پاسخنامه:

تمرین ها و مطالب سر کلاس مکمل کتاب های معرفی شده هستند و از محتوای هر دو برای طرح سوالات امتحانی استفاده میشود.
در مواردی که مطالب گفته شده در کتاب های معرفی شده موجود باشد به فصل های کتاب مورد نظر اشاره می شود.

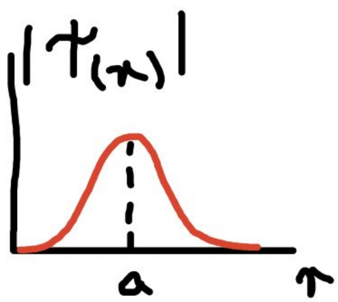
حد بالای رابطه ی عدم قطعیت (فصل ۴ شانکار + تمرین های سری دوم دانشگاه MIT OpenCourseWARE)

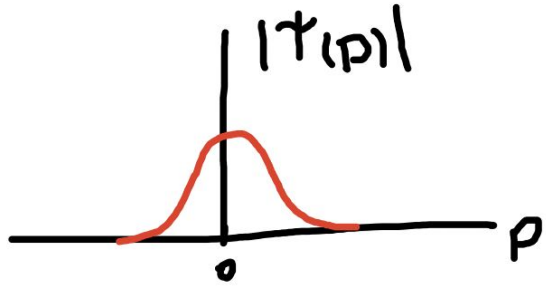
در این تمرین قصد داریم تابع موجی (تابع موج گاوسی) را بررسی کنیم که اصطلاحا رابطه عدم قطعیت را اشباع می کند:

The Gaussian happens to saturate the lower bound of the uncertainty relation

$\Delta X \,.\, \Delta P \;=\; \hbar/2$

این تابع موج همچنین تابعی شاخص است زیرا که هم پوزیشن به نسبت معینی دارد و هم مومنتوم به نسبت معین:





تابع موج گاوسی زیر را در نظر بگیرید و مراحل زیر را به ترتیب انجام دهید:

$$\psi(x) \;=\; A\; exp(-(x-a)^2/2\Delta^2)$$

الف) تابع بالا را نرمالایز کنید
ب) احتمال حضور الکترون در فاصله x تا x + dx یا $(P(x))\,dx$ را بدست آوردید.
ج) عدم قطعیت در مکان را حساب کنید: $\Delta x = sqrt(<x^2> \; - \; <x>^2)$

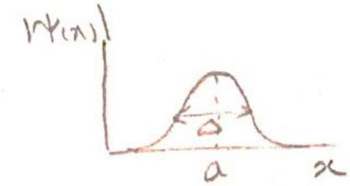
ولی p.w. در saturation مد p.v است، یعنی هم $|\psi(p)|$ هم $|\psi(x)|$

نامحدودی مشخص هستند. همینطور p.w. به صورت کلاسیک حرکت می کند (سرعت

envelope آن $\frac{p}{m}$ است!

Example 4.2.4 Shankar

Gaussian wfc



$$\psi(x) = A e^{\frac{-(x-a)^2}{2\Delta^2}}$$

Normalize:

$$1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \underbrace{\langle\psi|x\rangle}_{\psi^*(x)} \underbrace{\langle x|\psi\rangle}_{\psi(x)} dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$

$$= \int_{-\infty}^{\infty} A^2 e^{\frac{-(x-a)^2}{2\Delta^2} - \frac{(x-a)^2}{2\Delta^2}} dx = \int_{-\infty}^{\infty} A^2 e^{\frac{-(x-a)^2}{\Delta^2}} dx$$

نه! e^{ix} برای complex conj آن می شود e^{-ix}، اینجا برای e^{a}، a چون real است complex conj آن

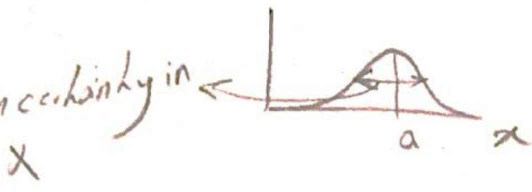
$$\int_{-\infty}^{\infty} A^2 e^{\frac{-(x-a)^2}{\Delta^2}} dx = A^2 (\pi\Delta^2)^{\frac{1}{2}}$$

نامحدودش برابر است.
انتگرال های گاوسی

$$\Rightarrow \boxed{\psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-(x-a)^2/2\Delta^2}}$$

$$P(x)\,dx = |\psi(x)|^2 dx = \frac{1}{(\pi\Delta^2)^{1/2}} e^{-(x-a)^2/\Delta^2} dx$$

که باهم همان شکل گوسی را دارد!



