

Algebraic Number Theory

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0 Motivation

Lecture 1
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Theorem 0.1 (Lagrange). *Let p be an odd prime. Then*

$$p = x^2 + y^2 \text{ with } x, y \in \mathbb{Z} \text{ if and only if } p \equiv 1 \pmod{4}.$$

Proof. For any integer x we have $x^2 \equiv 0, 1 \pmod{4}$, hence $x^2 + y^2 \equiv 0, 1$ or $2 \pmod{4}$ for all $x, y \in \mathbb{Z}$, hence $p \not\equiv 3 \pmod{4}$.

Conversely, assume that $p \equiv 1 \pmod{4}$. Then \mathbb{F}_p^\times is a cyclic group of order $p - 1$, so there exists some $\bar{m} \in \mathbb{F}_p^\times$ of order 4. Thus there is $m \in \mathbb{Z}$ with $m^2 \equiv -1 \pmod{p}$, i.e. $p \mid m^2 + 1 = (m + i)(m - i) \in \mathbb{Z}[i]$. Since the Gaussian integers form a Euclidean ring, it is in particular a PID.

Consider its norm $N : \mathbb{Z}[i] \rightarrow \mathbb{Z}$, $\alpha = a + bi \mapsto \alpha\bar{\alpha} = a^2 + b^2$, which is a multiplicative function. Suppose that $p \mid m + i$. Then $p \mid m - i$ as well, hence $p \mid 2i$, which is clearly wrong. Hence p is not a prime element in $\mathbb{Z}[i]$. Since we are in a PID, p is reducible in $\mathbb{Z}[i]$, i.e. there exist non-units $\alpha = x + yi, \beta = x' + y'i \in \mathbb{Z}[i]$ such that $p = \alpha\beta$. Now we see $p^2 = N(\alpha)N(\beta) = (x^2 + y^2)(x'^2 + y'^2)$. Since α, β aren't units, each factor is > 1 , hence $p = x^2 + y^2 = x'^2 + y'^2$. \square

Definition 0.2. A finite extension K of \mathbb{Q} is called a *number field*.

Example 0.3. $\mathbb{Q}(i)$ is a number field of degree 2. In the above example, we worked in $\mathbb{Z}[i] \subseteq \mathbb{Q}(i)$. We want to generalize this.

Definition 0.4. Let K/\mathbb{Q} be a number field. Then

$$\mathcal{O}_K := \{\alpha \in K \mid \exists f \in \mathbb{Z}[x] \text{ normalized s.t. } f(\alpha) = 0\},$$

i.e. the integral closure of \mathbb{Z} in K , is called the *ring of integers* in K .

We will show: \mathcal{O}_K is a Dedekind domain.

Example 0.5. (i) For $K = \mathbb{Q}(i)$ we have $\mathcal{O}_K = \mathbb{Z}[i]$

(ii) For $K = \mathbb{Q}(\sqrt{2})$ one gets $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$

(iii) For $K = \mathbb{Q}(\sqrt{-6})$ we have $\mathcal{O}_K = \mathbb{Z}[\sqrt{-6}]$

(iv) (Exercise) More generally, for $d \in \mathbb{Z} \setminus \{0, 1\}$ squarefree, the ring of integers of $K = \mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\omega]$, where

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \pmod{4}, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

Theorem 0.6. *Let p be an odd prime. Then*

$$p = x^2 - 2y^2 \text{ with } x, y \in \mathbb{Z} \text{ if and only if } p \equiv \pm 1 \pmod{8}.$$

Proof. The forward direction follows as in the first theorem. For the converse, we work in $\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{Q}(\sqrt{2})$. Consider the norm $N : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}$, $\alpha = x + y\sqrt{2} \mapsto \alpha\sigma(\alpha) = x^2 - 2y^2$, where $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \langle \sigma \rangle$. We will see later (Quadratic Reciprocity) that $p \equiv \pm 1 \pmod{8}$ is equivalent to $\left(\frac{2}{p}\right) = 1$, i.e. 2 being a square mod p .

Hence there exists $m \in \mathbb{Z}$ with $p \mid m^2 - 2 = (m - \sqrt{2})(m + \sqrt{2})$. As before, we see that p is not prime, hence reducible ($\mathbb{Z}[\sqrt{2}]$ is again Euclidean) and we finish as before. \square

The main difference between theorems 0.1 and 0.6 is that the unit group of $\mathbb{Z}[i]$ is finite, while $\mathbb{Z}[\sqrt{2}]^\times = \{\pm 1\} \times (1 + \sqrt{2})^\mathbb{Z}$ is infinite¹. This implies that $p = x^2 - 2y^2$ has infinitely many solutions for $p \equiv \pm 1 \pmod{8}$, for $N((1 + \sqrt{2})^{2k}\alpha) = N(\alpha)$ for all $k \in \mathbb{Z}$.

In this vein, an important goal of this lecture is

Theorem 0.7 (Dirichlet's unit theorem). *Let K/\mathbb{Q} be a number field. Let s be the number of real embeddings and let t be the number of pairs of complex embeddings of K . Then \mathcal{O}_K^\times is a finitely generated abelian group of rank $r = s + t - 1$, i.e. there exist fundamental units $\varepsilon_1, \dots, \varepsilon_r$ and $\zeta \in \mu_K = \{\text{roots of unity in } K\}$ such that each $\varepsilon \in \mathcal{O}_K^\times$ can be uniquely written in the form*

$$\varepsilon = \zeta^l \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r}$$

with $a_i \in \mathbb{Z}$ and $l \in \mathbb{Z}/\text{ord}(\zeta)\mathbb{Z}$.

Example 0.8. For $K = \mathbb{Q}(\sqrt{2})$ we have $\mu_K = \{\pm 1\}$, $\varepsilon_1 = 1 + \sqrt{2}$ and $r = 2 + 0 - 1 = 1$, since both embeddings $\sqrt{2} \mapsto \sqrt{2}$ and $\sqrt{2} \mapsto -\sqrt{2}$ are real.

Let K/\mathbb{Q} be a number field. We choose the algebraic closure \mathbb{Q}^c of \mathbb{Q} that sits inside of \mathbb{C} , so we may, and will, always assume $K \subseteq \mathbb{C}$. K/\mathbb{Q} is separable, so we may write $K = \mathbb{Q}(\alpha)$ for some $\alpha \in K$. Let $f \in \mathbb{Q}(\alpha)$ be the minimal polynomial of α . Then we have embeddings $\sigma : K \hookrightarrow \mathbb{C}$ corresponding to the zeroes $\alpha = \alpha_1, \dots, \alpha_n$ of f , i.e. the conjugates of α . σ is called a real embedding if $\sigma(K) \subseteq \mathbb{R}$, or equivalently if the corresponding $\alpha_i \in \mathbb{R}$. Otherwise it is called a complex embedding. These come in pairs, because if α_i is a conjugate of α , so is $\overline{\alpha_i}$.

Example 0.9. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field. If $d > 0$ we find as before that $s = 2, t = 0$, so $r = 1$. If, on the other hand, $d < 0$, then $s = 0, t = 1$, hence $r = 0$ and \mathcal{O}_K^\times is finite.

Question Which odd primes p can be written in the form $p = x^2 + 6y^2$ with $x, y \in \mathbb{Z}$? As in the previous theorems, we write this as $(x + y\sqrt{-6})(x - y\sqrt{-6}) = N(x + y\sqrt{-6})$ in the number field $K = \mathbb{Q}(\sqrt{-6})$ with ring of integers $\mathbb{Z}[\sqrt{-6}]$. However, our previous proof strategy does *not* work, because $\mathbb{Z}[\sqrt{-6}]$ is not a PID (e.g. $2 \cdot 3 = -\sqrt{-6} \cdot \sqrt{-6}$ are two essentially different factorizations of 6 into irreducibles).

This leads naturally to the question when \mathcal{O}_K is a PID. To investigate this, we will introduce the *class group*: The nonzero ideals of \mathcal{O}_K form a monoid w.r.t. multiplication.

Definition 0.10. Write I_K for the group of fractional nonzero ideals and $P_K = \{\alpha\mathcal{O}_K \mid \alpha \in K^\times\}$ the subgroup of principal fractional ideals. The quotient $\text{cl}_K = I_K/P_K$ is called the *ideal class group*.

One sees directly that $\text{cl}_K = 1$ if and only if \mathcal{O}_K is a PID. We will prove

Theorem 0.11. $|\text{cl}_K| < \infty$.

In any case \mathcal{O}_K is Dedekind, which is equivalent to prime factorization of *ideals*, i.e. each ideal $(0) \neq \mathfrak{a} \subseteq \mathcal{O}_K$ can be uniquely written as a product of prime ideals

$$\mathfrak{a} = \prod_{\substack{\mathfrak{p} \in \text{Spec}(\mathcal{O}_K) \\ \mathfrak{p} \neq 0}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}, \quad v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}_{\geq 0}, \text{ almost all } v_{\mathfrak{p}}(\mathfrak{a}) = 0.$$

¹ \supseteq is easy by direct computation, which is all we use here. We will see how to prove \subseteq later.

Example 0.12. In $\mathbb{Z}[\sqrt{-6}]$ we have $2\mathcal{O}_K = \mathfrak{p}_2^2$ with $\mathfrak{p}_2 = \langle 2, \sqrt{-6} \rangle_{\mathbb{Z}}$, $3\mathcal{O}_K = \mathfrak{p}_3^2$ with $\mathfrak{p}_3 = \langle 3, \sqrt{-6} \rangle_{\mathbb{Z}}$ and $\sqrt{-6}\mathcal{O}_K = \mathfrak{p}_2\mathfrak{p}_3$, so the "problematic" factorization $2 \cdot 3 = -\sqrt{-6}^2$ becomes $\mathfrak{p}_2^2\mathfrak{p}_3^2 = (\mathfrak{p}_2\mathfrak{p}_3)^2$ when passing to ideals.

Given an extension of number fields L/K , and a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, by the above the ideal $\mathfrak{p}\mathcal{O}_L$ splits into a product of prime ideals $\mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ in \mathcal{O}_L . A further goal of this lecture is to understand and compute this factorization. Denoting $f_i = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$, we will for example be able to show $[L : K] = \sum_{i=1}^r e_i f_i$.

Definition 0.13. Let p be a prime and $a \in \mathbb{Z}$ with $p \nmid a$. Then the *Legendre symbol* is defined as

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution in } \mathbb{Z}, \\ -1 & \text{otherwise.} \end{cases}$$

Also set $\left(\frac{a}{p}\right) = 0$ if $p \mid a$.

We will show: Let $K = \mathbb{Q}(\sqrt{d})$. Let $p \neq 2$. Then

$$p\mathcal{O}_K = \begin{cases} \mathfrak{p}\bar{\mathfrak{p}}, \mathfrak{p} \neq \bar{\mathfrak{p}} \text{ prime} & \text{if } \left(\frac{d}{p}\right) = 1, \\ \mathfrak{p}, \mathfrak{p} \text{ prime} & \text{if } \left(\frac{d}{p}\right) = -1, \\ \mathfrak{p}^2, \mathfrak{p} \text{ prime} & \text{if } p \mid d. \end{cases} \quad (*)$$

Law of quadratic reciprocity Let p, q be odd primes. Then

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \text{ and } q \equiv 3 \pmod{4} \end{cases}.$$

Further, we have the two supplements $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$ and $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$. This theorem allows quick computation of Legendre symbols.

Using the above, we will be able to generalize the theorems from the beginning:

Lecture 2
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Corollary 0.14. Let d be a squarefree integer. A prime $p \neq 2$ can be written in the form $p = x^2 - dy^2$ for $x, y \in \mathbb{Z}$ if and only if $\left(\frac{d}{p}\right) = 1$ and \mathfrak{p} is a principal ideal, where \mathfrak{p} is as in $(*)$.

1 Integrality

Rings are always commutative and contain a multiplicative unit, unless explicitly stated otherwise.

Definition 1.1. Let $A \subseteq B$ be a ring extension. An element $b \in B$ is *integral* over A if there exists a normalized polynomial $f(X) = X^m + a_{m-1}X^{m-1} + \dots + a_1X + a_0 \in A[X]$ such that $f(b) = 0$. B is *integral* over A if every $b \in B$ is integral over A .

Example 1.2. Let K be a number field. Then \mathcal{O}_K is integral (over \mathbb{Z}).

If B/A is a field extension, then B is integral over A if and only if B is algebraic over A .

We want to show that the set of all integral elements form a ring, i.e. that given integral elements $b_1, b_2 \in B$, $b_1 + b_2$ and b_1b_2 are integral as well.

Theorem 1.3. Let $b_1, \dots, b_n \in B$. Then b_1, \dots, b_n are integral over A if and only if $A[b_1, \dots, b_n]$ is a finitely generated A -module.

Proof. " \Rightarrow ": By induction. For $n = 1$ let $b \in B$ be integral over A . Let $f(b) = 0$. Then $b^m = -\sum_{i=0}^{m-1} a_i b^i$, so $A[b]$ is generated by $1, b, \dots, b^{m-1}$ as a A -module.

More explicitly: Let $g(b) \in A[b]$ be some element. Since f is normalized, we can perform division with remainder to write $g = qf + r$ with $q, r \in A[x]$ with $\deg(r) < m$. Hence $g(b) = q(b)f(b) + r(b) = r(b)$, which is a linear combination of b^i , $i < m$.

