

# Algebraic Number Theory

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Ludwig-Maximilians-Universität München – winter term 2025/26

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## 0 Motivation

**Theorem 0.1** (Lagrange). *Let  $p$  be an odd prime. Then*

$$p = x^2 + y^2 \text{ with } x, y \in \mathbb{Z} \text{ if and only if } p \equiv 1 \pmod{4}.$$

*Proof.* For any integer  $x$  we have  $x^2 \equiv 0, 1 \pmod{4}$ , hence  $x^2 + y^2 \equiv 0, 1 \text{ or } 2 \pmod{4}$  for all  $x, y \in \mathbb{Z}$ , hence  $p \not\equiv 3 \pmod{4}$ .

Conversely, assume that  $p \equiv 1 \pmod{4}$ . Then  $\mathbb{F}_p^\times$  is a cyclic group of order  $p - 1$ , so there exists some  $\bar{m} \in \mathbb{F}_p^\times$  of order 4. Thus there is  $m \in \mathbb{Z}$  with  $m^2 \equiv -1 \pmod{p}$ , i.e.  $p \mid m^2 + 1 = (m+i)(m-i) \in \mathbb{Z}[i]$ . Since the Gaussian integers form a Euclidean ring, it is in particular a PID.

Consider its norm  $N : \mathbb{Z}[i] \rightarrow \mathbb{Z}$ ,  $\alpha = a + bi \mapsto \alpha\bar{\alpha} = a^2 + b^2$ , which is a multiplicative function. Suppose that  $p \mid m + i$ . Then  $p \mid m - i$  as well, hence  $p \mid 2i$ , which is clearly wrong. Hence  $p$  is not a prime element in  $\mathbb{Z}[i]$ . Since we are in a PID,  $p$  is reducible in  $\mathbb{Z}[i]$ , i.e. there exist non-units  $\alpha = x + yi, \beta = x' + y'i \in \mathbb{Z}[i]$  such that  $p = \alpha\beta$ . Now we see  $p^2 = N(\alpha)N(\beta) = (x^2 + y^2)(x'^2 + y'^2)$ . Since  $\alpha, \beta$  aren't units, each factor is  $> 1$ , hence  $p = x^2 + y^2 = x'^2 + y'^2$ .  $\square$

**Definition 0.2.** A finite extension  $K$  of  $\mathbb{Q}$  is called a *number field*.

**Example 0.3.**  $\mathbb{Q}(i)$  is a number field of degree 2. In the above example, we worked in  $\mathbb{Z}[i] \subseteq \mathbb{Q}(i)$ . We want to generalize this.

**Definition 0.4.** Let  $K/\mathbb{Q}$  be a number field. Then

$$\mathcal{O}_K := \{\alpha \in K \mid \exists f \in \mathbb{Z}[x] \text{ normalized s.t. } f(\alpha) = 0\},$$

i.e. the integral closure of  $\mathbb{Z}$  in  $K$ , is called the *ring of integers* in  $K$ .

We will show:  $\mathcal{O}_K$  is a Dedekind domain.

**Example 0.5.** (i) For  $K = \mathbb{Q}(i)$  we have  $\mathcal{O}_K = \mathbb{Z}[i]$

(ii) For  $K = \mathbb{Q}(\sqrt{2})$  one gets  $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$

(iii) For  $K = \mathbb{Q}(\sqrt{-6})$  we have  $\mathcal{O}_K = \mathbb{Z}[\sqrt{-6}]$

(iv) (Exercise) More generally, for  $d \in \mathbb{Z} \setminus \{0, 1\}$  squarefree, the ring of integers of  $K = \mathbb{Q}(\sqrt{d})$  is  $\mathbb{Z}[\omega]$ , where

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \pmod{4}, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

**Theorem 0.6.** *Let  $p$  be an odd prime. Then*

$$p = x^2 - 2y^2 \text{ with } x, y \in \mathbb{Z} \text{ if and only if } p \equiv \pm 1 \pmod{8}.$$

*Proof.* The forward direction follows as in the first theorem. For the converse, we work in  $\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{Q}(\sqrt{2})$ . Consider the norm  $N : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}$ ,  $\alpha = x + y\sqrt{2} \mapsto \alpha\bar{\alpha} = x^2 - 2y^2$ , where  $\text{Gal}(\mathbb{Q}(\sqrt{2}) \mid \mathbb{Q}) = \langle \sigma \rangle$ . We will see later (Quadratic Reciprocity) that  $p \equiv \pm 1 \pmod{8}$  is equivalent to  $(\frac{2}{p}) = 1$ , i.e. 2 being a square mod  $p$ .

Hence there exists  $m \in \mathbb{Z}$  with  $p \mid m^2 - 2 = (m - \sqrt{2})(m + \sqrt{2})$ . As before, we see that  $p$  is not prime, hence reducible ( $\mathbb{Z}[\sqrt{2}]$  is again Euclidean) and we finish as before.  $\square$

The main difference between theorems 0.1 and 0.6 is that the unit group of  $\mathbb{Z}[i]$  is finite, while  $\mathbb{Z}[\sqrt{2}]^\times = \{\pm 1\} \times (1 + \sqrt{2})^\mathbb{Z}$  is infinite<sup>1</sup>. This implies that  $p = x^2 - 2y^2$  has infinitely many solutions for  $p \equiv \pm 1 \pmod{8}$ , for  $N((1 + \sqrt{2})^{2k}\alpha) = N(\alpha)$  for all  $k \in \mathbb{Z}$ .

In this vein, an important goal of this lecture is

**Theorem 0.7** (Dirichlet's unit theorem). *Let  $K/\mathbb{Q}$  be a number field. Let  $s$  be the number of real embeddings and let  $t$  be the number of pairs of complex embeddings of  $K$ . Then  $\mathcal{O}_K^\times$  is a finitely generated abelian group of rank  $r = s + t - 1$ , i.e. there exist fundamental units  $\varepsilon_1, \dots, \varepsilon_r$  and  $\zeta \in \mu_K = \{\text{roots of unity in } K\}$  such that each  $\varepsilon \in \mathcal{O}_K^\times$  can be uniquely written in the form*

$$\varepsilon = \zeta^l \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r}$$

with  $a_i \in \mathbb{Z}$  and  $l \in \mathbb{Z}/\text{ord}(\zeta)\mathbb{Z}$ .

**Example 0.8.** For  $K = \mathbb{Q}(\sqrt{2})$  we have  $\mu_K = \{\pm 1\}$ ,  $\varepsilon_1 = 1 + \sqrt{2}$  and  $r = 2 + 0 - 1 = 1$ , since both embeddings  $\sqrt{2} \mapsto \sqrt{2}$  and  $\sqrt{2} \mapsto -\sqrt{2}$  are real.

Let  $K/\mathbb{Q}$  be a number field. We choose the algebraic closure  $\mathbb{Q}^c$  of  $\mathbb{Q}$  that sits inside of  $\mathbb{C}$ , so we may, and will, always assume  $K \subseteq \mathbb{C}$ .  $K/\mathbb{Q}$  is separable, so we may write  $K = \mathbb{Q}(\alpha)$  for some  $\alpha \in K$ . Let  $f \in \mathbb{Q}(\alpha)$  be the minimal polynomial of  $\alpha$ . Then we have embeddings  $\sigma : K \hookrightarrow \mathbb{C}$  corresponding to the zeroes  $\alpha = \alpha_1, \dots, \alpha_n$  of  $f$ , i.e. the conjugates of  $\alpha$ .  $\sigma$  is called a real embedding if  $\sigma(K) \subseteq \mathbb{R}$ , or equivalently if the corresponding  $\alpha_i \in \mathbb{R}$ . Otherwise it is called a complex embedding. These come in pairs, because if  $\alpha_i$  is a conjugate of  $\alpha$ , so is  $\overline{\alpha_i}$ .

**Example 0.9.** Let  $K = \mathbb{Q}(\sqrt{d})$  be a quadratic number field. If  $d > 0$  we find as before that  $s = 2, t = 0$ , so  $r = 1$ . If, on the other hand,  $d < 0$ , then  $s = 0, t = 1$ , hence  $r = 0$  and  $\mathcal{O}_K^\times$  is finite.

**Question** Which odd primes  $p$  can be written in the form  $p = x^2 + 6y^2$  with  $x, y \in \mathbb{Z}$ ? As in the previous theorems, we write this as  $(x + y\sqrt{-6})(x - y\sqrt{-6}) = N(x + y\sqrt{-6})$  in the number field  $K = \mathbb{Q}(\sqrt{-6})$  with ring of integers  $\mathbb{Z}[\sqrt{-6}]$ . However, our previous proof strategy does not work, because  $\mathbb{Z}[\sqrt{-6}]$  is not a PID (e.g.  $2 \cdot 3 = -\sqrt{-6} \cdot \sqrt{-6}$  are two essentially different factorizations of 6 into irreducibles).

This leads naturally to the question when  $\mathcal{O}_K$  is a PID. To investigate this, we will introduce the *class group*: The nonzero ideals of  $\mathcal{O}_K$  form a monoid w.r.t. multiplication.

**Definition 0.10.** Write  $I_K$  for the group of fractional nonzero ideals and  $P_K = \{\alpha \mathcal{O}_K \mid \alpha \in K^\times\}$  the subgroup of principal fractional ideals. The quotient  $\text{cl}_K = I_K/P_K$  is called the *ideal class group*

One sees directly that  $\text{cl}_K = 1$  if and only if  $\mathcal{O}_K$  is a PID. We will prove

**Theorem 0.11.**  $|\text{cl}_K| < \infty$ .

In any case  $\mathcal{O}_K$  is Dedekind, which is equivalent to prime factorization of *ideals*, i.e. each ideal  $(0) \neq \mathfrak{a} \trianglelefteq \mathcal{O}_K$  can be uniquely written as a product of prime ideals

$$\mathfrak{a} = \prod_{\substack{\mathfrak{p} \in \text{Spec}(\mathcal{O}_K) \\ \mathfrak{p} \neq 0}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}, \quad v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}_{\geq 0}, \text{ almost all } v_{\mathfrak{p}}(\mathfrak{a}) = 0.$$

<sup>1</sup>  $\supseteq$  is easy by direct computation, which is all we use here. We will see how to prove  $\subseteq$  later.

**Example 0.12.** In  $\mathbb{Z}[\sqrt{-6}]$  we have  $2\mathcal{O}_K = \mathfrak{p}_2^2$  with  $\mathfrak{P}_2 = \langle 2, \sqrt{-6} \rangle_{\mathbb{Z}}$ ,  $3\mathcal{O}_K = \mathfrak{p}_3^2$  with  $\mathfrak{p}_3 = \langle 3, \sqrt{-6} \rangle_{\mathbb{Z}}$  and  $\sqrt{-6}\mathcal{O}_K = \mathfrak{p}_2\mathfrak{p}_3$ , so the "problematic" factorization  $2 \cdot 3 = -\sqrt{-6}^2$  becomes  $\mathfrak{p}_2^2\mathfrak{p}_3^2 = (\mathfrak{p}_2\mathfrak{p}_3)^2$  when passing to ideals.

Given an extension of number fields  $L/K$ , and a prime ideal  $\mathfrak{p} \subseteq \mathcal{O}_K$ , by the above the ideal  $\mathfrak{p}\mathcal{O}_L$  splits into a product of prime ideals  $\mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$  in  $\mathcal{O}_L$ . A further goal of this lecture is to understand and compute this factorization. Denoting  $f_i = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$ , we will for example be able to show  $[L : K] = \sum_{i=1}^r e_i f_i$ .

**Definition 0.13.** Let  $p$  be a prime and  $a \in \mathbb{Z}$  with  $p \nmid a$ . Then the *Legendre symbol* is defined as

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution in } \mathbb{Z}, \\ -1 & \text{otherwise.} \end{cases}$$

Also set  $(\frac{a}{p}) = 0$  if  $p \mid a$ .

We will show: Let  $K = \mathbb{Q}(\sqrt{d})$ . Let  $p \neq 2$ . Then

$$p\mathcal{O}_K = \begin{cases} \mathfrak{p}\bar{\mathfrak{p}}, \mathfrak{p} \neq \bar{\mathfrak{p}} \text{ prime} & \text{if } \left(\frac{d}{p}\right) = 1, \\ \mathfrak{p}, \mathfrak{p} \text{ prime} & \text{if } \left(\frac{d}{p}\right) = -1, \\ \mathfrak{p}^2, \mathfrak{p} \text{ prime} & \text{if } p \mid d. \end{cases} \quad (*)$$

**Law of quadratic reciprocity** Let  $p, q$  be odd primes. Then

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -1 & \text{if } p \equiv 3 \pmod{4} \text{ and } q \equiv 3 \pmod{4} \end{cases}.$$

Further, we have the two supplements  $(\frac{-1}{p}) = (-1)^{(p-1)/2}$  and  $(\frac{2}{p}) = (-1)^{(p^2-1)/8}$ . This theorem allows quick computation of Legendre symbols.

Using the above, we will be able to generalize the theorems from the beginning:

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**Corollary 0.14.** Let  $d$  be a squarefree integer. A prime  $p \neq 2$  can be written in the form  $p = x^2 - dy^2$  for  $x, y \in \mathbb{Z}$  if and only if  $(\frac{d}{p}) = 1$  and  $\mathfrak{p}$  is a principal ideal, where  $\mathfrak{p}$  is as in (\*).

# 1 Integrality

Rings are always commutative and contain a multiplicative unit, unless explicitly stated otherwise.

**Definition 1.1.** Let  $A \subseteq B$  be a ring extension. An element  $b \in B$  is *integral* over  $A$  if there exists a normalized polynomial  $f(X) = X^m + a_{m-1}X^{m-1} + \dots + a_1X + a_0 \in A[X]$  such that  $f(b) = 0$ .  $B$  is *integral* over  $A$  if every  $b \in B$  is integral over  $A$ .

**Example 1.2.** Let  $K$  be a number field. Then  $\mathcal{O}_K$  is integral (over  $\mathbb{Z}$ ).

If  $B/A$  is a field extension, then  $B$  is integral over  $A$  if and only if  $B$  is algebraic over  $A$ .

We want to show that the set of all integral elements form a ring, i.e. that given integral elements  $b_1, b_2 \in B$ ,  $b_1 + b_2$  and  $b_1b_2$  are integral as well.

**Theorem 1.3.** Let  $b_1, \dots, b_n \in B$ . Then  $b_1, \dots, b_n$  are integral over  $A$  if and only if  $A[b_1, \dots, b_n]$  is a finitely generated  $A$ -module.

*Proof.* " $\Rightarrow$ ": By induction. For  $n = 1$  let  $b \in B$  be integral over  $A$ . Let  $f(b) = 0$ . Then  $b^m = -\sum_{i=0}^{m-1} a_i b^i$ , so  $A[b]$  is generated by  $1, b, \dots, b^{m-1}$  as a  $A$ -module.

More explicitly: Let  $g(b) \in A[b]$  be some element. Since  $f$  is normalized, we can perform division with remainder to write  $g = qf + r$  with  $q, r \in A[x]$  with  $\deg(r) < m$ . Hence  $g(b) = q(b)f(b) + r(b) = r(b)$ , which is a linear combination of  $b^i$ ,  $i < m$ .

For the inductive step, we have to prove that  $A \subseteq A[b_1, \dots, b_n] \subseteq A[b_1, \dots, b_{n+1}]$  is finitely generated, knowing that the first extension is finitely generated. Since  $b_{n+1}$  is integral over  $A$ , it is also finitely generated over  $A[b_1, \dots, b_n]$ , hence  $A[b_1, \dots, b_n] \subseteq A[b_1, \dots, b_{n+1}]$  is finitely generated by the  $n = 1$  case, hence we are done.

" $\Leftarrow$ ": Let  $\omega_1, \dots, \omega_r$  be a set of  $A$ -generators of  $A[b_1, \dots, b_n]$ . For  $b \in A[b_1, \dots, b_n]$  we have

$$b\omega_i = \sum_{j=1}^r a_{ij}\omega_j \quad \text{with } a_{ij} \in A.$$

Hence  $(bE - M)(\omega_1, \dots, \omega_r)^t = 0$ , where  $M = (a_{ij})_{ij} \in A^{r \times r}$ . By cofactor expansion, see lemma 1.4, this implies that  $\det(bE - M)\omega_i = 0$  for all  $i = 1, \dots, r$ , hence  $\det(bE - M) = 0$  since the  $\omega_i$  generate  $A[b_1, \dots, b_n]$ . Hence  $\det(XE - M) \in A[X]$  is a normalized equation for  $b$ , i.e.  $b$  is integral over  $A$ .  $\square$

**Lemma 1.4.** Let  $A$  a ring and  $M \in A^{r \times r}$ . If  $Mx = 0$ , then  $\det(M)x = 0$ .

*Proof.* Let  $M^*$  be the adjoint matrix, i.e.  $(M^*)_{ij}$  is  $(-1)^{i+j}$  times the determinant of the matrix  $M$  with the  $j$ -th row and  $i$ -th column removed. Then  $M^*M = MM^* = \det(M)E$ . From  $Mx = 0$  we then get  $0 = M^*Mx = \det(M)x$ .  $\square$

**Example 1.5.**  $K = \mathbb{Q}(\sqrt{2}) \supseteq \mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ . Proceeding as in the proof, we can compute an integral equation for, say,  $\alpha = 1 + 2\sqrt{2}$ : Take  $\omega_1 = 1, \omega_2 = \sqrt{2}$ . Consider

$$T_\alpha : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}[\sqrt{2}], \quad x \mapsto \alpha x,$$

which has matrix representation w.r.t. the  $\omega_i$  as  $M = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$ . Now  $\det(XE - M) = X^2 - 2X - 7$  is the desired relation.

In the exercises, we will show the following slight generalization of proposition 1.3.

**Proposition 1.6.** Let  $A$  be a ring. Then the following are equivalent:

- (i)  $b$  is integral over  $A$ .
- (ii)  $A[b]$  is finitely generated as an  $A$ -module.
- (iii) There exists an  $A[b]$ -module  $M$  that is finitely generated as an  $A$ -module.

**Theorem 1.7.** Let  $A \subseteq B \subseteq C$  be extensions of rings. Let  $B/A$  be integral and let  $c \in C$  be integral over  $B$ . Then  $c$  is also integral over  $A$ .

*Proof.* Let  $c^n + b_{n-1}c^{n-1} + \dots + b_0$  with  $b_i \in B$ . Then  $A \subseteq A[b_0, \dots, b_{n-1}] \subseteq A[b_0, \dots, b_{n-1}][c]$  is a composition of finitely generated ring extensions by theorem 1.3, hence finitely generated. Again by theorem 1.3, we are done.  $\square$

**Definition 1.8.** Let  $A \subseteq B$  be a ring extension.

- (a) Then  $\overline{A} = \mathcal{O}_{A,B} := \{b \in B \mid b \text{ integral over } A\}$  is called the *integral closure* of  $A$  in  $B$ .
- (b)  $A$  is called *integrally closed* in  $B$  if  $\mathcal{O}_{A,B} = A$ .

Note that by theorem 1.3, the integral closure of  $A$  in  $B$  is a ring. In particular, the ring of integers  $\mathcal{O}_K$  of a number field  $K$  is indeed a ring.

**Example 1.9.**  $\mathcal{O}_{A,B}$  is integrally closed in  $B$ .

$\mathbb{Z}$  is integrally closed in  $\mathbb{Q}$ . More generally,  $\mathcal{O}_K$  is integrally closed in  $K$ , for if  $\alpha \in K$  is integral over  $\mathcal{O}_K$ , by transitivity 1.7 it is then integral over  $\mathbb{Z}$ , hence  $\alpha \in \mathcal{O}_K$ .

$R = \mathbb{Z}[\sqrt{-3}] \subseteq K = \mathbb{Q}(\sqrt{-3})$  is not integrally closed in  $K$ , because  $\frac{1}{2}(1 + \sqrt{-3}) \notin R$  is integral (even over  $\mathbb{Z}$ ).

**Theorem 1.10.** Let  $R$  be a UFD and  $K = \text{Quot}(R)$ . Then  $R$  is integrally closed in  $K$ .

*Proof.* Let  $\frac{a}{b} \in K$  be integral over  $R$ , with  $a, b \in R$  coprime. Let

$$X^n + c_{n-1}X^{n-1} + \dots + c_1X + c_0 = 0 \quad \text{with } c_i \in R$$

be an integral relation for  $\frac{a}{b}$ . Multiplying by  $b^n$ , we get

$$a^n + c_{n-1}ba^{n-1} + \dots + c_1ab^{n-1} + c_0b^n = 0.$$

Suppose  $b \notin R^\times$ , then there exists a prime element  $\pi \in R$  dividing  $b$ . Looking at the equation mod  $\pi$ , we see that  $\pi \mid a^n$ ; i.e.  $\pi \mid a$ , contradicting the coprime assumption.  $\square$

Let  $A$  be an integral domain which is integrally closed in  $K = \text{Quot}(A)$ . Let  $L/K$  be a finite field extension and let  $B = \mathcal{O}_{A,L}$  be the integral closure of  $A$  in  $L$ .

$$\begin{array}{ccc} L & \longleftrightarrow & B \\ | & & | \\ K & \longleftrightarrow & A \end{array}$$

Then, by transitivity,  $B$  is integrally closed in  $L$ .

**Lemma 1.11.** In the above situation,  $L = \text{Quot}(B)$ . More precisely, each  $\beta \in L$  can be written in the form  $\frac{b}{a}$  with  $b \in B$  and  $a \in A$ .

*Proof.* For  $\beta \in L$ , let  $a_n\beta^n + \dots + a_1\beta + a_0 = 0$  with  $a_i \in A$ . Multiplying by  $a_n^{n-1}$ , we obtain

$$(a_n\beta)^n + a_{n-1}(a_n\beta)^{n-1} + \dots + a_1a_n^{n-2}(a_n\beta) + a_0a_n^{n-1} = 0.$$

Thus  $a_n\beta$  is integral over  $A$ , and  $\beta = \frac{a_n\beta}{a_n}$  has the desired form.  $\square$

**Lemma 1.12.** One has  $\beta \in B$  if and only if its minimal polynomial  $\mu = \text{mipo}_{\beta, K}$  over  $K$  has coefficients in  $A$ .