$\Rightarrow$ particle is more likely to be found around $x=a$

Uncertainty for x ?

↓

chances to find the particle away from $x=a$ drop beyond Δ → let's quantify it by U. for X

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$$

$$\langle X\rangle = \langle\psi|X|\psi\rangle =$$

$$\int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|X|\psi\rangle dx = \int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x) =$$

$x|\psi\rangle$

$\langle x|x|\psi\rangle$

$x\langle x|\psi\rangle$

$\psi^*(x) \quad x\,\psi(x)$

$$= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-a)^2/\Delta^2}\, x\, dx$$

$\Rightarrow$ If $y = x-a \Rightarrow \langle x\rangle = a$

$$\Delta X = \sqrt{\langle X^2\rangle - a^2}$$

3

$$\langle x^2 \rangle = \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-a)^2/2\Delta^2} \cdot x^2 \cdot e^{-(x-a)^2/2\Delta^2}$$

$$\boxed{I_{2n}(\alpha) = \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx} \quad I_0(\alpha) = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$= \frac{1}{(\pi\Delta^2)^{1/2}} \int_{-\infty}^{\infty} x^2 e^{-(x-a)^2/\Delta^2} dx$$

$\alpha = \frac{1}{\Delta^2} \Rightarrow \Delta^2 = \frac{1}{\alpha}$

$x - a = y$

$y + a = x$

$$= \frac{1}{\left(\frac{\pi}{\alpha}\right)^{1/2}} \int_{-\infty}^{\infty} (y+a)^2 e^{-\alpha y^2} =$$

$$\left(\frac{\alpha}{\pi}\right)^{1/2} \left[\underbrace{\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2} dy}_{-\frac{\partial}{\partial\alpha}\int e^{-\alpy^2} dy} + \underbrace{\int_{-\infty}^{\infty} a^2 e^{-\alpha y^2} dy}_{a^2\left(\frac{\pi}{\alpha}\right)^{1/2}} + \underbrace{\cancel{\int_{-\infty}^{\infty} 2ay\, e^{-\alpha y^2}}}_{0 \text{ odd func.}} \right]$$

$$-\frac{\partial}{\partial\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2}$$

$-3/2$

$-\frac{1}{2}\alpha$

$-3/2$

$$\alpha^{-1/2} \quad \frac{\pi^{1/2}\alpha}{2} \quad \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2} + a^2\left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$\Rightarrow \langle X^2 \rangle = a^2 + \frac{1}{2\alpha} = a^2 + \frac{1}{2}\Delta^2$$

$$\Rightarrow \Delta X = \sqrt{\langle X^2 \rangle - a^2} = \sqrt{\cancel{a^2} + \frac{1}{2}\Delta^2 - \cancel{a^2}} = \frac{\Delta}{\sqrt{2}}$$

د) حال قصد داریم مراحل بالا (احتمال وجود تکانه ای به خصوص، عدم قطعیت در تکانه) را برای متغیر تکانه تکرار کنیم ولی پیش از آن با استفاده از روابط فوریه و تعاریف داده شده، روابط کاربردی زیر را ثابت کنید و با استفاده از آنها مراحل فوق را برای تکانه تکرار کنید.

