite subset). On the other hand, if K is finite (loosing algebraic closedness), then every function $\mathbb{A}^1 \to \mathbb{A}^1$ is regular.

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Remark 2.8. Let X be an algebraic set, and let $f : X \to \mathbb{A}^1$. Then f is a regular map if and only if f is a regular function. Note that the composition of regular maps is regular, since compositions of polynomials are polynomials.

Definition 2.9. Let X, Y be algebraic sets and $F : X \to Y$ be regular. Then we set $F^* : A(Y) \to A(X)$, $g \mapsto g \circ F$. This is well-defined by remark 2.8, and F^* clearly preserves addition and multiplication, so it is a morphism of K-algebras.

Remark 2.10. Let $F = (f_1, \dots, f_m) : X \to Y$, $f_i \in K[x_1, \dots, x_n]$, then F^* is given by the K-algebra homomorphism $A(Y) \cong K[y_1, \dots, y_m]/I(Y) \to K[x_1, \dots, x_n]/I(X) \cong A(X)$ (see remark 2.2) defined by $y_i \mapsto f_i$. Hence $F(x) = (F_1^*(y_1), \dots, F_m^*(y_m))$.

Theorem 2.11. *(i) There is a bijection* $\mathrm{Mor}(X,Y) \to \mathrm{Hom}_{K\text{-}Alg}(A(Y), A(X))$ *given by* $F \mapsto F^*$.

(ii) If $F : X \to Y$ *and* $H : Y \to Z$ *are regular, then* $(H \circ F)^* = F^* \circ H^*$. *Further,* $\mathrm{id}_X^* = \mathrm{id}_{A(X)}$.

(iii) Let $F : X \to Y$ *be regular. Then* F *is an isomorphism of affine sets if and only if* F^* *is an isomorphism of* K*-algebras*

Proof. Injectivity in (i) follows from remark 2.10. For surjectivity, let $\varphi : A(Y) \to A(X)$ be a K-algebra homomorphism and define $F : X \to Y$ by $F = (\varphi(y_1), \ldots, \varphi(y_m))$. We need to check that this is well-defined, i.e. that the image of F lies in Y. Then it is clear that F is regular and that $F^* = \varphi$, again by remark 2.10.

So let $g \in I(Y)$, we need to show $g \circ F = 0$. But this is exactly the statement $\varphi([g]) = \varphi(0) = 0$.

For (ii), $\mathrm{id}_X^* = \mathrm{id}_{A(X)}$ is clear, and for $f \in A(Z)$ one has

$$(H \circ F)^*(f) = f \circ H \circ F = H^*(f) \circ F = (F^* \circ H^*)(f),$$

so $(H \circ F)^* = F^* \circ H^*$. Then (iii) follows from (i) and (ii). $\square$

Example 2.12. Looking again at the maps from example 2.5, we see that $f : \mathbb{A}^1 \to V(y-x^2), t \mapsto (t, t^2)$ is an isomorphism, because $f^* : K[x,y]/(y - x^2) \to K[t]$, $x \mapsto t, y \mapsto t^2$ clearly is. On the other hand, let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3)$, $t \mapsto (t^2, t^3)$. We saw that this is a bijective regular map and gave intuitive reasoning for why this map isn't an isomorphism. But now we can prove it: We have

$$f^* : K[x,y]/(y^2 - x^3) \to K[t], \quad x \mapsto t^2, y \mapsto t^3$$

is not surjective, for the image does not contain t.

Remark 2.13. In categorical terms, theorem 2.11 says that

$$\begin{aligned} \left\{\begin{matrix}\text{algebraic sets}\\ \text{regular maps}\end{matrix}\right\} &\to \left\{\begin{matrix}\text{finitely generated reduced } K\text{-algebras}\\ K\text{-algebra homomorphisms}\end{matrix}\right\} \\ X &\mapsto A(X) \\ F &\mapsto F^* \end{aligned}$$

is a contravariant functor, and even an equivalence of categories: For essential surjectivity, note that every finitely generated K-algebra can be written as a quotient $K[x_1, \ldots, x_n]/I$ by choosing generators. Then consider $X = V(I)$.

Proposition 2.14. *Let* X, Y *be algebraic sets, and let* $f : X \to Y$ *be a regular map. Then*

(i) $f^* : A(Y) \to A(X)$ *is surjective if and only if* $\overline{f(X)} = Y$, *i.e. if the image of* f *is dense in* Y.

(ii) f^* *is injective if and only if* $f(X) \subseteq Y$ *is closed and* $f : X \to f(X)$ *is an isomorphism.*

Proof. Exercise. $\square$

2.2 Rational Maps of Varieties

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety. Then $I(X)$ is prime, so $A(X) \cong K[x_1, \dots, x_n]/I(X)$ is an integral domain. Hence we can define its field of fractions $K(X) := \operatorname{Frac} A(X)$.

Definition 2.15. An element $\varphi \in K(X)$ is called regular at $x \in X$ if there exist $f, g \in A(X)$ with $\varphi = \frac{f}{g}$ and $g(x) \neq 0$.

Example 2.16. Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$ and $x = (0, 0, 1)$. Consider $\varphi = \frac{y}{x} \in K(X)$. Even though it may look like φ might not be regular at x, one can note that $\frac{y}{x} = \frac{x}{z}$ in $K(X)$, so actually $\varphi(x)$ can be defined and φ is regular at x.

Proposition 2.17. *Let $\varphi \in K(X)$. Then φ is regular at every $x \in X$ if and only if $\varphi \in A(X)$* Lecture 5 Nov 3, 2025

Remark 2.18. If $X \subseteq \mathbb{A}^n$ is an affine algebraic variety, the closed sets are exactly of the form $V_X(I) := \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$ for ideals $I \subseteq A(X)$, and V_X is still an inclusion-reversing bijection between radical ideals and closed subsets, compare exercises.

Proof. Assume $\varphi \in K(X)$ is regular at every point $x \in X$. Consider $I := \{f \in A(X) \mid f\varphi \in A(X)\}$. Then the claim is equivalent to $I = A(X)$, hence to $V_X(I) = \emptyset$ by remark 2.18. Assume there exists $x \in V_X(I)$. Since φ is regular at x, we can write $\varphi = \frac{g}{h}$ with $g, h \in A(X)$ and $h(x) \neq 0$. Hence $h \in I$, and $h(x) = 0$ by choice of x, contradiction. □

Definition 2.19. Let $X \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ be open. Denote $\mathcal{O}_X(U) := \{\varphi \in K(X) \mid \varphi \text{ regular at all } x \in U\}$. For $\varphi \in K(X)$, its domain is $\operatorname{dom}(\varphi) := \{a \in X \mid \varphi \text{ is regular at } a\}$. In other words, $\mathcal{O}_X(U) = \{\varphi \in K(X) \mid U \subseteq \operatorname{dom}(\varphi)\}$.

By proposition 2.17, $\mathcal{O}_X(X) = A(X)$.

Example.
(i) $\varphi = \frac{y}{x}$ on $X = V(y - x^2)$ is regular, since $\varphi = x$. Hence $\operatorname{dom}(\varphi) = X$.
(ii) $\varphi = \frac{y}{x}$ on $X = V(y^2 - x^3)$ has $\operatorname{dom}(\varphi) = X \setminus \{(0, 0)\}$.

Proposition 2.20. *Let $\varphi \in K(X)$. Then $\operatorname{dom}(\varphi)$ is an open non-empty set in X.*

Proof. Define $I := \{f \in A(X) \mid f\varphi \in A(X)\}$. As before, we have φ is regular at x if and only if $x \notin V_X(I)$, so $\operatorname{dom}\varphi = X \setminus V_X(I)$ is open. □

Remark 2.21. Let X be an irreducible topological space. Then

(i) Every non-empty open subset $U \subseteq X$ is dense in X.
(ii) If $U_1, U_2 \subseteq X$ are open and non-empty, then $U_1 \cap U_2 \neq \emptyset$.

Hence, if X is an affine variety and $f \in A(X)$ evaluates to zero on some non-empty open, then already $f = 0$.

Remark 2.22. Let $U \subseteq X$ be a non-empty open. Any regular $\varphi \in \mathcal{O}_X(U) \subseteq K(X)$ defines a set-theoretical function $\varphi : U \to K$, by sending $a \in U$ to $\frac{f(a)}{g(a)}$, where $\varphi = \frac{f}{g}$ with $f, g \in A(X)$ and $g(a) \neq 0$. This is well-defined, for if $\varphi = \frac{f_1}{g_1}$ with $g_1(a) \neq 0$, then $fg_1 - f_1g = 0$ in $A(X)$.

Conversely, let $\varphi : U \to K$ be a (set-theoretical) function. Then φ defines a regular function on U if for every $a \in U$ there is an open neighbourhood $a \in V \subseteq U$ such that $\varphi(b) = \frac{f(b)}{g(b)}$ for all $b \in V$, where $f, g \in K[x_1, \dots, x_n]$ and $g(b) \neq 0$ for all $b \in B$.

These assignments $(\varphi \in \mathcal{O}_X(U)) \mapsto (\varphi : U \to K)$ and $(\varphi : U \to K) \mapsto [\frac{f}{g}]$ are clearly well-defined and mutually inverse, so this is an equivalent view on regular functions on U.

One sees easily that the composition of regular maps is again regular.

Remark 2.23. Let $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ be two regular functions, and let $U \subseteq V$ be nonempty open. If $\varphi_1|_U = \varphi_2|_U$ then $\varphi_1 = \varphi_2$.

Definition 2.24. (i) A *quasi-affine variety* is an open subset of an affine algebraic variety.

(ii) A regular map between quasi-affine varieties $U \subseteq \mathbb{A}^n, V \subseteq \mathbb{A}^m$ is a map $\varphi : U \to V$ given by $\varphi = (\varphi_1, \dots, \varphi_m)$ with φ_i regular on U. φ is an isomorphism if there is a regular inverse.

Remark 2.25. For affine varieties, by remark 2.13 all information on regular maps $f : X \to Y$ could be obtained from their induced coordinate maps $f^* : A(Y) \to A(X)$. This is no longer true for quasi-affine varieties: for example, $\mathbb{A}^2 \setminus 0 \hookrightarrow \mathbb{A}^2$ induces an isomorphism of coordinate rings

Definition 2.26. Let X be an affine variety and $f \in A(X)$. Then $D(f) := X \setminus V_X(f)$ is called the *distinguished open subset* of f in X. Lecture 6 Nov 5, 2025

Remark 2.27. Since $D(f) \cap D(g) = D(fg)$, finite intersections of distinguished opens are again distinguished open. Any open $U \in X$ is a finite union of distinguished open subsets. Indeed, $U = X \setminus V_X(f_1, \dots, f_n) = \bigcup_i D(f_i)$.