*Proof.* Let  $g(\beta) = 0$  with  $g \in A[X]$  normalized. Then  $\mu \mid g$  in  $K[X]$ . Thus all zeroes of  $\mu$  (in some algebraic closure of  $K$ ) are integral over  $A$ . Since the coefficients of  $\mu$  are the elementary symmetric functions in its zeroes, the coefficients of  $\mu$  are integral over  $A$ . Since by assumption  $A$  is integrally closed in  $K$ , it follows that  $\mu \in A[X]$ .  $\square$

We recall from Algebra the notions of trace and norm. Let  $L/K$  be a finite field extension of degree  $n$ , and let  $x \in L$ . Let  $T_x : L \rightarrow L$ ,  $y \mapsto xy$ .

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**Definition 1.13.** We define  $\text{Tr}_{L/K}(x) := \text{Tr}(T_x)$  and  $\text{N}_{L/K}(x) := \det(T_x)$ .

**Lemma 1.14.** (i) Let  $\chi_x(t) = \det(tE - T_x) \in K[t]$  be the characteristic polynomial of  $T_x$ . Let  $\chi_x(t) = t^n - a_1t^{n-1} + \dots + (-1)^n a_n$ . Then  $a_1 = \text{Tr}_{L/K}(x)$  and  $a_n = \text{N}_{L/K}(x)$ .

(ii)  $\text{Tr}_{L/K}$  is  $K$ -linear.

(iii)  $\text{N}_{L/K}$  is multiplicative

*Proof.* Everything follows from linear algebra once translated to the linear maps  $T_x$ .  $\square$

**Theorem 1.15.** Let  $L/K$  be separable. Let  $G = G(L/K, K^c/K)$  be the set of all homomorphisms  $\sigma : L \rightarrow K^c$  that fix  $K$ . (By separability we have  $|G| = [L : K]$ .) Then

$$(i) \quad \chi_x(t) = \prod_{\sigma \in G} (t - \sigma(x))$$

$$(ii) \quad \text{Tr}_{L/K}(x) = \sum_{\sigma \in G} \sigma(x)$$

$$(iii) \quad \text{N}_{L/K}(x) = \prod_{\sigma \in G} \sigma(x)$$

*Proof.* (ii) and (iii) follow from (i) using lemma 1.14(i). Let  $\mu_x(t)$  be the minimal polynomial of  $T_x$ . Then  $\mu_x(T_x) = 0$ , hence also  $\mu_x(x) = 0$  in  $L$ . Further  $\mu_x(\sigma(x)) = \sigma(\mu_x(x)) = 0$ , so  $\mu_x(t) = \prod_{\sigma \in G(K(x)/K, K^c/K)} (t - \sigma(x))$ . We conclude with

$$\chi_x(t) = \mu_x(t)^{[L:K(x)]} = \prod_{\sigma \in G} (t - \sigma(x)),$$

where both steps need further explanation: Let  $\sigma \in G(K(x)/K, K^c/K)$ . Then there are  $[L : K(x)]$  extensions  $\tilde{\sigma}$  of  $\sigma$ , which thus all have the same value at  $x$ . This explains the second equality. For the first, choose bases  $\omega_1, \dots, \omega_m$  and  $1, x, \dots, x^{n-1}$  of  $L/K(x)$  and  $K(x)/K$ , respectively. Then  $\omega_i x^j$  is a basis of  $L/K$ , and  $T_x$  w.r.t. this basis has as matrix representation a block-diagonal matrix with each block equal to the matrix representation of  $\mu_x$  w.r.t. the basis  $1, x, \dots, x^{n-1}$ .  $\square$

**Example 1.16.** (i)  $K = \mathbb{Q}(\sqrt{d})$  is a quadratic extension with  $G = \{\text{id}, \sigma : \sqrt{d} \mapsto -\sqrt{d}\}$ . Hence for  $\alpha = a + b\sqrt{d}$  one has  $\text{Tr}_{K/\mathbb{Q}}(\alpha) = 2a$  and  $\text{N}_{K/\mathbb{Q}}(\alpha) = a^2 - b^2d$ .

(ii) Let  $L/K$  be a finite field extension of degree  $m$ . Let  $\alpha \in K$ . Then  $\text{Tr}_{L/K}(\alpha) = m\alpha$  and  $\text{N}_{L/K}(\alpha) = \alpha^m$ .

(iii) Let  $L = \mathbb{Q}(\alpha)/K = \mathbb{Q}$ , where  $\alpha^3 = 2$ ,  $\alpha \in \mathbb{R}$ . In the exercises we will see  $\mathcal{O}_L = \mathbb{Z}[\alpha]$ . Let  $x = 1 + \alpha$ . We have

$$(1 + \alpha) \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} 1 + \alpha \\ \alpha + \alpha^2 \\ \alpha^2 + 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}}_{=:M} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix},$$

so  $\text{Tr}_{L/K}(1 + \alpha) = \text{Tr}(M) = 3$  and  $\text{N}_{L/K}(1 + \alpha) = \det(M) = 3$ . Alternatively, we could have calculated

$$\text{Tr}_{L/\mathbb{Q}}(1 + \alpha) = \text{Tr}_{L/\mathbb{Q}}(1) + \text{Tr}_{L/\mathbb{Q}}(\alpha) = 3 + 0 = 3,$$

since the minimal polynomial  $t^3 - 2$  of  $\alpha$  has no  $t^2$ -term.

**Corollary 1.17.** *Let  $M/L/K$  be a tower of finite field extensions. Then for  $\alpha \in M$  one has*

$$\text{Tr}_{M/K}(\alpha) = \text{Tr}_{L/K}(\text{Tr}_{M/L}(\alpha)) \quad \text{and} \quad \text{N}_{M/K}(\alpha) = \text{N}_{L/K}(\text{N}_{M/L}(\alpha)).$$

*Proof.* For  $\sigma_i : L/K \rightarrow K^c/K$ , we have  $[M : L]$  extensions  $\sigma_{ij} : M \rightarrow K^c$ . Fix one such extension  $\widehat{\sigma}_i$ .

$$\begin{array}{ccccc} & & \sigma_{ij} & & \\ & M & \xrightarrow{\widehat{\sigma}_i} & \widehat{\sigma}_i(M) & \longrightarrow K^c \\ & \downarrow & & \downarrow & \downarrow \\ L & \xrightarrow{\sigma_i} & \sigma_i(L) & \xrightarrow{\text{id}} & \sigma_i(L) \\ & \downarrow & & & \downarrow \\ K & \xrightarrow{\sigma_i} & \sigma_i(K) = K & & \end{array}$$

Then

$$\text{Tr}_{M/K}(\alpha) = \sum_{i,j} \sigma_{ij}(\alpha) = \sum_i \text{Tr}_{\widehat{\sigma}_i M / \sigma_i L}(\widehat{\sigma}_i(\alpha)). \quad (*)$$

Let  $\omega = (\omega_1, \dots, \omega_m)^t$  be a  $L$ -basis of  $M$ . Then  $\widehat{\sigma}_i(\omega_1), \dots, \widehat{\sigma}_i(\omega_m)$  is a  $\sigma_i(L)$ -basis of  $\widehat{\sigma}_i(M)$ . Let  $\alpha\omega = M_\alpha\omega$  with  $M_\alpha \in L^{m \times m}$ . Then  $\widehat{\sigma}_i(\alpha)\widehat{\sigma}_i(\omega) = \sigma_i(M_\alpha)\widehat{\sigma}_i(\omega)$ , where the actions on vectors and matrices is understood to be component-wise. Therefore,

$$\text{Tr}_{\widehat{\sigma}_i(M) / \sigma_i(L)}(\widehat{\sigma}_i(\alpha)) = \text{Tr}(\sigma_i(M_\alpha)) = \sigma_i(\text{Tr}(M_\alpha)) = \sigma_i(\text{Tr}_{M/L}(\alpha)).$$

Continuing from  $(*)$  we get

$$\text{Tr}_{M/K}(\alpha) = \sum_i \sigma_i(\text{Tr}_{M/L}(\alpha)) = \text{Tr}_{L/K}(\text{Tr}_{M/L}(\alpha)).$$

The same proof works for the norm, with all sums replaced by products.  $\square$

Let  $L/K$  be a finite separable extension of fields. Let  $\alpha_1, \dots, \alpha_n$  be  $[L : K]$ -many elements of  $L$ .

**Definition 1.18.** The discriminant of  $\alpha_1, \dots, \alpha_n$  is defined as

$$d(\alpha_1, \dots, \alpha_n) := \det(\sigma_i(\alpha_j))_{i,j=1,\dots,n}^2,$$

where  $\{\sigma_1, \dots, \sigma_n\} = G(L/K, K^c/K)$ .

**Lemma 1.19.** (i)  $d(\alpha_1, \dots, \alpha_n) = \det(\text{Tr}_{L/K}(\alpha_i \alpha_j))_{1 \leq i,j \leq n}$ .

(ii) For  $\theta \in L$  we have  $d(1, \theta, \theta^2, \dots, \theta^{n-1}) = \prod_{i < j} (\theta_i - \theta_j)^2$ , where  $\theta_i := \sigma_i(\theta)$ .

*Proof.* One calculates

$$(\sigma_k(\alpha_i))_{k,i}^t (\sigma_k(\alpha_j))_{kj} = \left( \sum_{k=1}^n \sigma_k(\alpha_i \alpha_j) \right)_{i,j} = (\text{Tr}_{L/K}(\alpha_i \alpha_j))_{i,j}$$

and takes determinants for the first part. For the second, the matrix in the definition 1.18 of  $d$  is the Vandermonde matrix of the  $\theta_i$ .  $\square$

**Theorem 1.20.** *Let  $L/K$  be a finite separable field extension of degree  $n$ . Let  $\alpha_1, \dots, \alpha_n \in L$ . Then*

- (i)  $\alpha_1, \dots, \alpha_n$  is a  $K$ -basis of  $L$  if and only if  $d(\alpha_1, \dots, \alpha_n) \neq 0$ .
- (ii) The bilinear map  $\langle -, - \rangle : L \times L \rightarrow K$ ,  $(x, y) \mapsto \text{Tr}_{L/K}(xy)$  (called trace form) is nondegenerate.

*Proof.* For (ii), separability of  $L/K$  implies that  $L = K(\theta)$  for some  $\theta \in L$ . The structure matrix of the bilinear form is given by

$$M = (\langle \theta^i, \theta^j \rangle)_{i,j} = (\text{Tr}_{L/K}(\theta^i \theta^j))_{i,j}.$$

Thus  $\det(M) = d(1, \theta, \dots, \theta^{n-1}) = \prod_{i < j} (\theta_i - \theta_j)^2 \neq 0$  by lemma 1.19.

Now let  $\alpha_1, \dots, \alpha_n$  be elements of  $L$ . Let  $S$  be the transition matrix from  $1, \theta, \dots, \theta^{n-1}$  to  $\alpha_1, \dots, \alpha_n$ . Then  $S^t M S$  is the structure matrix of  $\langle -, - \rangle$  w.r.t. the  $\alpha_i$ , so

$$d(\alpha_1, \dots, \alpha_n) = \det(S^t M S) = \det(S)^2 \det(M).$$

Hence  $d(\alpha_1, \dots, \alpha_n) = 0$  iff  $\det(S) = 0$  iff  $\alpha_1, \dots, \alpha_n$  is not a basis.  $\square$

As before, let  $A$  be an integral domain which is integrally closed in  $K = \text{Quot}(A)$ . Let  $L/K$  be a finite separable extension and  $B = \mathcal{O}_{A,L} \subseteq L$  the integral closure of  $A$  in  $L$ .

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**Lemma 1.21.** *For  $b \in B$ , one has  $\text{Tr}_{L/K}(b), \text{N}_{L/K}(b) \in A$ . Further,  $b \in B$  is a unit if and only if  $\text{N}_{L/K}(b) \in A^\times$ .*

*Proof.* If  $b$  is integral, so is  $\sigma(b)$  for all  $\sigma \in G = G(L/K, K^c/K)$ . Thus  $\text{Tr}_{L/K}(b) = \sum_\sigma \sigma(b)$   $\text{Norm}_{L/K}(b) = \prod_\sigma \sigma(b) \in K \cap B = A$ , since  $A$  is integrally closed.

Let  $b \in B^\times$ , then  $bc = 1$  for some  $c \in B$ . It follows that

$$1 = \text{N}_{L/K}(1) = \text{N}_{L/K}(bc) = \text{N}_{L/K}(b) \text{N}_{L/K}(c),$$

so  $\text{N}_{L/K}(b) \in A^\times$ .

Conversely, let  $a = \text{N}_{L/K}(b) \in A^\times$ . Then

$$1 = a^{-1} \text{N}_{L/K}(b) = a^{-1} \prod_{\sigma \in G} \sigma(b) = b a^{-1} \underbrace{\prod_{\substack{\text{id} \neq \sigma \in G \\ \in L, \text{ integral}}} \sigma(b)}$$

$\square$

**Example 1.22.** Let  $L = \mathbb{Q}(\alpha) \subseteq \mathbb{R}$ ,  $\alpha^3 = 2$ . Then

$$d(1, \alpha, \alpha^2) = \det(\text{Tr}_{L/\mathbb{Q}}(\alpha^i \alpha^j))_{0 \leq i,j \leq 2} = \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix} = -108.$$

In the exercises we will use this to prove  $\mathcal{O}_L = \mathbb{Z}[\alpha]$ .

Further we compute

$$\mathrm{N}_{L/\mathbb{Q}}(1-\alpha) = (1-\alpha)(1-\zeta_3\alpha)(1-\zeta_3^2\alpha) = -1,$$

so by the above lemma  $1-\alpha \in \mathcal{O}_L^\times$ . (Alternatively, we could have noticed that  $(\alpha-1)^{-1} = \frac{\alpha^3-1}{\alpha-1} = 1+\alpha+\alpha^2 \in \mathcal{O}_L$ .) Actually, we have  $\mathcal{O}_L^\times = \{\pm 1\} \times (1-\alpha)^\mathbb{Z}$ , which agrees with the result of Dirichlet's unit theorem 0.7, since there is one real and one pair of complex embeddings.

**Lemma 1.23.** *Let  $\alpha_1, \dots, \alpha_n \in B$  be a  $K$ -basis of  $L$ . Let  $d = d_{L/K}(\alpha_1, \dots, \alpha_n) \in A$ . Then*

$$dB \subseteq A\alpha_1 \oplus \dots \oplus A\alpha_n.$$

*Proof.* Let  $B \ni \alpha = a_1\alpha_1 + \dots + a_n\alpha_n$  with  $a_i \in K$ . Then  $\mathrm{Tr}_{L/K}(\alpha_i\alpha) = \sum_{j=1}^n a_j \mathrm{Tr}_{L/K}(\alpha_i\alpha_j)$ , hence  $(a_1, \dots, a_n)$  is a solution of

$$\sum_{j=1}^n \underbrace{\mathrm{Tr}_{L/K}(\alpha_i\alpha_j)}_{=:A} x_j = \mathrm{Tr}_{L/K}(\alpha_i\alpha), \quad i = 1, \dots, n.$$

Cramer's rule shows that  $a_j = \frac{\det A_j}{\det A} = \frac{\det A_j}{d}$ , where  $A_j$  is the matrix  $A$  with  $j$ -th column replaced by the vector  $(\mathrm{Tr}_{L/K}(\alpha_i\alpha))_i$ . Hence  $d(a_1, \dots, a_n) \in A^n$   $\square$

Recall that for  $R$  a PID, each finitely generated torsion-free  $R$ -module  $M$  is free of finite rank, i.e.  $M \cong R^n$ ,  $n < \infty$ . Further, if  $M$  is a free  $R$ -module and  $N \subseteq M$  is an  $R$ -submodule, then  $N$  is free of rank at most the rank of  $M$ .

**Theorem 1.24.** *Assume further that  $A$  is a PID. Then any finitely generated  $B$ -submodule  $0 \neq M \subseteq L$  is a free  $A$ -module of rank  $n = [L : K]$ . In particular,  $B$  has an integral basis over  $A$ , i.e. there exist  $\omega_1, \dots, \omega_n \in B$  such that  $B = A\omega_1 \oplus \dots \oplus A\omega_n$ .*

*Proof.* Let  $\alpha_1, \dots, \alpha_n \in B$  be a  $K$ -basis of  $L$ . Let  $\mu_1, \dots, \mu_r \in M \subseteq L$  be a  $B$ -generating system of  $M$ . Let  $0 \neq a \in A$  such that  $a\mu_i \in B$  (possible by lemma 1.11). Let  $d = d_{L/K}(\alpha_1, \dots, \alpha_n)$ , which is nonzero by theorem 1.20. Then  $daM \subseteq dB \subseteq A\alpha_1 \oplus \dots \oplus A\alpha_n \cong A^n$  by lemma 1.23. It follows that  $daM \cong A^m$  with  $m \leq n$ , hence also  $M \cong A^m$ .

Let  $0 \neq \mu \in M$ . Then  $\mu\alpha_1, \dots, \mu\alpha_n \in M$  are a  $K$ -basis of  $L$ , so they are certainly linearly independent in  $M$  as well, hence  $m \geq n$ .  $\square$

**Example 1.25.** (i)  $L = \mathbb{Q}(\sqrt{d})$ ,  $\omega = \sqrt{d}$  for  $d \equiv 2, 3 \pmod{4}$  or  $\omega = \frac{1+\sqrt{d}}{2}$  for  $d \equiv 1 \pmod{4}$  as before. Then  $1, \omega$  is an integral basis of  $\mathcal{O}_L$ .

(ii)  $L = \mathbb{Q}(\alpha)$ ,  $\alpha^3 = 2$ . In the exercises we will see that  $1, \alpha, \alpha^2$  is an integral basis of  $\mathcal{O}_L$ .

(iii) Let  $K$  be a number field. Let  $0 \neq \mathfrak{a} \trianglelefteq \mathcal{O}_K$ . Then  $\mathfrak{a}$  has a  $\mathbb{Z}$ -basis, equivalently  $\mathfrak{a}$  is free over  $\mathbb{Z}$  of rank  $n$ .

**Remark 1.26.** Let  $L/K/\mathbb{Q}$  be number fields. Then  $\mathcal{O}_K$  is in general not a PID, so theorem 1.24 is not applicable to  $\mathcal{O}_L/\mathcal{O}_K$ . However, one can look at the localization  $\mathcal{O}_{L,\mathfrak{p}} = S^{-1}\mathcal{O}_L$  at  $S = \mathcal{O}_K \setminus \mathfrak{p}$  for a prime ideal  $\mathfrak{p} \subseteq \mathcal{O}_K$ . Then  $\mathcal{O}_{L,\mathfrak{p}} = \mathcal{O}_{\mathcal{O}_K, \mathfrak{p}, L}$  is an  $\mathcal{O}_{K,\mathfrak{p}}$ -module and a DVR, so the theorem can be applied to this ring extension.

**Definition 1.27.** Let  $L/\mathbb{Q}$  be a number field. Let  $\alpha_1, \dots, \alpha_n \in \mathcal{O}_L$  be an integral basis, i.e.  $\mathcal{O}_L = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$ . Then  $d_L = d_{L/\mathbb{Q}} := d_{L/\mathbb{Q}}(\alpha_1, \dots, \alpha_n)$  is called the *discriminant* of  $L$  (over  $\mathbb{Q}$ ). More generally, if  $0 \neq M \subseteq L$  is a finitely generated  $\mathcal{O}_L$ -module, then  $d_L(M) = d_{L/\mathbb{Q}}(M) := d(m_1, \dots, m_n)$  for some integral basis  $m_1, \dots, m_n$  of  $M$ .

$d_L$  is well-defined: Let  $\beta_1, \dots, \beta_n$  be another integral basis. Let  $S \in \mathrm{GL}_n(\mathbb{Z})$  be the transition matrix from the  $\alpha_i$  to the  $\beta_i$ . Then

$$\begin{aligned} d_{L/\mathbb{Q}}(\beta_1, \dots, \beta_n) &= \det(\mathrm{Tr}_{L/\mathbb{Q}}(\beta_i \beta_j)) = \det(S^t (\mathrm{Tr}_{L/\mathbb{Q}}(\alpha_i \alpha_j))_{ij} S) \\ &= \det(S)^2 \det(\mathrm{Tr}_{L/K}(\alpha_i \alpha_j)) = d_{L/\mathbb{Q}}(\alpha_1, \dots, \alpha_n). \end{aligned}$$

**Example 1.28.**  $L = \mathbb{Q}(\sqrt{d})$ ,  $d \equiv 2, 3 \pmod{4}$ . Then

$$d_{L/\mathbb{Q}} = d_{L/\mathbb{Q}}(1, \sqrt{d}) = \det(\mathrm{Tr}_{L/\mathbb{Q}}(\sqrt{d}^{i+j}))_{0 \leq i, j \leq 1} = \det \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix} = 4d.$$

Similarly one computes  $d_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} = d$  for  $d \equiv 1 \pmod{4}$ .

**Remark 1.29.** (i) We will show that a prime  $p$  is ramified in  $L/\mathbb{Q}$  if and only if  $p \mid d_{L/\mathbb{Q}}$  (where  $p$  is called ramified if the factorization  $p\mathcal{O}_L = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$  has some  $e_i > 1$ ).

(ii) If  $L/K$  are number fields. One can easily define a "relative" discriminant  $d_{L/K}$  if  $\mathcal{O}_K$  is a PID by the same procedure as above, except that it is only well-defined up to units, i.e. the ideal  $d_{L/K} := (d_{L/K}(\alpha_1, \dots, \alpha_n))$  for an integral basis  $\alpha_i$  is well-defined.