روابط فوریه:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, \bar{f}(k)$$

$$\bar{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$$

تعاریف:

$$< x > = \int dx\, P(x)\, x$$

$$< \hat{p}^{\,n} > = \int dx\, \psi^*(x)\, \hat{p}^{\,n}\, \psi(x)$$

$$\hat{p} = -i\hbar \frac{\delta}{\delta x}$$

د-۱) ثابت کنید:

$$< \hat{p} > = \int dk\, |\bar{\psi}(k)|^2\, \hbar k$$

$$< \hat{p}^{\,2} > = \int dk\, |\bar{\psi}(k)|^2 (\hbar k)^2$$

$$< f(\hat{p}) > = \int dk\, |\bar{\psi}(k)|^2 f(\hbar k)$$

(a) **(5 points)** By definition of the expectation value of an operator, we have

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, \hat{p}\, \psi(x)\, dx, \tag{25}$$

where the hat on $\hat{p}$ emphasizes the fact that $\hat{p}$ is an *operator* and not a number. An immediate consequence of this is that except for in certain special cases, we can't swap the order of the terms in Equation 25, *i.e.* $\psi^*(x)\, \hat{p}\, \psi(x) \neq \hat{p}\, \psi^*(x)\, \psi(x)$. We can see this explicitly for this particular problem by plugging in the form of the momentum operator:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, \frac{\hbar}{i} \frac{\partial}{\partial x}\, \psi(x)\, dx. \tag{26}$$

We can see that the derivative acts only on the second copy of ψ, whereas if we had written $\hat{p}\, \psi^*(x)\, \psi(x)$ it would've acted on the product of $\psi(x)$ and $\psi^*(x)$, which we know gives a different result from the product rule in calculus.

Let us now proceed by substituting into Equation 26 the definition of the Fourier transform:

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} \tilde{\psi}(q) dq \right)^* \frac{\hbar}{i} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{\psi}(k) dk \right) dx & \text{(27a)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-iqx} \tilde{\psi}^*(q) dq \right) \left(\int_{-\infty}^{\infty} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{ikx} \tilde{\psi}(k) dk \right) dx & \text{(27b)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\psi}^*(q) \tilde{\psi}(k) e^{-iqx} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{ikx} dx\, dq\, dk & \text{(27c)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\psi}^*(q) \tilde{\psi}(k) \left(\int_{-\infty}^{\infty} e^{-iqx} \frac{\hbar}{i} \frac{\partial}{\partial x} e^{ikx} dx \right) dq\, dk. & \text{(27d)}
\end{aligned}$$

A lot happened in the last few lines, and here are some subtleties to be aware of:

- In Equation 26, there are two copies of $\psi(x)$. When substituting these for their Fourier space representations, it is crucially important to use different variables for the Fourier variables (*i.e.* to *not* use k for both of them). To see this, recall that the integrals are really just fancy summations with q or k as dummy summation variables, and consider the following example, which serves as an analogy. Suppose we're trying to find the product of $a \equiv \sum_{i=1}^{2} i = 1 + 2 = 3$ and $b \equiv \sum_{j=1}^{2} j^2 = 1 + 4 = 5$. The answer is of course 15, but let's do this formally:

$$ab = \left(\sum_{i=1}^{2} i \right) \left(\sum_{j=1}^{2} j^2 \right) = \sum_{i=1}^{2} \sum_{j=1}^{2} i j^2 = 1 + 4 + 2 + 8 = 15. \tag{28}$$

If we (incorrectly) use the same dummy index for the two summations, it is easy to forget that we're doing two sums, which leads to missing many of the terms in the sum:

$$ab = \left(\sum_{i=1}^{2} i\right)\left(\sum_{i=1}^{2} i^2\right) \to \sum_{i=1}^{2} ii^2 = 1 + 8 = 9 \quad \text{(Wrong!)} \tag{29}$$

where the right arrow signifies an incorrect logical step.

- In going from Equation 27a to 27b, we brought the momentum operator and the complex conjugate inside the Fourier integrals. This is allowed because differentiation and taking the complex conjugate of something are both linear operations acting on integrals, which are just sums. By definition, linear operations are ones where the same answer is obtained regardless of whether we sum (*i.e.* integrate) first and then "operate" or "operate" first and then sum.
- In going from Equation 27b to 27c, we switched the order of integration. This is allowed by Fubini's theorem (a purely mathematical result) thanks to the independence of our three integration variables x, q, and k.
- In going from Equation 27c to 27d, $\tilde{\psi}(k)$ passed right through the derivative, because it is a function of k and not of x.

