Nonlinear Asset Pricing Model

UdeM

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Outline

- In modern finance, the price of any asset is obtained by the expected discounted payoffs :

$$p_t^i = \mathbb{E}_t(SDF_{t+1} CF_{t+1}^i)$$

- The stochastic discount factor SDF_{t+1} is equal to the growth in the marginal value of wealth :

$$SDF_{t+1} = \frac{VW_{t+1}}{VW_t}$$

- The traditional theories of finance, CAPM, ICAPM, and APT, measure the marginal utility of wealth by the behavior of large portfolios of assets.
  1. **CAPM** : return on the market portfolio.
  2. **Multifactor models** : returns on multiple portfolios.

- In this paper, we rely on the multi-factor model framework.
- The SDF_{t+1} is specified as follows :

$$SDF_t = 1 - \lambda'(F_t - \mu)$$

  where $\mu = E(F_t)$ and λ is a vector of factor loadings.
- **IDEA of this paper :** Use the **nonlinear PCs as factors** to estimate the SDF instead of the linear PCs as it is usually done. (See Kozak et al. (2020))
- **Advantages :**
  1. Factors are truly independent as opposed to linear PCs which are merely uncorrelated. **As a consequence, we capture truly different risk measure.**
  2. We **likely** need less factors when using nonlinear PCs than when using linear counterpart. **As a consequence, we have parsimonious model.**
  3. The increasing number of anomalies is **likely** not a problem anymore.

- **Nonlinear factors** : Chen et al. (2009), Lawrence (2012), Gunsilius and Schennach (2019), Damianou et al. (2021)
- **Machine learning asset pricing models** : Feng et al. (2018), Nakagawa et al. (2019), Chen et al. (2020), and Fang and Taylor (2021).
- **Stochastic discount factor estimation** : Fama and Kenneth (1993), Hou et al. (2015), Fama and French (2015), Barillas and Shanken (2018) and Kozak et al. (2018).

- Anomalies considered : 50 anomaly characteristics (same as Kozak et al.(2020)) ;
- Daily returns data from November 1973 to December 2019 (2017 for Kozak et al.(2020)) ;
- Follow the same anomalies definition as Kozak et al.(2020) to construct the anomalies.

Optimization problems

- Let $r_t = (r_{1,t}, ..., r_{N,t})'$ be the vector of excess returns of N portfolios, t=1,...,T
- Let Z_t be a N-by-k matrix of asset anomaly characteristics ;
- Let $F_t = Z'_{t-1} r_t$ be a k-by-1 vector of factors (raw characteristic returns or linear PCs or nonlinear PCs) ;
- Let $\Sigma = Cov(F)$ be a k-by-k variance-covariance matrix of the factors ;
- Let $\mu = \mathbb{E}(F)$ be a k-by-1 vector of expected factor returns ;
- $SDF_t = 1 - \lambda'(F_t - \mathbb{E}F_t)$

Optimization problems

We impose two kind of penalties to estimate the SDF coefficients :

$$L2pen : \hat{\lambda} = arg \min_{\lambda} (\mu - \Sigma\lambda)'\Sigma^{-1}(\mu - \Sigma\lambda) + \gamma\lambda'\lambda \tag{1}$$

$$L1L2pen : \hat{\lambda} = arg \min_{\lambda} (\mu - \Sigma\lambda)'\Sigma^{-1}(\mu - \Sigma\lambda) + \gamma_1 \sum_{i=1}^{k} |\lambda_i| + \gamma_2\lambda'\lambda \tag{2}$$

- Estimate the parameter $\hat{\lambda}$ via Ridge or Elastic net using LAR-EN ;
- choose optimally the tuning parameters γ or (γ_1 and γ_2). Σ is a k-by-k matrix, μ is a k-by-1 vector and λ is a k-by-1 vector.

LARS-EN(1/2)

- For each γ_2, the problem (2) is equivalent to a lasso problem (3) ;
- So, for each γ_2 we use the modified LARS algorithm to solve the problem (3) equivalently the problem (2).

$$\hat{\lambda} = arg \min_{\lambda} (\mu^* - \Sigma^* \lambda)'(\mu^* - \Sigma^* \lambda) + \gamma_1 \sum_{i=1}^{k} |\lambda_i| \qquad (3)$$

where $\mu^* = (\Sigma^{-\frac{1}{2}} \mu, 0)'$ and $\Sigma^* = (\Sigma^{\frac{1}{2}}, \sqrt{\gamma_2} I)'$

- For each γ_2, we execute the algorithm described in the next slide to estimate $\hat{\lambda}$.

