# Algebraic Geometry I

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## Contents

Roughly speaking, the goal of algebraic geometry is to study systems of polynomial equations $F_1(X) = \ldots = F_n(X) = 0$ for polynomials $F_i \in K[X_1, \ldots, X_m]$ over a field (or ring) $K$. The set of solutions of this system is a geometric object, which we try to understand using algebraic methods, for example considering the ideal $I = (F_1, \ldots, F_n)$ in $K[X_1, \ldots, X_m]$ or the quotient $K[X_1, \ldots, X_m]/I$. Lecture 1 Oct 13, 2025

There is a very strong relation between these objects in the case that $K = \overline{K}$ is algebraically closed (e.g. $\mathbb{C}$). If $K$ is not algebraically closed, or some generic ring, things get more complicated: For example, there are many equations over $\mathbb{R}$ with no solutions, like $x^2+y^2+1 = 0$, which behave differently when considered over $\mathbb{C}$. The wish to still study these equations geometrically leads to the idea of spectra (the set of all prime ideals of a ring), and later the theory of sheaves and schemes.

# 1 Algebraic Sets and Affine Varieties

Let $K$ be an algebraically closed field.

**Definition 1.1.** For $n \in \mathbb{N}$ define *affine $n$-space* over $K$ as

$$\mathbb{A}^n := \mathbb{A}^n_K := K^n.$$

**Definition 1.2.** Let $I \subset K[x_1, \ldots, x_n]$ be a subset. The associated *(affine) algebraic set* is

$$V(I) := \{x \in \mathbb{A}^n_K \mid f(x) = 0 \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n$ is called *algebraic* if $X = V(I)$ for some $I \subset K[x_1, \ldots, x_n]$.

**Remark 1.3.** By definition $V(I) = V(\langle I \rangle) = V(f_1, \ldots, f_m)$ where $\langle I \rangle = (f_1, \ldots, f_m)$ is finitely generated because $K[x_1, \ldots, x_n]$ is Noetherian. Therefore, $X \subseteq \mathbb{A}^n$ is algebraic if and only if $X = V(I)$ for some ideal $I$ if and only if $X = V(f_1, \ldots, f_m)$ for a finite number of polynomials $f_i$.

**Example 1.4.** The following sets are algebraic:

- A parabola $\{(x, x^2) \mid x \in K\} = V(y - x^2)$
- $\emptyset = V(K[x_1, \ldots, x_n])$
- $\mathbb{A}^n = V(0)$
- Points: $\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, \ldots, x_n - a_n)$

**Lemma 1.5.** *Let $I, J \triangleleft K[x_1, \ldots, x_n]$ be ideals. Then*

*(a) If $I \subseteq J$, then $V(I) \supseteq V(J)$.*

*(b) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$*

*(c) For any family $(I_t)_{t \in T}$ of ideals, $\bigcap_t V(I_t) = V(\bigcup_t I_t) = V(\sum_t I_t)$*

*Proof.* (a) is clear.

For (b), part (a) yields $V(I \cap J) \subseteq V(IJ)$ and $V(I), V(J) \subseteq V(I \cap J)$, so it remains to show $V(IJ) \subseteq V(I) \cup V(J)$. Let $a \in V(IJ)$. Assume $a \notin V(I)$, i.e. there is $f \in I$ such that $f(a) \neq 0$. Let $g \in J$. Then $fg \in IJ$, so $0 = (fg)(a) = f(a)g(a)$. Since $f(a) \neq 0$, we conclude $g(a) = 0$.

The first equation of (c) is tautological, the second one is remark 1.3, □

**Definition 1.6.** The *Zariski topology* on $\mathbb{A}^n$ is the topology whose closed subsets are exactly the algebraic sets. That is, $U \subseteq \mathbb{A}^n$ is open iff its complement is algebraic.

**Remark 1.7.** This is indeed a topology by example 1.4 and lemma 1.5. Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set $X \subseteq \mathbb{A}^n$, which is also called the Zariski topology.

Recall from general topology that a topological space $X \neq \emptyset$ is called irreducible if $X \neq X_1 \cup X_2$ with $X_i \subsetneq X$ closed. $\emptyset$ is not considered irreducible.

For example, $V(xy) = V(x) \cup V(y)$ (the union of the coordinate axes in $\mathbb{A}^2$) is not irreducible, while a parabola $V(y - x^2)$ is irreducible (we will see how to check this later).

**Definition 1.8.** An *affine algebraic variety* is an irreducible closed subset of $\mathbb{A}^n$.

**Definition 1.9.** Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. We define the *vanishing ideal* of $X$ as

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

**Lemma 1.10.** *Let $X \subseteq \mathbb{A}^n$ and $S \subseteq K[x_1, \dots, x_n]$. Then*

*(a) $X \subseteq V(I(X))$ and $S \subseteq I(V(S))$.*

*(b) $V(I(X)) = \overline{X}$ is the closure of $X$ (w.r.t. the Zariski topology).*

*Proof.* (a) is clear, (b) is left as an exercise. □

**Proposition 1.11.** *An affine algebraic set $X \subseteq \mathbb{A}^n$ is a variety if and only if $I(X)$ is a prime ideal.*

*Proof.* Let $X$ be a variety and let $fg \in I(X)$ for $f, g \in K[x_1, \dots, x_n]$. We have $X \subseteq V(fg) \overset{1.5}{=} V(f) \cup V(g)$. Hence we can write $X = (X \cap V(f)) \cup (X \cap V(g))$ as the union of two closed subsets. By irreducibility, wlog we have $X = X \cap V(f)$, i.e. $X \subseteq V(f)$, which is equivalent to $f \in I(X)$.

Conversely, suppose that $X = A \cup B$ is not irreducible. Choose points $a \in A \setminus B$ and $b \in B \setminus A$. By Lemma 1.10 and since $A, B$ are closed, we get $V(I(A)) = A$ and $V(I(B)) = B$. Hence there exist $f \in I(A)$ and $g \in I(B)$ with $f(b) \neq 0$ and $g(a) \neq 0$. Thus $fg \in I(X)$, but both $f, g \notin I(X)$ □

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**Remark 1.12.** If $X = V(I)$ is an affine variety, this does not necessarily imply that $I$ is prime: Consider $V((x^2)) \subseteq \mathbb{A}^1$: $V((x^2)) = \{0\}$ is irreducible, but $(x^2)$ is not prime.

Note that $\mathbb{A}^n$ is irreducible since $K$ is infinite. However, this is no longer true if one considers finite fields, since then $\mathbb{A}^n$ is the union of its finitely many points. For example, $I(A^1_{\mathbb{F}_p}) = (X^p - X)$ is not prime.

We use the following result from commutative algebra without proof:

**Theorem 1.13** (Hilbert Nullstellensatz)**.** *Let $J \lhd K[x_1, \dots, x_n]$. Then*

*(a) $V(J) = \emptyset$ if and only if $J = K[x_1, \dots, x_n]$.*

*(b) $I(V(J)) = \sqrt{J} = \{f \in K[x_1, \dots, x_n] \mid f^n \in J \text{ for some } n\}$*

*(c) If $J$ is a maximal ideal, then $J = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in K$.*

**Corollary 1.14.** *There are inclusion-reversing bijections*

$$\{\textit{affine algebraic sets } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{radical ideals in } K[x_1, \ldots, x_n]\}$$
$$\{\textit{affine algebraic varieties } X \subseteq \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{prime ideals in } K[x_1, \ldots, x_n]\}$$
$$\{\textit{points } a \in \mathbb{A}^n\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{maximal ideals in } K[x_1, \ldots, x_n]\}$$

*Proof.* Clear from 1.13, 1.10 and 1.11. □

**Example 1.15.** Let $f$ be irreducible in $K[x_1, \ldots, x_n]$. Then $V(f)$ is an affine variety. Varieties of this form are called hypersurfaces in $\mathbb{A}^n$ (curves for $n = 2$, surfaces for $n = 3$).

**Remark 1.16.** If $X \subseteq \mathbb{A}^n$ is a variety, by proposition 1.11 $I(X)$ is prime, and $K[x_1, \ldots, x_n]/I$ is an integral domain. We can consider its fraction field $\mathrm{Frac}(K[x_1, \ldots, x_n]/I)$.

**Theorem 1.17.** *Any affine algebraic set can be uniquely written as a finite union of affine varieties.*

For the proof, we need some preparations.

**Definition 1.18.** A topological space $X$ is called *Noetherian* if any chain of descending closed subsets $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$ becomes stationary, i.e. there exists $n$ s.t. $X_m = X_n$ for all $m > n$.

**Lemma 1.19.** *Affine space $\mathbb{A}^n$ is Noetherian.*

*Proof.* Let $\mathbb{A}^n \supseteq X_1 \supseteq X_2 \supseteq \ldots$ be a chain of closed subsets. Applying $I(-)$ yields an ascending chain $(0) \subseteq I(X_1) \subseteq I(X_2) \subseteq \ldots$ of ideals in $K[x_1, \ldots, x_n]$. This is a Noetherian ring, so there is some $m$ such that $I(X_n) = I(X_{n+1})$ for all $n \geq m$. By corollary 1.14(a), $I$ is injective on closed subsets, so we are done. □

More generally,

**Corollary 1.20.** *Any affine algebraic space $X \subseteq \mathbb{A}^n$ is Noetherian.*

*Proof.* Any chain in $X$ is also a chain in $\mathbb{A}^n$. □

**Proposition 1.21.** *Let $X \neq \emptyset$ be a Noetherian topological space.*

*(a) Then $X$ can be written as a finite union of irreducible closed subspaces.*

*(b) Moreover, if we assume that $X_i \not\subseteq X_j$ for $i \neq j$, then the above decomposition is unique up to permutation. In this case, the $X_i$ are called irreducible components of $X$.*

*Proof.* Assume that (a) fails for $X$. Consider $S = \{Y \subseteq X \mid Y$ closed, cannot be written as a finite union of irreducible closed subsets$\}$. Since $X$ is Noetherian, $S$ must have some minimal element $Y$ w.r.t. inclusion. $Y$ is not irreducible, so we can write $Y = Y_1 \cup Y_2$ with $Y_{1,2}$ proper closed subspaces. By minimality, $Y_1$ and $Y_2$ can be written as finite unions of irreducible closed subsets, thus so can $Y$, contradicting $Y \in S$.

To check uniqueness, assume we have two decompositions $X = X_1 \cup \ldots X_r = X_1' \cup \ldots X_s'$ as in (b). Then $X_1' = \bigcup_i (X_i \cap X_1')$. Since $X_1'$ is irreducible, wlog $X_1' \subseteq X_1$. By the same argument, $X_1 \subseteq X_i'$ for some $i$. If $i \neq 1$, then $X_1' \subseteq X_i'$, contradicting our assumption. Hence $i = 1$ and $X_1 = X_1'$. Proceed inductively with $\overline{X \setminus X_1} = X_2 \cup \ldots \cup X_r = X_2' \cup \ldots \cup X_s'$. □

Combining 1.20 and 1.21 yields theorem 1.17.

**Remark 1.22.** The proof strategy for (a) can be summarized as follows: Let $X$ be a Noetherian space and $P$ a property of closed subsets. To show that $P$ holds for all subsets of $X$ (thus in particular for $X$), it suffices to show that for all $Y \subseteq X$ closed, if $P$ holds for all proper closed subsets of $Y$, then it also holds for $Y$. This is called *Noether induction* (a special case of well-founded induction).

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**Example 1.23.** Let $f \in K[x_1, \ldots, x_n]$. This is a factorial ring, so we may write $f = g_1^{k_1} \cdots g_r^{k_r}$ with $g_i$ irreducible and pairwise different. Then

$$V(f) = V(g_1^{k_1}) \cup \cdots V(g_r^{k_r}) = V(g_1) \cup \cdots \cup V(g_r)$$

is the decomposition of $V(f)$ into irreducible subsets: $V(g_i)$ is irreducible by proposition 1.11, since $I(V(g_i)) = (g_i)$ is prime.

In general, finding this composition for $V(f_1, \ldots, f_r)$ is not easy.

**Example 1.24.** What is the Zariski topology on $\mathbb{A}^1$? By definition, a closed/algebraic set is of the form $V(I)$ for some ideal $I \subseteq K[x]$. Since $K[x]$ is a PID, $I = (f)$ for some $f = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \in K[x]$. If $f$ is not constant, we see as in example 1.23 that

$$X = V(f) = \bigcup_i V(x - a_i) = \{a_1, \ldots, a_r\}.$$

Hence the closed sets are exactly $V(0) = \mathbb{A}^1, V(1) = \emptyset$, and finite unions of points. In other words, the Zariski topology coincides with the cofinite topology on $\mathbb{A}^1$. The affine varieties on $\mathbb{A}^1$ are therefore either $\mathbb{A}^1$ itself or a single point.

