

Problem III - A Stack of Sums

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5 Points

$$\mathcal{S}_n^\infty := \sum_{\substack{x_i = x_{i-1} \\ x_1 = 0}}^{\infty} \sum_{i=1}^n \frac{1}{2^{x_i}} := \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \cdots \sum_{x_n=x_{n-1}}^{\infty} \frac{1}{2^{x_n}}$$

What are the values of \mathcal{S}_1^∞ , \mathcal{S}_2^∞ and \mathcal{S}_3^∞ ? Does \mathcal{S}_n^∞ converge for all $n \in \mathbb{N}$? If so, can you give a general answer for \mathcal{S}_n^∞ ?

$$\begin{aligned} \mathcal{S}_1^\infty &= \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 2^n \sum_{x=0}^{\infty} \frac{1}{2^x \cdot 2^n} = 2^n \sum_{x=0}^{\infty} \frac{1}{2^{x+n}} = 2^n \sum_{x=n}^{\infty} \frac{1}{2^x} \quad \forall n \in \mathbb{N} \\ &= \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 1 + \sum_{x_1=1}^{\infty} \frac{1}{2^{x_1}} = 2 \cdot \sum_{x_1=1}^{\infty} \frac{1}{2^{x_1}} \Rightarrow \sum_{x_1=1}^{\infty} \frac{1}{2^{x_1}} = 1 \Rightarrow \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 2 \cdot 1 = 2^1 \end{aligned}$$

$$\begin{aligned} \mathcal{S}_2^\infty &= \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \frac{1}{2^{x_2}} = \sum_{x_1=0}^{\infty} \left(\frac{1}{2^{x_1}} \sum_{x_2=0}^{\infty} \frac{1}{2^{x_2}} \right) = \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} \cdot 2 = \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1-1}} = \frac{1}{2^{-1}} \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} \\ &= 2^1 \cdot 2 = 4 = 2^2 \end{aligned}$$

$$\begin{aligned} \mathcal{S}_3^\infty &= \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \sum_{x_3=x_2}^{\infty} \frac{1}{2^{x_3}} = \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \frac{1}{2^{x_2-1}} = \frac{1}{2^{-1}} \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \frac{1}{2^{x_2}} \\ &= 2^1 \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1-1}} = 2^2 \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 2^2 \cdot 2 = 8 = 2^3 \end{aligned}$$

$$\begin{aligned} \mathcal{S}_n^\infty &= \sum_{x_1=0}^{\infty} \cdots \sum_{x_{n-1}=x_{n-2}}^{\infty} \sum_{x_n=x_{n-1}}^{\infty} \frac{1}{2^{x_n}} = \sum_{x_1=0}^{\infty} \cdots \sum_{x_{n-1}=x_{n-2}}^{\infty} \frac{1}{2^{x_{n-1}-1}} = 2^1 \sum_{x_1=0}^{\infty} \cdots \sum_{x_{n-1}=x_{n-2}}^{\infty} \frac{1}{2^{x_{n-1}}} \\ &= 2^1 \cdot \mathcal{S}_{n-1}^\infty \end{aligned}$$

$$\begin{aligned} \mathcal{S}_1^\infty &= 2^1 \\ \mathcal{S}_n^\infty &= 2^1 \cdot \mathcal{S}_{n-1}^\infty \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{S}_n^\infty &= 2^1 \cdot \mathcal{S}_{n-1}^\infty = 2^1 \cdot 2^1 \cdot \mathcal{S}_{n-2}^\infty = \cdots = \overbrace{2^1 \cdot 2^1 \cdots 2^1}^{n-1 \text{ times}} \cdot \mathcal{S}_1^\infty = 2^{n-1} \cdot 2^1 = 2^n \\ \Rightarrow \mathcal{S}_n^\infty &\text{ converges for every } n \in \mathbb{N}, \text{ whereas } \lim_{n \rightarrow \infty} \mathcal{S}_n^\infty \text{ diverges.} \end{aligned}$$