Now assume  $\mathcal{O}_K$  is arbitrary. As in remark 1.26, consider the extensions  $\mathcal{O}_{L,\mathfrak{p}}/\mathcal{O}_{K,\mathfrak{p}}$  for prime ideals  $\mathfrak{p} \trianglelefteq \mathcal{O}_K$ . As above, we may define thus "local" discriminant ideals  $d_{L/K,\mathfrak{p}} \trianglelefteq \mathcal{O}_{K,\mathfrak{p}}$ . One can then prove that there exists a unique ideal  $\mathfrak{D} \trianglelefteq \mathcal{O}_K$  such that  $\mathfrak{D}_{\mathfrak{p}} = d_{L/K,\mathfrak{p}}$  called the relative discriminant.

**Theorem 1.30.** Let  $L/\mathbb{Q}$  be a number field. Let  $0 \neq \mathfrak{a} \subseteq \mathfrak{a}'$  be  $\mathcal{O}_L$ -submodules of  $L$ . Then

$$d_L(\mathfrak{a}) = [\mathfrak{a}' : \mathfrak{a}]^2 d_L(\mathfrak{a}').$$

In particular,  $[\mathfrak{a}' : \mathfrak{a}]$  is finite.

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*Proof.* Let  $\alpha'_1, \dots, \alpha'_n$  be a  $\mathbb{Z}$ -basis of  $\mathfrak{a}'$  and  $\alpha_1, \dots, \alpha_n$  be a  $\mathbb{Z}$ -basis of  $\mathfrak{a}$ . Let  $T$  be the transition matrix, i.e.  $\alpha_i = \sum_{j=1}^n t_{ij} \alpha'_j$ ,  $t_{ji} \in \mathbb{Z}$ . As before, we see that  $d(\mathfrak{a}) = \det(T)^2 d(\mathfrak{a}')$ . So it remains to show that  $|\det(T)| = [\mathfrak{a}' : \mathfrak{a}]$ . By the elementary divisor theorem, we may assume that  $T$  is a diagonal matrix, from where the claim follows easily.  $\square$

**Corollary 1.31.** Let  $\alpha_1, \dots, \alpha_n \in \mathcal{O}_L$ . If  $d_L(\alpha_1, \dots, \alpha_n)$  is squarefree, then  $\alpha_1, \dots, \alpha_n$  is an integral basis.

**Remark 1.32.** This is not a necessary condition: In example 1.28 we saw  $4 \mid d_{\mathbb{Q}(\sqrt{d})}$  for  $d \equiv 2, 3 \pmod{4}$ .

## 2 Ideals

**Noetherian Rings** Let  $R$  be a ring. Recall from commutative algebra that an  $R$ -module  $M$  is called *Noetherian* if all submodules of  $M$  are finitely generated. In particular,  $M$  is finitely generated. For  $M = R$  this says that  $R$  is Noetherian if all ideals of  $R$  are finitely generated. For example, PIDs, finite rings, or finite modules are clearly Noetherian.

Further recall that if  $R$  is noetherian and  $M$  a finitely generated  $R$ -module, then  $M$  is noetherian; as well as the following

**Theorem 2.1.** The following are equivalent:

- (i)  $M$  is Noetherian

- (ii) Each ascending chain  $M_1 \subseteq M_2 \subseteq \dots$  of submodules of  $M$  stabilizes, i.e. there exists  $n_0 \in \mathbb{N}$  s.t.  $M_i = M_{n_0}$  for all  $i \geq n_0$ .
- (iii) Every non-empty family of  $R$ -submodules of  $M$  contains maximal elements.

**Theorem 2.2.** Let  $K/\mathbb{Q}$  be a number field. Then  $\mathcal{O}_K$  is Noetherian, integrally closed and of dimension 1, i.e. each non-zero prime ideal is maximal.

*Proof.* Each ideal  $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$  has a finite  $\mathbb{Z}$ -basis by theorem 1.24, hence in particular finitely generated. Thus  $\mathcal{O}_K$  is noetherian.  $\mathcal{O}_K$  is integrally closed by definition and transitivity 1.7.

Finally, for  $0 \neq \mathfrak{p}$  prime,  $\mathcal{O}_K/\mathfrak{p}$  is an integral domain which is finite by theorem 1.30, hence a field. Therefore,  $\mathfrak{p}$  is maximal.  $\square$

**Definition 2.3.** A noetherian, integrally closed integral domain of dimension 1 is called a *Dedekind* domain.

**Example 2.4.** By theorem 2.2,  $\mathcal{O}_K$  is a Dedekind domain. Further, any PID is clearly Dedekind.

Our next goal will be to show that in a Dedekind domain  $\mathcal{O}$ , every ideal factors uniquely as a product of prime ideals.

**Definition 2.5.** Let  $R$  be a ring and  $\mathfrak{a}, \mathfrak{b}$  be ideals.

- (i) We write  $\mathfrak{a} | \mathfrak{b}$  for  $\mathfrak{b} \subseteq \mathfrak{a}$ .
- (ii) The ideal sum  $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$  is also called the gcd of  $\mathfrak{a}$  and  $\mathfrak{b}$ .
- (iii) The intersection  $\mathfrak{a} \cap \mathfrak{b}$  is also called the lcm of  $\mathfrak{a}$  and  $\mathfrak{b}$ .

**Theorem 2.6.** Let  $\mathcal{O}$  be a Dedekind domain and  $\mathfrak{a} \subseteq \mathcal{O}$  an ideal,  $\mathfrak{a} \neq (0), (1)$ . Then there exists a unique presentation (up to order) of  $\mathfrak{a}$  in the form

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r \quad (*)$$

with prime ideals  $\mathfrak{p}_i \neq (0)$ . If we write  $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$  with pairwise distinct primes  $\mathfrak{p}_j$ , then also  $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cap \cdots \cap \mathfrak{p}_s^{e_s}$

*Proof.* We start with the second statement: In general, one has  $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$  for coprime ideals  $\mathfrak{a}, \mathfrak{b} \subseteq R$  for any ring  $R$ . Also, if  $\mathfrak{p}, \mathfrak{q}$  are coprime, then so are  $\mathfrak{p}^e$  and  $\mathfrak{q}^f$ .

For the main statement, we will need the following lemmas:

**Lemma 2.7.** Let  $0 \neq \mathfrak{a} \subseteq \mathcal{O}$  be an ideal. Then there are non-zero prime ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ ,  $r \geq 1$ , s.t.  $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{a}$

*Proof.* Let

$$\mathcal{M} := \{0 \neq \mathfrak{a} \subseteq \mathcal{O} \text{ ideal} \mid \mathfrak{a} \text{ does not satisfy the statement of the lemma}\}.$$

Suppose  $\mathcal{M} \neq \emptyset$ . Since  $\mathcal{O}$  is noetherian, by theorem 2.1 there exists a maximal element  $\mathfrak{a} \in \mathcal{M}$ . Then  $\mathfrak{a}$  is not a prime ideal, so there exist  $b_1, b_2 \in \mathcal{O}$  such that  $b_1 b_2 \in \mathfrak{a}$ , but  $b_1, b_2 \notin \mathfrak{a}$ . Let  $\mathfrak{a}_i := \mathfrak{a} + (b_i)$ . By choice of  $\mathfrak{a}$ , we have  $\mathfrak{a}_i \notin \mathcal{M}$ , hence we can write

$$\mathfrak{a}_1 \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_s, \quad \mathfrak{a}_2 \supseteq \mathfrak{q}_1 \cdots \mathfrak{q}_r$$

for nonzero prime ideals  $\mathfrak{p}_i, \mathfrak{q}_j$ . But then

$$\mathfrak{p}_1 \cdots \mathfrak{p}_s \mathfrak{q}_1 \cdots \mathfrak{q}_r \subseteq \mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a} + (b_1 b_2) \subseteq \mathfrak{a},$$

contradicting  $\mathfrak{a} \in \mathcal{M}$ .  $\square$

**Lemma 2.8.** Let  $0 \neq \mathfrak{p} \subseteq \mathcal{O}$  be a prime ideal. Let  $K := \text{Quot}(\mathcal{O})$  and

$$\mathfrak{p}^{-1} := \{x \in K \mid x\mathfrak{p} \subseteq \mathcal{O}\} \subseteq K.$$

Then  $\mathfrak{p}^{-1} \supseteq \mathcal{O}$  is a non-zero  $\mathcal{O}$ -module, and for any ideal  $0 \neq \mathfrak{a} \subseteq \mathcal{O}$  one has  $\mathfrak{a}\mathfrak{p}^{-1} \supsetneq \mathfrak{a}$ .

*Proof.* Everything is clear but the strictness of the final inclusion. Let  $0 \neq a \in \mathfrak{p}$ . By lemma 2.7 there exists a product of nonzero prime ideals  $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq (a)$  with  $r$  minimal. Since  $\mathfrak{p}_1 \cdots \mathfrak{p}_r \subseteq \mathfrak{p}$  and all these ideals are maximal, we have  $\mathfrak{p}_1 = \mathfrak{p}$ , say. By minimality of  $r$ ,  $\mathfrak{p}_2 \cdots \mathfrak{p}_r \not\subseteq (a)$ , so there exists  $b \in \mathfrak{p}_2 \cdots \mathfrak{p}_r \setminus (a)$ , hence  $a^{-1}b \notin \mathcal{O}$ . On the other hand  $b\mathfrak{p} \subseteq (a)$ , so  $a^{-1}b\mathfrak{p} \subseteq \mathcal{O}$ , i.e.  $a^{-1}b \in \mathfrak{p}^{-1}$ . Hence  $\mathfrak{p}^{-1} \supsetneq \mathcal{O}$ .

Let now  $0 \neq \mathfrak{a} \subseteq \mathcal{O}$  be an ideal. Let  $\mathfrak{a} = (\alpha_1, \dots, \alpha_n)$  and suppose  $\mathfrak{a}\mathfrak{p}^{-1} = \mathfrak{a}$ . Let  $x \in \mathfrak{p}^{-1}$ . Then

$$x\alpha_i = \sum_{j=1}^n a_{ji}\alpha_j, \quad a_{ji} \in \mathcal{O}.$$

Let  $A = (xE - (a_{ji}))$ . Then  $A(\alpha_1, \dots, \alpha_n)^t = 0$ , so by lemma 1.4,  $\det(A)\alpha_i = 0$ , so  $x$  is a zero of the normalized polynomial  $\det(tE - (\alpha_{ji})) \in \mathcal{O}[t]$ , hence  $x$  is integral over  $\mathcal{O}$ . But  $\mathcal{O}$  is integrally closed by definition, so  $x \in \mathcal{O}$ . Thus we have shown  $\mathfrak{p}^{-1} \subseteq \mathcal{O}$ , contradicting the previous paragraph.  $\square$

Now we can return to the proof of theorem 2.6. Let

$$\mathcal{M} := \{\mathfrak{a} \subseteq \mathcal{O} \text{ ideal } \mid \mathfrak{a} \neq (0), (1); \mathfrak{a} \text{ cannot be written as in } (*)\}.$$

Suppose  $\mathcal{M} \neq \emptyset$ . Since  $\mathcal{O}$  is Noetherian, by theorem 2.1 there exists a maximal element  $\mathfrak{a} \subseteq \mathcal{M}$ . Let  $\mathfrak{p} \supseteq \mathfrak{a}$  be a maximal ideal containing  $\mathfrak{a}$ . By lemma 2.8,  $\mathfrak{a} \subsetneq \mathfrak{a}\mathfrak{p}^{-1}$  and  $\mathfrak{p} \subsetneq \mathfrak{p}\mathfrak{p}^{-1} \subseteq \mathcal{O}$ . Since  $\mathfrak{p}$  is maximal,  $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}$ . By choice of  $\mathfrak{a}$ , we know that  $\mathfrak{a}\mathfrak{p}^{-1} \not\subseteq \mathcal{O}$ , so there is a factorization

$$\mathfrak{a}\mathfrak{p}^{-1} = \mathfrak{p}_1 \cdots \mathfrak{p}_s \implies \mathfrak{a} = \mathfrak{a}\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_s\mathfrak{p}.$$

This contradicts  $\mathfrak{a} \in \mathcal{M}$ , showing the existence of ideal factorizations.

For uniqueness, suppose  $\mathfrak{p}_1 \cdots \mathfrak{p}_r = \mathfrak{q}_1 \cdots \mathfrak{q}_s$ . Then  $\mathfrak{q}_1 \cdots \mathfrak{q}_s \subseteq \mathfrak{p}_1$ , so one of the factors is already contained in  $\mathfrak{p}_1$ , wlog  $\mathfrak{q}_1 \subseteq \mathfrak{p}_1$ . Since  $\mathfrak{q}_1$  is maximal,  $\mathfrak{q}_1 = \mathfrak{p}_1$ . Then multiply the original equation by  $\mathfrak{p}_1^{-1}$  and proceed inductively.  $\square$

For convenience, we will often write prime ideal factorizations in the form  $\mathfrak{a} = \prod_{\mathfrak{p} \neq 0} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$ , where  $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{N}_0$  is zero for almost all  $\mathfrak{p}$ . By the Chinese Remainder Theorem, we have Lecture 6  
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$$\mathcal{O}/\mathfrak{a} \cong \prod_{\mathfrak{p} \neq 0} \mathcal{O}/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}.$$

**Definition 2.9.** A *fractional ideal* in  $K = \text{Quot}(\mathcal{O})$  is a nonzero finitely generated  $\mathcal{O}$ -submodule of  $K$ .

**Example 2.10.** (i) For  $a \in K^\times$ ,  $(a) = a\mathcal{O}$  is a principal fractional ideal.  
(ii) More generally,  $c\mathfrak{a}$  is a fractional ideal for  $0 \neq \mathfrak{a} \subseteq \mathcal{O}$  an ideal and  $c \in K^\times$ .

**Lemma 2.11.**  $\mathfrak{a} \subseteq K$  be a fractional ideal if and only if there exists  $c \in \mathcal{O} \setminus \{0\}$  such that  $c\mathfrak{a}$  is an ideal of  $\mathcal{O}$ .

*Proof.* The backwards direction is clear. Let  $\mathfrak{a} = (\alpha_1, \dots, \alpha_s)$  be a fractional ideal. Write  $\alpha_1 = \frac{b_1}{c_1}$  with  $b_i, c_i \in \mathcal{O}$ . Then  $\prod c_i \mathfrak{a} \subseteq \mathcal{O}$  is an ideal of  $\mathcal{O}$ .  $\square$

To better distinguish fractional ideals and ideals contained in  $\mathcal{O}$ , we will often call the latter "integral ideals".

**Theorem 2.12.** *Let  $J_{\mathcal{O}}$  be the set of fractional ideals. Then  $J_{\mathcal{O}}$  is an abelian group w.r.t. multiplication of ideals. The identity element is  $\mathcal{O}$ , and the inverse of  $\mathfrak{a}$  is given by  $\mathfrak{a}^{-1} = (\mathcal{O} : \mathfrak{a})$ , where*

$$(\mathfrak{b} : \mathfrak{c}) := \{x \in K \mid x\mathfrak{c} \subseteq \mathfrak{b}\}$$

*Proof.* In the proof of theorem 2.6 we have seen  $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}$ . Let  $\mathfrak{a}$  be an integral ideal. For  $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ , we have the inverse  $\mathfrak{b} = \mathfrak{p}_1^{-1} \cdots \mathfrak{p}_r^{-1}$ . By lemma 2.11, each fractional ideal has an inverse.

Let now  $\mathfrak{a}$  be a fractional ideal and  $\mathfrak{b}$  its inverse, we want to show  $\mathfrak{b} = (\mathcal{O} : \mathfrak{a})$ . The inclusion  $\mathfrak{b} \subseteq (\mathcal{O} : \mathfrak{a})$  is clear from the definition of inverse. If  $x \in (\mathcal{O} : \mathfrak{a})$ . Then  $x\mathfrak{a} \subseteq \mathcal{O}$ , so  $x\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}$ , i.e.  $x \in \mathfrak{b}$ , finishing the proof.  $\square$

**Corollary 2.13.** *Let  $\mathfrak{a} \in J_{\mathcal{O}}$  be a fractional ideal. Then we have a unique representation of  $\mathfrak{a}$  in the form*

$$\mathfrak{a} = \prod_{\mathfrak{p} \neq 0} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$$

with  $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$  and almost all  $v_{\mathfrak{p}}(\mathfrak{a}) = 0$ . Further, we can uniquely write  $\mathfrak{a} = \mathfrak{b}\mathfrak{c}^{-1} =: \frac{\mathfrak{b}}{\mathfrak{c}}$  with  $\mathfrak{b}, \mathfrak{c} \subseteq \mathcal{O}$  integral ideals s.t.  $(\mathfrak{b}, \mathfrak{c}) = 1$ .

**Lemma 2.14.** *Let  $0 \neq \mathfrak{a} \subseteq \mathcal{O}$  be an integral ideal, and let  $\mathfrak{p} \neq 0$  be a prime ideal. Let  $\mathfrak{a} = \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}\mathfrak{b}$  with  $v_{\mathfrak{p}}(\mathfrak{a}) \geq 0$  and  $\mathfrak{p} \nmid \mathfrak{b}$ . Then  $v_{\mathfrak{p}}(\mathfrak{a}) = n$  if and only if  $\mathfrak{a} \subseteq \mathfrak{p}^n$  and  $\mathfrak{a} \not\subseteq \mathfrak{p}^{n+1}$ , i.e.  $v_{\mathfrak{p}}(\mathfrak{a})$  is the highest power of  $\mathfrak{p}$  dividing  $\mathfrak{a}$ .*

*Proof.* If  $\mathfrak{a} = \mathfrak{p}^n\mathfrak{b}$ , it is clear that  $\mathfrak{a} \subseteq \mathfrak{p}^n$ , and if  $\mathfrak{a} \subseteq \mathfrak{p}^{n+1}$ , then we would have  $\mathfrak{b} \subseteq \mathfrak{p}$ .

Conversely, suppose  $\mathfrak{a} \subseteq \mathfrak{p}^n$ . Then  $\mathfrak{b} := \mathfrak{a}\mathfrak{p}^{-n} \subseteq \mathcal{O}$  is an ideal, and  $\mathfrak{a} = \mathfrak{b}\mathfrak{p}^n$  shows  $v_{\mathfrak{p}}(\mathfrak{a}) \geq n$ . Suppose  $\mathfrak{p} \mid \mathfrak{b}$ , i.e.  $\mathfrak{b} \subseteq \mathfrak{p}$ . Then  $\mathfrak{a} = \mathfrak{p}^n\mathfrak{b} \subseteq \mathfrak{p}^{n+1}$ , contradicting the assumption.  $\square$

**Definition 2.15.** Let  $\mathcal{O}$  be a Dedekind domain and  $K = \text{Quot}(\mathcal{O})$ . Set  $P_{\mathcal{O}} = \{x\mathcal{O} \mid x \in K^{\times}\} \subseteq J_{\mathcal{O}}$  be the subgroup of principal fractional ideals. Then  $\text{cl}_{\mathcal{O}} := J_{\mathcal{O}}/P_{\mathcal{O}}$  is called the *ideal class group* of  $\mathcal{O}$ .

In the case of a number field  $K/\mathbb{Q}$  with ring of integers  $\mathcal{O}_K$ , write  $\text{cl}_K = \text{cl}_{\mathcal{O}_K}$  and similarly for  $J_K$  and  $P_K$ . Our next aim is to prove that  $\text{cl}_K$  is a finite group. This is not true for general Dedekind domains.

**Remark 2.16.** From the definition it is clear that a Dedekind domain  $\mathcal{O}$  is a PID if and only if  $|\text{cl}_{\mathcal{O}}| = 1$ . In general, we have the following exact sequence

$$1 \rightarrow \mathcal{O}^{\times} \hookrightarrow K^{\times} \xrightarrow{a \mapsto (a)} J_{\mathcal{O}} \xrightarrow{\mathfrak{a} \mapsto [\mathfrak{a}]} \text{cl}_{\mathcal{O}} \rightarrow 1$$

**Theorem 2.17.** *Let  $\mathcal{O}$  be a Dedekind domain with finitely many prime ideals. Then  $\mathcal{O}$  is a PID.*

*Proof.* Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  be the nonzero prime ideals of  $\mathcal{O}$ . It suffices to show that each  $\mathfrak{p}_i$  is principal, the result then follows from the prime ideal factorization 2.6. Let  $a_1 \in \mathfrak{p}_1 \setminus \mathfrak{p}_1^2$ . By the Chinese Remainder Theorem, there exists  $a \in \mathcal{O}$  such that  $a \equiv a_1 \pmod{\mathfrak{p}_1^2}$  and  $a \equiv 1 \pmod{\mathfrak{p}_i}$  for  $i > 1$ .

Then  $\mathfrak{p}_1 = a\mathcal{O}$ . Indeed, let  $a\mathcal{O} = \mathfrak{p}_1^{\nu_1} \cdots \mathfrak{p}_n^{\nu_n}$ . Since  $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_1^2$  and  $a \in \mathcal{O} \setminus \mathfrak{p}_i$ , lemma 2.14 shows  $\nu_1 = 1$  and  $\nu_i = 0$  for  $i > 1$ .  $\square$

Let  $\mathcal{O} \subseteq K = \text{Quot}(\mathcal{O})$  be a Dedekind domain and  $S \subseteq \mathcal{O}$  be a multiplicative subset. Then  $S^{-1}\mathcal{O}$  is still Dedekind: It is clearly a noetherian integral domain of dimension 1, by the correspondence of ideals in  $\mathcal{O}$  and  $S^{-1}\mathcal{O}$ . For integrally closed check in general that  $S^{-1}\mathcal{O}_{B,C} = \mathcal{O}_{S^{-1}B, S^{-1}C}$ .

Now take a prime  $\mathfrak{p} \neq 0$  and  $S = S_{\mathfrak{p}} := \mathcal{O} \setminus \mathfrak{p}$ . Then  $\mathcal{O}_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}\mathcal{O}$  is a Dedekind domain with exactly one prime  $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ , hence a PID by theorem 2.17, even a DVR.