We also see that any two non-empty open subsets have nontrivial intersection, so $\mathbb{A}^1$ with the Zariski topology is not Hausdorff.

**Definition 1.25.** Let $X$ be a nonempty topological space. We define the dimension of $X$ as the supremum of all $n \in \mathbb{N}$ such that there is a chain of irreducible subspaces $\emptyset \neq Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n \subseteq X$

**Example 1.26.** By example 1.24, a maximal chain of affine varieties in $\mathbb{A}^1$ is $\{0\} \subsetneq \mathbb{A}^1$, hence $\dim \mathbb{A}^1 = 1$.

**Definition 1.27.** Let $R$ be a (commutative) ring. The Krull dimension of $R$ is the supremum over all $l$ such that there is a chain of prime ideals $\mathfrak{p}_l \subsetneq \mathfrak{p}_{l-1} \subsetneq \ldots \subsetneq \mathfrak{p}_0 \subsetneq R$.

Recall from corollary 1.14 that there is an inclusion-reversing correspondence between prime ideals of $K[x_1, \ldots, x_n]$ and affine algebraic varieties in $\mathbb{A}^n$. Fixing some variety $X$, it follows that subvarieties correspond bijectively to prime ideals that contain $I(X)$, i.e. prime ideals of $K[x_1, \ldots, x_n]/I(X)$. Hence

**Proposition 1.28.** *If $X$ is an affine algebraic variety, then* $\dim X = \dim K[x_1, \ldots, x_n]/I(X)$.

# 2 Morphisms of Affine Varieties

## 2.1 Regular Morphisms

**Definition 2.1.** Let $X \subseteq \mathbb{A}^n_K$ be an algebraic set. A function $f : X \to K$ is *regular* if there is a polynomial $F \in K[x_1, \dots, x_n]$ such that $f : F|_X$, i.e. $f(x) = F(x)$ for all $x \in X$. Write $A(X)$ for the set of regular functions on $X$.

**Remark 2.2.** $A(X)$ is a ring (and even a $K$-algebra) in a natural way, with addition and multiplication defined pointwise. Moreover, there is a homomorphism of $K$-algebras

$$K[x_1, \dots, x_n] \twoheadrightarrow A(X), \qquad F \mapsto F|_X.$$

The kernel of this morphism is exactly $I(X)$, so that $A(X) \cong K[x_1, \dots, x_n]/I(X)$ canonically.

**Remark 2.3.** By corollary 1.14, $A(X)$ is always reduced, $A(X)$ is integral iff $X$ is a variety, and $A(X)$ is a field iff $X$ is a point (in which case $A(X) \cong K$).

**Definition 2.4.** Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A map $\varphi : X \to Y$ is called *regular* if $\varphi = (f_1, \dots, f_m)$ for some regular $f_1, \dots, f_m \in A(X)$. A regular map $\varphi$ is an isomorphism if it has in anverse which is also regular.

**Example 2.5.**

(i) $f : \mathbb{A}^1 \to V(y - x^2) \subseteq \mathbb{A}^2$, $t \mapsto (t, t^2)$ is a regular map. It has inverse $(x, y) \mapsto x$, which is also regular, hence $\mathbb{A}^1 \cong V(y - x^2)$.

(ii) $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subseteq \mathbb{A}^2$, $t \mapsto (t^2, t^3)$ is regular and bijective as well, but its inverse $(x, y) \mapsto \frac{y}{x}$ is not regular, so $\varphi$ is not an isomorphism.

**Proposition 2.6.** *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets, and let $\varphi : X \to Y$ be a regular map. Then $\varphi$ is continuous (w.r.t. the Zariski topology on $X$ and $Y$).*

*Proof.* Let $\varphi = (f_1, \dots, f_m)$ and $J = \langle F_1, \dots, F_k \rangle \subseteq K[x_1, \dots, x_m]$ with $V(J) \subseteq Y$. Then

$$\varphi^{-1}(V(J)) = \varphi^{-1}(V(F_1, \dots, F_k)) = \{x \in X \mid F_j(f_1(x), \dots, f_m(x)) = 0,\ j = 1, \dots, k\}$$

Now $F_j(f_1(x), \dots, f_m(x))$ is a composition of polynomials, hence a polynomial, call it $\widetilde{F}_j$. We conlcude $\varphi^{-1}(V(J)) = X \cap V(\widetilde{F}_1, \dots, \widetilde{F}_k)$ as desired. □

**Remark 2.7.** The converse is false. For example, one easily concludes from example 1.24 that every bijective map $\mathbb{A}^1 \to \mathbb{A}^1$ is continuous, but there are way more bijections than polynomials (say because polynomials are defined by their values on any infinite subset). On the other hand, if $K$ is finite (loosing algebraic closedness), then every function $\mathbb{A}^1 \to \mathbb{A}^1$ is regular.

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**Remark 2.8.** Let $X$ be an algebraic set, and let $f : X \to \mathbb{A}^1$. Then $f$ is a regular map if and only if $f$ is a regular function. Note that the composition of regular maps is regular, since compositions of polynomials are polynomials.

**Definition 2.9.** Let $X, Y$ be algebraic sets and $F : X \to Y$ be regular. Then we set $F^* : A(Y) \to A(X)$, $g \mapsto g \circ F$. This is well-defined by remark 2.8, and $F^*$ clearly preserves addition and multiplication, so it is a morphism of $K$-algebras.

**Remark 2.10.** Let $F = (f_1, \dots, f_m) : X \to Y$, $f_i \in K[x_1, \dots, x_n]$, then $F^*$ is given by the $K$-algebra homomorphism $A(Y) \cong K[y_1, \dots, y_m]/I(Y) \to K[x_1, \dots, x_n]/I(X) \cong A(X)$ (see remark 2.2) defined by $y_i \mapsto f_i$. Hence $F(x) = (F_1^*(y_1), \dots, F_m^*(y_m))$.

**Theorem 2.11.** *(i) There is a bijection* $\mathrm{Mor}(X,Y) \to \mathrm{Hom}_{K\text{-}Alg}(A(Y), A(X))$ *given by* $F \mapsto F^*$.

*(ii) If* $F : X \to Y$ *and* $H : Y \to Z$ *are regular, then* $(H \circ F)^* = F^* \circ H^*$. *Further,* $\mathrm{id}_X^* = \mathrm{id}_{A(X)}$.

*(iii) Let* $F : X \to Y$ *be regular. Then* $F$ *is an isomorphism of affine sets if and only if* $F^*$ *is an isomorphism of* $K$*-algebras*

*Proof.* Injectivity in (i) follows from remark 2.10. For surjectivity, let $\varphi : A(Y) \to A(X)$ be a $K$-algebra homomorphism and define $F : X \to Y$ by $F = (\varphi(y_1), \dots, \varphi(y_m))$. We need to check that this is well-defined, i.e. that the image of $F$ lies in $Y$. Then it is clear that $F$ is regular and that $F^* = \varphi$, again by remark 2.10.

So let $g \in I(Y)$, we need to show $g \circ F = 0$. But this is exactly the statement $\varphi([g]) = \varphi(0) = 0$.

For (ii), $\mathrm{id}_X^* = \mathrm{id}_{A(X)}$ is clear, and for $f \in A(Z)$ one has

$$(H \circ F)^*(f) = f \circ H \circ F = H^*(f) \circ F = (F^* \circ H^*)(f),$$

so $(H \circ F)^* = F^* \circ H^*$. Then (iii) follows from (i) and (ii). □

**Example 2.12.** Looking again at the maps from example 2.5, we see that $f : \mathbb{A}^1 \to V(y - x^2), t \mapsto (t, t^2)$ is an isomorphism, because $f^* : K[x,y]/(y - x^2) \to K[t]$, $x \mapsto t, y \mapsto t^2$ clearly is. On the other hand, let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3)$, $t \mapsto (t^2, t^3)$. We saw that this is a bijective regular map and gave intuitive reasoning for why this map isn't an isomorphism. But now we can prove it: We have

$$f^* : K[x,y]/(y^2 - x^3) \to K[t], \quad x \mapsto t^2, y \mapsto t^3$$

is not surjective, for the image does not contain $t$.

**Remark 2.13.** In categorical terms, theorem 2.11 says that

$$\begin{aligned} \left\{ \begin{matrix} \text{algebraic sets} \\ \text{regular maps} \end{matrix} \right\} &\to \left\{ \begin{matrix} \text{finitely generated reduced } K\text{-algebras} \\ K\text{-algebra homomorphisms} \end{matrix} \right\} \\ X &\mapsto A(X) \\ F &\mapsto F^* \end{aligned}$$

is a contravariant functor, and even an equivalence of categories: For essential surjectivity, note that every finitely generated $K$-algebra can be written as a quotient $K[x_1, \dots, x_n]/I$ by choosing generators. Then consider $X = V(I)$.

**Proposition 2.14.** *Let* $X, Y$ *be algebraic sets, and let* $f : X \to Y$ *be a regular map. Then*

*(i)* $f^* : A(Y) \to A(X)$ *is surjective if and only if* $\overline{f(X)} = Y$, *i.e. if the image of* $f$ *is dense in* $Y$.

*(ii)* $f^*$ *is injective if and only if* $f(X) \subseteq Y$ *is closed and* $f : X \to f(X)$ *is an isomorphism.*

*Proof.* Exercise. □

## 2.2 Rational Maps of Varieties

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety. Then $I(X)$ is prime, so $A(X) \cong K[x_1, \dots, x_n]/I(X)$ is an integral domain. Hence we can define its field of fractions $K(X) := \operatorname{Frac} A(X)$.

**Definition 2.15.** An element $\varphi \in K(X)$ is called regular at $x \in X$ if there exist $f, g \in A(X)$ with $\varphi = \frac{f}{g}$ and $g(x) \neq 0$.

**Example 2.16.** Let $X = V(x^2 - yz) \subseteq \mathbb{A}^3$ and $x = (0, 0, 1)$. Consider $\varphi = \frac{y}{x} \in K(X)$. Even though it may look like $\varphi$ might not be regular at $x$, one can note that $\frac{y}{x} = \frac{x}{z}$ in $K(X)$, so actually $\varphi(x)$ can be defined and $\varphi$ is regular at $x$.

**Proposition 2.17.** *Let $\varphi \in K(X)$. Then $\varphi$ is regular at every $x \in X$ if and only if $\varphi \in A(X)$* Lecture 5 Nov 3, 2025

**Remark 2.18.** If $X \subseteq \mathbb{A}^n$ is an affine algebraic variety, the closed sets are exactly of the form $V_X(I) := \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}$ for ideals $I \subseteq A(X)$, and $V_X$ is still an inclusion-reversing bijection between radical ideals and closed subsets, compare exercises.

*Proof.* Assume $\varphi \in K(X)$ is regular at every point $x \in X$. Consider $I := \{f \in A(X) \mid f\varphi \in A(X)\}$. Then the claim is equivalent to $I = A(X)$, hence to $V_X(I) = \emptyset$ by remark 2.18. Assume there exists $x \in V_X(I)$. Since $\varphi$ is regular at $x$, we can write $\varphi = \frac{g}{h}$ with $g, h \in A(X)$ and $h(x) \neq 0$. Hence $h \in I$, and $h(x) = 0$ by choice of $x$, contradiction. $\square$

**Definition 2.19.** Let $X \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ be open. Denote $\mathcal{O}_X(U) := \{\varphi \in K(X) \mid \varphi \text{ regular at all } x \in U\}$. For $\varphi \in K(X)$, its domain is $\operatorname{dom}(\varphi) := \{a \in X \mid \varphi \text{ is regular at } a\}$. In other words, $\mathcal{O}_X(U) = \{\varphi \in K(X) \mid U \subseteq \operatorname{dom}(\varphi)\}$.

By proposition 2.17, $\mathcal{O}_X(X) = A(X)$.

**Example.**

(i) $\varphi = \frac{y}{x}$ on $X = V(y - x^2)$ is regular, since $\varphi = x$. Hence $\operatorname{dom}(\varphi) = X$.

(ii) $\varphi = \frac{y}{x}$ on $X = V(y^2 - x^3)$ has $\operatorname{dom}(\varphi) = X \setminus \{(0, 0)\}$.

**Proposition 2.20.** *Let $\varphi \in K(X)$. Then $\operatorname{dom}(\varphi)$ is an open non-empty set in $X$.*

*Proof.* Define $I := \{f \in A(X) \mid f\varphi \in A(X)\}$. As before, we have $\varphi$ is regular at $x$ if and only if $x \notin V_X(I)$, so $\operatorname{dom}\varphi = X \setminus V_X(I)$ is open. $\square$

**Remark 2.21.** Let $X$ be an irreducible topological space. Then

(i) Every non-empty open subset $U \subseteq X$ is dense in $X$.

(ii) If $U_1, U_2 \subseteq X$ are open and non-empty, then $U_1 \cap U_2 \neq \emptyset$.

