The Hardness of “Lemmings”

Another NP-complete Game

Philip Geißler

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Lemmings - an Overview

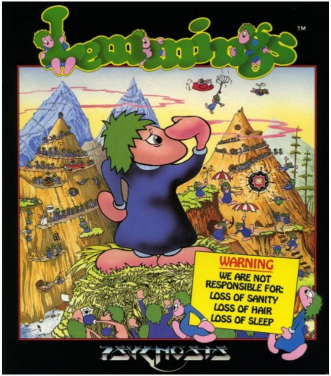

Figure 1: the game cover[1]

- released in 1991
- published by Psygnosis
- developed by DMA Design

- single player
- puzzle genre
- real time strategy

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[1] `https://goo.gl/1fnsdu` [2]

Lemmings - the Basics

Figure 2: an exemplary level[2]

[2] https://classicreload.com/lemmings.html

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Formalization

Formalization

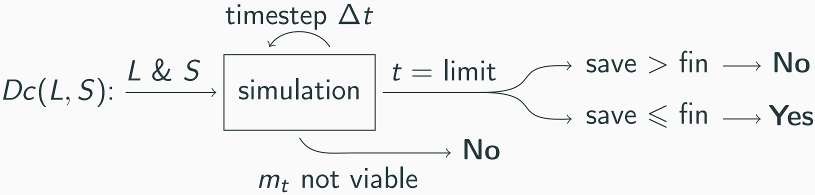
- time and space is discrete in "Lemmings"
- all other parameters are also discrete

→ level and solution can be formalized

$$\begin{aligned}
\texttt{level:}\quad & L = (\text{limit}^3, \text{save}, \text{lems}, \text{start}, \text{wdt}, \text{hgt}, \text{grid}, \text{exit}, \text{skills}) \\
\texttt{level size:}\quad & |L| \approx |grid| = constant \cdot wdt \cdot hgt \\
\texttt{solution:}\quad & S = \{m_1, m_2, m_3, \cdots\} \\
\texttt{move:}\quad & m_j = (\text{time}, \text{x}, \text{y}, \#\text{lemming}, \#\text{skill})
\end{aligned}$$

[3] Assumption: bounded by polynomial time in the size of the level

The Decision Problem



- $\forall\,(L \in \text{LEMMINGS})\ \exists S_x$ with $Dc(L, S_x) = \mathit{Yes}$

$\rightarrow$ LEMMINGS comprises all solvable levels

$\hookrightarrow$ 1-LEMMINGS comprises all solvable levels with 1 Lemming

- only LEMMINGS & 1-LEMMINGS will be investigated further

3 SAT

$$\texttt{3SAT:}\ \ F = (\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_3} \vee x_3 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge \cdots$$

$$\texttt{exact 3SAT:}\ \ F = (\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_4} \vee \overline{x_2} \vee x_3) \wedge \cdots$$

- 3SAT is NP-hard[4]

$$(x_i) \to (x_i \vee x_i \vee x_i) \qquad () \to (x_i \vee x_i \vee x_i) \wedge (\overline{x_i} \vee \overline{x_i} \vee \overline{x_i})$$

$$(x_i \vee x_j) \to (x_i \vee x_j \vee x_i)$$

$\hookrightarrow$ exact 3SAT is also NP-hard

[4] `https://dl.acm.org/citation.cfm?coll=GUIDE&dl=GUIDE&id=805047` [3]

Complexity Proofs

LEMMINGS $\in$ *NP*

"[An input that returns 'yes' must] be verifiable by deterministic computations that can be performed in polynomial time[5]"

$\rightarrow$ LEMMINGS be verified in polynomial time iff:

- timesteps are evaluatable in time polynomial in the input size
- moves are verifiable in time polynomial in the input size
- timesteps are polynomial in the input size[6]
- moves are polynomial in the input size