**Theorem 2.18.** *Let  $0 \neq \mathfrak{m} \subseteq \mathcal{O}$  be an ideal. Let  $c \in \text{cl}_{\mathcal{O}}$  be an ideal class. Then  $c$  contains an integral ideal  $\mathfrak{a} \subseteq \mathcal{O}$  with  $(\mathfrak{a}, \mathfrak{m}) = 1$ .*

*Proof.* If there are only finitely many primes, then  $\text{cl}_{\mathcal{O}} = 1$  by theorem 2.17, so we may take  $\mathfrak{a} = \mathcal{O}$ . Suppose now we have infinitely many primes. Let  $\mathfrak{m} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_s^{f_s}$  be the unique prime ideal factorization of  $\mathfrak{m}$  and  $c = [\mathfrak{a}]$ , wlog  $\mathfrak{a} \subseteq \mathcal{O}$ . Let  $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \mathfrak{b}$ ,  $r \leq s$  and  $(\mathfrak{b}, \mathfrak{m}) = 1$ . Choose  $\alpha_i \in \mathfrak{p}_i^{e_i} \setminus \mathfrak{p}_i^{e_i+1}$  for  $i = 1, \dots, r$ . By the Chinese Remainder Theorem, there is  $\alpha \in \mathcal{O}$  such that

$$\begin{aligned} \alpha &\equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i+1}} & \text{for } i = 1, \dots, r, \\ \alpha &\equiv 1 \pmod{\mathfrak{p}_i} & \text{for } i = r+1, \dots, s. \end{aligned}$$

Then by lemma 2.14  $\alpha\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \mathfrak{c}$  for an integral ideal  $\mathfrak{c}$  with  $(\mathfrak{c}, \mathfrak{m}) = 1$ .  $\square$

In general,  $\mathcal{O}$  is not a PID, but

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**Theorem 2.19.** *Each ideal  $\mathfrak{a} \in J_{\mathcal{O}}$  can be generated by two elements. In fact, given  $0 \neq \alpha \in \mathfrak{a}$ , then there exists  $\beta \in \mathfrak{a}$  with  $\mathfrak{a} = (\alpha, \beta)$ .*

*Proof.* Suffices to consider  $\mathfrak{a} \subseteq \mathcal{O}$ . Claim: If  $0 \neq \mathfrak{b} \subseteq \mathcal{O}$  is an ideal, then every ideal of  $\mathcal{O}/\mathfrak{b}$  is principal.

Given this, let  $0 \neq \alpha \in \mathfrak{a}$  and let  $\pi : \mathcal{O} \rightarrow \mathcal{O}/(\alpha)$  be the canonical projection. Then the image of  $\mathfrak{a}$  under  $\pi$  is principal by the claim, say  $\bar{\mathfrak{a}} = (\bar{\beta})$ . Hence  $\mathfrak{a} = \pi^{-1}((\bar{\beta})) = (\alpha, \beta)$ .

Hence it remains to prove the claim. Write  $\mathfrak{b} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$  with  $e_i \geq 1$  and  $(\mathfrak{p}_i, \mathfrak{p}_j) = 1$ . Let  $\bar{\mathfrak{c}} \subseteq \mathcal{O}/\mathfrak{b}$  be an ideal, with  $\mathfrak{c} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_r^{f_r}$ ,  $f_i \leq e_i$  the corresponding ideal in  $\mathcal{O}$ . By the Chinese Remainder Theorem,  $\mathcal{O}/\mathfrak{b} \cong \mathcal{O}/\mathfrak{p}_1^{e_1} \times \dots \times \mathcal{O}/\mathfrak{p}_r^{e_r}$ , let  $\mathfrak{q}_1 \times \dots \times \mathfrak{q}_r$  be the image of  $\mathfrak{p}_i$  under this isomorphism. It suffices to show that the  $\mathfrak{q}_j$  are principal. But  $\mathfrak{q}_j = 1$  for  $i \neq j$ , and  $\mathfrak{q}_i = \mathfrak{p}_i/\mathfrak{p}_i^{e_i}$ .

More generally,  $\mathfrak{p}^i/\mathfrak{p}^e$  is principal in  $\mathcal{O}/\mathfrak{p}^e$ : Take  $\alpha \in \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$ , then  $\alpha\mathcal{O} + \mathfrak{p}^e = \mathfrak{p}^i$  by lemma 2.14, so  $(\bar{\alpha}) = \mathfrak{p}^i/\mathfrak{p}^e$ .  $\square$

In general, computing integral bases is difficult. However, sometimes they can be pieced together from smaller rings: Let  $K, L$  be number fields of degree  $n, m$ , respectively. Let  $M = KL$  be their composite. Then  $\mathcal{O}_K \mathcal{O}_L \subseteq \mathcal{O}_M$ .

**Theorem 2.20.** *Assume that  $[M : \mathbb{Q}] = mn$ . Let  $d := \gcd(d_K, d_L)$ . Then  $\mathcal{O}_M \subseteq \frac{1}{d}\mathcal{O}_K \mathcal{O}_L$ .*

**Corollary 2.21.** *If  $[M : \mathbb{Q}] = mn$  and  $\gcd(d_K, d_L) = 1$ , then  $\mathcal{O}_M = \mathcal{O}_L \mathcal{O}_K$ . In addition,  $d_M = d_L^n d_K^m$ .*

**Example 2.22.** For  $m \in \mathbb{N}$  let  $\zeta_m$  be a primitive  $m$ -th root of unity. Then  $\mathbb{Q}(\zeta_m)/\mathbb{Q}$  is a number field, called *cyclotomic field*, of degree  $\varphi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$ , and a Galois extension with  $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ , where the isomorphism is given by  $k \mapsto \sigma_k : \zeta_m \mapsto \zeta_m^k$ .

We will show  $\mathcal{O}_{\mathbb{Q}(\zeta_{p^n})} = \mathbb{Z}[\zeta_{p^n}]$  and that  $d_{\mathbb{Q}(\zeta_{p^n})}$  is a power of  $p$ . Further it is easy to see that  $\mathbb{Q}(\zeta_m)\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{mn})$  for  $m, n$  coprime. So corollary 2.21 implies  $\mathcal{O}_{\mathbb{Q}(\zeta_m)} = \mathbb{Z}[\zeta_m]$  and gives a formula for the discriminant of  $\mathbb{Q}(\zeta_m)$ .

*Proof.* Claim: Let  $\sigma : K \rightarrow \mathbb{C}$ ,  $\tau : L \rightarrow \mathbb{C}$  be embeddings. Then there exists a unique embedding  $\kappa : M \rightarrow \mathbb{C}$  such that  $\kappa|_K = \sigma$  and  $\kappa|_L = \tau$ . For the restriction map  $\text{Hom}_{\mathbb{Q}}(M, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \times \text{Hom}_{\mathbb{Q}}(L, \mathbb{C})$  is clearly injective and between finite sets of the same size  $nm$ , so bijective.

Let  $\alpha_1, \dots, \alpha_n$  be an integral basis of  $\mathcal{O}_K$ , and  $\beta_1, \dots, \beta_m$  an integral basis of  $\mathcal{O}_L$ . Then  $\alpha_i \beta_j$  form a  $\mathbb{Z}$ -basis of  $\mathcal{O}_K \mathcal{O}_L$ . Any  $\alpha \in \mathcal{O}_N$  can be written in the form  $\alpha = \sum_{i,j} \frac{m_{ij}}{r} \alpha_i \beta_j$  with  $m_{ij}, r \in \mathbb{Z}$  and  $\gcd(r, \gcd(m_{ij})) = 1$ . To show:  $r \mid d$ .

By symmetry, it suffices to show  $r \mid d_K$ . By the claim, for each  $\sigma : K \rightarrow \mathbb{C}$  there exists a unique  $\tilde{\sigma} : M \rightarrow \mathbb{C}$  such that  $\tilde{\sigma}|_K = \sigma$  and  $\tilde{\sigma}|_L = \text{id}_L$ . Then

$$\tilde{\sigma}(\alpha) = \sum_{i,j} \frac{m_{ij}}{r} \tilde{\sigma}(\alpha_i \beta_j) = \sum_{i,j} \frac{m_{ij}}{r} \sigma(\alpha_i) \beta_j.$$

Set  $x_i = \sum_{j=1}^m \frac{m_{ij}}{r} \beta_j$ . Then we have  $n$  equations  $\tilde{\sigma}(\alpha) = \sum_{i=1}^n \sigma(\alpha_i) x_i$ , one for each  $\sigma$ . By Cramer's rule,  $x_i = \frac{\gamma_i}{\delta}$ , where  $\delta = \det(\sigma(\alpha_i))_{\sigma,i}$ . Clearly,  $\gamma_i, \delta_i \in \mathcal{O}_M$ , and by definition  $\delta^2 = d_K$ . Hence  $d_K x_i = \delta \gamma_i$ , so  $d_K x_i = \sum_j \frac{d_K m_{ij}}{r} \beta_j \in \mathcal{O}_N \cap L = \mathcal{O}_L$ . But this means  $r \mid d_K m_{ij}$  for all  $i, j$ , so  $r \mid d_K$  by the coprimality assumption.

For the discriminant formula in the corollary, we now know that  $\alpha_i \beta_j$  is a  $\mathbb{Z}$ -basis of  $\mathcal{O}_M$ , hence

$$\begin{aligned} d_N &= d(\alpha_i \beta_j) = \det(\text{Tr}_{M/\mathbb{Q}}(\alpha_i \beta_j \alpha_k \beta_l)) = \det(\text{Tr}_{K/\mathbb{Q}}(\text{Tr}_{M/K}(\alpha_i \beta_j \alpha_k \beta_l))) \\ &= \det(\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k \text{Tr}_{M/K}(\beta_j \beta_l))) = \det(\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k \text{Tr}_{L/\mathbb{Q}}(\beta_j \beta_l))) \\ &= \det(\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k) \text{Tr}_{L/\mathbb{Q}}(\beta_j \beta_l)) = \det((\text{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_k)) \otimes (\text{Tr}_{L/\mathbb{Q}}(\beta_j \beta_l))) \\ &= d_K^m d_L^n, \end{aligned}$$

where we used the fact from linear algebra that  $A \otimes B = (a_{ij}B) \in R^{nm \times nm}$  for  $A \in R^{n \times n}$ ,  $B \in R^{m \times m}$  satisfies  $\det(A \otimes B) = \det(A)^m \det(B)^n$   $\square$

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### 3 Lattices

**Definition 3.1.** Let  $V$  be an  $n$ -dimensional  $\mathbb{R}$ -vector space. A *lattice* in  $V$  is a subgroup  $\Gamma$  of  $V$  of the form  $\Gamma = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_m$  with linearly independent vectors  $v_1, \dots, v_m \in V$ ,  $m \leq n$ . The set  $\Phi = \{x_1 v_1 + \dots + x_m v_m \mid 0 \leq x_i < 1\}$  is called a *fundamental domain* of  $\Gamma$ . Further,  $\Gamma$  is a *full lattice* if  $m = n$ .

**Definition 3.2.** A subgroup  $\Gamma$  of  $V$  is called discrete if for each  $\gamma \in \Gamma$  there exists a neighbourhood  $U$  such that  $\Gamma \cap U = \{\gamma\}$

**Lemma 3.3.** If  $\Gamma$  is a discrete subgroup of  $V$ , then  $\Gamma$  is closed.

*Proof.* Claim: Each  $a \in V \setminus \Gamma$  has an open neighbourhood  $U$  with  $|\Gamma \cap U| < \infty$ .

Then since  $V$  is Hausdorff, there exists an open neighbourhood  $\tilde{U}$  of  $a$  that avoids these finitely many points, so  $(U \cap \tilde{U}) \cap \Gamma = \emptyset$ , i.e.  $U \cap \tilde{U}$  is a neighbourhood of  $a$  in  $V \setminus \Gamma$ .

To prove the claim, let  $a \in V \setminus \Gamma$ . By assumption, there exists an open  $\tilde{U} \subseteq V$  such that  $\tilde{U} \cap \Gamma = \{0\}$ . Since  $V \times V \rightarrow V$ ,  $(a, b) \mapsto a - b$  is continuous, there exists an open neighbourhood  $U$  of 0 such that  $U - U \subseteq \tilde{U}$ . Then  $a + U$  is an open neighbourhood of  $a$ , suppose there are  $\gamma_1, \gamma_2 \in \Gamma \cap (a + U)$ . But then  $\gamma_1 - \gamma_2 \in \tilde{U}$ , so  $\gamma_1 = \gamma_2$ .  $\square$

**Lemma 3.4.** Let  $\Gamma$  be a subgroup of  $V$ . Then  $\Gamma$  is discrete if and only if for all bounded  $C \subseteq V$  one has  $|C \cap \Gamma| < \infty$ .

*Proof.* Let  $\Gamma$  be discrete. Wlog  $C$  is compact. If  $C \cap \Gamma$  were infinite, then by Bolzano-Weierstrass, there is an accumulation point  $\gamma \in C \cap \Gamma$  (by lemma 3.3), contradicting the definition.

Conversely, let  $\gamma \in \Gamma$ . Choose an open ball around  $\gamma$ . By assumption, this ball contains only finitely many  $\gamma_i \in \Gamma$ , which, as before, can be separated from  $\gamma$  using the Hausdorff property.  $\square$

**Example 3.5.** Let  $K = \mathbb{Q}(\sqrt{2}) \subseteq \mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}[\sqrt{2}]$  is not a lattice in  $V = \mathbb{R}$ , but  $\mathcal{O}_K$  becomes a lattice in  $\mathbb{R}^2$  via

$$j : \mathcal{O}_K \hookrightarrow \mathbb{R}^2, \quad a + b\sqrt{2} \mapsto (a + b\sqrt{2}, a - b\sqrt{2}).$$

We will prove soon (in general) that  $j(\mathcal{O}_K) \subseteq \mathbb{R}^2$  is a lattice.

**Theorem 3.6.** Let  $\Gamma \subseteq V$  be a subgroup. Then  $\Gamma$  is a lattice if and only if  $\Gamma$  is discrete.

*Proof.* Let  $\Gamma = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_m$  be a lattice. Choose a basis  $v_1, \dots, v_m, v_{m+1}, \dots, v_n$  of  $V$ . Let  $\gamma = a_1v_1 + \dots + a_mv_m$ . Consider

$$U := \{x_1v_1 + \dots + x_nv_n \mid x_i \in \mathbb{R} \mid |a_i - x_i| < 1 \text{ for } i \leq m\}.$$

Then  $U$  is open and  $U \cap \Gamma = \{\gamma\}$ .

Conversely, let  $\Gamma$  be discrete. Let  $V_0$  be the  $\mathbb{R}$ -subspace of  $V$  generated by  $\Gamma$  and denote  $m := \dim_{\mathbb{R}} V_0$ . Choose a  $\mathbb{R}$ -basis  $u_1, \dots, u_m$  of  $V_0$  with  $u_i \in \Gamma$ . Consider  $\Gamma_0 := \mathbb{Z}u_1 \oplus \dots \oplus \mathbb{Z}u_m \subseteq V_0$ , which is a lattice by definition.

Claim:  $q := (\Gamma : \Gamma_0) < \infty$ . Then  $\Gamma_0 \subseteq \Gamma \subseteq \frac{1}{q}\Gamma_0$  is a subgroup of a free abelian group, so is itself free (of rank  $m$ ).

To prove the claim, let  $\{\gamma_i\}_{i \in I}$  be a set of representatives of  $\Gamma/\Gamma_0$ . Let  $\Phi_0 = \{x_1u_1 + \dots + x_mu_m \mid 0 \leq x_i < 1\}$  be a fundamental domain of  $\Gamma_0$ . Then  $\bigcup_{\gamma \in \Gamma_0} (\gamma + \Phi_0) = V$ , hence  $\gamma_i = \gamma_{0i} + \mu_i$  with  $\gamma_{0i} \in \Gamma_0$  and  $\mu_i \in \Phi_0$ . Then the bounded  $\Phi_0$  contains all the  $\mu_i = \gamma_i - \gamma_{0i} \in \Gamma$ , hence  $I$  is finite by lemma 3.4.  $\square$

**Lemma 3.7.** Let  $\Gamma \subseteq V$  be a lattice. Then  $\Gamma$  is full if and only if there exists a bounded subset  $M \subseteq V$  such that  $\bigcup_{\gamma \in \Gamma} \gamma \in \Gamma(\gamma + M) = V$ .

*Proof.* If  $\Gamma$  is full, take  $M$  to be a fundamental domain. Conversely, let  $V_0$  be the  $\mathbb{R}$ -span of  $\Gamma$ . Let  $v \in V$ . For  $\nu \in \mathbb{N}$  write  $\nu v = \gamma_\nu + a_\nu$  with  $\gamma_\nu \in \Gamma$  and  $a_\nu \in M$ . Since  $M$  is bounded,  $\frac{a_\nu}{\nu} \xrightarrow{\nu \rightarrow \infty} 0$ . Hence

$$v = \lim_{\nu \rightarrow \infty} \frac{\gamma_\nu + a_\nu}{\nu} = \lim_{\nu \rightarrow \infty} \frac{\gamma_\nu}{\nu} \in V_0,$$

since  $V_0 \subseteq V$  is closed.  $\square$

Now let  $V$  be an euclidean vector space with inner product  $\langle -, - \rangle : V \times V \rightarrow \mathbb{R}$ . Let  $e_1, \dots, e_n$  be an orthonormal basis. Then we define a volume. For the "unit cube"

$$E := \left\{ \sum_i \alpha_i e_i \mid 0 \leq \alpha_i \leq 1 \right\},$$

we set  $\text{Vol}(E) = 1$ . More generally, let  $v_1, \dots, v_n$  be an  $\mathbb{R}$ -basis of  $V$  and let  $\Phi := \{\sum_i x_i v_i \mid 0 \leq x_i \leq 1\}$ . Let  $A = (a_{ji}) \in \text{GL}_n(\mathbb{R})$  be the transition matrix,  $v_i = \sum_j a_{ji} e_j$ .

**Lemma 3.8.** One has  $\text{Vol}(\Phi) = |\det(A)| = \sqrt{\det(\langle v_i, v_j \rangle_{ij})}$ .

*Proof.*

$$\text{Vol}(\Phi) = \int_{\Phi} dx = \int_E |\det(A)| dx = |\det(A)| \text{Vol}(E) = |\det(A)|.$$

The second equality follows from  $\langle v_i, v_j \rangle_{ij} = A^t \langle e_i, e_j \rangle_{ij} A = A^t A$ .  $\square$

**Definition 3.9.** Let  $\Gamma \subseteq V$  be a full lattice. Then we define  $\text{Vol}(\Gamma) := \text{Vol}(\Phi)$  for any fundamental domain  $\Phi$  for  $\Gamma$ .

This is well-defined, i.e. independent of the choice of  $\Phi$ , since different  $\mathbb{Z}$ -bases of  $\Gamma$  differ by a transition matrix  $T \in \text{GL}_n(\mathbb{Z})$ , i.e.  $\det(T) = \pm 1$ , so the absolute value of the determinant does not change.

**Definition 3.10.** Let  $X \subseteq V$  be a subset.  $X$  is called *central-symmetric* if for all  $x \in X$  we have  $-x \in X$ .  $X$  is *convex* if for all  $x, y \in X$  also  $tx + (1-t)y \in X$  for  $0 \leq t \leq 1$ .

For example, a ball centered around 0 is both central-symmetric and convex.

**Theorem 3.11** (Minkowski's Lattice Point Theorem). *Let  $\Gamma \subseteq V$  be a full lattice in an euclidean vector space of dimension  $\dim_{\mathbb{R}}(V) = n$ . Let  $X \subseteq V$  be a central-symmetric, convex subset with  $\text{Vol}(X) > 2^n \text{Vol}(\Gamma)$ . Then there exists a  $0 \neq \gamma \in \Gamma$  with  $\gamma \in X$ .*

*Proof.* It suffices to show that there are  $\gamma_1 \neq \gamma_2 \in \Gamma$  such that  $(\frac{1}{2}X + \gamma_1) \cap (\frac{1}{2}X + \gamma_2) \neq \emptyset$ . Indeed, let  $v = \frac{1}{2}x_1 + \gamma_1 + \frac{1}{2}x_2 + \gamma_2$  be an element of the intersection. Then

$$\gamma_1 - \gamma_2 = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}x_2 + \left(1 - \frac{1}{2}\right)(-x_1) \in X$$

by central-symmetry and convexity.

To prove the claim, suppose that the sets  $(\frac{1}{2}X + \gamma)$ ,  $\gamma \in \Gamma$  are pairwise disjoint. Then so are the sets  $\Phi \cap (\frac{1}{2}X + \gamma)$  for a fundamental domain  $\Phi$  of  $\Gamma$ . Hence

$$\text{Vol}(\Gamma) = \text{Vol}(\Phi) \geq \sum_{\gamma \in \Gamma} \text{Vol}(\Phi \cap (\gamma + \frac{1}{2}X)) = \sum_{\gamma \in \Gamma} \text{Vol}((\Phi - \gamma) \cap \frac{1}{2}X).$$

Since  $\Phi - \gamma$  covers all of  $X$ , cf. lemma 3.7. Therefore

$$\text{Vol}(\Gamma) \geq \text{Vol}(\frac{1}{2}X) = 2^{-n} \text{Vol}(X),$$

contradicting our assumption. □

## 4 Minkowski Theory

Let  $K/\mathbb{Q}$  be a number field of degree  $n$ . Of the embeddings  $\tau : K \rightarrow \mathbb{C}$ , we distinguish real embeddings  $\rho_1, \dots, \rho_r : K \rightarrow \mathbb{R}$  and pairs of complex embeddings  $\sigma_1, \bar{\sigma}_1, \dots, \sigma_s, \bar{\sigma}_s : K \rightarrow \mathbb{C}$  with image not contained in  $\mathbb{R}$ , with  $n = r + 2s$ .