Hence, if $X$ is an affine variety and $f \in A(X)$ evaluates to zero on some non-empty open, then already $f = 0$.

**Remark 2.22.** Let $U \subseteq X$ be a non-empty open. Any regular $\varphi \in \mathcal{O}_X(U) \subseteq K(X)$ defines a set-theoretical function $\varphi : U \to K$, by sending $a \in U$ to $\frac{f(a)}{g(a)}$, where $\varphi = \frac{f}{g}$ with $f, g \in A(X)$ and $g(a) \neq 0$. This is well-defined, for if $\varphi = \frac{f_1}{g_1}$ with $g_1(a) \neq 0$, then $fg_1 - f_1 g = 0$ in $A(X)$.

Conversely, let $\varphi : U \to K$ be a (set-theoretical) function. Then $\varphi$ defines a regular function on $U$ if for every $a \in U$ there is an open neighbourhood $a \in V \subseteq U$ such that $\varphi(b) = \frac{f(b)}{g(b)}$ for all $b \in V$, where $f, g \in K[x_1, \dots, x_n]$ and $g(b) \neq 0$ for all $b \in B$.

These assignments $(\varphi \in \mathcal{O}_X(U)) \mapsto (\varphi : U \to K)$ and $(\varphi : U \to K) \mapsto [\frac{f}{g}]$ are clearly well-defined and mutually inverse, so this is an equivalent view on regular functions on $U$.

One sees easily that the composition of regular maps is again regular.

**Remark 2.23.** Let $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ be two regular functions, and let $U \subseteq V$ be nonempty open. If $\varphi_1|_U = \varphi_2|_U$ then $\varphi_1 = \varphi_2$.

**Definition 2.24.** (i) A *quasi-affine variety* is an open subset of an affine algebraic variety.

(ii) A regular map between quasi-affine varieties $U \subseteq \mathbb{A}^n, V \subseteq \mathbb{A}^m$ is a map $\varphi : U \to V$ given by $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_i$ regular on $U$. $\varphi$ is an isomorphism if there is a regular inverse.

**Remark 2.25.** For affine varieties, by remark 2.13 all information on regular maps $f : X \to Y$ could be obtained from their induced coordinate maps $f^* : A(Y) \to A(X)$. This is no longer true for quasi-affine varieties: for example, $\mathbb{A}^2 \setminus 0 \hookrightarrow \mathbb{A}^2$ induces an isomorphism of coordinate rings

**Definition 2.26.** Let $X$ be an affine variety and $f \in A(X)$. Then $D(f) := X \setminus V_X(f)$ is called the *distinguished open subset* of $f$ in $X$. Lecture 6 Nov 5, 2025

**Remark 2.27.** Since $D(f) \cap D(g) = D(fg)$, finite intersections of distinguished opens are again distinguished open. Any open $U \in X$ is a finite union of distinguished open subsets. Indeed, $U = X \setminus V_X(f_1, \dots, f_n) = \bigcup_i D(f_i)$.

**Proposition 2.28.** *Let $X$ be an affine variety and $0 \neq f \in A(X)$. Then $\mathcal{O}_X(D(f)) = A(X)_f = \{\frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N}\} \subseteq K(X)$. In particular, on a distinguished open subset a regular function is always globally the quotient of two elements from $A(X)$.*

*Proof.* $\supseteq$ is clear. So let $\varphi \in \mathcal{O}_X(D(f))$ and consider

$$I = \{h \in A(X) \mid h\varphi \in A(X)\} \subseteq A(X).$$

This is an ideal which clearly satisfies $V_X(I) \cap D(f) = \emptyset$. Hence $V_x(I) \subseteq V_X(f)$, and by the Nullstellensatz 1.13 we see that $f \in \sqrt{I}$, i.e. $f^n \in I$ for some $n$. $\square$

**Example 2.29.** Consider $D(x) = \mathbb{A}^1 \setminus 0 \to V(xy-1) \subseteq \mathbb{A}^2, x \mapsto (x, \frac{1}{x})$. This is an isomorphism (with inverse $(x, y) \mapsto x$ between the quasi-affine $\mathbb{A}^1 \setminus 0$ and the affine variety $V(xy-1)$. Note that this is not true in general: not every quasi-affine variety is isomorphic to an affine variety. For example, $\mathbb{A}^2 \setminus 0$ isn't isomorphic to any affine variety. However, we have

**Proposition 2.30.** *Let $X$ be an affine variety and $f \in A(X)$. Then $D(f)$ is isomorphic to an affine variety $Y$ with $A(Y) \cong A(X)_f$.*

*Proof.* Set

$$Y := \{(x, t) \in X \times \mathbb{A}^1 \mid tf(x) = 1\} \subseteq X \times \mathbb{A}^1 \subseteq \mathbb{A}^{n+1}.$$

Then as in example 1.28, $D(f) \to Y$, $x \mapsto (x, \frac{1}{f(x)}$ is an isomorphism with inverse $(x, y) \mapsto x$, so $D(f) \cong Y$ and $A(Y) \cong \mathcal{O}_X(D(f)) \cong A(X)_f$. $\square$

We have seen that for $X$ an algebraic set and $f \in A(X)$ regular, $V_X(f)$ is closed in $X$. The same is true for quasi-affine varieties:

**Lemma 2.31.** *Let $X$ be an affine variety and $U \subseteq X$ open. Let $\varphi \in \mathcal{O}_X(U)$. Then $V_U(\varphi) := V(\varphi) := \{x \in U \mid \varphi(x) = 0\}$ is closed in $U$.*

*Proof.* Let $a \in U$. Then there exists an open neighbourhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that $\varphi = \frac{f_a}{g_a}$ on $U_a$. Then

$$U_a \setminus V(\varphi) = \{x \in U_a \mid \varphi(x) \neq 0\} = \{x \in U_a \mid f_a(x) \neq 0\} = U_a \setminus V(f_a)$$

is open in $X$, hence $U \setminus V(\varphi) = \bigcup_a U_a \setminus V(\varphi)$ is open. $\square$

**Proposition 2.32.** *Let $X$ be a quasi-affine variety and $U \subseteq X$ be open. Let $\varphi, \psi$ be two regular functions on $X$ such that $\varphi|_U = \psi|_U$. Then $\varphi = \psi$ on $X$.*

*Proof.* $V_X(\varphi - \psi)$ contains the open, hence dense by 2.21, set $U$. □

**Proposition 2.33.** *Let $X, Y$ be algebraic sets and $U \subseteq X$ be open. Then any regular map $\varphi : U \to Y$ is continuous (w.r.t. the Zariski topology). In particular, $\varphi \in \mathcal{O}_X(U)$ is a continuous map $U \to \mathbb{A}^1$.*

*Proof.* Let $\varphi = (\varphi_1, \dots, \varphi_m)$ and let $Z = V_Y(g_1, \dots, g_m) \subseteq Y$ be a closed subset. Then $\varphi^{-1}(Z) = \{x \in U \mid g_i(\varphi_1(x), \dots, \varphi_m(x)) = 0 \text{ for all } i\}$, which is closed by lemma 2.31. □

Let $\varphi : U \to V$ be regular. For any regular map $f \in \mathcal{O}(V)$, the composition $f \circ \varphi \in \mathcal{O}(U)$ is well-defined, hence we get as before a $K$-algebra homomorphism Lecture 7 Nov 10, 2025

$$\varphi^* : \mathcal{O}(V) \to \mathcal{O}(U), \quad f \mapsto f \circ \varphi.$$

The assignment $U \mapsto \mathcal{O}(U)$, $\varphi \mapsto \varphi^*$ is a contravariant functor as before, but no longer an equivalence of categories, see exercises.

Let $X, Y$ be affine algebraic subsets. We know that regular maps $X \to Y$ are given by polynomial functions. It may happen that we do not have any "interesting" polynomial maps. For example, over $K = \mathbb{C}$ consider $X = \mathbb{A}^1$ and $Y = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$. Then the only regular maps $X \to Y$ are constant. However, the nontrivial map $t \mapsto (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$ induces an isomorphism $\mathbb{A}^1 \setminus \{\pm i\} \to Y \setminus \{(1,0)\}$.

Let $X$ be an affine algebraic variety. Then $\varphi \in K(X)$ is a regular function on $\operatorname{dom}\varphi$. Moreso, given $\varphi_1, \dots, \varphi_m \in K(X)$, we get a regular map on the open set $\bigcap_i \dim \varphi_i \to \mathbb{A}^m$.

**Definition 2.34.** Let $X$ be an affine algebraic variety and $Y$ an affine algebraic set. A *rational map* $\varphi : X \dashrightarrow Y \subseteq \mathbb{A}^m$ is given by $\varphi = (\varphi_1, \dots, \varphi_m)$ with $\varphi_i \in K(X)$ such that $\varphi(x) \in Y$ for every $x \in \operatorname{dom}\varphi := \bigcap_i \operatorname{dom}\varphi_i$. A rational map $\varphi : X \dashrightarrow Y$ is called *dominant* if the image of $\varphi$ is dense in $Y$, i.e. if $\overline{\varphi(\operatorname{dom}\varphi)} = Y$.

A rational map $\varphi : X \dashrightarrow Y$ induces a regular map $\operatorname{dom}\varphi \to Y$. Let $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow Z$ be rational maps. Then $\psi$ might not be defined on $\operatorname{im}\varphi$. But if $\varphi$ is dominant, then $\psi \circ \varphi$ is well-defined on the non-empty open $\varphi^{-1}(\operatorname{dom}\psi)$.

**Definition 2.35.** Let $X$ be an affine algebraic variety and $Y$ an affine algebraic set. A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs $(U, \varphi_U)$, where $U \subseteq X$ is nonempty open, $\varphi_U : U \to Y$ is regular, and $(U, \varphi_U) \sim (V, \varphi_V)$ if and only if $\varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}$. The rational map is dominant if for some (and therefore all) $(U, \varphi_U)$ one has $\overline{\varphi_U(U)} = Y$.

**Remark 2.36.** The relation in definition 2.35 is an equivalence relation. Indeed, if $(U, \varphi_U) \sim (V, \varphi_V) \sim (W, \varphi_W)$, then $\varphi_U|_{U \cap W}$ and $\varphi_W|_{U \cap W}$ are regular maps that agree on the non-empty open $U \cap V \cap W$, hence they are equal by proposition 2.32.

The two above definitions are equivalent: If $\varphi$ is regular in the sense of 2.34, then $[(\operatorname{dom}\varphi, \varphi)]$ defines a regular map as in 2.35. Conversely, if an equivalence class $\{(U_i, \varphi_i)\}_i$ is given, then the map $\bigcup_i U_i$, $x \mapsto \varphi_i(x)$ for any $i$ with $x \in U_i$ is regular, i.e. a rational map as in 2.34. Clearly, the notion of dominance is preserved by these identifications.

One can compose dominant rational maps $\varphi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow Z$ by setting

$$[(U, \varphi_U)] \circ [(V, \varphi_V)] := [(\varphi_U^{-1}(V), \psi_V \circ \varphi_U|_{\varphi_U^{-1}(V)})]$$

Write $\operatorname{Mor}_{\text{rat}}(X, Y)$ for the set of rational morphisms $X \dashrightarrow Y$. Lecture 8 Nov 12, 2025

**Definition 2.37.** Let $\varphi : X \dashrightarrow Y$ be dominant. In the same way as for regular maps, we define

$$\varphi^* : \mathrm{Mor}_{\mathrm{rat}}(Y, \mathbb{A}^1) \to \mathrm{Mor}_{\mathrm{rat}}(X, \mathbb{A}^1), \qquad f \mapsto f \circ \varphi.$$

**Proposition 2.38.** *Let $X$ be an affine algeraic variety. Then $\mathrm{Mor}_{rat}(X, \mathbb{A}^1)$ is a field with the operations $(U, f) * (V, g) := (U \cap V, f|_{U \cap V} + g|_{U \cap V})$ for $* \in \{+, -, \cdot\}$ and $(U, f)^{-1} = (U \setminus V(f), \frac{1}{f})$. Moreover, $\mathrm{Mor}_{rat}(X, \mathbb{A}^1) \cong K(X)$ as fields.*

*Proof.* It is clear that the given operations are well-defined and make $\mathrm{Mor}_{\mathrm{rat}}(X, \mathbb{A}^1)$ a field. The equivalence of definitions 2.34 and 2.35 provides a field isomorphism $K(X) \to \mathrm{Mor}_{\mathrm{rat}}(X, \mathbb{A}^1)$, $f \mapsto (\operatorname{dom} f, f)$. □

**Corollary 2.39.** *If $\varphi : X \dashrightarrow Y$ is a dominant rational map between affine varieties, we get a $K$-homomorphism of fields $\varphi^* : K(Y) \to K(X)$, $f \mapsto f \circ \varphi$.*

Recall that for regular maps, we had in 2.13 an equivalence between algebraic sets + regular maps, and reduced f.g. $K$-algebras + $K$-algebra homomorphisms. In the case of rational maps, we get similarly

**Theorem 2.40.** *$\varphi \mapsto \varphi^*$ is a bijection $\{\varphi \in \mathrm{Mor}_{rat}(X, Y) \mid \varphi$ dominant$\}$ to $\mathrm{Hom}_K(K(Y), K(X))$. This assignment is functorial, and induces an equivalence of categories*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{affine algebraic varieties +}\\ \text{dominant rational maps}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{field extensions } L/K \text{ of finite transcendence degree +}\\ K\text{-linear homomorphisms}\end{array}\right\} \\
X & \mapsto & K(X) \\
\varphi & \mapsto & \varphi^*
\end{array}$$

*Proof.* To show that $\varphi \mapsto \varphi^*$ is a bijection, define an inverse by assigning to $f : K(Y) \to K(X)$ the morphism $(f(y_1), \ldots, f(y_m))$. Everything else is clear. □

**Definition 2.41.** A dominant rational map $\varphi : X \dashrightarrow Y$ is called a *birational equivalence* (and $X$ and $Y$ are called *birational* or *birationally equivalent*) if there exists a rational dominant map $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$ as rational maps.

