Topology I

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Contents

1 Preliminaries from Group Theory

Lecture 1
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Free groups usually do not appear in the beginner courses even though they are fundamental objects in group theory. E.g. every group is a quotient of a free group. They play an important role in geometric group theory or (low-dimensional) topology. They can be constructed as *free products* (of copies of $\mathbb{Z}$), an equally fundamental construction.

1.1 The free product of groups

Intuitively, the free product of a family of groups is the "largest group generated by them". As many basic constructions in algebra or topology, it can be elegantly characterized by a *universal property*, namely as the coproduct in the category of groups.

Definition 1.1. The *free product* of a family of groups G_ι, $\iota \in I$, is a group G together with a family of group homomorphisms $\varphi_\iota : G_\iota \to G$ such that the following universal property holds:

$$\begin{tikzcd} G_\iota \arrow[r, "\varphi_\iota"] \arrow[dr, "\psi_\iota"'] & G \arrow[d, dashed, "\exists!\psi"] \\ & H \end{tikzcd}$$

For every family of homomorphisms $\psi_\iota : G_\iota \to H$ into some group H, there exists a unique group homomorphism $\psi : G \to H$ such that $\psi_\iota = \psi \circ \varphi_\iota$ for all $\iota \in I$.

Notation The free product will be denoted as $\ast_{\iota\in I}\, G_\iota$, or $G_1 * \ldots * G_n$ in the finite case.

The *uniqueness* of the free product of groups up to (unique) isomorphism follows from general arguments (sometimes referred to as *general* or *abstract nonsense*), independent of the category (applying to the coproduct in any category):

Consider two free products $(\varphi_\iota : G_\iota \to G)_{\iota\in I}$ and $(\varphi'_\iota : G_\iota \to G')_{\iota\in I}$ of the family $(G_\iota)_{\iota\in I}$.

$$\begin{tikzcd} & G \arrow[d, "\psi"] \arrow[dd, bend left, dashed, "\mathrm{id}_G"] \\ G_\iota \arrow[ur, "\varphi_\iota"] \arrow[r, "\varphi'_\iota"] \arrow[dr, "\varphi_\iota"] & G' \arrow[d, "\psi'"] \\ & G \end{tikzcd}$$

By the universal properties, we obtain maps ψ and ψ' as in the diagram. Applying uniqueness to the big triangle, we see that $\psi' \circ \psi$ is the *unique* map satisfying the universal property for $G \to G$. But the identity map clearly does as well, hence $\psi' \circ \psi = \mathrm{id}_G$, and in the same way one sees $\psi \circ \psi' = \mathrm{id}_{G'}$. Hence $G \cong G'$.

However, the *existence* of the free product is nontrivial (The existence of a coproduct depends on the category).

Immediate requirements for the free product as a consequence of the universal property:

- The φ_ι are injective (i.e. embeddings), so we can think of the G_ι as subgroups of G.
- The subgroups G_ι generate G.

Thus every element in the free product has a representation as a product of the form $g_1 \cdots g_n$ with $n \in \mathbb{N}_0$ and $g_i \in G_{\iota_i} - \{1\}$ such that $\iota_i \neq \iota_{i+1}$ for all $1 \leq i \leq n-1$. This form is called *reduced*.

Not clear right away: The free product being the "largest gorup generated by its factors" should mean that the factors are "algebraically independent" in the sense that group elements have *unique representations* as reduced products.

First proof of existence For a family of families

$$(\psi_{\iota\kappa} G_\iota \to H_\kappa)_{\iota \in I}, \quad \kappa \in K$$

the mapping problem is solved by the family of induced homomorphisms into the direct product of the groups H_κ.

$$\begin{tikzcd}
G_\iota \arrow[r, "(\varphi_{\iota\kappa})_\kappa"] \arrow[dr, "\varphi_{\iota\kappa_0}"'] & \prod_{\kappa \in K} H_\kappa \arrow[d, "\mathrm{proj}_{\kappa_0}"] \\
& H_{\kappa_0}
\end{tikzcd}$$

To achieve uniqueness, replace $\prod_\kappa H_\kappa$ by the subgroup generated by the images of all $(\varphi_{\iota\kappa})_\kappa$. This family can be regarded as an "approximation" of the free product of the G_ι.

Now note that the universal property needs only be checked for families $(\psi_\iota : G_\iota \to H)_\iota$ whose images generate H. The equivalence classes of such families form a set, because the size of H is restricted in terms of the sizes of I and the G_ι. Apply the above-mentioned construction to a family of representatives of these equivalence classes. □ Lecture 2 Oct 15, 2025

This proof, while quite general, reveals very little about the structure of the free product. We now give an explicit construction of the free product, which clarifies its structure.

Second proof of existence One can construct the free product as an abstract group with underlying set the reduced words, as defined above, and concatenation plus reduction as the group operation. However, verifying associativity is complicated due to possible cancelations.

It is simpler to onstruct the free product as a group of symmetries of a combinatorial object (as a permutation group). Take W to be the set of reduced words $(g_1, \ldots, g_n)$ with $n \in \mathbb{N}_0$, $g_i \in G_{\iota_i} - \{1\}$ and $\iota_i \neq \iota_{i+1}$ for all $1 \leq i < n$. For every $\iota \in I$, define an action of G_ι on W by defining $g \cdot (g_1, \ldots, g_n)$ as the reduction of the word $(g, g_1, \ldots, g_n)$. It is easy to see that this is indeed an action.

These actions are clearly faithful (effective), even free[1], and yield embeddings $\varphi_\iota : G_\iota \hookrightarrow S(W)$ into the symmetric group of W. Take $G < S(W)$ to be the subgroup generated by the images of the φ_ι. Observe that for the action G on W it holds that $(g_1 \cdots g_n) \cdot () = (g_1 \cdots g_n)$ for a reduced product in G. This shows that different reduced products act by different permutations of W, and therefore are different group elements. In other words, the elements in G have *unique* representations as reduced products. In this sense, the G_i are "algebraically independent".

The universal property is now a direct consequence of the uniqueness of reduced product representations. With the notation as in definition 1.1, the only possibility to define ψ on reduced words is as $\psi(g_1 \cdots g_n) := \prod_{i=1}^n \psi_{\iota_i}(g_i)$, where $g_i \in G_{\iota_i}$. By the uniquenss of reduced representations, this is well-defined and clearly makes the necessary diagram commute. It remains to see that the map ψ is multiplicative and hence a group homomorphism.

Let $g_1 \cdots g_n$ and $g'_k \cdots g'_1 \in W$. There is a maximal index m with $0 \leq m \leq n, k$ s.t. $\iota'_j = \iota_j$ and $g'_j g_j = 1$ for $1 \leq j \leq m$. Then either the product $g'_k \cdots g'_{m+1} g_{m+1} \cdots g_n$ obtained from the full unreduced product by m cancellations is reduced (i.e. $\iota'_{m+1} \neq \iota_{m+1}$ or $m = \min(n,k)$), or

[1] faithful = nontrivial elements act nontrivially, full = nontrivial elements have no fixed points

$m < \min(n, k)$ and $\iota'_{m+1} = \iota_{m+1}$, $g'_{m+1}g_{m+1} \neq 1$, in which case $g'_k \cdots (g'_{m+1}g_{m+1}) \cdots g_n$ is reduced. In both cases, multiplicativity is clear. □

Remark 1.2. The action of G on W is simply transitive, because the empty word has trivial stabilizer and point stabilizers along orbits are conjugate to each other. It extends to an action of G on a coloured graph with vertices W. To construct it, connect $()$ to (g) for $g \in G_\iota - \{1\}$ by an edge of colour ι. Extend this in a G-invariant way by connecting $(g_1, \dots, g_n)$ to $(g_1, \dots, g_{n-1}, g)$ with $g \in G_{\iota_n} - \{1, g_n\}$, and $(g_1, \dots, g_{n-1})$ by edges of colour ι_n, and to $(g_1, \dots, g_n, g)$ for all $g \in G_\iota - \{1\}$, $\iota \neq \iota_n$ by an edge of colour ι.

This homogeneous graph is in general not a tree, but it has tree-like structure, since vertices disconnect, e.g. when removing the empty word, the connected components correspond to the factors G_ι, determined by the colour" of the first letter of their vertices.

