

Proof of the Rightfulness of $\varphi_p(q)$ as a Counterexample

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$$\varphi_p(q) := \begin{cases} 2^y & \text{if } q = x + y\pi \mid x, y \in \mathbb{Q} \\ 0 & \text{else} \end{cases} \quad \varphi_p(q_1) \cdot \varphi_p(q_2) = \varphi_p(q_1 + q_2)$$

Part I: $\varphi_p(q)$ is a well-defined function

$$\begin{aligned} & \text{to be shown: } \forall q_1, q_2 \in \mathbb{R}, \ x_1, y_1, x_2, y_2, x_s, y_s \in \mathbb{Q} \\ & \left. \begin{matrix} 2^y & \text{if } q_1 + q_2 = x_s + y_s\pi \\ 0 & \text{else} \end{matrix} \right\} = \left\{ \begin{matrix} 2^y & \text{if } q_1 = x_1 + y_1\pi \\ 0 & \text{else} \end{matrix} \right\} \cdot \left\{ \begin{matrix} 2^y & \text{if } q_2 = x_2 + y_2\pi \\ 0 & \text{else} \end{matrix} \right\} \\ & \varphi_p(q_1 + q_2) = \varphi_p(q_1) \cdot \varphi_p(q_2) \end{aligned}$$

$$\begin{aligned} & q_1, q_2 \in \mathbb{R}, \ x_1, y_1, x_2, y_2 \in \mathbb{Q} \\ \text{Case I: } & q_1 = x_1 + y_1\pi \wedge q_2 = x_2 + y_2\pi \quad i_1, i_2 \neq x_i + y_i\pi \ \forall x_i, y_i \in \mathbb{Q} \\ & q_1 + q_2 = (x_1 + x_2) + (y_1 + y_2)\pi \\ & \Rightarrow \varphi_p(q_1 + q_2) = 2^{y_1+y_2} = 2^{y_1} \cdot 2^{y_2} = \varphi_p(q_1) \cdot \varphi_p(q_2) \\ \text{Case II: } & q_1 \neq x_1 + y_1\pi \wedge q_2 = x_2 + y_2\pi \vee q_1 = x_1 + y_1\pi \wedge q_2 \neq x_2 + y_2\pi \\ & q_1 + q_2 = x_a + y_a\pi + i_a = i'_a \\ & \Rightarrow \varphi_p(q_1 + q_2) = 0 = 0 \cdot 2^{y_a} = \varphi_p(q_1) \cdot \varphi_p(q_2) \\ \text{Case III: } & q_1 \neq x_1 + y_1\pi \wedge q_2 \neq x_2 + y_2\pi \\ & q_1 + q_2 = i_1 + i_2 \\ & \Rightarrow \varphi_p(q_1 + q_2) = 0 = 0 \cdot 0 = \varphi_p(q_1) \cdot \varphi_p(q_2) \\ & \Rightarrow \varphi_p(q_1 + q_2) = \varphi_p(q_1) \cdot \varphi_p(q_2) \end{aligned}$$

Part II: $\varphi_p(q)$ is not continuous

$$\text{continuity} \iff \forall x_0 \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists \omega > 0 \ \forall x \in \mathbb{R} : |x - x_0| < \omega \implies |\varphi_p(x) - \varphi_p(x_0)| < \varepsilon$$

$$\begin{aligned} & \implies \exists x_0 \in \mathbb{R} \ \exists \varepsilon > 0 \ \nexists \omega > 0 \ \forall x \in \mathbb{R} : |x - x_0| < \omega \implies |\varphi_p(x) - \varphi_p(x_0)| < \varepsilon \\ & \Leftrightarrow \exists x_0 \in \mathbb{R} \ \exists \varepsilon > 0 \ \forall \omega > 0 \ \exists x \in \mathbb{R} : |x - x_0| < \omega \implies |\varphi_p(x) - \varphi_p(x_0)| \geq \varepsilon \\ & \Leftrightarrow \text{noncontinuity} \end{aligned}$$

$$\begin{aligned} \varepsilon_0 &:= 0.17491 && (\text{because } 0.17491 < 1) \\ x_0 &:= \pi, \quad x := \lim_{n \rightarrow \infty} \frac{\lfloor 24^n \pi \rfloor}{24^n} = q \in \mathbb{Q}, \quad \forall \omega > 0 : |q - \pi| < \omega && (\text{because } 24 \gg 1) \\ \varphi_p(\pi) &= 2, \quad \varphi_p(q) = 1 && \implies \forall \omega > 0 : |\varphi_p(x) - \varphi_p(x_0)| = 1 > \varepsilon_0 \end{aligned}$$

$\implies \varphi_p(q)$ is a noncontinuous function