**Proposition 2.42.** *Let $X, Y$ be affine algebraic varieties. The following statements are equivalent:*

*(i) $X$ and $Y$ are birational.*

*(ii) $K(X) \cong K(Y)$.*

*(iii) There exist non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$ are isomorphic (in the sense of regular maps).*

*Proof.* (i)⇔(ii) follows from 2.40 and (iii)⇒(i) from the definition of rational function as regular functions on some open. Now assume (i), i.e. that there exists a birational equivalence $\varphi = (U, \varphi_U) : X \dashrightarrow Y$ with inverse $\psi = (V, \psi_V)$. Then $\varphi = (U \cap \psi^{-1}(V), \varphi_U|_{\ldots})$ and $\psi = (V \cap \varphi^{-1}(U), \psi_V|_{\ldots})$ are the required isomorphisms $U \cap \psi^{-1}(V) \cong V \cap \varphi^{-1}(U)$. □ Lecture 9 Nov 17, 2025

**Remark 2.43.** An affine algebraic variety $X$ is called *rational*, if $X$ is birational to $\mathbb{A}^k$ for some $k$. Equivalently, $K(X)/K$ is a purely transcendental field extension. For example, in the exercises we proved that $S = V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$ is rational[1].

**Theorem 2.44.** *Every affine algebraic variety $X$ is birational to some hypersurface, i.e. a variety $V(f) \subseteq \mathbb{A}^n$ for some irreducible $f \in K[x_1, \ldots, x_n]$.*

[1]Our proof works in char $K \neq 2$, but otherwise $\sqrt{(x^2 + y^2 - 1)} = (x + y - 1)$, so even $A(S) \cong K[x]$

*Proof.* For simplicity, we only consider the case $\operatorname{char} K = 0$. Since $K(X)/K$ is finitely generated, by basic algebra $K(X)/K$ factors as a purely transcendental extension followed by a finite one $K(X)/K(t_1, \ldots, t_d)/K$. Since everything is separable, $K(X)/K(t_1, \ldots, t_d)$ is generated by a primitive element, i.e. $K(X) = K(t_1, \ldots, t_d, \alpha)$ with $\alpha$ algebraic over $K(t_1, \ldots, t_d)$. Let wlog $f \in K[t_1, \ldots, t_d]$ be the minimal polynomial of $\alpha$. Then $K(X) \cong \operatorname{Frac} K[t_1, \ldots, t_d, s]/(f(s)) \cong K(V(f))$ as desired. □

**Remark 2.45.** Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. Then $X \times Y \subseteq \mathbb{A}^{n+m}$ is also affine algebraic, given by the same equations, now considered in $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Furthermore, if $X, Y$ are irreducible, then so is $X \times Y$ (Exercise). This is the product in the category of affine algebraic sets (resp. varieties), i.e. for regular maps $\varphi : Z \to X$, $\psi : Z \to Y$, there exists a unique regular map $Z \to X \times Y$. Therefore $A(X \times Y) = A(X) \otimes_K A(Y)$.

# 3 Projective Varieties

**Definition 3.1.** *Projective $n$-space* over $K$ is given by $\mathbb{P}^n_K := \mathbb{A}^{n+1} \setminus \{0\}/\sim$, where $x \sim y$ if $x = \lambda y$ for some $\lambda \in K$. We notate the equivalence class of $x$ by $x = [x_0 : x_1 : \ldots : x_n]$, called the *homogeneous coorinates* of $x$.

Note that points in $\mathbb{P}^n$ correspond to one-dimensional linear subspaces of $\mathbb{A}^{n+1}$.

**Remark 3.2.** We would like to define projective algebraic sets as zeroes of polynomials as in the affine case. But this is not well-defined, because evaluation of a polynomial need not respect the equivalence relation of 3.1. For example, let $f = x_1^2 - x_0 \in K[x_0, x_1]$. Then $f(1,1) = 0$ and $f(-1,-1) = 2$, but $[1 : 1] = [-1 : -1] \in \mathbb{P}^1_K$.

This problem can be solved by only considering *homogeneous polynomials*. For such a polynomial

$$f = \sum_{k_0+\ldots+k_n=d} a_{k_0,\ldots,k_n} x_0^{k_0} \cdots x_n^{k_n},$$

we have $f(\lambda x) = \lambda^d f(x)$, so $f(x) = 0$ is well-defined for $x \in \mathbb{P}^n$.

**Definition 3.3.** An ideal $I \subseteq K[x_0, \ldots, x_n]$ is called homogeneous if it can be generated by homogeneous polynomials.

**Remark 3.4.**
(i) If $I$ is homogeneous and $f \in K[x_0, \ldots, x_n]$, write $f = f_0 + f_1 + \ldots + f_d$ with $f_i$ homogeneous of degree $i$. Then $f \in I$ if and only if $f_i \in I$ for all $i$. (Say $I = (g_1, \ldots, g_n)$ with $g_i$ homogeneous, write $f = \sum g_i h_i$. Then $f_d = \sum g_i (h_i)_{d-\deg g_i} \in I$.)
(ii) If $I_1, I_2$ are homogeneous ideals, then so are $I_1 + I_2, I_1 I_2, I_1 \cap I_2, \sqrt{I_1}$. (For "$\cap$", find an arbitrary generating set and then use (i), $\sqrt{-}$ is exercise.)

**Definition 3.5.** Let $f_1, \ldots, f_k \in K[x_0, \ldots, x_n]$ be homogeneous. Then

$$V(f_1, \ldots, f_k) := V^p(f_1, \ldots, f_k) := \{x \in \mathbb{P}^n_K \mid f_i(x) = 0 \text{ for all } i\}$$

is called a *projective algebraic set*. In the same way, for a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$, set

$$V(I) := V^p(I) := \{x \in \mathbb{P}^n_K \mid f(x) = 0 \text{ for all } f \in I \text{ homogeneous}\}.$$

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**Example 3.6.** We have $V^p(0) = \mathbb{P}^n$, $V^p(1) = \emptyset$. Further, every point $x = [x_0 : \ldots : x_n]$ forms a projective algebraic set, since $V^p(a_i x_j - a_j x_i)_{i,j} = \{x\}$.

**Remark 3.7.** Just as in 1.5, projective algebraic sets are closed under arbitrary intersections and finite unions.

**Definition 3.8.** The *Zariski topology* on $\mathbb{P}^n$ is defined as the topology which closed sets the projective algebraic sets. On a projective algebraic set $X \subseteq \mathbb{P}^n$, the induced subspace topology is also called the Zariski topology on $X$.

**Definition 3.9.** A projective algebraic variety is an irreducible projective algebraic set.

For a subset $X \subseteq \mathbb{P}^n$ we may set

$$I^p(X) := \{f \in K[x_0, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

$V^p$ and $I^p$ enjoy many of the same properties as in the affine case. In particular

**Proposition 3.10.** *(i) For a subset $X \subseteq \mathbb{P}^n$, $V^p(I^p(X)) = \overline{X}$.*
*(ii) For a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ with $(x_0, \ldots, x_n) \not\subseteq I$, $I^p(V^p(I)) = \sqrt{I}$.*

*(iii) A projective algebraic set $X$ is a variety if and only if $I^p(X)$ is a prime ideal.*

*Proof.* (i) and (ii) as in the affine case. For (iii), we need the following

Claim: A homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ is prime if and only if for all homogeneous $f, g \in L[x_0, \ldots, x_n]$ with $fg \in I$, one has $f \in I$ or $g \in I$.

Indeed, suppose $I$ were not prime, and let $f, g \notin I$ such that $fg \in I$. Let $d_0, e_0$ be maximal w.r.t. $f_{d_0}, g_{e_0} \notin I$. Then $(fg)_{d_0+e_0} = f_{d_0}g_{e_0} + \sum_{i+j=d_0+e_0, i\neq d_0} f_i g_j$. The left hand side is in $I$ by remark 3.4, and the sum by the maximality assumption. Hence $f_{d_0}g_{e_0} \in I$. □

**Definition 3.11.** Let $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the canonical projection.

(i) An algebraic set $X \subseteq \mathbb{A}^{n+1}$ is called a *cone* if $0 \in X$ and $x \in X$ implies $\lambda x \in X$ for all $\lambda \in K$.
(ii) Given a cone $X \subseteq \mathbb{A}^{n+1}$, its *projectivization* is $\mathbb{P}(X) := \pi(X \setminus \{0\})$.
(iii) For a projective algebraic set $X \subseteq \mathbb{P}^n$, its *cone* is $C(X) := \{0\} \cup \pi^{-1}(X)$

Note that $\mathbb{P}(X)$ and $C(X)$ are projective resp. affine algebraic sets. Indeed, for a homogeneous ideal $S \subseteq K[x_0, \ldots, x_n]$ we have $\mathbb{P}(V(S)) = V^p(S)$ and $C(V^p(S)) = V(S)$. It remains to show that all cones are of this form, which is

**Proposition 3.12.** *Let $X \subseteq \mathbb{A}^{n+1}$ be a cone. Then $I(X)$ is a homogeneous ideal.*

*Proof.* For $f = f_0 + \ldots + f_d$ and $x \in X$ we have $0 = f(\lambda x) = \sum_i \lambda^i f_i(x)$. As the 0 polynomial function in $\lambda$, since $K$ is infinite we must have $f_i(x) = 0$ for all $i$. □

The next goal is to prove a projective version of the Nullstellensatz 1.13. Lecture 11 Nov 24, 2025

**Definition 3.13.** The (homogeneous maximal) ideal $I_0 := (x_0, \ldots, x_n) \subseteq K[x_0, \ldots, x_n]$ is called the *irrelevant* ideal.

Note that $V^p(I_0) = \emptyset$, so in general $I^p(V^p(I)) \neq I$ for radical homogeneous ideals $I$. But in some sense this is the only problematic case:

**Proposition 3.14** (Projective Nullstellensatz)**.** *For any homogeneous ideal $J \subseteq K[x_0, \ldots, x_n]$ with $\sqrt{J} \neq I_0$ we have $I^p(V^p(J)) = \sqrt{J}$.*

*Proof.* The inclustion "$\supseteq$" is clear. We have

$$\begin{aligned} I^p(V^p(J)) &= \langle f \in K[x_0, \ldots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in V^p(J) \rangle \\ &= \langle f \in K[x_0, \ldots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in \overline{V(J) \setminus \{0\}} \rangle \end{aligned}$$

Now $V(J) \neq \{0\}$, otherwise $\sqrt{J} = I(V(J)) = I_0$, hence $\overline{V(J) \setminus \{0\}} = V(J)$ (since then either $V(J) = \emptyset$ or $V(J)$ contains a line through 0). Then $I^p(V^p(J))$ is generated by homogeneous polynomials in $I(V(J)) = \sqrt{J}$. But $\sqrt{J}$ is homogeneous itself, so $I^p(V^p(J)) = \sqrt{J}$ as well. □

**Corollary 3.15.** *(i) If $I \subseteq K[x_0, \ldots, x_n]$ is a homogeneous ideal, then $V^p(I) = \emptyset$ if and only if $I_0 \subseteq \sqrt{I}$, if and only if $\sqrt{I} = I_0$ or $I = (1)$.*
*(ii) If $V^p(J) \neq \emptyset$, then $I^p(V^p(J)) = \sqrt{J}$.*
*(iii) $I^p$ and $V^p$ define inclusion-reversing bijections*

$$\begin{aligned} \{\textit{projective algebraic sets in } \mathbb{P}^n\} &\rightleftarrows \{\textit{radical hom. ideals } I_0 \neq J \subseteq K[x_0, \ldots, x_n]\} \\ \{\textit{projective algebraic varieties in } \mathbb{P}^n\} &\rightleftarrows \{\textit{prime hom. ideals } I_0 \neq J \subseteq K[x_0, \ldots, x_n]\} \\ \{\textit{points in } \mathbb{P}^n\} &\rightleftarrows \{\textit{maximal hom. ideals } I_0 \neq J \subseteq K[x_0, \ldots, x_n]\} \end{aligned}$$

*(iv)* $I^p(\mathbb{P}^n) = 0$*, and* $\mathbb{P}^n$ *is a variety.*

**Remark 3.16.** Let $U_i := D(x_i) = \{x \in \mathbb{P}^n \mid x_i \neq 0\} = \{x \in \mathbb{P}^n \mid x_i = 1\}$. Leaving out the $i$-th coordinate in the last presentation yields a homeomorphism $\iota_i : \mathbb{A}^n \to U_i$ (even an isomorphism of varieties, cf. later).

