تمرین دوم مکانیک کوانتومی ۱ — به نام او — عرفان عابدی، 9631427

1.

$a)\ AB - BA = \frac{d}{dx} - x\frac{d^2}{dx^2}$ b) $AB - BA = \frac{d^2}{dx^2} + 1 - x\frac{d}{dx} - x^2 - \frac{d^2}{dx^2} + 1 - x\frac{d}{dx} + x^2 = 2 - 2x\frac{d}{dx}$

c) $AB - BA = 1 - 1 = 0$ d) $\frac{d^3}{dx^3} + 2 - x\frac{d}{dx} - x^3 - \frac{d^3}{dx^3} + 1 - x^2\frac{d}{dx} + x^3 = 3 - x\frac{d}{dx} - x^2\frac{d}{dx}$

2.

به نقل از کتاب زتیلی، سرعت گروه سرعتی‌ست که کل ذره با آن سرعت حرکت می‌کند و سرعت فاز سرعتی‌ست که اگر موج‌های درون wavepacket ذره را به صورت مستقل نگاه کنیم، در درون ذره (یعنی مکان‌هایی که حد احتمال آن بیش‌تر است) با این سرعت حرکت می‌کنند و بسته به گذرایی محیط دارند، پس سرعتی که الکترون در اصل دارد حرکت می‌کند همان سرعت گروه‌ش است، اما موج‌های درون الکترون با سرعت فاز حرکت می‌کنند که تغییرات فاز موج الکترون را با این سرعت تغییر می‌دهند.

3.

3) a) $1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|\psi\rangle dx$

$= \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} A^2 e^{-(x-a)^2/\Delta^2} dx = A^2(\pi\Delta^2)^{\frac{1}{2}} = 1$

$\Longrightarrow$ Normalized : $\psi(x) = \frac{1}{(\pi\Delta^2)^{\frac{1}{4}}} e^{-(x-a)^2/2\Delta^2}$

3) b) $P(x)\,dx = |\psi(x)|^2 dx = \frac{1}{\sqrt{\pi\Delta^2}} e^{-(x-a)^2/\Delta^2} dx$

3.c) $\langle X\rangle = \langle\psi|X|\psi\rangle = \int_{-\infty} \langle\psi|x\rangle\langle x|X|\psi\rangle dx$

Knowing that $\langle x|X|\psi\rangle = \int_{-\infty}^{+\infty} \langle x|X|x'\rangle\langle x'|\psi\rangle dx'$

$= \int_{-\infty}^{+\infty} x\,\delta(x-x')\psi(x')dx' = x\psi(x)$,

$\langle X\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\,x\,\psi(x)\,dx = \frac{1}{\sqrt{\pi\Delta^2}}\int_{-\infty}^{+\infty} e^{-(x-a)^2/\Delta^2} x\,dx$

ادامه‌ی قسمت C در صفحه‌ی بعد:

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Change of Variable from $y = x-a$ and $dy = dx$,

$$\langle x\rangle = \frac{1}{\sqrt{\pi\Delta^2}}\int_{-\infty}^{+\infty}(y+a)\,e^{-y^2/\Delta^2}\,dy = a$$

$$\Delta x = \left[\langle\psi|(x-\langle x\rangle)^2|\psi\rangle\right]^{1/2} = \left[\langle\psi|x^2-2x\langle x\rangle+\langle x\rangle^2|\psi\rangle\right]^{1/2}$$

$$= \left[\langle\psi|x^2-\langle x\rangle^2|\psi\rangle\right]^{1/2} \quad (\text{Since } \langle\psi|x|\psi\rangle = \langle x\rangle)$$

$$= \left[\langle x^2\rangle-\langle x\rangle^2\right]^{1/2} = \left[\langle x^2\rangle-a^2\right]^{1/2}.$$

$$\langle x^2\rangle = \frac{1}{(\pi\Delta^2)^{1/2}}\int_{-\infty}^{+\infty} e^{-(x-a)^2/2\Delta^2}\cdot x^2\cdot e^{-(x-a)^2/2\Delta^2}\,dx$$

$$= \frac{1}{\sqrt{(\pi\Delta)^2}}\int_{-\infty}^{+\infty} e^{-y^2/\Delta^2}(y^2+2ya+a^2)\,dy = \frac{\Delta^2}{2}+a^2$$

$$\Rightarrow \Delta x = \frac{\Delta}{\sqrt{2}}$$

قسمت د-۱ در صفحه‌ی بعد:

