Complex Geometry

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0 Introduction

Lecture 1
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Definition 0.1. (i) A *complex manifold* is a second countable Hausdorff topological space X together with an equivalence class of holomorphic atlases.

(ii) An *atlas* (on a topological space X) is a collection of charts $\{U_i, \varphi_i\}_i$ such that $X = \bigcup_i U_i$.

(iii) A *chart* (on a topological space X) is a pair (U, φ), where $U \subseteq X$ is an open set, and $\varphi : U \to V \subseteq \mathbb{C}^N$ is a homeomorphism, where V is open.

(iv) An atlas is called *holomorphic* if for any two of its charts (U, φ), (V, ψ), the map $\varphi \circ \psi^{-1}$ is a holomorphic map (where defined)

(v) Two holomorphic atlases $\mathcal{A}, \mathcal{A}'$ are equivalent if $\mathcal{A} \cup \mathcal{A}'$ is again a holomorphic atlas.

(vi) For $U \subseteq \mathbb{C}^N$, $V \subseteq \mathbb{C}^M$ open, a C^1-map $f : U \to V$ is called *holomorphic* if $Df(z)$ is $\mathbb{C}$-linear for any $z \in U$.

Example 0.2. (0) Open subsets of $\mathbb{C}^N$ are complex manifolds.

(i) Complex projective space $\mathbb{CP}^N = \mathbb{C}^{N+1} \setminus \{0\}/(\mathbb{C} \setminus \{0\})$

(ii) Complex tori $\mathbb{C}^N/(\mathbb{Z}[i])^N$ or more generally $\mathbb{C}^N/\Lambda$ for any discrete subgroup $\Lambda \subseteq \mathbb{C}^N$ s.t. $\Lambda \otimes \mathbb{R} \to \mathbb{C}^N$ is an isomorphism of $\mathbb{R}$-vector spaces.

Why should one study Complex Geometry? Holomorphic functions have many "surprising" features (e.g. the maximum principle: If $f : X \to \mathbb{C}$ is a holomorphic map from a complex manifold, s.t. $|f|$,$\operatorname{Re} f$,$\operatorname{Im} f$ have a maximum, then f is constant), which are interesting to study intrinsically. In particular, if X is compact and connected, all holomorphic functions $f : X \to \mathbb{C}$ are constant. Hence the only compact complex submanifolds of $\mathbb{C}^N$ are points. Indeed, $X \hookrightarrow \mathbb{C}^N \xrightarrow{\mathrm{pr}_i} \mathbb{C}$ is holomorphic, hence constant.

Furthermore, Complex Geometry is connected to many other subjects, like the theory of holomorphic functions (e.g. meromorphic functions on $U \subseteq \mathbb{C}$ correspond to holomorphic maps $U \to \mathbb{CP}^1$. Analytic functions on $U \subseteq \mathbb{C}$ can be extended to a "maximal domain of definition" by analytic continuation, which is naturally an abstract complex manifold), algebraic geometry (smooth quasiprojective varieties are complex manifolds), differential geometry (every complex manifold has an underlying smooth manifold. Notorious open question: Which smooth manifolds admit a (compatible) structure of a complex manifold? (e.g. unknown for S^6)), Riemannian geometry (on complex manifolds, there is a correspondence between certain Riemannian metrics and certain 2-forms), representation theory (By the Borel-Weil-theorem one can construct irreducible representations of semisimple Lie groups in a geometric way), and physics (string theory: spacetime is modelled as $\mathbb{R}^4 \times M$, where M is a Calabi-Yau 3d compact complex manifold),

Example 0.3 (Complex geometry in dimension 1)**.** The uniformization theorem says that any simply connected complex manifold of dimension 1 is biholomorphic to one of $\mathbb{CP}^1, \mathbb{C}, \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. By covering theory, every 1-dimensional connected complex manifold is a quotient of one of these spaces by a "nice" subgroup.

Literature Huybrechts, Griffiths&Harris

1 Local Theory

Lecture 2
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1.1 Linear Algebra

Let V be a complex vector space of $\mathbb{C}$-dimension n. Then V is a $\mathbb{R}$-vector space of dimension $2n$, and multiplication by i gives a natural $\mathbb{R}$-linear endomorphism $I \in \operatorname{End}(V)$ such that $I^2 = -\operatorname{id}$. Conversely, given such a pair (V, I), one can define a complex structure on V by letting multiplication by i act as I. Hence

Lemma 1.1. *There is an equivalence of categories*

$$\{\mathbb{C}\text{-vector spaces of dimension } n\} \xleftrightarrow{\simeq} \{(V, I) \mid \dim_{\mathbb{R}}(V) = 2n, I \in \operatorname{End}(V), I^2 = -\operatorname{id}\}$$

Furthermore, given $V \simeq (V, I)$ a $\mathbb{C}$-vector space of dimension n, $I \otimes \operatorname{id}$ is a $\mathbb{C}$-linear endomorphism of $V \otimes_R \mathbb{C} =: V_{\mathbb{C}}$, which we still denote by I. Still $I^2 = -\operatorname{id}$, so I has eigenvalues $\pm i$ and is diagonalizable. Hence there is a decomposition in eigenspaces $V_{\mathbb{C}} = V_i \oplus V_{-i}$