Therefore $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ is an open cover of projective space by $n+1$ copies of $\mathbb{A}^n$.

**Definition 3.17.** (i) For a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$, its dehomogenization is $f^i := f(1, x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$. For a homogeneous ideal $J \subseteq K[x_0, \ldots, x_n]$, write $J^i = \{f^i \mid f \in J\} \subseteq K[x_1, \ldots, x_n]$. In other words, these are the images of $f$, resp. $J$, under the natural map $K[x_0, \ldots, x_n] \mapsto K[x_0, \ldots, x_n]/(x_0 - 1) \cong K[x_1, \ldots, x_n]$.

(ii) For $0 \neq f \in K[x_1, \ldots, x_n]$ of $\deg f = d$, its homogenization is $f^h := x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in K[x_0, \ldots, x_n]$. For an ideal $J \subseteq K[x_0, \ldots, x_n]$, write $J^h$ for the ideal of $K[x_0, \ldots, x_n]$. generated by $f^h$, $f \in J$

For example, if $f = 1 + X_1 + X_2 + X_1^2$, then $f^h = X_0^2 + X_0X_1 + X_0X_2 + X_1^2$., and $(f^h)^i = f$. Note that in general, $J^h$ is not generated by homogenizations of generators of $J$, e.g. $J = \langle 1 + x_1 - x_2, x_1 - x_2 \rangle$. Lecture 12 Nov 26, 2025

**Proposition 3.18.** *The bijection* $\iota : \mathbb{A}^n \to U_0$ *as in remark 3.16 is a homeomorphism.*

*Proof.* It is clear that $i$ is bijective, so we need to check that $i, i^{-1}$ are continuous. Let $Z \subseteq U_0$ be a closed subset, say $Z = U_0 \cap V^p(f_1, \ldots, f_m)$ for homogeneous polynomials $f_1, \ldots, f_m \in K[x_0, \ldots, x_n]$. Then

$$i^{-1}(Z) = \{x \in \mathbb{A}^n \mid 0 = f_j(i(x)) = f_j((1 : x]) = f_j^i(x)\, \forall j\} = V(f_1^i, \ldots, f_m^i).$$

Hence $i$ is continuous. Now let $W = V(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$ be closed. Then $i(W)$ consists of those $x \in U_0$ with $f_j(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = 0$ for all $j$. Since $x_0 \neq 0$, this is exactly the case if all $x_0^{\deg f_j} f_j(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = 0$, i.e. if $f_j^h(x) = 0$. Hence $i(W) = V^p(f_1^h, \ldots, f_m^h) \cap U_0$ is closed in $U_0$. ☐

**Remark 3.19.** We identify $\mathbb{A}^n$ with $U_0 \subseteq \mathbb{P}^n$. Then as subsets of $\mathbb{P}^n$ we have $\overline{V^a(I)} = V^p(I^h)$ and $V^p(I^h) \cap U_0 = V(I)$ (Exercise).

**Definition 3.20.** A *quasi-projective* set (variety) is an open subset of a projective algebraic set (variety).

**Remark 3.21.** Any (quasi-)affine algebraic set $V$ is quasi-projective, since the closure of $V$ in $\mathbb{P}^n \supseteq U_0 \cong \mathbb{A}^n$ may be computed by first taking the closure in $\mathbb{A}^n$ and then the closure of the resulting algebraic set in $\mathbb{P}^n$.

For an affine algebraic set $X$, an important construction was the ring of regular functions $A(X)$. We now want to consider the corresponding projective variant.

**Definition 3.22.** Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set. The $K$-algebra

$$S(X) := K[x_0, \ldots, x_n]/I^p(X)$$

is called the *homogeneous coordinate ring* of $X$. It is a graded ring, namely

$$S(X) = \bigoplus_{d \in \mathbb{N}} K[x_0, \ldots, x_n]_d/(I^p(X) \cap K[x_0, \ldots, x_n]_d),$$

where $K[x_0, \ldots, x_n]_d$ denotes the homogeneous polynomials of degree $d$.

Homogeneous elements of $S(X)$ can be considered as functions $X \to K$.

**Definition 3.23.** Let $X$ be a quasi-projective algebraic set. and let $\varphi : X \to K$ be a map. Then $\varphi$ is *regular* at $a \in X$ if locally at $a$ one has $\varphi = \frac{f}{g}$, $f, g$ homogeneous, $\deg(f) = \deg(g)$ and $g(a) \neq 0$. $\varphi$ is regular on $X$ if it is regular at every point of $X$.

**Definition 3.24.** Let $X \subseteq \mathbb{P}^n, Y \subseteq \mathbb{P}^m$ be quasi-projective algebraic sets. A map $\varphi : X \to Y$ is called *regular* if for every $a \in X$ there exists $a \in U_a \subseteq X$ open such that $\varphi|_{U_a} = (F_0, \dots, F_m)$ with $F_i$ homogeneous of the same degree, such that for any $x \in U_a$, not all $F_i(x) = 0$.

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**Example 3.25.** $\mathbb{P}^2 \to \mathbb{P}^1$, $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ is not a morphism, because the image of $[0 : 0 : 1]$ is not defined. However, the restriction of the above rule to $V(x_2)$ is a well-defined regular map.

A regular map is an isomorphism if it has a regular inverse, see exercises for (counter)examples.

For a quasi-projective set $X$, write $A(X)$ for the $K$-algebra of regular functions. Then $A(\mathbb{P}^n) = K$ (see exercises). If $U \subseteq X$ is open, set $\mathcal{O}_X(U)$ for the $K$-algebra of regular functions on $U$. Intuitively, we should have $\mathcal{O}_{\mathbb{P}^n}(U_0) \cong A(\mathbb{A}^n) \cong K[x_1, \dots, x_n]$.

**Remark 3.26.** Similarly as in the affine case, one can show that a regular map is continuous.

**Theorem 3.27.** *Let $X \subseteq \mathbb{A}^n$ be a quasi-affine algebraic set. Write $\widetilde{X} := i(X) \subseteq U_0 \subseteq \mathbb{P}^n$ for the corresponding quasi-projective algebraic set. Then for $V \subseteq \widetilde{X}$ open, we have an isomorphism of $K$-algebras*

$$i^* : \mathcal{O}_{\widetilde{X}}(V) \to \mathcal{O}_X(i^{-1}(V)), \quad f \mapsto f \circ i,$$

*and for $W \subseteq X$ open, we have a $K$-algebra isomorphism*

$$j^* : \mathcal{O}_X(W) \to \mathcal{O}_{\widetilde{X}}(j^{-1}(W)), \quad f \mapsto f \circ j.$$

*Proof.* It is clear that these maps are mutually inverse, so one only has to check well-definedness, i.e. that $i^*(f), j^*(f)$ are regular. Let $h \in \mathcal{O}_{\widetilde{X}}(V)$ be regular, and $P \in i^{-1}(V)$. Then $h = \frac{f}{g}$ around $i(P)$, so $i^*(h) = \frac{f^i}{g^i}$ around $P$, and $h$ is regular at $P$.

Similarly, let $h \in \mathcal{O}_X(W)$ and $Q \in j^{-1}(W)$. Then $h$ is regular at $j(Q)$, so $h = \frac{f}{g}$ around $j(Q)$. Then near $Q$ one has

$$h \circ j = \frac{f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})} = \frac{x_0^{\deg f + \deg g}}{\deg f + \deg g} \frac{f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{g(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})} = \frac{x_0^{\deg g} f^h}{x_0^{\deg f} g^h},$$

so $h$ is regular at $Q$. $\square$

In particular for $X = \mathbb{A}^n$ we see $\mathcal{O}_{\mathbb{P}^n}(U_0) = \mathcal{O}_{U_0}(U_0) \cong A(\mathbb{A}^n) = K[x_1, \dots, x_n]$.

**Remark 3.28.** Let $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ be the standard open cover, $X \subseteq \mathbb{P}^n$ quasi-projective, and let $\varphi : X \to K$ be a function. Then $\varphi$ is regular if and only if all $\varphi|_{U_i}$ are regular.

We give a second, equivalent definition of regular maps.

**Definition 3.29.** Let $X, Y$ be qausi-projective algebraic sets. A map $\varphi : X \to Y$ is called a morphism or regular map if $\varphi$ is continuous and preserves regular functions, that is for every $U \subseteq Y$ open and $f \in \mathcal{O}_Y(U)$, the function $\varphi^*(f) := f \circ \varphi$ is regular.

In particular, we get a $k$-algebra morphism $\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$. Note that by Theorem 3.27, the maps $i, j$ are regular in the sense of this definition.

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**Remark 3.30.** (i) $\mathrm{id} : X \to X$ is a morphism.

(ii) It follows from the definition that the composition of morphisms is a morphism.

(iii) Being a morphism is a local property: Let $\varphi : X \to Y$ be a map. Then $\varphi$ is a morphism if and only if for every $P \in X$ there is an open neighbourhood $P \in U_P \subseteq X$ s.t. $\varphi|_{U_P} \to Y$ is a morphism. In particular, if $X = \bigcup_i U_i$ is an open cover, then $\varphi$ is a morphism if and only if all the $\varphi|_{U_i}$ are.

We now have to convince ourselves that this definition agrees with our previous notions of regular maps.

**Theorem 3.31.** *Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be quasi-projective algebraic sets, and let $\varphi : X \to Y$ be a map. Then the following are equivalent:*

*(i) $\varphi$ is a morphism (in the sense of definition 3.29)*

*(ii) $\varphi$ is locally given by regular functions, i.e. for every $p \in X$ there is an open neighbourhood $p \in U \subseteq X$ and $h_1, \ldots, h_m \in \mathcal{O}_X(U)$ with no common zero on $U$, s.t. $\varphi = [h_0 : \ldots : h_m]$ on $U$.*

*(iii) $\varphi$ is locally given by homogeneous polynomials of the same degree and no common zeroes (cf. definition 3.24).*

*Proof.* (1)⇒(2): Let $p \in X$, and consider $\varphi(p) =: y =: [y_0 : \ldots : y_m]$. Wlog we may assume $y_0 \neq 0$. Then $y \in U_0$, and

$$\varphi(x) = [1 : \varphi^*(\tfrac{y_1}{y_0})(x) : \ldots : \varphi^*(\tfrac{y_m}{y_0})]$$

on $\varphi^{-1}(U_0)$, with $\varphi^*(\frac{y_i}{y_0})$ regular by assumption.

(2)⇒(3): By possibly shringking $U$, we may assume $h_i = \frac{F_i}{G_i}$ on $U$, with $F_i, G_i$ homogeneous polynomials of the same degree. Multiplying all functions with $G_1 \cdots G_m$ clears denominators, so yields the desired presentation.

(3)⇒(1): $\varphi$ is continuous by remark 3.26. Let $U \subseteq Y$ be open, and $h : U \to K$ regular. Let $P \in \varphi^{-1}(U)$. There exists a neighbourhood $\varphi(P) \in V \subseteq U$ such that $h = \frac{f}{g}$ on $V$, with $f, g$ homogeneous polynomials of the same degree, and there is $W \subseteq X$ s.t. $\varphi|_W = [F_0 : \ldots : F_m]$ with homogeneous polynomials of the same degree $F_i$. For $x \in W \cap \varphi^{-1}(V)$ one has

$$\varphi^*(h)(x) = h([F_0(x) : \ldots : F_m(x)]) = \frac{f(F_0(x), \ldots, F_m(x))}{g(F_0(x), \ldots, F_m(x))},$$

which is again a quotient of two homogeneous polynomials of the same degree, and $\varphi^*(h)$ is regular at $p$. □

**Proposition 3.32.** *Let $X \subseteq \mathbb{P}^n$ be quasi-projective and $Y \subseteq \mathbb{A}^m$ quasi-affine. Then $\varphi : X \to Y$ is a morphism if and only if $\varphi = (f_1, \ldots, f_m)$ for $f_1, \ldots, f_m \in A(X)$.*

*Proof.* If $\varphi$ is a morphism, then, as before, $\varphi = (\varphi^*(y_1), \ldots, \varphi^*(y_n))$. For the converse, by remark 3.30(iii) and theorem 3.27 we may assume that $X$ is quasi-affine. Then we know that $\varphi$ is continuous, and that the composition of regular maps is regular. □

**Corollary 3.33.** *Let $X$ be quasi-projective. Then $f : X \to \mathbb{A}^1$ is a morphism if and only if $f$ is a regular function on $X$.*

# 4 The Segre Embedding and Closed Morphisms

Our next goal is to understand the product of quasi-projective algebraic sets. Lecture 15 Dec 8, 2025

**Remark 4.1.** The obvious map $\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^{n+m}$ is regular and bijective, but not a homeomorphism. However, it allows us to define products of (quasi-)affine algebraic sets, see exercises. Similarly, if $U \subseteq \mathbb{A}^n, V \subseteq \mathbb{A}^m$ are open, then so is $U \times V \subseteq \mathbb{A}^{n+m}$, by taking complements.