For the inductive step, we have to prove that $A \subseteq A[b_1, \dots, b_n] \subseteq A[b_1, \dots, b_{n+1}]$ is finitely generated, knowing that the first extension is finitely generated. Since b_{n+1} is integral over A , it is also finitely generated over $A[b_1, \dots, b_n]$, hence $A[b_1, \dots, b_n] \subseteq A[b_1, \dots, b_{n+1}]$ is finitely generated by the $n = 1$ case, hence we are done.

" \Leftarrow ": Let $\omega_1, \dots, \omega_r$ be a set of A -generators of $A[b_1, \dots, b_n]$. For $b \in A[b_1, \dots, b_n]$ we have

$$b\omega_i = \sum_{j=1}^r a_{ij}\omega_j \quad \text{with } a_{ij} \in A.$$

Hence $(bE - M)(\omega_1, \dots, \omega_r)^t = 0$, where $M = (a_{ij})_{ij} \in A^{r \times r}$. By cofactor expansion, see lemma 1.4, this implies that $\det(bE - M)\omega_i = 0$ for all $i = 1, \dots, r$, hence $\det(bE - M) = 0$ since the ω_i generate $A[b_1, \dots, b_n]$. Hence $\det(XE - M) \in A[X]$ is a normalized equation for b , i.e. b is integral over A . \square

Lemma 1.4. Let A a ring and $M \in A^{r \times r}$. If $Mx = 0$, then $\det(M)x = 0$.

Proof. Let M^* be the adjoint matrix, i.e. $(M^*)_{ij}$ is $(-1)^{i+j}$ times the determinant of the matrix M with the j -th row and i -th column removed. Then $M^*M = MM^* = \det(M)E$. From $Mx = 0$ we then get $0 = M^*Mx = \det(M)x$. \square

Example 1.5. $K = \mathbb{Q}(\sqrt{2}) \supseteq \mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. Proceeding as in the proof, we can compute an integral equation for, say, $\alpha = 1 + 2\sqrt{2}$: Take $\omega_1 = 1, \omega_2 = \sqrt{2}$. Consider

$$T_\alpha : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}[\sqrt{2}], \quad x \mapsto \alpha x,$$

which has matrix representation w.r.t. the ω_i as $M = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$. Now $\det(XE - M) = X^2 - 2X - 7$ is the desired relation.

In the exercises, we will show the following slight generalization of proposition 1.3.

Proposition 1.6. Let A be a ring. Then the following are equivalent:

- (i) b is integral over A .
- (ii) $A[b]$ is finitely generated as an A -module.
- (iii) There exists an $A[b]$ -module M that is finitely generated as an A -module.

Theorem 1.7. Let $A \subseteq B \subseteq C$ be extensions of rings. Let B/A be integral and let $c \in C$ be integral over B . Then c is also integral over A .

Proof. Let $c^n + b_{n-1}c^{n-1} + \dots + b_0$ with $b_i \in B$. Then $A \subseteq A[b_0, \dots, b_{n-1}] \subseteq A[b_0, \dots, b_{n-1}][c]$ is a composition of finitely generated ring extensions by theorem 1.3, hence finitely generated. Again by theorem 1.3, we are done. \square

Definition 1.8. Let $A \subseteq B$ be a ring extension.

- (a) Then $\overline{A} = \mathcal{O}_{A,B} := \{b \in B \mid b \text{ integral over } A\}$ is called the *integral closure* of A in B .
- (b) A is called *integrally closed* in B if $\mathcal{O}_{A,B} = A$.

Note that by theorem 1.3, the integral closure of A in B is a ring. In particular, the ring of integers \mathcal{O}_K of a number field K is indeed a ring.

Example 1.9. $\mathcal{O}_{A,B}$ is integrally closed in B .

\mathbb{Z} is integrally closed in \mathbb{Q} . More generally, \mathcal{O}_K is integrally closed in K , for if $\alpha \in K$ is integral over \mathcal{O}_K , by transitivity 1.7 it is then integral over \mathbb{Z} , hence $\alpha \in \mathcal{O}_K$.

$R = \mathbb{Z}[\sqrt{-3}] \subseteq K = \mathbb{Q}(\sqrt{-3})$ is not integrally closed in K , because $\frac{1}{2}(1 + \sqrt{-3}) \notin R$ is integral (even over \mathbb{Z}).

Theorem 1.10. Let R be a UFD and $K = \text{Quot}(R)$. Then R is integrally closed in K .

Proof. Let $\frac{a}{b} \in K$ be integral over R , with $a, b \in R$ coprime. Let

$$X^n + c_{n-1}X^{n-1} + \dots + c_1X + c_0 = 0 \quad \text{with } c_i \in R$$

be an integral relation for $\frac{a}{b}$. Multiplying by b^n , we get

$$a^n + c_{n-1}ba^{n-1} + \dots + c_1ab^{n-1} + c_0b^n = 0.$$

Suppose $b \notin R^\times$, then there exists a prime element $\pi \in R$ dividing b . Looking at the equation mod π , we see that $\pi \mid a^n$; i.e. $\pi \mid a$, contradicting the coprime assumption. \square

Let A be an integral domain which is integrally closed in $K = \text{Quot}(A)$. Let L/K be a finite field extension and let $B = \mathcal{O}_{A,L}$ be the integral closure of A in L .

$$\begin{array}{ccc} L & \longleftrightarrow & B \\ | & & | \\ K & \longleftrightarrow & A \end{array}$$

Then, by transitivity, B is integrally closed in L .

Lemma 1.11. In the above situation, $L = \text{Quot}(B)$. More precisely, each $\beta \in L$ can be written in the form $\frac{b}{a}$ with $b \in B$ and $a \in A$.

Proof. For $\beta \in L$, let $a_n\beta^n + \dots + a_1\beta + a_0 = 0$ with $a_i \in A$. Multiplying by a_n^{n-1} , we obtain

$$(a_n\beta)^n + a_{n-1}(a_n\beta)^{n-1} + \dots + a_1a_n^{n-2}(a_n\beta) + a_0a_n^{n-1} = 0.$$

Thus $a_n\beta$ is integral over A , and $\beta = \frac{a_n\beta}{a_n}$ has the desired form. \square

Lemma 1.12. *One has $\beta \in B$ if and only if its minimal polynomial $\mu = \text{mipo}_{\beta, K}$ over K has coefficients in A .*

Proof. Let $g(\beta) = 0$ with $g \in A[X]$ normalized. Then $\mu \mid g$ in $K[X]$. Thus all zeroes of μ (in some algebraic closure of K) are integral over A . Since the coefficients of μ are the elementary symmetric functions in its zeroes, the coefficients of μ are integral over A . Since by assumption A is integrally closed in K , it follows that $\mu \in A[X]$. \square

We recall from Algebra the notions of trace and norm. Let L/K be a finite field extension of degree n , and let $x \in L$. Let $T_x : L \rightarrow L, y \mapsto xy$.

Lecture 3
Oct 22, 2025

Definition 1.13. We define $\text{Tr}_{L/K}(x) := \text{Tr}(T_x)$ and $N_{L/K}(x) := \det(T_x)$.

Lemma 1.14. (i) *Let $\chi_x(t) = \det(tE - T_x) \in K[t]$ be the characteristic polynomial of T_x .*

Let $\chi_x(t) = t^n - a_1 t^{n-1} + \dots + (-1)^n a_n$. Then $a_1 = \text{Tr}_{L/K}(x)$ and $a_n = N_{L/K}(x)$.

(ii) $\text{Tr}_{L/K}$ is K -linear.

(iii) $N_{L/K}$ is multiplicative

Proof. Everything follows from linear algebra once translated to the linear maps T_x . \square

Theorem 1.15. *Let L/K be separable. Let $G = G(L/K, K^c/K)$ be the set of all homomorphisms $\sigma : L \rightarrow K^c$ that fix K . (By separability we have $|G| = [L : K]$.) Then*

$$(i) \quad \chi_x(t) = \prod_{\sigma \in G} (t - \sigma(x))$$

$$(ii) \quad \text{Tr}_{L/K}(x) = \sum_{\sigma \in G} \sigma(x)$$

$$(iii) \quad N_{L/K}(x) = \prod_{\sigma \in G} \sigma(x)$$

Proof. (ii) and (iii) follow from (i) using lemma 1.14(i). Let $\mu_x(t)$ be the minimal polynomial of T_x . Then $\mu_x(T_x) = 0$, hence also $\mu_x(x) = 0$ in L . Further $\mu_x(\sigma(x)) = \sigma(\mu_x(x)) = 0$, so $\mu_x(t) = \prod_{\sigma \in G(K(x)/K, K^c/K)} (t - \sigma(x))$. We conclude with

$$\chi_x(t) = \mu_x(t)^{[L:K(x)]} = \prod_{\sigma \in G} (t - \sigma(x)),$$

where both steps need further explanation: Let $\sigma \in G(K(x)/K, K^c/K)$. Then there are $[L : K(x)]$ extensions $\tilde{\sigma}$ of σ , which thus all have the same value at x . This explains the second equality. For the first, choose bases $\omega_1, \dots, \omega_m$ and $1, x, \dots, x^{n-1}$ of $L/K(x)$ and $K(x)/K$, respectively. Then $\omega_i x^j$ is a basis of L/K , and T_x w.r.t. this basis has as matrix representation a block-diagonal matrix with each block equal to the matrix representation of μ_x w.r.t. the basis $1, x, \dots, x^{n-1}$. \square

Example 1.16. (i) $K = \mathbb{Q}(\sqrt{d})$ is a quadratic extension with $G = \{\text{id}, \sigma : \sqrt{d} \mapsto -\sqrt{d}\}$. Hence for $\alpha = a + b\sqrt{d}$ one has $\text{Tr}_{K/\mathbb{Q}}(\alpha) = 2a$ and $N_{K/\mathbb{Q}}(\alpha) = a^2 - b^2d$.

(ii) Let L/K be a finite field extension of degree m . Let $\alpha \in K$. Then $\text{Tr}_{L/K}(\alpha) = m\alpha$ and $N_{L/K}(\alpha) = \alpha^m$.

(iii) Let $L = \mathbb{Q}(\alpha)/K = \mathbb{Q}$, where $\alpha^3 = 2, \alpha \in \mathbb{R}$. In the exercises we will see $\mathcal{O}_L = \mathbb{Z}[\alpha]$. Let $x = 1 + \alpha$. We have

$$(1 + \alpha) \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} 1 + \alpha \\ \alpha + \alpha^2 \\ \alpha^2 + 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}}_{=: M} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix},$$

so $\text{Tr}_{L/K}(1 + \alpha) = \text{Tr}(M) = 3$ and $N_{L/K}(1 + \alpha) = \det(M) = 3$. Alternatively, we could have calculated

$$\text{Tr}_{L/\mathbb{Q}}(1 + \alpha) = \text{Tr}_{L/\mathbb{Q}}(1) + \text{Tr}_{L/\mathbb{Q}} = 3 + 0 = 3,$$

since the minimal polynomial $t^3 - 2$ of α has no t^2 -term.

Corollary 1.17. *Let $M/L/K$ be a tower of finite field extensions. Then for $\alpha \in M$ one has*

$$\text{Tr}_{M/K}(\alpha) = \text{Tr}_{L/K}(\text{Tr}_{M/L}(\alpha)) \quad \text{and} \quad N_{M/K}(\alpha) = N_{L/K}(N_{M/L}(\alpha)).$$

Proof. For $\sigma_i : L/K \rightarrow K^c/K$, we have $[M : L]$ extensions $\sigma_{ij} : M \rightarrow K^c$. Fix one such extension $\hat{\sigma}_i$.

$$\begin{array}{ccccc} & & \sigma_{ij} & & \\ & \nearrow & & \searrow & \\ M & \xrightarrow{\hat{\sigma}_i} & \hat{\sigma}_i(M) & \xrightarrow{\quad} & K^c \\ | & & | & & | \\ L & \xrightarrow{\sigma_i} & \sigma_i(L) & \xrightarrow{\text{id}} & \sigma_i(L) \\ | & & | & & | \\ K & \xrightarrow{\sigma_i} & \sigma_i(K) = K & & \end{array}$$

Then

$$\text{Tr}_{M/K}(\alpha) = \sum_{i,j} \sigma_{ij}(\alpha) = \sum_i \text{Tr}_{\hat{\sigma}_i M / \sigma_i L}(\hat{\sigma}_i(\alpha)). \quad (*)$$

Let $\omega = (\omega_1, \dots, \omega_m)^t$ be a L -basis of M . Then $\hat{\sigma}_i(\omega_1), \dots, \hat{\sigma}_i(\omega_m)$ is a $\sigma_i(L)$ -basis of $\hat{\sigma}_i(M)$. Let $\alpha\omega = M_\alpha\omega$ with $M_\alpha \in L^{m \times m}$. Then $\hat{\sigma}_i(\alpha)\hat{\sigma}_i(\omega) = \sigma_i(M_\alpha)\hat{\sigma}_i(\omega)$, where the actions on vectors and matrices is understood to be component-wise. Therefore,

$$\text{Tr}_{\hat{\sigma}_i(M)/\sigma_i(L)}(\hat{\sigma}_i(\alpha)) = \text{Tr}(\sigma_i(M_\alpha)) = \sigma_i(\text{Tr}(M_\alpha)) = \sigma_i(\text{Tr}_{M/L}(\alpha)).$$

Continuing from (*) we get

$$\text{Tr}_{M/K}(\alpha) = \sum_i \sigma_i(\text{Tr}_{M/L}(\alpha)) = \text{Tr}_{L/K}(\text{Tr}_{M/L}(\alpha)).$$

The same proof works for the norm, with all sums replaced by products. \square

Let L/K be a finite separable extension of fields. Let $\alpha_1, \dots, \alpha_n$ be $[L : K]$ -many elements of L .