Proceeding with the algebra from Equation 27d, we have

$$\begin{aligned}
\langle p \rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(q)\tilde{\psi}(k)\left(\int_{-\infty}^{\infty} e^{-iqx}\hbar k\, e^{ikx}dx\right) dq\, dk && \text{(30a)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(q)\tilde{\psi}(k)\,\hbar k \underbrace{\left(\int_{-\infty}^{\infty} e^{i(k-q)x}dx\right)}_{=2\pi\delta(k-q)} dq\, dk && \text{(30b)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(q)\tilde{\psi}(k)\,\hbar k\,\delta(k-q)dq\, dk && \text{(30c)}\\
&= \int_{-\infty}^{\infty} \tilde{\psi}^*(k)\,\hbar k\,\tilde{\psi}(k)\, dk && \text{(30d)}\\
&= \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2\,\hbar k\, dk, && \text{(30e)}
\end{aligned}$$

which is our desired result. Note that Equation 30d is very similar in form to Equation 25, except we have k instead of x and $\tilde{\psi}$ instead of ψ. This suggests that while the momentum operator $\hat{p}$ takes the form of a derivative in real coordinate space, in Fourier space it is simply a multiplicative operator.

(b) **(5 points)** The steps to follow are the same as in part (a), except for the operator $\hat{p}$

which must be applied twice:

$$\begin{aligned}
\langle \hat{p}^2 \rangle &= \int_{-\infty}^{\infty} \mathrm{dx} \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x}\right) \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x) && (31a) \\
&= -\hbar^2 \int_{-\infty}^{\infty} \mathrm{dx} \psi^*(x) \frac{\partial^2}{\partial x^2} \psi(x) && (31b) \\
&= -\hbar^2 \int_{-\infty}^{\infty} \mathrm{dx} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{dq} e^{iqx} \tilde{\psi}(q)\right)^* \frac{\partial^2}{\partial x^2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{dk} e^{ikx} \tilde{\psi}(k)\right) && (31c) \\
&= -\frac{\hbar^2}{2\pi} \int_{-\infty}^{\infty} \mathrm{dx} \left(\int_{-\infty}^{\infty} \mathrm{dq} e^{-iqx} \tilde{\psi}^*(q)\right) \left(\int_{-\infty}^{\infty} \mathrm{dk} \left(\frac{\partial^2}{\partial x^2} e^{ikx}\right) \tilde{\psi}(k)\right) && (31d) \\
&= -\frac{\hbar^2}{2\pi} \int_{-\infty}^{\infty} \mathrm{dx} \left(\int_{-\infty}^{\infty} \mathrm{dq} e^{-iqx} \tilde{\psi}^*(q)\right) \left(\int_{-\infty}^{\infty} \mathrm{dk} \left(-k^2\right) e^{ikx} \tilde{\psi}(k)\right) && (31e) \\
&= \hbar^2 \int_{-\infty}^{\infty} \mathrm{dq} \tilde{\psi}^*(q) \int_{-\infty}^{\infty} \mathrm{dk} k^2 \tilde{\psi}(k) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{dx}\ e^{i(k-q)x}}_{\delta(k-q)} && (31f) \\
&= \int_{-\infty}^{\infty} \mathrm{dk} |\tilde{\psi}(k)|^2 (\hbar k)^2. && (31g)
\end{aligned}$$

Again we see that the operator $\hat{p}$ becomes a multiplicative operator in the momentum space.