LARS-EN(2/2)

1. Initialize $\hat{\lambda}^{(0)} = 0$, $\mathcal{A} = argmax_j |\Sigma_j' \mu|$, $\nabla \hat{\lambda}_{\mathcal{A}}^{(0)} = -sign(\Sigma_{\mathcal{A}}' \mu), \nabla \hat{\lambda}_{I}^{(0)} = 0$, $n = 0$.
2. While $\mathcal{I} \neq \emptyset$ do ;
3. $\delta_j = \min^+_{j \in \mathcal{A}} - \frac{\hat{\lambda}^{(n)}}{\nabla \hat{\lambda}_j^{(n)}}$
4. $\delta_i = \min^+_{i \in \mathcal{I}} \left\{ \frac{(\Sigma_i + \Sigma_j)'(\mu - X\hat{\lambda}^{(n)})}{(\Sigma_i + \Sigma_j)'(\Sigma \nabla \hat{\lambda}^{(n)})}, \frac{(\Sigma_i - \Sigma_j)'(\mu - \Sigma\hat{\lambda}^{(n)})}{(\Sigma_i - \Sigma_j)'(\Sigma \nabla \hat{\lambda}^{(n)})} \right\}$ where j is any index in $\mathcal{A}$.
5. $\delta = \min(\delta_j, \delta_i)$
6. if $\delta = \delta_j$ then move j from $\mathcal{A}$ to $\mathcal{I}$ else move i from $\mathcal{I}$ to $\mathcal{A}$.
7. $\hat{\lambda}^{(n+1)} = \hat{\lambda}^{(n)} + \delta \nabla \hat{\lambda}^{(n)}$
8. $\nabla \hat{\lambda}_{\mathcal{A}}^{(n+1)} = -\frac{1}{2} (\Sigma_{\mathcal{A}} + \gamma_2 I)^{-1} .sign(\hat{\lambda}_{\mathcal{A}}^{(n+1)})$
9. Update the value of n=n+1
10. end while
11. Output the series of coefficients $\Lambda = (\hat{\lambda}^{(0)}, \hat{\lambda}^{(1)}, ..., \hat{\lambda}^{(k)})$

L2pen : Raw characteristics and linear PCs

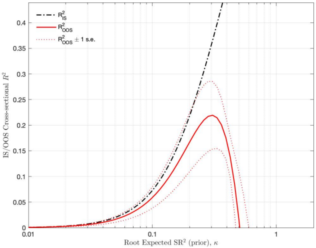


FIGURE – 50 raw characteristics

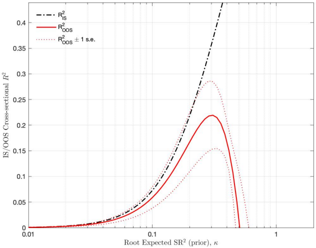
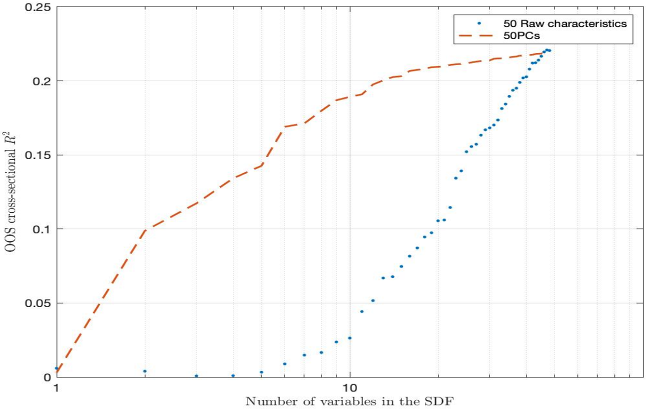


FIGURE – 50 linear PCs

Sparsity


0.25
0.2
0.15
0.1
0.05
0
1
10
OOS cross-sectional R^2
Number of variables in the SDF
50 Raw characteristics
50PCs