Definition 1.18. The discriminant of $\alpha_1, \dots, \alpha_n$ is defined as

$$d(\alpha_1, \dots, \alpha_n) := \det(\sigma_i(\alpha_j))_{i,j=1,\dots,n}^2,$$

where $\{\sigma_1, \dots, \sigma_n\} = G(L/K, K^c/K)$.

Lemma 1.19. (i) $d(\alpha_1, \dots, \alpha_n) = \det(\text{Tr}_{L/K}(\alpha_i \alpha_j))_{1 \leq i,j \leq n}$.

(ii) For $\theta \in L$ we have $d(1, \theta, \theta^2, \dots, \theta^{n-1}) = \prod_{i < j} (\theta_i - \theta_j)^2$, where $\theta_i := \sigma_i(\theta)$.

Proof. One calculates

$$(\sigma_k(\alpha_i))_{k,i}^t (\sigma_k(\alpha_j))_{k,j} = \left(\sum_{k=1}^n \sigma_k(\alpha_i \alpha_j) \right)_{i,j} = (\text{Tr}_{L/K}(\alpha_i \alpha_j))_{i,j}$$

and takes determinants for the first part. For the second, the matrix in the definition 1.18 of d is the Vandermonde matrix of the θ_i . \square

Theorem 1.20. *Let L/K be a finite separable field extension of degree n . Let $\alpha_1, \dots, \alpha_n \in L$. Then*

- (i) $\alpha_1, \dots, \alpha_n$ is a K -basis of L if and only if $d(\alpha_1, \dots, \alpha_n) \neq 0$.
- (ii) The bilinear map $\langle -, - \rangle : L \times L \rightarrow K, (x, y) \mapsto \text{Tr}_{L/K}(xy)$ (called trace form) is nondegenerate.

Proof. For (ii), separability of L/K implies that $L = K(\theta)$ for some $\theta \in L$. The structure matrix of the bilinear form is given by

$$M = (\langle \theta^i, \theta^j \rangle)_{i,j} = (\text{Tr}_{L/K}(\theta^i \theta^j))_{i,j}.$$

Thus $\det(M) = d(1, \theta, \dots, \theta^{n-1}) = \prod_{i < j} (\theta_i - \theta_j)^2 \neq 0$ by lemma 1.19.

Now let $\alpha_1, \dots, \alpha_n$ be elements of L . Let S be the transition matrix from $1, \theta, \dots, \theta^{n-1}$ to $\alpha_1, \dots, \alpha_n$. Then $S^t M S$ is the structure matrix of $\langle -, - \rangle$ w.r.t. the α_i , so

$$d(\alpha_1, \dots, \alpha_n) = \det(S^t M S) = \det(S)^2 \det(M).$$

Hence $d(\alpha_1, \dots, \alpha_n) = 0$ iff $\det(S) = 0$ iff $\alpha_1, \dots, \alpha_n$ is not a basis. \square

As before, let A be an integral domain which is integrally closed in $K = \text{Quot}(A)$. Let L/K be a finite separable extension and $B = \mathcal{O}_{A,L} \subseteq L$ the integral closure of A in L .

Lemma 1.21. *For $b \in B$, one has $\text{Tr}_{L/K}(b), N_{L/K}(b) \in A$. Further, $b \in B$ is a unit if and only if $N_{L/K}(b) \in A^\times$.*

Proof. If b is integral, so is $\sigma(b)$ for all $\sigma \in G = G(L/K, K^c/K)$. Thus $\text{Tr}_{L/K}(b) = \sum_{\sigma} \sigma(b)$ and $N_{L/K}(b) = \prod_{\sigma} \sigma(b) \in K \cap B = A$, since A is integrally closed.

Let $b \in B^\times$, then $bc = 1$ for some $c \in B$. It follows that

$$1 = N_{L/K}(1) = N_{L/K}(bc) = N_{L/K}(b) N_{L/K}(c),$$

so $N_{L/K}(b) \in A^\times$.

Conversely, let $a = N_{L/K}(b) \in A^\times$. Then

$$1 = a^{-1} N_{L/K}(b) = a^{-1} \prod_{\sigma \in G} \sigma(b) = b a^{-1} \underbrace{\prod_{\text{id} \neq \sigma \in G} \sigma(b)}_{\in L, \text{ integral} \Rightarrow \in B}$$

\square

Example 1.22. Let $L = \mathbb{Q}(\alpha) \subseteq \mathbb{R}, \alpha^3 = 2$. Then

$$d(1, \alpha, \alpha^2) = \det(\text{Tr}_{L/\mathbb{Q}}(\alpha^i \alpha^j))_{0 \leq i, j \leq 2} = \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix} = -108.$$

In the exercises we will use this to prove $\mathcal{O}_L = \mathbb{Z}[\alpha]$.

Further we compute

$$N_{L/\mathbb{Q}}(1 - \alpha) = (1 - \alpha)(1 - \zeta_3\alpha)(1 - \zeta_3^2\alpha) = -1,$$

so by the above lemma $1 - \alpha \in \mathcal{O}_L^\times$. (Alternatively, we could have noticed that $(\alpha - 1)^{-1} = \frac{\alpha^3 - 1}{\alpha - 1} = 1 + \alpha + \alpha^2 \in \mathcal{O}_L$.) Actually, we have $\mathcal{O}_L^\times = \{\pm 1\} \times (1 - \alpha)^\mathbb{Z}$, which agrees with the result of Dirichlet's unit theorem 0.7, since there is one real and one pair of complex embeddings.

Lemma 1.23. *Let $\alpha_1, \dots, \alpha_n \in B$ be a K -basis of L . Let $d = d_{L/K}(\alpha_1, \dots, \alpha_n) \in A$. Then*

$$dB \subseteq A\alpha_1 \oplus \dots \oplus A\alpha_n.$$

Proof. Let $B \ni \alpha = a_1\alpha_1 + \dots + a_n\alpha_n$ with $a_i \in K$. Then $\text{Tr}_{L/K}(\alpha_i\alpha) = \sum_{j=1}^n a_j \text{Tr}_{L/K}(\alpha_i\alpha_j)$, hence (a_1, \dots, a_n) is a solution of

$$\sum_{j=1}^n \underbrace{\text{Tr}_{L/K}(\alpha_i\alpha_j)}_{=: A_{ij}} x_j = \text{Tr}_{L/K}(\alpha_i\alpha), \quad i = 1, \dots, n.$$

Cramer's rule shows that $a_j = \frac{\det A_j}{\det A} = \frac{\det A_j}{d}$, where A_j is the matrix A with j -th column replaced by the vector $(\text{Tr}_{L/K}(\alpha_i\alpha))_i$. Hence $d(a_1, \dots, a_n) \in A^n$. \square

Recall that for R a PID, each finitely generated torsion-free R -module M is free of finite rank, i.e. $M \cong R^n$, $n < \infty$. Further, if M is a free R -module and $N \subseteq M$ is an R -submodule, then N is free of rank at most the rank of M .

Theorem 1.24. *Assume further that A is a PID. Then any finitely generated B -submodule $0 \neq M \subseteq L$ is a free A -module of rank $n = [L : K]$. In particular, B has an integral basis over A , i.e. there exist $\omega_1, \dots, \omega_n \in B$ such that $B = A\omega_1 \oplus \dots \oplus A\omega_n$.*

Proof. Let $\alpha_1, \dots, \alpha_n \in B$ be a K -basis of L . Let $\mu_1, \dots, \mu_r \in M \subseteq L$ be a B -generating system of M . Let $0 \neq a \in A$ such that $a\mu_i \in B$ (possible by lemma 1.11). Let $d = d_{L/K}(\alpha_1, \dots, \alpha_n)$, which is nonzero by theorem 1.20. Then $daM \subseteq dB \subseteq A\alpha_1 \oplus \dots \oplus A\alpha_n \cong A^n$ by lemma 1.23. It follows that $daM \cong A^m$ with $m \leq n$, hence also $M \cong A^m$.

Let $0 \neq \mu \in M$. Then $\mu\alpha_1, \dots, \mu\alpha_n \in M$ are a K -basis of L , so they are certainly linearly independent in M as well, hence $m \geq n$. \square

Example 1.25. (i) $L = \mathbb{Q}(\sqrt{d})$, $\omega = \sqrt{d}$ for $d \equiv 2, 3 \pmod{4}$ or $\omega = \frac{1+\sqrt{d}}{2}$ for $d \equiv 1 \pmod{4}$ as before. Then $1, \omega$ is an integral basis of \mathcal{O}_L .

(ii) $L = \mathbb{Q}(\alpha)$, $\alpha^3 = 2$. In the exercises we will see that $1, \alpha, \alpha^2$ is an integral basis of \mathcal{O}_L .

(iii) Let K be a number field. Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$. Then \mathfrak{a} has a \mathbb{Z} -basis, equivalently \mathfrak{a} is free over \mathbb{Z} of rank n .

Remark 1.26. Let $L/K/\mathbb{Q}$ be number fields. Then \mathcal{O}_K is in general not a PID, so theorem 1.24 is not applicable to $\mathcal{O}_L/\mathcal{O}_K$. However, one can look at the localization $\mathcal{O}_{L,\mathfrak{p}} = S^{-1}\mathcal{O}_L$ at $S = \mathcal{O}_K \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$. Then $\mathcal{O}_{L,\mathfrak{p}} = \mathcal{O}_{\mathcal{O}_{K,\mathfrak{p}},L}$ is an $\mathcal{O}_{K,\mathfrak{p}}$ -module and a DVR, so the theorem can be applied to this ring extension.

Definition 1.27. Let L/\mathbb{Q} be a number field. Let $\alpha_1, \dots, \alpha_n \in \mathcal{O}_L$ be an integral basis, i.e. $\mathcal{O}_L = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$. Then $d_L = d_{L/\mathbb{Q}} := d_{L/\mathbb{Q}}(\alpha_1, \dots, \alpha_n)$ is called the *discriminant* of L (over \mathbb{Q}). More generally, if $0 \neq M \subseteq L$ is a finitely generated \mathcal{O}_L -module, then $d_L(M) = d_{L/\mathbb{Q}}(M) := d(m_1, \dots, m_n)$ for some integral basis m_1, \dots, m_n of M .

d_L is well-defined: Let β_1, \dots, β_n be another integral basis. Let $S \in \mathrm{GL}_n(\mathbb{Z})$ be the transition matrix from the α_i to the β_i . Then

$$\begin{aligned} d_{L/\mathbb{Q}}(\beta_1, \dots, \beta_n) &= \det(\mathrm{Tr}_{L/\mathbb{Q}}(\beta_i \beta_j)) = \det(S^t (\mathrm{Tr}_{L/\mathbb{Q}}(\alpha_i \alpha_j))_{ij} S) \\ &= \det(S)^2 \det(\mathrm{Tr}_{L/\mathbb{Q}}(\alpha_i \alpha_j)) = d_{L/\mathbb{Q}}(\alpha_1, \dots, \alpha_n). \end{aligned}$$

Example 1.28. $L = \mathbb{Q}(\sqrt{d})$, $d \equiv 2, 3 \pmod{4}$. Then

$$d_{L/\mathbb{Q}} = d_{L/\mathbb{Q}}(1, \sqrt{d}) = \det(\mathrm{Tr}_{L/\mathbb{Q}}(\sqrt{d}^{i+j}))_{0 \leq i, j \leq 1} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} = 4d.$$

Similarly one computes $d_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} = d$ for $d \equiv 1 \pmod{4}$.

Remark 1.29. (i) We will show that a prime p is ramified in L/\mathbb{Q} if and only if $p \mid d_{L/\mathbb{Q}}$ (where p is called ramified if the factorization $p\mathcal{O}_L = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ has some $e_i > 1$).

(ii) If L/K are number fields. One can easily define a "relative" discriminant $d_{L/K}$ if \mathcal{O}_K is a PID by the same procedure as above, except that it is only well-defined up to units, i.e. the ideal $d_{L/K} := (d_{L/K}(\alpha_1, \dots, \alpha_n))$ for an integral basis α_i is well-defined.

Now assume \mathcal{O}_K is arbitrary. As in remark 1.26, consider the extensions $\mathcal{O}_{L, \mathfrak{p}}/\mathcal{O}_{K, \mathfrak{p}}$ for prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$. As above, we may define thus "local" discriminant ideals $d_{L/K, \mathfrak{p}} \subseteq \mathcal{O}_{K, \mathfrak{p}}$. One can then prove that there exists a unique ideal $\mathfrak{D} \subseteq \mathcal{O}_K$ such that $\mathfrak{D}_{\mathfrak{p}} = d_{L/K, \mathfrak{p}}$ called the relative discriminant.

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Theorem 1.30. Let L/\mathbb{Q} be a number field. Let $0 \neq \mathfrak{a} \subseteq \mathfrak{a}'$ be \mathcal{O}_L -submodules of L . Then

$$d_L(\mathfrak{a}) = [\mathfrak{a}' : \mathfrak{a}]^2 d_L(\mathfrak{a}').$$

In particular, $[\mathfrak{a}' : \mathfrak{a}]$ is finite.

