

# Problem II - Implicit Function

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4 Points

$$\begin{array}{ll} \varphi_{\mathbb{Q}} : \mathbb{Q} \mapsto \mathbb{Q} & \varphi_{\mathbb{Q}}(p) \cdot \varphi_{\mathbb{Q}}(q) = \varphi_{\mathbb{Q}}(p+q) \\ \varphi_{\mathbb{R}} : \mathbb{R} \mapsto \mathbb{R} & \varphi_{\mathbb{R}}(p) \cdot \varphi_{\mathbb{R}}(q) = \varphi_{\mathbb{R}}(p+q) \end{array}$$

Which function do both functions  $\varphi(x)$  represent? Are they continuous?

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Let's start with  $\varphi_{\mathbb{Q}}(c), c \in \mathbb{Q}$ :

$$\begin{aligned} \varphi(0) &= \varphi(0+0) = \varphi(0) \cdot \varphi(0) = \varphi(0)^2 \\ \implies \varphi(0) &= \text{id}_{(\mathbb{R}, \cdot)} = 1 \\ &(\text{ or } \varphi(c) = 0 \implies \varphi(q) = 0 \quad \forall b \in \mathbb{Q} ) \end{aligned}$$

$$\varphi(c) = \varphi\left(\frac{1}{n} \cdot c\right) \cdot \varphi\left(\frac{n-1}{n} \cdot c\right) = \dots \quad n \in \mathbb{N}$$

$$= \overbrace{\varphi\left(\frac{1}{n} \cdot c\right) \cdot \varphi\left(\frac{1}{n} \cdot c\right) \cdots \varphi\left(\frac{1}{n} \cdot c\right)}^{n \text{ times}} = \varphi\left(\frac{c}{n}\right)^n \quad n \in \mathbb{N}$$

$$\varphi(cm) = \varphi(c)^m \implies \varphi(c) = \varphi(cm)^{\frac{1}{m}} \quad m \in \mathbb{N}^+$$

$$\varphi(c) = \varphi\left(\frac{m}{n} \cdot c\right)^{\frac{n}{m}} \quad n \in \mathbb{N}, m \in \mathbb{N}^+$$

$$\implies \varphi\left(\frac{m}{n} \cdot c\right) = \varphi(c)^{\frac{m}{n}} = \varphi(qc) = \varphi(c)^q \quad q \in \mathbb{Q}^+, n \in \mathbb{N}, m \in \mathbb{N}^+$$

$$\begin{aligned} \varphi(0) &= \varphi((q-q)c) = \varphi(qc) \cdot \varphi(-qc) \\ &= \varphi(c)^q \cdot \varphi(-qc) = 1 \\ \implies \varphi(-qc) &= \frac{1}{\varphi(c)^q} = \varphi(c)^{-q} \implies q \in \mathbb{Q} \end{aligned}$$

$$\begin{aligned} \varphi(1) &:= b & b \in ? \\ \varphi(q) &= \varphi(1)^q = b^q & \implies b \in \mathbb{Q}^+ \text{ so } b^q \in \mathbb{Q} \end{aligned}$$

$$\implies \varphi_{\mathbb{Q}}(q) = b^q \quad \forall b \in \mathbb{Q}^+ \implies \varphi_{\mathbb{Q}}(q) \text{ is continuous}$$

$$\implies \varphi_{\mathbb{R}}(q) = b^q \quad \forall q \in \mathbb{Q} \quad \forall b \in \mathbb{R}^+ \quad \text{analogous}$$

So can we extend the definition of  $\varphi_{\mathbb{Q}}$  to  $\varphi_{\mathbb{R}}$ ? No!

$$\varphi_{\mathbb{R}}(q) := \begin{cases} 2^{y\pi} & \text{if } q = x + y\pi \mid x, y \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$$