[5] `https://goo.gl/5BZjRf` about problems in NP

[6] see assumptions

LEMMINGS $\in NP$

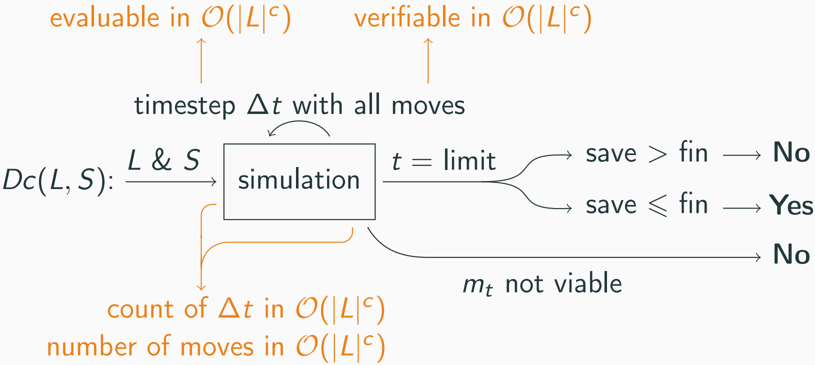


Figure 3: S is verifiable for L in $\mathcal{O}\left(|L|^{c'}\right)$

LEMMINGS $\in$ *NP-Hard*

goal: reduce LEMMINGS to 3SAT

$\rightarrow$ for each exact 3SAT problem, one can construct an equivalent problem in LEMMINGS

- clause conjunction $\rightarrow$ level L_x
- literals & clauses $\rightarrow$ gadgets
- assignment of values to literals $\rightarrow$ strategy S_x (1st part)
- evaluation of the attempt for specific literals in each clause $\rightarrow$ strategy S_x (2nd part)

- for every satisfiable 3SAT, a strategy S_x solving L_x exists
- for every solution of L_x, solution of 1 specific clause is found

$\hookrightarrow$ if reduction is possible: $L_x \in$ LEMMINGS

reduction example

`level:` $(\overline{x_1} \vee x_2 \vee x_4) \wedge (\overline{x_4} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_3} \vee x_3 \vee \overline{x_4})$

`strategy pt.1:` $x_1 = \mathit{false} \qquad x_2 = \mathit{true}$

$x_3 = \mathit{false} \qquad x_4 = \mathit{false}$

`strategy pt.2:` $\left(\underline{\underline{\overline{x_1}}} \vee x_2 \vee x_4\right) \wedge \left(\underline{\underline{\overline{x_4}}} \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_3} \vee \underline{\underline{x_3}} \vee \overline{x_4}\right)$

$\Rightarrow \overline{x_1} \wedge \overline{x_4} \wedge x_3$

$= \mathit{true} \wedge \mathit{true} \wedge \mathit{false}$

$= \underline{\underline{\mathit{false}}}$

LEMMINGS $\in$ *NP-Hard*

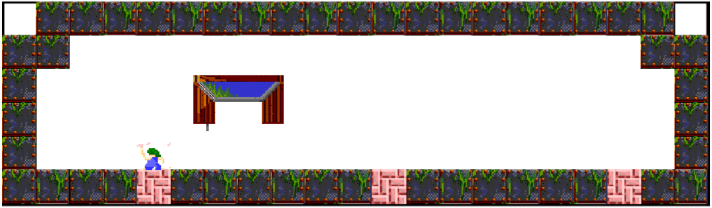


Figure 4: the clause gadget [1]

- the lemming can escape through digging a hole
- each hole represents a check for the *truthness* of the corresponding literal in the clause