Proposition 2.28. *Let X be an affine variety and $0 \neq f \in A(X)$. Then $\mathcal{O}_X(D(f)) = A(X)_f = \{\frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N}\} \subseteq K(X)$. In particular, on a distinguished open subset a regular function is always globally the quotient of two elements from $A(X)$.*

Proof. $\supseteq$ is clear. So let $\varphi \in \mathcal{O}_X(D(f))$ and consider

$$I = \{h \in A(X) \mid h\varphi \in A(X)\} \subseteq A(X).$$

This is an ideal which clearly satisfies $V_X(I) \cap D(f) = \emptyset$. Hence $V_x(I) \subseteq V_X(f)$, and by the Nullstellensatz 1.13 we see that $f \in \sqrt{I}$, i.e. $f^n \in I$ for some n. $\square$

Example 2.29. Consider $D(x) = \mathbb{A}^1 \setminus 0 \to V(xy-1) \subseteq \mathbb{A}^2, x \mapsto (x, \frac{1}{x})$. This is an isomorphism (with inverse $(x,y) \mapsto x$ between the quasi-affine $\mathbb{A}^1 \setminus 0$ and the affine variety $V(xy-1)$. Note that this is not true in general: not every quasi-affine variety is isomorphic to an affine variety. For example, $\mathbb{A}^2 \setminus 0$ isn't isomorphic to any affine variety. However, we have

Proposition 2.30. *Let X be an affine variety and $f \in A(X)$. Then $D(f)$ is isomorphic to an affine variety Y with $A(Y) \cong A(X)_f$.*

Proof. Set

$$Y := \{(x,t) \in X \times \mathbb{A}^1 \mid tf(x) = 1\} \subseteq X \times \mathbb{A}^1 \subseteq \mathbb{A}^{n+1}.$$

Then as in example 1.28, $D(f) \to Y$, $x \mapsto (x, \frac{1}{f(x)}$ is an isomorphism with inverse $(x,y) \mapsto x$, so $D(f) \cong Y$ and $A(Y) \cong \mathcal{O}_X(D(f)) \cong A(X)_f$. $\square$

We have seen that for X an algebraic set and $f \in A(X)$ regular, $V_X(f)$ is closed in X. The same is true for quasi-affine varieties:

Lemma 2.31. *Let X be an affine variety and $U \subseteq X$ open. Let $\varphi \in \mathcal{O}_X(U)$. Then $V_U(\varphi) := V(\varphi) := \{x \in U \mid \varphi(x) = 0\}$ is closed in U.*

Proof. Let $a \in U$. Then there exists an open neighbourhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that $\varphi = \frac{f_a}{g_a}$ on U_a. Then

$$U_a \setminus V(\varphi) = \{x \in U_a \mid \varphi(x) \neq 0\} = \{x \in U_a \mid f_a(x) \neq 0\} = U_a \setminus V(f_a)$$

is open in X, hence $U \setminus V(\varphi) = \bigcup_a U_a \setminus V(\varphi)$ is open. $\square$

Proposition 2.32. *Let X be a quasi-affine variety and $U \subseteq X$ be open. Let φ, ψ be two regular functions on X such that $\varphi|_U = \psi|_U$. Then $\varphi = \psi$ on X.*

Proof. $V_X(\varphi - \psi)$ contains the open, hence dense by 2.21, set U. □

Proposition 2.33. *Let X, Y be algebraic sets and $U \subseteq X$ be open. Then any regular map $\varphi : U \to Y$ is continuous (w.r.t. the Zariski topology). In particular, $\varphi \in \mathcal{O}_X(U)$ is a continuous map $U \to \mathbb{A}^1$.*

Proof. Let $\varphi = (\varphi_1, \dots, \varphi_m)$ and let $Z = V_Y(g_1, \dots, g_m) \subseteq Y$ be a closed subset. Then $\varphi^{-1}(Z) = \{x \in U \mid g_i(\varphi_1(x), \dots, \varphi_m(x)) = 0 \text{ for all } i\}$, which is closed by lemma 2.31. □

Let $\varphi : U \to V$ be regular. For any regular map $f \in \mathcal{O}(V)$, the composition $f \circ \varphi \in \mathcal{O}(U)$ is well-defined, hence we get as before a K-algebra homomorphism Lecture 7 Nov 10, 2025

$$\varphi^* : \mathcal{O}(V) \to \mathcal{O}(U), \quad f \mapsto f \circ \varphi.$$

The assignment $U \mapsto \mathcal{O}(U)$, $\varphi \mapsto \varphi^*$ is a contravariant functor as before, but no longer an equivalence of categories, see exercises.

Let X, Y be affine algebraic subsets. We know that regular maps $X \to Y$ are given by polynomial functions. It may happen that we do not have any "interesting" polynomial maps. For example, over $K = \mathbb{C}$ consider $X = \mathbb{A}^1$ and $Y = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$. Then the only regular maps $X \to Y$ are constant. However, the nontrivial map $t \mapsto (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$ induces an isomorphism $\mathbb{A}^1 \setminus \{\pm i\} \to Y \setminus \{(1,0)\}$.

Let X be an affine algebraic variety. Then $\varphi \in K(X)$ is a regular function on $\operatorname{dom}\varphi$. Moreso, given $\varphi_1, \dots, \varphi_m \in K(X)$, we get a regular map on the open set $\bigcap_i \dim \varphi_i \to \mathbb{A}^m$.

Definition 2.34. Let X be an affine algebraic variety and Y an affine algebraic set. A *rational map* $\varphi : X \dashrightarrow Y \subseteq \mathbb{A}^m$ is given by $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_i \in K(X)$ such that $\varphi(x) \in Y$ for every $x \in \operatorname{dom}\varphi := \bigcap_i \operatorname{dom}\varphi_i$. A rational map $\varphi : X \dashrightarrow Y$ is called *dominant* if the image of φ is dense in Y, i.e. if $\overline{\varphi(\operatorname{dom}\varphi)} = Y$.

A rational map $\varphi : X \dashrightarrow Y$ induces a regular map $\operatorname{dom}\varphi \to Y$. Let $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ be rational maps. Then ψ might not be defined on $\operatorname{im}\varphi$. But if φ is dominant, then $\psi \circ \varphi$ is well-defined on the non-empty open $\varphi^{-1}(\operatorname{dom}\psi)$.

Definition 2.35. Let X be an affine algebraic variety and Y an affine algebraic set. A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs (U, φ_U), where $U \subseteq X$ is nonempty open, $\varphi_U : U \to Y$ is regular, and $(U, \varphi_U) \sim (V, \varphi_V)$ if and only if $\varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}$. The rational map is dominant if for some (and therefore all) (U, φ_U) one has $\overline{\varphi_U(U)} = Y$.

Remark 2.36. The relation in definition 2.35 is an equivalence relation. Indeed, if $(U, \varphi_U) \sim (V, \varphi_V) \sim (W, \varphi_W)$, then $\varphi_U|_{U \cap W}$ and $\varphi_W|_{U \cap W}$ are regular maps that agree on the non-empty open $U \cap V \cap W$, hence they are equal by proposition 2.32.

The two above definitions are equivalent: If φ is regular in the sense of 2.34, then $[(\operatorname{dom}\varphi, \varphi)]$ defines a regular map as in 2.35. Conversely, if an equivalence class $\{(U_i, \varphi_i)\}_i$ is given, then the map $\bigcup_i U_i$, $x \mapsto \varphi_i(x)$ for any i with $x \in U_i$ is regular, i.e. a rational map as in 2.34. Clearly, the notion of dominance is preserved by these identifications.

One can compose dominant rational maps $\varphi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow Z$ by setting

$$[(U, \varphi_U)] \circ [(V, \varphi_V)] := [(\varphi_U^{-1}(V), \psi_V \circ \varphi_U|_{\varphi_U^{-1}(V)})]$$

Write $\operatorname{Mor}_{\text{rat}}(X, Y)$ for the set of rational morphisms $X \dashrightarrow Y$. Lecture 8 Nov 12, 2025

Definition 2.37. Let $\varphi : X \dashrightarrow Y$ be dominant. In the same way as for regular maps, we define

$$\varphi^* : \mathrm{Mor}_{\mathrm{rat}}(Y, \mathbb{A}^1) \to \mathrm{Mor}_{\mathrm{rat}}(X, \mathbb{A}^1), \qquad f \mapsto f \circ \varphi.$$

Proposition 2.38. *Let X be an affine algeraic variety. Then $\mathrm{Mor}_{rat}(X, \mathbb{A}^1)$ is a field with the operations $(U, f) * (V, g) := (U \cap V, f|_{U\cap V} + g|_{U\cap V})$ for $* \in \{+, -, \cdot\}$ and $(U, f)^{-1} = (U \setminus V(f), \frac{1}{f})$. Moreover, $\mathrm{Mor}_{rat}(X, \mathbb{A}^1) \cong K(X)$ as fields.*

Proof. It is clear that the given operations are well-defined and make $\mathrm{Mor}_{\mathrm{rat}}(X, \mathbb{A}^1)$ a field. The equivalence of definitions 2.34 and 2.35 provides a field isomorphism $K(X) \to \mathrm{Mor}_{\mathrm{rat}}(X, \mathbb{A}^1)$, $f \mapsto (\mathrm{dom}\, f, f)$. □

Corollary 2.39. *If $\varphi : X \dashrightarrow Y$ is a dominant rational map between affine varieties, we get a K-homomorphism of fields $\varphi^* : K(Y) \to K(X)$, $f \mapsto f \circ \varphi$.*

Recall that for regular maps, we had in 2.13 an equivalence between algebraic sets + regular maps, and reduced f.g. K-algebras + K-algebra homomorphisms. In the case of rational maps, we get similarly

Theorem 2.40. *$\varphi \mapsto \varphi^*$ is a bijection $\{\varphi \in \mathrm{Mor}_{rat}(X, Y) \mid \varphi \text{ dominant}\}$ to $\mathrm{Hom}_K(K(Y), K(X))$. This assignment is functorial, and induces an equivalence of categories*

$$\begin{array}{ccc} \left\{\begin{array}{c}\textit{affine algebraic varieties +}\\ \textit{dominant rational maps}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\textit{field extensions } L/K \textit{ of finite transcendence degree +}\\ K\textit{-linear homomorphisms}\end{array}\right\} \\ X & \mapsto & K(X) \\ \varphi & \mapsto & \varphi^* \end{array}$$

Proof. To show that $\varphi \mapsto \varphi^*$ is a bijection, define an inverse by assigning to $f : K(Y) \to K(X)$ the morphism $(f(y_1), \dots, f(y_m))$. Everything else is clear. □

Definition 2.41. A dominant rational map $\varphi : X \dashrightarrow Y$ is called a *birational equivalence* (and X and Y are called *birational* or *birationally equivalent*) if there exists a rational dominant map $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$ as rational maps.

Proposition 2.42. *Let X, Y be affine algebraic varieties. The following statements are equivalent:*

(i) X and Y are birational.

(ii) $K(X) \cong K(Y)$.

(iii) There exist non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$ are isomorphic (in the sense of regular maps).

Proof. (i)⇔(ii) follows from 2.40 and (iii)⇒(i) from the definition of rational function as regular functions on some open. Now assume (i), i.e. that there exists a birational equivalence $\varphi = (U, \varphi_U) : X \dashrightarrow Y$ with inverse $\psi = (V, \psi_V)$. Then $\varphi = (U \cap \psi^{-1}(V), \varphi_U|_{\dots})$ and $\psi = (V \cap \varphi^{-1}(U), \psi_V|_{\dots})$ are the required isomorphisms $U \cap \psi^{-1}(V) \cong V \cap \varphi^{-1}(U)$. □ Lecture 9 Nov 17, 2025

Remark 2.43. An affine algebraic variety X is called *rational*, if X is birational to $\mathbb{A}^k$ for some k. Equivalently, $K(X)/K$ is a purely transcendental field extension. For example, in the exercises we proved that $S = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$ is rational[1].

