

Algebraic Geometry I

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Roughly speaking, the goal of algebraic geometry is to study systems of polynomial equations $F_1(X) = \dots = F_n(X) = 0$ for polynomials $F_i \in K[X_1, \dots, X_m]$ over a field (or ring) K . The set of solutions of this system is a geometric object, which we try to understand using algebraic methods, for example considering the ideal $I = (F_1, \dots, F_n)$ in $K[X_1, \dots, X_m]$ or the quotient $K[X_1, \dots, X_m]/I$.

There is a very strong relation between these objects in the case that $K = \overline{K}$ is algebraically closed (e.g. \mathbb{C}). If K is not algebraically closed, or some generic ring, things get more complicated: For example, there are many equations over \mathbb{R} with no solutions, like $x^2 + y^2 + 1 = 0$, which behave differently when considered over \mathbb{C} . The wish to still study these equations geometrically leads to the idea of spectra (the set of all prime ideals of a ring), and later the theory of sheaves and schemes.

1 Algebraic Sets and Affine Varieties

Let K be an algebraically closed field.

Definition 1.1. For $n \in \mathbb{N}$ define *affine n -space* over K as

$$\mathbb{A}^n := \mathbb{A}_K^n := K^n.$$

Definition 1.2. Let $I \subset K[x_1, \dots, x_n]$ be a subset. The associated (*affine*) *algebraic set* is

$$V(I) := \{x \in \mathbb{A}_K^n \mid f(x) = 0 \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n$ is called *algebraic* if $X = V(I)$ for some $I \subset K[x_1, \dots, x_n]$.

Remark 1.3. By definition $V(I) = V(\langle I \rangle) = V(f_1, \dots, f_m)$ where $\langle I \rangle = (f_1, \dots, f_m)$ is finitely generated because $K[x_1, \dots, x_n]$ is Noetherian. Therefore, $X \subseteq \mathbb{A}^n$ is algebraic if and only if $X = V(I)$ for some ideal I if and only if $X = V(f_1, \dots, f_m)$ for a finite number of polynomials f_i .

Example 1.4. The following sets are algebraic:

- A parabola $\{(x, x^2) \mid x \in K\} = V(y - x^2)$
- $\emptyset = V(K[x_1, \dots, x_n])$
- $\mathbb{A}^n = V(0)$
- Points: $\{(a_1, \dots, a_n)\} = V(x_1 - a_1, \dots, x_n - a_n)$

Lemma 1.5. Let $I, J \triangleleft K[x_1, \dots, x_n]$ be ideals. Then

- (a) If $I \subseteq J$, then $V(I) \supseteq V(J)$.
- (b) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$
- (c) For any family $(I_t)_{t \in T}$ of ideals, $\bigcap_t V(I_t) = V(\bigcup_t I_t) = V(\sum_t I_t)$

Proof. (a) is clear.

For (b), part (a) yields $V(I \cap J) \subseteq V(IJ)$ and $V(I), V(J) \subseteq V(I \cap J)$, so it remains to show $V(IJ) \subseteq V(I) \cup V(J)$. Let $a \in V(IJ)$. Assume $a \notin V(I)$, i.e. there is $f \in I$ such that $f(a) \neq 0$. Let $g \in J$. Then $fg \in IJ$, so $0 = (fg)(a) = f(a)g(a)$. Since $f(a) \neq 0$, we conclude $g(a) = 0$.

The first equation of (c) is tautological, the second one is remark 1.3, □

Definition 1.6. The *Zariski topology* on \mathbb{A}^n is the topology whose closed subsets are exactly the algebraic sets. That is, $U \subseteq \mathbb{A}^n$ is open iff its complement is algebraic.

Remark 1.7. This is indeed a topology by example 1.4 and lemma 1.5. Note that the Zariski topology induces (via the subspace topology) a topology on any algebraic set $X \subseteq \mathbb{A}^n$, which is also called the Zariski topology.

Recall from general topology that a topological space $X \neq \emptyset$ is called irreducible if $X \neq X_1 \cup X_2$ with $X_i \subsetneq X$ closed. \emptyset is not considered irreducible.

For example, $V(xy) = V(x) \cup V(y)$ (the union of the coordinate axes in \mathbb{A}^2) is not irreducible, while a parabola $V(y - x^2)$ is irreducible (we will see how to check this later).

Definition 1.8. An *affine algebraic variety* is an irreducible closed subset of \mathbb{A}^n .