**Definition 4.1.** We define *Minkowski Space* of  $K$  as

$$K_{\mathbb{R}} := \left\{ (z_{\tau}) \in \prod_{\tau: K \rightarrow \mathbb{C}} \mathbb{C} \mid z_{\rho} \in \mathbb{R}, z_{\bar{\sigma}} = \overline{z_{\sigma}} \right\}$$

**Remark 4.2.**  $K_{\mathbb{R}}$  is an  $\mathbb{R}$ -vector space of dimension  $r + 2s = n$ .

**Example 4.3.** Let  $K = \mathbb{Q}(\sqrt{d})$ . If  $d > 0$ , then  $K_{\mathbb{R}} = \mathbb{R}\rho_1 + \mathbb{R}\rho_2$ . If, on the other hand,  $d < 0$ , then  $K_{\mathbb{R}} = \{(\beta, \bar{\beta}) \mid \beta \in \mathbb{C}\} \subseteq \mathbb{C}^2$ .

**Example 4.4.** For  $K = \mathbb{Q}(\omega)$  with  $\omega \in \mathbb{R}$ ,  $\omega^3 = 2$ , we have  $K_{\mathbb{R}} = \{(\alpha, \beta, \bar{\beta}) \mid \alpha \in \mathbb{R}, \beta \in \mathbb{C}\}$ .

On  $K_{\mathbb{R}}$  we define the inner product

$$\langle x, y \rangle := \sum_{\tau} x_{\tau} \overline{y_{\tau}}.$$

It is clear that this is bilinear and positive definite; we check that the image is contained in  $\mathbb{R}$ :

$$\langle x, y \rangle = \sum_{\rho} x_{\rho} y_{\rho} + \sum_{\sigma} (x_{\sigma} \overline{y_{\sigma}} + \underbrace{x_{\bar{\sigma}} \overline{y_{\bar{\sigma}}}}_{= \bar{x}_{\sigma} y_{\sigma}}).$$

Hence the terms of the last sum are stable under conjugation, thus they lie in  $\mathbb{R}$ .

**Theorem 4.5.** *Consider the isomorphism of  $\mathbb{R}$ -vector spaces*

$$f : K_{\mathbb{R}} \rightarrow \prod_{\tau} \mathbb{R}, \quad (z_{\tau})_{\tau} \mapsto (z_{\rho_1}, \dots, z_{\rho_r}, \operatorname{Re}(z_{\sigma_1}), \operatorname{Im}(z_{\sigma_1}), \dots, \operatorname{Re}(z_{\sigma_s}), \operatorname{Im}(z_{\sigma_s})).$$

For  $(-, -) : (\prod_{\tau} \mathbb{R})^2 \rightarrow \mathbb{R}$  defined by  $(x, y) := \sum \alpha_{\tau} x_{\tau} y_{\tau}$  with  $\alpha_{\tau} = 1$  if  $\tau$  is real and  $\alpha_{\tau} = 2$  if  $\tau$  is complex, we have  $\langle x, y \rangle = (f(x), f(y))$  for all  $x, y \in K_{\mathbb{R}}$ .

*Proof.* Exercise. □

**Remark 4.6.** By the above theorem,  $\operatorname{Vol}_{(-, -)} = 2^s \operatorname{Vol}_{\text{Lebesgue}}$ , since an orthonormal basis w.r.t.  $(-, -)$  is given by  $e_1, \dots, e_r, \frac{1}{\sqrt{2}}e_{r+1}, \dots, \frac{1}{\sqrt{2}}e_{r+2s}$ .

Generalizing example 3.5, define

$$j : K \hookrightarrow K_{\mathbb{R}}, \quad \alpha \mapsto (\tau(\alpha))_{\tau: K \rightarrow \mathbb{C}}.$$

This is a  $\mathbb{Q}$ -linear embedding.

**Theorem 4.7.** *Let  $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$  be an ideal. Then  $\Gamma := j(\mathfrak{a})$  is a full lattice in  $K_{\mathbb{R}}$  with  $\operatorname{Vol}(\Gamma) = \sqrt{|d_K|}[\mathcal{O}_K : \mathfrak{a}]$ .*

*Proof.* Let  $\alpha_1, \dots, \alpha_n$  be a  $\mathbb{Z}$ -basis of  $\mathfrak{a}$ . Consider  $A = (\tau_l(\alpha_i))_{il}$ . Then  $d(\mathfrak{a}) = \det(A)^2 = [\mathcal{O}_K : \mathfrak{a}]^2 d_K$ . On the other hand,  $j(\alpha_1), \dots, j(\alpha_n)$  is a  $\mathbb{Z}$ -basis of  $\Gamma$ . We have  $j(\alpha_i) = (\tau_l(\alpha_i))_l$ , so that

$$\langle j(\alpha_i), j(\alpha_k) \rangle = \sum_l \tau_l(\alpha_i) \overline{\tau_l(\alpha_k)}.$$

Hence the structure matrix of  $\langle -, - \rangle$  is  $(\langle j(\alpha_i), j(\alpha_k) \rangle)_{i,k} = A \overline{A}^t$ , so

$$\operatorname{Vol}(\Gamma) = \sqrt{\det(A \overline{A}^t)} = |\det(A)| = [\mathcal{O}_K : \mathfrak{a}] \sqrt{|d_K|}.$$

In particular, the volume of a fundamental domain is nonzero, so the lattice is full. □

**Theorem 4.8.** *Let  $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$  be an ideal. Let  $c_{\tau} \in \mathbb{R}_{>0}$  such that  $c_{\tau} = c_{\bar{\tau}}$ . Assume that  $\prod_{\tau} c_{\tau} > (\frac{2}{\pi})^s \sqrt{|d_K|}[\mathcal{O}_K : \mathfrak{a}]$ . Then there exists  $0 \neq a \in \mathfrak{a}$  with  $|\tau(a)| < c_{\tau}$  for all  $\tau$ .*

*Proof.* Look at  $X := \{(z_{\tau})_{\tau} \in K_{\mathbb{R}} \mid |z_{\tau}| < c_{\tau}\}$ . Then  $X$  is convex and central-symmetric. One computes

$$\operatorname{Vol}(X) = 2^{r+s} \pi^s \prod_{\tau} c_{\tau} > 2^n \operatorname{Vol}(j(\mathfrak{a})).$$

Therefore, the conditions of Minkowski's Lattice Point Theorem 3.11 are satisfied, so there exists  $0 \neq j(a) \in j(\mathfrak{a}) \cap X$ . This is the desired  $a \in \mathfrak{a}$ . □

**Definition 4.9.** For  $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$  we define its norm

$$N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}].$$

In the exercises we saw that this ideal norm is multiplicative, hence we may extend it to a multiplicative function  $N : I_K \rightarrow \mathbb{Z}$  on all fractional ideals.

**Lemma 4.10.** Let  $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$ . Then there exists  $0 \neq a \in \mathfrak{a}$  with  $|N_{K/\mathbb{Q}}(a)| \leq (\frac{2}{\pi})^s \sqrt{|d_K|} N(\mathfrak{a})$ .

*Proof.* For  $\varepsilon > 0$  choose  $c_\tau \in \mathbb{R}_{>0}$ ,  $c_{\bar{\tau}} = c_\tau$  such that  $\prod_\tau c_\tau = (\frac{2}{\pi})^s \sqrt{|d_K|} N(\mathfrak{a}) + \varepsilon$ . By theorem 4.8, there exists  $0 \neq a \in \mathfrak{a}$  such that  $|\tau(a)| < c_\tau$ , hence

$$|N_{K/\mathbb{Q}}(a)| = \prod_\tau |\tau(a)| < \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} N(\mathfrak{a}) + \varepsilon.$$

Since the norm is an integer, for small enough  $\varepsilon$  we get the claim.  $\square$

**Theorem 4.11.** The class number is finite:  $h_K := |\text{cl}_K| < \infty$ .

*Proof.* For each  $M > 0$ , there are only finitely many integral ideals  $\mathfrak{a} \subseteq \mathcal{O}_K$  with  $N(\mathfrak{a}) < M$ . Indeed, since each such integral ideal factors into prime ideals, it suffices to show that there are only finitely many prime ideals of bounded norm. But by the exercises,  $N(\mathfrak{p})$  is a  $p$ -power, where  $p \in \mathbb{Z}$  is the prime such that  $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$ . Since there are only finitely many prime ideals containing each prime  $p$ , we are done.

Hence it suffices to show that each ideal class  $c \in \text{cl}_K$  contains an integral ideal  $\mathfrak{a} \subseteq \mathcal{O}_K$  such that  $N(\mathfrak{a}) \leq (\frac{2}{\pi})^s \sqrt{|d_K|}$ . So let  $\mathfrak{b} \in c$  be a representative. Choose  $\gamma \in \mathcal{O}_K$  such that  $\gamma\mathfrak{b}^{-1} \subseteq \mathcal{O}_K$  is an integral ideal. Then by the previous lemma, there exists  $0 \neq \alpha \in \gamma\mathfrak{b}^{-1}$  such that

$$|N_{K/\mathbb{Q}}(\alpha)| \leq \left(\frac{2}{\pi}\right)^s \sqrt{|d_K|} N(\gamma\mathfrak{b}^{-1}).$$

Using the following lemma, the integral ideal  $\alpha\gamma^{-1}\mathfrak{b}$  satisfies  $N(\alpha\gamma^{-1}\mathfrak{b}) \leq (\frac{2}{\pi})^s \sqrt{|d_K|}$ .  $\square$

**Lemma 4.12.** For  $0 \neq \alpha \in K$  one has  $N(\alpha\mathcal{O}_K) = |N_{K/\mathbb{Q}}(\alpha)|$ .

*Proof.* Let  $\omega_1, \dots, \omega_n$  be an integral basis of  $\mathcal{O}_K$ . Let  $\alpha(\omega_1, \dots, \omega_n)^t = A(\omega_1, \dots, \omega_n)^t$  for  $A \in M_n(\mathbb{Z})$ . Then  $[\mathcal{O}_K : \alpha\mathcal{O}_K] = |\det(A)| = |N_{K/\mathbb{Q}}(\alpha)|$ .  $\square$

**Remark 4.13.** The proof of theorem 4.11 yields a finite generating set for the class group:

$$\text{cl}_K = \langle [\mathfrak{a}] \mid 0 \neq \mathfrak{a} \subseteq \mathcal{O}_K, N(\mathfrak{a}) \leq (\frac{2}{\pi})^s \sqrt{|d_K|} \rangle.$$

In fact, the bound on  $N(\mathfrak{a})$  can be improved: Let

$$X = \left\{ (z_\tau)_\tau \in K_{\mathbb{R}} \mid \sum_\tau |z_\tau| < t \right\}.$$

Then  $X$  is central-symmetric, convex, and  $\text{Vol}(X) = 2^r \pi^s \frac{t^n}{n!}$ . Repeating the above proofs with this set, one obtains

**Theorem 4.14.** Let  $0\mathfrak{a} \subseteq \mathcal{O}_K$  be an integral ideal. Then there exists  $0 \neq a \in \mathfrak{a}$  such that  $|N_{K/\mathbb{Q}}(a)| \leq M \cdot N(\mathfrak{a})$  with the Minkowski constant

$$M := \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_K|}.$$

Therefore,  $\text{cl}_K$  is generated by classes of ideals  $\mathfrak{a}$  with  $N(\mathfrak{a}) \leq M$ .

*Proof.* Exercise. □

**Example 4.15.** (i) Let  $K = \mathbb{Q}(\sqrt{2})$ . Then  $M = \sqrt{2} \approx 1.41$ . But the only integral ideal with norm 1 is  $\mathcal{O}_K$ , so  $\text{cl}_K = 1$ .

(ii) Let  $K = \mathbb{Q}(\sqrt{-5})$ . Then  $M = \frac{4}{\pi}\sqrt{5} \approx 2.84$ . Hence  $\text{cl}_K$  is generated by the classes of integral ideals of norm  $\leq 2$ . By factoring (2), one can compute directly that the only such ideals are  $\mathcal{O}_K$  and  $\mathfrak{p} = \langle 2, 1 + \sqrt{-5} \rangle_{\mathbb{Z}}$  with  $\text{ord}([\mathfrak{p}]) \leq 2$  since  $\mathfrak{p}^2 = (2)$ . So  $\text{cl}_K = 1$  if  $\mathfrak{p}$  is principal, or  $\text{cl}_K = \mathbb{Z}/2\mathbb{Z}$  otherwise. Here, the latter is the case, so  $h_K = 2$ .

Generalizing the last example, we can give a general procedure for computing the class group:  
List all prime ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  with  $N(\mathfrak{p}_i) \leq M$ , by factoring  $p\mathcal{O}_K$  for  $p \leq M$  prime. Then  $\text{cl}_K$  is generated by their classes (it suffices to consider prime ideals by prime ideal factorization). Let  $\pi : \mathbb{Z}^m \rightarrow \text{cl}_K$ ,  $a \mapsto \prod_i [\mathfrak{p}_i]^{a_i}$  and  $\Lambda := \ker \pi$ . This gives an exact sequence

$$0 \rightarrow \Lambda \rightarrow \mathbb{Z}^m \xrightarrow{\pi} \text{cl}_K \rightarrow 1.$$

Every relation comes from an equation  $\alpha\mathcal{O}_K = \prod_i \mathfrak{p}_i^{a_i}$ ,  $\alpha \in K^\times$ . Finding sufficiently many of such relations, one can determine the class group.

**Lemma 4.16.** *Let  $0 \neq \mathfrak{a} \subseteq \mathcal{O}_K$  be an integral ideal. Then  $\mathfrak{a}$  is a principal ideal if and only if  $N(\mathfrak{a}) = |N_{K/\mathbb{Q}}(\alpha)|$  for some  $\alpha \in \mathfrak{a}$ .*

*Proof.* One direction was proven in lemma 4.12. Conversely, suppose  $\alpha$  as in the statement exists. Then  $\alpha\mathcal{O}_K \subseteq \mathfrak{a}$  is a submodule, but again by lemma 4.12, their indices in  $\mathcal{O}_K$  are equal. □

## 5 Dirichlet's Unit Theorem

Lecture 11  
Nov 19, 2025

### 5.1 Statement and Proof

The next goal is to understand the unit group of a number ring. Let

$$K_{\mathbb{R}}^\times := \{(z_\tau)_{\tau} \in K_{\mathbb{R}} \mid z_\tau \neq 0 \text{ for all } \tau\}$$

and consider the map

$$l : K_{\mathbb{R}}^\times \rightarrow \mathbb{R}^{r+s}, \quad (z_\tau)_{\tau} \mapsto (\log|x_{\rho_1}|, \dots, \log|x_{\rho_r}|, 2\log|x_{\sigma_1}|, \dots, 2\log|x_{\sigma_s}|).$$

Then clearly  $l(xy) = l(x) + l(y)$  for  $x, y \in K_{\mathbb{R}}^\times$ . Further define a norm  $N : K_{\mathbb{R}}^\times \rightarrow \mathbb{R}^\times$  and trace  $\text{Tr} : \mathbb{R}^{r+s} \rightarrow \mathbb{R}$  by  $N((z_\tau)_{\tau}) := \prod_{\tau} z_\tau$  and  $\text{Tr}(x) = \sum_{i=1}^{r+s} x_i$ . Putting everything together, we have the following commutative diagram of group homomorphisms:

$$\begin{array}{ccccc} K^\times & \xrightarrow{j} & K_{\mathbb{R}}^\times & \xrightarrow{l} & \mathbb{R}^{r+s} \\ \downarrow N_{K/\mathbb{Q}} & & \downarrow N & & \downarrow \text{Tr} \\ \mathbb{Q}^\times & \hookrightarrow & \mathbb{R}^\times & \xrightarrow{\log|\cdot|} & \mathbb{R} \end{array}$$

Also let  $\lambda := l \circ j$ . Since units in  $\mathcal{O}_K$  are characterized by their norm being  $\pm 1$ , further define

$$S := \{y \in K_{\mathbb{R}}^\times \mid N(y) = \pm 1\}, \quad H = \{x \in \mathbb{R}^{r+s} \mid \text{Tr}(x) = 0\}.$$

Then  $j(\mathcal{O}_K^\times) \subseteq S$  and  $l(S) = H$ .

**Definition 5.1.**  $\Gamma := \lambda(\mathcal{O}_K^\times) \subseteq H$

**Theorem 5.2.** (i) There is a short exact sequence  $1 \rightarrow \mu_K \rightarrow \mathcal{O}_K^\times \xrightarrow{\lambda} \Gamma \rightarrow 0$ , where  $\mu_K := \{x \in K^\times \mid \text{ord}(x) < \infty\}$ .  
(ii)  $|\mu_K| < \infty$ .

*Proof.* We have to show that  $\ker(\lambda) = \mu_K$ . " $\supseteq$ " is clear, either by direct computation or by noticing that  $H$  has no torsion elements. Conversely, let  $\alpha \in \ker \lambda$ . Then  $|\tau(\alpha)| = 1$  for all  $\tau$ , hence also  $|\tau(\alpha^n)| = 1$  for all  $n \geq 1$ . Looking at  $\chi_{\alpha^n}(t) = \prod_\tau (t - \tau(\alpha^n))$ , we see that all coefficients are bounded. Hence there are only finitely many possible characteristic polynomials, each with finitely many zeroes, among the  $\alpha^n$ . So  $\alpha^i = \alpha^j$  for some  $i > j$ , i.e.  $\alpha^{i-j} = 1$ . The same argument also shows (ii).  $\square$

**Lemma 5.3.** Up to multiplication by elements in  $\mathcal{O}_K^\times$ , there are only finitely many  $\alpha \in \mathcal{O}_K$  with  $|\text{N}_{K/\mathbb{Q}}(\alpha)| = a$ , where  $a \in \mathbb{N}$  is given.

*Proof.* We have  $|\text{N}_{K/\mathbb{Q}}(\alpha)| = \text{N}(\alpha \mathcal{O}_K)$  by lemma 4.12, but in the proof of theorem 4.11, we already showed that there are only finitely many ideals of bounded norm.  $\square$

**Theorem 5.4.**  $\Gamma$  is a full lattice in  $H$ , i.e.  $\Gamma \cong \mathbb{Z}^{r+s-1}$ .

*Proof.* To show that  $\Gamma$  is a lattice, we may show by theorem 3.6 that it is discrete. By lemma 3.4, it suffices to show that for all  $c \in \mathbb{R}_{>0}$  one has  $B_c \cap \Gamma$  finite, where  $B_c := \{(x_\tau) \in \mathbb{R}^{r+s} \mid |x_\tau| < c\}$ . But by definition of the map  $l$ , this is the same as requiring  $e^{-c} < x_\rho < e^c$  for real embeddings and  $e^{-\frac{1}{2}c} < x_\sigma < e^{\frac{1}{2}c}$  for complex embeddings, which is finite again by lemma 3.4, since  $j(\mathcal{O}_K) \supseteq j(\mathcal{O}_K^\times)$  is a lattice by theorem 4.7.

We now have to show that  $\Gamma$  has full rank. We want to apply lemma 3.7, i.e. construct a bounded set  $M \subseteq H$  s.t.  $\bigcup_{\gamma \in \Gamma} \gamma + H = H$ . For this, we construct a bounded set  $T \subseteq S$  s.t.  $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} T j(\varepsilon)$ . Then by surjectivity of  $\lambda$  the translates of  $M := l(T)$  clearly cover  $H$ . Let  $x \in T$ . Then  $|x_\tau|$  is bounded from above since  $T$  is bounded. But then  $|x_\tau|$  is also bounded from below (away from 0) because  $\prod_\tau |x_\tau| = 1$ . Therefore,  $\log |x_\tau|$ , and hence  $M$ , are bounded, and the theorem follows.

To construct  $T$ , choose  $c_\tau > 0$  with  $c_{\bar{\tau}} = c_\tau$  and  $C := \prod_\tau c_\tau > (\frac{2}{\pi})^s \sqrt{|d_K|}$ . Consider  $X = \{(z_\tau)_\tau \in K_\mathbb{R} \mid |z_\tau| < c_\tau\}$ . For  $y \in S$  one has  $Xy = \{z \in K_\mathbb{R} \mid |z_\tau| < c'_\tau\}$  with  $c'_\tau = c_\tau |y_\tau|$ ,  $c'_{\bar{\tau}} = c'_\tau$  and  $\prod_\tau c'_\tau = C \prod_\tau |y_\tau| = C$ . Hence by lemma 4.10 there exists  $0 \neq a \in \mathcal{O}_K$  with  $j(a) \in Xy$ .

By lemma 5.3 there exist  $\alpha_1, \dots, \alpha_N \in \mathcal{O}_K$  such that each  $a \in \mathcal{O}_K$  with  $0 < |\text{N}_{K/\mathbb{Q}}(a)| < C$  is associated to one of the  $\alpha_i$ . Now set

$$T := S \cap \bigcup_{i=1}^N X j(\alpha_i)^{-1}.$$

We claim this  $T$  has the required properties. It is clear that  $T$  is bounded, since  $X$  is. So let  $y \in S$ . By the previous paragraph, there exists  $0 \neq a \in \mathcal{O}_K$  with  $j(a) \in Xy^{-1}$ , i.e.  $j(a) = xy^{-1}$  for some  $x \in X$ . Since  $|\text{N}_{K/\mathbb{Q}}(a)| = |\text{N}(xy^{-1})| = |\text{N}(x)| < C$ , there exists  $\alpha_i$  such that  $\alpha_i = \varepsilon a$ ,  $\varepsilon \in \mathcal{O}_K^\times$ . Then  $y = xj(a)^{-1} = xj(\alpha_i\varepsilon)^{-1} = xj(\alpha_i)^{-1}j(\varepsilon)^{-1}$ , hence  $xj(\alpha_i)^{-1} \in S \cap X j(\alpha_i)^{-1} \subseteq T$ . Therefore  $y \in T j(\varepsilon)^{-1}$ , which finishes the proof.  $\square$

Combining theorems 5.2 and 5.4, let  $s : \Gamma \rightarrow \mathcal{O}_K^\times$  be a splitting of  $1 \rightarrow \mu_K \rightarrow \mathcal{O}_K^\times \rightarrow \Gamma \rightarrow 0$ , which exists since  $\Gamma$  is free. Then  $\mu_K \times \Gamma \cong \mathcal{O}_K^\times$ ,  $(\varepsilon, \gamma) \mapsto \varepsilon \cdot s(\gamma)$  is an isomorphism. That is, we have proven

**Theorem 5.5.**  $\mathcal{O}_K^\times$  is a finitely generated group of rank  $t := r + s - 1$ . Explicitly, there exist so-called fundamental units  $\varepsilon_1, \dots, \varepsilon_t \in \mathcal{O}_K^\times$  such that each  $\varepsilon \in \mathcal{O}_K^\times$  has a unique representation of the form

$$\varepsilon = \zeta \varepsilon_1^{k_1} \cdots \varepsilon_t^{k_t}$$

with  $k_i \in \mathbb{Z}$  and  $\zeta \in \mu_K$ .