Theorem 2.44. *Every affine algebraic variety X is birational to some hypersurface, i.e. a variety $V(f) \subseteq \mathbb{A}^n$ for some irreducible $f \in K[x_1, \dots, x_n]$.*

[1] Our proof works in char $K \neq 2$, but otherwise $\sqrt{(x^2 + y^2 - 1)} = (x + y - 1)$, so even $A(S) \cong K[x]$

Proof. For simplicity, we only consider the case $\operatorname{char} K = 0$. Since $K(X)/K$ is finitely generated, by basic algebra $K(X)/K$ factors as a purely transcendental extension followed by a finite one $K(X)/K(t_1, \ldots, t_d)/K$. Since everything is separable, $K(X)/K(t_1, \ldots, t_d)$ is generated by a primitive element, i.e. $K(X) = K(t_1, \ldots, t_d, \alpha)$ with α algebraic over $K(t_1, \ldots, t_d)$. Let wlog $f \in K[t_1, \ldots, t_d]$ be the minimal polynomial of α. Then $K(X) \cong \operatorname{Frac} K[t_1, \ldots, t_d, s]/(f(s)) \cong K(V(f))$ as desired. □

Remark 2.45. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. Then $X \times Y \subseteq \mathbb{A}^{n+m}$ is also affine algebraic, given by the same equations, now considered in $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Furthermore, if X, Y are irreducible, then so is $X \times Y$ (Exercise). This is the product in the category of affine algebraic sets (resp. varieties), i.e. for regular maps $\varphi : Z \to X$, $\psi : Z \to Y$, there exists a unique regular map $Z \to X \times Y$. Therefore $A(X \times Y) = A(X) \otimes_K A(Y)$.

3 Projective Varieties

Definition 3.1. *Projective n-space* over K is given by $\mathbb{P}^n_K := \mathbb{A}^{n+1} \setminus \{0\}/\sim$, where $x \sim y$ if $x = \lambda y$ for some $\lambda \in K$. We notate the equivalence class of x by $x = [x_0 : x_1 : \ldots : x_n]$, called the *homogeneous coorinates* of x.

Note that points in $\mathbb{P}^n$ correspond to one-dimensional linear subspaces of $\mathbb{A}^{n+1}$.

Remark 3.2. We would like to define projective algebraic sets as zeroes of polynomials as in the affine case. But this is not well-defined, because evaluation of a polynomial need not respect the equivalence relation of 3.1. For example, let $f = x_1^2 - x_0 \in K[x_0, x_1]$. Then $f(1,1) = 0$ and $f(-1,-1) = 2$, but $[1:1] = [-1:-1] \in \mathbb{P}^1_K$.

This problem can be solved by only considering *homogeneous polynomials*. For such a polynomial

$$f = \sum_{k_0+\ldots+k_n=d} a_{k_0,\ldots,k_n} x_0^{k_0} \cdots x_n^{k_n},$$

we have $f(\lambda x) = \lambda^d f(x)$, so $f(x) = 0$ is well-defined for $x \in \mathbb{P}^n$.

Definition 3.3. An ideal $I \subseteq K[x_0, \ldots, x_n]$ is called homogeneous if it can be generated by homogeneous polynomials.

Remark 3.4.
(i) If I is homogeneous and $f \in K[x_0, \ldots, x_n]$, write $f = f_0 + f_1 + \ldots + f_d$ with f_i homogeneous of degree i. Then $f \in I$ if and only if $f_i \in I$ for all i. (Say $I = (g_1, \ldots, g_n)$ with g_i homogeneous, write $f = \sum g_i h_i$. Then $f_d = \sum g_i (h_i)_{d-\deg g_i} \in I$.)
(ii) If I_1, I_2 are homogeneous ideals, then so are $I_1 + I_2, I_1 I_2, I_1 \cap I_2, \sqrt{I_1}$. (For "$\cap$", find an arbitrary generating set and then use (i), $\sqrt{-}$ is exercise.)

Definition 3.5. Let $f_1, \ldots, f_k \in K[x_0, \ldots, x_n]$ be homogeneous. Then

$$V(f_1, \ldots, f_k) := V^p(f_1, \ldots, f_k) := \{x \in \mathbb{P}^n_K \mid f_i(x) = 0 \text{ for all } i\}$$

is called a *projective algebraic set*. In the same way, for a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$, set

$$V(I) := V^p(I) := \{x \in \mathbb{P}^n_K \mid f(x) = 0 \text{ for all } f \in I \text{ homogeneous}\}.$$

Example 3.6. We have $V^p(0) = \mathbb{P}^n$, $V^p(1) = \emptyset$. Further, every point $x = [x_0 : \ldots : x_n]$ forms a projective algebraic set, since $V^p(a_i x_j - a_j x_i)_{i,j} = \{x\}$. Lecture 10 Nov 19, 2025

Remark 3.7. Just as in 1.5, projective algebraic sets are closed under arbitrary intersections and finite unions.

Definition 3.8. The *Zariski topology* on $\mathbb{P}^n$ is defined as the topology which closed sets the projective algebraic sets. On a projective algebraic set $X \subseteq \mathbb{P}^n$, the induced subspace topology is also called the Zariski topology on X.

Definition 3.9. A projective algebraic variety is an irreducible projective algebraic set.

For a subset $X \subseteq \mathbb{P}^n$ we may set

$$I^p(X) := \{f \in K[x_0, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

V^p and I^p enjoy many of the same properties as in the affine case. In particular

Proposition 3.10.
(i) For a subset $X \subseteq \mathbb{P}^n$, $V^p(I^p(X)) = \overline{X}$.
(ii) For a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ with $(x_0, \ldots, x_n) \not\subseteq I$, $I^p(V^p(I)) = \sqrt{I}$.

(iii) A projective algebraic set X is a variety if and only if $I^p(X)$ is a prime ideal.

Proof. (i) and (ii) as in the affine case. For (iii), we need the following

Claim: A homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$ is prime if and only if for all homogeneous $f, g \in L[x_0, \dots, x_n]$ with $fg \in I$, one has $f \in I$ or $g \in I$.

Indeed, suppose I were not prime, and let $f, g \notin I$ such that $fg \in I$. Let d_0, e_0 be maximal w.r.t. $f_{d_0}, g_{e_0} \notin I$. Then $(fg)_{d_0+e_0} = f_{d_0} g_{e_0} + \sum_{i+j=d_0+e_0, i \neq d_0} f_i g_j$. The left hand side is in I by remark 3.4, and the sum by the maximality assumption. Hence $f_{d_0} g_{e_0} \in I$. □

Definition 3.11. Let $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the canonical projection.

(i) An algebraic set $X \subseteq \mathbb{A}^{n+1}$ is called a *cone* if $0 \in X$ and $x \in X$ implies $\lambda x \in X$ for all $\lambda \in K$.
(ii) Given a cone $X \subseteq \mathbb{A}^{n+1}$, its *projectivization* is $\mathbb{P}(X) := \pi(X \setminus \{0\})$.
(iii) For a projective algebraic set $X \subseteq \mathbb{P}^n$, its *cone* is $C(X) := \{0\} \cup \pi^{-1}(X)$

Note that $\mathbb{P}(X)$ and $C(X)$ are projective resp. affine algebraic sets. Indeed, for a homogeneous ideal $S \subseteq K[x_0, \dots, x_n]$ we have $\mathbb{P}(V(S)) = V^p(S)$ and $C(V^p(S)) = V(S)$. It remains to show that all cones are of this form, which is

Proposition 3.12. *Let $X \subseteq \mathbb{A}^{n+1}$ be a cone. Then $I(X)$ is a homogeneous ideal.*

Proof. For $f = f_0 + \ldots + f_d$ and $x \in X$ we have $0 = f(\lambda x) = \sum_i \lambda^i f_i(x)$. As the 0 polynomial function in λ, since K is infinite we must have $f_i(x) = 0$ for all i. □

The next goal is to prove a projective version of the Nullstellensatz 1.13. Lecture 11 Nov 24, 2025

Definition 3.13. The (homogeneous maximal) ideal $I_0 := (x_0, \dots, x_n) \subseteq K[x_0, \dots, x_n]$ is called the *irrelevant* ideal.

Note that $V^p(I_0) = \emptyset$, so in general $I^p(V^p(I)) \neq I$ for radical homogeneous ideals I. But in some sense this is the only problematic case:

Proposition 3.14 (Projective Nullstellensatz)**.** *For any homogeneous ideal $J \subseteq K[x_0, \dots, x_n]$ with $\sqrt{J} \neq I_0$ we have $I^p(V^p(J)) = \sqrt{J}$.*

Proof. The inclustion "$\supseteq$" is clear. We have

$$\begin{aligned} I^p(V^p(J)) &= \langle f \in K[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in V^p(J) \rangle \\ &= \langle f \in K[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in \overline{V(J) \setminus \{0\}} \rangle \end{aligned}$$

Now $V(J) \neq \{0\}$, otherwise $\sqrt{J} = I(V(J)) = I_0$, hence $\overline{V(J) \setminus \{0\}} = V(J)$ (since then either $V(J) = \emptyset$ or $V(J)$ contains a line through 0). Then $I^p(V^p(J))$ is generated by homogeneous polynomials in $I(V(J)) = \sqrt{J}$. But $\sqrt{J}$ is homogeneous itself, so $I^p(V^p(J)) = \sqrt{J}$ as well. □

Corollary 3.15. *(i) If $I \subseteq K[x_0, \dots, x_n]$ is a homogeneous ideal, then $V^p(I) = \emptyset$ if and only if $I_0 \subseteq \sqrt{I}$, if and only if $\sqrt{I} = I_0$ or $I = (1)$.*
(ii) If $V^p(J) \neq \emptyset$, then $I^p(V^p(J)) = \sqrt{J}$.
(iii) I^p and V^p define inclusion-reversing bijections

$$\begin{aligned} \{\textit{projective algebraic sets in } \mathbb{P}^n\} &\rightleftarrows \{\textit{radical hom. ideals } I_0 \neq J \subseteq K[x_0, \dots, x_n]\} \\ \{\textit{projective algebraic varieties in } \mathbb{P}^n\} &\rightleftarrows \{\textit{prime hom. ideals } I_0 \neq J \subseteq K[x_0, \dots, x_n]\} \\ \{\textit{points in } \mathbb{P}^n\} &\rightleftarrows \{\textit{maximal hom. ideals } I_0 \neq J \subseteq K[x_0, \dots, x_n]\} \end{aligned}$$

(iv) $I^p(\mathbb{P}^n) = 0$, *and* $\mathbb{P}^n$ *is a variety.*

Remark 3.16. Let $U_i := D(x_i) = \{x \in \mathbb{P}^n \mid x_i \neq 0\} = \{x \in \mathbb{P}^n \mid x_i = 1\}$. Leaving out the i-th coordinate in the last presentation yields a homeomorphism $\iota_i : \mathbb{A}^n \to U_i$ (even an isomorphism of varieties, cf. later).

Therefore $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ is an open cover of projective space by $n+1$ copies of $\mathbb{A}^n$.

Definition 3.17. (i) For a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$, its dehomogenization is $f^i := f(1, x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$. For a homogeneous ideal $J \subseteq K[x_0, \ldots, x_n]$, write $J^i = \{f^i \mid f \in J\} \subseteq K[x_1, \ldots, x_n]$. In other words, these are the images of f, resp. J, under the natural map $K[x_0, \ldots, x_n] \mapsto K[x_0, \ldots, x_n]/(x_0 - 1) \cong K[x_1, \ldots, x_n]$.