Proposition 1.2. *Let V be a $\mathbb{C}$-vector space of dimension n. There is a splitting $V_{\mathbb{C}} = V_i \oplus V_{-i}$ into $\pm i$-eigenspaces of I with $\dim_{\mathbb{C}} V_{\pm i} = n$. The composition $V \hookrightarrow V_{\mathbb{C}} \twoheadrightarrow V_i$ is a $\mathbb{C}$-linear isomorphism.*

Proof. Conjugation $\operatorname{id} \otimes \sigma$ exchanges V_i and V_{-i}, so they have the same dimension. The composition is $\mathbb{C}$-linear since $I \equiv (\cdot i)$ in V_i. It is injective since $V \hookrightarrow V_{\mathbb{C}}$ is a real subspace (fixed by conjugation), hence an isomorphism. $\square$

Also note that the map in the proposition is explicitly given as $v \mapsto \frac{1}{2}(v - iIv)$.

Notation We will write $V_i = V^{1,0}$ and $V_{-i} = V^{0,1}$.

1.2 Holomorphic Maps

Fix the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $(z_1, \dots, z_n) \mapsto (\operatorname{Re}(z_1), \dots, \operatorname{Re}(z_n), \operatorname{Im}(z_1), \dots, \operatorname{Im}(z_n))$. Under this identification, multiplication by i on the left corresponds to $I = \left(\begin{smallmatrix} 0 & -\operatorname{id} \\ \operatorname{id} & 0 \end{smallmatrix}\right)$. This determines an embedding $\operatorname{Mat}_{n\times n}(\mathbb{C}) \hookrightarrow \operatorname{Mat}_{2n\times 2n}(\mathbb{R})$ with image the matrices such that $IA = AI$.

Let $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$.

Definition 1.3. A C^1-map $f : U \to V$ is called holomorphic if for all $p \in U$, $Df(u) : T_pU \cong \mathbb{C}^m \to T_{f(p)}V \cong \mathbb{C}^n$ is $\mathbb{C}$-linear.

We have $T_p\mathbb{C}^n \cong T_p\mathbb{R}^{2n} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \rangle_i$.

Definition 1.4. The Wirtinger operators are

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \overline{x}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$$

By proposition 1.2 we have $T_p\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C} \cong (T_p\mathbb{C}^n)^{(1,0)} \oplus (T_p\mathbb{C}^n)^{(0,1)}$, where the summands are spanned by $\langle \frac{\partial}{\partial z_i} \rangle_i$ and $\langle \frac{\partial}{\partial \overline{z}_i} \rangle_i$, respectively. Since an $\mathbb{R}$-linear map of complex vector spaces is $\mathbb{C}$-linear iff it respects the eigenspace decomposition, we have

Proposition 1.5 (Cauchy-Riemann equations)**.** *$f : U \to V$ as above is holomorphic if and only if*

$$\frac{\partial f}{\partial \overline{z}_i} = 0 \qquad \textit{for all } i = 1, \dots, n.$$

Proof. At any point $p \in U$, $Df(p) \otimes \mathrm{id} = Df(p)_{\mathbb{C}} : T_pU \otimes \mathbb{C} \to T_{f(p)}V \otimes \mathbb{C}$ has the following form w.r.t. the bases given by the $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}$:

$$\begin{pmatrix} \left(\frac{\partial f_i}{\partial z_j}\right)_{ij} & \left(\frac{\partial f_i}{\partial \overline{z}_j}\right)_{ij} \\ \left(\frac{\partial \overline{f}_i}{\partial z_j}\right)_{ij} & \left(\frac{\partial \overline{f}_i}{\partial \overline{z}_j}\right)_{ij} \end{pmatrix},$$

which respects the eigenspaces iff the off-diagonal blocks are 0. □

Proposition 1.6 (Cauchy-Riemann equations, second form)**.** *$f : U \to V$ as above is holomorphic iff for all i, j one has*

$$\frac{\partial \operatorname{Re}(f_i)}{\partial x_j} = \frac{\partial \operatorname{Im}(f_i)}{\partial y_j} \qquad \text{and} \qquad \frac{\partial \operatorname{Re}(f_i)}{\partial y_j} = -\frac{\partial \operatorname{Im}(f_i)}{\partial x_j}$$

Definition 1.7. For $f : U \to V$ holomorphic we call $Jf(p) = (\frac{\partial f_i}{\partial z_j})_{ij}$ the *complex Jacobi matrix.*

Corollary 1.8. *If $n = m$ and $f : U \to V$ is holomorphic, then* $\det Df(p) \geq 0$.

Proof. $\det Df(p) = \det Df(p)_{\mathbb{C}} = \det Jf(p) \cdot \det \overline{Jf(p)} = |\det Jf(p)|^2 \geq 0$. □

Corollary 1.9. *Any complex manifold is canonically oriented.*

Next, we want to phrase some standard analysis results in this new context. Observe that if $f : U \to V$ is holomorphic, a point $p \in U$ is regular ($Df(p)$ is surjective) if and only if $Jf(p)$ is surjective.