**Example 5.6.** Let  $K = \mathbb{Q}(\sqrt{d})$ ,  $d > 1$  squarefree. Then  $t = 1$  and  $\mu_K = \{\pm 1\}$ , hence there is a single fundamental unit  $\varepsilon \in \mathcal{O}_K^\times$  such that  $\mathcal{O}_K^\times = \{\pm \varepsilon^n \mid n \in \mathbb{Z}\}$ .

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## 5.2 The Regulator

Let  $K$  be a number field and  $\varepsilon_1, \dots, \varepsilon_t$  be fundamental units. Let  $\lambda_0 := \frac{1}{\sqrt{r+s}}(1, \dots, 1)^t \in \mathbb{R}^{r+s}$  so that  $\|\lambda_0\| = 1$  and  $\lambda_0 \perp H$ . Hence the  $t$ -dimensional volume of  $\Gamma$  is equal to the  $(r+s)$ -dimensional volume of the  $\mathbb{Z}$ -span of  $\lambda_0, \lambda(\varepsilon_1), \dots, \lambda(\varepsilon_t)$ , i.e. of the matrix

$$M = \begin{pmatrix} \frac{1}{\sqrt{r+s}} & \log |\tau_1(\varepsilon_1)| & \cdots & \log |\tau_1(\varepsilon_t)| \\ \vdots & \vdots & & \vdots \\ \frac{1}{\sqrt{r+s}} & \log |\tau_{r+s}(\varepsilon_1)| & \cdots & \log |\tau_{r+s}(\varepsilon_t)| \end{pmatrix}$$

Let  $\Phi$  be a fundamental domain of  $\Gamma = \lambda(\mathcal{O}_K^\times)$ . Then  $\text{Vol}(\Phi) = |\det(M)|$ .

**Theorem 5.7.**  $\text{Vol}(\Gamma) = \sqrt{r+s}R$ , where  $R$  is an arbitrary  $t \times t$ -minor of  $(\lambda(\varepsilon_1), \dots, \lambda(\varepsilon_t))$ .

**Definition 5.8.**  $R_K := R$  as in theorem 5.7 is called the *regulator* of  $K$  (Exercise: It is independent of the choice of fundamental units).

*Proof.* Fix some  $i$  and add all rows to the  $i$ -th row. Then this row becomes  $(\sqrt{r+s}, 0, \dots, 0)$ .  $\square$

**Lemma 5.9.** Let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space. Let  $\Gamma \subseteq V$  be a full lattice, and  $\Gamma' \subseteq V$  be a sublattice. Then  $\Gamma'$  is full if and only if  $\text{Vol}(\Gamma') \neq 0$ , and in this case  $[\Gamma : \Gamma'] = \text{Vol}(\Gamma')/\text{Vol}(\Gamma)$ .

*Proof.* Let  $\omega_1, \dots, \omega_n$  be a  $\mathbb{Z}$ -basis of  $\Gamma$ , and  $\omega'_1, \dots, \omega'_n$  a  $\mathbb{Z}$ -basis of  $\Gamma'$ . Let  $\Phi, \Phi'$  be the corresponding fundamental domains, and let  $\omega'_i = \sum_j t_{ji} \omega_j$ , with  $t_{ji} \in \mathbb{Z}$ . Then  $T = (t_{ji}) \in \text{GL}_n(\mathbb{Q})$  and

$$\text{Vol}(\Gamma') = \text{Vol}(\Phi') = \int_{\Phi'} dx = \int_{\Phi} |\det(T)| dx = |\det(T)| \text{Vol}(\Gamma) = [\Gamma : \Gamma'] \text{Vol}(\Gamma).$$

The other direction is clear.  $\square$

**Theorem 5.10.** Let  $\eta_1, \dots, \eta_t \in \mathcal{O}_K^\times$ . Then the  $\eta_i$  are independent (i.e.  $[\mathcal{O}_K^\times : \langle \eta_1, \dots, \eta_t \rangle] < \infty$ ) if and only if  $R(\eta_1, \dots, \eta_t) \neq 0$ , where  $R(\eta_1, \dots, \eta_t)$  is defined as a  $t \times t$ -minor of the matrix  $(\lambda(\eta_1), \dots, \lambda(\eta_t))$  as before. Further,  $[\mathcal{O}_K^\times / \mu_K : \langle \eta_1, \dots, \eta_t \rangle \mu_K / \mu_K] = R(\eta_1, \dots, \eta_t)/R_K$ .

*Proof.* Exercise.  $\square$

**Remark 5.11.** Regulators are in general transcendental numbers.

Let  $\zeta_K(s) := \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - \frac{1}{N(\mathfrak{p})^s})^{-1}$  for  $\text{Re}(s) > 1$  be the Dedekind  $L$ -function of  $K$ . In the special case  $K = \mathbb{Q}$ , this is the usual Riemann zeta function. As in this special case,  $\zeta_K$  can be analytically extended to a meromorphic function on all of  $\mathbb{C}$ , with a simple pole only at  $s = 1$ . Further,  $\zeta_K$  satisfies a functional equation of the form

$$\zeta_K(1-z) = 2|d_K|^{s-1/2} \cos(\frac{\pi z}{2})^{r+s} \sin(\frac{\pi z}{2})^s (2\pi)^{-z} \Gamma(z) \zeta_K(z).$$

$\zeta_K(s)$  has a zero at  $s = 0$  of order  $t$  and the leading term is  $\pm h_K R_K$ . This is the so-called *analytic class number formula*, which is proved in analytic number theory.

- Example 5.12.**
- (i) For imaginary quadratic fields, one has  $t = 0$ , so  $\mathcal{O}_K^\times$  is finite.
  - (ii) For real quadratic fields, one has  $t = 1$ , and we will see how to compute a fundamental unit.
  - (iii) Let  $K = \mathbb{Q}(\sqrt[3]{m})$  for  $m$  cubefree. Then  $t = 1 + 1 - 1 = 1$ .
  - (iv) Let  $K = \mathbb{Q}(\zeta_m)$  be a cyclotomic field. Then  $K/\mathbb{Q}$  is a Galois extension with Galois group  $G = (\mathbb{Z}/m\mathbb{Z})^\times$ . We have  $\varphi(m)$  complex embeddings  $\sigma_a : \zeta_m \mapsto \zeta_m^a$ , so  $t_K = \frac{1}{2}\varphi(m) - 1$ . Let  $K^+ := \mathbb{Q}(\zeta_m + \zeta_m^{-1}) = K \cap \mathbb{R}$  be the fixed field of  $\sigma_{-1}$ . This is the largest totally real subfield of  $K$  of degree  $\frac{1}{2}\varphi(m)$ , so  $t_{K^+} = \frac{1}{2}\varphi(K) - 1 = t_K =: t$ . Let  $\mathcal{O}_{K^+}^\times = \pm \varepsilon_1^{\mathbb{Z}} \cdots \varepsilon_t^{\mathbb{Z}}$ , then  $[\mathcal{O}_K^\times : \mathcal{O}_{K^+}] < \infty$ . Actually the index is small, cf. exercises.

For  $m = p$ ,  $p$  an odd prime, the element  $\frac{\zeta_p^a - 1}{\zeta_p^a + 1}$ ,  $(a, p) = 1$  is a unit, since if  $ab = 1 \pmod{p}$ , then

$$\frac{\zeta_p - 1}{\zeta_p^a - 1} = \frac{\zeta_p^{ab} - 1}{\zeta_p^a - 1} = \sum_{i=0}^{b-1} \zeta_p^{ai} \in \mathbb{Z}[\zeta_p] \subseteq \mathcal{O}_K.$$

Hence we have  $p - 2$  nontrivial units (for  $a = 2, \dots, p - 1$ ), called *cyclotomic units*. One can show that the index of the subgroup generated by them depends explicitly on  $h_K$ . (The above can be generalized to  $\mathbb{Q}(\zeta_m)$  for  $m$  not a prime.) See Washington, Cyclotomic Units for details.

### 5.3 The fundamental unit in a real quadratic field

Let  $d > 1$  be squarefree, and  $K = \mathbb{Q}(\sqrt{d})$ . Then  $\mathcal{O}_K \ni \alpha = \frac{1}{2}(x + y\sqrt{d})$ ,  $x, y \in \mathbb{Z}$ , is a unit if and only if  $N_{K/\mathbb{Q}}(\alpha) = \pm 1$ .

**Corollary 5.13.** *The units of  $\mathcal{O}_K$  are in 1-1 correspondence with the solutions  $(x, y) \in \mathbb{Z}^2$  of the (generalized) Pell's equation  $x^2 - dy^2 = \pm 4$ .*

Let  $\pm 1 \neq \eta \in \mathcal{O}_K^\times$ . Then  $\eta, \eta^{-1}, -\eta, -\eta^{-1}$  are four different units. Let  $\tau$  be the nontrivial Galois automorphism of  $K$ , then  $\tau(\eta) = \pm\eta^{-1}$ . Then if  $\eta = x + y\sqrt{d}$ , then

$$\{\eta, \eta^{-1}, -\eta, -\eta^{-1}\} = \{\eta, \tau(\eta), -\eta, -\tau(\eta)\} = \{\pm x \pm y\sqrt{d}\},$$

so there is exactly one fundamental unit  $\varepsilon > 0$  with  $\varepsilon > 1$ . Such a unit  $\frac{1}{2}(a + b\sqrt{d})$  with  $a, b > 0$  is called *normalized*.

**Theorem 5.14.**  $\eta = a + b\sqrt{d}$  with  $a, b \in \frac{1}{2}\mathbb{Z}$ ,  $a, b > 0$  is the normalized fundamental unit if and only if for any unit  $\varepsilon = c + d\sqrt{d} > 1$  we have  $a < c$ .

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*Proof.* Let  $u = p + q\sqrt{d}$  be the normalized fundamental unit. Let  $\varepsilon = c + d\sqrt{d} > 1$  be a unit. Then  $\varepsilon = u^m =: p_m + q_m\sqrt{d}$  for some  $m > 0$ . Hence it suffices to show that  $(p_m)_m$  is strictly increasing. Note that  $p_{m+1} = pp_m + dq_m$ ,  $q_{m+1} = pq_m + qp_m$ , so if  $p \geq 1$  (in particular if  $d \not\equiv 1 \pmod{4}$ ) we immediately see  $p_{m+1} > p_m$ . Otherwise, assume that  $p = \frac{1}{2}$ . Then  $\frac{1}{4} - q^2d = \pm 1$  by Pell's equation, which immediately implies  $q = \frac{1}{2}$ ,  $d = 5$ . Here we have  $u = \frac{1+\sqrt{5}}{2}$ .  $\square$

**Example 5.15.** Using the above theorem, we can algorithmically calculate the normalized fundamental unit by finding the solution of  $x^2 - dy^2 = \pm 4$  with smallest  $x > 0$ . We find

$$\begin{array}{c|cc|cc|cc} d & 2 & 3 & 5 & 13 & 46 \\ \hline \varepsilon & 1 + \sqrt{2} & 2 + \sqrt{3} & \frac{1+\sqrt{5}}{2} & \frac{3+\sqrt{13}}{2} & 24335 + 3588\sqrt{46} \end{array}$$

This last example shows that fundamental units can be very large compared to  $d$ , so our naive algorithm can be very inefficient. There are better algorithms, for example using continued fractions.

## 6 Extensions of Dedekind Domains

Let  $L/K$  be an extension of number fields. Let  $\mathfrak{p} \subseteq \mathcal{O}_K$  be a prime ideal. We want to understand how the ideal  $\mathfrak{p}\mathcal{O}_L$  of  $\mathcal{O}_L$  factors.

Note first that  $\mathfrak{p}\mathcal{O}_L \subsetneq \mathcal{O}_L$ . Indeed, let  $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ . Then  $\pi\mathcal{O}_K = \mathfrak{p}\mathfrak{a}$  with  $\mathfrak{a} + \mathfrak{p} = \mathcal{O}_K$ . Write  $1 = b + s$  with  $s \in \mathfrak{a}, b \in \mathfrak{p}$ . Then  $s\mathfrak{p} \subseteq \mathfrak{a}\mathfrak{p} = \pi\mathcal{O}_K$ . Suppose now  $\mathfrak{p}\mathcal{O}_L = \mathcal{O}_L$ . Then  $s\mathcal{O}_L = s\mathfrak{p}\mathcal{O}_L \subseteq \pi\mathcal{O}_L$ , i.e.  $s = \pi x$  with  $x \in \mathcal{O}_L \cap K = \mathcal{O}_K$ . But then  $s \in \mathfrak{p}$ , contradiction.

So let  $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$  be the nonempty prime factorization of  $\mathfrak{p}\mathcal{O}_L$ . Then the  $\mathfrak{P}_i$  are precisely the primes of  $\mathcal{O}_L$  lying over  $\mathfrak{p}$ , i.e.  $\mathfrak{P}_i \cap \mathcal{O}_L = \mathfrak{p}$ . (Exercise) In this case we also write  $\mathfrak{P} \mid \mathfrak{p}$ , and call  $\mathfrak{P}$  a prime divisor of  $\mathfrak{p}$ .

The exponents  $e_i$  are called *ramification indices*. Furthermore,  $f_i := [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$  is called the residue field degree or *inertia degree*. Here, the field extension is induced by the natural map  $\mathcal{O}_K \hookrightarrow \mathcal{O}_L \rightarrow \mathcal{O}_L/\mathfrak{P}_i$ .

**Theorem 6.1.** *Let  $L/K$  be separable<sup>2</sup>. Then  $\sum_{i=1}^r e_i f_i = n := [L : K]$ .*

*Proof.* By the Chinese Remainder Theorem, we have

$$\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \cong \mathcal{O}_L/\mathfrak{P}_1^{e_1} \times \cdots \times \mathcal{O}_L/\mathfrak{P}_r^{e_r}.$$

Let  $k := \mathcal{O}_K/\mathfrak{p}$ . It suffices to show  $\dim_k \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L = n$  and  $\dim_k \mathcal{O}_L/\mathfrak{P}_i^{e_i} = e_i f_i$ .

Let  $\omega_1, \dots, \omega_m \in \mathcal{O}_L$  such that  $\bar{\omega}_1, \dots, \bar{\omega}_m \in \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$  are a  $k$ -basis. We will show that  $\omega_1, \dots, \omega_m$  are a  $K$ -basis of  $L$  (hence  $m = n$ ). Let  $a_1\omega_1 + \dots + a_m\omega_m = 0$  with  $a_i \in \mathcal{O}_K$  not all 0. Then  $\mathfrak{a} = (a_1, \dots, a_m) \neq 0$ . Choose  $a \in \mathfrak{a}^{-1} \setminus \mathfrak{a}^{-1}\mathfrak{p}$ . Then  $a\mathfrak{a} \not\subseteq \mathfrak{p}$ , hence  $aa_1, \dots, aa_m \in \mathcal{O}_K$  are not all contained in  $\mathfrak{p}$ . But then  $a(a_1\omega_1 + \dots + a_m\omega_m) \equiv 0 \pmod{\mathfrak{p}\mathcal{O}_L}$  contradicts the independence of the  $\bar{\omega}_i$ . Hence  $\omega_1, \dots, \omega_m$  are linearly independent.

Consider  $M = \mathcal{O}_K\omega_1 + \dots + \mathcal{O}_K\omega_n \subseteq \mathcal{O}_L$  and  $N = \mathcal{O}_L/M$ . Then  $\mathfrak{p}N = (\mathfrak{p}\mathcal{O}_L + M)/M = N$ . Let  $\alpha_1, \dots, \alpha_s \in N$  be a set of generators of  $N$  over  $\mathcal{O}_K$ . Then we find relations  $\alpha_i = \sum_{j=1}^s a_{ij}\alpha_j$  with  $a_{ij} \in \mathfrak{p}$ . Let  $A = (a_{ij}) - E_s$ . Then  $A(\alpha_1, \dots, \alpha_s)^t = 0$ , so by 1.4 we find  $\det(A)N = 0$  with  $\det(A) \equiv \det(-E_s) = \pm 1 \pmod{p}$ , in particular  $\det(A) \neq 0$ . Hence  $\det(A)\mathcal{O}_L \subseteq M = \mathcal{O}_K\omega_1 + \dots + \mathcal{O}_K\omega_n$ , so  $L = K\omega_1 + \dots + K\omega_n$ .

Thus  $\dim_k \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L = n$ . For  $\dim_k \mathcal{O}_L/\mathfrak{P}_i^{e_i}$ , look at the filtration  $\mathcal{O}_L \supseteq \mathfrak{P} \supseteq \mathfrak{P}^2 \supseteq \dots \supseteq \mathfrak{P}^e$ . By induction, it suffices to prove  $\dim_k (\mathfrak{P}^i/\mathfrak{P}^{i+1}) = f_i$ , which was done in the exercises.  $\square$

Note that for  $L/\mathbb{Q}$  we can give a much simpler proof: By definition of the  $f_i$  we have

$$p^n = N_{L/\mathbb{Q}}(p) = N(p\mathcal{O}_L) = \prod_i N(\mathfrak{P}_i)^{e_i} = \prod_i p^{e_i f_i} = p^{\sum_i e_i f_i}$$

Assume  $L/K$  separable and  $L = K(\theta)$  with  $\theta \in \mathcal{O}_L$ . Let  $f \in \mathcal{O}_K[X]$  be the minimal polynomial of  $\theta$ . Set  $\mathcal{O} := \mathcal{O}_K[\theta] \subseteq \mathcal{O}_L$ .

**Definition 6.2.**  $\mathfrak{f} := \{\alpha \in \mathcal{O}_L \mid \alpha\mathcal{O}_L \subseteq \mathcal{O}\}$  is called the *conductor* of  $\mathcal{O}$  in  $\mathcal{O}_L$ .

Note that  $\mathfrak{f}$  is the largest  $\mathcal{O}_L$ -ideal which is contained in  $\mathcal{O}$ . In particular,  $\mathcal{O} = \mathcal{O}_L$  if and only if  $\mathfrak{f} = 1$ .

**Lemma 6.3.** *Let  $\mathfrak{p} \subseteq \mathcal{O}$  be a prime ideal. Then  $\mathfrak{p}$  is not invertible if and only if  $\mathfrak{f} \subseteq \mathfrak{p}$*

*Proof.* Exercise.  $\square$

<sup>2</sup>That is, more generally, we may consider a finite separable field extension  $L/K$  with Dedekind domains  $R \subseteq K$ ,  $\mathcal{O} \subseteq L$  such that  $\mathcal{O} = \mathcal{O}_{R,L}$  and  $\text{Quot}(R) = K$

**Theorem 6.4.** Let  $\mathfrak{p} \subseteq \mathcal{O}_K$  be a prime ideal with  $\mathfrak{p}\mathcal{O}_L + \mathfrak{f} = \mathcal{O}_L$ . Let

$$\bar{f}(x) = \bar{f}_1(x)^{e_1} \cdots \bar{f}_r(x)^{e_r} \quad \text{in } k[x] := \mathcal{O}_K/\mathfrak{p}[x]$$

with pairwise distinct irreducible polynomials  $\bar{f}_i \in k[x]$ . Then the  $\mathfrak{P}_i := \mathfrak{p}\mathcal{O}_L + f_i(\theta)\mathcal{O}_L$ ,  $i = 1, \dots, r$  are exactly the primes of  $\mathcal{O}_L$  over  $\mathfrak{p}$ . We have  $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$  and  $f_i = \deg(\bar{f}_i)$ .

**Example 6.5.** Let  $L = \mathbb{Q}(\sqrt{d})$  with  $d \equiv 2, 3 \pmod{4}$  squarefree. Then  $f(X) = X^2 - d$  and  $\mathfrak{f} = 1$ . If  $p \mid d$ , then  $\bar{f} = X^2$ , so  $p\mathcal{O}_L = (p, \sqrt{d})^2 = \langle p, \sqrt{d} \rangle_{\mathbb{Z}}$ . If  $p \nmid d$  and  $d$  is a square mod  $p$ , then  $X^2 - d \equiv (X - \bar{a})(X + \bar{a}) \pmod{p}$  for some  $a$ , so  $p\mathcal{O}_L = \mathfrak{P}_1\mathfrak{P}_2$  with  $\mathfrak{P}_{1,2} = (p, a \pm \sqrt{d})$ . Finally, if  $p \nmid d$  and  $d$  is not a square mod  $p$ , then  $\bar{f}$  is irreducible and  $p\mathcal{O}_L = (p)$ . Similar calculations work for  $d \equiv 1 \pmod{4}$  as well, except that  $\mathfrak{f} = 2\mathcal{O}_L$ , so  $p = 2$  has to be treated separately.