Definition 1.9. Let $X \subseteq \mathbb{A}^n$ be an arbitrary set. We define the *vanishing ideal* of X as

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

Lemma 1.10. Let $X \subseteq \mathbb{A}^n$ and $S \subseteq K[x_1, \dots, x_n]$. Then

- (a) $X \subseteq V(I(X))$ and $S \subseteq I(V(S))$.
- (b) $V(I(X)) = \overline{X}$ is the closure of X (w.r.t. the Zariski topology).

Proof. (a) is clear, (b) is left as an exercise. □

Proposition 1.11. An affine algebraic set $X \subseteq \mathbb{A}^n$ is a variety if and only if $I(X)$ is a prime ideal.

Proof. Let X be a variety and let $fg \in I(X)$ for $f, g \in K[x_1, \dots, x_n]$. We have $X \subseteq V(fg) \stackrel{1.5}{=} V(f) \cup V(g)$. Hence we can write $X = (X \cap V(f)) \cup (X \cap V(g))$ as the union of two closed subsets. By irreducibility, wlog we have $X = X \cap V(f)$, i.e. $X \subseteq V(f)$, which is equivalent to $f \in I(X)$.

Conversely, suppose that $X = A \cup B$ is not irreducible. Choose points $a \in A \setminus B$ and $b \in B \setminus A$. By Lemma 1.10 and since A, B are closed, we get $V(I(A)) = A$ and $V(I(B)) = B$. Hence there exist $f \in I(A)$ and $g \in I(B)$ with $f(b) \neq 0$ and $g(a) \neq 0$. Thus $fg \in I(X)$, but both $f, g \notin I(X)$ □

Remark 1.12. If $X = V(I)$ is an affine variety, this does not necessarily imply that I is prime: Consider $V((x^2)) \subseteq \mathbb{A}^1$: $V((x^2)) = \{0\}$ is irreducible, but (x^2) is not prime.

Note that \mathbb{A}^n is irreducible since K is infinite. However, this is no longer true if one considers finite fields, since then \mathbb{A}^n is the union of its finitely many points. For example, $I(A_{\mathbb{F}_p}^1) = (X^p - X)$ is not prime.

We use the following result from commutative algebra without proof:

Theorem 1.13 (Hilbert Nullstellensatz). Let $J \triangleleft K[x_1, \dots, x_n]$. Then

- (a) $V(J) = \emptyset$ if and only if $J = K[x_1, \dots, x_n]$.
- (b) $I(V(J)) = \sqrt{J} = \{f \in K[x_1, \dots, x_n] \mid f^n \in J \text{ for some } n\}$
- (c) If J is a maximal ideal, then $J = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in K$.

Corollary 1.14. *There are inclusion-reversing bijections*

$$\begin{aligned} \{\text{affine algebraic sets } X \subseteq \mathbb{A}^n\} &\xrightleftharpoons[V]{I} \{\text{radical ideals in } K[x_1, \dots, x_n]\} \\ \{\text{affine algebraic varieties } X \subseteq \mathbb{A}^n\} &\xrightleftharpoons[V]{I} \{\text{prime ideals in } K[x_1, \dots, x_n]\} \\ \{\text{points } a \in \mathbb{A}^n\} &\xrightleftharpoons[V]{I} \{\text{maximal ideals in } K[x_1, \dots, x_n]\} \end{aligned}$$

Proof. Clear from 1.13, 1.10 and 1.11. \square

Example 1.15. Let f be irreducible in $K[x_1, \dots, x_n]$. Then $V(f)$ is an affine variety. Varieties of this form are called hypersurfaces in \mathbb{A}^n (curves for $n = 2$, surfaces for $n = 3$).

Remark 1.16. If $X \subseteq \mathbb{A}^n$ is a variety, by proposition 1.11 $I(X)$ is prime, and $K[x_1, \dots, x_n]/I$ is an integral domain. We can consider its fraction field $\text{Frac}(K[x_1, \dots, x_n]/I)$.

Theorem 1.17. *Any affine algebraic set can be uniquely written as a finite union of affine varieties.*

For the proof, we need some preparations.

Definition 1.18. A topological space X is called *Noetherian* if any chain of descending closed subsets $X \supseteq X_1 \supseteq X_2 \supseteq \dots$ becomes stationary, i.e. there exists n s.t. $X_m = X_n$ for all $m > n$.

