Topology I

read by Prof. Dr. Bernhard Leeb

Ludwig-Maximilians-Universität München
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notes by Stefan Albrecht

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I Preliminaries from Group Theory

Lecture 1
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Free groups usually do not appear in the beginner courses even though they are fundamental objects in group theory. E.g. every group is a quotient of a free group. They play an important role in geometric group theory or (low-dimensional) topology. They can be constructed as *free products* (of copies of $\mathbb{Z}$), an equally fundamental construction.

I.1 The free product of groups

Intuitively, the free product of a family of groups is the "largest group generated by them". As many basic constructions in algebra or topology, it can be elegantly characterized by a *universal property*, namely as the coproduct in the category of groups.

Definition I.1. The *free product* of a family of groups G_ι, $\iota \in I$, is a group G together with a family of group homomorphisms $\varphi_\iota : G_\iota \to G$ such that the following universal property holds:

$$\begin{tikzcd} G_\iota \arrow[r, "\varphi_\iota"] \arrow[dr, "\psi_\iota"'] & G \arrow[d, dashed, "\exists!\psi"] \\ & H \end{tikzcd}$$

For every family of homomorphisms $\psi_\iota : G_\iota \to H$ into some group H, there exists a unique group homomorphism $\psi : G \to H$ such that $\psi_\iota = \psi \circ \varphi_\iota$ for all $\iota \in I$.

Notation The free product will be denoted as $\ast_{\iota \in I} G_\iota$, or $G_1 * \ldots * G_n$ in the finite case.

The *uniqueness* of the free product of groups up to (unique) isomorphism follows from general arguments (sometimes referred to as *general* or *abstract nonsense*), independent of the category (applying to the coproduct in any category):

Consider two free products $(\varphi_\iota : G_\iota \to G)_{\iota \in I}$ and $(\varphi'_\iota : G_\iota \to G')_{\iota \in I}$ of the family $(G_\iota)_{\iota \in I}$.

$$\begin{tikzcd} & G \arrow[d, "\psi"] \arrow[dd, dashed, bend left, "\mathrm{id}_G"] \\ G_\iota \arrow[ur, "\varphi_\iota"] \arrow[r, "\varphi'_\iota"] \arrow[dr, "\varphi_\iota"] & G' \arrow[d, "\psi'"] \\ & G \end{tikzcd}$$

By the universal properties, we obtain maps ψ and ψ' as in the diagram. Applying uniqueness to the big triangle, we see that $\psi' \circ \psi$ is the *unique* map satisfying the universal property for $G \to G$. But the identity map clearly does as well, hence $\psi' \circ \psi = \mathrm{id}_G$, and in the same way one sees $\psi \circ \psi' = \mathrm{id}_{G'}$. Hence $G \cong G'$.

However, the *existence* of the free product is nontrivial (The existence of a coproduct depends on the category).

Immediate requirements for the free product as a consequence of the universal property:

- The φ_ι are injective (i.e. embeddings), so we can think of the G_ι as subgroups of G.
- The subgroups G_ι generate G.

Thus every element in the free product has a representation as a product of the form $g_1 \cdots g_n$ with $n \in \mathbb{N}_0$ and $g_i \in G_{\iota_i} - \{1\}$ such that $\iota_i \neq \iota_{i+1}$ for all $1 \leq i \leq n-1$. This form is called *reduced*.

Not clear right away: The free product being the "largest gorup generated by its factors" should mean that the factors are "algebraically independent" in the sense that group elements have *unique representations* as reduced products.

First proof of existence For a family of families

$$(\psi_{\iota\kappa} G_\iota \to H_\kappa)_{\iota \in I}, \quad \kappa \in K$$

the mapping problem is solved by the family of induced homomorphisms into the direct product of the groups H_κ.

$$\begin{tikzcd} G_\iota \arrow[r, "(\varphi_{\iota\kappa})_\kappa"] \arrow[dr, "\varphi_{\iota\kappa_0}"'] & \prod_{\kappa \in K} H_\kappa \arrow[d, "\mathrm{proj}_{\kappa_0}"] \\ & H_{\kappa_0} \end{tikzcd}$$

To achieve uniqueness, replace $\prod_\kappa H_\kappa$ by the subgroup generated by the images of all $(\varphi_{\iota\kappa})_\kappa$. This family can be regarded as an "approximation" of the free product of the G_ι.