(ii) For $0 \neq f \in K[x_1, \ldots, x_n]$ of $\deg f = d$, its homogenization is $f^h := x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in K[x_0, \ldots, x_n]$. For an ideal $J \subseteq K[x_0, \ldots, x_n]$, write J^h for the ideal of $K[x_0, \ldots, x_n]$. generated by f^h, $f \in J$

For example, if $f = 1 + X_1 + X_2 + X_1^2$, then $f^h = X_0^2 + X_0X_1 + X_0X_2 + X_1^2$., and $(f^h)^i = f$. Note that in general, J^h is not generated by homogenizations of generators of J, e.g. $J = \langle 1 + x_1 - x_2, x_1 - x_2 \rangle$. Lecture 12 Nov 26, 2025

Proposition 3.18. *The bijection* $\iota : \mathbb{A}^n \to U_0$ *as in remark 3.16 is a homeomorphism.*

Proof. It is clear that i is bijective, so we need to check that i, i^{-1} are continuous. Let $Z \subseteq U_0$ be a closed subset, say $Z = U_0 \cap V^p(f_1, \ldots, f_m)$ for homogeneous polynomials $f_1, \ldots, f_m \in K[x_0, \ldots, x_n]$. Then

$$i^{-1}(Z) = \{x \in \mathbb{A}^n \mid 0 = f_j(i(x)) = f_j((1 : x]) = f_j^i(x) \, \forall j\} = V(f_1^i, \ldots, f_m^i).$$

Hence i is continuous. Now let $W = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$ be closed. Then $i(W)$ consists of those $x \in U_0$ with $f_j(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = 0$ for all j. Since $x_0 \neq 0$, this is exactly the case if all $x_0^{\deg f_j} f_j(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = 0$, i.e. if $f_j^h(x) = 0$. Hence $i(W) = V^p(f_1^h, \ldots, f_m^h) \cap U_0$ is closed in U_0. ☐

Remark 3.19. We identify $\mathbb{A}^n$ with $U_0 \subseteq \mathbb{P}^n$. Then as subsets of $\mathbb{P}^n$ we have $\overline{V^a(I)} = V^p(I^h)$ and $V^p(I^h) \cap U_0 = V(I)$ (Exercise).

Definition 3.20. A *quasi-projective* set (variety) is an open subset of a projective algebraic set (variety).

Remark 3.21. Any (quasi-)affine algebraic set V is quasi-projective, since the closure of V in $\mathbb{P}^n \supseteq U_0 \cong \mathbb{A}^n$ may be computed by first taking the closure in $\mathbb{A}^n$ and then the closure of the resulting algebraic set in $\mathbb{P}^n$.

For an affine algebraic set X, an important construction was the ring of regular functions $A(X)$. We now want to consider the corresponding projective variant.

Definition 3.22. Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set. The K-algebra

$$S(X) := K[x_0, \ldots, x_n]/I^p(X)$$

is called the *homogeneous coordinate ring* of X. It is a graded ring, namely

$$S(X) = \bigoplus_{d \in \mathbb{N}} K[x_0, \ldots, x_n]_d / (I^p(X) \cap K[x_0, \ldots, x_n]_d),$$

where $K[x_0, \ldots, x_n]_d$ denotes the homogeneous polynomials of degree d.

Homogeneous elements of $S(X)$ can be considered as functions $X \to K$.

Definition 3.23. Let X be a quasi-projective algebraic set. and let $\varphi : X \to K$ be a map. Then φ is *regular* at $a \in X$ if locally at a one has $\varphi = \frac{f}{g}$, f, g homogeneous, $\deg(f) = \deg(g)$ and $g(a) \neq 0$. φ is regular on X if it is regular at every point of X.

Definition 3.24. Let $X \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ be quasi-projective algebraic sets. A map $\varphi : X \to Y$ is called *regular* if for every $a \in X$ there exists $a \in U_a \subseteq X$ open such that $\varphi|_{U_a} = (F_0, \dots, F_m)$ with F_i homogeneous of the same degree, such that for any $x \in U_a$, not all $F_i(x) = 0$.

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Example 3.25. $\mathbb{P}^2 \to \mathbb{P}^1$, $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ is not a morphism, because the image of $[0 : 0 : 1]$ is not defined. However, the restriction of the above rule to $V(x_2)$ is a well-defined regular map.

A regular map is an isomorphism if it has a regular inverse, see exercises for (counter)examples.

For a quasi-projective set X, write $A(X)$ for the K-algebra of regular functions. Then $A(\mathbb{P}^n) = K$ (see exercises). If $U \subseteq X$ is open, set $\mathcal{O}_X(U)$ for the K-algebra of regular functions on U. Intuitively, we should have $\mathcal{O}_{\mathbb{P}^n}(U_0) \cong A(\mathbb{A}^n) \cong K[x_1, \dots, x_n]$.

Remark 3.26. Similarly as in the affine case, one can show that a regular map is continuous.

Theorem 3.27. *Let $X \subseteq \mathbb{A}^n$ be a quasi-affine algebraic set. Write $\widetilde{X} := i(X) \subseteq U_0 \subseteq \mathbb{P}^n$ for the corresponding quasi-projective algebraic set. Then for $V \subseteq \widetilde{X}$ open, we have an isomorphism of K-algebras*

$$i^* : \mathcal{O}_{\widetilde{X}}(V) \to \mathcal{O}_X(i^{-1}(V)), \quad f \mapsto f \circ i,$$

and for $W \subseteq X$ open, we have a K-algebra isomorphism

$$j^* : \mathcal{O}_X(W) \to \mathcal{O}_{\widetilde{X}}(j^{-1}(W)), \quad f \mapsto f \circ j.$$

Proof. It is clear that these maps are mutually inverse, so one only has to check well-definedness, i.e. that $i^*(f), j^*(f)$ are regular. Let $h \in \mathcal{O}_{\widetilde{X}}(V)$ be regular, and $P \in i^{-1}(V)$. Then $h = \frac{f}{g}$ around $i(P)$, so $i^*(h) = \frac{f^i}{g^i}$ around P, and h is regular at P.

Similarly, let $h \in \mathcal{O}_X(W)$ and $Q \in j^{-1}(W)$. Then h is regular at $j(Q)$, so $h = \frac{f}{g}$ around $j(Q)$. Then near Q one has

$$h \circ j = \frac{f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})} = \frac{x_0^{\deg f + \deg g}}{\deg f + \deg g} \frac{f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})} = \frac{x_0^{\deg g} f^h}{x_0^{\deg f} g^h},$$

so h is regular at Q. $\square$

In particular for $X = \mathbb{A}^n$ we see $\mathcal{O}_{\mathbb{P}^n}(U_0) = \mathcal{O}_{U_0}(U_0) \cong A(\mathbb{A}^n) = K[x_1, \dots, x_n]$.

Remark 3.28. Let $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ be the standard open cover, $X \subseteq \mathbb{P}^n$ quasi-projective, and let $\varphi : X \to K$ be a function. Then φ is regular if and only if all $\varphi|_{U_i}$ are regular.

We give a second, equivalent definition of regular maps.

Definition 3.29. Let X, Y be qausi-projective algebraic sets. A map $\varphi : X \to Y$ is called a morphism or regular map if φ is continuous and preserves regular functions, that is for every $U \subseteq Y$ open and $f \in \mathcal{O}_Y(U)$, the function $\varphi^*(f) := f \circ \varphi$ is regular.

In particular, we get a k-algebra morphism $\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$. Note that by Theorem 3.27, the maps i, j are regular in the sense of this definition.

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Remark 3.30. (i) $\mathrm{id} : X \to X$ is a morphism.

(ii) It follows from the definition that the composition of morphisms is a morphism.

(iii) Being a morphism is a local property: Let $\varphi : X \to Y$ be a map. Then φ is a morphism if and only if for every $P \in X$ there is an open neighbourhood $P \in U_P \subseteq X$ s.t. $\varphi|_{U_P} \to Y$ is a morphism. In particular, if $X = \bigcup_i U_i$ is an open cover, then φ is a morphism if and only if all the $\varphi|_{U_i}$ are.

We now have to convince ourselves that this definition agrees with our previous notions of regular maps.

Theorem 3.31. *Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be quasi-projective algebraic sets, and let $\varphi : X \to Y$ be a map. Then the following are equivalent:*

(i) φ is a morphism (in the sense of definition 3.29)

(ii) φ is locally given by regular functions, i.e. for every $p \in X$ there is an open neighbourhood $p \in U \subseteq X$ and $h_1, \dots, h_m \in \mathcal{O}_X(U)$ with no common zero on U, s.t. $\varphi = [h_0 : \dots : h_m]$ on U.

(iii) φ is locally given by homogeneous polynomials of the same degree and no common zeroes (cf. definition 3.24).

Proof. (1)⇒(2): Let $p \in X$, and consider $\varphi(p) =: y =: [y_0 : \dots : y_m]$. Wlog we may assume $y_0 \neq 0$. Then $y \in U_0$, and

$$\varphi(x) = [1 : \varphi^*(\tfrac{y_1}{y_0})(x) : \dots : \varphi^*(\tfrac{y_m}{y_0})]$$

on $\varphi^{-1}(U_0)$, with $\varphi^*(\frac{y_i}{y_0})$ regular by assumption.

(2)⇒(3): By possibly shringking U, we may assume $h_i = \frac{F_i}{G_i}$ on U, with F_i, G_i homogeneous polynomials of the same degree. Multiplying all functions with $G_1 \cdots G_m$ clears denominators, so yields the desired presentation.

(3)⇒(1): φ is continuous by remark 3.26. Let $U \subseteq Y$ be open, and $h : U \to K$ regular. Let $P \in \varphi^{-1}(U)$. There exists a neighbourhood $\varphi(P) \in V \subseteq U$ such that $h = \frac{f}{g}$ on V, with f, g homogeneous polynomials of the same degree, and there is $W \subseteq X$ s.t. $\varphi|_W = [F_0 : \dots : F_m]$ with homogeneous polynomials of the same degree F_i. For $x \in W \cap \varphi^{-1}(V)$ one has

$$\varphi^*(h)(x) = h([F_0(x) : \dots : F_m(x)]) = \frac{f(F_0(x), \dots, F_m(x))}{g(F_0(x), \dots, F_m(x))},$$

which is again a quotient of two homogeneous polynomials of the same degree, and $\varphi^*(h)$ is regular at p. □

Proposition 3.32. *Let $X \subseteq \mathbb{P}^n$ be quasi-projective and $Y \subseteq \mathbb{A}^m$ quasi-affine. Then $\varphi : X \to Y$ is a morphism if and only if $\varphi = (f_1, \dots, f_m)$ for $f_1, \dots, f_m \in A(X)$.*

Proof. If φ is a morphism, then, as before, $\varphi = (\varphi^*(y_1), \dots, \varphi^*(y_n))$. For the converse, by remark 3.30(iii) and theorem 3.27 we may assume that X is quasi-affine. Then we know that φ is continuous, and that the composition of regular maps is regular. □

Corollary 3.33. *Let X be quasi-projective. Then $f : X \to \mathbb{A}^1$ is a morphism if and only if f is a regular function on X.*

4 The Segre Embedding and Closed Morphisms

Lecture 15
Dec 8, 2025

Our next goal is to understand the product of quasi-projective algebraic sets.

Remark 4.1. The obvious map $\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^{n+m}$ is regular and bijective, but not a homeomorphism. However, it allows us to define products of (quasi-)affine algebraic sets, see exercises. Similarly, if $U \subseteq \mathbb{A}^n, V \subseteq \mathbb{A}^m$ are open, then so is $U \times V \subseteq \mathbb{A}^{n+m}$, by taking complements.

