Problem I - Commutative Exponentiation

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$$p^q = q^p \qquad p, q \in \mathbb{R}^+$$

Find the functions $p(q)$ which satisfy the given equation. Where (p, q) do they intersect?

$$\begin{aligned} p^p &= p^p & p, q &\in \mathbb{R}^+ \\ p^q &= q^p & \text{with } q &= p = p(q) \end{aligned}$$

$$\Rightarrow p_1(q) = q \tag{1}$$

$$p_1(c) = c \text{ and } q_1(c) = c \tag{2}$$

If $p(q) = q$, then obviously $p(q)^q = p^p = q^{p(q)}$. But $2^4 = 4^2$ and $2 \neq 4$, so what if $p \neq q$?

$$\begin{aligned} p &\neq q & p, q &\in \mathbb{R}^+ \\ \Rightarrow p(q) &= c \cdot q & c &\in \mathbb{R}^+ \setminus \{1\} \end{aligned}$$

$$\begin{aligned} q^{p(q)} = (cq)^q = (q^c)^q = q^{cq} &= p(q)^q \\ c \cdot q &= q^c \\ c &= q^{c-1} \\ c^{\frac{1}{c-1}} &= q = q(c) \end{aligned}$$

$$\Rightarrow p_2(c) = c \cdot c^{\frac{1}{c-1}} = c^{\frac{c}{c-1}} \text{ and } q_2(c) = c^{\frac{1}{c-1}} \tag{3}$$

We know from (1) that $p = q$ in the first function, so to get the intersection of the 2 possible functions, we would just set $c = 1$, but both functions are undefined in that case. So instead we are looking at limit of c going to 1.

$$\begin{aligned} \lim_{c \to 1} p_2(x) &= \lim_{c \to 1} c \cdot \lim_{c \to 1} c^{\frac{1}{c-1}} = 1 \cdot \lim_{c \to 1} c^{\frac{1}{c-1}} = \lim_{c \to 1} q_2(c) \\ &= \lim_{c \to 1} e^{\log c^{\frac{1}{c-1}}} = \lim_{c \to 1} e^{\frac{1}{c-1} \log c} \\ &= e^{\lim_{c \to 1} \frac{\log(c)}{c-1}} = e^{\lim_{c \to 1} \frac{1}{c}} \\ &= e^1 = e \end{aligned}$$

$$\Rightarrow P_i = (e, e) \approx (2.71,\ 2.71) \tag{4}$$