Note that in general, G is *not* the full group of symmetries of this coloured graph.

Example 1.3. Consider the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty = \mathrm{Isom}(\mathbb{Z})$: Let a, b be the reflections of $\mathbb{Z}$ around 0 and $\frac{1}{2}$, respectively. Send the generators of the two copies of $\mathbb{Z}/2\mathbb{Z}$ to a and b to get a surjective map $\alpha : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to D_\infty$, which one checks to be injective by computing the images of 0 and 1.

1.2 Free Groups

Lecture 3
Oct 20, 2025

Intuitively, the free group on a given set is the "largest group generated by it". It is defined via a universal property:

Definition 1.4. The *free group* on a set S is a group $F = F(S)$ together with a map $\varphi : S \to F$ such that the following universal property holds:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & F \arrow[d, dashed, "\exists!\widehat{\psi}"] \\ & H \end{tikzcd}$$

For any map $\psi : S \to H$ into some group H, there exists a unique group homomorphism $\widehat{\psi}$: $F(S) \to H$ such that $\psi = \widehat{\psi} \circ \varphi$.

In other words, the free group constitutes a left adjoint to the forgetful functor from groups to sets. As before, this defines the free group uniquely up to isomorphism(, provided it exists). The existence can be established by constructing them as free products:

Existence Let S be a set. Consider the map $S \to *_{s \in S} \mathbb{Z}_s =: F(S)$, $s \mapsto 1_s$, where $\mathbb{Z}_s$ is a copy of $\mathbb{Z}$ indexed by s and 1_s is the unit in that copy. The universal property of free products implies that φ satisfies the universal property of a free group: A map $\psi : S \to H$ corresponds to a family of group homomorphisms $\psi_s \mathbb{Z}_s \to H, 1_s \mapsto \psi(s)$. The universal property of free products yields a map $\widehat{\psi} : F(S) \to H$ such that $\psi_s = \widehat{\psi} \circ \varphi_s$. Restricting those maps to S shows $\psi = \widehat{\psi} \circ \varphi$, as desired. □

We also obtain the following information on the structure of free groups: $\varphi : S \to F(S)$ is injective, so we can think of S as a subset of $F(S)$, and the elements of S generate $F(S)$. By our analysis of free products, the elements in $F(S)$ have *unique reduced normal forms* $s_1^{n_1} \cdots s_k^{n_k}$ with $s_i \in S, n_i \in \mathbb{Z} - \{0\}$ and $s_i \neq s_{i+1}$ for $1 \leq i < k$. In this sense, the generators $s \in S$ are "algebraically independent" in $F(S)$.

The free group can be realized (and could in fact also be constructed) as the *full* group of symmetries of a combinatorial object, namely of a coloured oriented tree. Consider the graph

$G = G(S)$ with vertex set $F = F(S)$ and edges $(\gamma, \gamma s)$ for all $\gamma \in F$ and $s \in S$ with colour s, as in remark 1.2. At any point $\gamma \in F$, there will be $|S|$ edges going away and towards γ each. Left multiplication gives a natural action of F on G, which is simply transitive on vertices. One easily sees that this is the full group of symmetries (respecting orientation and colours). An (unoriented) path $\gamma_0, \gamma_1 = \gamma_0 s_1^{\varepsilon_1}, \ldots, \gamma_n = \gamma_0 s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ is "locally embedded" in the sense that $\gamma_{i-1} \neq \gamma_{i+1}$ for all $1 \leq i \leq n-1$ if and only if $s_i^{\varepsilon_i} s_{i+1}^{\varepsilon_{i+1}}$, and in this case $s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ is a reduced product (i.e. no cancellations can occur). The uniqueness of reduced product representations in the free group implies that there are no cycles in the graph G, i.e. G is indeed a tree.

Remark 1.5. This gives an alternate geometric way of proving the existence of free groups: First construct the tree (either inductively or via covering theory as the universal cover of a bouquet of circles), then look at its automorphism group and prove that it satisfies the universal property. We will come back to this later.

Next we will define the *rank* of a free group by abelianizing it. We recall that the *abelianization* of a group G is the left adjoint of the inclusion from abelian groups to groups, obtained by dividing out its commutator subgroup $G^{\mathrm{ab}} := G/[G,G]$, thereby introducing commutation relations between all elements. It is the largest abelian quotient of G.

Abelianizing the free group $F(S)$ on a set S yields the free abelian group with basis S, $F^{\mathrm{ab}}(S) = \bigoplus_{s \in S} \mathbb{Z}_s$. We will think of elements of this direct sum as finite sums $\sum_{s \in S} m_s s$. Indeed, the homomorphism $\ast\, \mathbb{Z}_s \to \bigoplus \mathbb{Z}_s$ defined by $1_s \mapsto s$, i.e. $s_1^{m_1} \cdots s_n^{m_n} \mapsto \sum m_i s_i$ descends to an isomorphism $F(S)^{\mathrm{ab}} \to F^{\mathrm{ab}}(S)$, whose inverse is given by $\sum m_i s_i \mapsto [s_1^{m_1} \cdots s_n^{m_n}]$.

We know that the rank free abelian groups is well-defined as the size of a basis. Therefore, the following definition yields a well-defined invariant of free groups:

Definition 1.6. The rank of a free group on a set S is defined to be the cardinality $|S|$.

1.3 Group Presentations

A presentation of a group is a description in terms of generators and relations. Presentations always exist, but they are highly non-unique and in general it is difficult to derive actual information about the group from them. Lecture 4 Oct 22, 2025

Every group Γ is a quotient of a free group: Take $S \subseteq \Gamma$ to be a generating set of Γ. Then the morphism $f : F(S) \twoheadrightarrow \Gamma$ extending id_S is surjective. In other words, f is the map that sends a word in the $s_i \in S$ to its product evaluated in Γ. An element $r = s_1^{m_1} \cdots s_n^{m_n} \in \ker f$ is called a *relation* between the generators in S. It corresponds to an equation $f(r) = s_1^{m_1} \cdots s_n^{m_n} = 1$ in Γ. Hence the free group provides a natural space for parametrizing possible relations between the elements of S in Γ.

If $R \subseteq \ker f \triangleleft F(S)$ is a set of relations, then an element r' in the normal subgroup $N(R) \triangleleft F(S)$ generated by R is called a *consequence* of the relations in R, since the equation in Γ corresponding to r' can be algebraically deduced from the equations corresponding to R. A set $R \subseteq \ker f$ of relations which generates $\ker f$ as a normal subgroup of $F(S)$ is called a *complete* set of relations for Γ. The homomorphism f then descends to the isomorphism $F(S)/N(R) \xrightarrow{\cong} \Gamma$.

This gives a description of Γ in terms of generators $S \subseteq \Gamma$ and relations $R \subseteq F(S)$ called a presentation.

Definition 1.7. For sets S and $R \subseteq F(S)$ define the group $\langle S \mid R \rangle := F(S)/N(R)$ *presented* by generators S and relations R. Given a group Γ, an isomorphism $\Gamma \cong \langle S \mid R \rangle$ for S, R as above is called a *presentation* of Γ.

A group admitting a presentation with finitely many generators and relations is called *finitely presented.*

Every group admits many different presentations. Unfortunately, some basic questions are often hard or impossible to answer, e.g. it is undecidable whether two given presentations describe the same group, or even whether a given presentation describes the trivial group. Also, given a presentation and a word in the generators, it is undecidable in general whether the corresponding group element is trivial.

Example 1.8. (i) $F(S) = \langle S \mid \emptyset \rangle$.

(ii) $\langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle \cong \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \rangle =: G$ canonically: For $i = 1, 2$ we have a morphism $F(S_i) \to G$ by the universal property, which sends R_i to 1. Hence we get morphisms $\varphi_i : \langle S_i \mid R_i \rangle \to G$ by the universal property of quotients.

Let $\psi_i : \langle S_i \mid R_i \rangle \to H$ be some morphisms. Similarly to before, let $F(S_1 \sqcup S_2) \to H$ be defined as $s_i \mapsto \psi_i(s_i)$ for $s_i \in S_i$. Again this map sends $R_i \mapsto 1$, so it descends to a morphism $G \to H$. Since we had no choice in the definition of this map, it is unique. (This gives an alternative way of proving the existence of free products, based on the existence of free groups).