Proof. Let $\alpha'_1, \dots, \alpha'_n$ be a \mathbb{Z} -basis of \mathfrak{a}' and $\alpha_1, \dots, \alpha_n$ be a \mathbb{Z} -basis of \mathfrak{a} . Let T be the transition matrix, i.e. $\alpha_i = \sum_{j=1}^n t_{ij} \alpha'_j$, $t_{ji} \in \mathbb{Z}$. As before, we see that $d(\mathfrak{a}) = \det(T)^2 d(\mathfrak{a}')$. So it remains to show that $|\det(T)| = [\mathfrak{a}' : \mathfrak{a}]$. By the elementary divisor theorem, we may assume that T is a diagonal matrix, from where the claim follows easily. \square

Corollary 1.31. Let $\alpha_1, \dots, \alpha_n \in \mathcal{O}_L$. If $d_L(\alpha_1, \dots, \alpha_n)$ is squarefree, then $\alpha_1, \dots, \alpha_n$ is an integral basis.

Remark 1.32. This is not a necessary condition: In example 1.28 we saw $4 \mid d_{\mathbb{Q}(\sqrt{d})}$ for $d \equiv 2, 3 \pmod{4}$.

2 Ideals

Noetherian Rings Let R be a ring. Recall from commutative algebra that an R -module M is called *Noetherian* if all submodules of M are finitely generated. In particular, M is finitely generated. For $M = R$ this says that R is Noetherian if all ideals of R are finitely generated. For example, PIDs, finite rings, or finite modules are clearly Noetherian.

Further recall that if R is noetherian and M a finitely generated R -module, then M is noetherian; as well as the following

Theorem 2.1. The following are equivalent:

(i) M is Noetherian

- (ii) Each ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M stabilizes, i.e. there exists $n_0 \in \mathbb{N}$ s.t. $M_i = M_{n_0}$ for all $i \geq n_0$.
- (iii) Every non-empty family of R -submodules of M contains maximal elements.

Theorem 2.2. Let K/\mathbb{Q} be a number field. Then \mathcal{O}_K is Noetherian, integrally closed and of dimension 1, i.e. each non-zero prime ideal is maximal.

Proof. Each ideal $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ has a finite \mathbb{Z} -basis by theorem 1.24, hence in particular finitely generated. Thus \mathcal{O}_K is noetherian. \mathcal{O}_K is integrally closed by definition and transitivity 1.7.

Finally, for $0 \neq \mathfrak{p}$ prime, $\mathcal{O}_K/\mathfrak{p}$ is an integral domain which is finite by theorem 1.30, hence a field. Therefore, \mathfrak{p} is maximal. \square

Definition 2.3. A noetherian, integrally closed integral domain of dimension 1 is called a *Dedekind domain*.

Example 2.4. By theorem 2.2, \mathcal{O}_K is a Dedekind domain. Further, any PID is clearly Dedekind.

Our next goal will be to show that in a Dedekind domain \mathcal{O} , every ideal factors uniquely as a product of prime ideals.

Definition 2.5. Let R be a ring and $\mathfrak{a}, \mathfrak{b}$ be ideals.

- (i) We write $\mathfrak{a} \mid \mathfrak{b}$ for $\mathfrak{b} \subseteq \mathfrak{a}$.
- (ii) The ideal sum $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$ is also called the gcd of \mathfrak{a} and \mathfrak{b} .
- (iii) The intersection $\mathfrak{a} \cap \mathfrak{b}$ is also called the lcm of \mathfrak{a} and \mathfrak{b} .

Theorem 2.6. Let \mathcal{O} be a Dedekind domain and $\mathfrak{a} \subseteq \mathcal{O}$ an ideal, $\mathfrak{a} \neq (0), (1)$. Then there exists a unique presentation (up to order) of \mathfrak{a} in the form

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r \quad (*)$$

with prime ideals $\mathfrak{p}_i \neq (0)$. If we write $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$ with pairwise distinct primes \mathfrak{p}_j , then also $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cap \cdots \cap \mathfrak{p}_s^{e_s}$

Proof. We start with the second statement: In general, one has $\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$ for coprime ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$ for any ring R . Also, if $\mathfrak{p}, \mathfrak{q}$ are coprime, then so are \mathfrak{p}^e and \mathfrak{q}^f .

For the main statement, we will need the following lemmas:

Lemma 2.7. Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}$ be an ideal. Then there are non-zero prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, $r \geq 1$, s.t. $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{a}$

Proof. Let

$$\mathcal{M} := \{0 \neq \mathfrak{a} \subseteq \mathcal{O} \text{ ideal} \mid \mathfrak{a} \text{ does not satisfy the statement of the lemma}\}.$$

Suppose $\mathcal{M} \neq \emptyset$. Since \mathcal{O} is noetherian, by theorem 2.1 there exists a maximal element $\mathfrak{a} \in \mathcal{M}$. Then \mathfrak{a} is not a prime ideal, so there exist $b_1, b_2 \in \mathcal{O}$ such that $b_1 b_2 \in \mathfrak{a}$, but $b_1, b_2 \notin \mathfrak{a}$. Let $\mathfrak{a}_i := \mathfrak{a} + (b_i)$. By choice of \mathfrak{a} , we have $\mathfrak{a}_i \notin \mathcal{M}$, hence we can write

$$\mathfrak{a}_1 \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_s, \quad \mathfrak{a}_2 \supseteq \mathfrak{q}_1 \cdots \mathfrak{q}_r$$

for nonzero prime ideals $\mathfrak{p}_i, \mathfrak{q}_j$. But then

$$\mathfrak{p}_1 \cdots \mathfrak{p}_s \mathfrak{q}_1 \cdots \mathfrak{q}_r \subseteq \mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a} + (b_1 b_2) \subseteq \mathfrak{a},$$

contradicting $\mathfrak{a} \in \mathcal{M}$. \square

Lemma 2.8. Let $0 \neq \mathfrak{p} \subseteq \mathcal{O}$ be a prime ideal. Let $K := \text{Quot}(\mathcal{O})$ and

$$\mathfrak{p}^{-1} := \{x \in K \mid x\mathfrak{p} \subseteq \mathcal{O}\} \subseteq K.$$

Then $\mathfrak{p}^{-1} \supseteq \mathcal{O}$ is a non-zero \mathcal{O} -module, and for any ideal $0 \neq \mathfrak{a} \subseteq \mathcal{O}$ one has $\mathfrak{a}\mathfrak{p}^{-1} \supsetneq \mathfrak{a}$.

Proof. Everything is clear but the strictness of the final inclusion. Let $0 \neq a \in \mathfrak{p}$. By lemma 2.7 there exists a product of nonzero prime ideals $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq (a)$ with r minimal. Since $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{p}$ and all these ideals are maximal, we have $\mathfrak{p}_1 = \mathfrak{p}$, say. By minimality of r , $\mathfrak{p}_2 \cdots \mathfrak{p}_r \not\subseteq (a)$, so there exists $b \in \mathfrak{p}_2 \cdots \mathfrak{p}_r \setminus (a)$, hence $a^{-1}b \notin \mathcal{O}$. On the other hand $b\mathfrak{p} \subseteq (a)$, so $a^{-1}b\mathfrak{p} \subseteq \mathcal{O}$, i.e. $a^{-1}b \in \mathfrak{p}^{-1}$. Hence $\mathfrak{p}^{-1} \supsetneq \mathcal{O}$.

Let now $0 \neq \mathfrak{a} \subseteq \mathcal{O}$ be an ideal. Let $\mathfrak{a} = (\alpha_1, \dots, \alpha_n)$ and suppose $\mathfrak{a}\mathfrak{p}^{-1} = \mathfrak{a}$. Let $x \in \mathfrak{p}^{-1}$. Then

$$x\alpha_i = \sum_{j=1}^n a_{ji}\alpha_j, \quad a_{ji} \in \mathcal{O}.$$

Let $A = (xE - (a_{ji}))$. Then $A(\alpha_1, \dots, \alpha_n)^t = 0$, so by lemma 1.4, $\det(A)\alpha_i = 0$, so x is a zero of the normalized polynomial $\det(tE - (a_{ji})) \in \mathcal{O}[t]$, hence x is integral over \mathcal{O} . But \mathcal{O} is integrally closed by definition, so $x \in \mathcal{O}$. Thus we have shown $\mathfrak{p}^{-1} \subseteq \mathcal{O}$, contradicting the previous paragraph. \square

Now we can return to the proof of theorem 2.6. Let

$$\mathcal{M} := \{\mathfrak{a} \subseteq \mathcal{O} \text{ ideal} \mid \mathfrak{a} \neq (0), (1); \mathfrak{a} \text{ cannot be written as in } (*)\}.$$

Suppose $\mathcal{M} \neq \emptyset$. Since \mathcal{O} is Noetherian, by theorem 2.1 there exists a maximal element $\mathfrak{a} \subseteq \mathcal{M}$. Let $\mathfrak{p} \supseteq \mathfrak{a}$ be a maximal ideal containing \mathfrak{a} . By lemma 2.8, $\mathfrak{a} \subsetneq \mathfrak{a}\mathfrak{p}^{-1}$ and $\mathfrak{p} \subsetneq \mathfrak{p}\mathfrak{p}^{-1} \subseteq \mathcal{O}$. Since \mathfrak{p} is maximal, $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}$. By choice of \mathfrak{a} , we know that $\mathfrak{a}\mathfrak{p}^{-1} \notin \mathcal{M}$, so there is a factorization

$$\mathfrak{a}\mathfrak{p}^{-1} = \mathfrak{p}_1 \cdots \mathfrak{p}_s \implies \mathfrak{a} = \mathfrak{a}\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_s\mathfrak{p}.$$

This contradicts $\mathfrak{a} \in \mathcal{M}$, showing the existence of ideal factorizations.

For uniqueness, suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_r = \mathfrak{q}_1 \cdots \mathfrak{q}_s$. Then $\mathfrak{q}_1 \cdots \mathfrak{q}_s \subseteq \mathfrak{p}_1$, so one of the factors is already contained in \mathfrak{p}_1 , wlog $\mathfrak{q}_1 \subseteq \mathfrak{p}_1$. Since \mathfrak{q}_1 is maximal, $\mathfrak{q}_1 = \mathfrak{p}_1$. Then multiply the original equation by \mathfrak{p}_1^{-1} and proceed inductively. \square

For convenience, we will often write prime ideal factorizations in the form $\mathfrak{a} = \prod_{\mathfrak{p} \neq 0} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$, where $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{N}_0$ is zero for almost all \mathfrak{p} . By the Chinese Remainder Theorem, we have

$$\mathcal{O}/\mathfrak{a} \cong \prod_{\mathfrak{p} \neq 0} \mathcal{O}/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}.$$

Definition 2.9. A fractional ideal in $K = \text{Quot}(\mathcal{O})$ is a nonzero finitely generated \mathcal{O} -submodule of K .

Example 2.10. (i) For $a \in K^\times$, $(a) = a\mathcal{O}$ is a principal fractional ideal.

(ii) More generally, $c\mathfrak{a}$ is a fractional ideal for $0 \neq \mathfrak{a} \subseteq \mathcal{O}$ an ideal and $c \in K^\times$.

Lemma 2.11. $\mathfrak{a} \subseteq K$ be a fractional ideal if and only if there exists $c \in \mathcal{O} \setminus \{0\}$ such that $c\mathfrak{a}$ is an ideal of \mathcal{O} .

Proof. The backwards direction is clear. Let $\mathfrak{a} = (\alpha_1, \dots, \alpha_s)$ be a fractional ideal. Write $\alpha_1 = \frac{b_1}{c_1}$ with $b_i, c_i \in \mathcal{O}$. Then $\prod c_i \mathfrak{a} \subseteq \mathcal{O}$ is an ideal of \mathcal{O} . \square

To better distinguish fractional ideals and ideals contained in \mathcal{O} , we will often call the latter "integral ideals".

Theorem 2.12. *Let $J_{\mathcal{O}}$ be the set of fractional ideals. Then $J_{\mathcal{O}}$ is an abelian group w.r.t. multiplication of ideals. The identity element is \mathcal{O} , and the inverse of \mathfrak{a} is given by $\mathfrak{a}^{-1} = (\mathcal{O} : \mathfrak{a})$, where*

$$(\mathfrak{b} : \mathfrak{c}) := \{x \in K \mid x\mathfrak{c} \subseteq \mathfrak{b}\}$$

Proof. In the proof of theorem 2.6 we have seen $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}$. Let now \mathfrak{a} be an integral ideal. For $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$, we have the inverse $\mathfrak{b} = \mathfrak{p}_1^{-1} \cdots \mathfrak{p}_r^{-1}$. By lemma 2.11, each fractional ideal has an inverse.

Let now \mathfrak{a} be a fractional ideal and \mathfrak{b} its inverse, we want to show $\mathfrak{b} = (\mathcal{O} : \mathfrak{a})$. The inclusion $\mathfrak{b} \subseteq (\mathcal{O} : \mathfrak{a})$ is clear from the definition of inverse. If $x \in (\mathcal{O} : \mathfrak{a})$. Then $x\mathfrak{a} \subseteq \mathcal{O}$, so $x\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}$, i.e. $x \in \mathfrak{b}$, finishing the proof. \square

Corollary 2.13. *Let $\mathfrak{a} \in J_{\mathcal{O}}$ be a fractional ideal. Then we have a unique representation of \mathfrak{a} in the form*

$$\mathfrak{a} = \prod_{\mathfrak{p} \neq 0} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$$

with $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$ and almost all $v_{\mathfrak{p}}(\mathfrak{a}) = 0$. Further, we can uniquely write $\mathfrak{a} = \mathfrak{b}\mathfrak{c}^{-1} =: \frac{\mathfrak{b}}{\mathfrak{c}}$ with $\mathfrak{b}, \mathfrak{c} \subseteq \mathcal{O}$ integral ideals s.t. $(\mathfrak{b}, \mathfrak{c}) = 1$.