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P-1) $\langle \hat{p} \rangle = \int_{-\infty}^{+\infty} \varphi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \varphi(x) dx$

Putting this into the definition of FT

$$\Rightarrow \langle p \rangle = \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iqx} \tilde{\varphi}(q) dq\right) \frac{\hbar}{i} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{\varphi}(k) dk\right) dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-iqx} \tilde{\varphi}^*(q) dq\right) \left(\int_{-\infty}^{\infty} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{ikx} \tilde{\varphi}(k) dk\right) dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\varphi}^*(q) \tilde{\varphi}(k) e^{-iqx} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{ikx} dx\, dq\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\varphi}^*(q) \tilde{\varphi}(k) \left(\int_{-\infty}^{+\infty} e^{-iqx} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{ikx} dx\right) dq\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\varphi}^*(q) \tilde{\varphi}(k) \underbrace{\left(\int_{-\infty}^{+\infty} \hbar k\, e^{-iqx} e^{ikx} dx\right)}_{= 2\pi \delta(k-q) \hbar k} dq\, dk$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{\varphi}^*(q) \tilde{\varphi}(k) \hbar k\, \delta(k-q)\, dq\, dk$$

$$\langle \hat{p} \rangle = \int_{-\infty}^{+\infty} \tilde{\varphi}^*(k) \hbar k\, \tilde{\varphi}(k) dk = \int_{-\infty}^{+\infty} |\tilde{\varphi}(k)|^2 \hbar k\, dk$$

$$\langle \hat{p}^2 \rangle = \int_{-\infty}^{+\infty} dx\, \varphi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right)\left(-i\hbar \frac{\partial}{\partial x}\right) \varphi(x)$$

$$= -\hbar^2 \int_{-\infty}^{+\infty} dx\, \varphi^*(x) \frac{\partial^2}{\partial x^2} \varphi(x) =$$

$$= \hbar^2 \int_{-\infty}^{+\infty} dx \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq\, e^{iqx} \tilde{\varphi}(q)\right)^* \frac{\partial^2}{\partial x^2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{ikx} \tilde{\varphi}(k)\right)$$

$$= \frac{\hbar^2}{2\pi} \int_{-\infty}^{+\infty} dx \left(\int_{-\infty}^{+\infty} dq\, e^{-iqx} \tilde{\varphi}^*(q)\right) \left(\int_{-\infty}^{+\infty} dk \left(\frac{\partial^2}{\partial x^2} e^{ikx}\right) \tilde{\varphi}(k)\right)$$

$$= -\frac{\hbar^2}{2\pi} \int_{-\infty}^{+\infty} dx \left(\int_{-\infty}^{+\infty} dq\, e^{-iqx} \tilde{\varphi}^*(q)\right) \left(\int_{-\infty}^{+\infty} dk\, (-k^2)\, e^{ikx} \tilde{\varphi}(k)\right)$$

$$= \hbar^2 \int_{-\infty}^{+\infty} dq\, \tilde{\varphi}^*(q) \int_{-\infty}^{+\infty} dk\, k^2 \tilde{\varphi}(k) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, e^{i(k-q)x}}_{= \delta(k-q)}$$

$$\langle \hat{p}^2 \rangle = \int_{-\infty}^{+\infty} dk\, |\tilde{\varphi}(k)|^2 (\hbar k)^2$$

Taylor expansion $= \langle f(0) + \hat{p} f'(0) + \frac{\hat{p}^2}{2!} f''(0) + \cdots \rangle$

$$= f(0) + \langle \hat{p} \rangle f'(0) + \frac{\langle \hat{p} \rangle^2}{2!} f''(0) + \cdots$$