(c) **(5 points)** We wish to prove that

$$\langle f(\hat{p}) \rangle = \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 f(\hbar k) dk. \tag{32}$$

The left hand side can be Taylor expanded to give

$$\begin{aligned}
\langle f(\hat{p}) \rangle &= \langle f(0) + \hat{p} f'(0) + \frac{\hat{p}^2}{2!} f''(0) + \ldots \rangle && (33a) \\
&= f(0) + \langle \hat{p} \rangle f'(0) + \frac{\langle \hat{p}^2 \rangle}{2!} f''(0) + \ldots && (33b)
\end{aligned}$$

To evaluate this, we need to know how to work out expectation values of powers of $\hat{p}$, *i.e.* $\langle \hat{p}^n \rangle$. One way to do this would be to go all the way back to $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$ and to find $\langle \hat{p}^n \rangle$ by brute force, but an easier way would be to use our result from the previous part. There, we showed that in Fourier space, the momentum operator takes the simple form of multiplication by $\hbar k$. It follows, then, that powers of $\hat{p}$ correspond to powers of $\hbar k$ in Fourier space, which means

$$\begin{aligned}
\langle f(\hat{p}) \rangle &= f(0) + f'(0) \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 \hbar k \, dk + \frac{1}{2!} f''(0) \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 (\hbar k)^2 \, dk + \ldots && (34a) \\
&= \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 \left(f(0) + \hbar k f'(0) + \frac{(\hbar k)^2}{2!} f''(0) + \ldots\right) dk && (34b) \\
&= \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 f(\hbar k) \, dk. && (34c)
\end{aligned}$$

د-۲) حال با توجه به روابط بالا ، $\overline{\psi}(k)$ ، $P(p) = |\overline{\psi}(k)|^2$ ، $<\widehat{p}>$ و Δp را حساب کنید.

$$\langle p\rangle = \int_{-\infty}^{\infty} \underbrace{|\tilde{\psi}(k)|^2}_{P(p)} \underbrace{\hbar k}_{p}\, dk$$

$p = \hbar k$

$dp = \hbar\, dk$

$$P(p) = |\langle k|\psi\rangle|^2 \quad [illegible] \quad [illegible] \quad |\langle p|\psi\rangle|^2$$

Shankar -> example 4.2.4

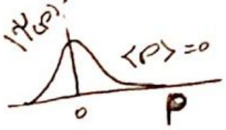
$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \psi(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \frac{e^{-(x-a)^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}} =$$

$p = \hbar k$

$$\boxed{= \left(\frac{\Delta^2}{\pi\hbar^2}\right)^{1/4} e^{-ipa/\hbar}\, e^{-p^2\Delta^2/2\hbar^2}} = \tilde{\psi}(k)$$

$$\Rightarrow \langle p\rangle = \int_{-\infty}^{\infty} \underbrace{\left(\frac{\Delta^2}{\pi\hbar^2}\right)^{\frac{1}{2}} e^{-p^2\Delta^2/2\hbar^2}}_{|\tilde{\psi}(k)|^2} \frac{p}{\hbar}\, dp = 0$$



$$\Rightarrow \Delta P = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\quad - 0}$$

$$\langle p^2\rangle = \int dk\, |\tilde{\psi}(k)|^2\, p^2 =$$

همان بالایی‌ها با جای p, p^2

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \left(\frac{\Delta^2}{\pi\hbar^2}\right)^{1/2} e^{-p^2\Delta^2/2\hbar^2}\, p^2\, dp$$

$$\underbrace{\quad}_{\frac{2\alpha}{\pi}}$$

$$\boxed{\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2} dy = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2}}$$

$\alpha = \Delta^2 / 2\hbar^2$

$$\langle p^2 \rangle = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2} \times \left(\frac{2\alpha}{\pi}\right)^{1/2} = \frac{\sqrt{2}}{2\alpha}$$

$$\frac{\sqrt{2}\,\hbar^2}{\Delta^2}$$

$$\Delta p = \frac{\hbar}{\sqrt{2}\,\Delta} \qquad \Delta x \cdot \Delta p = \hbar/2$$

We are solvrati

$\Delta x \cdot \Delta p \geq \hbar/2$

آیا حد بالای رابطه ی عدم قطعیت برای $\Delta x \,.\, \Delta p$ به دست آمد؟

ه) نشان دهید که اگر تابع موج حقیقی باشد $<\hat{p}>=0$
راهنمایی: نشان دهید تابع احتمال برای $\pm p$ برابر است.