**Proposition 4.2.** *If $X, Y$ are quasi-affine, then $X \times Y$ as in remark 4.1 satisfies the universal property of products, that is: The projections $p_1, p_2 : X \times Y \to X, Y$ are morphisms, and for any morphisms $f_1, f_2 : Z \to X, Y$, there exists a unique morphism $f : Z \to X \times Y$ s.t. $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.*

*Proof.* Everything is clear from proposition 3.32. □

**Remark 4.3.** An analogous construction for projective spaces does not work, since the naive map $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+1}$ is not well-defined.

**Definition 4.4** (Segre Embedding)**.** Let $N = (n+1)(m+1) - 1$ and denote coordinates of $\mathbb{P}^N$ by $z_{ij}$, $0 \leq i \leq n$, $0 \leq j \leq m$. The map

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N, \quad (x, y) \mapsto [x_i y_j]_{ij}$$

is called the *Segre embedding*

One easily checks that this is a well-defined map.

**Proposition 4.5.** *The Segre embedding $\sigma$ is injective and $\Sigma := \sigma(\mathbb{P}^n \times \mathbb{P}^m)$ is closed in $\mathbb{P}^N$. (That is, we can and will identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma$.)*

*Proof.* Let $z = \sigma(x, y) \in \Sigma$. Say $z_{ij} \neq 0$, then $x_i, y_j \neq 0$. Hence $x = [z_{kj}]_k$ and $y = [z_{ik}]_k$, so $\sigma$ is injective. Note that $z \in \Sigma$ if and only if the rank of the matrix $M_z = (z_{ij})_{ij}$ (defined up to scalar multiples) is 1, if and only if all $2 \times 2$-minors of $M_z$ are 0. But this last condition is given by the zero set of polynomials of $z_{ij}$: $\Sigma = V^p(\{z_{ij} z_{kl} - z_{il} z_{jk}\}_{0 \leq i,k \leq n, 0 \leq j,l \leq m})$ □

**Definition 4.6.** We endow $\mathbb{P}^n \times \mathbb{P}^m$ with the Zariski topology induced from $\sigma : \mathbb{P}^n \times \mathbb{P}^m \cong \Sigma$.

**Proposition 4.7.** *The closed subsets $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ are precisely the sets of zeroes of polynomials $\{f_i\}_{i \in I}$ with $f_i \in K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ bihomogeneous, i.e. homogeneous w.r.t. $x_i$ and homogeneous w.r.t. $y_j$.*

Note that these are exactly the types of polynomials such that being 0 is well-defined on $\mathbb{P}^n \times \mathbb{P}^m$, so this condition makes sense.

*Proof.* Assume $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed, i.e. $\sigma(Z) \subseteq \Sigma \subseteq \mathbb{P}^N$ is closed. Hence $\sigma(Z) = V^p(f_\alpha(z_{ij}))_\alpha$ with $f_\alpha \in K[z_{ij}]$ homogeneous. Then

$$Z = \sigma^{-1}(V^p(f_\alpha(z_{ij})) = \{(x, y) \mid f_\alpha((x_i y_j)_{ij}) = 0, \ \forall \alpha\}$$

is of the desired form. Conversely, assume $Z = \{(x, y) \mid f_\alpha(x, y) = 0, \ \forall \alpha\}$ with $f_\alpha$ bihomogeneous. Then $\sigma(Z) = \{z \mid z_{ij} = x_i y_j, f_\alpha(x, y) = 0\}$. If $\deg_x f_\alpha = \deg_y f_\alpha$, then $f_\alpha$ is already a homogeneous polynomial in the $z_{ij}$ (non-uniquely); otherwise, say if $\deg_x f_\alpha > \deg_y f_\alpha$, then $f_\alpha = 0$ if and only if all of $y_j^{\deg_x f_\alpha - \deg_y f_\alpha} f_\alpha = 0$, so we are reduced to the first case. □

**Corollary 4.8.** *If $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ are closed, then $X \times Y$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$.* Lecture 16 Dec 10, 2025

*Proof.* $X \times Y = (\mathbb{P}^n \times Y) \cap (X \times \mathbb{P}^m)$. If $X = V^p(f_1, \ldots, f_n)$, then $X \times \mathbb{P}^m = V(f_1, \ldots, f_n)$ with $f_i$ considered in $K[x_i, y_j]$. $\square$

Similarly as in the affine case (remark 4.1), it follows that the product of opens is open, and that products of quasi-projective sets are quasi-projective.

Consider products of standard opens $U_i \times U_j \subseteq \mathbb{P}^n \times \mathbb{P}^m$. On the one hand, we should have $U_i \times U_j \cong \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$, on the other hand $\sigma(U_i \times U_j) = \Sigma_{ij} := \Sigma \cap U_{ij}$. These two descriptions indeed agree:

**Proposition 4.9.** *The composition of bijections $\mathbb{A}^n \times \mathbb{A}^m \cong U_i \times U_k \cong \Sigma_{ik}$ is an isomorphism of algebraic sets.*

*Proof.* Identify $\Sigma_{ik}$ with its image under the isomorphism $j : U_{ik} \to \mathbb{A}^N$. It is enough to show that the composition $\mathbb{A}^n \times \mathbb{A}^m \to j(\Sigma_{lk})$, and its inverse, are regular. Wlog take $l = k = 0$. Then the map is given by

$$(x, y) \mapsto ([1 : x_1 : \ldots : x_n], [1 : y_1 : \ldots : y_n] \mapsto [1 : y_1 : \ldots : x_n y_n] \mapsto [y_1 : \ldots : x_n y_n].$$

Now we see that all coordinates are given by polynomials, so the map is regular. Moreover, all the $x_i$ and $y_j$ occur as coordinates in the image, so the inverse is given by a collection of projections, and is regular as well. $\square$

**Corollary 4.10.** *$\Sigma$ has an open cover by subsets $\Sigma_{ij} \cong \mathbb{A}^{n+m}$, corresponding to $\mathbb{P}^n \times \mathbb{P}^m = \bigcup_{i,j} U_i \times U_j$.*

**Remark 4.11.** $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ quasi-affine, then we defined $X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m$ in remark 4.1. As in the proposition, this algebraic set can be identified with its image under $\mathbb{A}^n \times \mathbb{A}^m \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m \to \Sigma$ (since this map is just a restriction of the isomorphism above, hence still an isomorphism).

**Proposition 4.12.** *If $X, Y$ are quasi-projective algebraic sets, then $p_{1,2} : X \times Y \to X, Y$ are morphisms, and for any quasi-projective $Z$ and morphisms $f_1, f_2 : Z \to X, Y$, there exists a unique morphism $(f_1, f_2) : Z \to X \times Y$ s.t. $p_i \circ (f_1, f_2) = f_i$.*

*Proof.* Let wlog $x_0 = 1$. Then $\sigma(x, y) = [y_0 : \ldots : y_n : \ldots]$, so $p_2(x, y) = [y_0 : \ldots : y_n]$ is locally just the projection onto the first $n + 1$ coordinates, thus regular, and similarly for $p_1$.

So let $f_1, f_2$ be as in the statement. Since $X \times Y$ is the set-theoretic product, the only choice is $(f_1, f_2)(a) = (f_1(a), f_2(a))$, we have to check that this is a morphism. But since $\sigma, f_1, f_2$ are all locally given by homogeneous polynomials, it is clear that the same is true for $a \mapsto \sigma(f_1(a), f_2(a))$. $\square$

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**Theorem 4.13.** *Let $\varphi : X \to X', \psi : Y \to Y'$ be morphisms of algebraic sets. Then $(\varphi, \psi) : X \times Y \to X' \times Y'$ is also a morphism. If $\varphi$ and $\psi$ are isomorphisms, then so is $(\varphi, \psi)$.*

*Proof.* Abstract Nonsense. $\square$

**Definition 4.14.** Let $X$ be an algebraic set. The diagonal of $X$ is $\Delta_X := \{(a, a) \in X \times X \mid a \in X\} = \operatorname{im}(\mathrm{id}_X, \mathrm{id}_X)$, with diagonal morphism is $S_X : X \to \Delta_X$.

**Proposition 4.15.** *$\Delta_X \subseteq X \times X$ is a closed subset, and $S_X$ is an isomorphism.*

*Proof.* Let $X \subseteq \mathbb{P}^n$. Since $\Delta_X = \Delta_{\mathbb{P}^n} \cap X \times X$, it suffices to show that $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. Under the Segre embedding, we have $\sigma(\Delta_{\mathbb{P}^n}) = \Sigma \cap V^p(\{z_{ij} - z_{ji}\}_{ij})$, hence we are done. Further clearly $S_X^{-1} = p_1|_{\Delta_X}$. $\square$

**Definition 4.16.** Let $f : X \to Y$ be a morphism of algebraic sets. The *graph* of $f$ is $\Gamma_f := \{(a, f(a)) \in X \times Y \mid a \in X\} = \operatorname{im}(\mathrm{id}_X, f)$.

**Proposition 4.17.** *$\Gamma_f \subseteq X \times Y$ is a closed subset, and the natural map $X \to \Gamma_f$ is an isomorphism.*

*Proof.* $\Gamma_f = (f, \mathrm{id}_Y)^{-1}(\Delta_Y)$. □

**Definition 4.18.** A map $\varphi : X \to Y$ between topological spaces is called *closed* if for every closed subset $Z \subseteq X$ the image $\varphi(Z) \subseteq Y$ is closed in $Y$. An algebraic set $X$ is called *complete* if $p_2 : X \times Y \to Y$ is a closed map for every algebraic set $Y$.

**Theorem 4.19.** *Let $\varphi : X \to Y$ be a morphism where $X$ is projective. Then $\varphi$ is closed, in particular $\varphi(X) \subseteq Y$ is closed.*

**Remark 4.20.** The projective assumption is essential: Let $p_1 : \mathbb{A}^2 \supseteq V(XY - 1) \to \mathbb{A}^1$ be the projection onto the $X$-coordinate, then $\operatorname{im} p_1 = \mathbb{A}^1 \setminus \{0\}$ is not closed.

**Corollary 4.21.** *Let $X$ be a connected projective algebraic set. Then $A(X) = K$.*

*Proof.* By corollary 3.33 a regular function $\varphi \in A(X)$ can be seen as a morphism $\varphi : X \to \mathbb{A}^1$. By theorem 4.19 $\operatorname{im} \varphi$ is closed in $\mathbb{A}^1$. But $\operatorname{im} \varphi = \mathbb{A}^1$ is impossible, since then $X \to \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ would have non-closed image, contradicting theorem 4.19. Therefore, $\operatorname{im} \varphi$ is finite, and in fact a singleton, since $X$ is connected. □

**Corollary 4.22.** *More generally, let $X$ be a projective algebraic set and $Y$ quasi-affine. Then any morphism $\varphi : X \to Y$ is constant.*

*Proof.* By corollary 4.21, all coordinate functions $X \xrightarrow{\varphi} Y \xrightarrow{p_i} \mathbb{A}^1$ are constant. □

**Corollary 4.23.** *The image of a morphism $\mathbb{A}^1 \to \mathbb{A}^n$ is closed.*

*Proof.* Exercise. □

**Example 4.24.** By corollary 4.23, the sets $\{(t^2, t^3) \mid t \in K\}$, $\{(t^3, t^4, t^5) \mid t \in K\}$ from exercise sheet 1 are affine.

*Proof.* (of theorem 4.19) Let $Z \subseteq X$ be closed. Then $Z$ is projective, and $Z \hookrightarrow X \to Y$ is regular. Hence, it is enough to show that $\varphi(X) = p_2(\Gamma_\varphi)$ is closed in $Y$. By proposition 4.17, the result thus follows from □

**Theorem 4.25.** *Every projective algebraic set $X \subseteq \mathbb{P}^n$ is complete.* Lecture 18 Dec 17, 2025

*Proof.* Let $Y \subseteq \mathbb{P}^m$ be an algebraic set and $Z \subseteq X \times Y$ be closed. Then $Z$ is also closed in $\mathbb{P}^n \times Y$, and if $p_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is the projection, then $p_2(Z) = p_2|_{X \times Y}(Z)$. So it suffices to prove the theorem for $X = \mathbb{P}^n$.