Lemma 2.14. *Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}$ be an integral ideal, and let $\mathfrak{p} \neq 0$ be a prime ideal. Let $\mathfrak{a} = \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}\mathfrak{b}$ with $v_{\mathfrak{p}}(\mathfrak{a}) \geq 0$ and $\mathfrak{p} \nmid \mathfrak{b}$. Then $v_{\mathfrak{p}}(\mathfrak{a}) = n$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}^n$ and $\mathfrak{a} \not\subseteq \mathfrak{p}^{n+1}$, i.e. $v_{\mathfrak{p}}(\mathfrak{a})$ is the highest power of \mathfrak{p} dividing \mathfrak{a} .*

Proof. If $\mathfrak{a} = \mathfrak{p}^n\mathfrak{b}$, it is clear that $\mathfrak{a} \subseteq \mathfrak{p}^n$, and if $\mathfrak{a} \subseteq \mathfrak{p}^{n+1}$, then we would have $\mathfrak{b} \subseteq \mathfrak{p}$.

Conversely, suppose $\mathfrak{a} \subseteq \mathfrak{p}^n$. Then $\mathfrak{b} := \mathfrak{a}\mathfrak{p}^{-n} \subseteq \mathcal{O}$ is an ideal, and $\mathfrak{a} = \mathfrak{b}\mathfrak{p}^n$ shows $v_{\mathfrak{p}}(\mathfrak{a}) \geq n$. Suppose $\mathfrak{p} \mid \mathfrak{b}$, i.e. $\mathfrak{b} \subseteq \mathfrak{p}$. Then $\mathfrak{a} = \mathfrak{p}^n\mathfrak{b} \subseteq \mathfrak{p}^{n+1}$, contradicting the assumption. \square

Definition 2.15. Let \mathcal{O} be a Dedekind domain and $K = \text{Quot}(\mathcal{O})$. Set $P_{\mathcal{O}} = \{x\mathcal{O} \mid x \in K^{\times}\} \subseteq J_{\mathcal{O}}$ be the subgroup of principal fractional ideals. Then $\text{cl}_{\mathcal{O}} := J_{\mathcal{O}}/P_{\mathcal{O}}$ is called the *ideal class group* of \mathcal{O} .

In the case of a number field K/\mathbb{Q} with ring of integers \mathcal{O}_K , write $\text{cl}_K = \text{cl}_{\mathcal{O}_K}$ and similarly for J_K and P_K . Our next aim is to prove that cl_K is a finite group. This is not true for general Dedekind domains.

Remark 2.16. From the definition it is clear that a Dedekind domain \mathcal{O} is a PID if and only if $|\text{cl}_{\mathcal{O}}| = 1$. In general, we have the following exact sequence

$$1 \rightarrow \mathcal{O}^{\times} \hookrightarrow K^{\times} \xrightarrow{a \mapsto (a)} J_{\mathcal{O}} \xrightarrow{a \mapsto [a]} \text{cl}_{\mathcal{O}} \rightarrow 1$$

Theorem 2.17. *Let \mathcal{O} be a Dedekind domain with finitely many prime ideals. Then \mathcal{O} is a PID.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the nonzero prime ideals of \mathcal{O} . It suffices to show that each \mathfrak{p}_i is principal, the result then follows from the prime ideal factorization 2.6. Let $a_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_1^2$. By the Chinese Remainder Theorem, there exists $a \in \mathcal{O}$ such that $a \equiv a_1 \pmod{\mathfrak{p}_1^2}$ and $a \equiv 1 \pmod{\mathfrak{p}_i}$ for $i > 1$.

Then $\mathfrak{p}_1 = a\mathcal{O}$. Indeed, let $a\mathcal{O} = \mathfrak{p}_1^{\nu_1} \cdots \mathfrak{p}_n^{\nu_n}$. Since $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_1^2$ and $a \in \mathcal{O} \setminus \mathfrak{p}_i$, lemma 2.14 shows $\nu_1 = 1$ and $\nu_i = 0$ for $i > 1$. \square

Let $\mathcal{O} \subseteq K = \text{Quot}(\mathcal{O})$ be a Dedekind domain and $S \subseteq \mathcal{O}$ be a multiplicative subset. Then $S^{-1}\mathcal{O}$ is still Dedekind: It is clearly a noetherian integral domain of dimension 1, by the correspondence of ideals in \mathcal{O} and $S^{-1}\mathcal{O}$. For integrally closed check in general that $S^{-1}\mathcal{O}_{B,C} = \mathcal{O}_{S^{-1}B, S^{-1}C}$.

Now take a prime $\mathfrak{p} \neq 0$ and $S = S_{\mathfrak{p}} := \mathcal{O} \setminus \mathfrak{p}$. Then $\mathcal{O}_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}\mathcal{O}$ is a Dedekind domain with exactly one prime $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$, hence a PID by theorem 2.17, even a DVR.

Theorem 2.18. *Let $0 \neq \mathfrak{m} \subseteq \mathcal{O}$ be an ideal. Let $c \in \text{cl}_{\mathcal{O}}$ be an ideal class. Then c contains an integral ideal $\mathfrak{a} \subseteq \mathcal{O}$ with $(\mathfrak{a}, \mathfrak{m}) = 1$.*

Proof. If there are only finitely many primes, then $\text{cl}_{\mathcal{O}} = 1$ by theorem 2.17, so we may take $\mathfrak{a} = \mathcal{O}$. Suppose now we have infinitely many primes. Let $\mathfrak{m} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_s^{f_s}$ be the unique prime ideal factorization of \mathfrak{m} and $c = [\mathfrak{a}]$, wlog $\mathfrak{a} \subseteq \mathcal{O}$. Let $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \mathfrak{b}$, $r \leq s$ and $(\mathfrak{b}, \mathfrak{m}) = 1$. Choose $\alpha_i \in \mathfrak{p}_i^{e_i} \setminus \mathfrak{p}_i^{e_i+1}$ for $i = 1, \dots, r$. By the Chinese Remainder Theorem, there is $\alpha \in \mathcal{O}$ such that

$$\begin{aligned} \alpha &\equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i+1}} && \text{for } i = 1, \dots, r, \\ \alpha &\equiv 1 \pmod{\mathfrak{p}_i} && \text{for } i = r+1, \dots, s. \end{aligned}$$

Then by lemma 2.14 $\alpha\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \mathfrak{c}$ for an integral ideal \mathfrak{c} with $(\mathfrak{c}, \mathfrak{m}) = 1$. □

In general, \mathcal{O} is not a PID, but

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Theorem 2.19. *Each ideal $\mathfrak{a} \in J_{\mathcal{O}}$ can be generated by two elements. In fact, given $0 \neq \alpha \in \mathfrak{a}$, then there exists $\beta \in \mathfrak{a}$ with $\mathfrak{a} = (\alpha, \beta)$.*

Proof. Suffices to consider $\mathfrak{a} \subseteq \mathcal{O}$. Claim: If $0 \neq \mathfrak{b} \subseteq \mathcal{O}$ is an ideal, then every ideal of \mathcal{O}/\mathfrak{b} is principal.

Given this, let $0 \neq \alpha \in \mathfrak{a}$ and let $\pi : \mathcal{O} \rightarrow \mathcal{O}/(\alpha)$ be the canonical projection. Then the image of \mathfrak{a} under π is principal by the claim, say $\bar{\mathfrak{a}} = (\bar{\beta})$. Hence $\mathfrak{a} = \pi^{-1}((\bar{\beta})) = (\alpha, \beta)$.

Hence it remains to prove the claim. Write $\mathfrak{b} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ with $e_i \geq 1$ and $(\mathfrak{p}_i, \mathfrak{p}_j) = 1$. Let $\bar{\mathfrak{c}} \subseteq \mathcal{O}/\mathfrak{b}$ be an ideal, with $\mathfrak{c} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_r^{f_r}$, $f_i \leq e_i$ the corresponding ideal in \mathcal{O} . By the Chinese Remainder Theorem, $\mathcal{O}/\mathfrak{b} \cong \mathcal{O}/\mathfrak{p}_1^{e_1} \times \cdots \times \mathcal{O}/\mathfrak{p}_r^{e_r}$, let $\mathfrak{q}_1 \times \cdots \times \mathfrak{q}_r$ be the image of \mathfrak{p}_i under this isomorphism. It suffices to show that the \mathfrak{q}_j are principal. But $\mathfrak{q}_j = 1$ for $i \neq j$, and $\mathfrak{q}_i = \mathfrak{p}_i/\mathfrak{p}_i^{e_i}$.

More generally, $\mathfrak{p}^i/\mathfrak{p}^e$ is principal in $\mathcal{O}/\mathfrak{p}^e$: Take $\alpha \in \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$, then $\alpha\mathcal{O} + \mathfrak{p}^e = \mathfrak{p}^i$ by lemma 2.14, so $(\bar{\alpha}) = \mathfrak{p}^i/\mathfrak{p}^e$. □

In general, computing integral bases is difficult. However, sometimes they can be pieced together from smaller rings: Let K, L be number fields of degree n, m , respectively. Let $M = KL$ be their composite. Then $\mathcal{O}_K\mathcal{O}_L \subseteq \mathcal{O}_M$.

Theorem 2.20. *Assume that $[M : \mathbb{Q}] = mn$. Let $d := \gcd(d_K, d_L)$. Then $\mathcal{O}_M \subseteq \frac{1}{d}\mathcal{O}_K\mathcal{O}_L$.*

Corollary 2.21. *If $[M : \mathbb{Q}] = mn$ and $\gcd(d_K, d_L) = 1$, then $\mathcal{O}_M = \mathcal{O}_L\mathcal{O}_K$. In addition, $d_M = d_L^m d_K^n$.*

Example 2.22. For $m \in \mathbb{N}$ let ζ_m be a primitive m -th root of unity. Then $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ is a number field, called *cyclotomic field*, of degree $\varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$, and a Galois extension with $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$, where the isomorphism is given by $k \mapsto \sigma_k : \zeta_m \mapsto \zeta_m^k$.

We will show $\mathcal{O}_{\mathbb{Q}(\zeta_{p^n})} = \mathbb{Z}[\zeta_{p^n}]$ and that $d_{\mathbb{Q}(\zeta_{p^n})}$ is a power of p . Further it is easy to see that $\mathbb{Q}(\zeta_m)\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{mn})$ for m, n coprime. So corollary 2.21 implies $\mathcal{O}_{\mathbb{Q}(\zeta_m)} = \mathbb{Z}[\zeta_m]$ and gives a formula for the discriminant of $\mathbb{Q}(\zeta_m)$.

Proof. Claim: Let $\sigma : K \rightarrow \mathbb{C}$, $\tau : L \rightarrow \mathbb{C}$ be embeddings. Then there exists a unique embedding $\kappa : M \rightarrow \mathbb{C}$ such that $\kappa|_K = \sigma$ and $\kappa|_L = \tau$. For the restriction map $\text{Hom}_{\mathbb{Q}}(M, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \times \text{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ is clearly injective and between finite sets of the same size nm , so bijective.

Let $\alpha_1, \dots, \alpha_n$ be an integral basis of \mathcal{O}_K , and β_1, \dots, β_m an integral basis of \mathcal{O}_L . Then $\alpha_i \beta_j$ form a \mathbb{Z} -basis of $\mathcal{O}_K \mathcal{O}_L$. Any $\alpha \in \mathcal{O}_N$ can be written in the form $\alpha = \sum_{i,j} \frac{m_{ij}}{r} \alpha_i \beta_j$ with $m_{ij}, r \in \mathbb{Z}$ and $\gcd(r, \gcd(m_{ij})_{i,j}) = 1$. To show: $r \mid d$.

By symmetry, it suffices to show $r \mid d_K$. By the claim, for each $\sigma : K \rightarrow \mathbb{C}$ there exists a unique $\tilde{\sigma} : M \rightarrow \mathbb{C}$ such that $\tilde{\sigma}|_K = \sigma$ and $\tilde{\sigma}|_L = \text{id}_L$. Then

$$\tilde{\sigma}(\alpha) = \sum_{i,j} \frac{m_{ij}}{r} \tilde{\sigma}(\alpha_i \beta_j) = \sum_{i,j} \frac{m_{ij}}{r} \sigma(\alpha_i) \beta_j.$$

Set $x_i = \sum_{j=1}^m \frac{m_{ij}}{r} \beta_j$. Then we have n equations $\tilde{\sigma}(\alpha) = \sum_{i=1}^n \sigma(\alpha_i) x_i$, one for each σ . By Cramer's rule, $x_i = \frac{\gamma_i}{\delta}$, where $\delta = \det(\sigma(\alpha_i))_{\sigma,i}$. Clearly, $\gamma_i, \delta_i \in \mathcal{O}_M$, and by definition $\delta^2 = d_K$. Hence $d_K x_i = \delta \gamma_i$, so $d_K x_i = \sum_j \frac{d_K m_{ij}}{r} \beta_j \in \mathcal{O}_N \cap L = \mathcal{O}_L$. But this means $r \mid d_K m_{ij}$ for all i, j , so $r \mid d_K$ by the coprimality assumption.

