Exercise 1

Since $\frac{Y}{X} = \frac{X}{Z}$, we certainly have $\operatorname{dom}(\varphi) \supseteq D(X) \cup D(Z)$. We claim we have equality. Indeed, suppose $(0, y, 0) \in \operatorname{dom}(\varphi)$. Then by continuity

$$\varphi(0, y, 0) = \lim_{\varepsilon \to 0} \varphi(\underbrace{\sqrt{y}\varepsilon, y, \varepsilon^2}_{\in D(Z)}) = \lim_{\varepsilon \to 0} \frac{\sqrt{y}}{\varepsilon} = \infty$$

for $y \neq 0$, and similarly

$$\varphi(0, 0, 0) = \lim_{\varepsilon \to 0} \varphi(\varepsilon^2, \varepsilon, \varepsilon^3) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}$$

for $y = 0$, contradiction.

Exercise 2

(1) $\varphi(0, 1) = 0 = \varphi(0, 0)$, so φ is not injective.

(2) $\varphi^* : K(x, y) \to K(x, y)$ is the morphism induced by $x \mapsto x$, $y \mapsto xy$. This is clearly surjective, since $\varphi^*(\frac{y}{x}) = y$, hence an iso.

Exercise 3

(1) By Eistenstein with $Y - 1$, the polynomial is irreducible in $K(X)[Y]$, hence by Gauss also in $K[X, Y]$.

(2) Assume there exists a birational equivalence $\varphi : \mathbb{A}^1 \to Y$. This corresponds to an isomorphism $\varphi^* : K(Y) \to K(x)$. Let $F = \varphi^*(x)$, $G = \varphi^*(y)$. Then $F^3 + G^3 = 1$, and multiplying by a common denominator we obtain an equation of the form $f^3 + g^3 = h^3$ for $f, g, h \in K[x]$. Our goal is to show that this equation has no solutions in non-constant polynomials.

By dividing out common factors, we may assume $\gcd(f, g, h) = 1$. In fact, then f, g, h are already pairwise prime, since any prime factor dividing two of them would also divide the third. Suppose wlog that $\deg(f) \geq \deg(g)$. Then $3\deg(h) = \deg(f^3 + g^3) \leq \deg(f^3) = 3\deg(f)$, so $\deg(f) \geq \deg(h)$

Take formal derivatives and multiply by h to obtain $f^2 f'h + g^2 g'h = (f^3 + g^3)h'$. Then we see $f^2 \mid f^2 f'h - f^3 h' = g^3 h' - g^2 g'h = g^2(gh' - g'h)$. But since f, g are coprime, it follows that $f^2 \mid gh' - hg'$. If $gh' - g'h \neq 0$, then in particular, $2\deg(f) \leq \deg(g) + \deg(h) - 1 < \deg(g) + \deg(f)$, contradicting the assumption $\deg(f) \geq \deg(g)$.

Hence $gh' - g'h = 0$, i.e. $(\frac{g}{h})' = 0$. But that means $\frac{g}{h}$ is constant, say $g = kh$ for some $k \in K$. But this contradicts the coprimality assumption.