*Proof.* Consider the maps

$$k[x]/(\bar{f}(x)) \xrightarrow{\bar{\alpha}} \mathcal{O}/\mathfrak{p}\mathcal{O} \xrightarrow{\bar{\beta}} \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$$

induced by  $\alpha(\bar{g}) := g(\theta) + \mathfrak{p}\mathcal{O}$  and  $\beta(\alpha) = \alpha + \mathfrak{p}\mathcal{O}_L$ .

By universal properties,  $\bar{\alpha}$  and  $\bar{\beta}$  are well-defined. Further  $\alpha$  is surjective since any element  $g(\theta) + \mathfrak{p}\mathcal{O}$  is hit by  $\bar{g}(x)$ . If  $\bar{g} \in \ker \alpha$ , then  $g(\theta) \in \mathfrak{p}\mathcal{O} = \mathfrak{p}[\theta]$ . Hence  $g(\theta) = h(\theta)$  for some  $h \in \mathfrak{p}[\theta]$ , i.e.  $f \mid g - h$  in  $\mathcal{O}_K[x]$ . But then  $\bar{f} \mid \bar{g}$ , so  $\bar{\alpha}$  is an isomorphism.

By assumption,  $\mathfrak{p}\mathcal{O}_L + \mathfrak{f} = \mathcal{O}_L$ , hence also  $\mathfrak{p}\mathcal{O}_L + \mathcal{O} = \mathcal{O}_L$ , so  $\beta$  is surjective. Clearly  $\mathfrak{p}\mathcal{O} \subseteq \ker \beta$ . For the converse, we will show

$$\mathfrak{p} + (\mathfrak{f} \cap \mathcal{O}_K) = \mathcal{O}_K. \tag{*}$$

Then  $\ker \beta = \mathcal{O} \cap \mathfrak{p}\mathcal{O}_L \subseteq (\mathfrak{p} + \mathfrak{f})(\mathcal{O} \cap \mathfrak{p}\mathcal{O}_L) \subseteq \mathfrak{p}\mathcal{O}$ . To prove (\*), suppose  $\mathfrak{f} \cap \mathcal{O}_K \subseteq \mathfrak{p}$ . Let  $\mathfrak{f} = \mathfrak{q}_1^{s_1} \cdots \mathfrak{q}_m^{s_m}$ . Then  $\mathfrak{f} \cap \mathcal{O}_K = (\mathfrak{q}_1^{s_1} \cap \mathcal{O}_K) \cap \dots \cap (\mathfrak{q}_m^{s_m} \cap \mathcal{O}_K)$ , so say  $\mathfrak{q}_1^{s_1} \cap \mathcal{O}_K \subseteq \mathfrak{p}$ . Hence  $\mathfrak{p} \mid (\mathfrak{q}_1 \cap \mathcal{O}_K)^{s_1}$ , and since  $\mathfrak{q}_1 \cap \mathcal{O}_K$  is a prime ideal of  $\mathcal{O}_K$ , we have  $\mathfrak{p} = \mathfrak{q}_1 \cap \mathcal{O}_K$ . Hence  $\mathfrak{q}_1$  occurs in the prime decomposition of both  $\mathfrak{f}$  and  $\mathfrak{p}$ , in contradiction to the coprime assumption.

Hence  $\bar{\beta} \circ \bar{\alpha}$  is an isomorphism. In particular, prime ideals of  $k[x]$  above  $\bar{f}(x)$  correspond bijectively to prime ideals of  $\mathcal{O}_L$  above  $\mathfrak{p}\mathcal{O}_L$ . But the former are exactly of the form  $(\bar{f}_i(x))$ , which are mapped to  $\mathfrak{p}\mathcal{O}_L + f_i(\theta)\mathcal{O}_L$ .  $\square$

Warning: There are extensions  $L/K$  such that there exists no  $\theta \in \mathcal{O}_L$  with  $\mathcal{O}_L = \mathcal{O}_K[\theta]$ .

**Definition 6.6.** In the notation of the theorem,  $\mathfrak{p}$  is called *completely or totally split* if  $r = n$ , and *unramified* if  $e_i = 1$  for all  $i$ .

**Theorem 6.7.** There are only finitely many ramified prime ideals.

*Proof.* Let  $L = K(\theta)$ ,  $\theta \in \mathcal{O}_L$ . Let  $d := d(\theta) = \prod_{1 \leq i < j \leq n} (\theta_i - \theta_j)^2 \in \mathcal{O}_K$ . Let  $\mathfrak{f}$  be the conductor of  $\mathcal{O} = \mathcal{O}_K[\theta]$  in  $\mathcal{O}_L$ . Then every prime ideal  $\mathfrak{p} \subseteq \mathcal{O}_L$  prime to  $d\mathfrak{f}$  is unramified.

Indeed, by theorem 6.4, the decomposition of  $\mathfrak{p}\mathcal{O}_L$  corresponds to the decomposition of  $\bar{f} \in k[x]$ . Since  $\bar{d} \neq 0 \in k$ , the polynomial  $\bar{f}$  has no multiple roots. Hence all  $e_i = 1$ .  $\square$

This result can be improved considerably. Without proof, we mention the following

**Theorem 6.8.**  $\mathfrak{p}$  is ramified in  $L/K$  if and only if  $\mathfrak{p} \mid d_{L/K}$ .

Here,  $d_{L/K}$  is defined as the integral ideal in  $\mathcal{O}_K$  such that for every prime ideal  $\mathfrak{p}$ , its image in  $\mathcal{O}_{K,\mathfrak{p}}$  is the discriminant  $d_{L/K,\mathfrak{p}}$  of  $\mathcal{O}_{L,\mathfrak{p}}/\mathcal{O}_{K,\mathfrak{p}}$ , cf. remark 1.29. Such an ideal exists since  $d_{L/K,\mathfrak{p}} = 1$  for almost all  $\mathfrak{p}$ , which follows from the following

**Lemma 6.9.** With the notation as before, if  $\mathfrak{p}\mathcal{O}_L + \mathfrak{f} = \mathcal{O}_L$ , then  $d_{L/K,\mathfrak{p}} = 1$ .

*Proof.* Exercise. □

**Remark 6.10.** If  $K = \mathbb{Q}$ , then  $\mathfrak{f} \mid d(\theta)$ , for  $d(\theta)\mathcal{O}_L \subseteq \mathcal{O}_K[\theta]$  by lemma 1.23.

We prove one direction of theorem 6.8 in the most interesting case  $K = \mathbb{Q}$ :

**Theorem 6.11.** Let  $K = \mathbb{Q}$ . Then  $p$  is ramified only if  $p \mid d_L$ .

*Proof.* Assume  $p\mathcal{O}_L = \mathfrak{p}^e\mathfrak{a}$  with  $\mathfrak{p} + \mathfrak{a} = \mathcal{O}_L$  and  $e > 1$ . Let  $\mathfrak{b} = \mathfrak{p}^{e-1}\mathfrak{a}$ , then all primes above  $p$  occur in  $\mathfrak{b}$ . Let  $\sigma_1, \dots, \sigma_n$  be the embeddings  $L \rightarrow \mathbb{C}$ , and let  $M$  be the normal closure of  $L/\mathbb{Q}$ . Let  $\widehat{\sigma}_1, \dots, \widehat{\sigma}_n$  be extensions of the  $\sigma_i$  to  $L$ . Let  $\mathcal{O}_K = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$ . Take  $\alpha \in \mathfrak{b} \setminus p\mathcal{O}_K$ . Then  $\alpha$  is contained in all primes in  $\mathcal{O}_L$  above  $p$ . Let  $\alpha = m_1\alpha_1 + \dots + m_n\alpha_n$  with  $m_i \in \mathbb{Z}$ , wlog  $p \nmid m_1$ . Then  $d(\alpha, \alpha_2, \dots, \alpha_n) = d(m_1\alpha_1, \alpha_2, \dots, \alpha_n) = m_1^2 d_L$ . Hence it suffices to show  $p \mid d(\alpha, \alpha_2, \dots, \alpha_n) = \det(\sigma_i(\alpha | \alpha_j))_{ij}$ . Let  $\mathfrak{P}$  be a prime of  $\mathcal{O}_L$  lying above  $\mathfrak{p}$ . Then  $\widehat{\sigma}_i^{-1}(\mathfrak{P})$  is lying above  $p$ , so  $\alpha \in \widehat{\sigma}_i^{-1}(\mathfrak{P})$  and  $\sigma_i(\alpha) = \widehat{\sigma}_i(\alpha) \in \mathfrak{P}$ . Hence the first column of  $(\sigma_i(\alpha | \alpha_j))_{ij}$  is contained in  $\mathfrak{P}$ , so  $d(\alpha, \alpha_2, \dots, \alpha_n) \in \mathfrak{P} \cap \mathbb{Z} = (p)$ . □

## 7 Hilbert's Ramification Theory

We now assume that the extension  $L/K$  is Galois with Galois group  $G$ . Then  $G$  acts on all of  $L, \mathcal{O}_L, I_L, \mathcal{O}_L^\times, \text{cl}_L$ .

By a theorem of algebra, there exists a *normal basis element*  $\alpha \in L$ , s.t.  $\{\sigma(\alpha) \mid \sigma \in G\}$  is a  $K$ -basis of  $L$ . In other words: Consider the (commutative iff  $G$  abelian) group ring  $K[G] := \{\sum_{\sigma \in G} a_\sigma \sigma \mid a_\sigma \in K\}$  (with the obvious addition and multiplication). Then  $L$  becomes a  $K[G]$ -module via the natural action  $\sum_\sigma a_\sigma \sigma \cdot \beta = \sum_\sigma a_\sigma \sigma(\beta)$ , and  $L \cong K[G]$  as  $K[G]$ -modules via  $K[G] \ni \lambda \mapsto \lambda(\alpha) \in L$ .

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In the same way,  $\mathcal{O}_K[G]$  acts on  $\mathcal{O}_L$ . Hence one may ask the same question: Is  $\mathcal{O}_L \cong \mathcal{O}_K[G]$  as  $\mathcal{O}_K[G]$ -modules? The answer is negative, in general there is no integral normal basis.

**Example 7.1.** Let  $L/K$  be *tame*. Then  $\mathcal{O}_L$  is  $\mathcal{O}_K[G]$ -projective, hence  $\mathcal{O}_L$  defines a class in  $K_0(\mathcal{O}_K[G])$ .<sup>3</sup>

$\mathcal{O}_L^\times$  is a  $\mathbb{Z}[G]$ -module via  $\sum_\sigma a_\sigma \sigma \cdot u := \prod_\sigma \sigma(u)^{a_\sigma}$ . Almost nothing is known about the  $\mathbb{Z}[G]$ -module structure of  $\mathcal{O}_L^\times$ .

Now we look at the action of  $G$  on  $I_L$ . If  $\mathfrak{a} \in I_L$ , then  $\sigma(\mathfrak{a}) \in I_L$  for  $\sigma \in G$ . If  $\mathfrak{P} \mid \mathfrak{p}$  and  $\sigma \in G$ , then  $\sigma(\mathfrak{P}) \mid \mathfrak{p}$ .

**Theorem 7.2.**  $G$  acts transitively on the set of prime ideals above a prime ideal  $\mathfrak{p} \subseteq \mathcal{O}_K$ .

*Proof.* Let  $\mathfrak{P}, \mathfrak{P}' \mid \mathfrak{p}\mathcal{O}_L$ . Assume  $\mathfrak{P}' \neq \sigma\mathfrak{P}$  for all  $\sigma \in G$ . By the Chinese Remainder Theorem, there exists  $x \in \mathcal{O}_L$  with  $x \equiv 0 \pmod{\mathfrak{P}'}$  and  $x \equiv 1 \pmod{\sigma\mathfrak{P}}$  for all  $\sigma \in G$ . Then  $N_{L/K}(x) = \prod_\sigma \sigma(x) \in \mathfrak{P}' \cap \mathcal{O}_K = \mathfrak{p}$ . On the other hand,  $x \notin \sigma(\mathfrak{P})$  for all  $\sigma \in G$ . Hence  $N_{L/K}(x) \notin \mathfrak{P} \cap \mathcal{O}_K = \mathfrak{p}$ . □

**Definition 7.3.** For a prime ideal  $\mathfrak{P} \subseteq \mathcal{O}_L$ , the subgroup  $G_{\mathfrak{P}} := \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\} \subseteq G$  is called the *decomposition group* of  $\mathfrak{P}$ . Let  $Z_{\mathfrak{P}} := L^{G_{\mathfrak{P}}}$  be the fixed field of  $G_{\mathfrak{P}}$ .

Then  $\sigma \mapsto \sigma(\mathfrak{P})$  induces a bijection from  $G/G_{\mathfrak{P}}$  to the set of primes above  $\mathfrak{p}$  by the orbit-stabilizer theorem.

<sup>3</sup>For more in this direction, look up Fröhlich's conjecture, proven by M. Taylor, 1985.

**Lemma 7.4.** (i) There are  $|G/G_{\mathfrak{P}}|$  many primes of  $\mathcal{O}_L$  above  $\mathfrak{p}$ .

(ii)  $G_{\mathfrak{P}} = 1 \iff Z_{\mathfrak{P}} = L \iff \mathfrak{p}$  is completely split.

(iii)  $G_{\mathfrak{P}} = G \iff Z_{\mathfrak{P}} = K \iff \mathfrak{p}$  is fully inert, i.e. there is exactly one  $\mathfrak{P}$  above  $\mathfrak{p}$ .

(iv)  $G_{\sigma(\mathfrak{P})} = \sigma G_{\mathfrak{P}} \sigma^{-1}$ .

*Proof.* (i)-(iii) are clear. For (iv), we have  $\tau \in G_{\sigma(\mathfrak{P})}$  if and only if  $\tau\sigma(\mathfrak{P}) = \sigma(\mathfrak{P})$ , i.e.  $\sigma^{-1}\tau\sigma \in G_{\mathfrak{P}}$ .  $\square$

Let  $\mathfrak{p} \subseteq \mathcal{O}_K$  be prime. Recall that for its factorization in  $\mathcal{O}_L$ , we had the formula  $[L : K] = n = \sum_{i=1}^r e_i f_i$ .

**Proposition 7.5.** In the above formula, one has  $f := f_1 = \dots = f_r$  and  $e := e_1 = \dots = e_r$ , hence  $n = ref$  and  $\mathfrak{p} = \prod_{\sigma \in G/G_{\mathfrak{P}}} \sigma(\mathfrak{P})^e$ .

*Proof.* Let  $\mathfrak{P}, \mathfrak{P}'$  be above  $\mathfrak{p}$ . Let  $\mathfrak{P}' = \sigma(\mathfrak{P})$ ,  $\sigma \in G$ . Then  $\sigma$  induces an isomorphism  $\mathcal{O}_L/\mathfrak{P} \rightarrow \mathcal{O}_L/\mathfrak{P}'$  of  $\mathcal{O}_K/\mathfrak{p}$ -extensions, thus  $f_{\mathfrak{P}} = f_{\mathfrak{P}'}$ .

Since  $\sigma(\mathfrak{p}\mathcal{O}_L) = \mathfrak{p}\mathcal{O}_L$ , we see  $\mathfrak{P}'^\nu \mid \mathfrak{p}\mathcal{O}_L$  if and only if  $(\mathfrak{P}')^\nu \mid \mathfrak{p}\mathcal{O}_L$ . Since the ramification index is characterized as the highest power satisfying this divisibility, all these indices are equal.  $\square$

**Theorem 7.6.** Let  $\mathfrak{P} \mid \mathfrak{p}$ . Let  $\mathfrak{P}_Z := \mathfrak{P} \cap Z_{\mathfrak{P}}$ . Then

- (i) There is exactly one prime of  $L$  above  $\mathfrak{P}_Z$ , namely  $\mathfrak{P}$ .
- (ii)  $\mathfrak{P}$  has ramification degree  $e$  and inertia degree  $f$  in  $L/Z_{\mathfrak{P}}$ , i.e.  $\mathfrak{P}_Z\mathcal{O}_L = \mathfrak{P}^e$  and  $[\mathcal{O}_L/\mathfrak{P} : \mathcal{O}_{Z_{\mathfrak{P}}}/\mathfrak{P}_Z] = f$ .
- (iii) The ramification index and inertia degree of  $\mathfrak{P}_Z/\mathfrak{p}$  are 1.

In addition, if  $G_{\mathfrak{P}}$  is a normal subgroup, then  $\mathfrak{p}$  is completely split in  $Z_{\mathfrak{P}}$ .

*Proof.* (i)  $L/Z_{\mathfrak{P}}$  is Galois with  $\text{Gal}(L/Z_{\mathfrak{P}}) = G_{\mathfrak{P}}$ . This group acts transitively on the primes of  $L$  above  $\mathfrak{P}_Z$  by lemma 7.2, yet fixes  $\mathfrak{P}$ .

(ii) and (iii) Let  $e', f'$  be the ramification index and inertia degree of  $\mathfrak{P}_Z/\mathfrak{p}$ , and  $e'', f''$  be the corresponding numbers for  $\mathfrak{P}/\mathfrak{P}_Z$ . We know  $e = e'e'', f = f'f''$  (cf. Exercises),  $|G| = n = ref$  and  $r = |G/G_{\mathfrak{P}}| = [Z_{\mathfrak{P}} : K]$ . Hence  $[L : Z_{\mathfrak{P}}] = ef$ . On the other hand,  $[L : Z_{\mathfrak{P}}] = 1e''f''$ . Since  $e'' \leq e$  and  $f'' \leq f$ , we get  $e'' = e$ ,  $f'' = f$ , and therefore  $e' = 1 = f'$ .  $\square$

We want to characterize  $e$  group-theoretically. Since we already have  $ef = |G_{\mathfrak{P}}|$ , this automatically yields a group-theoretic characterization of  $f$  as well.

Let  $\sigma \in G_{\mathfrak{P}}$ . Then  $\sigma$  induces an isomorphism  $\bar{\sigma} : \mathcal{O}_L/\mathfrak{P} \rightarrow \mathcal{O}_L/\mathfrak{P}$ ,  $\alpha + \mathfrak{P} \mapsto \sigma(\alpha) + \mathfrak{P}$  with  $\bar{\sigma}|_{\mathcal{O}_K/\mathfrak{p}} = \text{id}$ .

**Definition 7.7.** For a prime ideal  $\mathfrak{p}$  of a number field  $K$ , write  $\kappa(\mathfrak{p}) := \mathcal{O}_K/\mathfrak{p}$  for the residue field.

**Theorem 7.8.** The map  $G_{\mathfrak{P}} \rightarrow \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ ,  $\sigma \mapsto \bar{\sigma}$  is surjective.

Recall that  $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$  is an extension of finite fields. Since there is only one such field of each degree, we know that any such extension is cyclic, with Galois group generated by the Frobenius  $\varphi(\alpha) = \alpha^{|\kappa(\mathfrak{p})|}$ .

**Definition 7.9.**  $I_{\mathfrak{P}} := \ker(\sigma \mapsto \bar{\sigma})$  is called the *inertia group* or *ramification group*.

By the above theorem, one has  $G_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ . Therefore,  $|I_{\mathfrak{P}}| = e$ .

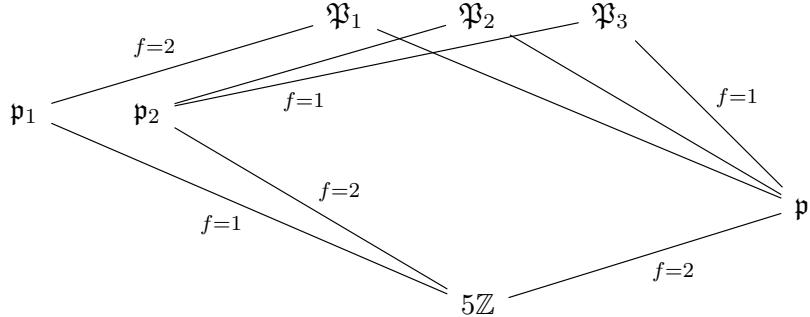
*Proof.* Since  $\mathcal{O}_{Z_{\mathfrak{P}}}/\mathfrak{P}_Z = \mathcal{O}_K/\mathfrak{p}$  by theorem 7.6, we may wlog assume  $Z_{\mathfrak{P}} = K$ . Hence  $G = G_{\mathfrak{P}}$ . Let  $\bar{\theta}$  be a primitive element for  $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ . Let  $f \in K[X]$  be the minimal polynomial of  $\theta \in L$ , and  $\bar{g} \in \kappa(\mathfrak{p})[X]$  be the minimal polynomial of  $\bar{\theta}$ . Then  $\bar{f}(\bar{\theta}) = 0$ , so  $\bar{g} \mid \bar{f}$  in  $\kappa(\mathfrak{p})[X]$ . Let  $f(X) = \prod_i (X - \theta_i)$  be the factorization of  $f$  in  $L[X]$ . Then  $\bar{f} = \prod_i (X - \bar{\theta}_i)$ , with  $\bar{\theta}_i \in \kappa(\mathfrak{P})$ .

