Problem I - Commutative Exponentiation

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3 Points

$$p^q = q^p \qquad \qquad p, q \in \mathbb{R}^+$$

Find the functions p(q) which satisfy the given equation. Where (p,q) do they intersect?

$$p^{p} = p^{p}$$

$$p^{q} = q^{p}$$

$$\text{with } q = p = p(q)$$

$$\Rightarrow p_{1}(q) = q$$

$$p_{1}(c) = c \text{ and } q_{1}(c) = c$$

$$(1)$$

If p(q) = q, then obviously $p(q)^q = p^p = q^{p(q)}$. But $2^4 = 4^2$ and $2 \neq 4$, so what if $p \neq q$?

$$p \neq q \qquad p, q \in \mathbb{R}^+$$

$$\Rightarrow p(q) = c \cdot q \qquad c \in \mathbb{R}^+ \setminus 1$$

$$q^{p(q)} = (cq)^{q} = (q^{c})^{q} = q^{cq} = p(q)^{q}$$

$$c \cdot q = q^{c}$$

$$c = q^{c-1}$$

$$c^{\frac{1}{c-1}} = q = q(c)$$

$$\Rightarrow p_{2}(c) = c \cdot c^{\frac{1}{c-1}} = c^{\frac{c}{c-1}} \text{ and } q_{2}(c) = c^{\frac{1}{c-1}}$$
(3)

We know from (1) that p = q in the first function, so to get the intersection of the 2 possible functions, we would just set c = 1, but both functions are undefined in that case. So instead we are looking at limit of c going to 1.

$$\lim_{c \to 1} p_2(x) = \lim_{c \to 1} c \cdot \lim_{c \to 1} c^{\frac{1}{c-1}} = 1 \cdot \lim_{c \to 1} c^{\frac{1}{c-1}} = \lim_{c \to 1} q_2(c)$$

$$= \lim_{c \to 1} e^{\log c^{\frac{1}{c-1}}} = \lim_{c \to 1} e^{\frac{1}{c-1} \log c}$$

$$= e^{\lim_{c \to 1} \frac{\log(c)}{c-1}} = e^{\lim_{c \to 1} \frac{1}{c}}$$

$$= e^1 = e$$

$$\Rightarrow P_i = (e, e) \approx (2.71, 2.71)$$
 (4)