So let $Z \subseteq \mathbb{P}^n \times Y$ be closed and write $Z = (\mathbb{P}^n \times Y) \cap \widetilde{Z}$ with $\widetilde{Z} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ closed. It suffices to show that $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed, for then

$$p_2|\mathbb{P}^n \times Y(Z) = p_2(Z) = p_2(\mathbb{P}^n \times Y) \cap p_2(\widetilde{Z}) = Y \cap p_2(\widetilde{Z})$$

is closed in $Y$.

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be closed. To show that $p_2(Z)$ is closed, we use the following

**Lemma 4.26.** *Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal. Then $V^p(I) = \emptyset$ if and only if $K[x_0, \ldots, x_n]_s \subseteq I$ for some $s \geq 0$.*

*Proof.* Indeed, by the projective Nullstellensatz, both conditions are equivalent to $(x_0, \ldots, x_n) \subseteq \sqrt{I}$. □

By proposition 4.7, we may write $Z$ as the sof of common zeroes of biholomorphic polynomials $f_1, \ldots, f_r$, by the proof of the proposition we may assume that all $f_i$ have the same degree $d$ in both $x_i$'s and $y_j$'s. Fix a point $a \in \mathbb{P}^m$. Then $a \in p_2(Z)$ iff $V^p(f_1(x,a), \ldots, f_r(x,a)) \neq \emptyset$, which by the lemma is equivalent to $K[x_0, \ldots, x_n]_s \nsubseteq (f_1(x,a), \ldots, f_r(x,a))$ for all $s \geq 0$. This is clear for $s < d$, for $s \geq d$ define $T_s := \{a \in \mathbb{P}^m \mid K[x_0, \ldots, x_n]_s \nsubseteq (f_1(x,a), \ldots, f_r(x,a))\}$. Then $p_2(Z) = \bigcap_{s \geq d} T_s$ and it is enough to show that all $T_s$, $s \geq d$ are closed in $\mathbb{P}^m$.

Note that $y \in T_s$ is equivalent to the $K$-linear map

$$\varphi : K[x_0, \ldots, x_n]_{s-d}^r \to K[x_0, \ldots, x_n]_s, \quad (h_1, \ldots, h_n) \mapsto f_1(x,y)h_1(x) + \ldots + f_r(x,y)h_r(x)$$

not being surjective. Let $v_1, \ldots, v_l$ be all monomials of degre $s - d$, and $w_1, \ldots, w_t$ all monomials of degree $s$, so that the $w_j$ form a basis of $K[x_0, \ldots, x_n]_s$, and $v_i e_j$ form a basis of $K[x_0, \ldots, x_n]_{s-d}^r$. Now $\varphi$ is not surjective iff $\dim \operatorname{im} \varphi < \dim K[x_0, \ldots, x_n]_s = \binom{n+s}{n} = t$, hence iff all $t \times t$ minors of the representative matrix of $\varphi$ vanish. But this is a system of polynomial equations in $y$, so $T_s$ is closed. □

**Example 4.27** (The Veronese Embedding)**.** Let $n, d > 0$ and $M_0, \ldots, M_N \in K[x_0, \ldots, x_n]$ all monomials of degree $d$, with $N = \binom{n+d}{d} - 1$. The Veronese embedding of degree $d$ is Lecture 19 Dec 22, 2025

$$v_d : \mathbb{P}^n \to \mathbb{P}^N, \qquad x \mapsto [M_0(x) : \ldots : M_N(x)].$$

For instance, if $n = 1$, then $v_d(x) = [x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d]$. $v_d$ is a well-defined morphism, so by theorem 4.19, $\operatorname{im} v_d$ is a projective subvariety of $\mathbb{P}^N$. We claim that $v_d : \mathbb{P}^n \to v_d(\mathbb{P}^n)$ is an isomorphism.

Indeed, for any $l$ there is a unique $i_l$ such that $M_{i_l}(x) = x_l^d$. Then $v_d(\mathbb{P}^n) = \bigcup_l v_d(\mathbb{P}^n) \cap \{y_{i_l} \neq 0\}$, and the inverse map consists of suitable projections from $\mathbb{P}^n$ to $n+1$ coordinates.

**Example 4.28.** In sheet 6, exercise 2 we prove that $v_2 : \mathbb{P}^1 \to V^p(Y^2 - XZ)$ is an isomorphism.

**Definition 4.29.** Let $A \in \mathrm{GL}_{n+1}(K)$ be an invertible matrix. We define the map

$$\varphi_A : \mathbb{P}^n \to \mathbb{P}^n, x \mapsto Ax.$$

Note $Ax = 0$ iff $x = 0$, so this is a well-defined isomorphism (with inverse $\varphi_{A^{-1}}$), called a projective transformation. One can show that projective transformations are the only automorphisms of $\mathbb{P}^n$.

**Remark 4.30.** Let $X$ be a connected projective algebraic set. We proved in corollary 4.22 that if $\varphi : X \to Y \subseteq \mathbb{A}^n$ is regular, then $\varphi$ is constant. It follows that if $X$ is isomorphic to a quasi-affine algebraic set, then $X$ is a point.

**Proposition 4.31.** *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and $F \in K[x_0, \ldots, x_n]$ a homogeneous non-constant polynomial. Then $X \setminus V^p(F)$ is isomorphic to a affine algebraic set.*

*Proof.* We already know that this holds if $F = X_0$ (then $X \setminus U_0 \cong \mathbb{A}^n$), hence also if $F$ is linear, since then there exists a projective transformation mapping $V^p(F)$ onto $V^p(X_0)$.

So let $\deg(F) = d > 1$. Under the Veronese embedding $v_d$, $V^p(F)$ gets mapped to the zero set of a linear polynomial, so the result follows again from the above. □

**Corollary 4.32.** *Let $X \subseteq \mathbb{P}^n$ be a connected projective algebraic set, containing more than one point. Let $F \in K[x_0, \ldots, x_n]$ be a nonconstant homogeneous polynomial. Then $X \cap V^p(F) \neq \emptyset$.*

*Proof.* Combine proposition 4.31 and remark 4.30. □

**Remark 4.33.** In the affine case, the above statement is not true: Consider two parallel lines in $\mathbb{A}^2$.

# 5 Dimension

**Remark 5.1.** Recall from section 1 that for an affine algebraic set $X \subseteq \mathbb{A}^n$, we have $\dim X = \dim A(X)$.

From commutative algebra, we use the following facts:

**Theorem 5.2.** *(i)* $\dim A(X) < \infty$.
*(ii) If $R$ is a finitely generated integral $K$-algebra and $0 \neq f \in R$ is not a unit, then*

$$\dim(R/(f)) = \dim R - 1.$$

**Proposition 5.3.** $\dim \mathbb{A}^n = n$.

*Proof.* By induction. The case $n = 1$ was dealt with before, and

$$\dim \mathbb{A}^{n-1} = \dim K[x_1, \ldots, x_{n-1}, x_n]/(x_n) = \dim \mathbb{A}^n - 1.$$

□

**Lemma 5.4.** *Let $X$ be a Noetherian topological space, and $X = \bigcup_{i=1}^n U_i$ an open cover of $X$. Then $\dim X = \max_i \dim U_i$. In particular, $\dim \mathbb{P}^n = n$.*

*Proof.* Exercise. □

Lecture 20
Jan 7, 2026

**Remark 5.5.** We record the following general properties of dimension, see also the exercises. Let $X$ be a topological space and $Y \subseteq X$.

(i) $\dim Y \leq \dim X$
(ii) If $X$ is irreducible and $\dim X = \dim Y < \infty$, then $\overline{Y} = X$.
(iii) Let $X = \bigcup_i U_i$ be an open cover. Then $\dim X = \sup\{\dim U_i\}_i$.
(iv) Let $X$ be Noetherian. We proved that $X$ can be written as a finite union $X = X_1 \cup \ldots \cup X_n$ of irreducible components. Then $\dim(X) = \max(\dim(X_i))$. (In particular, any quasi-projective set is noetherian.)

Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then $\dim(X) = \dim A(X) = \operatorname{trdeg} K(X)$.

**Corollary 5.6.** *Let $F \in K[X_1, \ldots, X_n]$, $\deg F > 0$. Then $\dim V(F) = n - 1$. Moreover, every irreducible component of $V(F)$ has dimension $n - 1$.*

*Proof.* Write $F = F_1^{e_1} \cdots F_k^{e_k}$ with $F_i$ pairwise different and irreducible. Then the irreducible decomposition of $V(F)$ is $V(F) = V(F_1) \cup \ldots \cup V(F_k)$, so wlog $F$ is irreducible. We get

$$\dim V(F) = \dim K[X_1, \ldots, X_n]/(F) \overset{5.2}{=} \dim K[X_1, \ldots, X_n] - 1 \overset{5.3}{=} n - 1.$$

□

**Definition 5.7.** Let $X$ be a Noetherian topological space. We say that $X$ is of *pure* dimension $n$ if every irreducible component of $X$ has dimension $n$. A closed algebraic set $X$ is called a curve if it is of pure dimension 1, a surface if it is of pure dimension 2, and a hypersurface in a pure-dimensional algebraic set $Y$ if $X \subseteq Y$ and $X$ is of pure dimension $\dim(Y) - 1$.

**Proposition 5.8.** *Let $X$ be a hypersurface in $\mathbb{A}^n$. Then $X = V(F)$ for some polynomial $F \in K[X_1, \ldots, X_n]$.*

*Proof.* Wlog $X$ is irreducible and $\dim(X) = n - 1$. Then the prime ideal $I(X)$ has height $\dim(A^n) - \dim(X) = 1$, hence is principal. $\square$

**Proposition 5.9.** *Let $\emptyset \neq U \subseteq X$ be an open subset of an affine variety. Then $\dim U = \dim X$.*

*Proof.* $U$ contains a basic open $D(f)$, it suffices to prove the claim for the latter open set. By proposition 2.31, $D(f)$ is birational to $X$. In particular, they have the same function field, so by the above remark, we have $\dim(D(f)) = \dim X$. $\square$

**Proposition 5.10.** *Let $X$ be a projective algebraic variety, and $\emptyset \neq U \subseteq X$ be open. Then $\dim U = \dim X$.*

*Proof.* Exercise. $\square$

# 6 Hilbert Polynomial and Bezout's Theorem

Let $X$ be a projective variety. We will associate to it a polynomial, the so called Hilbert polynomial, which encodes interesting structure of $X$. In particular, it will help us understand intersecions of two projective algebraic sets. For example, we will be able to prove that two distinct curves in $\mathbb{P}^2$ intersect, and the number of intersection points is bounded by the product of their degrees.

Lecture 21
Jan 12, 2026

**Definition 6.1.** (i) Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal. Then $K[x_0, \dots, x_n]/I$ is a graded $K$-algebra, with degree $d$-part $S_d := K[x_0, \dots, x_n]_d / K[x_0, \dots, x_n]_d \cap I$. The Hilbert function of $I$ is

$$h_I : \mathbb{N} \to \mathbb{N}, \qquad d \mapsto \dim_K S_d.$$

(ii) For a projective algebraic set $X \subseteq \mathbb{P}^n$ we set $h_X := h_{I(X)}$

**Remark 6.2.** Note that the Hilbert function of $X \subseteq \mathbb{P}^n$ is invariant under projective linear automorphisms $\varphi : x \mapsto Ax$. Indeed, $\varphi$ induces a grading-preserving automorphism of $K[x_0, \dots, x_n]$

**Example 6.3.** (i) $h_{\mathbb{P}^n}(d) = \dim_K K[x_0, \dots, x_n]_d = \binom{n+d}{n}$

(ii) Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal s.t. $V^p(I) = \emptyset$. By lemma 4.26 we have $h_I(d) = 0$ for all $d \gg 0$.

(iii) Let $X = \{a\} \subseteq \mathbb{P}^n$ a point. By remark 6.2 we can assume $a = [1 : 0 : \ldots : 0]$. Then $I(X) = (x_1, \dots, x_n)$, hence $S(X) \cong K[x_0]$ and $h_X(d) = \dim_K K[x_0]_d = 1$ for all $d \in \mathbb{N}$.

**Proposition 6.4.** *Let $I, J \subseteq R := K[x_0, \dots, x_n]$ be two homogeneous ideals. Then $h_{I\cap J} + h_{I+J} = h_I + h_J$.*

*Proof.* Follows immediately from the exact sequence of graded $K$-algebras

$$0 \to R/I \cap J \to R/I \times R/J \to R/(I+J) \to 0$$

and the fact that $\dim_K$ is additive on exact sequences. □

**Example 6.5.** (i) Let $X, Y \subseteq \mathbb{P}^n$ be disjoint projective algebraic sets. By example 6.3(ii) and proposition 6.4, we have $h_{X\cup Y} = h_X + h_Y$ for $d \gg 0$. In particular, for a finite set of $r$ points, we have $h(d) = r$ for $d \gg 0$.