Theorem 1.10 (Inverse Function Theorem)**.** *Let $f : U \to V$ be holomorphic, $n = m$. If $p \in U$ is regular, there exist open neighbourhoods $p \in U' \subseteq U$ of p and $f(p) \in V' \subseteq V$ of $f(p)$ and a holomorphic map $g : V' \to U'$ such that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$.*

Proof. The usual inverse function theorem yields the existence of U', V' and g as a C^1-map, and the formula $Dg(f(p)) = (Df(p))^{-1}$, so $Dg(f(p))$ is also $\mathbb{C}$-linear. □

Theorem 1.11 (Implicit Function Theorem)**.** *Let $f : U \to V$ be holomorphic and $m \geq n$. Suppose $p \in U$ is such that $\det(\frac{\partial f_i}{\partial z_j})_{i,j=1,\ldots,n} \neq 0$. Then, there exists an open $p \in U_1 \times U_2 \subseteq U$ with $U_1 \subseteq \mathbb{C}^n$, $U_2 \subseteq \mathbb{C}^{m-n}$ and a holomorphic $g : U_2 \to U_1$ such that for all $z \in U_1 \times U_2$ we have $f(z) = f(p)$ if and only if $g(z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n)$.*

Proof. Again, the usual implicit function theorem gives everything but the holomorphicity of g. Note that the composition $f \circ (g \times \mathrm{id}) : U_2 \to U_1 \times U_2 \to V$ is constant, so $D(f \circ (g \times \mathrm{id})) = 0$. But

$$D(f \circ g \times \mathrm{id}) = Df \circ D(g \times \mathrm{id}) = \begin{pmatrix} \left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} & 0 \\ 0 & \cdots \end{pmatrix} \begin{pmatrix} \left(\frac{\partial g_i}{\partial z_j}\right)_{\substack{i=1,\ldots,n \\ j=n+1,\ldots,m}} & \frac{\partial g_i}{\partial \overline{z}_j} \\ \mathrm{id} & 0 \\ \frac{\partial \overline{g}_i}{\partial z_j} & \frac{\partial \overline{g}_i}{\partial \overline{z}_j} \\ 0 & \mathrm{id} \end{pmatrix}$$

Since $(\frac{\partial f_i}{\partial z_j})$ is invertible, the only way for this product to be 0 is for $(\frac{\partial g_i}{\partial \overline{z}_j})$ to vanish. □

Corollary 1.12 (Immersion-Submersion)**.** *Let $p \in U$, $f : U \to V$ such that the rank of $Jf(p)$ is maximal.*

(i) If $m \geq n$, there is $h : U' \to U$ which is a biholomorphism onto some open $p \in h(U') \subseteq U$ such that $(f \circ h)(z_1, \dots, z_m) = (z_1, \dots, z_n)$.

(ii) If $m \leq n$ there exists an open $f(p) \in V' \subseteq V$ and a biholomorphism $g : V' \to g(V') \subseteq \mathbb{C}^n$ such that $g \circ f(z_1, \dots, z_m) = (z_1, \dots, z_m, 0, \dots, 0)$.

Theorem 1.13. *Let $f : \mathbb{C}^n \supseteq U \to \mathbb{C}$ be a C^1 function. The following are equivalent:*

(i) f is holomorphic.

(ii) f is partially holomorphic, i.e. the maps $z \mapsto f(z_1, \dots, z_{i-1}, z, z_{i+1}, \dots, z_n)$ with z_i constant are holomorphic where defined.

(iii) (Cauchy integral formula) For any $\varepsilon \in \mathbb{R}^n$ such that $B_\varepsilon(p) = \{z \in \mathbb{C}^n \mid |z_i - p_i| < \varepsilon_i\} \subseteq U$

$$f(p) = (2\pi i)^{-n} \int_{\prod \partial B_{\varepsilon_i}(p_i)} \frac{f(\xi)}{\prod_i (\xi_i - p_i)} d\xi_1 \cdots d\xi_n$$

(iv) f is analytic, i.e. around every p there exists a power series expansion

$$f(z) = \sum_{I \in \mathbb{N}^n} a_I (z-p)^I$$

on some $B_\varepsilon(p) \subseteq U$.

(v) f satisfies the Cauchy-Riemann equations 1.5 or 1.6.

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) follows from Fubini and the one-dimensional case.

(iii) $\Rightarrow$(iv): For $|z| < |\xi|$ one has $\frac{1}{\xi - z} = \frac{1}{\xi} \frac{1}{1 - \frac{z}{\xi}} = \sum \frac{z^\alpha}{\xi^{\alpha+1}}$. Denote $\mathbf{1} = (1, \dots, 1)$, Then

$$f(p) = (2\pi i)^{-n} \int_{\prod \partial B_{\varepsilon_i}(p_i)} \sum \frac{z^I}{\xi^{I+\mathbf{1}}} f(\xi) d\xi_1 \cdots d\xi_n$$

Interchanging sum and integral yields the result. (iv)$\Rightarrow$(i): Exchange $\sum$ and $\frac{\partial}{\partial \bar{z}_i}$. $\square$