Exercise 1

Since $V(I(X))$ is closed by definition and onctains X by 1.10(a), the inclusion $\overline{X} \subseteq V(I(X))$ is clear. For the converse, let $\overline{X} = V(J)$ for some ideal J. Then $J \subseteq I(V(J))$ by 1.10(a), and applying V to this yields

$$V(I(X)) \subseteq V(I(\overline{X})) = V(I(V(J))) \subseteq V(J) = \overline{X}.$$

Exercise 2

We proceed by induction. For $n = 1$, if we had $f(x) = 0$ for all $x \in K$, then $X - x$ would need to be a factor, which is absurd. Now let n be arbitrary and let $0 \neq f = \sum_{i=0}^{m} f_i(x_1, \ldots, x_{n-1})x_n^i \in K[x_1, \ldots, x_n]$ be some nonzero polynomial such that $f(c) = 0$ for all $c \in K^n$. Fix $c_1, \ldots, c_{n-1} \in K$. Then $f(c_1, \ldots, c_{n-1}, x_n)$ is a polynomial in one variable that vanishes for all $x_n \in K$, hence by the $n = 1$ case $f_i(c_1, \ldots, c_{n-1}) = 0$. This is true for all $c_1, \ldots, c_{n-1}$, so by induction $f_i = 0$ and thus $f = 0$.

The contrapositive of the proven statement is that if f vanishes everywhere, then $f = 0$. This is exactly what $I(\mathbb{A}_K^n) = (0)$ says.

Exercise 3

We claim that

$$X = V(X^2 - YZ, XZ - X) = \underbrace{V(X, Y)}_{=:X_1} \cup \underbrace{V(X, Z)}_{=:X_2} \cup \underbrace{V(Z - 1, X^2 - Y)}_{=:X_3}$$

is the desired decomposition into irreducible components. First we show the above equality. Let $(x, y, z) \in X$. Then $0 = xz - x = x(z - 1)$, so either $x = 0$ or $z = 1$. In the first case, we have $0 = x^2 - yz = -yz$, so either $y = 0$ or $z = 0$, corresponding to X_1 and X_2, respectively. If, on the other hand, $z = 1$, then $0 = x^2 - y$, and $(x, y, z) \in X_3$. The converse inclusions $X_i \subseteq X$ are clear.

Now we have to show that the X_i are irreducible. Let $I_1 = (X, Y)$, $I_2 = (X, Z)$ and $I_3 = (Z - 1, X^2 - Y) \triangleleft K[X, Y, Z]$. We claim that $I(X_i) = I_i$ and that all the I_i are prime, which shows irreducibility according to the lecture. By Hilbert's Nullstellensatz we have $I(V(I_i)) = \sqrt{I_i}$, so it suffices to show primality, since prime ideals are radical. For that, note that $K[X, Y, Z]/I_1 \cong K[Z]$ and $K[X, Y, Z]/I_2 \cong K[Y]$ are clearly integral domains. For I_3, consider the ring morphism defined by

$$\varphi : K[X, Y, Z] \mapsto K[X], \qquad X \mapsto X, Y \mapsto X^2, Z \mapsto 1.$$

It is clearly surjective, so it remains to show that $\ker \varphi = I_3$, where $\supseteq$ is clear. So let $f \in \ker \varphi$. Using division with remainder twice, we may write $f = g \cdot (Z - 1) + h \cdot (Y - X^2) + k$ with $g \in K[X, Y, Z]$, $h \in K[X, Y]$ and $k \in K[X]$. Now $0 = \varphi(f) = \varphi(k) = k$, hence $f \in I_3$.

Geometrically, X_1 is the Z-axis, X_2 the Y-axis, and X_3 is a standard parabola $y = x^2$ over $z = 1$.

Exercise 4

(a) $Y_1 = \{(t, t^2) \mid t \in K\} = V(Y - X^2)$. Further, $Y - X^2 \in K[X, Y]$ is irreducible (e.g. by Eisenstein), hence $(Y - X^2)$ is a prime ideal. As before, this implies $I(Y_1) = I(V(Y - X^2)) = \sqrt{(Y - X^2)} = (Y - X^2)$.

(b) $Y_2 = \{(t^2, t^3) \mid t \in K\} = V(Y^2 - X^3)$: "$\subseteq$" is clear, for the converse let $(x, y) \in V(Y^2 - X^3)$. Since K is algebraically closed, there is $t \in K$ such that $t^2 = x$. Then $y^2 = x^3 = t^6$, so $y = \pm t^3$, and either $(x, y) = (t^2, t^3)$ or $(x, y) = ((-t)^2, (-t)^3) \in Y_2$. Again it suffices to show that $Y^2 - X^3$ is irreducible in $K[X, Y]$ to conclude $I(Y_2) = (Y^2 - X^3)$. So let $Y^2 - X^3 = fg$. If $\deg_Y f = 0$, then comparing coefficients of Y^2 we must have $f \mid 1$, i.e. f is a unit. The only other possibility is $\deg_Y f = \deg_Y g = 1$. As before the leading coefficients must be units, so wlog we may take them to be one. Hence write $f = Y - h(X)$, $g = Y - k(X)$ for $h, k \in K[X]$. Comparing linear terms we see $h = k$, and thus $X^3 = h^2$, which is impossible since the right hand side has even degree.

(c) We will show $Y_3 = \{(t^3, t^4, t^5) \mid t \in K\} = V(X^4 - Y^3, X^5 - Z^3, Y^5 - Z^4)$. Again "$\subseteq$" is clear. Let $(x, y, z) \in V(X^4 - Y^3, X^5 - Z^3, Y^5 - Z^4)$. Let t be such that $x = t^3$. Then $y^3 = x^4 = t^{12}$, so $y = \omega t^4$ for some third root of unity ω. Similarly, $z = \zeta t^5$. Now from

$$0 = y^5 - z^4 = \omega^2 t^{20} - \zeta t^{20} = (\omega^2 - \zeta) t^{20}$$

we see that $\zeta = \omega^2$, hence

$$(x, y, z) = ((\omega t)^3, (\omega t)^4, (\omega t)^5) \in Y_3.$$

Further, the map $\mathbb{A}^1 \to Y_3,\ t \mapsto (t^3, t^4, t^5)$ is regular, hence continuous, hence maps irreducible spaces to irreducible spaces. Since $\mathbb{A}^1$ is irreducible and the map is surjective, it follows that Y_3 is a variety.