Let  $\bar{\sigma} \in \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ . Then  $\bar{\sigma}(\bar{\theta}) = \bar{\theta}_i$  for some  $i$ . There is  $\sigma'_1 \in G(K(\theta)/K, K^c/K)$  with  $\sigma'_1(\theta) = \theta_i$ . Let  $\sigma_1 \in G$  be an extension of  $\sigma'_1$ . Then  $\bar{\sigma}_1 = \bar{\sigma}$ , since they agree on the generator  $\bar{\theta}$ .  $\square$

**Example 7.10.** Consider the number fields

$$\begin{array}{ccccc} & & L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3) & & \\ & \swarrow 2 & & \searrow 3 & \\ K = \mathbb{Q}(\sqrt[3]{2}) & & & & \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \\ & \searrow 3 & & \swarrow 2 & \\ & \langle \sigma \rangle & & \langle \tau \rangle & \end{array}$$

By theorem 6.11 and a short calculation, one sees that 5 is unramified in  $L/\mathbb{Q}$ . From 6.4, we see  $5\mathbb{Z} = \mathfrak{p}_1\mathfrak{p}_2$  in  $\mathcal{O}_K$ , with  $f_1 = 1, f_2 = 2$ . Then  $\mathfrak{p}_1\mathcal{O}_L = \mathfrak{P}_1$ , because  $X^2 + X + 1$  is irreducible in  $\mathcal{O}_K/\mathfrak{p}_1[X] \cong \mathbb{Z}/5\mathbb{Z}[X]$ . Hence  $G_{\mathfrak{P}_1|5} = \langle \tau \rangle$ . Now we see by 7.5 that there are two prime ideals above  $\mathfrak{p}_2$ , both with inertia degree 1. By example 6.5,  $5\mathbb{Z}$  is inert in  $\mathbb{Q}(\sqrt{-3})$ , so we have the following diagram of primes above 5:



Note that by definition  $I_{\mathfrak{P}} = \{\sigma \in G_{\mathfrak{P}} \mid \sigma(\alpha) \equiv \alpha \pmod{\mathfrak{P}}, \forall \alpha \in \mathcal{O}_L\}$ . In the exercises we will show that one can replace  $\sigma \in G_{\mathfrak{P}}$  by  $\sigma \in G$  in the last set. Let  $T_{\mathfrak{P}} = L^{I_{\mathfrak{P}}}$  be the fixed field of  $I_{\mathfrak{P}}$ .

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**Theorem 7.11.** (i)  $T_{\mathfrak{P}}/Z_{\mathfrak{P}}$  is a Galois extension with Galois group  $G_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ .

Recall that this is a cyclic group generated by the Frobenius  $\bar{\varphi}_{\mathfrak{P}}$ .

(ii)  $|I_{\mathfrak{P}}| = e, |G_{\mathfrak{P}}/I_{\mathfrak{P}}| = f$ .

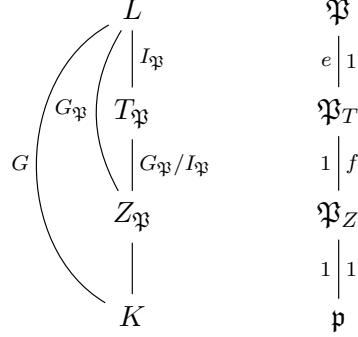
(iii)  $e(\mathfrak{P}|\mathfrak{P}_T) = e$  and  $e(\mathfrak{P}_T|\mathfrak{P}_Z) = e(\mathfrak{P}_Z|\mathfrak{p}) = 1$ , as well as  $f(\mathfrak{P}_T|\mathfrak{P}_Z) = f$  and  $f(\mathfrak{P}_Z|\mathfrak{p}) = f(\mathfrak{P}|\mathfrak{P}_Z) = 1$ .

*Proof.* (i) and (ii) are clear. For (iii), by multiplicativity of  $e$  and  $f$ , as well as propositions 7.5 and 6.1, it suffices to show  $\kappa(\mathfrak{P}_T) = \kappa(\mathfrak{P})$ . Consider the inertia group of  $\mathfrak{P}$  in  $L/T_{\mathfrak{P}}$ ,

$$I_{\mathfrak{P}}(L/T_{\mathfrak{P}}) = \{\sigma \in I_{\mathfrak{P}} \mid \sigma(\alpha) \equiv \alpha \pmod{\mathfrak{P}}, \forall \alpha \in \mathcal{O}_L\} = I_{\mathfrak{P}},$$

so  $f(\mathfrak{P}|\mathfrak{P}_T) = 1$ , because then the surjective map  $I_{\mathfrak{P}} \rightarrow \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{P}_T))$  has full kernel, so its image is trivial.  $\square$

In total, we have the following diagram of fields (with Galois groups) and primes between  $\mathfrak{P}$  and  $\mathfrak{p}$ , with ramification indices indicated on the left, and inertia degrees on the right:



**Theorem 7.12.** Let  $L/K$  be Galois. Let  $\mathfrak{P} \subseteq \mathcal{O}_L$  be unramified. There is a unique element  $\varphi_{\mathfrak{P}} \in G$  with

$$\varphi_{\mathfrak{P}}(\alpha) \equiv \alpha^q \pmod{\mathfrak{P}}$$

for all  $\alpha \in \mathcal{O}_L$ , where  $q = |\kappa(\mathfrak{p})|$ . In addition,  $G_{\mathfrak{P}} = \langle \varphi_{\mathfrak{P}} \rangle$ .

*Proof.* Since  $I_{\mathfrak{P}} = 1$ , we have  $G_{\mathfrak{P}} \cong \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})) = \langle \bar{\varphi}_q \rangle$ . Taking a preimage of  $\bar{\varphi}_q : \bar{\alpha} \mapsto \bar{\alpha}^q$  yields the desired element.  $\square$

**Remark 7.13.** If  $L/K$  is abelian, then  $\varphi_{\mathfrak{P}}$  only depends on  $\mathfrak{p}$ , then denoted  $\varphi_{\mathfrak{p}}$ , since the same is true for  $G_{\mathfrak{P}}$  by lemma 7.4. Similarly,  $\varphi_{\sigma(\mathfrak{P})} = \sigma\varphi_{\mathfrak{P}}\sigma^{-1}$ . This Frobenius plays a crucial role in class field theory.

**Corollary 7.14.** If  $L/K$  is Galois, but not cyclic, then there are at most finitely many primes  $\mathfrak{p} \subseteq \mathcal{O}_K$  which do not split.

*Proof.* If  $\mathfrak{p}$  is not split and unramified, then  $G = G_{\mathfrak{P}}$  since  $\mathfrak{p}$  is non-split, and by the previous theorem,  $G$  would be cyclic.  $\square$

We will apply this theory to the study of cyclotomic fields: A cyclotomic field is a field of the form  $K = \mathbb{Q}(\zeta_m)$ , with  $\zeta_m$  a primitive  $m$ -th root of unity. Without loss we may take  $\zeta_m = \exp(2\pi i/m)$ , for example. These fields play an important role in algebraic number theory. For example, by a famous theorem of Kronecker-Weber, every abelian number field is contained in some cyclotomic extension.<sup>4</sup>

Recall the following facts from Algebra:  $K = \mathbb{Q}(\zeta_m)/\mathbb{Q}$  is the splitting field of  $X^m - 1$ , in particular a Galois extension. We have  $\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ , given by  $\bar{a} \mapsto (\sigma_a : \zeta_m \mapsto \zeta_m^a)$ . Its order is  $\varphi(m) := (\mathbb{Z}/m\mathbb{Z})^\times = [K : \mathbb{Q}]$ . If  $p$  is an odd prime, then  $\mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}$  is a cyclic extension, because  $(\mathbb{Z}/p^\nu\mathbb{Z})^\times$  is a cyclic group of  $\varphi(p) = (p-1)p^{\nu-1}$ . Let  $\Phi_m(X)$  be the minimal polynomial of  $\zeta_m$ , hence  $\Phi_m(X) = \prod_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} (X - \zeta_m^a)$ . This is the  $m$ -th cyclotomic polynomial. We have the formula  $X^m - 1 = \prod_{d|m} \Phi_d(X)$ . In particular,

$$\Phi_{p^\alpha} = \frac{X^{p^\alpha} - 1}{\prod_{i=0}^{\alpha-1} \Phi_{p^i}(X)} = \frac{X^{p^\alpha} - 1}{X^{p^{\alpha-1}} - 1} = \left( X^{p^{\alpha-1}} \right)^{p-1} + \dots + X^{p^{\alpha-1}} + 1.$$

**Lemma 7.15.** (i) Let  $K = \mathbb{Q}(\zeta_{p^\nu})/\mathbb{Q}$ . Set  $\pi := 1 - \zeta_{p^\nu}$ . Then  $\pi\mathcal{O}_K$  is a prime ideal of degree 1 (i.e.  $f = 1$ ) and we have  $p\mathcal{O}_K = (\pi)^{\varphi(p^\nu)}$ . In other words,  $p$  is totally ramified in  $K$ .

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<sup>4</sup>The goal of Class Field Theory is to describe in more generality, given a number field  $K$ , all abelian extension  $L/K$  with data in  $K$ , see lectures next term.

(ii)  $d(\zeta_{p^\nu}) = \pm p^s$  with  $s = p^{\nu-1}(\nu p - \nu - 1)$ .

*Proof.* (i) Write  $\zeta = \zeta_{p^\nu}$ . Setting  $X = 1$  in  $\Phi_{p^\nu}(X)$  yields

$$p = \prod_{a \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times} (1 - \zeta^a).$$

Recall that, for general  $(m, a) = 1$ , we have  $\frac{1-\zeta_m^a}{1-\zeta_m} \in \mathbb{Z}[\zeta]^\times \subseteq \mathcal{O}_{\mathbb{Q}(\zeta_m)}^\times$ , see example 5.12(iv). Hence  $p = u(1 - \zeta)^{\varphi(p^\nu)}$  for some unit  $u$ , and  $p\mathcal{O}_K = (1 - \zeta)^{\varphi(p^\nu)}$ . Now everything follows from  $[K : \mathbb{Q}] = \varphi(p^\nu) = efr$ .

(ii) By definition,

$$d(\zeta) = d(1, \zeta, \dots, \zeta^{\varphi(p^\nu)-1}) = \prod_{i \neq j} (\zeta_i - \zeta_j) = \prod_{i=1}^{\varphi(p^\nu)} \Phi'_{p^\nu}(\zeta_i),$$

since by the product rule,  $\Phi_{p^\nu}(X) = \prod_{j \neq i} (X - \zeta_j) + (X - \zeta_i)g$  for some polynomial  $g$ . Hence  $d(\zeta) = N_{K/\mathbb{Q}}(\Phi'_{p^\nu}(\zeta))$ . Differentiate  $(X^{p^{\nu-1}} - 1)\Phi_{p^\nu}(X) = X^{p^\nu} - 1$  to obtain at  $X = \zeta$

$$(\zeta^{p^{\nu-1}} - 1)\Phi'_{p^\nu}(\zeta) = p^\nu \zeta^{p^{\nu-1}} \quad \text{and} \quad N_{K/\mathbb{Q}}(p^\nu \zeta^{p^{\nu-1}}) = \pm p^{\nu \varphi(p^\nu)}$$

So it remains to show that  $N_{K/\mathbb{Q}}(\zeta^{p^{\nu-1}} - 1) = p^{p^{\nu-1}}$ . But  $\zeta^{p^{\nu-1}}$  is a primitive  $p$ -th root of unity  $\zeta_p$ , so  $p\mathcal{O}_{\mathbb{Q}(\zeta_p)} = (1 - \zeta_p)^{p-1}$  by (i), hence

$$p^{p-1} = N(p\mathcal{O}_{\mathbb{Q}(\zeta_p)}) = |N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(1 - \zeta_p)|^{p-1}.$$

Thus  $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(1 - \zeta_p) = p$ , and  $N_{K/\mathbb{Q}}(1 - \zeta_p) = p^{[K:\mathbb{Q}(\zeta_p)]} = p^{p^{\nu-1}}$  □

**Theorem 7.16.** Let  $n \in \mathbb{N}$  and  $K = \mathbb{Q}(\zeta_n)$ . Then  $\mathcal{O}_K = \mathbb{Z}[\theta_n]$

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*Proof.* First assume  $n = p^\nu$  is a prime power. Then  $p^s\mathcal{O}_K \subseteq \mathbb{Z}[\zeta] \subseteq \mathcal{O}_K$ . Let  $\pi = 1 - \zeta$ . Then  $\mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{Z}/p\mathbb{Z}$  by lemma 7.15(i), hence  $\mathcal{O}_K = \mathbb{Z} + \pi\mathcal{O}_K$ .

We claim  $\pi^t\mathcal{O}_K + \mathbb{Z}[\zeta] = \mathcal{O}_K$  for all  $t \geq 1$ . By the above,  $t = 1$  is clear. Proceeding inductively, multiply  $\pi^t\mathcal{O}_K + \mathbb{Z}[\zeta] = \mathcal{O}_K$  by  $\pi$  to obtain

$$\pi^{t+1}\mathcal{O}_K + \pi\mathbb{Z}[\zeta] = \pi\mathcal{O}_K \implies \mathcal{O}_K = \mathbb{Z}[\zeta] + \pi^{t+1}\mathcal{O}_K.$$

Now take  $t = s\varphi(p^\nu)$ , with  $s$  as in lemma 7.15. Then  $\mathcal{O}_K = \pi^t\mathcal{O}_K + \mathbb{Z}[\zeta] = p^s\mathcal{O}_K + \mathbb{Z}[\zeta] = \mathbb{Z}[\zeta]$  by the first observation.

Let  $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$  be arbitrary. Then  $\mathbb{Q}(\zeta_{p_i^{\nu_i}}) \cap \mathbb{Q}(\zeta_{p_j^{\nu_j}}) = \mathbb{Q}$  (for example, because  $p_i$  is totally ramified in the first, but unramified in the second extension), so  $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{p_1^{\nu_1}}) \cdots \mathbb{Q}(\zeta_{p_r^{\nu_r}})$ , and the result follows by induction from corollary 2.21, also cf. example 2.22. □

**Theorem 7.17.** Let  $n = \prod_p p^{\nu_p}$ ,  $\nu_p \in \mathbb{Z}_{\geq 0}$ , almost all 0. Let  $p$  be prime. Let  $f_p \in \mathbb{N}$  be minimal with  $p^{f_p} \equiv 1 \pmod{n/p^{\nu_p}}$ . Then

$$p\mathcal{O}_K = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^{\varphi(p^{\nu_p})}$$

and each  $\mathfrak{p}_i$  has inertia degree  $f_p$ . Hence  $\varphi(n) = r\varphi(p^{\nu_p})f_p$

**Corollary 7.18.** Precisely the divisors of  $n$  are ramified in  $\mathbb{Q}(\zeta_n)$ , unless  $p = 2 = (4, n)$ . A prime  $p \neq 2$  is totally split iff  $p \equiv 1 \pmod{n}$ .

*Proof.*  $\mathcal{O}_K = \mathbb{Z}[\zeta_n]$ , so  $\mathfrak{f} = 1$ . Hence we may apply the polynomial decomposition law 6.4 for all  $p$ . So we have to factor  $\Phi_n(X) = (\bar{p}_1(X) \cdots \bar{p}_r(X))^e \pmod{p}$ . First consider  $p \nmid n$ . Then  $X^n - 1$  is separable  $\pmod{p}$ , hence so is  $\Phi_n$ . In other words, if  $\mathfrak{p} \mid p$ , then  $\mathcal{O}_K^\times \rightarrow (\mathcal{O}_K/\mathfrak{p})^\times$  is injective on  $\mu_n$ . But then  $\mu_n$  is contained in  $\mathcal{O}_K/\mathfrak{p}$ . One has  $\mu_n \subseteq \mathbb{F}_{p^f}^\times$  iff  $p^f \equiv 1 \pmod{n}$ , hence  $\mathbb{F}_{p^f}$  is the splitting field of  $\bar{\Phi}_n \in \mathbb{F}_p[X]$ . If  $\bar{\Phi}_n(X) = \bar{p}_1(X) \cdots \bar{p}_r(X)$  with  $\bar{p}_i(X) \in \mathbb{F}_p[X]$  irreducible and pairwise distinct, then each  $\bar{p}_i$  is a minimal polynomial of a primitive root of unity in  $\mathbb{F}_{p^f}$ , hence of degree  $f_p$ .

Now let  $n$  be general. Write  $n = p^\nu m$ ,  $p \nmid m$ . Let  $\xi_i, \eta_j$  denote the primitive  $m$ -th, and  $p^\nu$ -th roots of unity, respectively. Then  $\Phi_n(X) = \prod_{i,j} (X - \xi_i \eta_j)$ . Because  $X^{p^\nu} - 1 = (X - 1)^{p^\nu} \pmod{p}$  we have  $(\eta_j - 1)^{p^\nu} \equiv 0 \pmod{\mathfrak{p}}$  for all  $j$  and  $\mathfrak{p} \mid p$ . Hence  $\eta_j \equiv 1 \pmod{\mathfrak{p}}$ . Then

$$\Phi_n(X) \equiv \prod_{i,j} (X - \xi_i) = \prod_i (X - \xi_i)^{\varphi(p^\nu)} = \Phi_m(X)^{\varphi(p^\nu)} \pmod{\mathfrak{p}}$$

and the result follows from the first case, since  $p \nmid m$ .  $\square$

## 8 Valuations

Let  $\mathcal{O}$  be a Dedekind domain with  $K = \text{Quot}(\mathcal{O})$ . Let  $\mathfrak{p} \subseteq \mathcal{O}$  be a maximal ideal. Then  $0 \neq \mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$ ,  $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$ .

**Definition 8.1.**  $v_{\mathfrak{p}} : K \rightarrow \mathbb{Z} \cup \{\infty\}$ ,  $K^\times \ni x \mapsto v_{\mathfrak{p}}(x\mathcal{O})$  and  $0 \mapsto \infty$  is called the *valuation at  $\mathfrak{p}$* .

**Lemma 8.2.**  $v_{\mathfrak{p}}$  is a valuation, that is

- (i)  $v_{\mathfrak{p}}(ab) = v_{\mathfrak{p}}(a) + v_{\mathfrak{p}}(b)$ ,
- (ii)  $v_{\mathfrak{p}}(a+b) \geq \min(v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b))$ , with equality if  $v_{\mathfrak{p}}(a) \neq v_{\mathfrak{p}}(b)$ .

*Proof.* (i) is clear, for (ii) wlog  $a, b \in \mathcal{O}$ , then write  $a\mathcal{O} = \mathfrak{p}^{v_{\mathfrak{p}}(a)}\mathfrak{a}$ ,  $\mathfrak{p} \nmid \mathfrak{a}$  and  $b\mathcal{O} = \mathfrak{p}^{v_{\mathfrak{p}}(b)}\mathfrak{b}$ ,  $\mathfrak{p} \nmid \mathfrak{b}$ . Assume  $v_{\mathfrak{p}}(a) \leq v_{\mathfrak{p}}(b)$ . Now  $\mathfrak{p}^{v_{\mathfrak{p}}(a)} \mid a\mathcal{O}, b\mathcal{O}$ , hence also  $(a+b)\mathcal{O}$ .

Assume  $v_{\mathfrak{p}}(a) < v_{\mathfrak{p}}(b)$ , Then  $a \in \mathfrak{p}^{v_{\mathfrak{p}}(a)} \setminus \mathfrak{p}^{v_{\mathfrak{p}}(a)+1}$  and  $b \in \mathfrak{p}^{v_{\mathfrak{p}}(b)} \setminus \mathfrak{p}^{v_{\mathfrak{p}}(b)+1}$ , so  $a+b \notin \mathfrak{p}^{v_{\mathfrak{p}}(a)+1}$ .  $\square$

**Definition 8.3.** Let  $K$  be a number field and  $\mathfrak{p}$  a maximal ideal of  $\mathcal{O}_K$ . Let  $x \in K^\times$ . Then  $|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$  is called the  *$\mathfrak{p}$ -adic value* of  $x$ . Further  $|0|_{\mathfrak{p}} := 0$ .

**Example 8.4.** For  $K = \mathbb{Q}$ ,  $\mathcal{O} = \mathbb{Z}$  one has  $v_3(27) = 3$ ,  $v_3(10) = 0$  and  $|27|_3 = 3^{-3}$ ,  $|10|_3 = 1$ .

For  $K = \mathbb{Q}(\sqrt{2})$ ,  $2\mathcal{O}_K = \mathfrak{p}^2$  with  $\mathfrak{p} = \sqrt{2}\mathcal{O}_K$ .  $|2|_{\mathfrak{p}} = 2^{-2} = \frac{1}{4}$ . Note that in addition to the  $\mathfrak{p}$ -adic values, we also have two archimedean values given by the usual absolute value  $|\cdot|$ , and  $|\cdot| \circ \tau$ , where  $\tau$  denotes conjugation.

In general, if  $K/\mathbb{Q}$  is a number field, one has  $\mathfrak{p}$ -adic values, and  $r+s$  archimedean values  $|\cdot|_{\rho} = |\cdot| \circ \rho$ ,  $|\cdot|_{\sigma} = |\cdot| \circ \sigma$  for each real embedding  $\rho : K \rightarrow \mathbb{R}$  and pairs of complex embeddings  $\sigma, \bar{\sigma} : K \rightarrow \mathbb{C}$ . One can show that these are all values on  $K/\mathbb{Q}$  up to equivalence. Just as  $\mathbb{R}$  can be thought of as the completion of  $\mathbb{Q}$  w.r.t. the usual absolute value, we want to construct "completions" for the  $\mathfrak{p}$ -adic values.

Motivation: Let  $L/K$  be number fields. Given an ideal factorization  $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ , one can pass to the localization at  $\mathfrak{p}$ . Then  $\mathcal{O}_{K,\mathfrak{p}}$  is a discrete valuation ring, and  $\mathcal{O}_{L,\mathfrak{p}}$  is a Dedekind domain with primes  $S_{\mathfrak{p}}^{-1}\mathfrak{P}_i$ , hence a PID by lemma 2.17. One still has  $\mathfrak{p}\mathcal{O}_{L,\mathfrak{p}} = (S_{\mathfrak{p}}^{-1}\mathfrak{P}_1)^{e_1} \cdots (S_{\mathfrak{p}}^{-1}\mathfrak{P}_r)^{e_r}$ , so all the information is still present in these easier rings.

In another direction, we will define a completion at  $\mathfrak{P}$ , which yields a field extension  $L_{\mathfrak{P}}/K_{\mathfrak{p}}$  with corresponding discrete valuation rings. For the corresponding prime ideals, one has  $\widehat{\mathfrak{P}} \mid \widehat{\mathfrak{p}}$  and  $\widehat{\mathfrak{p}}\mathcal{O}_{L,\mathfrak{P}} = \widehat{\mathfrak{P}}^e$ . So completion is an even finer construction than localization, such that everything becomes local and therefore often easier to deal with.