Exercise 1

$\dim X_i \leq \dim X$ is clear. Conversely, let $Y_0 \subsetneq \ldots \subsetneq Y_n \subseteq X$ be a chain of irreducible closed subsets. Then $Y_n = \bigcup(Y_n \cap X_i)$, so by irreducibility $Y_n = Y_n \cap X_i$, i.e. $Y_n \subseteq X_i$. But then the whole chain is contained in X_i, so $n \leq \dim X_i \leq \max_i \dim X_i$. Since this holds for all chains, $\dim X \leq \max_i \dim X_i$.

Exercise 2

We have $X = \bigcup(X \cap U_i)$ and $U = \bigcup U \cap U_i$, with each $X \cap U_i$ affine. Hence by the lecture $\dim X \cap U_i = \dim U \cap U_i$ ($U \cap U_i$ is nonempty since opens are dense.) and $\dim X = \sup_i \dim X \cap U_i = \sum_i \dim U \cap U_i = \dim U$.

Exercise 3

(1) First let $X = V^p(F)$ for some homogeneous non-constant polynomial F. Then $X = \bigcup(X \cap U_i) = \bigcup V^a(F^i)$ and $\dim X = \dim V^a(F^i) = n - 1$ by the lecture.

Conversely, assume $\dim X = n - 1$, and consider the affine sub-variety $X \cap U_0$. Then $\dim X \cap U_0 = n - 1$, so $X \cap U_0 = V^a(F)$ for some polynomial F, and $X = \overline{X \cap U_0} = \overline{V^a(F)} = V^p(F^h)$.

(2) follows directly from (1) and sheet 9, exercise 1.

Exercise 4

Let $0 \neq x \in \mathfrak{p}$. Write $x = p_1 \cdots p_n$ with p_i prime elements. Since $\mathfrak{p}$ is prime, one of the $p_i \in \mathfrak{p}$. But then $0 \subsetneq (p_i) \subseteq \mathfrak{p}$ is a chain of prime ideals, so $\mathfrak{p} = (p_i)$.

Exercise 5

Write $F = F_1^{e_1} \cdots F_r^{e_r}$. Then $X \cap V(F) = \bigcup_i (X \cap V(F_i))$, so (2) follows from (1) and for (1) we may assume that F is irreducible. Now $A(X \cap V(F)) = K[X_1, \ldots, X_n]/(I(X), F) \cong A(X)/(F)$ has dimension $\dim A(X) - 1 = \dim X - 1$.