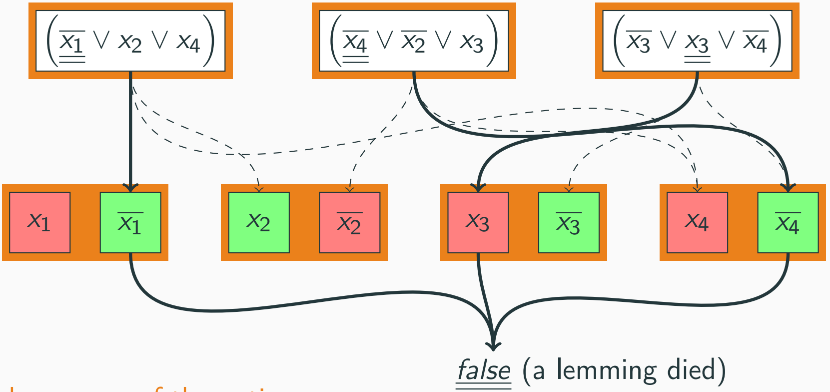
$\hookrightarrow$ if the literal is true, the entire clause evaluates to true

Figure 5: the variable gadget [1]

- the lemming can escape through bashing and building a bridge
- one side corresponds to $x_j = \textit{true}$, the other to $x_j = \textit{false}$
- the clause lemming falls through the chosen hole and dies if he chose wrong

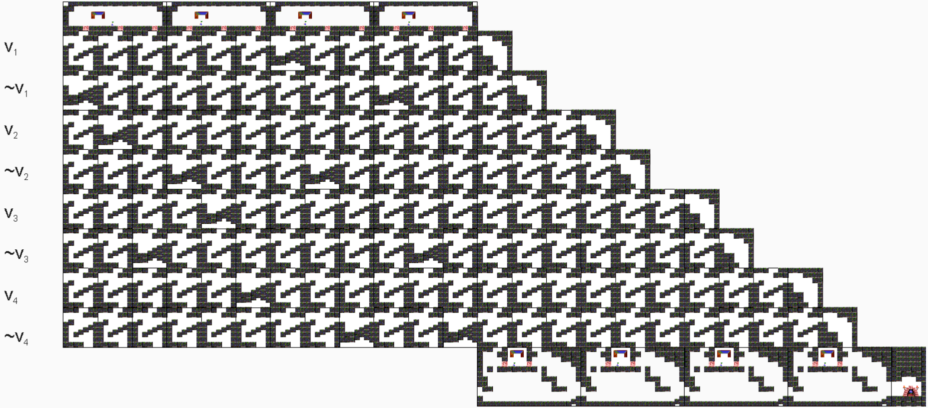
$\hookrightarrow$ if all clause lemmings can choose right, the variables satisfy the clause

algorithm example



choose one of the options

one combination of options leads to “true” if the problem is satisfiable



$$(\overline{v_1} \vee v_2 \vee \overline{v_3}) \wedge (\overline{v_2} \vee v_3 \vee v_4) \wedge (v_1 \vee \overline{v_2} \vee \overline{v_4}) \wedge (\overline{v_1} \vee \overline{v_3} \vee \overline{v_4}) \quad [1]$$

LEMMINGS $\in$ *NP-Complete*

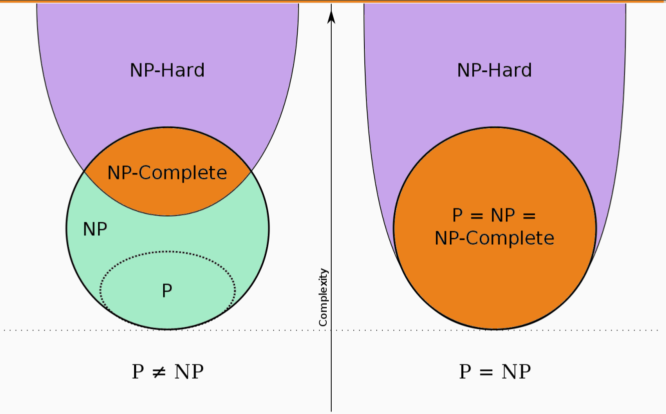


Figure 6: *NP-Complete* as intersection of *NP* and *NP-Hard*

- LEMMINGS $\in$ *NP* $\wedge$ LEMMINGS $\in$ *NP-Hard*

$\rightarrow$ LEMMINGS $\in$ *NP-Complete*

Sources

Quellen

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1-LEMMINGS $\in$ *NP-Complete*

- also in NP-Complete
- much harder construction: