Algebraic Geometry I

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Roughly speaking, the goal of algebraic geometry is to study systems of polynomial equations $F_1(X) = \ldots = F_n(X) = 0$ for polynomials $F_i \in K[X_1, \ldots, X_m]$ over a field (or ring) K. The set of solutions of this system is a geometric object, which we try to understand using algebraic methods, for example considering the ideal $I = (F_1, \ldots, F_n)$ in $K[X_1, \ldots, X_m]$ or the quotient $K[X_1, \ldots, X_m]/I$. Lecture 1 Oct 13, 2025

There is a very strong relation between these objects in the case that $K = \overline{K}$ is algebraically closed (e.g. $\mathbb{C}$). If K is not algebraically closed, or some generic ring, things get more complicated: For example, there are many equations over $\mathbb{R}$ with no solutions, like $x^2+y^2+1 = 0$, which behave differently when considered over $\mathbb{C}$. The wish to still study these equations geometrically leads to the idea of spectra (the set of all prime ideals of a ring), and later the theory of sheaves and schemes.

1 Algebraic Sets and Affine Varieties

Let K be an algebraically closed field.

Definition 1.1. For $n \in \mathbb{N}$ define *affine n-space* over K as

$$\mathbb{A}^n := \mathbb{A}^n_K := K^n.$$

Definition 1.2. Let $I \subset K[x_1, \ldots, x_n]$ be a subset. The associated *(affine) algebraic set* is

$$V(I) := \{x \in \mathbb{A}^n_K \mid f(x) = 0 \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n$ is called *algebraic* if $X = V(I)$ for some $I \subset K[x_1, \ldots, x_n]$.

Remark 1.3. By definition $V(I) = V(\langle I \rangle) = V(f_1, \ldots, f_m)$ where $\langle I \rangle = (f_1, \ldots, f_m)$ is finitely generated because $K[x_1, \ldots, x_n]$ is Noetherian. Therefore, $X \subseteq \mathbb{A}^n$ is algebraic if and only if $X = V(I)$ for some ideal I if and only if $X = V(f_1, \ldots, f_m)$ for a finite number of polynomials f_i.

Example 1.4. The following sets are algebraic:

- A parabola $\{(x, x^2) \mid x \in K\} = V(y - x^2)$
- $\emptyset = V(K[x_1, \ldots, x_n])$
- $\mathbb{A}^n = V(0)$
- Points: $\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$

Lemma 1.5. *Let $I, J \lhd K[x_1, \ldots, x_n]$ be ideals. Then*

(a) If $I \subseteq J$, then $V(I) \supseteq V(J)$.

(b) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$

(c) For any family $(I_t)_{t \in T}$ of ideals, $\bigcap_t V(I_t) = V(\bigcup_t I_t) = V(\sum_t I_t)$

Proof. (a) is clear.

For (b), part (a) yields $V(I \cap J) \subseteq V(IJ)$ and $V(I), V(J) \subseteq V(I \cap J)$, so it remains to show $V(IJ) \subseteq V(I) \cup V(J)$. Let $a \in V(IJ)$. Assume $a \notin V(I)$, i.e. there is $f \in I$ such that $f(a) \neq 0$. Let $g \in J$. Then $fg \in IJ$, so $0 = (fg)(a) = f(a)g(a)$. Since $f(a) \neq 0$, we conclude $g(a) = 0$.

The first equation of (c) is tautological, the second one is remark 1.3, □

Definition 1.6. The *Zariski topology* on $\mathbb{A}^n$ is the topology whose closed subsets are exactly the algebraic sets. That is, $U \subseteq \mathbb{A}^n$ is open iff its complement is algebraic.

Remark 1.7. This is indeed a topology by example 1.4 and lemma 1.5. Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set $X \subseteq \mathbb{A}^n$, which is also called the Zariski topology.

Recall from general topology that a topological space $X \neq \emptyset$ is called irreducible if $X \neq X_1 \cup X_2$ with $X_i \subsetneq X$ closed. $\emptyset$ is not considered irreducible.

For example, $V(xy) = V(x) \cup V(y)$ (the union of the coordinate axes in $\mathbb{A}^2$) is not irreducible, while a parabola $V(y - x^2)$ is irreducible (we will see how to check this later).

Definition 1.8. An *affine algebraic variety* is an irreducible closed subset of $\mathbb{A}^n$.

Definition 1.9. Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. We define the *vanishing ideal* of X as

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

Lemma 1.10. *Let $X \subseteq \mathbb{A}^n$ and $S \subseteq K[x_1, \dots, x_n]$. Then*

(a) $X \subseteq V(I(X))$ and $S \subseteq I(V(S))$.

(b) $V(I(X)) = \overline{X}$ is the closure of X (w.r.t. the Zariski topology).

Proof. (a) is clear, (b) is left as an exercise.

[Since $V(I(X))$ is closed by definition and onctains X by (a), the inclusion $\overline{X} \subseteq V(I(X))$ is clear. For the converse, let $\overline{X} = V(J)$ for some ideal J. Then $J \subseteq I(V(J))$ by (a), and applying V to this yields

$$V(I(X)) \subseteq V(I(\overline{X})) = V(I(V(J))) \subseteq V(J) = \overline{X}.]$$

□

Proposition 1.11. *An affine algebraic set $X \subseteq \mathbb{A}^n$ is a variety if and only if $I(X)$ is a prime ideal.*

Proof. Let X be a variety and let $fg \in I(X)$ for $f, g \in K[x_1, \dots, x_n]$. We have $X \subseteq V(fg) \overset{1.5}{=} V(f) \cup V(g)$. Hence we can write $X = (X \cap V(f)) \cup (X \cap V(g))$ as the union of two closed subsets. By irreducibility, wlog we have $X = X \cap V(f)$, i.e. $X \subseteq V(f)$, which is equivalent to $f \in I(X)$.

