Proof of the Rightfulness of $\varphi_p(q)$ as a Counterexample

Philip Geißler

$$\varphi_p(q) \coloneqq \begin{cases} 2^y & \text{if } q = x + y\pi \mid x, y \in \mathbb{Q} \\ 0 & \text{else} \end{cases} \qquad \varphi_p(q_1) \cdot \varphi_p(q_2) = \varphi_p(q_1 + q_2)$$

Part I: $\varphi_p(q)$ is a well-defined function

to be shown: $\quad \forall q_1, q_2 \in \mathbb{R},\ x_1, y_1, x_2, y_2, x_s, y_s \in \mathbb{Q}$

$$\left.\begin{matrix} 2^y & \text{if } q_1 + q_2 = x_s + y_s\pi \\ 0 & \text{else} \end{matrix}\right\} = \begin{cases} 2^y & \text{if } q_1 = x_1 + y_1\pi \\ 0 & \text{else} \end{cases} \cdot \begin{cases} 2^y & \text{if } q_2 = x_2 + y_2\pi \\ 0 & \text{else} \end{cases}$$

$$\varphi_p(q_1 + q_2) = \varphi_p(q_1) \cdot \varphi_p(q_2)$$

$$q_1, q_2 \in \mathbb{R},\ x_1, y_1, x_2, y_2 \in \mathbb{Q}$$
$$i_1, i_2 \neq x_i + y_i\pi \quad \forall x_i, y_i \in \mathbb{Q}$$

$$\begin{aligned}
\text{Case I:}\ & q_1 = x_1 + y_1\pi \ \wedge\ q_2 = x_2 + y_2\pi \\
& q_1 + q_2 = (x_1 + x_2) + (y_1 + y_2)\pi \\
\Rightarrow\ & \varphi_p(q_1 + q_2) = 2^{y_1 + y_2} = 2^{y_1} \cdot 2^{y_2} = \varphi_p(q_1) \cdot \varphi_p(q_2) \\
\text{Case II:}\ & q_1 \neq x_1 + y_1\pi \ \wedge\ q_2 = x_2 + y_2\pi \ \vee\ q_1 = x_1 + y_1\pi \ \wedge\ q_2 \neq x_2 + y_2\pi \\
& q_1 + q_2 = x_a + y_a\pi + i_a = i'_a \\
\Rightarrow\ & \varphi_p(q_1 + q_2) = 0 = 0 \cdot 2^{y_a} = \varphi_p(q_1) \cdot \varphi_p(q_2) \\
\text{Case III:}\ & q_1 \neq x_1 + y_1\pi \ \wedge\ q_2 \neq x_2 + y_2\pi \\
& q_1 + q_2 = i_1 + i_2 \\
\Rightarrow\ & \varphi_p(q_1 + q_2) = 0 = 0 \cdot 0 = \varphi_p(q_1) \cdot \varphi_p(q_2) \\
\Longrightarrow\ & \varphi_p(q_1 + q_2) = \varphi_p(q_1) \cdot \varphi_p(q_2)
\end{aligned}$$

Part II: $\varphi_p(q)$ is not continuous

$$\text{continuity} \iff \forall x_0 \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \omega > 0\ \forall x \in \mathbb{R} : |x - x_0| < \omega \Longrightarrow |\varphi_p(x) - \varphi_p(x_0)| < \varepsilon$$

$$\begin{aligned}
\Longrightarrow\ & \exists x_0 \in \mathbb{R}\ \exists \varepsilon > 0\ \nexists \omega > 0\ \forall x \in \mathbb{R} : |x - x_0| < \omega \Longrightarrow |\varphi_p(x) - \varphi_p(x_0)| < \varepsilon \\
\Leftrightarrow\ & \exists x_0 \in \mathbb{R}\ \exists \varepsilon > 0\ \forall \omega > 0\ \exists x \in \mathbb{R} : |x - x_0| < \omega \Longrightarrow |\varphi_p(x) - \varphi_p(x_0)| \geqslant \varepsilon \\
\Leftrightarrow\ & \text{noncontinuity}
\end{aligned}$$

$$\varepsilon_0 \coloneqq 0.17491 \qquad \text{(because } 0.17491 < 1\text{)}$$
$$x_0 \coloneqq \pi, \quad x \coloneqq \lim_{n \to \infty} \frac{\lfloor 24^n \pi \rfloor}{24^n} = q \in \mathbb{Q}, \quad \forall \omega > 0 : |q - \pi| < \omega \qquad \text{(because } 24 \gg 1\text{)}$$
$$\varphi_p(\pi) = 2, \quad \varphi_p(q) = 1 \qquad \Longrightarrow \forall \omega > 0 : |\varphi_p(qb) - \varphi_p(x_0)| = 1 > \varepsilon_0$$

$\Longrightarrow \varphi_p(q)$ is a noncontinuous function