(iii) There is a canonical isomorphism $\langle S \mid R \cup R' \rangle \cong \langle S \mid R \rangle / N(\pi(R'))$, where $\pi : F(S) \to \langle S \mid R \rangle$ is the natural quotient map: Both morphisms $F(S) \to \langle S \mid R_1 \cup R_2 \rangle, \langle S \mid R \rangle / N(\pi(R'))$ send $R \cup R'$ to 1, so we get induced morphisms in both directions which are clearly inverse to each other.

(iv) $\langle S \mid R \rangle^{\text{ab}} \cong \langle S \mid R \cup \{[s_1, s_2] \in F(S) \mid s_1, s_2 \in S\} \rangle$ by (iii).

(v) $\mathbb{Z}/n\mathbb{Z} \cong \langle a \mid a^n \rangle$

(vi) $\mathbb{Z}^2 \cong \langle a, b \mid [a, b] \rangle$

(vii) $D_n \cong \langle a, b \mid a^n, b^2, baba \rangle =: \Gamma$ where a represents a rotation by $\frac{2\pi}{n}$, and b a reflection: Indeed, by universal properties we get a surjective map $\Gamma \to D_n$. In Γ it holds that $ba^k = a^{-k}b$, so every element can be written as $a^i b^j$ with $0 \le i < n$ and $0 \le j < 2$, i.e. $|\Gamma| \le 2n$.

1.4 Free products with amalgamations

With regard to the Seifert-van Kampen theorem for fundamental groups we discuss a generalization of the free product of (two) groups. Consider two group homomorphisms $\alpha_i : H \to G_i$. We want to construct the "largest group generated by G_1 and G_2 where G_1 and G_2 are amalgamated along H", meaning that $\alpha_1(H) \subseteq G_1$ and $\alpha_2(H) \subseteq G_2$ should be identified. More precisely, we have a universal property. By a solution of (α_1, α_2) we mean an extension to a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] & K \\ H \arrow[u, "\alpha_1"] \arrow[r, "\alpha_2"] & G_2 \arrow[u, "\varphi_2"'] \end{tikzcd}$$

Definition 1.9. The amalgamated product (or pushout, or fibered coproduct) $G_1 *_{(H,\alpha_1,\alpha_2)} G_2 = G_1 *_H G_2$ of α_1, α_2 is a solution $(K, \varphi_1, \varphi_2)$ such that for every solution (K', ψ_1, ψ_2) there exists a unique morphism $\psi : K \to K'$ such that $\psi_i = \psi \circ \varphi_i$.

If $H = 1$, this is exactly the free product. In general, the free product with amalgamation can easily be constructed using presentations (see exercise).

2 Fundamental Group and Covering Spaces

2.1 Homotopy

Lecture 5
Oct 27, 2025

Definition 2.1. Let $f, g : X \to Y$ be continuous maps of topologial spaces. A *homotopy* from f to g is a continuous map $H : X \times [0,1] \to Y$ s.t. $H(-,0) = f$ and $H(-,1) = g$. If $A \subseteq X$ is a subspace and $f|_A = g|_A$, then a homotopy H is a called a homotopy from f to g *relative* A, if in addition it is stationary on A, i.e. $H(a,-)$ is constant for all $a \in A$. If a homotopy (relative to A) exists, the maps f, g are called *homotopic* (relative to A), write $f \simeq g$ (or $f \simeq_A g$).

We may think of a homotopy as a "continuous deformation" of maps in the sense that it connecets $f = f_0$ and $g = f_1$ by the "continuous family" of maps $f_t = H(-,t)$.

Remark 2.2. (i) Compositions of homotopic maps are homotopic: Let $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$. (Exercise)

(ii) Being homotopic is an equivalence relation on the set of continuous maps $X \to Y$: Reflexivity is clear, for symmetry take $H^- : x, t \mapsto H(x, 1-t)$. Let $f \simeq g \simeq h : X \to Y$ be homotopic, with homotopies H_1 and H_2, respectively. Then

$$H : X \times [0,1] \to Y, \qquad (x,t) \mapsto \begin{cases} H(x, 2t) & \text{if } t \le \frac{1}{2}, \\ H(x, 2t-1) & \text{otherwise.} \end{cases}$$

is a homotopy from f to h. The equivalence classes are called *homotopy classes*.

Definition 2.3. Continuous maps homotopic to a constant map are called *nullhomotopic*.

Remark 2.4. A map $X \to Y$ is nullhomotopic if and only if it continuously extends to the cone $CX := X \times [0,1]/X \times \{1\}$ of X. For example, if $X = S^n$, then $CX \cong D^{n+1}$

Definition 2.5. A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there exists a *homotopy inverse*, i.e. a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Two topological spaces X, Y are *homotopy equivalent* if there exists a homotopy equivalence $X \to Y$, write $X \simeq Y$.

Remark 2.6. (i) Being homotopy equivalent is an equivalence relation on topological spaces by remark 2.2. Equivalence classes are called *homotopy types*.

(ii) Homeomorphisms are homotopy equivalences.

Definition 2.7. A topological space is *contractible* if it is homotopy equivalent to "the" point. Equivalently, id_X is nullhomotopic.

Example 2.8. If $W \subseteq \mathbb{R}^n$ is star-shaped relative to $w_0 \in W$ (that is, for every $w \in W$, the line segment connecting w_0 and w lies completely in W), then id_W is homotopic to the constant map c_{w_0} with value w_0 via the homotopy $H(w,t) = (1-t)w + tw_0$. Hence W is contractible.

Definition 2.9. A *retraction* of a topological space X onto a subspace $A \subseteq X$ is a continuous map $r : X \to A$ with $r|_A = \mathrm{id}_A$. Equivalently, it is a continuous map $r : X \to X$ with $r \circ r = r$ and $r(X) = A$.

Retractions are the topological version of projections. Intuitively, a topological space is topologically "at least as complicated" as any of its retracts. A retract is a sbuspace whose position inside the ambient space is "as simple as possible".

Definition 2.10. A (strong) *deformation retraction* of a topological space X onto a subspace $A \subseteq X$ is a homotopy relative A from id_X to a retraction onto A. In this case, one calls A a (strong) deformation retract of X.

Note that if a retraction is (the final map of) a deformation retraction, then it is a homotopy inverse to the inclusion $A \hookrightarrow X$, in particular a homotopy equivalence.

Remark 2.11. (i) If X (strongly) deformation retracts to a point, then X is contractible. Note that the converse fails, see exercises.

(ii) Consider the punctured disk $D^n \setminus \{0\}$. It (strongly) deformation retracts to $\partial D^n = S^{n-1}$. We will see later that D^n does not deformation retract to its boundary.

(iii) The mapping cylinder M_f of a continuous map $f : X \to X$,

$$M_f := (X \times [0,1] \sqcup Y)/((x,1) \sim f(x))$$

naturally deformation retracts to Y.

2.2 Homotopy of Paths and the Fundamental Group

A *path* in a topological space X is a continuous map from an interval into X. We call a path $\gamma : [a,b] \to X$ a path from $\gamma(a)$ to $\gamma(b)$. We will now consider paths parametrized by $I = [0,1]$.

Two paths $\gamma_0, \gamma_1 : I \to X$ from x to y are called homotopic if as maps they are homotopic relative to ∂I, i.e. if there eists a continuous map $H : I \times I \to X$ such that $\gamma_0 = H(-,0)$, $\gamma_1 = H(-,1)$, $H(0,-) = x$ and $H(1,-) = y$. Homotopy of paths is an equivalence relation (as above).