Conversely, suppose that $X = A \cup B$ is not irreducible. Choose points $a \in A \setminus B$ and $b \in B \setminus A$. By Lemma 1.10 and since A, B are closed, we get $V(I(A)) = A$ and $V(I(B)) = B$. Hence there exist $f \in I(A)$ and $g \in I(B)$ with $f(b) \neq 0$ and $g(a) \neq 0$. Thus $fg \in I(X)$, but both $f, g \notin I(X)$ □

Remark 1.12. If $X = V(I)$ is an affine variety, this does not necessarily imply that I is prime: Consider $V((x^2)) \subseteq \mathbb{A}^1$: $V((x^2)) = \{0\}$ is irreducible, but (x^2) is not prime. Lecture 2 Oct 15, 2025

Note that $\mathbb{A}^n$ is irreducible since K is infinite. However, this is no longer true if one considers finite fields, since then $\mathbb{A}^n$ is the union of its finitely many points. For example, $I(A^1_{\mathbb{F}_p}) = (X^p - X)$ is not prime.

We use the following result from commutative algebra without proof:

Theorem 1.13 (Hilbert Nullstellensatz)**.** *Let $J \triangleleft K[x_1, \dots, x_n]$. Then*

(a) $V(J) = \emptyset$ if and only if $J = K[x_1, \dots, x_n]$.

(b) $I(V(J)) = \sqrt{J} = \{f \in K[x_1, \dots, x_n] \mid f^n \in J \text{ for some } n\}$

(c) *If J is a maximal ideal, then $J = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in K$.*

Corollary 1.14. *There are inclusion-reversing bijections*

$$\{\textit{affine algebraic sets } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{radical ideals in } K[x_1, \dots, x_n]\}$$

$$\{\textit{affine algebraic varieties } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{prime ideals in } K[x_1, \dots, x_n]\}$$

$$\{\textit{points } a \in \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{maximal ideals in } K[x_1, \dots, x_n]\}$$

Proof. Clear from 1.13, 1.10 and 1.11. □

Example 1.15. Let f be irreducible in $K[x_1, \dots, x_n]$. Then $V(f)$ is an affine variety. Varieties of this form are called hypersurfaces in $\mathbb{A}^n$ (curves for $n = 2$, surfaces for $n = 3$).

Remark 1.16. If $X \subseteq \mathbb{A}^n$ is a variety, by proposition 1.11 $I(X)$ is prime, and $K[x_1, \dots, x_n]/I$ is an integral domain. We can consider its fraction field $\mathrm{Frac}(K[x_1, \dots, x_n]/I)$.

Theorem 1.17. *Any affine algebraic set can be uniquely written as a finite union of affine varieties.*

For the proof, we need some preparations.

Definition 1.18. A topological space X is called *Noetherian* if any chain of descending closed subsets $X \supseteq X_1 \supseteq X_2 \supseteq \dots$ becomes stationary, i.e. there exists n s.t. $X_m = X_n$ for all $m > n$.

Lemma 1.19. *Affine space $\mathbb{A}^n$ is Noetherian.*

Proof. Let $\mathbb{A}^n \supseteq X_1 \supseteq X_2 \supseteq \dots$ be a chain of closed subsets. Applying $I(-)$ yields an ascending chain $(0) \subseteq I(X_1) \subseteq I(X_2) \subseteq \dots$ of ideals in $K[x_1, \dots, x_n]$. This is a Noetherian ring, so there is some m such that $I(X_n) = I(X_{n+1})$ for all $n \geq m$. By corollary 1.14(a), I is injective on closed subsets, so we are done. □

More generally,

Corollary 1.20. *Any affine algebraic space $X \subseteq \mathbb{A}^n$ is Noetherian.*

Proof. Any chain in X is also a chain in $\mathbb{A}^n$. □

Proposition 1.21. *Let $X \neq \emptyset$ be a Noetherian topological space.*

(a) *Then X can be written as a finite union of irreducible closed subspaces.*

(b) *Moreover, if we assume that $X_i \not\subseteq X_j$ for $i \neq j$, then the above decomposition is unique up to permutation. In this case, the X_i are called irreducible components of X.*

Proof. Assume that (a) fails for X. Consider $S = \{Y \subseteq X \mid Y$ closed, cannot be written as a finite union of irreducible closed subsets$\}$. Since X is Noetherian, S must have some minimal element Y w.r.t. inclusion. Y is not irreducible, so we can write $Y = Y_1 \cup Y_2$ with $Y_{1,2}$ proper closed subspaces. By minimality, Y_1 and Y_2 can be written as finite unions of irreducible closed subsets, thus so can Y, contradicting $Y \in S$.

To check uniqueness, assume we have two decompositions $X = X_1 \cup \dots X_r = X_1' \cup \dots X_s'$ as in (b). Then $X_1' = \bigcup_i (X_i \cap X_1')$. Since X_1' is irreducible, wlog $X_1' \subseteq X_1$. By the same argument, $X_1 \subseteq X_i'$ for some i. If $i \neq 1$, then $X_1' \subseteq X_i'$, contradicting our assumption. Hence $i = 1$ and $X_1 = X_1'$. Proceed inductively with $\overline{X \setminus X_1} = X_2 \cup \dots \cup X_r = X_2' \cup \dots \cup X_s'$. □

Remark 1.22. The proof strategy for (a) can be summarized as follows: Let X be a Noetherian space and P a property of closed subsets. To show that P holds for all subsets of X (thus in particular for X), it suffices to show that for all $Y \subseteq X$ closed, if P holds for all proper closed subsets of Y, then it also holds for Y. This is called *Noether induction* (a special case of well-founded induction).