Now note that the universal property needs only be checked for families $(\psi_\iota : G_\iota \to H)_\iota$ whose images generate H. The equivalence classes of such families form a set, because the size of H is restricted in terms of the sizes of I and the G_ι. Apply the above-mentioned construction to a family of representatives of these equivalence classes. □ Lecture 2 Oct 15, 2025

This proof, while quite general, reveals very little about the structure of the free product. We now give an explicit construction of the free product, which clarifies its structure.

Second proof of existence One can construct the free product as an abstract group with underlying set the reduced words, as defined above, and concatenation plus reduction as the group operation. However, verifying associativity is complicated due to possible cancelations.

It is simpler to onstruct the free product as a group of symmetries of a combinatorial object (as a permutation group). Take W to be the set of reduced words $(g_1, \ldots, g_n)$ with $n \in \mathbb{N}_0$, $g_i \in G_{\iota_i} - \{1\}$ and $\iota_i \neq \iota_{i+1}$ for all $1 \leq i < n$. For every $\iota \in I$, define an action of G_ι on W by defining $g \cdot (g_1, \ldots, g_n)$ as the reduction of the word $(g, g_1, \ldots, g_n)$. It is easy to see that this is indeed an action.

These actions are clearly faithful (effective), even free[1], and yield embeddings $\varphi_\iota : G_\iota \hookrightarrow S(W)$ into the symmetric group of W. Take $G < S(W)$ to be the subgroup generated by the images of the φ_ι. Observe that for the action G on W it holds that $(g_1 \cdots g_n) \cdot () = (g_1 \cdots g_n)$ for a reduced product in G. This shows that different reduced products act by different permutations of W, and therefore are different group elements. In other words, the elements in G have *unique* representations as reduced products. In this sense, the G_i are "algebraically independent".

The universal property is now a direct consequence of the uniqueness of reduced product representations. With the notation as in definition I.1, the only possibility to define ψ on reduced words is as $\psi(g_1 \cdots g_n) := \prod_{i=1}^n \psi_{\iota_i}(g_i)$, where $g_i \in G_{\iota_i}$. By the uniquenss of reduced representations, this is well-defined and clearly makes the necessary diagram commute. It remains to see that the map ψ is multiplicative and hence a group homomorphism.

Let $g_1 \cdots g_n$ and $g'_k \cdots g'_1 \in W$. There is a maximal index m with $0 \leq m \leq n, k$ s.t. $\iota'_j = \iota_j$ and $g'_j g_j = 1$ for $1 \leq j \leq m$. Then either the product $g'_k \cdots g'_{m+1} g_{m+1} \cdots g_n$ obtained from the full unreduced product by m cancellations is reduced (i.e. $\iota'_{m+1} \neq \iota_{m+1}$ or $m = \min(n, k)$), or

[1] faithful = nontrivial elements act nontrivially, full = nontrivial elements have no fixed points

$m < \min(n,k)$ and $\iota'_{m+1} = \iota_{m+1}$, $g'_{m+1}g_{m+1} \neq 1$, in which case $g'_k \cdots (g'_{m+1}g_{m+1}) \cdots g_n$ is reduced. In both cases, multiplicativity is clear. □

Remark I.2. The action of G on W is simply transitive, because the empty word has trivial stabilizer and point stabilizers along orbits are conjugate to each other. It extends to an action of G on a coloured graph with vertices W. To construct it, connect $()$ to (g) for $g \in G_\iota - \{1\}$ by an edge of colour ι. Extend this in a G-invariant way by connecting $(g_1, \ldots, g_n)$ to $(g_1, \ldots, g_{n-1}, g)$ with $g \in G_{\iota_n} - \{1, g_n\}$, and $(g_1, \ldots, g_{n-1})$ by edges of colour ι_n, and to $(g_1, \ldots, g_n, g)$ for all $g \in G_\iota - \{1\}$, $\iota \neq \iota_n$ by an edge of colour ι.

This homogeneous graph is in general not a tree, but it has tree-like structure, since vertices disconnect, e.g. when removing the empty word, the connected components correspond to the factors G_ι, determined by the colour" of the first letter of their vertices.

Note that in general, G is *not* the full group of symmetries of this coloured graph.

Example I.3. Consider the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong D_\infty = \mathrm{Isom}(\mathbb{Z})$: Let a, b be the reflections of $\mathbb{Z}$ around 0 and $\frac{1}{2}$, respectively. Send the generators of the two copies of $\mathbb{Z}/2\mathbb{Z}$ to a and b to get a surjective map $\alpha : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to D_\infty$, which one checks to be injective by computing the images of 0 and 1.