Lemma 1.19. *Affine space \mathbb{A}^n is Noetherian.*

Proof. Let $\mathbb{A}^n \supseteq X_1 \supseteq X_2 \supseteq \dots$ be a chain of closed subsets. Applying $I(-)$ yields an ascending chain $(0) \subseteq I(X_1) \subseteq I(X_2) \subseteq \dots$ of ideals in $K[x_1, \dots, x_n]$. This is a Noetherian ring, so there is some m such that $I(X_n) = I(X_{n+1})$ for all $n \geq m$. By corollary 1.14(a), I is injective on closed subsets, so we are done. \square

More generally,

Corollary 1.20. *Any affine algebraic space $X \subseteq \mathbb{A}^n$ is Noetherian.*

Proof. Any chain in X is also a chain in \mathbb{A}^n . \square

Proposition 1.21. *Let $X \neq \emptyset$ be a Noetherian topological space.*

- (a) *Then X can be written as a finite union of irreducible closed subspaces.*
- (b) *Moreover, if we assume that $X_i \not\subseteq X_j$ for $i \neq j$, then the above decomposition is unique up to permutation. In this case, the X_i are called irreducible components of X .*

Proof. Assume that (a) fails for X . Consider $S = \{Y \subseteq X \mid Y \text{ closed, cannot be written as a finite union of irreducible closed subsets}\}$. Since X is Noetherian, S must have some minimal element Y w.r.t. inclusion. Y is not irreducible, so we can write $Y = Y_1 \cup Y_2$ with $Y_{1,2}$ proper closed subspaces. By minimality, Y_1 and Y_2 can be written as finite unions of irreducible closed subsets, thus so can Y , contradicting $Y \in S$.

To check uniqueness, assume we have two decompositions $X = X_1 \cup \dots \cup X_r = X'_1 \cup \dots \cup X'_s$ as in (b). Then $X'_1 = \bigcup_i (X_i \cap X'_1)$. Since X'_1 is irreducible, wlog $X'_1 \subseteq X_1$. By the same argument, $X_1 \subseteq X'_i$ for some i . If $i \neq 1$, then $X'_1 \subseteq X'_i$, contradicting our assumption. Hence $i = 1$ and $X_1 = X'_1$. Proceed inductively with $X \setminus X_1 = X_2 \cup \dots \cup X_r = X'_2 \cup \dots \cup X'_s$. \square

Combining 1.20 and 1.21 yields theorem 1.17.

Remark 1.22. The proof strategy for (a) can be summarized as follows: Let X be a Noetherian space and P a property of closed subsets. To show that P holds for all subsets of X (thus in particular for X), it suffices to show that for all $Y \subseteq X$ closed, if P holds for all proper closed subsets of Y , then it also holds for Y . This is called *Noether induction* (a special case of well-founded induction).

Example 1.23. Let $f \in K[x_1, \dots, x_n]$. This is a factorial ring, so we may write $f = g_1^{k_1} \cdots g_r^{k_r}$ with g_i irreducible and pairwise different. Then

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$$V(f) = V(g_1^{k_1}) \cup \cdots \cup V(g_r^{k_r}) = V(g_1) \cup \cdots \cup V(g_r)$$

is the decomposition of $V(f)$ into irreducible subsets: $V(g_i)$ is irreducible by proposition 1.11, since $I(V(g_i)) = (g_i)$ is prime.

In general, finding this composition for $V(f_1, \dots, f_r)$ is not easy.

Example 1.24. What is the Zariski topology on \mathbb{A}^1 ? By definition, a closed/algebraic set is of the form $V(I)$ for some ideal $I \subseteq K[x]$. Since $K[x]$ is a PID, $I = (f)$ for some $f = (x - a_1)^{k_1} \cdots (x - a_r)^{k_r} \in K[x]$. If f is not constant, we see as in example 1.23 that

$$X = V(f) = \bigcup_i V(x - a_i) = \{a_1, \dots, a_r\}.$$

Hence the closed sets are exactly $V(0) = \mathbb{A}^1$, $V(1) = \emptyset$, and finite unions of points. In other words, the Zariski topology coincides with the cofinite topology on \mathbb{A}^1 . The affine varieties on \mathbb{A}^1 are therefore either \mathbb{A}^1 itself or a single point.

We also see that any two non-empty open subsets have nontrivial intersection, so \mathbb{A}^1 with the Zariski topology is not Hausdorff.