Proposition 4.2. *If X, Y are quasi-affine, then $X \times Y$ as in remark 4.1 satisfies the universal property of products, that is: The projections $p_1, p_2 : X \times Y \to X, Y$ are morphisms, and for any morphisms $f_1, f_2 : Z \to X, Y$, there exists a unique morphism $f : Z \to X \times Y$ s.t. $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.*

Proof. Everything is clear from proposition 3.32. □

Remark 4.3. An analogous construction for projective spaces does not work, since the naive map $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+1}$ is not well-defined.

Definition 4.4 (Segre Embedding). Let $N = (n+1)(m+1) - 1$ and denote coordinates of $\mathbb{P}^N$ by z_{ij}, $0 \le i \le n$, $0 \le j \le m$. The map

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N, \quad (x, y) \mapsto [x_i y_j]_{ij}$$

is called the *Segre embedding*

One easily checks that this is a well-defined map.

Proposition 4.5. *The Segre embedding σ is injective and $\Sigma := \sigma(\mathbb{P}^n \times \mathbb{P}^m)$ is closed in $\mathbb{P}^N$. (That is, we can and will identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ.)*

Proof. Let $z = \sigma(x, y) \in \Sigma$. Say $z_{ij} \neq 0$, then $x_i, y_j \neq 0$. Hence $x = [z_{kj}]_k$ and $y = [z_{ik}]_k$, so σ is injective. Note that $z \in \Sigma$ if and only if the rank of the matrix $M_z = (z_{ij})_{ij}$ (defined up to scalar multiples) is 1, if and only if all 2×2-minors of M_z are 0. But this last condition is given by the zero set of polynomials of z_{ij}: $\Sigma = V^p(\{z_{ij}z_{kl} - z_{il}z_{jk}\}_{0 \le i,k \le n, 0 \le j,l \le m})$ □

Definition 4.6. We endow $\mathbb{P}^n \times \mathbb{P}^m$ with the Zariski topology induced from $\sigma : \mathbb{P}^n \times \mathbb{P}^m \cong \Sigma$.

Proposition 4.7. *The closed subsets $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ are precisely the sets of zeroes of polynomials $\{f_i\}_{i \in I}$ with $f_i \in K[x_0, \dots, x_n, y_0, \dots, y_m]$ bihomogeneous, i.e. homogeneous w.r.t. x_i and homogeneous w.r.t. y_j.*

Note that these are exactly the types of polynomials such that being 0 is well-defined on $\mathbb{P}^n \times \mathbb{P}^m$, so this condition makes sense.

Proof. Assume $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed, i.e. $\sigma(Z) \subseteq \Sigma \subseteq \mathbb{P}^N$ is closed. Hence $\sigma(Z) = V^p(f_\alpha(z_{ij}))_\alpha$ with $f_\alpha \in K[z_{ij}]$ homogeneous. Then

$$Z = \sigma^{-1}(V^p(f_\alpha(z_{ij})) = \{(x, y) \mid f_\alpha((x_i y_j)_{ij}) = 0, \ \forall \alpha\}$$

is of the desired form. Conversely, assume $Z = \{(x, y) \mid f_\alpha(x, y) = 0, \ \forall \alpha\}$ with f_α bihomogeneous. Then $\sigma(Z) = \{z \mid z_{ij} = x_i y_j, f_\alpha(x, y) = 0\}$. If $\deg_x f_\alpha = \deg_y f_\alpha$, then f_α is already a homogeneous polynomial in the z_{ij} (non-uniquely); otherwise, say if $\deg_x f_\alpha > \deg_y f_\alpha$, then $f_\alpha = 0$ if and only if all of $y_j^{\deg_x f_\alpha - \deg_y f_\alpha} f_\alpha = 0$, so we are reduced to the first case. □

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Corollary 4.8. *If $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ are closed, then $X \times Y$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$.*

Proof. $X \times Y = (\mathbb{P}^n \times Y) \cap (X \times \mathbb{P}^m)$. If $X = V^p(f_1, \dots, f_n)$, then $X \times \mathbb{P}^m = V(f_1, \dots, f_n)$ with f_i considered in $K[x_i, y_j]$. □

Similarly as in the affine case (remark 4.1), it follows that the product of opens is open, and that products of quasi-projective sets are quasi-projective.

Consider products of standard opens $U_i \times U_j \subseteq \mathbb{P}^n \times \mathbb{P}^m$. On the one hand, we should have $U_i \times U_j \cong \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$, on the other hand $\sigma(U_i \times U_j) = \Sigma_{ij} := \Sigma \cap U_{ij}$. These two descriptions indeed agree:

Proposition 4.9. *The composition of bijections $\mathbb{A}^n \times \mathbb{A}^m \cong U_i \times U_k \cong \Sigma_{ik}$ is an isomorphism of algebraic sets.*

Proof. Identify Σ_{ik} with its image under the isomorphism $j : U_{ik} \to \mathbb{A}^N$. It is enough to show that the composition $\mathbb{A}^n \times \mathbb{A}^m \to j(\Sigma_{lk})$, and its inverse, are regular. Wlog take $l = k = 0$. Then the map is given by

$$(x, y) \mapsto ([1 : x_1 : \dots : x_n], [1 : y_1 : \dots : y_n] \mapsto [1 : y_1 : \dots : x_n y_n] \mapsto [y_1 : \dots : x_n y_n].$$

Now we see that all coordinates are given by polynomials, so the map is regular. Moreover, all the x_i and y_j occur as coordinates in the image, so the inverse is given by a collection of projections, and is regular as well. □

Corollary 4.10. *Σ has an open cover by subsets $\Sigma_{ij} \cong \mathbb{A}^{n+m}$, corresponding to $\mathbb{P}^n \times \mathbb{P}^m = \bigcup_{i,j} U_i \times U_j$.*

Remark 4.11. $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ quasi-affine, then we defined $X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m$ in remark 4.1. As in the proposition, this algebraic set can be identified with its image under $\mathbb{A}^n \times \mathbb{A}^m \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m \to \Sigma$ (since this map is just a restriction of the isomorphism above, hence still an isomorphism).

Proposition 4.12. *If X, Y are quasi-projective algebraic sets, then $p_{1,2} : X \times Y \to X, Y$ are morphisms, and for any quasi-projective Z and morphisms $f_1, f_2 : Z \to X, Y$, there exists a unique morphism $(f_1, f_2) : Z \to X \times Y$ s.t. $p_i \circ (f_1, f_2) = f_i$.*

Proof. Let wlog $x_0 = 1$. Then $\sigma(x, y) = [y_0 : \dots : y_n : \dots]$, so $p_2(x, y) = [y_0 : \dots : y_n]$ is locally just the projection onto the first $n + 1$ coordinates, thus regular, and similarly for p_1.

So let f_1, f_2 be as in the statement. Since $X \times Y$ is the set-theoretic product, the only choice is $(f_1, f_2)(a) = (f_1(a), f_2(a))$, we have to check that this is a morphism. But since σ, f_1, f_2 are all locally given by homogeneous polynomials, it is clear that the same is true for $a \mapsto \sigma(f_1(a), f_2(a))$. □

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Theorem 4.13. *Let $\varphi : X \to X', \psi : Y \to Y'$ be morphisms of algebraic sets. Then $(\varphi, \psi) : X \times Y \to X' \times Y'$ is also a morphism. If φ and ψ are isomorphisms, then so is (φ, ψ).*

Proof. Abstract Nonsense. □

Definition 4.14. Let X be an algebraic set. The diagonal of X is $\Delta_X := \{(a, a) \in X \times X \mid a \in X\} = \mathrm{im}(\mathrm{id}_X, \mathrm{id}_X)$, with diagonal morphism is $S_X : X \to \Delta_X$.

Proposition 4.15. *$\Delta_X \subseteq X \times X$ is a closed subset, and S_X is an isomorphism.*

Proof. Let $X \subseteq \mathbb{P}^n$. Since $\Delta_X = \Delta_{\mathbb{P}^n} \cap X \times X$, it suffices to show that $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. Under the Segre embedding, we have $\sigma(\Delta_{\mathbb{P}^n}) = \Sigma \cap V^p(\{z_{ij} - z_{ji}\}_{ij})$, hence we are done. Further clearly $S_X^{-1} = p_1|_{\Delta_X}$. □

Definition 4.16. Let $f : X \to Y$ be a morphism of algebraic sets. The *graph* of f is $\Gamma_f := \{(a, f(a)) \in X \times Y \mid a \in X\} = \operatorname{im}(\operatorname{id}_X, f)$.

Proposition 4.17. *$\Gamma_f \subseteq X \times Y$ is a closed subset, and the natural map $X \to \Gamma_f$ is an isomorphism.*

Proof. $\Gamma_f = (f, \operatorname{id}_Y)^{-1}(\Delta_Y)$. □

Definition 4.18. A map $\varphi : X \to Y$ between topological spaces is called *closed* if for every closed subset $Z \subseteq X$ the image $\varphi(Z) \subseteq Y$ is closed in Y. An algebraic set X is called *complete* if $p_2 : X \times Y \to Y$ is a closed map for every algebraic set Y.

Theorem 4.19. *Let $\varphi : X \to Y$ be a morphism where X is projective. Then φ is closed, in particular $\varphi(X) \subseteq Y$ is closed.*

Remark 4.20. The projective assumption is essential: Let $p_1 : \mathbb{A}^2 \supseteq V(XY - 1) \to \mathbb{A}^1$ be the projection onto the X-coordinate, then $\operatorname{im} p_1 = \mathbb{A}^1 \setminus \{0\}$ is not closed.

Corollary 4.21. *Let X be a connected projective algebraic set. Then $A(X) = K$.*

Proof. By corollary 3.33 a regular function $\varphi \in A(X)$ can be seen as a morphism $\varphi : X \to \mathbb{A}^1$. By theorem 4.19 $\operatorname{im} \varphi$ is closed in $\mathbb{A}^1$. But $\operatorname{im} \varphi = \mathbb{A}^1$ is impossible, since then $X \to \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ would have non-closed image, contradicting theorem 4.19. Therefore, $\operatorname{im} \varphi$ is finite, and in fact a singleton, since X is connected. □

Corollary 4.22. *More generally, let X be a projective algebraic set and Y quasi-affine. Then any morphism $\varphi : X \to Y$ is constant.*

Proof. By corollary 4.21, all coordinate functions $X \xrightarrow{\varphi} Y \xrightarrow{p_i} \mathbb{A}^1$ are constant. □

Corollary 4.23. *The image of a morphism $\mathbb{A}^1 \to \mathbb{A}^n$ is closed.*

Proof. Exercise. □

Example 4.24. By corollary 4.23, the sets $\{(t^2, t^3) \mid t \in K\}$, $\{(t^3, t^4, t^5) \mid t \in K\}$ from exercise sheet 1 are affine.

Proof. (of theorem 4.19) Let $Z \subseteq X$ be closed. Then Z is projective, and $Z \hookrightarrow X \to Y$ is regular. Hence, it is enough to show that $\varphi(X) = p_2(\Gamma_\varphi)$ is closed in Y. By proposition 4.17, the result thus follows from □

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Dec 17, 2025

Theorem 4.25. *Every projective algebraic set $X \subseteq \mathbb{P}^n$ is complete.*

Proof. Let $Y \subseteq \mathbb{P}^m$ be an algebraic set and $Z \subseteq X \times Y$ be closed. Then Z is also closed in $\mathbb{P}^n \times Y$, and if $p_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is the projection, then $p_2(Z) = p_2|_{X \times Y}(Z)$. So it suffices to prove the theorem for $X = \mathbb{P}^n$.