For an arbitrary wave function $|\psi\rangle$, if we know its position-space form, we can find its momentum-space version as follows:

$$\langle p|\psi\rangle = \int \langle p|x\rangle\langle x|\psi\rangle\, dx \tag{11}$$

$$= \int \psi_p^*(x)\langle x|\psi\rangle\, dx \tag{12}$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi(x)\, dx \tag{13}$$

This has an interesting consequence if the position-space function $\psi(x)$ is real. The probability density for finding a particle in a state with momentum p is $|\langle p|\psi\rangle|^2$, which we can write as

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^*\langle p|\psi\rangle \tag{14}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{ip(x-x')/\hbar}\psi(x)\,\psi(x')\, dx\, dx' \tag{15}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{-ip(x'-x)/\hbar}\psi(x)\,\psi(x')\, dx\, dx' \tag{16}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{-ip(x-x')/\hbar}\psi(x')\,\psi(x)\, dx\, dx' \tag{17}$$

$$= |\langle -p|\psi\rangle|^2 \tag{18}$$

In the fourth line, since x and x' are dummy integration variables, both of which are integrated over the same range, we can simply swap them without changing anything. Note that the derivation relies on $\psi(x)$ being real, since if it were complex we would have

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^*\langle p|\psi\rangle \tag{19}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{ip(x-x')/\hbar}\psi(x)\,\psi^*(x')\, dx\, dx' \tag{20}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{-ip(x'-x)/\hbar}\psi(x)\,\psi^*(x')\, dx\, dx' \tag{21}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{-ip(x-x')/\hbar}\psi(x')\,\psi^*(x)\, dx\, dx' \tag{22}$$

$$\neq |\langle -p|\psi\rangle|^2 \tag{23}$$

since

و) نشان دهید اگر میانگین تکانه برای $\psi(x)$ برابر با $<\widehat{p}>$ باشد، میانگین تکانه برای $\psi(x)exp(ip_0x/\hbar)$ ، برابر با $<\widehat{p}>+p_0$ میباشد.

$$|\langle -p|\psi\rangle|^2 = \frac{1}{2\pi\hbar}\int\int e^{-ip(x-x')/\hbar}\psi(x)\psi^*(x')\,dx\,dx' \tag{24}$$

That is, for $|\langle -p|\psi\rangle|^2$ the position x' that is the argument of the $\psi^*(x')$ factor appears as the positive term ipx' in the exponential, but in 22 the argument of the complex conjugate wave function is x, which appears as the negative term $-ipx$ in the exponential.

Thus for any real wave function, the probability of the particle having momentum $+p$ is equal to the probability of it having $-p$, so for such wave functions, the mean momentum is always $\langle P\rangle = 0$.

As another example, suppose we have a wave function $\psi(x)$ with a mean momentum $\bar{p}$, so that

$$\langle\psi|P|\psi\rangle = \bar{p} \tag{25}$$

If we now multiply ψ by $e^{ip_0x/\hbar}$ where p_0 is a constant momentum, we can calculate the new mean momentum using 5:

$$\begin{aligned}
\langle P\rangle &= \left\langle e^{ip_0x/\hbar}\psi\,|P|\,e^{ip_0x/\hbar}\psi\right\rangle && (26)\\
&= -i\hbar\int e^{-ip_0x/\hbar}\psi^*(x)\frac{d}{dx}\left(e^{ip_0x/\hbar}\psi(x)\right)dx && (27)\\
&= -i\hbar\int e^{-ip_0x/\hbar}\psi^*\left[\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi(x)+e^{ip_0x/\hbar}\frac{d}{dx}\psi(x)\right]dx && (28)\\
&= \int p_0\psi^*\psi dx - i\hbar\int\psi^*(x)\frac{d}{dx}\psi(x)\,dx && (29)\\
&= p_0+\bar{p} && (30)
\end{aligned}$$

The first integral in the fourth line uses the fact that p_0 is constant and ψ is normalized so that

$$\int\psi^*\psi dx = 1 \tag{31}$$

محاسبات خوبی داشته باشید!

نگار اشعری