Exercise 1

i) Let $\pi : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]/I(X) \cong A(X)$ be the canonical projection. Last time we saw that $V_X(J) = V(\pi^{-1}(J))$. Similarly, I claim that $I_X(Z) = (\pi(I(Z)))$ is the ideal generated by the image of $I(Z)$ under π, for any subset $Z \subseteq X$. Indeed, for $f \in A(X)$ and $z \in X$ one has $f(z) = 0$ if and only if $F(z) = 0$ for some, or all, $F \in K[x_1, \ldots, x_n]$ such that $\pi(F) = f$.

Therefore,

$$I_X(V_X(I)) = \pi(I(V(\pi^{-1}(I))) = \pi(\sqrt{\pi^{-1}(I)}) = \pi(\pi^{-1}(\sqrt{I})) = \sqrt{I},$$

where the second to last equality may require further justification: Let $f : R \to S$ be a ring homomorphism and $I \subseteq S$ an ideal. Then $f^{-1}(\sqrt{I}) = \sqrt{f^{-1}(I)}$, since

$$\begin{aligned} x \in f^{-1}(\sqrt{I}) &\iff f(x) \in \sqrt{I} \iff f(x)^n = f(x^n) \in I \text{ for some } n > 0 \\ &\iff x^n \in f^{-1}(I) \text{ for some } n > 0 \iff x \in \sqrt{f^{-1}(I)} \end{aligned}$$

ii)

$$A(Z) \cong K[x_1, \ldots, x_n]/I(Z) \cong (K[x_1, \ldots, x_n]/I(X))/(\pi(I(Z))) \cong A(X)/I_X(Z)$$

iii) Z is a variety iff $A(Z) \cong A(X)/I_X(Z)$ is integral iff $I_X(Z)$ is prime.

Exercise 2

i) Let $\varphi \in \mathcal{O}_{\mathbb{A}^2}(U) \subseteq K(\mathbb{A}^2) \cong K(x, y)$. In particular, φ is given globally as a rational function $\varphi = \frac{f}{g}$ in x, y, wlog with $f, g \in K[x, y]$ coprime.Suppose $g \notin K^\times$. Then $V(g)$ is infinite (If $g = \sum a_i(y)x^i \notin K[y]$, then there are only finitely many y that make all of the $a_i, i > 0$ vanish simultaneously, hence for any other y, $g(-, y)$ is a nonconstant polynomial in x and therefore has at least one zero.) But this contradicts $V(g) \subseteq \mathbb{A}^2 \setminus U = \{0\}$. Hence g is a unit and $\varphi \in K[x, y] = A(\mathbb{A}^2)$.

ii) Assume that U were isomorphic to some affine variety and consider the inclusion $i : U \to \mathbb{A}^2$. Then $i^* : K[x_1, x_2] \cong A(\mathbb{A}^2) \to A(U) \cong K[x_1, x_2]$ is the identity, in particular an isomorphism. By the lecture, this should imply that i is an isomorphism as well, but it clearly isn't surjective.

Exercise 3

i) Since $x^2 + y^2 - 1$ is irreducible (Eisenstein with $p = y - 1$), $I(Y) = (x^2 + y^2 - 1)$. Hence regular maps $X \to Y$ correspond naturally to morphisms

$$K[x, y]/(x^2 + y^2 - 1) \cong A(Y) \to A(X) \cong K[t].$$

Such a morphism is uniquely determined by the images $\varphi(t), \psi(t)$ of x and y, respectively, which have to satisfy $1 = \varphi(t)^2 + \psi(t)^2 = (\varphi(t) + i\psi(t))(\varphi(t) - i\psi(t))$. Thus both factors are units, hence so are φ, ψ. Hence all maps $X \to Y$ are constant.

ii) $t^2 + 1 \neq 0$ for $t \neq \pm i$ implies that φ is regular on U, and $(t^2 - 1)^2 + (-2t)^2 = (t^2 + 1)^2$, as well as $t^2 - 1 \neq t^2 + 1$ show that the image of φ is contained in $V \setminus \{(1, 0)\}$.

iii) Let

$$\psi : V \to U, \qquad (x, y) \mapsto \frac{y}{x - 1}.$$

This is certainly regular on V, and $\frac{y}{x-1} = \pm i$ would imply $y = \pm i(x-1)$, hence $1 - x^2 = y^2 = -(x-1)^2$, i.e. $x = 1$. So ψ is well-defined, and one calculates

$$\begin{aligned}
\psi \circ \varphi(t) &= \frac{-2t/(t^2+1)}{(t^2-1)/(t^2+1)-1} = \frac{-2t}{t^2-1-(t^2+1)} = \frac{-2t}{-2} = t, \\
\varphi \circ \psi(x,y) &= \left(\frac{y^2/(x-1)^2-1}{y^2/(x-1)^2+1}, \frac{-2y/(x-1)}{y^2/(x-1)^2+1}\right) = \left(\frac{y^2-(x-1)^2}{y^2+(x-1)^2}, \frac{-2y(x-1)}{y^2+(x-1)^2}\right) \\
&= \left(\frac{(y^2+x^2-1)-2x^2+2x}{(y^2+x^2-1)-2x+2}, \frac{-2(x-1)y}{(x^2+y^2-1)-2x+2}\right) \\
&= \left(\frac{-2x^2+2x}{-2x+2}, \frac{-2(x-1)y}{-2(x-1)}\right) = (x,y)
\end{aligned}$$

Therefore, φ and ψ are mutually inverse regular maps, i.e. isomorphisms.