Using our Previous Results:

$$\langle f(\hat{p}) \rangle = f(0) + f'(0) \int_{-\infty}^{\infty} |\tilde{\varphi}(k)|^2 \hbar k \, dk + \frac{1}{2!} f''(0) \int_{-\infty}^{\infty} |\tilde{\varphi}(k)|^2 (\hbar k)^2 dk + \cdots$$

$$= \int_{-\infty}^{\infty} |\tilde{\varphi}(k)|^2 \left(f(0) + \hbar k f'(0) + \frac{(\hbar k)^2}{2!} f''(0) + \cdots \right) dk$$

$$= \int_{-\infty}^{\infty} |\tilde{\varphi}(k)|^2 f(\hbar k) \, dk$$

قسمت د-۲:

D-2) $\psi(x) = \frac{1}{\sqrt[4]{\pi \Delta^2}} e^{-(x-a)^2/2\Delta^2}$

$$\tilde{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, e^{-ikx} \frac{1}{\sqrt[4]{\pi \Delta^2}} e^{-(x-a)^2/2\Delta^2}$$

$$= \frac{1}{\sqrt{2\pi} \times \sqrt[4]{\pi \Delta^2}} \int_{-\infty}^{\infty} e^{-ikx} e^{-(x-a)^2/2\Delta^2} dx$$

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First, we consider $a = 0$ and $\Delta = 1$ for integration, Calculating $F.T(e^{-x^2})$

$\frac{d}{dk}\hat{f}(k) = \int_{-\infty}^{+\infty}(-ix)e^{-ikx}dx\, e^{-x^2} = \frac{1}{2}i\int_{-\infty}^{+\infty}\left(\frac{d}{dx}e^{-x^2}\right)e^{-ikx}dx$

Integrating by Parts we have: $\frac{d}{dk}\hat{f}(k) = -\frac{1}{2}k.\int_{-\infty}^{+\infty}e^{-x^2}e^{-ikx}dx = -\frac{1}{2}k\hat{f}(k)$

$\frac{d}{dk}\hat{f}(k) = -\frac{1}{2}k\hat{f}(k) \Longrightarrow \hat{f}(k) = c_0 . e^{-\frac{k^2}{4}}$ $\quad c_0 = \int_{-\infty}^{+\infty}f(x)dx = \sqrt{\pi} = \hat{f}(0)$ Gaussian

$\widehat{f(x-a)} = e^{-ika}\hat{f}(k)$

$\widehat{f(bx)} = \frac{1}{|b|}\hat{f}(k/b)$ Using these two Properties of the F.T,

$\hat{f}(k) = \sqrt{\pi}\, e^{-\frac{k^2}{4}}$ $\quad \widehat{f(x-a)} = \sqrt{\pi}\, e^{-ika}e^{-\frac{k^2}{4}}$

$\widehat{f\left(\frac{x-a}{\sqrt{2}\Delta}\right)} = \frac{1}{|\sqrt{2}\Delta|}\sqrt{\pi}\, e^{\frac{-ika}{\sqrt{2}\Delta}}e^{-\frac{k^2}{8\Delta^2}} =$

Returning the coefficients, we have: $\boxed{\tilde{\varphi}(k) = \frac{\sqrt{\pi}}{\sqrt{2\pi}\sqrt[4]{\pi\Delta^2}|\sqrt{2}\Delta|}e^{\frac{-ika}{\sqrt{2}\Delta}}e^{\frac{-k^2}{8\Delta^2}}}$

$|\tilde{\varphi}(k)|^2 = \tilde{\varphi}^*(k)\times\tilde{\varphi}(k) = \frac{1}{4\Delta^2\sqrt{\pi\Delta^2}}e^{\frac{-ika}{\sqrt{2}\Delta}}e^{\frac{-k^2}{8\Delta^2}}e^{\frac{ika}{\sqrt{2}\Delta}}e^{\frac{-k^2}{8\Delta^2}}$

