Problem II - Implicit Function

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$$\varphi_{\mathbb{Q}}: \ \mathbb{Q} \longmapsto \mathbb{Q} \qquad \varphi_{\mathbb{Q}}(p) \cdot \varphi_{\mathbb{Q}}(q) = \varphi_{\mathbb{Q}}(p+q)$$
$$\varphi_{\mathbb{R}}: \ \mathbb{R} \longmapsto \mathbb{R} \qquad \varphi_{\mathbb{R}}(p) \cdot \varphi_{\mathbb{R}}(q) = \varphi_{\mathbb{R}}(p+q)$$

Which function do both functions $\varphi(x)$ represent? Are they continuous?

Let's start with $\varphi_{\mathbb{Q}}(c), c \in \mathbb{Q}$:

$$\begin{aligned} \varphi(0) &= \varphi(0+0) = \varphi(0) \cdot \varphi(0) = \varphi(0)^2 \\ \Longrightarrow \varphi(0) &= \mathrm{id}_{(\mathbb{R},\cdot)} = 1 \\ &\quad (\text{ or } \varphi(c) = 0 \Longrightarrow \varphi(q) = 0 \ \ \forall b \in \mathbb{Q} \,\text{or}\, \forall b \in \mathbb{R}) \end{aligned}$$

$$\begin{aligned} \varphi(c) &= \varphi\left(\frac{1}{n} \cdot c\right) \cdot \varphi\left(\frac{n-1}{n} \cdot c\right) && n \in \mathbb{N} \\ &= \overbrace{\varphi\left(\frac{1}{n} \cdot c\right) \cdot \varphi\left(\frac{1}{n} \cdot c\right) \cdots \varphi\left(\frac{1}{n} \cdot c\right)}^{n \text{ times}} = \varphi\left(\frac{c}{n}\right)^n && n \in \mathbb{N} \\ \varphi(cm) &= \varphi\left(c\right)^m \Longrightarrow \varphi(c) = \varphi\left(cm\right)^{\frac{1}{m}} && m \in \mathbb{N}^+ \\ \varphi\left(c\right) &= \varphi\left(\frac{m}{n} \cdot c\right)^{\frac{n}{m}} && n \in \mathbb{N}, m \in \mathbb{N}^+ \\ \Longrightarrow \varphi\left(\frac{m}{n} \cdot c\right) &= \varphi\left(c\right)^{\frac{m}{n}} = \varphi\left(qc\right) = \varphi\left(c\right)^q && q \in \mathbb{Q}^+, \ n \in \mathbb{N}, m \in \mathbb{N}^+ \end{aligned}$$

$$\begin{aligned} \varphi(0) &= \varphi((q-q)c) = \varphi(qc) \cdot \varphi(-qc) \\ &= \varphi(c)^q \cdot \varphi(-qc) = 1 \\ \Longrightarrow \varphi(-qc) &= \frac{1}{\varphi(c)^q} = \varphi(c)^{-q} && \Longrightarrow q \in \mathbb{Q} \end{aligned}$$

$$\begin{aligned} \varphi(1) &\coloneqq b && b \in \, ? \\ \varphi(q) &= \varphi(1)^q = b^q && \Longrightarrow b \in \mathbb{Q}^+ \text{ so } b^q \in \mathbb{Q} \end{aligned}$$

$$\begin{aligned} &\Longrightarrow \varphi_{\mathbb{Q}}(q) = b^q \ \ \forall b \in \mathbb{Q}^+ \ \Longrightarrow \varphi_{\mathbb{Q}}(q) \text{ is continuous} \\ &\Longrightarrow \varphi_{\mathbb{R}}(q) = b^q \ \ \forall q \in \mathbb{Q} \ \forall b \in \mathbb{R}^+ && \text{analogous} \end{aligned}$$

So can we extend the definition of $\varphi_{\mathbb{Q}}$ to $\varphi_{\mathbb{R}}$? No!

$$\varphi_{\mathbb{R}}(q) \coloneqq \begin{cases} 2^{y\pi} & \text{if } q = x + y\pi \mid x, y \in \mathbb{Q} \\ 1 & \text{else} \end{cases}$$