So let $Z \subseteq \mathbb{P}^n \times Y$ be closed and write $Z = (\mathbb{P}^n \times Y) \cap \widetilde{Z}$ with $\widetilde{Z} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ closed. It suffices to show that $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed, for then

$$p_2|\mathbb{P}^n \times Y(Z) = p_2(Z) = p_2(\mathbb{P}^n \times Y) \cap p_2(\widetilde{Z}) = Y \cap p_2(\widetilde{Z})$$

is closed in Y.

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be closed. To show that $p_2(Z)$ is closed, we use the following

Lemma 4.26. *Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal. Then $V^p(I) = \emptyset$ if and only if $K[x_0, \ldots, x_n]_s \subseteq I$ for some $s \geq 0$.*

Proof. Indeed, by the projective Nullstellensatz, both conditions are equivalent to $(x_0, \dots, x_n) \subseteq \sqrt{I}$. $\square$

By proposition 4.7, we may write Z as the sof of common zeroes of biholomorphic polynomials $f_1, \dots, f_r$, by the proof of the proposition we may assume that all f_i have the same degree d in both x_i's and y_j's. Fix a point $a \in \mathbb{P}^m$. Then $a \in p_2(Z)$ iff $V^p(f_1(x,a), \dots, f_r(x,a)) \neq \emptyset$, which by the lemma is equivalent to $K[x_0, \dots, x_n]_s \not\subseteq (f_1(x,a), \dots, f_r(x,a))$ for all $s \geq 0$. This is clear for $s < d$, for $s \geq d$ define $T_s := \{a \in \mathbb{P}^m \mid K[x_0, \dots, x_n]_s \not\subseteq (f_1(x,a), \dots, f_r(x,a))\}$. Then $p_2(Z) = \bigcap_{s \geq d} T_s$ and it is enough to show that all T_s, $s \geq d$ are closed in $\mathbb{P}^m$.

Note that $y \in T_s$ is equivalent to the K-linear map

$$\varphi : K[x_0, \dots, x_n]_{s-d}^r \to K[x_0, \dots, x_n]_s, \quad (h_1, \dots, h_n) \mapsto f_1(x,y)h_1(x) + \ldots + f_r(x,y)h_r(x)$$

not being surjective. Let $v_1, \dots, v_l$ be all monomials of degre $s - d$, and $w_1, \dots, w_t$ all monomials of degree s, so that the w_j form a basis of $K[x_0, \dots, x_n]_s$, and $v_i e_j$ form a basis of $K[x_0, \dots, x_n]_{s-d}^r$. Now φ is not surjective iff $\dim \operatorname{im} \varphi < \dim K[x_0, \dots, x_n]_s = \binom{n+s}{n} = t$, hence iff all $t \times t$ minors of the representative matrix of φ vanish. But this is a system of polynomial equations in y, so T_s is closed. $\square$

Lecture 19
Dec 22, 2025

Example 4.27 (The Veronese Embedding)**.** Let $n, d > 0$ and $M_0, \dots, M_N \in K[x_0, \dots, x_n]$ all monomials of degree d, with $N = \binom{n+d}{d} - 1$. The Veronese embedding of degree d is

$$v_d : \mathbb{P}^n \to \mathbb{P}^N, \qquad x \mapsto [M_0(x) : \ldots : M_N(x)].$$

For instance, if $n = 1$, then $v_d(x) = [x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d]$. v_d is a well-defined morphism, so by theorem 4.19, $\operatorname{im} v_d$ is a projective subvariety of $\mathbb{P}^N$. We claim that $v_d : \mathbb{P}^n \to v_d(\mathbb{P}^n)$ is an isomorphism.

Indeed, for any l there is a unique i_l such that $M_{i_l}(x) = x_l^d$. Then $v_d(\mathbb{P}^n) = \bigcup_l v_d(\mathbb{P}^n) \cap \{y_{i_l} \neq 0\}$, and the inverse map consists of suitable projections from $\mathbb{P}^n$ to $n+1$ coordinates.

Example 4.28. In sheet 6, exercise 2 we prove that $v_2 : \mathbb{P}^1 \to V^p(Y^2 - XZ)$ is an isomorphism.

Definition 4.29. Let $A \in \mathrm{GL}_{n+1}(K)$ be an invertible matrix. We define the map

$$\varphi_A : \mathbb{P}^n \to \mathbb{P}^n, x \mapsto Ax.$$

Note $Ax = 0$ iff $x = 0$, so this is a well-defined isomorphism (with inverse $\varphi_{A^{-1}}$), called a projective transformation. One can show that projective transformations are the only automorphisms of $\mathbb{P}^n$.

Remark 4.30. Let X be a connected projective algebraic set. We proved in corollary 4.22 that if $\varphi : X \to Y \subseteq \mathbb{A}^n$ is regular, then φ is constant. It follows that if X is isomorphic to a quasi-affine algebraic set, then X is a point.

Proposition 4.31. *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and $F \in K[x_0, \dots, x_n]$ a homogeneous non-constant polynomial. Then $X \setminus V^p(F)$ is isomorphic to a affine algebraic set.*

Proof. We already know that this holds if $F = X_0$ (then $X \setminus U_0 \cong \mathbb{A}^n$), hence also if F is linear, since then there exists a projective transformation mapping $V^p(F)$ onto $V^p(X_0)$.

So let $\deg(F) = d > 1$. Under the Veronese embedding v_d, $V^p(F)$ gets mapped to the zero set of a linear polynomial, so the result follows again from the above. $\square$

Corollary 4.32. *Let $X \subseteq \mathbb{P}^n$ be a connected projective algebraic set, containing more than one point. Let $F \in K[x_0, \dots, x_n]$ be a nonconstant homogeneous polynomial. Then $X \cap V^p(F) \neq \emptyset$.*

Proof. Combine proposition 4.31 and remark 4.30. $\square$

Remark 4.33. In the affine case, the above statement is not true: Consider two parallel lines in $\mathbb{A}^2$.

5 Dimension

Remark 5.1. Recall from section 1 that for an affine algebraic set $X \subseteq \mathbb{A}^n$, we have $\dim X = \dim A(X)$.

From commutative algebra, we use the following facts:

Theorem 5.2. *(i)* $\dim A(X) < \infty$.
(ii) If R is a finitely generated integral K-algebra and $0 \neq f \in R$ is not a unit, then

$$\dim(R/(f)) = \dim R - 1.$$

Proposition 5.3. $\dim \mathbb{A}^n = n$.

Proof. By induction. The case $n = 1$ was dealt with before, and

$$\dim \mathbb{A}^{n-1} = \dim K[x_1, \ldots, x_{n-1}, x_n]/(x_n) = \dim \mathbb{A}^n - 1.$$

□

Lemma 5.4. *Let X be a Noetherian topological space, and $X = \bigcup_{i=1}^n U_i$ an open cover of X. Then $\dim X = \max_i \dim U_i$. In particular, $\dim \mathbb{P}^n = n$.*

Proof. Exercise. □

Lecture 20
Jan 7, 2026

Remark 5.5. We record the following general properties of dimension, see also the exercises. Let X be a topological space and $Y \subseteq X$.

(i) $\dim Y \leq \dim X$
(ii) If X is irreducible and $\dim X = \dim Y < \infty$, then $\overline{Y} = X$.
(iii) Let $X = \bigcup_i U_i$ be an open cover. Then $\dim X = \sup\{\dim U_i\}_i$.
(iv) Let X be Noetherian. We proved that X can be written as a finite union $X = X_1 \cup \ldots \cup X_n$ of irreducible components. Then $\dim(X) = \max(\dim(X_i))$. (In particular, any quasi-projective set is noetherian.)

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then $\dim(X) = \dim A(X) = \operatorname{trdeg} K(X)$.

Corollary 5.6. *Let $F \in K[X_1, \ldots, X_n]$, $\deg F > 0$. Then $\dim V(F) = n - 1$. Moreover, every irreducible component of $V(F)$ has dimension $n - 1$.*

Proof. Write $F = F_1^{e_1} \cdots F_k^{e_k}$ with F_i pairwise different and irreducible. Then the irreducible decomposition of $V(F)$ is $V(F) = V(F_1) \cup \ldots \cup V(F_k)$, so wlog F is irreducible. We get

$$\dim V(F) = \dim K[X_1, \ldots, X_n]/(F) \overset{5.2}{=} \dim K[X_1, \ldots, X_n] - 1 \overset{5.3}{=} n - 1.$$

□

Definition 5.7. Let X be a Noetherian topological space. We say that X is of *pure* dimension n if every irreducible component of X has dimension n. A closed algebraic set X is called a curve if it is of pure dimension 1, a surface if it is of pure dimension 2, and a hypersurface in a pure-dimensional algebraic set Y if $X \subseteq Y$ and X is of pure dimension $\dim(Y) - 1$.

Proposition 5.8. *Let X be a hypersurface in $\mathbb{A}^n$. Then $X = V(F)$ for some polynomial $F \in K[X_1, \ldots, X_n]$.*

Proof. Wlog X is irreducible and $\dim(X) = n - 1$. Then the prime ideal $I(X)$ has height $\dim(A^n) - \dim(X) = 1$, hence is principal. $\square$

Proposition 5.9. *Let $\emptyset \neq U \subseteq X$ be an open subset of an affine variety. Then $\dim U = \dim X$.*

Proof. U contains a basic open $D(f)$, it suffices to prove the claim for the latter open set. By proposition 2.31, $D(f)$ is birational to X. In particular, they have the same function field, so by the above remark, we have $\dim(D(f)) = \dim X$. $\square$

Proposition 5.10. *Let X be a projective algebraic variety, and $\emptyset \neq U \subseteq X$ be open. Then $\dim U = \dim X$.*

Proof. Exercise. $\square$

6 Hilbert Polynomial and Bezout's Theorem

Let X be a projective variety. We will associate to it a polynomial, the so called Hilbert polynomial, which encodes interesting structure of X. In particular, it will help us understand intersecions of two projective algebraic sets. For example, we will be able to prove that two distinct curves in $\mathbb{P}^2$ intersect, and the number of intersection points is bounded by the product of their degrees.

Lecture 21
Jan 12, 2026

Definition 6.1. (i) Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal. Then $K[x_0, \dots, x_n]/I$ is a graded K-algebra, with degree d-part $S_d := K[x_0, \dots, x_n]_d / K[x_0, \dots, x_n]_d \cap I$. The Hilbert function of I is

$$h_I : \mathbb{N} \to \mathbb{N}, \qquad d \mapsto \dim_K S_d.$$

(ii) For a projective algebraic set $X \subseteq \mathbb{P}^n$ we set $h_X := h_{I(X)}$

Remark 6.2. Note that the Hilbert function of $X \subseteq \mathbb{P}^n$ is invariant under projective linear automorphisms $\varphi : x \mapsto Ax$. Indeed, φ induces a grading-preserving automorphism of $K[x_0, \dots, x_n]$

Example 6.3. (i) $h_{\mathbb{P}^n}(d) = \dim_K K[x_0, \dots, x_n]_d = \binom{n+d}{n}$

(ii) Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal s.t. $V^p(I) = \emptyset$. By lemma 4.26 we have $h_I(d) = 0$ for all $d \gg 0$.

(iii) Let $X = \{a\} \subseteq \mathbb{P}^n$ a point. By remark 6.2 we can assume $a = [1 : 0 : \ldots : 0]$. Then $I(X) = (x_1, \dots, x_n)$, hence $S(X) \cong K[x_0]$ and $h_X(d) = \dim_K K[x_0]_d = 1$ for all $d \in \mathbb{N}$.

