Exercise 1

(1) Let $X = V(f_1, \dots, f_k)$ and $Y = V(g_1, \dots, g_l)$ with $f_i \in K[x_1, \dots, x_n]$, $g_j \in K[y_1, \dots, y_m]$. Then, considering all polynomials as elements of $K[x_1, \dots, x_n, y_1, \dots, y_m]$, one has

$$X \times Y = V(f_1, \dots, f_k, g_1, \dots, g_l).$$

Indeed, let $(x, y) \in X \times Y$. Then $f_i(x, y) = f_i(x) = 0$ and $g_j(x, y) = g_j(y) = 0$ for all i, j. Conversely, let $(x, y) \in V(f_1, \dots, f_k, g_1, \dots, g_l)$. Then $0 = f_i(x, y) = f_i(x)$ for all i, so $x \in X$, and similarly $y \in Y$.

(2) Say $X \times Y = Z_1 \cup Z_2$ with Z_1, Z_2 closed. For $x \in X$, one has

$$Y \cong \{x\} \times Y = (\{x\} \times Y \cap Z_1) \cup (\{x\} \times Y \cap Z_2).$$

Since Y is irreducible, we have $\{x\} \times Y \subseteq Z_1$ or Z_2. For $y \in Y$, let $\iota_y : X \to X \times Y$, $x \mapsto (x, y)$. This is a regular map, hence continuous. Set

$$X_i := \{x \in X \mid \{x\} \times Y \subseteq Z_i\} = \bigcap_{y \in Y} \iota_y^{-1}(Z_i)$$

for $i = 1, 2$. By the first equation and the above, $X = X_1 \cup X_2$. By the second equality, the X_i are closed, so wlog $X = X_1$, but that means $X \times Y = Z_1$.

Exercise 2

(1) It suffices to convince oneself that $X \times Y$ is the categorical product in the category of affine algebraic sets, then it is clear that the contravariant equivalence $A(-)$ sends products to coproducts.

But this is easy: The projections $X \times Y \to X, Y$ are regular, and for regular maps $Z \to X, Y$ there is a unique set map $Z \to X \times Y$, and one quickly verifies that it is regular.

(2) Let X, Y be varieties with $A = A(X)$ and $B = A(Y)$. Then $A \otimes_K B = A(X \times Y)$ is integral, since $X \times Y$ is a variety.

(3) No:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2 + 1) \cong \mathbb{C}[x]/(x + 1) \times \mathbb{C}[x]/(x - 1) \cong \mathbb{C} \times \mathbb{C}.$$

Exercise 3

$V^p(J^h)$ is closed by definition and contains X, since $f^h([1 : x_1 : \dots : x_n]) = f(x_1, \dots, x_n) = 0$ for $x \in X$. Thus $\overline{X} \subseteq V^p(J^h)$. Conversely, let $f \in I^p(X)$. We may assume that $X_0 \nmid f$, since otherwise $f/X_0 \in I^p(X)$ as well. Then $f(1, x_1, \dots, x_n) = 0$ for all $x \in X$, so $f(1, X_1, \dots, X_n) \in \sqrt{J}$. But now $f = f(1, X_1, \dots, X_n)^h \in \sqrt{J}^h = \sqrt{J^h}$. So $I^p(X) \subseteq I^p(V^p(J^h))$, i.e. $V^p(J^h) \subseteq V^p(I^p(X)) = \overline{X}$.