For the discriminant formula in the corollary, we now know that $\alpha_i \beta_j$ is a \mathbb{Z} -basis of \mathcal{O}_M , hence

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$$\begin{aligned} d_N &= d(\alpha_i \beta_j) = \det(\text{Tr}_{M/\mathbb{Q}}(\alpha_i \beta_j \alpha_k \beta_l)) = \det(\text{Tr}_{K/\mathbb{Q}}(\text{Tr}_{M/K}(\alpha_i \beta_j \alpha_k \beta_l))) \\ &= \det(\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k \text{Tr}_{M/K}(\beta_j \beta_l))) = \det(\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k \text{Tr}_{L/\mathbb{Q}}(\beta_j \beta_l))) \\ &= \det(\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k) \text{Tr}_{L/\mathbb{Q}}(\beta_j \beta_l)) = \det((\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k)) \otimes (\text{Tr}_{L/\mathbb{Q}}(\beta_j \beta_l))) \\ &= d_K^m d_L^n, \end{aligned}$$

where we used the fact from linear algebra that $A \otimes B = (a_{ij} B) \in R^{nm \times nm}$ for $A \in R^{n \times n}$, $B \in R^{m \times m}$ satisfies $\det(A \otimes B) = \det(A)^m \det(B)^n$ \square

3 Lattices

Definition 3.1. Let V be an n -dimensional \mathbb{R} -vector space. A *lattice* in V is a subgroup Γ of V of the form $\Gamma = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_m$ with linearly independent vectors $v_1, \dots, v_m \in V$, $m \leq n$. The set $\Phi = \{x_1 v_1 + \dots + x_m v_m \mid 0 \leq x_i < 1\}$ is called a *fundamental domain* of Γ . Further, Γ is a *full lattice* if $m = n$.

Definition 3.2. A subgroup Γ of V is called *discrete* if for each $\gamma \in \Gamma$ there exists a neighbourhood U such that $\Gamma \cap U = \{\gamma\}$

Lemma 3.3. If Γ is a discrete subgroup of V , then Γ is closed.

Proof. Claim: Each $a \in V \setminus \Gamma$ has an open neighbourhood U with $|\Gamma \cap U| < \infty$.

Then since V is Hausdorff, there exists an open neighbourhood \tilde{U} of a that avoids these finitely many points, so $(U \cap \tilde{U}) \cap \Gamma = \emptyset$, i.e. $U \cap \tilde{U}$ is a neighbourhood of a in $V \setminus \Gamma$.

To prove the claim, let $a \in V \setminus \Gamma$. By assumption, there exists an open $\tilde{U} \subseteq V$ such that $\tilde{U} \cap \Gamma = \{0\}$. Since $V \times V \rightarrow V$, $(a, b) \mapsto a - b$ is continuous, there exists an open neighbourhood U of 0 such that $U - U \subseteq \tilde{U}$. Then $a + U$ is an open neighbourhood of a , suppose there are $\gamma_1, \gamma_2 \in \Gamma \cap (a + U)$. But then $\gamma_1 - \gamma_2 \in \tilde{U}$, so $\gamma_1 = \gamma_2$. \square

Lemma 3.4. Let Γ be a subgroup of V . Then Γ is discrete if and only if for all bounded $C \subseteq V$ one has $|C \cap \Gamma| < \infty$.

Proof. Let Γ be discrete. Wlog C is compact. If $C \cap \Gamma$ were infinite, then by Bolzano-Weierstrass, there is an accumulation point $\gamma \in C \cap \Gamma$ (by lemma 3.3), contradicting the definition.

Conversely, let $\gamma \in \Gamma$. Choose an open ball around γ . By assumption, this ball contains only finitely many $\gamma_i \in \Gamma$, which, as before, can be separated from γ using the Hausdorff property. \square

Example 3.5. Let $K = \mathbb{Q}(\sqrt{2}) \subseteq \mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}[\sqrt{2}]$ is not a lattice in $V = \mathbb{R}$, but \mathcal{O}_K becomes a lattice in \mathbb{R}^2 via

$$j : \mathcal{O}_K \hookrightarrow \mathbb{R}^2, \quad a + b\sqrt{2} \mapsto (a + b\sqrt{2}, a - b\sqrt{2}).$$

We will prove soon (in general) that $j(\mathcal{O}_K) \subseteq \mathbb{R}^2$ is a lattice.

Theorem 3.6. *Let $\Gamma \subseteq V$ be a subgroup. Then Γ is a lattice if and only if Γ is discrete.*

Proof. Let $\Gamma = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_m$ be a lattice. Choose a basis $v_1, \dots, v_m, v_{m+1}, \dots, v_n$ of V . Let $\gamma = a_1v_1 + \dots + a_mv_m$. Consider

$$U := \{x_1v_1 + \dots + x_nv_n \mid x_i \in \mathbb{R} \mid |a_i - x_i| < 1 \text{ for } i \leq m\}.$$

Then U is open and $U \cap \Gamma = \{\gamma\}$.

Conversely, let Γ be discrete. Let V_0 be the \mathbb{R} -subspace of V generated by Γ and denote $m := \dim_{\mathbb{R}} V_0$. Choose a \mathbb{R} -basis u_1, \dots, u_m of V_0 with $u_i \in \Gamma$. Consider $\Gamma_0 := \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_m \subseteq V_0$, which is a lattice by definition.

Claim: $q := (\Gamma : \Gamma_0) < \infty$. Then $\Gamma_0 \subseteq \Gamma \subseteq \frac{1}{q}\Gamma_0$ is a subgroup of a free abelian group, so is itself free (of rank m).

To prove the claim, let $\{\gamma_i\}_{i \in I}$ be a set of representatives of Γ/Γ_0 . Let $\Phi_0 = \{x_1u_1 + \dots + x_mu_m \mid 0 \leq x_i < 1\}$ be a fundamental domain of Γ_0 . Then $\bigcup_{\gamma \in \Gamma_0} (\gamma + \Phi_0) = V$, hence $\gamma_i = \gamma_{0i} + \mu_i$ with $\gamma_{0i} \in \Gamma_0$ and $\mu_i \in \Phi_0$. Then the bounded Φ_0 contains all the $\mu_i = \gamma_i - \gamma_{0i} \in \Gamma$, hence I is finite by lemma 3.4. \square

Lemma 3.7. *Let $\Gamma \subseteq V$ be a lattice. Then Γ is full if and only if there exists a bounded subset $M \subseteq V$ such that $\bigcup_{\gamma \in \Gamma} \gamma \in \Gamma(\gamma + M) = V$.*

Proof. If Γ is full, take M to be a fundamental domain. Conversely, let V_0 be the \mathbb{R} -span of Γ . Let $v \in V$. For $\nu \in \mathbb{N}$ write $\nu v = \gamma_\nu + a_\nu$ with $\gamma_\nu \in \Gamma$ and $a_\nu \in M$. Since M is bounded, $\frac{a_\nu}{\nu} \xrightarrow{\nu \rightarrow \infty} 0$. Hence

$$v = \lim_{\nu \rightarrow \infty} \frac{\gamma_\nu + a_\nu}{\nu} = \lim_{\nu \rightarrow \infty} \frac{\gamma_\nu}{\nu} \in V_0,$$

since $V_0 \subseteq V$ is closed. \square

Now let V be an euclidean vector space with inner product $\langle -, - \rangle : V \times V \rightarrow \mathbb{R}$. Let e_1, \dots, e_n be an orthonormal basis. Then we define a volume. For the "unit cube"

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$$E := \left\{ \sum_i \alpha_i e_i \mid 0 \leq \alpha_i \leq 1 \right\},$$

we set $\text{Vol}(E) = 1$. More generally, let v_1, \dots, v_n be an \mathbb{R} -basis of V and let $\Phi := \{\sum_i x_i v_i \mid 0 \leq x_i \leq 1\}$. Let $A = (a_{ji}) \in \text{GL}_n(\mathbb{R})$ be the transition matrix, $v_i = \sum_j a_{ji} e_j$.

Lemma 3.8. *One has $\text{Vol}(\Phi) = |\det(A)| = \sqrt{\det(\langle v_i, v_j \rangle_{ij})}$.*

Proof.

$$\text{Vol}(\Phi) = \int_{\Phi} dx = \int_E |\det(A)| dx = |\det(A)| \text{Vol}(E) = |\det(A)|.$$

The second equality follows from $\langle v_i, v_j \rangle_{ij} = A^t \langle e_i, e_j \rangle_{ij} A = A^t A$. \square

Definition 3.9. Let $\Gamma \subseteq V$ be a full lattice. Then we define $\text{Vol}(\Gamma) := \text{Vol}(\Phi)$ for any fundamental domain Φ for Γ .

This is well-defined, i.e. independent of the choice of Φ , since different \mathbb{Z} -bases of Γ differ by a transition matrix $T \in \text{GL}_n(\mathbb{Z})$, i.e. $\det(T) = \pm 1$, so the absolute value of the determinant does not change.

Definition 3.10. Let $X \subseteq V$ be a subset. X is called *central-symmetric* if for all $x \in X$ we have $-x \in X$. X is *convex* if for all $x, y \in X$ also $tx + (1 - t)y \in X$ for $0 \leq t \leq 1$.

For example, a ball centered around 0 is both central-symmetric and convex.

Theorem 3.11 (Minkowski's Lattice Point Theorem). *Let $\Gamma \subseteq V$ be a full lattice in an euclidean vector space of dimension $\dim_{\mathbb{R}}(V) = n$. Let $X \subseteq V$ be a central-symmetric, convex subset with $\text{Vol}(X) > 2^n \text{Vol}(\Gamma)$. Then there exists a $0 \neq \gamma \in \Gamma$ with $\gamma \in X$.*

Proof. It suffices to show that there are $\gamma_1 \neq \gamma_2 \in \Gamma$ such that $(\frac{1}{2}X + \gamma_1) \cap (\frac{1}{2}X + \gamma_2) \neq \emptyset$. Indeed, let $v = \frac{1}{2}x_1 + \gamma_1 + \frac{1}{2}x_2 + \gamma_2$ be an element of the intersection. Then

$$\gamma_1 - \gamma_2 = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}x_2 + \left(1 - \frac{1}{2}\right)(-x_1) \in X$$

by central-symmetry and convexity.

To prove the claim, suppose that the sets $(\frac{1}{2}X + \gamma)$, $\gamma \in \Gamma$ are pairwise disjoint. Then so are the sets $\Phi \cap (\frac{1}{2}X + \gamma)$ for a fundamental domain Φ of Γ . Hence

$$\text{Vol}(\Gamma) = \text{Vol}(\Phi) \geq \sum_{\gamma \in \Gamma} \text{Vol}(\Phi \cap (\gamma + \frac{1}{2}X)) = \sum_{\gamma \in \Gamma} \text{Vol}((\Phi - \gamma) \cap \frac{1}{2}X).$$

Since $\Phi - \gamma$ covers all of X , cf. lemma 3.7. Therefore

$$\text{Vol}(\Gamma) \geq \text{Vol}(\frac{1}{2}X) = 2^{-n} \text{Vol}(X),$$

contradicting our assumption. □

4 Minkowski Theory

Let K/\mathbb{Q} be a number field of degree n . Of the embeddings $\tau : K \rightarrow \mathbb{C}$, we distinguish real embeddings $\rho_1, \dots, \rho_r : K \rightarrow \mathbb{R}$ and pairs of complex embeddings $\sigma_1, \bar{\sigma}_1, \dots, \sigma_s, \bar{\sigma}_s : K \rightarrow \mathbb{C}$ with image not contained in \mathbb{R} , with $n = r + 2s$.

Definition 4.1. We define *Minkowski Space* of K as

$$K_{\mathbb{R}} := \left\{ (z_{\tau}) \in \prod_{\tau: K \rightarrow \mathbb{C}} \mathbb{C} \mid z_{\rho} \in \mathbb{R}, z_{\sigma} = \overline{z_{\bar{\sigma}}} \right\}$$

Remark 4.2. $K_{\mathbb{R}}$ is an \mathbb{R} -vector space of dimension $r + 2s = n$.

Example 4.3. Let $K = \mathbb{Q}(\sqrt{d})$. If $d > 0$, then $K_{\mathbb{R}} = \mathbb{R}\rho_1 + \mathbb{R}\rho_2$. If, on the other hand, $d < 0$, then $K_{\mathbb{R}} = \{(\beta, \bar{\beta}) \mid \beta \in \mathbb{C}\} \subseteq \mathbb{C}^2$.

Example 4.4. For $K = \mathbb{Q}(\omega)$ with $\omega \in \mathbb{R}$, $\omega^3 = 2$, we have $K_{\mathbb{R}} = \{(\alpha, \beta, \bar{\beta}) \mid \alpha \in \mathbb{R}, \beta \in \mathbb{C}\}$.

On $K_{\mathbb{R}}$ we define the inner product

$$\langle x, y \rangle := \sum_{\tau} x_{\tau} \overline{y_{\tau}}.$$

It is clear that this is bilinear and positive definite; we check that the image is contained in \mathbb{R} :

$$\langle x, y \rangle = \sum_{\rho} x_{\rho} y_{\rho} + \sum_{\sigma} (x_{\sigma} \overline{y_{\sigma}} + \underbrace{x_{\overline{\sigma}} \overline{y_{\overline{\sigma}}}}_{= \overline{x_{\sigma}} y_{\sigma}}).$$

Hence the terms of the last sum are stable under conjugation, thus they lie in \mathbb{R} .

Theorem 4.5. *Consider the isomorphism of \mathbb{R} -vector spaces*

$$f : K_{\mathbb{R}} \rightarrow \prod_{\tau} \mathbb{R}, \quad (z_{\tau})_{\tau} \mapsto (z_{\rho_1}, \dots, z_{\rho_r}, \operatorname{Re}(z_{\sigma_1}), \operatorname{Im}(z_{\sigma_1}), \dots, \operatorname{Re}(z_{\sigma_s}), \operatorname{Im}(z_{\sigma_s})).$$

For $(-, -) : (\prod_{\tau} \mathbb{R})^2 \rightarrow \mathbb{R}$ defined by $(x, y) := \sum \alpha_{\tau} x_{\tau} y_{\tau}$ with $\alpha_{\tau} = 1$ if τ is real and $\alpha_{\tau} = 2$ if τ is complex, we have $\langle x, y \rangle = (f(x), f(y))$ for all $x, y \in K_{\mathbb{R}}$.