Proposition 6.4. *Let $I, J \subseteq R := K[x_0, \dots, x_n]$ be two homogeneous ideals. Then $h_{I \cap J} + h_{I+J} = h_I + h_J$.*

Proof. Follows immediately from the exact sequence of graded K-algebras

$$0 \to R/I \cap J \to R/I \times R/J \to R/(I+J) \to 0$$

and the fact that $\dim_K$ is additive on exact sequences. □

Example 6.5. (i) Let $X, Y \subseteq \mathbb{P}^n$ be disjoint projective algebraic sets. By example 6.3(ii) and proposition 6.4, we have $h_{X \cup Y} = h_X + h_Y$ for $d \gg 0$. In particular, for a finite set of r points, we have $h(d) = r$ for $d \gg 0$.

(ii) $I = (x_1^2) \subseteq K[x_0, x_1]$. Then $V(I) = \{[1 : 0]\}$ is a point, but h_I detects the multiplicativity 2: For $d \geq 1$ one has

$$K[x_0, x_1]_d / I_d = Kx_0^d \oplus Kx_0^{d_1} x_1,$$

so $h_I(d) = 2$ for all $d \geq 1$.

Our next goal is to show that the Hilbert function $h_I(d)$ agrees with a polynomial for $d \gg 0$.

Lemma 6.6. *Let $I \subseteq R = K[x_0, \dots, x_n]$ be a homogeneous ideal. Let $f \in K[x_0, \dots, x_n]$ be homogeneous of degree e. Assume that there exists $d_0 \in \mathbb{N}$ s.t. for all homogeneous polynomials $g \in K[x_0, \dots, x_n]$ of $\deg g \geq d_0$ wih $fg \in I$ we have $g \in I$. Then $h_{I+(f)}(d) = h_I(d) - h_I(d-e)$ for $d \gg 0$.*

Proof. By assumption, multiplication with f induces an exact sequence

$$0 \to R_{d-e}/I_{d-e} \to R_d/I_d \to R_d/(I+(f))_d \to 0.$$

Taking dimensions yields the formula. □

Remark 6.7. Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set and $X = X_1 \cup \ldots \cup X_r$ its decomposition into irreducible components. Then $I(X) = \bigcap I(X_i)$ and the assumption of lemma 6.6 is satisfied for all $f \notin \bigcup I(X_i)$.

Picking any point $a_i \in X_i$, there exists a hyperplane H which does not intersect the set $\{a_1, \ldots, a_r\}$, hence does not contain any X_i. Then there exists a projective linear automorphism sending H to $V(x_0)$. Therefore, for any radical ideal I we may assume that x_0 satisfies the requirements of lemma 6.6. A similar idea, using primary decomposition, works for general ideals.

Lecture 22
Jan 14, 2026

Definition 6.8. Let $P \in \mathbb{Q}[X]$. P is called *numerical* if $P(n) \in \mathbb{Z}$ for all $n \gg 0$.

For example, $P(n) = \binom{n}{d}$ is a numerical polynomial of degree d for all d.

Proposition 6.9. *(i) Let $P \in \mathbb{Q}[X]$ be a numerical polynomial. Then $P(X) = \sum_i c_i \binom{X}{i}$ with $c_i \in \mathbb{Z}$.*

(ii) Let $f : \mathbb{Z} \to \mathbb{Z}$ s.t. there exists a numerical polynomial Q of degree r with $\Delta f(n) := f(n+1) - f(n) = Q(n)$ for $n \gg 0$. Then there exists a numerical polynomial P of degree $r+1$ s.t. $f(n) = P(n)$ for $n \gg 0$.

Proof. (i) Induction on $\deg P$. $\deg P = 0$ is clear. Assume $\deg P = r > 0$. The polynomials $1, \ldots, \binom{X}{r}$ form a $\mathbb{Q}$-basis of $\mathbb{Q}[X]_{\leq r}$. Therefore we can write $P(X) = \sum_{i=0}^{r} c_i \binom{X}{i}$ with $c_i \in \mathbb{Q}$. Then $\Delta P(x) = \sum_{i=0}^{r-1} c_{i+1} \binom{X}{i}$, so by induction $c_1, \ldots, c_r \in \mathbb{Z}$, and then $c_0 \in \mathbb{Z}$ as well.

(ii) By (i) we can write $Q(X) = \sum_{i=0}^{r} c_i \binom{X}{i}$. Set $P := \sum_{i=0}^{r} c_i \binom{X}{i+1}$. This is a numberical polynomial with $\Delta P = Q$, so $\Delta(f-P)(n) = 0$ for $n \gg 0$, i.e. $f - P$ is eventually constant. □

Theorem 6.10. *Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal. Then*

(i) There exists a unique (numerical) polynomial $\chi_I(X) \in \mathbb{Q}[X]$ such that $\chi_I(d) = h_I(d)$ for $d \gg 0$.

(ii) $\deg \chi_I = \dim V^p(I) =: m$.

(iii) If $V^p(I) \neq \emptyset$, the leading coefficient of χ_I is $\frac{1}{m!}$ times a positive integer.

χ_I is called the Hilbert polynomial of I. For a projective algebraic set $X \subseteq \mathbb{P}^n$, $\chi_X := \chi_{I(X)}$ is the Hilbert polynomial of X.

Proof. Uniqueness is clear. Induction on m: The cases with $m \leq 0$ are handled in example 6.3(ii) and 6.5(i). Now let $\dim V^p(I) = m > 0$ and let $V^p(I) = X_1 \cup \ldots \cup X_r$ be the decomposition into irreducible components. By remark 6.7, there exists a linear polynomial f s.t. $X_i \not\subseteq V^p(f)$ and the condition from lemma 6.6 holds. Then $h_{I+(f)}(d) = h_I(d) - h_I(d-1)$ for $d \gg 0$. Since $\dim V^p(I + (f)) = \dim V^p(I) - 1$ (see the following lemma), by the induction hypothesis $h_{I+(f)}$ agrees with a numerical polynomial of degree $m-1$ for $d \gg 0$, hence h_I agrees with a numerical polynomial of degree m for $d \gg 0$ by proposition 6.9(ii). This proves (i) and (ii). (iii) follows immediately from proposition 6.9(i) and the observation $h_I(d) \geq 0$. □

Lecture 23
Jan 19, 2026

Lemma 6.11. *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and $f \in K[x_0, \ldots, x_n]$ a homogeneous non-constant polynomial with $X_i \not\subseteq V(f)$ for all irreducible components X_i of X. Then $\dim(X \cap V^p(f)) = \dim X - 1$.*

Proof. Let $X = X_1 \cup \ldots \cup X_r$ be the decomposition into irreducible components. Then $V(I + (f)) = \bigcup_i (X_i \cap V(f))$, so wlog X is irreducible. Also, the case $\dim X = 0$ is clear, so let $\dim X > 0$.

By corollary 4.32, $X \cap V^p(f) \neq \emptyset$. There exists an open $U \in \{U_0, \ldots, U_n\}$ such that

$$\emptyset \neq U \cap X \cap V^p(f) = (U \cap X) \cap (U \cap V^p(f)) \subseteq U \cong \mathbb{A}^n.$$

Now the result follows from the corresponding result for affine varieties, which was shown in the exercises. □

Definition 6.12. The positive integer from theorem 6.10(iii) is called the degree $\deg(I)$ of I. For a projective algebraic set, we set $\deg X := \deg I(X)$.

Example 6.13. From examples 6.3 and 6.5 it follows that

- $\chi_{\mathbb{P}^n}(t) = \binom{t+n}{n}$ and $\deg(\mathbb{P}^n) = 1$,
- $\chi_I(t) = 0$ iff $V(I) = \emptyset$,
- If $S \subseteq \mathbb{P}^n$ is finite, then $\chi_S(t) = |S|$ and $\deg(S) = |S|$. Note the same is not necessarily true for homogeneous ideals I with $V(I)$ finite, cf. example 6.5(ii). In this case one only has $I \subseteq \sqrt{I}$, so $\deg I \geq |V(I)|$.

Example 6.14. Let $X \subseteq \mathbb{P}^n$ be a linear subspace of dimension r, i.e. X is the projectivization of some linear subspace of $\mathbb{A}^{n+1}$). Then $\deg X = 1$. Indeed, $\mathrm{GL}(K, n+1)$ acts transitively on the set of subspaces of dimension $r+1$, so by remark 6.2, we may assume $X = V^p(x_{r+1}, \ldots, x_n)$. Hence $S_d \cong K[x_0, \ldots, x_r]_d$ and $\chi_X = \chi_{\mathbb{P}^r}$.

In fact, every pure-dimensional projective algebraic set of degree 1 is a linear subspace in $\mathbb{P}^n$.

Lecture 24
Jan 21, 2026

Remark 6.15. If $X, Y \subseteq \mathbb{P}^n$ have the same dimension m and do not share a common irreducible component, then $\dim X \cap Y < m$. Observe $V(I(X) + I(Y)) = X \cap Y$. By proposition 6.4 $\chi_{X \cup Y} = \chi_X + \chi_Y - \chi_{I(X)+I(Y)}$, where the last term has smaller degree than the others, so comparing leading coefficients yields $\deg(X \cup Y) = \deg X + \deg Y$.

In particular, if X is pure-dimensional and $X = X_1 \cup \ldots \cup X_n$ is the decomposition into irreducible decomponents, then $\deg X = \sum_i \deg X_i$.

Proposition 6.17 (Bézout's Theorem). *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set,* $\dim X \geq 1$, *and $X = X_1 \cup \ldots \cup X_r$ the decomposition into irreducible components. Let $f \in K[x_0, \ldots, x_n]$ be a homogeneous polynomial with $X_i \not\subseteq V^p(f)$ for all i. Then*

$$\deg(I(X) + (f)) = \deg X \cdot \deg f.$$

Proof. By lemma 6.6 we have $\chi_{I(X)+(f)}(t) = \chi_X(t) - \chi_X(t-e)$, where $e = \deg f$. Let $\chi_X(t) = \frac{c_m}{m!} t^m + O(t^{m-1})$, then

$$\chi_X(t) - \chi_X(t-e) = \frac{c_m}{m!}(t^m - (t-e)^m) + O(t^{m-2}) = \frac{c_m m e}{m!} t^{m-1} + O(t^{m-2}).$$

Hence $\deg(I(X) + (f)) = (m-1)! \frac{c_m m e}{m!} = c_m e = \deg X \deg f$. □

Remark 6.18. Let $X = \mathbb{P}^n$. Then Bezout's Theorem reads $\deg((f)) = \deg f$, motivating the terminology "degree". If X is a hypersurface, by the exercises $I(X) = (f)$ for some homogeneous polynomial f. Then $\deg X = \deg f$.

Corollary 6.19. *(i) Let $X \subseteq \mathbb{P}^n$ be a curve, and let $f \in K[x_0, \ldots, x_n]$ be a nonconstant homogeneous polynomial s.t. $X_i \not\subseteq V^p(f)$ for any irreducible component $X_i \subseteq X$. Then $0 < |X \cap V^p(f)| \leq \deg X \deg f$.*

(ii) Let $X, Y \subseteq \mathbb{P}^2$ be curves without common irreducible components. Then $|X \cap Y| \leq \deg X \deg Y$.

Proof. (i) Combine Bézout's Theorem with example 6.13. (ii) follows directly from (i) with $n = 2$ and remark 6.18 □

Proposition 6.20 (Pascal's Theorem). *Let $X \subseteq \mathbb{P}^2$ be an irreducible conic, i.e. $X = V(f)$ with $\deg f = 2$ and f irreducible. Let $A, B, C, D, E, F \in X$. Then the three intersection points $P = \overline{AB} \cap \overline{DE}$, $Q = \overline{BC} \cap \overline{EF}$ and $R = \overline{AF} \cap \overline{CD}$ of opposite edges of the hexagon $ABCDEF$ lie on a line.*

Note that since lines have degree 1 by example 6.14, they intersect in exactly one point in $\mathbb{P}^2$ by corollary 6.19. So P, Q, R in the proposition exist and are uniquely specified.

