Exercise 1

(1) see last sheet
(2) By (1), it suffices to show that $(f)^h = (f^h)$. But this follows directly from $(gf)^h = g^h f^h$.

Exercise 2

(1) It is given as a tuple of homogeneous polynomials of the same degree, so one only has to check that they don't all vanish simultaneously. So assume $x^2 = xy = y^2 = 0$ for some $[x : y] \in \mathbb{P}^1$. Then it follows immediately that $x = 0 = y$, contradiction.

(2) Consider

$$\psi : Y \to \mathbb{P}^1, \quad [X : Y : Z] \mapsto \begin{cases} [X : Y] & \text{if } X \neq 0, \\ [Y : Z] & \text{if } Z \neq 0. \end{cases}$$

Now those two cases cover all of Y, for if $X = Z = 0$, then $Y^2 = XZ = 0$, so $Y = 0$, contradiction. Further they are clearly open, and in both cases the map is given as homogeneous polynomials of the same degree. It remains to check that they agree on the overlap: If $X, Z \neq 0$, then $[X : Y] = [Y : Y\frac{Y}{X}] = [Y : Z]$. Hence ψ is regular.

We now compute

$$\psi(\varphi([x : y])) = \psi([x^2 : xy : y^2]) = \begin{cases} [x^2 : xy] & \text{if } x \neq 0 \\ [xy : y^2] & \text{if } y \neq 0 \end{cases} = [x : y]$$

$$\varphi(\psi([x : y : z])) = \begin{cases} \varphi([x : y]) = [x^2 : xy : y^2] = [x : y : \frac{y^2}{x}] = [x : y : z] & \text{if } x \neq 0, \\ \varphi([y : z]) = [y^2 : yz : z^2] = [\frac{y^2}{z} : y : z] = [x : y : z] & \text{if } z \neq 0. \end{cases}$$

So φ, ψ are mutually inverse regular maps, hence isomorphisms.

Exercise 3

By exercise 1, we have $\overline{Y_1} = V^p(X_0X_2 - X_1^2)$ and $\overline{Y_2} = V^p(X_1X_2 - X_0^2)$, hence $[X_0 : X_1 : X_2] \mapsto [X_1 : X_0 : X_2]$ defines a self-inverse isomorphism $\overline{Y_1} \to \overline{Y_2}$. On the other hand, we have $A(Y_1) = K[X_1, X_2]/(X_2 - X_1^2) \cong K[X]$, while $A(Y_2) = K[X_1, X_2]/(X_1X_2 - 1) \cong K[X, \frac{1}{X}]$. These two coordinate rings are certainly not isomorphic as K-algebras, since $K[X]$ contains no units outside K, but X is invertible in $K[X, \frac{1}{X}]$.

Exercise 4

Let $\varphi : \mathbb{P}^n \to K$ be a regular map. For some $x \in \mathbb{P}^n$, we may in an open neighbourhood U of x write $\varphi = \frac{f}{g}$ for homogeneous $f, g \in K[x_0, \dots, x_n]$ of the same degree, and $g(y) \neq 0$ for all $y \in U$. Wlog we may take f, g to be coprime. We claim that $\varphi = \frac{f}{g}$ globally.

Indeed, assume there exists $V \subseteq \mathbb{P}^n$ open and $\varphi = \frac{f'}{g'}$ on V, with f', g' as above. Since $\mathbb{P}^n$ is irreducible, U and V intersect, so on this intersection we must have $\frac{f}{g} = \frac{f'}{g'}$. But if polynomials agree on infinitely many points, they agree everywhere. hence $fg' = f'g$. By the coprimality assumption, we immediately get $f' = \varepsilon f$ and $g' = \varepsilon g$ for some unit ε, and the claim follows.

But now for $\frac{f}{g}$ to be defined everywhere, g cannot have roots. Since K is algebraically closed, g is constant. Since $\deg(f) = \deg(g)$, f, and hence φ are constant as well.