(ii) $I = (x_1^2) \subseteq K[x_0, x_1]$. Then $V(I) = \{[1 : 0]\}$ is a point, but $h_I$ detects the multiplicativity 2: For $d \geq 1$ one has

$$K[x_0, x_1]_d / I_d = Kx_0^d \oplus Kx_0^{d_1} x_1,$$

so $h_I(d) = 2$ for all $d \geq 1$.

Our next goal is to show that the Hilbert function $h_I(d)$ agrees with a polynomial for $d \gg 0$.

**Lemma 6.6.** *Let $I \subseteq R = K[x_0, \dots, x_n]$ be a homogeneous ideal. Let $f \in K[x_0, \dots, x_n]$ be homogeneous of degree $e$. Assume that there exists $d_0 \in \mathbb{N}$ s.t. for all homogeneous polynomials $g \in K[x_0, \dots, x_n]$ of $\deg g \geq d_0$ wih $fg \in I$ we have $g \in I$. Then $h_{I+(f)}(d) = h_I(d) - h_I(d-e)$ for $d \gg 0$.*

*Proof.* By assumption, multiplication with $f$ induces an exact sequence

$$0 \to R_{d-e}/I_{d-e} \to R_d/I_d \to R_d/(I+(f))_d \to 0.$$

Taking dimensions yields the formula. □

**Remark 6.7.** Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set and $X = X_1 \cup \ldots \cup X_r$ its decomposition into irreducible components. Then $I(X) = \bigcap I(X_i)$ and the assumption of lemma 6.6 is satisfied for all $f \notin \bigcup I(X_i)$.

Picking any point $a_i \in X_i$, there exists a hyperplane $H$ which does not intersect the set $\{a_1, \ldots, a_r\}$, hence does not contain any $X_i$. Then there exists a projective linear automorphism sending $H$ to $V(x_0)$. Therefore, for any radical ideal $I$ we may assume that $x_0$ satisfies the requirements of lemma 6.6. A similar idea, using primary decomposition, works for general ideals.

Lecture 22
Jan 14, 2026

**Definition 6.8.** Let $P \in \mathbb{Q}[X]$. $P$ is called *numerical* if $P(n) \in \mathbb{Z}$ for all $n \gg 0$.

For example, $P(n) = \binom{n}{d}$ is a numerical polynomial of degree $d$ for all $d$.

**Proposition 6.9.** *(i) Let $P \in \mathbb{Q}[X]$ be a numerical polynomial. Then $P(X) = \sum_i c_i \binom{X}{i}$ with $c_i \in \mathbb{Z}$.*

*(ii) Let $f : \mathbb{Z} \to \mathbb{Z}$ s.t. there exists a numerical polynomial $Q$ of degree $r$ with $\Delta f(n) := f(n+1) - f(n) = Q(n)$ for $n \gg 0$. Then there exists a numerical polynomial $P$ of degree $r+1$ s.t. $f(n) = P(n)$ for $n \gg 0$.*

*Proof.* (i) Induction on $\deg P$. $\deg P = 0$ is clear. Assume $\deg P = r > 0$. The polynomials $1, \ldots, \binom{X}{r}$ form a $\mathbb{Q}$-basis of $\mathbb{Q}[X]_{\leq r}$. Therefore we can write $P(X) = \sum_{i=0}^{r} c_i \binom{X}{i}$ with $c_i \in \mathbb{Q}$. Then $\Delta P(x) = \sum_{i=0}^{r-1} c_{i+1} \binom{X}{i}$, so by induction $c_1, \ldots, c_r \in \mathbb{Z}$, and then $c_0 \in \mathbb{Z}$ as well.

(ii) By (i) we can write $Q(X) = \sum_{i=0}^{r} c_i \binom{X}{i}$. Set $P := \sum_{i=0}^{r} c_i \binom{X}{i+1}$. This is a numberical polynomial with $\Delta P = Q$, so $\Delta(f-P)(n) = 0$ for $n \gg 0$, i.e. $f - P$ is eventually constant. □

**Theorem 6.10.** *Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal. Then*

*(i) There exists a unique (numerical) polynomial $\chi_I(X) \in \mathbb{Q}[X]$ such that $\chi_I(d) = h_I(d)$ for $d \gg 0$.*

*(ii) $\deg \chi_I = \dim V^p(I) =: m$.*

*(iii) If $V^p(I) \neq \emptyset$, the leading coefficient of $\chi_I$ is $\frac{1}{m!}$ times a positive integer.*

$\chi_I$ is called the Hilbert polynomial of $I$. For a projective algebraic set $X \subseteq \mathbb{P}^n$, $\chi_X := \chi_{I(X)}$ is the Hilbert polynomial of $X$.

*Proof.* Uniqueness is clear. Induction on $m$: The cases with $m \leq 0$ are handled in example 6.3(ii) and 6.5(i). Now let $\dim V^p(I) = m > 0$ and let $V^p(I) = X_1 \cup \ldots \cup X_r$ be the decomposition into irreducible components. By remark 6.7, there exists a linear polynomial $f$ s.t. $X_i \not\subseteq V^p(f)$ and the condition from lemma 6.6 holds. Then $h_{I+(f)}(d) = h_I(d) - h_I(d-1)$ for $d \gg 0$. Since $\dim V^p(I + (f)) = \dim V^p(I) - 1$ (see the following lemma), by the induction hypothesis $h_{I+(f)}$ agrees with a numerical polynomial of degree $m-1$ for $d \gg 0$, hence $h_I$ agrees with a numerical polynomial of degree $m$ for $d \gg 0$ by proposition 6.9(ii). This proves (i) and (ii). (iii) follows immediately from proposition 6.9(i) and the observation $h_I(d) \geq 0$. □

Lecture 23
Jan 19, 2026

**Lemma 6.11.** *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set, and $f \in K[x_0, \ldots, x_n]$ a homogeneous non-constant polynomial with $X_i \not\subseteq V(f)$ for all irreducible components $X_i$ of $X$. Then $\dim(X \cap V^p(f)) = \dim X - 1$.*

*Proof.* Let $X = X_1 \cup \ldots \cup X_r$ be the decomposition into irreducible components. Then $V(I + (f)) = \bigcup_i (X_i \cap V(f))$, so wlog $X$ is irreducible. Also, the case $\dim X = 0$ is clear, so let $\dim X > 0$.

By corollary 4.32, $X \cap V^p(f) \neq \emptyset$. There exists an open $U \in \{U_0, \ldots, U_n\}$ such that

$$\emptyset \neq U \cap X \cap V^p(f) = (U \cap X) \cap (U \cap V^p(f)) \subseteq U \cong \mathbb{A}^n.$$

Now the result follows from the corresponding result for affine varieties, which was shown in the exercises. □

**Definition 6.12.** The positive integer from theorem 6.10(iii) is called the degree $\deg(I)$ of $I$. For a projective algebraic set, we set $\deg X := \deg I(X)$.

**Example 6.13.** From examples 6.3 and 6.5 it follows that

- $\chi_{\mathbb{P}^n}(t) = \binom{t+n}{n}$ and $\deg(\mathbb{P}^n) = 1$,
- $\chi_I(t) = 0$ iff $V(I) = \emptyset$,
- If $S \subseteq \mathbb{P}^n$ is finite, then $\chi_S(t) = |S|$ and $\deg(S) = |S|$. Note the same is not necessarily true for homogeneous ideals $I$ with $V(I)$ finite, cf. example 6.5(ii). In this case one only has $I \subseteq \sqrt{I}$, so $\deg I \geq |V(I)|$.

**Example 6.14.** Let $X \subseteq \mathbb{P}^n$ be a linear subspace of dimension $r$, i.e. $X$ is the projectivization of some linear subspace of $\mathbb{A}^{n+1}$). Then $\deg X = 1$. Indeed, $\mathrm{GL}(K, n+1)$ acts transitively on the set of subspaces of dimension $r+1$, so by remark 6.2, we may assume $X = V^p(x_{r+1}, \dots, x_n)$. Hence $S_d \cong K[x_0, \dots, x_r]_d$ and $\chi_X = \chi_{\mathbb{P}^r}$.

In fact, every pure-dimensional projective algebraic set of degree 1 is a linear subspace in $\mathbb{P}^n$.

Lecture 24
Jan 21, 2026

**Remark 6.15.** If $X, Y \subseteq \mathbb{P}^n$ have the same dimension $m$ and do not share a common irreducible component, then $\dim X \cap Y < m$. Observe $V(I(X) + I(Y)) = X \cap Y$. By proposition 6.4 $\chi_{X \cup Y} = \chi_X + \chi_Y - \chi_{I(X)+I(Y)}$, where the last term has smaller degree than the others, so comparing leading coefficients yields $\deg(X \cup Y) = \deg X + \deg Y$.

In particular, if $X$ is pure-dimensional and $X = X_1 \cup \ldots \cup X_n$ is the decomposition into irreducible decomponents, then $\deg X = \sum_i \deg X_i$.

**Proposition 6.17** (Bézout's Theorem)**.** *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set,* $\dim X \geq 1$, *and $X = X_1 \cup \ldots \cup X_r$ the decomposition into irreducible components. Let $f \in K[x_0, \dots, x_n]$ be a homogeneous polynomial with $X_i \not\subseteq V^p(f)$ for all $i$. Then*

$$\deg(I(X) + (f)) = \deg X \cdot \deg f.$$

*Proof.* By lemma 6.6 we have $\chi_{I(X)+(f)}(t) = \chi_X(t) - \chi_X(t-e)$, where $e = \deg f$. Let $\chi_X(t) = \frac{c_m}{m!} t^m + O(t^{m-1})$, then

$$\chi_X(t) - \chi_X(t-e) = \frac{c_m}{m!}(t^m - (t-e)^m) + O(t^{m-2}) = \frac{c_m m e}{m!} t^{m-1} + O(t^{m-2}).$$

Hence $\deg(I(X) + (f)) = (m-1)! \frac{c_m m e}{m!} = c_m e = \deg X \deg f$. □

**Remark 6.18.** Let $X = \mathbb{P}^n$. Then Bezout's Theorem reads $\deg((f)) = \deg f$, motivating the terminology "degree". If $X$ is a hypersurface, by the exercises $I(X) = (f)$ for some homogeneous polynomial $f$. Then $\deg X = \deg f$.

**Corollary 6.19.** *(i) Let $X \subseteq \mathbb{P}^n$ be a curve, and let $f \in K[x_0, \dots, x_n]$ be a nonconstant homogeneous polynomial s.t. $X_i \not\subseteq V^p(f)$ for any irreducible component $X_i \subseteq X$. Then $0 < |X \cap V^p(f)| \leq \deg X \deg f$.*

*(ii) Let $X, Y \subseteq \mathbb{P}^2$ be curves without common irreducible components. Then $|X \cap Y| \leq \deg X \deg Y$.*

*Proof.* (i) Combine Bézout's Theorem with example 6.13. (ii) follows directly from (i) with $n = 2$ and remark 6.18 □

**Proposition 6.20** (Pascal's Theorem). *Let $X \subseteq \mathbb{P}^2$ be an irreducible conic, i.e. $X = V(f)$ with $\deg f = 2$ and $f$ irreducible. Let $A, B, C, D, E, F \in X$. Then the three intersection points $P = \overline{AB} \cap \overline{DE}$, $Q = \overline{BC} \cap \overline{EF}$ and $R = \overline{AF} \cap \overline{CD}$ of opposite edges of the hexagon $ABCDEF$ lie on a line.*

Note that since lines have degree 1 by example 6.14, they intersect in exactly one point in $\mathbb{P}^2$ by corollary 6.19. So $P, Q, R$ in the proposition exist and are uniquely specified.

*Proof.* Consider the reducible cubics $X_1 = AB \cup CD \cup EF$, $X_2 = BC \cup DE \cup AF$. Let $I(X_i) = (f_i)$ and pick $S \in X \setminus \{A, B, C, D, E, F\}$. We can find $0 \neq \alpha, \beta \in K$ s.t. $(\alpha f_1 + \beta f_2)(S) = 0$. Observe that $\alpha f_1 + \beta f_2 \neq 0$, since $V(f_1) \neq V(f_2)$. Set $X' = V(\alpha f_1 + \beta f_2)$, $\deg X' \leq 3$. But now $X' \cap X$ contains $A, B, C, D, E, F$ and $S$, contradicting Bezout's theorem unless $X'$ and $X$ share an irreducible component. But $X$ is irreducible, so $X \subseteq X'$ and $X' = X \cup L$ for some line $L$. We conclude $P, Q, R \in X' \setminus X = L$. $\square$