Proof. Consider the reducible cubics $X_1 = AB \cup CD \cup EF$, $X_2 = BC \cup DE \cup AF$. Let $I(X_i) = (f_i)$ and pick $S \in X \setminus \{A, B, C, D, E, F\}$. We can find $0 \neq \alpha, \beta \in K$ s.t. $(\alpha f_1 + \beta f_2)(S) = 0$. Observe that $\alpha f_1 + \beta f_2 \neq 0$, since $V(f_1) \neq V(f_2)$. Set $X' = V(\alpha f_1 + \beta f_2)$, $\deg X' \leq 3$. But now $X' \cap X$ contains A, B, C, D, E, F and S, contradicting Bezout's theorem unless X' and X share an irreducible component. But X is irreducible, so $X \subseteq X'$ and $X' = X \cup L$ for some line L. We conclude $P, Q, R \in X' \setminus X = L$. $\square$

7 Tangent Space and Smooth Varieties

Lecture 25
Jan 26, 2026

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set, $I(X) = (f_1, \ldots, f_n)$. We will define the tangent space of X at $b \in X$ as the union of all lines through b which are tangent to X. Assume for simplicity that $b = 0$. (Otherwise, one can shift b to 0 using the isomorphism $\mathbb{A}^n \to \mathbb{A}^n$, $x \mapsto x - b$.) Let $l = Ka$ be the line passing through 0 and $0 \neq a \in \mathbb{A}^n$. Then $f_i(ta_1, \ldots, ta_n)$ is a polynomial in one variable with a root at $t = 0$. Write $f_i(ta_1, \ldots, ta_n) = t^{m_i} g(t)$ with $t \nmid g(t)$.

Now define $i_b(X \cap l) := \min_i m_i \geq 1$.

Definition 7.1. A line l is called *tangent* to X at b if $i_b(X \cap l) \geq 2$. The *tangent space* $T_{X,b}$ is the union of all lines tangent to X at b.

Remark 7.2. Let again $b = 0$ for simplicity. Let $I(X) = (f_1, \ldots, f_r)$. Then let $f_i = \sum_{j=1}^d f_i^{(j)}$ be the decomposition into homogeneous parts. Then $f_i(ta_1, \ldots, ta_n) = tf_i^{(1)}(a_1, \ldots, a_n) + O(t^2)$, so $l = aK$ is tangent to X at $b = 0$ iff $f_i^{(1)}(a) = 0$ for all i. Hence $a \in T_{X,0}$ if and only if a is a root of the linear equations $f_1^{(1)}(x) = 0, \ldots, f_r^{(1)}(x) = 0$, i.e. $T_{X,0} = V(f_1^{(1)}, \ldots, f_r^{(1)})$. In particular, the tangent space is a linear subspace of $\mathbb{A}^n$.

Example 7.3.
(i) Let $X = V(y - x^2) \subseteq \mathbb{A}^2$. If $f = y - x^2$, then by the above remark, $T_{X,0} = V^p(f^{(1)}) = V^p(y)$ is the x-axis.
(ii) Let $f = y^2 - x^3$, $X = V(f) \subseteq \mathbb{A}^2$. Then $f^{(1)} = 0$, so $T_{X,0} = \mathbb{A}^2$.
(iii) More generally, let $X = V(f) \subseteq \mathbb{A}^n$ be a hypersurface. If $f^{(1)} \neq 0$, then $T_{X,0} = V(f^{(1)})$ is a hyperplane, so $\dim T_{X,0} = n - 1 = \dim X$.

Remark 7.4. Assume $b \in X$ is general, $I(X) = (f_1, \ldots, f_r)$. Then write

$$f_i(x) = f_i(\underbrace{x - b}_{=:y} + b) = f_i^{(1)}(y) + \ldots + f_i^{(d)}(y)$$

for the homogeneous decomposition w.r.t. the variables $y_i = x_i - b$. (This is exactly the Taylor expansion of f at b.) Then as before $T_{X,b} = V(f_1^{(1)}(y), \ldots, f_r^{(1)}(y))$.

Definition 7.5. With the notation above, the linear form $d_b f(x) := f^{(1)}(y)$ is called the *differential* of f at b. We have

$$d_b f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(b)(x_i - b_i)$$

From the two previous remarks, it is now clear that

Lemma 7.6. $T_{X,b} = V(d_b f_1, \ldots, d_b f_r) = b + V(\{d_b f_i(x - b)\}_i)$.

Geometrically, $T_{X,b}$ is the "best" linear approximation of $V(f_1, \ldots, f_r)$ around b.

Remark 7.7. d_b satisfies tue usual properties $d_b(f + g) = d_b f + d_b g$, $d_b(\lambda f) = \lambda d_b f$ and $d_b(fg) = f(b) d_b g + g(b) d_b f$.

Proposition 7.8. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then $T_{X,b} = V(\{d_b f \mid f \in I(X)\})$. In particular, $T_{X,b}$ does not depend on the choice of generators for $I(X)$.*

Proof. Let $I(X) = (f_1, \ldots, f_r)$. We have to show $V(d_b f_1, \ldots, d_b f_r) = V(\{d_b f \mid f \in I(X)\})$ by lemma 7.6. "$\subseteq$" is clear. Conversely, let $a \in V(d_b f_1, \ldots, d_b f_r)$ and $f \in I(X)$. Write $f = f_1 g_1 + \ldots + f_r g_r$ with $g_i \in K[x_1, \ldots, x_n]$. By remark 7.8 it immediately follows that $d_b f(a) = 0$, so we are done. ☐

Definition 7.9. A point $b \in X$ is called *smooth* or *non-singular* if $\dim T_{X,b} = \dim_b X$, where $\dim_b X$ is the maximum dimension of an irreducible component of X containing b. Otherwise b is called *singular*. X^{sm}, X^{sing} is the set of all smooth, resp. singular, points of X. X is called *smooth* or *nonsingular* if $X = X^{\text{sm}}$.

Example 7.10. Let $X = V(f) \subseteq \mathbb{A}^n$ be an irreducible hypersurface. From example 7.3(iii) we see that X is nonsingular at b if and only if $d_b f \neq 0$.

Remark 7.11. In the above example, X^{sm} is open dense in X, because it is the complement of $X^{\text{sing}} = V(f, \{\frac{\partial f}{\partial x_i}\}_i)$, and if $X^{\text{sing}} = X$, then $\frac{\partial f}{\partial x_i} \in I(X) = (f)$, which for degree reasons implies $\frac{\partial f}{\partial x_i} = 0$. But then f is a polynomial in the x_i^p, and hence $f = g^p$, contradicting the irreducibility.

Lecture 26
Jan 28, 2026

Example 7.12. Let $X = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$, $\operatorname{char} K = 0$. Then $(x, y) \in X$ is singular iff both derivates vanish at the point, i.e. iff $2y = 0 = -3x^2 - 2x$, so the only candidates are $(0, 0)$ and $(-\frac{2}{3}, 0)$, but the latter is not a point on X. Hence $X^{\text{sing}} = \{(0, 0)\}$.

Remark 7.13. In general, let $I(X) = (f_1, \dots, f_r)$ and consider the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \in K[x_1, \dots, x_n]^{r \times n}.$$

Then $T_{X,b} = b + \ker J(b)$ by lemma 7.6. Thus $\dim T_{X,b} = \dim \ker J(b) = n - \operatorname{rank} J(b)$, so b is nonsingular if and only if $\operatorname{rank} J(b) = n - \dim X_b$.

Note that for $X \neq \emptyset$, $\dim X = \dim V(I(X)) = \dim V(f_1, \dots, f_r) \geq n - r$ by exercise 10.5 Assume that $\dim X = n - r$ (such X is called a complete intersection). In this case, $b \in X$ is smooth iff $J(b)$ has full rank.

We now want to understand smoothness algebraically, i.e. translate the above to properties of $A(X)$. For $F \in K[x_1, \dots, x_n]$, $d_b F(x)$ is a linear functional, so we get a linear map $d_b : K[x_1, \dots, x_n] \to (K^n)^\vee$. We want to do something similar for $A(X)$.

Assume $F, G \in K[x_1, \dots, x_n]$ have the same class in $A(X)$, say $F = G + H$ with $H \in I(X)$. Then $d_b F = d_b G + d_b H$. Now if we restrict F, G, H to $T_{X,b} \subseteq \mathbb{A}^n$, we get $d_b H|_{T_{X,b}} = 0$ by proposition 7.8. Therefore $d_b F|_{T_{X,b}} = d_b G|_{T_{X,b}}$ and we get a well-defined linear map $A(X) \to (T_{X,b})^\vee$, $f \mapsto d_b f := d_b F|_{T_{X,b}}$, where F is some lift of f to $K[x_1, \dots, x_n]$.

Remark 7.14. Let $f, g \in A(X)$. The usual rules for differentials still hold for $d_b(f + g)$, $d_b(fg)$.

Note that $d_b g = d_b(g - g(b))$, where $(g - g(b))(b) = 0$. Therefore, restricting to such polynomials does not loose information. Consider the maximal ideals $\mathfrak{m}_b = I_X(\{b\}) = \{f \in A(X) \mid f(b) = 0\}$ and $M_b = I(\{b\}) = \{F \in K[x_1, \dots, x_n] \mid F(b) = 0\} = (x_1 - b_1, \dots, x_n - b_n)$. Obviously $\mathfrak{m}_b = M_b / I(X)$. Hence $\mathfrak{m}_b$ is generated by the classes of $x_i - b_i$ as well. Finally, we get a K-linear map $\varphi : \mathfrak{m}_b \to (T_{X,b})^\vee$, $f \mapsto d_b f$.

Theorem 7.15. *φ induces an isomorphism of K-vector spaces $\mathfrak{m}_b / \mathfrak{m}_b^2 \cong (T_{X,b})^\vee$.*

Proof. Note that $M_b \to (K^n)^\vee$ is clearly surjective, because $X_i - b_i$ maps to the standard dual basis. Hence so is $\mathfrak{m}_b \to (T_{X,b})^\vee$, since any linear map $T_{X,b} \to K$ can be extended to a linear map $\mathbb{A}^n \cong K^n \to K$.

It remains to show that $\ker \varphi = \mathfrak{m}_b^2$. For "$\supseteq$", one computes $d_b(f_1 f_2) = f_1(b) d_b f_2 + f_2(b) d_b f_1 = 0$ for $f_1, f_2 \in \mathfrak{m}_b$. Conversely, let $f \in \ker \varphi$, so that $d_b F|_{T_{X,b}} = 0$ for a lift F. Let $I(X) = (F_1, \dots, F_r)$, then $T_{X,b} = V(d_b F_1, \dots, d_b F_r)$. It follows from linear algebra that $d_b F = \lambda_1 d_b F_1 + \dots + \lambda_r d_b F_r$ for some $\lambda_i \in K$. Consider $G = F - \lambda_1 F_1 - \dots - \lambda_r F_r$, then $G(0) = 0$ and $d_b G = 0$, so in the Taylor expansion at b, $G^{(0)} = G^{(1)} = 0$, i.e. $G \in M_b^2$, hence $f = [F] = [G] \in \mathfrak{m}_b^2$. $\square$