Definition 1.25. Let X be a nonempty topological space. We define the dimension of X as the supremum of all $n \in \mathbb{N}$ such that there is a chain of irreducible subspaces $\emptyset \neq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \subseteq X$.

Example 1.26. By example 1.24, a maximal chain of affine varieties in \mathbb{A}^1 is $\{0\} \subsetneq \mathbb{A}^1$, hence $\dim \mathbb{A}^1 = 1$.

Definition 1.27. Let R be a (commutative) ring. The Krull dimension of R is the supremum over all l such that there is a chain of prime ideals $\mathfrak{p}_l \subsetneq \mathfrak{p}_{l-1} \subsetneq \cdots \subsetneq \mathfrak{p}_0 \subsetneq R$.

Recall from corollary 1.14 that there is an inclusion-reversing correspondence between prime ideals of $K[x_1, \dots, x_n]$ and affine algebraic varieties in \mathbb{A}^n . Fixing some variety X , it follows that subvarieties correspond bijectively to prime ideals that contain $I(X)$, i.e. prime ideals of $K[x_1, \dots, x_n]/I(X)$. Hence

Proposition 1.28. If X is an affine algebraic variety, then $\dim X = \dim K[x_1, \dots, x_n]/I(X)$.

2 Morphisms of Affine Varieties

Definition 2.1. Let $X \subseteq \mathbb{A}_K^n$ be an algebraic set. A function $f : X \rightarrow K$ is *regular* if there is a polynomial $F \in K[x_1, \dots, x_n]$ such that $f = F|_X$, i.e. $f(x) = F(x)$ for all $x \in X$. Write $A(X)$ for the set of regular functions on X .

Remark 2.2. $A(X)$ is a ring (and even a K -algebra) in a natural way, with addition and multiplication defined pointwise. Moreover, there is a homomorphism of K -algebras

$$K[x_1, \dots, x_n] \twoheadrightarrow A(X), \quad F \mapsto F|_X.$$

The kernel of this morphism is exactly $I(X)$, so that $A(X) \cong K[x_1, \dots, x_n]/I(X)$ canonically.

Remark 2.3. By corollary 1.14, $A(X)$ is always reduced, $A(X)$ is integral iff X is a variety, and $A(X)$ is a field iff X is a point (in which case $A(X) \cong K$).

Definition 2.4. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A map $\varphi : X \rightarrow Y$ is called *regular* if $\varphi = (f_1, \dots, f_m)$ for some regular $f_1, \dots, f_m \in A(X)$. A regular map φ is an isomorphism if it has an inverse which is also regular.

Example 2.5. (i) $\mathbb{A}^1 \rightarrow V(y - x^2) \subseteq \mathbb{A}^2$, $t \mapsto (t, t^2)$ is a regular map. It has inverse $(x, y) \mapsto x$, which is also regular, hence $\mathbb{A}^1 \cong V(y - x^2)$.

(ii) $\varphi : \mathbb{A}^1 \rightarrow V(y^2 - x^3) \subseteq \mathbb{A}^2$, $t \mapsto (t^2, t^3)$ is regular and bijective as well, but its inverse $(x, y) \mapsto \frac{y}{x}$ is not regular, so φ is not an isomorphism.

Proposition 2.6. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be algebraic sets, and let $\varphi : X \rightarrow Y$ be a regular map. Then φ is continuous (w.r.t. the Zariski topology on X and Y).

Proof. Let $\varphi = (f_1, \dots, f_m)$ and $J = \langle F_1, \dots, F_k \rangle \subseteq K[x_1, \dots, x_m]$ with $V(J) \subseteq Y$. Then

$$\varphi^{-1}(V(J)) = \varphi^{-1}(V(F_1, \dots, F_k)) = \{x \in X \mid F_j(f_1(x), \dots, f_m(x)) = 0, j = 1, \dots, k\}$$

Now $F_j(f_1(x), \dots, f_m(x))$ is a composition of polynomials, hence a polynomial, call it \tilde{F}_j . We conclude $\varphi^{-1}(V(J)) = X \cap V(\tilde{F}_1, \dots, \tilde{F}_k)$ as desired. \square

Remark 2.7. The converse is false. For example, one easily concludes from example 1.24 that every bijective map $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ is continuous, but there are way more bijections than polynomials (say because polynomials are defined by their values on any infinite subset).