$= \frac{1}{4\Delta^2\sqrt{\pi\Delta^2}}e^{\frac{-k^2}{4\Delta^2}}$

$\langle\hat{p}\rangle = \frac{1}{4\Delta^2\sqrt{\pi\Delta^2}}\int_{-\infty}^{+\infty}e^{\frac{-k^2}{4\Delta^2}}\hbar k\, dk = -\frac{\hbar}{8\Delta^2\sqrt{\pi\Delta^2}}4\Delta^2 e^{\frac{-k^2}{4\Delta^2}}$

$= \cancel{\frac{\hbar}{4\pi\sqrt{\pi}}e^{\frac{-k^2}{4\Delta^2}}} = \frac{-\hbar}{2\sqrt{\pi\Delta^2}}$

ادامه‌ی د-۲ و ه:

$$\Delta P = \left(\langle p^2\rangle - \langle p\rangle^2\right)^{1/2}$$ \ Same way we did for X)

$$\langle p^2\rangle = \frac{1}{4\Delta^2\sqrt{\pi\Delta^2}}\int_{-\infty}^{+\infty} e^{\frac{-k^2}{4\Delta^2}}\hbar^2 k^2\,dk$$ (Same with a = 0) $\Rightarrow$

$$= \frac{\hbar^2}{4\Delta^2\sqrt{4\pi\Delta^2}} \times \frac{4\Delta^2\sqrt{\pi 4\Delta^2}}{2} = \hbar^2$$

$$\left(\langle p^2\rangle - \langle p\rangle^2\right)^{1/2} = \left(\frac{\hbar^2}{1} - \frac{\hbar^2}{4\pi\Delta^2}\right)^{1/2} = \hbar\sqrt{\left(1 - \frac{1}{4\pi\Delta^2}\right)}$$

$$\Delta x \cdot \Delta P = \frac{\Delta}{\sqrt{2}} \times \hbar \times \sqrt{1 - \frac{1}{4\pi\Delta^2}} \neq \frac{\hbar}{2}$$

باید به مینیمم عدم قطعیت می‌رسیدیم. احتمالا در حسابات من اشتباهی رخ داده اما نمی‌توانم آن را بیابم.

ه) اگر موج حقیقی باشد، در انتگرال $\langle p\rangle$ فقط ترم k داریم که یک ترم فرد است، بقیه ترم‌ها انتگرال زوج هستند $\Leftarrow$ انتگرال صفر می‌شود از بازه‌ی $(-\infty, +\infty)$.

قسمت و:

In the x Basis, $\langle P \rangle = \int dx\, \psi^*(x) \, (-i\hbar \frac{d}{dx}) \, \psi(x)$

$$\Rightarrow -i\hbar \int dx\, \psi^*(x)\, e^{-\frac{iP_0x}{\hbar}} \frac{d}{dx} \psi(x)\, e^{+\frac{iP_0x}{\hbar}}$$

$$= -i\hbar \int dx\, \psi^*(x)\, e^{-\frac{iP_0x}{\hbar}} \left[+\frac{iP_0}{\hbar} \psi(x)\, e^{\frac{iP_0x}{\hbar}} + e^{\frac{iP_0x}{\hbar}} \frac{d}{dx} \psi(x) \right]$$

$$= -i\hbar \left[\int dx \left(\frac{iP_0}{\hbar} e^{-\frac{iP_0x}{\hbar}} \psi^*(x)\, e^{\frac{iP_0x}{\hbar}} \psi(x) \right) + \int dx \left(\psi^*(x)\, e^{-\frac{iP_0x}{\hbar}} e^{\frac{iP_0x}{\hbar}} \frac{d}{dx} \psi(x) \right) \right]$$

$$= P_0 \underbrace{\int dx\, \psi^*(x) \psi(x)}_{1} \underbrace{- i\hbar \int dx\, \psi^*(x) \frac{d}{dx} \psi(x)}_{\langle P \rangle}$$

$$= P_0 + \langle P \rangle$$