Exercise 1

(1) Assume $f(X) = Z_1 \cup Z_2$ with $Z_1, Z_2 \subseteq f(X)$ closed. Then $X = f^{-1}(Z_1) \cup f^{-1}(Z_2)$ with the $f^{-1}(Z_i)$ closed by continuity of f. By irreducibility of X, wlog $X = f^{-1}(Z_1)$. But then $f(X) = Z_1$, so $f(X)$ is irreducible.

Moreover, if $A \subseteq X$ is irreducible, then so is $\overline{A}$, for if $\overline{A} = Z_1 \cup Z_2$, then $A = (A \cap Z_1) \cup (A \cap Z_2)$, so wlog $A = A \cap Z_1$ by irreducibility. But this means $A \subseteq Z_1$, and since Z_1 is closed even $\overline{A} \subseteq Z_1$. So $\overline{f(X)}$ is irreducible.

(2) Consider the regular, hence continuous map $\varphi : \mathbb{A}^1 \to Y_3,\ t \mapsto (t^3, t^4, t^5)$. It is surjective and $\mathbb{A}^1$ is irreducible, so Y_3 is irreducible by part (1).

Exercise 2

Let $\pi : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]/I(X) \cong A(X)$ be the natural projection. Then we know from commutative algebra that $I \mapsto \pi(I)$ and $J \mapsto \pi^{-1}(J)$ define mutually inverse, inclusion-preserving bijections between the ideals containing I and the ideals of $A(X)$. We prove:

$$V_X(I) = V(\pi^{-1}(I)),$$

since $f(x) = \pi(f)(x)$ for $f \in A(X)$ and $x \in X$. With this identity, all parts are now easy:

(1) $V_X(I) = V(\pi^{-1}(I))$ is clearly closed. Conversely, if $V(J) \subseteq X$ is closed for some radical ideal J, then $I(X) \subseteq J$, so $V_X(\pi(J)) = V(J)$.

(2) $\emptyset = V_X(I) = V(\pi^{-1}(I)) \iff \pi^{-1}(I) = K[x_1, \ldots, x_n] \iff I = A(X)$

(3) $X = V_X(I) = V(\pi^{-1}(I)) \iff \sqrt{\pi^{-1}(I)} = I(X) \iff (0) = \sqrt{I} \iff I = (0)$

Exercise 3

(1) We have $\ker f^* = I(f(X))$, since both contain exactly the $g \in A(Y)$ with $f^*(g)(x) = g(f(x)) = 0$ for all $x \in X$. Further, $\ker f^* = 0$ iff f^* is injective, and $I(f(X)) = 0$ iff $\overline{f(X)} = V_Y(0) = Y$ by 2(3).

(2) If $f(X)$ is closed and $f : X \to f(X)$ an isomorphism, then $f^* : A(f(X)) \to A(X)$ is an isomorphism as well, in particular injective. But this morphism is just a quotient of $f^* : A(Y) \to A(X)$ to $A(f(X))$, which is thus surjective as well.

Conversely, suppose that f^* is surjective, let $I = \ker f^*$. Then f factors as the projection $A(Y) \to A(Y)/I$ followed by $\overline{f}^* : A(Y)/I \to A(X)$. On the level of varieties, this translates to $X \to V(I) \hookrightarrow Y$. But $\overline{f}^*$ is an isomorphism, hence so is $f : X \to V(I)$. In particular, $f(X) = V(I)$ is closed.

Exercise 4

(1) Considering f, g as elements in $K(X)[Y]$, they are still irreducible by Gauss's Lemma. But this ring is a PID, so by Bezout's identity there are $p, q \in K(X)[Y]$ such that $pf + qg = 1$. Clearing denominators, we get an equation of the form $\overline{p}f + \overline{q}g = d$, where $d \in K(X)$. Now if $(x, y) \in V(f, g)$, then this shows $d(x) = 0$. But d is some non-zero polynomial in one variable, so has only finitely many zeroes. Therefore, there are only finitely many possible x-values for points $(x, y) \in V(f, g)$. Switching x and y in the above argument shows that there are only finitely many possible y-values as well, hence in total only finitely many possible combinations (x, y).

(2) Let $X = V(I)$ be algebraic. First assume I is principal, say $I = (f)$. If $f \neq 0, 1$, write $f = f_1^{e_1} \cdots f_r^{e_r}$ with f_i irreducible. Then $X = \bigcup V(f_i)$ is a union of curves. Otherwise $I = (f_1, \dots, f_n)$ with $n \geq 1$, and the f_i have factorizations $f_i = \prod_j g_{ij}$, so that $X = \bigcup_{j_1,\dots,j_n} V(g_{1j_1}, \dots, g_{nj_n})$. Now each argument is finite by (1), so X is a finite union of finite sets, hence finite.

In conclusion, the affine sets of $\mathbb{A}^2$ are $\emptyset$, finite unions of points, finite unions of curves, and $\mathbb{A}^2$.

(3) Out of the affine sets characterized above, the irreducible ones are exactly single points, single curves, and $\mathbb{A}^2$, hence $V(x, y) \subseteq V(x) \subseteq V(0)$ is a maximal chain of irreducible subspaces, and $\dim \mathbb{A}^2 = 2$.