Proof. Exercise. □

Remark 4.6. By the above theorem, $\operatorname{Vol}_{(-,-)} = 2^s \operatorname{Vol}_{\text{Lebesgue}}$, since an orthonormal basis w.r.t. $(-, -)$ is given by $e_1, \dots, e_r, \frac{1}{\sqrt{2}}e_{r+1}, \dots, \frac{1}{\sqrt{2}}e_{r+2s}$.

Generalizing example 3.5, define

$$j : K \hookrightarrow K_{\mathbb{R}}, \quad \alpha \mapsto (\tau(\alpha))_{\tau: K \rightarrow \mathbb{C}}.$$

This is a \mathbb{Q} -linear embedding.

Theorem 4.7. *Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ be an ideal. Then $\Gamma := j(\mathfrak{a})$ is a full lattice in $K_{\mathbb{R}}$ with $\operatorname{Vol}(\Gamma) = \sqrt{|d_K|} [\mathcal{O}_K : \mathfrak{a}]$.*

Proof. Let $\alpha_1, \dots, \alpha_n$ be a \mathbb{Z} -basis of \mathfrak{a} . Consider $A = (\tau_l(\alpha_i))_{il}$. Then $d(\mathfrak{a}) = \det(A)^2 = [\mathcal{O}_K : \mathfrak{a}]^2 d_K$. On the other hand, $j(\alpha_1), \dots, j(\alpha_n)$ is a \mathbb{Z} -basis of Γ . We have $j(\alpha_i) = (\tau_l(\alpha_i))_l$, so that

$$\langle j(\alpha_i), j(\alpha_k) \rangle = \sum_l \tau_l(\alpha_i) \overline{\tau_l(\alpha_k)}.$$

Hence the structure matrix of $\langle -, - \rangle$ is $(\langle j(\alpha_i), j(\alpha_k) \rangle)_{i,k} = A \overline{A}^t$, so

$$\operatorname{Vol}(\Gamma) = \sqrt{\det(A \overline{A}^t)} = |\det(A)| = [\mathcal{O}_K : \mathfrak{a}] \sqrt{|d_K|}.$$

In particular, the volume of a fundamental domain is nonzero, so the lattice is full. □

Theorem 4.8. *Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ be an ideal. Let $c_{\tau} \in \mathbb{R}_{>0}$ such that $c_{\tau} = c_{\overline{\tau}}$. Assume that $\prod_{\tau} c_{\tau} > (\frac{2}{\pi})^s \sqrt{|d_K|} [\mathcal{O}_K : \mathfrak{a}]$. Then there exists $0 \neq a \in \mathfrak{a}$ with $|\tau(a)| < c_{\tau}$ for all τ .*

Proof. Look at $X := \{(z_{\tau})_{\tau} \in K_{\mathbb{R}} \mid |z_{\tau}| < c_{\tau}\}$. Then X is convex and central-symmetric. One computes

$$\operatorname{Vol}(X) = 2^{r+s} \pi^s \prod_{\tau} c_{\tau} > 2^n \operatorname{Vol}(j(\mathfrak{a})).$$

Therefore, the conditions of Minkowski's Lattice Point Theorem 3.11 are satisfied, so there exists $0 \neq j(a) \in j(\mathfrak{a}) \cap X$. This is the desired $a \in \mathfrak{a}$. □

Definition 4.9. For $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ we define its norm

$$N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}].$$

In the exercises we saw that this ideal norm is multiplicative, hence we may extend it to a multiplicative function $N : I_K \rightarrow \mathbb{Z}$ on all fractional ideals.

Lemma 4.10. Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$. Then there exists $0 \neq a \in \mathfrak{a}$ with $|N_{K/\mathbb{Q}}(a)| \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} N(\mathfrak{a})$.

Proof. For $\varepsilon > 0$ choose $c_\tau \in \mathbb{R}_{>0}$, $c_{\bar{\tau}} = c_\tau$ such that $\prod_\tau c_\tau = \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} N(\mathfrak{a}) + \varepsilon$. By theorem 4.8, there exists $0 \neq a \in \mathfrak{a}$ such that $|\tau(a)| < c_\tau$, hence

$$|N_{K/\mathbb{Q}}(a)| = \prod_\tau |\tau(a)| < \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} N(\mathfrak{a}) + \varepsilon.$$

Since the norm is an integer, for small enough ε we get the claim. \square

Theorem 4.11. The class number is finite: $h_K := |\text{cl}_K| < \infty$.

Proof. For each $M > 0$, there are only finitely many integral ideals $\mathfrak{a} \subseteq \mathcal{O}_K$ with $N(\mathfrak{a}) < M$. Indeed, since each such integral ideal factors into prime ideals, it suffices to show that there are only finitely many prime ideals of bounded norm. But by the exercises, $N(\mathfrak{p})$ is a p -power, where $p \in \mathbb{Z}$ is the prime such that $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$. Since there are only finitely many prime ideals containing each prime p , we are done.

Hence it suffices to show that each ideal class $c \in \text{cl}_K$ contains an integral ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ such that $N(\mathfrak{a}) \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$. So let $\mathfrak{b} \in c$ be a representative. Choose $\gamma \in \mathcal{O}_K$ such that $\gamma\mathfrak{b}^{-1} \subseteq \mathcal{O}_K$ is an integral ideal. Then by the previous lemma, there exists $0 \neq \alpha \in \gamma\mathfrak{b}^{-1}$ such that

$$|N_{K/\mathbb{Q}}(\alpha)| \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} N(\gamma\mathfrak{b}^{-1}).$$

Using the following lemma, the integral ideal $\alpha\gamma^{-1}\mathfrak{b}$ satisfies $N(\alpha\gamma^{-1}\mathfrak{b}) \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|}$. \square

Lemma 4.12. For $0 \neq \alpha \in K$ one has $N(\alpha\mathcal{O}_K) = |N_{K/\mathbb{Q}}(\alpha)|$.

Proof. Let $\omega_1, \dots, \omega_n$ be an integral basis of \mathcal{O}_K . Let $\alpha(\omega_1, \dots, \omega_n)^t = A(\omega_1, \dots, \omega_n)^t$ for $A \in M_n(\mathbb{Z})$. Then $[\mathcal{O}_K : \alpha\mathcal{O}_K] = |\det(A)| = |N_{K/\mathbb{Q}}(\alpha)|$. \square

Remark 4.13. The proof of theorem 4.11 yields a finite generating set for the class group:

$$\text{cl}_K = \langle [\mathfrak{a}] \mid 0 \neq \mathfrak{a} \subseteq \mathcal{O}_K, N(\mathfrak{a}) \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} \rangle.$$

In fact, the bound on $N(\mathfrak{a})$ can be improved: Let

$$X = \left\{ (z_\tau)_\tau \in K_{\mathbb{R}} \mid \sum_\tau |z_\tau| < t \right\}.$$

Then X is central-symmetric, convex, and $\text{Vol}(X) = 2^r \pi^s \frac{t^n}{n!}$. Repeating the above proofs with this set, one obtains

Theorem 4.14. Let $0\mathfrak{a} \subseteq \mathcal{O}_K$ be an integral ideal. Then there exists $0 \neq a \in \mathfrak{a}$ such that $|N_{K/\mathbb{Q}}(a)| \leq M \cdot N(\mathfrak{a})$ with the Minkowski constant

$$M := \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}.$$

Therefore, cl_K is generated by classes of ideals \mathfrak{a} with $N(\mathfrak{a}) \leq M$.

Proof. Exercise. □

Example 4.15. (i) Let $K = \mathbb{Q}(\sqrt{2})$. Then $M = \sqrt{2} \approx 1.41$. But the only integral ideal with norm 1 is \mathcal{O}_K , so $\text{cl}_K = 1$.

(ii) Let $K = \mathbb{Q}(\sqrt{-5})$. Then $M = \frac{4}{\pi}\sqrt{5} \approx 2.84$. Hence cl_K is generated by the classes of integral ideals of norm ≤ 2 . By factoring (2), one can compute directly that the only such ideals are \mathcal{O}_K and $\mathfrak{p} = \langle 2, 1 + \sqrt{-5} \rangle_{\mathbb{Z}}$ with $\text{ord}([\mathfrak{p}]) \leq 2$ since $\mathfrak{p}^2 = (2)$. So $\text{cl}_K = 1$ if \mathfrak{p} is principal, or $\text{cl}_K = \mathbb{Z}/2\mathbb{Z}$ otherwise. Here, the latter is the case, so $h_K = 2$.

Generalizing the last example, we can give a general procedure for computing the class group:

List all prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ with $N(\mathfrak{p}_i) \leq M$, by factoring $p\mathcal{O}_K$ for $p \leq M$ prime. Then cl_K is generated by their classes (it suffices to consider prime ideals by prime ideal factorization). Let $\pi : \mathbb{Z}^m \rightarrow \text{cl}_K$, $a \mapsto \prod_i [\mathfrak{p}_i]^{a_i}$ and $\Lambda := \ker \pi$. This gives an exact sequence

$$0 \rightarrow \Lambda \rightarrow \mathbb{Z}^m \xrightarrow{\pi} \text{cl}_K \rightarrow 1.$$

Every relation comes from an equation $\alpha\mathcal{O}_K = \prod_i \mathfrak{p}_i^{a_i}$, $\alpha \in K^\times$. Finding sufficiently many of such relations, one can determine the class group.

Lemma 4.16. *Let $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ be an integral ideal. Then \mathfrak{a} is a principal ideal if and only if $N(\mathfrak{a}) = |N_{K/\mathbb{Q}}(\alpha)|$ for some $\alpha \in \mathfrak{a}$.*

Proof. One direction was proven in lemma 4.12. Conversely, suppose α as in the statement exists. Then $\alpha\mathcal{O}_K \subseteq \mathfrak{a}$ is a submodule, but again by lemma 4.12, their indices in \mathcal{O}_K are equal. □

5 Dirichlet's Unit Theorem

The next goal is to understand the unit group of a number ring. Let

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$$K_{\mathbb{R}}^\times := \{(z_\tau)_\tau \in K_{\mathbb{R}} \mid z_\tau \neq 0 \text{ for all } \tau\}$$

and consider the map

$$l : K_{\mathbb{R}}^\times \rightarrow \mathbb{R}^{r+s}, \quad (z_\tau)_\tau \mapsto (\log |x_{\rho_1}|, \dots, \log |x_{\rho_r}|, 2 \log |x_{\sigma_1}|, \dots, 2 \log |x_{\sigma_s}|).$$

Then clearly $l(xy) = l(x) + l(y)$ for $x, y \in K_{\mathbb{R}}^\times$. Further define a norm $N : K_{\mathbb{R}}^\times \rightarrow \mathbb{R}^\times$ and trace $\text{Tr} : \mathbb{R}^{r+s} \rightarrow \mathbb{R}$ by $N((z_\tau)_\tau) := \prod_\tau z_\tau$ and $\text{Tr}(x) = \sum_{i=1}^{r+s} x_i$. Putting everything together, we have the following commutative diagram of group homomorphisms:

$$\begin{array}{ccccc} K^\times & \xrightarrow{j} & K_{\mathbb{R}}^\times & \xrightarrow{l} & \mathbb{R}^{r+s} \\ \downarrow N_{K/\mathbb{Q}} & & \downarrow N & & \downarrow \text{Tr} \\ \mathbb{Q}^\times & \hookrightarrow & \mathbb{R}^\times & \xrightarrow{\log |\cdot|} & \mathbb{R} \end{array}$$

Also let $\lambda := l \circ j$. Since units in \mathcal{O}_K are characterized by their norm being ± 1 , further define

$$S := \{y \in K_R^\times \mid N(y) = \pm 1\}, \quad H = \{x \in \mathbb{R}^{r+s} \mid \text{Tr}(x) = 0\}.$$

Then $j(\mathcal{O}_K^\times) \subseteq S$ and $l(S) = H$.

Definition 5.1. $\Gamma := \lambda(\mathcal{O}_K^\times) \subseteq H$

Theorem 5.2. (i) *There is a short exact sequence $1 \rightarrow \mu_K \rightarrow \mathcal{O}_K^\times \xrightarrow{\lambda} \Gamma \rightarrow 0$, where $\mu_K := \{x \in K^\times \mid \text{ord}(x) < \infty\}$.*
(ii) $|\mu_K| < \infty$.

Proof. We have to show that $\ker(\lambda) = \mu_K$. " \supseteq " is clear, either by direct computation or by noticing that H has no torsion elements. Conversely, let $\alpha \in \ker \lambda$. Then $|\tau(\alpha)| = 1$ for all τ , hence also $|\tau(\alpha^n)| = 1$ for all $n \geq 1$. Looking at $\chi_{\alpha^n}(t) = \prod_{\tau}(t - \tau(\alpha^n))$, we see that all coefficients are bounded. Hence there are only finitely many possible characteristic polynomials, each with finitely many zeroes, among the α^n . So $\alpha^i = \alpha^j$ for some $i > j$, i.e. $\alpha^{i-j} = 1$. The same argument also shows (ii). \square

Lemma 5.3. *Up to multiplication by elements in \mathcal{O}_K^\times , there are only finitely many $\alpha \in \mathcal{O}_K$ with $|\mathbf{N}_{K/\mathbb{Q}}(\alpha)| = a$, where $a \in \mathbb{N}$ is given.*

Proof. We have $|\mathbf{N}_{K/\mathbb{Q}}(\alpha)| = \mathbf{N}(\alpha \mathcal{O}_K)$ by lemma 4.12, but in the proof of theorem 4.11, we already showed that there are only finitely many ideals of bounded norm. \square

Theorem 5.4. Γ is a full lattice in H , i.e. $\Gamma \cong \mathbb{Z}^{r+s-1}$.