The *concatenation* $\alpha * \beta : I \to X$ of paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$ is defined as Lecture 6 Oct 29, 2025

$$\alpha * \beta(s) := \begin{cases} \alpha(2s) & \text{if } s \leq \frac{1}{2}, \\ \beta(2s-1) & \text{otherwise.} \end{cases}$$

This partially defined "product" on paths induces one on homotopy classes of paths due to

Lemma 2.12. *For paths $\alpha_0, \alpha_1 : I \to X$ from x to y and paths $\beta_0, \beta_1 : I \to X$ from y to z, it holds that if $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$, then $\alpha_0 * \beta_0 \simeq \alpha_1 * \beta_1$*

Proof. If $H_\alpha, H_\beta : I \times I \to X$ are the respective homotopies, then

$$H : I \times I \to X, \quad (s,t) \mapsto \begin{cases} H_\alpha(2s,t) & \text{if } s \leq \frac{1}{2}, \\ H_\beta(2s-1,t) & \text{otherwise} \end{cases}$$

is a homotopy from $\alpha_0 * \beta_0$ to $\alpha_1 * \beta_1$, which is continuous since homotopies of paths are relative to the endpoints by definition. □

The partial "product" on homotopy classes induced by concatenation is then defined as $[\alpha] \cdot [\beta] := [\alpha * \beta]$, if $\alpha(1) = \beta(0)$. In contrast with the concatenation, the product on homotopy classes has good algebraic properties, because reparametrization yields homotopic paths:

Lemma 2.13. *If $\psi : I \to I$ is a continuous map with $\psi(0) = 0$ and $\psi(1) = 1$, then for every path $\gamma : I \to X$, it holds that $\gamma \circ \psi \simeq \gamma$.*

Proof. By linear interpolation, $\psi \simeq \mathrm{id}_I$, hence $\gamma \circ \psi \simeq \gamma$ by remark 2.2(i) □

Let $c_x : I \to X$ denote the constant path with value $c_x(s) = x \in X$.

Proposition 2.14. *For the product on homotopy classes of paths it holds that*

(i) $[c_{\gamma(0)}][\gamma] = [\gamma] = [\gamma][c_{\gamma(1)}]$

(ii) $[\gamma] \cdot [\gamma^-] = [c_{\gamma(0)}]$ *and* $[\gamma^-] \cdot [\gamma] = [c_{\gamma(1)}]$, *where* $\gamma^- := \gamma(1 - \cdot)$

(iii) $([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]([\gamma_2][\gamma_3])$ *where defined.*

Proof. For (i), apply the lemma with $\psi(t) = \max(2t-1, 0)$ and $\min(2t, 1)$, respectively. For (ii), write $\gamma * \gamma^- = \gamma \circ \varphi$ with $\varphi(t) = -|2t-1| + 1$, which is affinely homotopic to the zero map. For (iii), we want to see that $(\gamma_1 * \gamma_2) * \gamma_3 = (\gamma_1 * (\gamma_2 * \gamma_3)) \circ \psi$ to again conclude with the lemma. Here

$$\psi(s) = \begin{cases} 2s & \text{if } s \leq \frac{1}{4}, \\ s + \frac{1}{4} & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2}, \\ \frac{1}{2}s + \frac{1}{2} & \text{otherwise.} \end{cases}$$

□

When restricting the product to homotopy classes of *loops* with a fixed base point $x_0 \in X$, then it is always defined and the above proposition shows that it enjoys all properties of a group multiplication. The neutral element is $[c_{x_0}]$, and the inverse of $[\gamma]$ is $[\gamma^-]$.

Definition 2.15. Let X be a topological space, and let $x_0 \in X$ a base point. The group $\pi_1(X, x_0)$ of homotopy classes of loops with base point x_0 is called the *fundamental group* of X with base point x_0.

We describe the dependence on the base point:

Proposition 2.16. *If α is a path from x_0 to x_1, then*

$$C_\alpha : \pi_1(X, x_1) \to \pi_0(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \alpha^-] = [\alpha][\gamma][\alpha]^{-1}$$

is a group isomorphism.

Proof. C_α is just conjugation by $[\alpha]$, so it is clearly a well-defined bijective homomorphism. □

Note that for $x_0 = x_1$, then the C_α are exactly the inner automorphisms of $\pi_1(X, x_0)$.

Thus the fundamental groups for base points in the same path component of X are isomorphic to each other. Among them, there are natural isomorphisms which are unique up to conjugation.

If X is path-connected, we may thus of "the" fundamental group of X, meaning the isomorphism type of any $\pi_1(X, x)$, and use the notation $\pi_1(X)$ without base point.

If $\pi_1(X)$ is abelian, then inner automorphisms are trivial, so the identifications C_α are independent of α and hence canonical.

Functoriality Given a continuous map $f : X \to Y$, and $x_0 \in X$, we can associate to it a morphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)), \qquad [\gamma] \mapsto [f \circ \gamma].$$

It is clear that $\mathrm{id}_* = \mathrm{id}$ and $(g \circ f)_* = g_* \circ f_*$, so the fundamental group, with this assignment of morphisms, is a functor $\mathbf{Top}_* \to \mathbf{Grp}$. In particular, it is a topological invariant.

More is true: The fundamental group is a homotopy invariant. To see this, assume we have Lecture 7 Nov 3, 2025

$H : X \times [0,1] \to Y$ be a homotopy from $f_0 = H(-,0)$ to $f_1 = H(-,1)$. Then for any point $x \in X$ we have the following diagram

$$\begin{tikzcd} & \pi_1(Y, f_0(x)) \arrow[dd, "\cong"', "C_{H(x,-)}"] \\ \pi_1(X,x) \arrow[ur, "(f_0)_*"] \arrow[dr, "(f_1)_*"'] & \\ & \pi_1(Y, f_1(x)) \end{tikzcd}$$

It commutes, since

$$H(x,-) * (f_1 \circ \gamma) * H(x,-)^- \simeq c_{f_0(x)} * (f_0 \circ \gamma) * c_{f_0(x)} \simeq f_0 \circ \gamma$$

via the homotopy $h(\cdot,t) = H(x,t\cdot) * (H(\cdot,t) \circ \gamma) * H(x,t\cdot)^-$. It follows that the fundamental group is homotopy invariant:

Theorem 2.17. *Let $f : X \to Y$ be a homotopy equivalence, and let $x \in X$. Then $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$ is an isomorphism.*

Proof. Let g be a homotopy inverse of f. Then by the previous observation, $g \circ f \simeq \mathrm{id}_X$ gives the commutative diagram

$$\begin{tikzcd} & \pi_1(X, x) \arrow[dd, "\cong"'] \\ \pi_1(X,x) \arrow[ur, "(\mathrm{id}_X)_*"] \arrow[dr, "(g\circ f)_*"'] & \\ & \pi_1(X, (g \circ f)(x)) \end{tikzcd}$$

Hence $g_* \circ f_* = (g \circ f)_*$ is the composition of two isomorphisms, hence an isomorphism. Thus f_* is injective and g_* is surjective. Repeating the argument with $f \circ g$ shows that f_*, g_* are isomorphisms. □

Next we will explore situations where the fundamental group is trivial.

Definition 2.18. A path connected (sometimes called 0-connected) space is called *simply connected* (or 1-connected) if its fundamental group is trivial.

Remark 2.19. For a path connected space X, the following properties are equivalent:

(i) X is simply connected

(ii) If $\gamma_0, \gamma_1 : I \to X$ are two paths with the same start- and endpoint, then they are homotopic.

(iii) Every continuous map $S^1 \to X$ is nullhomotopic.

(iv) Every continuous map $S^1 = \partial D^2 \to X$ extends to a continuous map $D^2 \to X$.

In some cases, simple connectivity is easy to detect:

Proposition 2.20. *If X is contractible, then X is simply connected.*

Proof. By homotopy invariance, $\pi_1(X) \cong \pi_1(\mathrm{pt}) = 1$. □

Theorem 2.21. *The spheres S^n, $n \geq 2$, are simply connected.*

Proof. Exercise. □

Proposition 2.22 (π_1 of products)**.** *For pointed spaces $(X,x), (Y,y)$, the natural homomorphism*

$$\pi_1(X,x) \times \pi_1(Y,y) \to \pi_1(X \times Y, (x,y))$$

is an isomorphism.

Proof. Clear. □

2.3 The Fundamental Group of the Circle

To verify the nontriviality of fundamental groups and to compute them is a greater challenge. For this one can often use covering space theory, which we will study later. We first treat a basic special case, the circle.

