Problem III - A Stack of Sums

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$$\mathcal{S}_n^\infty := \sum_{\substack{x_i = x_{i-1} \\ x_1 = 0}}^{\infty} {}^{\sum\limits_{i=1}^{n}} \frac{1}{2^{x_i}} := \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \cdots \sum_{x_n=x_{n-1}}^{\infty} \frac{1}{2^{x_n}}$$

What are the values of $\mathcal{S}_1^\infty$, $\mathcal{S}_2^\infty$ and $\mathcal{S}_3^\infty$? Does $\mathcal{S}_n^\infty$ converge for all $n \in \mathbb{N}$? If so, can you give a general answer for $\mathcal{S}_n^\infty$?

$$\mathcal{S}_1^\infty = \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 2^n \sum_{x=0}^{\infty} \frac{1}{2^x \cdot 2^n} = 2^n \sum_{x=0}^{\infty} \frac{1}{2^{x+n}} = 2^n \sum_{x=n}^{\infty} \frac{1}{2^x} \qquad \forall n \in \mathbb{N}$$
$$= \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 1 + \sum_{x_1=1}^{\infty} \frac{1}{2^{x_1}} = 2 \cdot \sum_{x_1=1}^{\infty} \frac{1}{2^{x_1}} \Rightarrow \sum_{x_1=1}^{\infty} \frac{1}{2^{x_1}} = 1 \Rightarrow \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 2 \cdot 1 = 2^1$$

$$\mathcal{S}_2^\infty = \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \frac{1}{2^{x_2}} = \sum_{x_1=0}^{\infty} \left(\frac{1}{2^{x_1}} \sum_{x_2=0}^{\infty} \frac{1}{2^{x_2}} \right) = \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} \cdot 2 = \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1-1}} = \frac{1}{2^{-1}} \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}}$$
$$= 2^1 \cdot 2 = 4 = 2^2$$

$$\mathcal{S}_3^\infty = \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \sum_{x_3=x_2}^{\infty} \frac{1}{2^{x_3}} = \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \frac{1}{2^{x_2-1}} = \frac{1}{2^{-1}} \sum_{x_1=0}^{\infty} \sum_{x_2=x_1}^{\infty} \frac{1}{2^{x_2}}$$
$$= 2^1 \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1-1}} = 2^2 \sum_{x_1=0}^{\infty} \frac{1}{2^{x_1}} = 2^2 \cdot 2 = 8 = 2^3$$

$$\mathcal{S}_n^\infty = \sum_{x_1=0}^{\infty} \cdots \cdot \sum_{x_{n-1}=x_{n-2}}^{\infty} \sum_{x_n=x_{n-1}}^{\infty} \frac{1}{2^{x_n}} = \sum_{x_1=0}^{\infty} \cdots \cdot \sum_{x_{n-1}=x_{n-2}}^{\infty} \frac{1}{2^{x_{n-1}-1}} = 2^1 \sum_{x_1=0}^{\infty} \cdots \cdot \sum_{x_{n-1}=x_{n-2}}^{\infty} \frac{1}{2^{x_{n-1}}}$$
$$= 2^1 \cdot \mathcal{S}_{n-1}^\infty$$

$$\mathcal{S}_1^\infty = 2^1$$
$$\mathcal{S}_n^\infty = 2^1 \cdot \mathcal{S}_{n-1}^\infty$$

$$\Longrightarrow \mathcal{S}_n^\infty = 2^1 \cdot \mathcal{S}_{n-1}^\infty = 2^1 \cdot 2^1 \cdot \mathcal{S}_{n-2}^\infty = \cdots = \overbrace{2^1 \cdot 2^1 \cdots 2^1}^{n-1 \text{ times}} \cdot \mathcal{S}_1^\infty = 2^{n-1} \cdot 2^1 = 2^n$$

$\Longrightarrow \mathcal{S}_n^\infty$ coverges for every $n \in \mathbb{N}$, whereas $\lim\limits_{n\to\infty} \mathcal{S}_n^\infty$ diverges.