Proof. To show that Γ is a lattice, we may show by theorem 3.6 that it is discrete. By lemma 3.4, it suffices to show that for all $c \in \mathbb{R}_{>0}$ one has $B_c \cap \Gamma$ finite, where $B_c := \{(x_\tau) \in \mathbb{R}^{r+s} \mid |x_\tau| < c\}$. But by definition of the map l , this is the same as requiring $e^{-c} < x_\rho < e^c$ for real embeddings and $e^{-\frac{1}{2}c} < x_\sigma < e^{\frac{1}{2}c}$ for complex embeddings, which is finite again by lemma 3.4, since $j(\mathcal{O}_K) \supseteq j(\mathcal{O}_K^\times)$ is a lattice by theorem 4.7.

We now have to show that Γ has full rank. We want to apply lemma 3.7, i.e. construct a bounded set $M \subseteq H$ s.t. $\bigcup_{\gamma \in \Gamma} \gamma + H = H$. For this, we construct a bounded set $T \subseteq S$ s.t. $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} Tj(\varepsilon)$. Then by surjectivity of λ the translates of $M := l(T)$ clearly cover H . Let $x \in T$. Then $|x_\tau|$ is bounded from above since T is bounded. But then $|x_\tau|$ is also bounded from below (away from 0) because $\prod_{\tau} |x_\tau| = 1$. Therefore, $\log |x_\tau|$, and hence M , are bounded, and the theorem follows.

To construct T , choose $c_\tau > 0$ with $c_{\bar{\tau}} = c_\tau$ and $C := \prod_{\tau} c_\tau > (\frac{2}{\pi})^s \sqrt{|d_K|}$. Consider $X = \{(z_\tau)_\tau \in K_\mathbb{R} \mid |z_\tau| < c_\tau\}$. For $y \in S$ one has $Xy = \{z \in K_\mathbb{R} \mid |z_\tau| < c'_\tau\}$ with $c'_\tau = c_\tau |y_\tau|$, $c'_{\bar{\tau}} = c'_\tau$ and $\prod_{\tau} c'_\tau = C \prod_{\tau} |y_\tau| = C$. Hence by lemma 4.10 there exists $0 \neq a \in \mathcal{O}_K$ with $j(a) \in Xy$.

By lemma 5.3 there exist $\alpha_1, \dots, \alpha_N \in \mathcal{O}_K$ such that each $a \in \mathcal{O}_K$ with $0 < |\mathbf{N}_{K/\mathbb{Q}}(a)| < C$ is associated to one of the α_i . Now set

$$T := S \cap \bigcup_{i=1}^N Xj(\alpha_i)^{-1}.$$

We claim this T has the required properties. It is clear that T is bounded, since X is. So let $y \in S$. By the previous paragraph, there exists $0 \neq a \in \mathcal{O}_K$ with $j(a) \in Xy^{-1}$, i.e. $j(a) = xy^{-1}$ for some $x \in X$. Since $|\mathbf{N}_{K/\mathbb{Q}}(a)| = |\mathbf{N}(xy^{-1})| = |\mathbf{N}(x)| < C$, there exists α_i such that $\alpha_i = \varepsilon a$, $\varepsilon \in \mathcal{O}_K^\times$. Then $y = xj(a)^{-1} = xj(\alpha_i \varepsilon)^{-1} = xj(\alpha_i)^{-1}j(\varepsilon)^{-1}$, hence $xj(\alpha_i)^{-1} \in S \cap Xj(\alpha_i)^{-1} \subseteq T$. Therefore $y \in Tj(\varepsilon)^{-1}$, which finishes the proof. \square

Combining theorems 5.2 and 5.4, let $s : \Gamma \rightarrow \mathcal{O}_K^\times$ be a splitting of $1 \rightarrow \mu_K \rightarrow \mathcal{O}_K^\times \rightarrow \Gamma \rightarrow 0$, which exists since Γ is free. Then $\mu_K \times \Gamma \cong \mathcal{O}_K^\times$, $(\varepsilon, \gamma) \mapsto \varepsilon \cdot s(\gamma)$ is an isomorphism. That is, we have proven

Theorem 5.5. \mathcal{O}_K^\times is a finitely generated group of rank $t := r + s - 1$. Explicitly, there exist so-called fundamental units $\varepsilon_1, \dots, \varepsilon_t \in \mathcal{O}_K^\times$ such that each $\varepsilon \in \mathcal{O}_K^\times$ has a unique representation of the form

$$\varepsilon = \zeta \varepsilon_1^{k_1} \cdots \varepsilon_t^{k_t}$$

with $k_i \in \mathbb{Z}$ and $\zeta \in \mu_K$.

Example 5.6. Let $K = \mathbb{Q}(\sqrt{d})$, $d > 1$ squarefree. Then $t = 1$ and $\mu_K = \{\pm 1\}$, hence there is a single fundamental unit $\varepsilon \in \mathcal{O}_K^\times$ such that $\mathcal{O}_K^\times = \{\pm \varepsilon^n \mid n \in \mathbb{Z}\}$.

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Let K be a number field and $\varepsilon_1, \dots, \varepsilon_t$ be fundamental units. Let $\lambda_0 := \frac{1}{\sqrt{r+s}}(1, \dots, 1)^t \in \mathbb{R}^{r+s}$ so that $\|\lambda_0\| = 1$ and $\lambda_0 \perp H$. Hence the t -dimensional volume of Γ is equal to the $(r+s)$ -dimensional volume of the \mathbb{Z} -span of $\lambda_0, \lambda(\varepsilon_1), \dots, \lambda(\varepsilon_t)$, i.e. of the matrix

$$M = \begin{pmatrix} \frac{1}{\sqrt{r+s}} & \log |\tau_1(\varepsilon_1)| & \cdots & \log |\tau_1(\varepsilon_t)| \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{r+s}} & \log |\tau_{r+s}(\varepsilon_1)| & \cdots & \log |\tau_{r+s}(\varepsilon_t)| \end{pmatrix}$$

Let Φ be a fundamental domain of $\Gamma = \lambda(\mathcal{O}_K^\times)$. Then $\text{Vol}(\Phi) = |\det(M)|$.

Theorem 5.7. $\text{Vol}(\Gamma) = \sqrt{r+s}R$, where R is an arbitrary $t \times t$ -minor of $(\lambda(\varepsilon_1), \dots, \lambda(\varepsilon_t))$.

Definition 5.8. $R_K := R$ as in theorem 5.7 is called the *regulator* of K (Exercise: It is independent of the choice of fundamental units).

Proof. Fix some i and add all rows to the i -th row. Then this row becomes $(\sqrt{r+s}, 0, \dots, 0)$. \square

Lemma 5.9. Let V be a finite-dimensional \mathbb{R} -vector space. Let $\Gamma \subseteq V$ be a full lattice, and $\Gamma' \subseteq V$ be a sublattice. Then Γ' is full if and only if $\text{Vol}(\Gamma') \neq 0$, and in this case $[\Gamma : \Gamma'] = \text{Vol}(\Gamma') / \text{Vol}(\Gamma)$.

Proof. Let $\omega_1, \dots, \omega_n$ be a \mathbb{Z} -basis of Γ , and $\omega'_1, \dots, \omega'_n$ a \mathbb{Z} -basis of Γ' . Let Φ, Φ' be the corresponding fundamental domains, and let $\omega'_i = \sum_j t_{ji} \omega_j$, with $t_{ji} \in \mathbb{Z}$. Then $T = (t_{ji}) \in \text{GL}_n(\mathbb{Q})$ and

$$\text{Vol}(\Gamma') = \text{Vol}(\Phi') = \int_{\Phi'} dx = \int_{\Phi} |\det(T)| dx = |\det(T)| \text{Vol}(\Gamma) = [\Gamma : \Gamma'] \text{Vol}(\Gamma).$$

The other direction is clear. \square

Theorem 5.10. Let $\eta_1, \dots, \eta_t \in \mathcal{O}_K^\times$. Then the η_i are independent (i.e. $[\mathcal{O}_K^\times : \langle \eta_1, \dots, \eta_t \rangle] < \infty$) if and only if $R(\eta_1, \dots, \eta_t) \neq 0$, where $R(\eta_1, \dots, \eta_t)$ is defined as a $t \times t$ -minor of the matrix $(\lambda(\eta_1), \dots, \lambda(\eta_t))$ as before. Further, $[\mathcal{O}_K^\times / \mu_K : \langle \eta_1, \dots, \eta_t \rangle \mu_K / \mu_K] = R(\eta_1, \dots, \eta_t) / R_K$.

Proof. Exercise. \square

Remark 5.11. Regulators are in general transcendental numbers.

Let $\zeta_K(s) := \sum_{0 \neq \mathfrak{a} \leq \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - \frac{1}{N(\mathfrak{p})^s})^{-1}$ for $\text{Re}(s) > 1$ be the Dedekind L -function of K . In the special case $K = \mathbb{Q}$, this is the usual Riemann zeta function. As in this special case, ζ_K can be analytically extended to a meromorphic function on all of \mathbb{C} , with a simple pole only at $s = 1$. Further, ζ_K satisfies a functional equation of the form

$$\zeta_K(1-z) = 2|d_K|^{s-1/2} \cos(\frac{\pi z}{2})^{r+s} \sin(\frac{\pi z}{2})^s (2\pi)^{-z} \Gamma(z) \zeta_K(z).$$

$\zeta_K(s)$ has a zero at $s = 0$ of order t and the leading term is $\pm h_K R_K$. This is the so-called *analytic class number formula*, which is proved in analytic number theory.

- Example 5.12.** (i) For imaginary quadratic fields, one has $t = 0$, so \mathcal{O}_K^\times is finite.
(ii) For real quadratic fields, one has $t = 1$, and we will see how to compute a fundamental unit.
(iii) Let $K = \mathbb{Q}(\sqrt[3]{m})$ for m cubefree. Then $t = 1 + 1 - 1 = 1$.
(iv) Let $K = \mathbb{Q}(\zeta_m)$ be a cyclotomic field. Then K/\mathbb{Q} is a Galois extension with Galois group $G = (\mathbb{Z}/m\mathbb{Z})^\times$. We have $\varphi(m)$ complex embeddings $\sigma_a : \zeta_m \mapsto \zeta_m^a$, so $t_K = \frac{1}{2}\varphi(m) - 1$. Let $K^+ := \mathbb{Q}(\zeta_m + \zeta_m^{-1}) = K \cap \mathbb{R}$ be the fixed field of σ_{-1} . This is the largest totally real subfield of K of degree $\frac{1}{2}\varphi(m)$, so $t_{K^+} = \frac{1}{2}\varphi(K) - 1 = t_K =: t$. Let $\mathcal{O}_{K^+}^\times = \pm \varepsilon_1^{\mathbb{Z}} \cdots \varepsilon_t^{\mathbb{Z}}$, then $[\mathcal{O}_K^\times : \mathcal{O}_{K^+}^\times] < \infty$. Actually the index is small, cf. exercises.

For $m = p$, p an odd prime, the element $\frac{\zeta_p^a - 1}{\zeta_p - 1}$, $(a, p) = 1$ is a unit, since if $ab = 1 \pmod{p}$, then

$$\frac{\zeta_p - 1}{\zeta_p^a - 1} = \frac{\zeta_p^{ab} - 1}{\zeta_p^a - 1} = \sum_{i=0}^{b-1} \zeta_p^{ai} \in \mathbb{Z}[\zeta_p] \subseteq \mathcal{O}_K.$$

Hence we have $p - 2$ nontrivial units (for $a = 2, \dots, p - 1$), called *cyclotomic units*. One can show that the index of the subgroup generated by them depends explicitly on h_K . (The above can be generalized to $\mathbb{Q}(\zeta_m)$ for m not a prime.) See Washington, Cyclotomic Units for details.

5.1 The fundamental unit in a real quadratic field

Let $d > 1$ be squarefree, and $K = \mathbb{Q}(\sqrt{d})$. Then $\mathcal{O}_K \ni \alpha = \frac{1}{2}(x + y\sqrt{d})$, $x, y \in \mathbb{Z}$, is a unit if and only if $N_{K/\mathbb{Q}}(\alpha) = \pm 1$.

Corollary 5.13. *The units of \mathcal{O}_K are in 1-1 correspondence with the solutions $(x, y) \in \mathbb{Z}^2$ of the (generalized) Pell's equation $x^2 - dy^2 = \pm 4$.*

Let $\pm 1 \neq \eta \in \mathcal{O}_K^\times$. Then $\eta, \eta^{-1}, -\eta, -\eta^{-1}$ are four different units. Let τ be the nontrivial Galois automorphism of K , then $\tau(\eta) = \pm \eta^{-1}$. Then if $\eta = x + y\sqrt{d}$, then

$$\{\eta, \eta^{-1}, -\eta, -\eta^{-1}\} = \{\eta, \tau(\eta), -\eta, -\tau(\eta)\} = \{\pm x \pm y\sqrt{d}\},$$

so there is exactly one fundamental unit $\varepsilon > 0$ with $\varepsilon > 1$. Such a unit $\frac{1}{2}(a + b\sqrt{d})$ with $a, b > 0$ is called *normalized*