The circle can be obtained as a quotient of $\mathbb{R}$ by a discrete group of translations. In other words, $p : \mathbb{R} \to S^1 \subseteq \mathbb{C}$, $t \mapsto \exp(2\pi i t)$ descends to a continuous bijection $\mathbb{R}/\mathbb{Z} \to S^1$. Recall that a continuous bijection from a compact to a Hausdorff space is a homeomorphism, hence $S^1 \cong \mathbb{R}/\mathbb{Z}$. In other words, p is a quotient map.

Observe that p is injective on subsets of $\mathbb{R}$ contained in open intervals I of length at most 1. For such a subset $O \subseteq I$ it holds that $f(O)$ is open in S^1 if and only if $p^{-1}(p(O)) = O+\mathbb{Z} = \bigsqcup_n O+n$ is open in $\mathbb{R}$, i.e. if and only if $O \subseteq \mathbb{R}$ is open. Hence $p|_I$ is a homeomorphism onto an open arc, so p is a local homeomorphism.

Define now the *winding number* of loops in S^1 by measuring the "total change of the argument" along the loop: Let $\gamma : [0,1] \to S^1$ be a loop, fix $0 < \varepsilon \ll 1$ and take a sufficiently fine subdivision $0 = s_0 < s_1 < \ldots < s_n = 1$ of $[0,1]$ such that $\gamma([s_{k-1}, s_k])$ is contained in an open arc $p(I_k)$ for an open interval $I_k \subseteq \mathbb{R}$ of length at most ε. Using the local invertability of p, we can inductively lift the sequence of subdivision points $\gamma(s_k)$ to $a_k \in \mathbb{R}$ such that $|a_k - a_{k-1}| < \varepsilon$. We define the winding number of γ as $w(\gamma) := a_n - a_0 \in \mathbb{Z}$.

To obtain a well-defined quantity, we have to check that $w(\gamma)$ does not depend on the subdivision. Given another subdivision, it suffices to compare both to a common refinement, hence by induction to the subdivision with a single point $s_{k-1} < s' < s_k$ added. Then the lift a' of s' lies in I_k, so one chooses I_k for both the intervals $[s_{k-1}, s']$ and $[s', s_k]$, for a total contribution of $a' - a_{k-1} + a_k - a' = a_k - a_{k-1}$. Therefore, the winding number is well-defined.

Furthermore, the winding number is invariant under homotopies: Let $H : I^2 \to S^1$ be a homotopy of loops. Let s_k, a_k, I_k be as in the construction above, for the definition of $w(H(-, t_0))$. Since H is continuous and $p(I_k)$ is open, there exists $\delta_k > 0$ such that $H([s_{k-1}, s_k] \times (t_0 - \delta_k, t_0 + \delta_k) \cap [0,1]) \subseteq p(I_k)$. Hence we can use this subdivision also for the loops $\gamma_t = H(-,t)$ with $|t - t_0| < \min_k(\delta_k)$. Using the local inverses $(p|_{I_k})^{-1}$ of p, we lift the sequences of subdivision points $\gamma_t(s_k)$ to $a_k(t) := (p|_{I_k})^{-1}(\gamma_t(s_k)) = (p|_{I_{k+1}})^{-1}(\gamma_t(s_k))$. Hence a_k, and thus w, varies continuously with t, and the latter takes values in the discrete space $\mathbb{Z}$, so w is constant. Lecture 8 Nov 5, 2025

Therefore, the winding number descends to a well-defined function on homotopy classes of loops $w([\gamma]) := w(\gamma)$. The winding number is clearly additive (combine subdivisions), i.e. $w(\gamma_1 * \gamma_2) = w(\gamma_1) + w(\gamma_2)$ for γ_1, γ_2 loops in S_1 with the same basepoint. Fixing the base point $1 \in S^1 \subseteq \mathbb{C}$, we thus obtain a well-defined homomorphism $w : \pi_1(S^1, 1) \to \mathbb{Z}$. It is clearly surjective, since $\gamma_n(t) = e^{2\pi i n t}$ has winding number n. We show that it is also injective:

Lemma 2.23. *Loops $\gamma : [0,1] \to S^1$ with $w(\gamma) = 0$ are nullhomotopic, $\gamma \simeq c_{\gamma(0)}$.*

Proof. Wlog $\gamma(0) = 1$. Returning to the definition of the winding number, we now lift not only the subdivision points, but the entire loop γ to a path $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ with $p \circ \widetilde{\gamma} = \gamma$.

Now assume $w(\gamma) = 0$, then $\widetilde{\gamma}$ is a loop in $\mathbb{R}$. Since loops in $\mathbb{R}$ are nullhomotopic, the same follows for the image loop γ. □

We have shown the following

Theorem 2.24. *The natural homomorphism $w : \pi_1(S^1, 1) \to \mathbb{Z}$ given by the winding number is an isomorphism. Thus $\pi_1(S^1) \cong \mathbb{Z}$ and the loop $\gamma(t) = e^{2\pi i t}$ represents a generator.*

Remark 2.25. (i) With the path integral from complex analysis, lifts $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ of paths $\gamma : [0,1] \to S^1$ may be written as

$$\widetilde{\gamma}(s) = \widetilde{\gamma}(0) + \frac{1}{2\pi i} \int_{\gamma|_{[0,s]}} \frac{dz}{z}$$

and the winding number of a loop $\gamma : [0,1] \to \mathbb{R}$ is given by $w(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}$.

(ii) There is an analogous isomorphism $\pi_1(S^1, \zeta) \to \mathbb{Z}$ provided by the winding number for every base point $\zeta \in S^1$. The winding numbers for different base points are consistent with the canonical identifications of fundamental groups under change of base point (which are induced by rotation).

(iii) Since C^* is homotopy equivalent to S^1 (via inclusion and radial projection/retraction), there are isomorphisms of fundamental groups $\pi_1(S^1, \zeta) \cong \pi_1(\mathbb{C}^*, \zeta)$, hence we can speak of the winding number of a loop in $\mathbb{C}^*$. The analytic formulas of (i) still hold in this case.

We give some examplatory applications of winding numbers:

Theorem 2.26 (Fundamental Theorem of Algebra). *Every non-constant complex polynomial* $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ *has a zero in* $\mathbb{C}$.

Proof. Assume P has no roots. It then yields a continuous map $P : \mathbb{C} \to \mathbb{C}^*$. This map is nullhomotopic, because $\mathbb{C}$ is contractible. Hence $P \circ \gamma$ is nullhomotopic in $\mathbb{C}^*$ for every loop γ in $\mathbb{C}$. On the other hand, choose $\gamma_R(t) := Re^{2\pi i t}$ for $R \gg 0$ and use that the leading term z^n of P dominates: For $|z| > R_0 := |a_0| + \ldots + |a_{n-1}| + 1$ we estimate $|P(z) - z^n| \leq (|a_0| + \ldots + |a_{n-1}|)|z^{n-1}| < |z|^n$, so the straight path between z^n and $P(z)$ avoids the origin. Consequently, for $R \geq R_0$, $P \circ \gamma_R \simeq \gamma_R^n$ in $\mathbb{C}^*$ via an affine homotopy. However, $w_{\mathbb{C}^*}[\gamma_R^n] = n \neq 0$. ☐

Theorem 2.27 (Invariance of dimension 2). *Nonempty open subsetes of* $\mathbb{R}^2$ *are not homeomorphic to open subsets of* $\mathbb{R}^{n\geq 3}$ *(Also true for* $n = 1$*, but clear in this case.)*

Proof. Suppose there exists a homeomorphism $\varphi : D^{n\geq 3} \to U \subseteq \mathbb{R}^2$ Removing some $p \in D^n$ and $\varphi(p) \in U$, we get $1 = \pi_1(S^{n-1}) = \pi_1(D^n \setminus p) \cong \pi_1(U \setminus \varphi(p))$, whereas $U \setminus \varphi(p)$ retracts onto a circle, so $\pi_1(U \setminus \varphi(p))$ surjects onto $\mathbb{Z}$. ☐

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Lemma 2.28. *There is no retraction* $\overline{D}^2 \to \partial D^2 = S^1$

Proof. Exercise. ☐

Theorem 2.29 (Brower Fixed Point Theorem for D^2). *Every continuous map* $f : \overline{D}^2 \to \overline{D}^2$ *has a fixed point.*

Proof. Assume otherwise. Then the map $r : \overline{D}^2 \to \partial D^2$ which sends x to the intersection of the ray from $f(x)$ to x with ∂D^2 is a retraction. ☐

2.4 The Seifert-van Kampen Theorem

Besides covering space theory, this theorem is the most important general tool for computing fundamental groups. For suitable open covers of a topological space, it establishes a purely algebraic relation of the fundamental groups of the space, the covering sets, and their intersections. For simplicity, we will restrict to covers by two open subsets. As a preparation, we need

Lemma 2.30. *Let* $n \geq 1$ *and* $f : \overline{D}^n \to Y$ *be a continuous map. Let* $y \in Y$ *be some point in the path-component of* $f(\overline{D}^n)$. *Then* f *is homotopic relative* ∂D^n *to a continuous map* $g : \overline{D}^n \to Y$ *with* $g(0) = y$.

Proof. Fix a path $\alpha : I \to Y$ with $\alpha(0) = f(0)$ and $\alpha(1) = y$. Define a homotopy $H : \overline{D} \times [0,1] \to Y$ by

$$H(x,t) := \begin{cases} \alpha(t - 2\|x\|) & \text{if } 0 \leq \|x\| \leq \frac{t}{2}, \\ f\left(\frac{2\|x\|-t}{2-t} \frac{x}{\|x\|}\right) & \text{otherwise} \end{cases}$$

that moves the action of f to an annulus, freeing space to move along α in the smaller disc. ☐

Remark 2.31. The proof works along rays of the disk, so analogous statements hold for maps from e.g. the half-disk.

Consider now the following situation: Let $X = U \cup V$ be a topological space with U, V open and path-connected, and let $x_0 \in U \cap V$. Our goal is to show that $\pi_1(X, x_0)$ is determined purely algebraically by the diagram of inclusion-induced group homomorphisms

$$\begin{tikzcd} \pi_1(U, x_0) \arrow[r] & \pi_1(X, x_0) \\ \pi_1(U \cap V, x_0) \arrow[u, "\alpha"] \arrow[r, "\beta"] & \pi_1(V, x_0) \arrow[u] \end{tikzcd} \tag{$*$}$$

as an amalgamated product (pushout).

First let $\gamma : [0,1] \to X$ be an arbitrary path. Then there exists a subdivision $0 = s_0 < s_1 < \ldots < s_n = 1$ such that $\gamma([s_i, s_i + 1])$ is fully contained in U or V, and $\gamma(s_k) \in U \cap V$. (Consider the open cover $[0,1] = \varphi^{-1}(U) \cup \varphi^{-1}(V)$. Then the claim is clear locally, and $[0,1]$ is compact.)

Now let $\gamma : I \to X$ be a loop based at x_0, and choose a subdivision as above. For each $1 \leq k \leq n-1$, let $\alpha_k : I \to U \cap V$ be a path from x_0 to $\gamma(s_k)$. Consider a small closed neighbourhood of s_k with image contained in $U \cap V$, and apply the previous lemma with this 1-disk and α_k, to homotopy γ rel ∂I such that in addition $\gamma(s_k) = x_0$ for all k. Then γ becomes a concatenation of loops which are contained in one of U or V.

This shows that the natural homomorphism

$$\varphi : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

extending the inclusion-induced morphisms is surjective. We now have to determine the kernel of φ. Since $*$ commutes, the kernel clearly contains the normal subgroup N generated by the elements $\alpha(h)\beta(h)^{-1}$ for $h \in \pi_1(U \cap V, x_0)$.

Let $g \in \ker \varphi$. Write g as a (not necessarily reduced) product $g = g_1 \cdots g_n$ with $g_k \in \pi_1(U, x_0)$ or $g_k \in \pi_1(V, x_0)$, and represent the g_k by loops based at x_0 in U or V, respectively. Their concatenation is a loop $\gamma : [0,1] \to X$ representing $\varphi(g) = 1$. By construction, we have a subdivision $0 = s_0 < s_1 < \ldots < s_n = 1$ such that $\gamma|_{[s_{i-i}, s_i]}$ are the representations of the g_i. By assumption, γ is nullhomotopic in X, so let $H : I^2 \to X$ be a nullhomotopy relative base point. Similarly to before, subdivide the square I^2 by rectangles via a subdivision $0 = \widetilde{s}_0 < \widetilde{s}_1 < \ldots < \widetilde{s}_{\widetilde{n}}$ of $(s_i)_i$ and $0 = t_0 < t_1 < \ldots < t_m = 1$ such that $H([\widetilde{s}_{k-1}, \widetilde{s}_k] \times [t_{l-1}, t_l])$ is contained in U or V, for all k, l.

Apply the lemma to small 2-disks around the vertices of the rectangles. If, at some vertex, all surrounding rectangles map into one of the covering sets, say U, we may choose to perform the homotopy inside U, otherwise we use $U \cap V$. Hence we can modify H such that in addition, $H(\widetilde{s}_k, t_l) = x_0$ for all k, l, and such that the change of γ has the effect that the product decomposition $g = g_1 \cdots g_n$ is refined without changing the element $g \in \pi_1(U, x_0) * \pi_1(V, x_0)$.

The oriented edges of the subdivision now map via H to loops based at x_0 and contained in U, V, or $U \cap V$, depending on where the adjacent rectangles map to. These loops represent elements in at least one of $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$. If they are in several of these groups, these elements correspond to each other under α, β. By projecting to $G := \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$, we therefore obtain well-defined labels in G for the oriented edges of the subdivision.

The product of labels around the oriented boundary of a rectangle σ is trivial. Indeed, wlog $H(\sigma) \subseteq U$. Then the labels of the oriented boundary edges of σ are images in G of elements in $\pi_1(U, x_0)$ whose product going once around $\partial\sigma$ equals 1, since $H|_\sigma$ provides a nullhomotopy.

Consider now arbitrary polygonal loops consisting of oriented edges of the subdivision. The product of labels around the loop is an element of G, well-defined up to conjugation. We claim that Lecture 10 Nov 12, 2025 this element is actually trivial. In particular, the loop along ∂I^2 maps to $1 \in G$, but by construction this is just the image of g in G, hence we are done.

Actually for our application it suffices to show that all shortest paths from 0 to $(1,1)$ yield the same element of G. Every such path can be tranformed to any other by going around a subsquare the other way, i.e. swapping a "north-east" segment with a "east-north" segment, or vice-versa. But this is fine by the previous claim. In total, we proved

Theorem 2.32 (Seifert-van Kampen). *Let $X = U \cup V$ be a topological space with U, V open and path-connected, and let $x_0 \in U \cap V$. Then the natural homomorphism*

$$\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$$

is an isomorphism.

We discuss various special situations and examples:

Example 2.33. If U and V are simply connected, then so is X. For example, $S^{n\geq 2}$ is covered by $S^n \setminus \{p\}$ and $S^n \setminus \{-p\}$ for any $p \in S^n$. Each cover is homeomorphic to $\mathbb{R}^n$, hence contractible. For $n \geq 2$ the intersection $\cong \mathbb{R}^n - \text{pt}$ is path-connected, so $\pi_1(S^{n\geq 2}) = 1$. Moreover, we see that $U \cap V$ being path-connected is essential, since $\pi_1(S^1) \neq 1$.

Example 2.34. (i) If V and $U \cap V$ are simply connected, then the natural map $\pi_1(X, x_0) \to \pi_1(U, x_0)$ is an isomorphism. Of course the same then holds for every base point of U.

(ii) (Attaching n-cells, $n \geq 3$) Let Y be a path-connected space and consider a gluing map $\varphi : \partial D^n = S^{n-1} \to Y$. Let $X := Y \cup_\varphi \overline{D}^n := Y \sqcup \overline{D}^n/(z \sim \varphi(z))$ (this is a pushout in the category of topological spaces). Then Y is embedded as a closed subset, and D^n embeds as an open subset.

Now choose $U = Y \cup_\varphi \overline{D}^n \setminus \{0\}$ and $V = D^n$, then V and $U \cap V = D^n \setminus \{0\} \simeq S^{n-1}$ are simply connected, so by (i) $\pi_1(X) \cong \pi_1(U) \cong \pi_1(Y)$ since U radially deformation retracts onto Y.

(iii) (Real projective space) $\mathbb{RP}^n \cong S^n/z \sim -z \cong S^n_+/z \sim -z$, where S^n_+ is the upper hemisphere and the identifications now only happen on the equator. Hence $\mathbb{RP}^n$ is obtained by attaching an n-cell to $\mathbb{RP}^{n-1}$ using the gluing map $\partial D^n = S^{n-1} \to S^{n-1}/\sim$. By (ii), $\pi_1(\mathbb{RP}^n) \cong \pi_1(\mathbb{RP}^2)$ for $n \geq 2$. We will determine the latter group later.

Example 2.35. (i) If $U \cap V$ is simply connected, then the amalgamated product is free, so that we have a natural isomorphism $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$. Lecture 11 Nov 17, 2025

(ii) Graphs:

(a) Let $X = S^1 \vee S^1$ be a bouquet of two circles. Label the circles A, B. Then $A \cap B = \{*\}$. Let $a \in A, b \in B$ be points with $a \neq x \neq b$. Then $U = X \setminus \{b\}$, $V = X \setminus \{a\}$ cover X, and $U \cap V = X \setminus \{a, b\}$ deformation retracts onto $\{*\}$, so that $\pi_1(X, *) \cong \pi_1(U, *) * \pi_1(V, *) \cong \pi_1(A, *) * \pi_1(B, *) \cong \mathbb{Z} * \mathbb{Z} \cong F_2$. Concretely, let α, β be generators of $\pi_1(A)$, $\pi_1(B)$ respectively (generated by a loop running around A or B once). Then $\pi_1(X, *)$ is naturally isomorphic to $\langle \alpha, \beta \rangle$.

(b) Inductively, we get a natural identification of F_n as the fundamental group of a bouquet of n circles.

(c) Let Γ be a connected finite graph with v vertices and e edges (with loops and multiple edges allowed). The following modifications preserve the hootopy type of Γ: Removing loose ends (by deformation retracting them to their initial point), and collapsing embedded edges (exercise). Doing these operations as long as possible, the result is a bouquet of circles. Note that both of these operations reduce both verteces and edges by 1, so there are $e - v + 1$ circles, and $\pi_1(\Gamma) \cong F_{e-v+1}$.

(iii) Consequently, if a space deformation retracts onto a graph, we can now calculate its fundamental group. For example, take an n-times punctured 2-sphere $S^2 \setminus \{p_1, \dots, p_n\}$. This deformation retracts to a suspension (double cone) of a n-point set: By a suitable homotopy (see exercises), we may assume that the points lie on the equator. Then embed the suspension into the sphere by identifying the poles with the vertices of the suspension, and leaving one point per cell. Hence $\pi_1(S^2 \setminus \{p_1, \dots, p_n\}) \cong F_{n-1}$.

(iv) Similarly, consider a punctured torus $X = (S^1 \times S^1) \setminus \{p\}$. It can be deformation retracted onto its 1-skeleton (in the standard gluing construction of the torus), so $\pi_1(X) \cong F_2$.

(v) (Connected Sum) Let M_1, M_2 be topological manifolds of dimension $n \geq 2$. Let $\overline{B}_i \subseteq M_i$ be a closed ball contained in some chart, and $p_i \in B_i$ its center (w.r.t. that chart). Turn $B_1 \setminus \{p_1\}$ inside out by reflecting around the sphere of radius $\frac{1}{2}$ and use this as a gluing map $B_2 \setminus \{p_2\} \to B_1 \setminus \{p_1\}$. This yields a new manifold $M_1 \# M_2$, called the connected sum. Note that the $M_i \setminus \{p_i\}$ embed into $M_1 \# M_2$, and their intersection is homeomorphic to $S^{n-1} \times (-1, 1)$.

Note that the homeomorphism type depends on the choices in the construction. However, for smooth manifolds and smooth charts, the diffeomorphism type is well-defined.

For $n \geq 3$, we now see $\pi_1(M_1 \# M_2) \cong \pi_1(M_1 \setminus \{p_1\}) * \pi_1(M_2 \setminus \{p_2\}) \cong \pi_1(M_1) * \pi_1(M_2)$, where the second isomorphism comes from 2.34(ii).

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Example 2.36. (i) Assume that V is simply connected. Then Seifert-van Kampen gives a natural isomorphism $\pi_1(X, x_0) \leftarrow \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} 1 \cong \pi_1(U, x_0)/N(\operatorname{im} \alpha)$. That is, $\pi_1(X, x_0)$ is generated by $\pi_1(U, x_0)$, with new relations given by $\alpha(\pi_1(U \cap V, x_0))$.

(ii) (Attaching 2-cells) As in 2.34(ii), given a gluing map $\varphi : \partial D^2 = S^1 \to Y$, we may form the space $X = \overline{D}^2 \cup_\varphi Y$. Assume that Y is path-connected. As a result of the gluing, the loop $\gamma(\exp(2\pi i \cdot))$ in Y based at $y_0 = \varphi(1)$ becomes nullhomotopic in X. Note that $[\gamma] \in \pi_1(Y, y_0)$ generates the image of $\varphi_* : \pi_1(S^1, 1) \to \pi_1(Y, y_0)$. We claim that the natural homomorphism $\pi_1(Y, y_0)/N([\gamma]) \to \pi_1(X, y_0)$ is an isomorphism.

Indeed, let $U = Y \cup_\varphi \overline{D}^2 \setminus \{0\}$ and $V = D^2$. Then $U \cap V \cong D^2 \setminus \{0\} \simeq S^1$. Fix $r \in (0, 1)$ and consider the loop $\gamma' := r \exp(2\pi i \cdot) \subset X$. Then $[\gamma']$ generates the image of $\pi_1(U \cap V, r)$ in $\pi_1(U, r)$. Connect $y_0 = \varphi(1)$ to r in U by the path $\alpha(t) = t(r-1) + 1$. Then $\alpha * \gamma' * \alpha^- \simeq \gamma$ in U. Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & N([\gamma']) \arrow[r, hook] \arrow[d, "C_\alpha"] & \pi_1(U, r) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "C_\alpha"] & \pi_1(X, r) \arrow[r] \arrow[d, "C_\alpha"] & 1 \\
1 \arrow[r] & N([\gamma]) \arrow[r, hook] & \pi_1(U, y_0) \arrow[r, "\mathrm{incl}_*"] & \pi_1(X, x_0) \arrow[r] & 1
\end{tikzcd}$$

This diagram commutes (the first square by the previous considerations, the second one by general facts). By Seifert-van Kampen (i.e. (i)), the first row is exact, and the vertical arrows are natural isomorphisms, so the second row is exact as well. Finally, $\pi_1(U, y_0) \cong \pi_1(Y, y_0)$ naturally, since U deformation retracts onto Y.

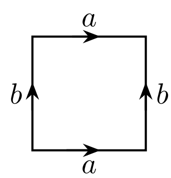
(iii) (Attaching a 2-cell to a circle) Consider a gluing map $\varphi : \partial D^2 = S^1 \to S^1$. With the notation of (ii), we have $\pi_1(X, y_0) \cong \pi_1(S^1, \varphi(1))/\langle[\gamma]\rangle \cong \mathbb{Z}/(\deg \varphi)\mathbb{Z}$.

Concretely, let $\varphi(z) = z^n$, $n \in \mathbb{N}$ and denote the corresponding space by X_n. Then $\pi_1(X) \cong \mathbb{Z}/n\mathbb{Z}$. Since φ is surjective, we see $X_n \cong \overline{D}^2/\sim$ with $z \sim e^{\frac{2\pi i}{n}} z$ for $z \in \partial D^2$. In particular, $X_2 \cong \mathbb{RP}^2$, so $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$ and by 2.34(iii) $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$ as well.

(iv) (Attaching 2-cells to a bouquet of circles) Write $B_n := S_1 \vee \ldots \vee S_1$ for a bouquet of $n \in \mathbb{N}$ circles. We know $\pi_1(B_n, *) \cong \langle a_1, \ldots, a_n \rangle \cong F_n$, where the a_i are the standard generators of the i-th $\pi_1(S^1)$. Glue on a 2-disk $\overline{D}_1^2$ via $\varphi_1 : \partial D_1^2 = S_1^1 \to B_n$ with $\varphi_1(1) = *$. Let $X_1 := B_n \cup_{\varphi_1} \overline{D}_1^2$, $\gamma_1 := \varphi_1(\exp(2\pi i \cdot))$ and $c_1 := [\gamma]$. By (ii), $\pi_1(X_1, *) \cong \langle a_1, \ldots, a_n \mid c_1 \rangle$.

Now glue another disk $\overline{D}_2^2$ via $\varphi_2 : \partial D_2^2 = S_2^1 \to B_n$ with $1 \mapsto *$. Let $\widehat{\varphi}_2$ be the composition of φ_2 with the inclusion of B_n into X_1. Let $X_2 := X_1 \cup_{\widehat{\varphi}_2} \overline{D}_2^2 \cong B_n \cup_{\varphi_1 \sqcup \varphi_2} (\overline{D}_1^2 \sqcup \overline{D}_2^2)$, γ_2, c_2 as above. Let $\widehat{\gamma}_2, \widehat{c}_2$ be the corresponding (homotopy class of) loop in X_1. Then $\pi_1(X_2, *) \cong \pi_1(X, *)/N(\widehat{c}_2) \cong \langle a_1, \ldots, a_n \mid c_1, c_2 \rangle$, by 1.8(iii). It is now clear we can proceed inductively: By gluing finitely many disks, one can realize every finitely presented group as a fundamental group.

(v) (Surfaces) We construct closed surfaces (that is compact topological 2-manifolds without boundary). Consider polygons in the plane as 2-dimensional objects. Note that they are homeomorphic to $\overline{D}^2$, for example because they can be triangulated. Let P be a $2n$-gon, $n \geq 2$. A side pairing on P consists of a fixed point free involution of its sides, together with a choice of orientation for every side. We indicate such side pairings graphically by using pairs of labels, and denote the orientation with arrows: Lecture 13 Nov 24, 2025



We now form a topological quotient $\Sigma = P/\sim$ by identifying corresponding sides by homeomorphisms preserving the chosen orientations. To see that Σ is a topological surface, note that P° embeds into Σ, so Σ is already locally euclidean away from the boundary of P. Now on an edge, we can find two matching semicircles that become a full open ball after gluing, and at the vertices, parametrize the interior angles by circular sectors of appropriate angle (depending on how many vertices get identified) to achieve the same effect.

In the same manner, one can generalize this construction to glue multiple (disjoint) polygons, and (by using spherical edges) admit bigons. Note that the surface constructed in this manner naturally carries the structure of a 2-dimensional CW-complex.

Alternatively, we could have glued a 2-cell $\overline{P}$ to the finite graph $\partial P/\sim$ via the natural map $\partial P \to \partial P/\sim$. Let v be a vertex of P and $\widehat{v}$ its image in σ. Similarly, let γ be the loop based at v going once around $\partial P \cong S^1$, and $\widehat{\gamma}$ its image loop. By Seifert-van Kampen, we compute $\pi_1(\Sigma, \widehat{v}) = \pi_1(\partial P/\sim, \widehat{v})/N([\gamma])$.

Now we look at concrete examples of such surfaces.

(a) Let $T = S^1 \times S^1$ be the torus. It is constructed via side pairings as in the diagram above, hence $\pi_1(T, *) \cong \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} \rangle \cong \mathbb{Z}^2$, where the generators α, β are exactly the image loops of the edges a, b in T.

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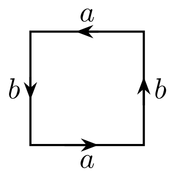
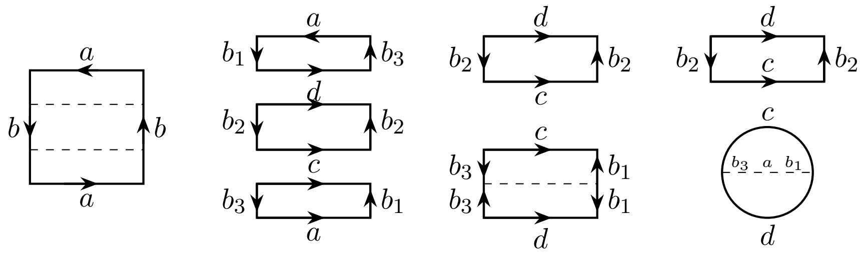
(b) Let $\Sigma = \mathbb{RP}^2$. We know it can be constructed from a disc by antipodal identifications along the boundary. Hence it can be constructed from side pairings as

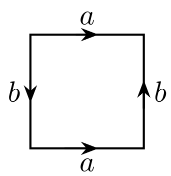


Then $\partial P/\sim$ consists of two points connected with edges a, b. The gluing map sends γ to $\widehat{\gamma} = a * b * a * b$ and $\pi_1(\mathbb{RP}^2) \cong \langle \alpha\beta \mid (\alpha\beta)^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.
Note that $\mathbb{RP}^2$ contains an embedded Möbius strip, which one easily sees from the above picture by restricting it to a small vertical or horizontal strip.



If one cuts along the edges of such a strip as in the picture, we see that one can obtain projective plane by gluing a Möbius strip and a disk along the boundary.

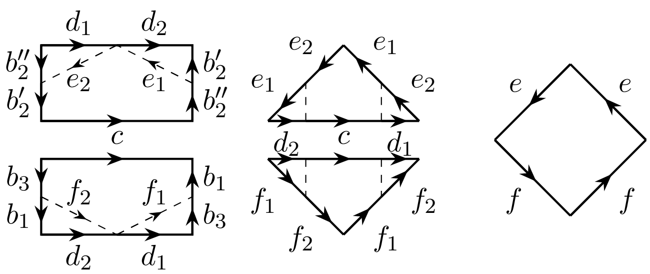
(c) Let $\Sigma = K$ be the Klein bottle, constructed as



As before, we compute $\pi_1(K) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta \rangle$. Note

$$\pi_1(K)^{\text{ab}} = \langle \alpha, \beta \mid [\alpha, \beta], \beta^2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

thus in particular K is not homeomorphic to S^2, T or $\mathbb{RP}^2$. We note that the Klein bottle contains an embedded Möbius strip. In fact, in the same way as for $\mathbb{RP}^2$, it can be constructed by gluing two Möbius strips M_1, M_2 along their boundaries, equivalently as the connected sum of two projective planes.
We can continue as in the next diagram to obtain a different representation of the Klein bottle by side pairings of the square:

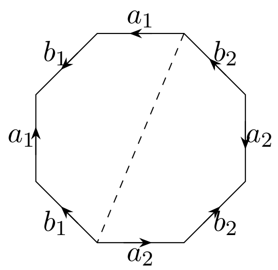


Hence we get a different representations $\pi_1(K) \cong \langle \eta, \phi \mid \eta^2\phi^2 \rangle$ and (by directly applying Seifert-van Kampen to the covering with two thickened Möbius strips)

$$\pi_1(K) \cong \pi_1(M_1) *_{\pi_1(\delta M_1 = \delta M_2)} \pi_1(M_2) \cong \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z},$$

where the amalgamation maps are given by $1 \mapsto 2$. One can write down isomorphisms between these different presentations by keeping track of where generating loops are sent to under the cuttings and gluings.

(d) Let Σ_2 be the two-holed torus, constructed via



All vertices are identified, so $P/\sim$ is a bouquet of four circles. Hence $\pi_1(\Sigma_2) \cong \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] \rangle$. We see that this group is non-abelian (e.g. because it surjects onto $\langle \alpha_1, \alpha_2 \rangle$) and $\pi_1(\Sigma_2)^{\text{ab}} \cong \mathbb{Z}^4$. Cutting along the dashed line, we obtain two tori with an open disk removed. Gluing these together, we see that $\Sigma_2 = T\#T$ is indeed a two-holed torus.

More generally, the g-holed torus Σ_g, $g \geq 1$ can be constructed from a $4g$-gon, or as the connected sum of g tori. We see $\pi_1(\Sigma_g) \cong \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_i [\alpha_i, \beta_i] = 1 \rangle$, and all these surfaces are different, because $\pi_1(\Sigma_g)^{\text{ab}} \cong \mathbb{Z}^{2g}$.