L1L2pen : Raw characteristics and linear PCs

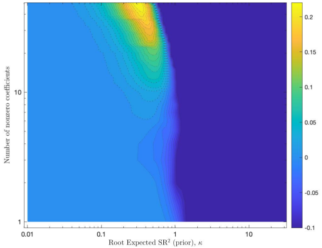


FIGURE – 50 raw characteristics

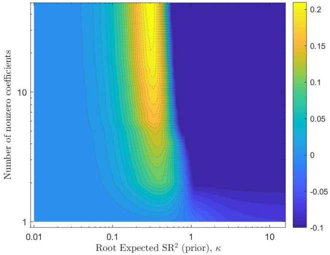


FIGURE – 50 linear PCs

L2pen : With interaction terms

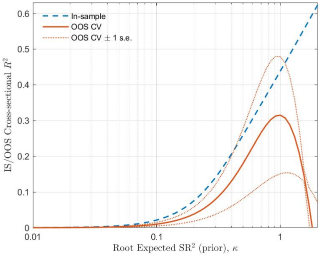


FIGURE – 2600 raw char.

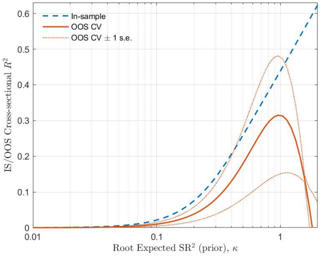


FIGURE – 2600 linear PCs

L1L2pen : With interaction terms

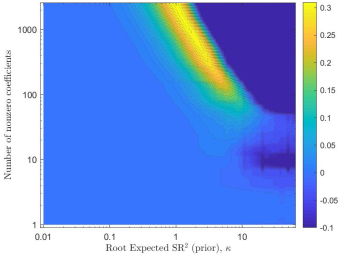


FIGURE – 2600 raw char.

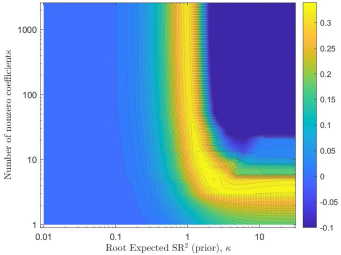


FIGURE – 2600 linear PCs

Takeaway 1

- From the previous slides, the results are quite similar to the one of Kozak et al. (2020) (replication) ;
- Let us turn to the second part of our analysis, which consist of integrating nonlinear factors.

- Let $r_t = (r_{1,t}, ..., r_{N,t})$ be the vector of excess returns of N portfolios, t=1,...,T
- r_t is orthogonalized with respect to the market and rescaled to have the same standard deviations as the market ;
- Nonlinear PCs construction : Follows Gunsilius and Schennach (2019)

- Extract N linear PCs from r denoted by $f_t = (f_t^1, ..., f_t^N)$;
- Extract the nonlinear PCs from the first k linear PCs : $y_t = (f_t^1, ..., f_t^k)$;
- y has a density function g.
- Find a map T transforming $g(y)$ into a target density $\Phi(x)$ where $x = T(y)$
- Change of variable formula gives :

$$g(y) = \Phi(T(y)) det(\frac{\partial T(y)}{\partial y'}) \tag{4}$$

- T minimizes $\int ||T(y) - y||^2 g(y) dy$
- $T(y) = \dfrac{\partial C(y)}{\partial y}$, where C is a convex function.
- C is determined by Gradient descent using equation (4)

- Compute

$$\tilde{J} = -\int g(y) ln \frac{\partial T(y)}{\partial y'} dy \tag{5}$$

- Extract k eigenvectors $e = (e_1, e_2, ..., e_k)$ corresponding to the k largest eigenvalues of $\tilde{J}$
- Therefore, the i^{th} nonlinear principal component is defined by : $\tilde{f}_i = T(y)e_i, \; i = 1, 2, ..., k.$

50 anomaly characteristcis

- Let $r_t = (r_{1,t}, ..., r_{50,t})$ be the raw characteristic excess returns ;
- Let y be the first k linear principal components ;
- Set a squared grid y with a size $MxMx...xM$ from -4 to 4 each variable ;
- Estimate the Brenier map T for the grid T(y) ;
- Calculate $\tilde{J}$ over the grid points, then the eigenvectors $e = (e_1, e_2, ..., e_k)$;
- Interpolate the Brenier map to have the full nonlinear transformation of the data : $T(f_1, f_2, ..., f_k)$;
- Let $\tilde{f}_t = (\tilde{f}_{1,t}, ..., \tilde{f}_{k,t})$ be the time series of the k nonlinear PCs ;
- Since the nonlinear factors are not tradable, we construct the corresponding mimicking portfolios.

- Approximation of the NLPCs using a piecewise linear function :

$$\tilde{f}_{j,t} = \beta_{0,j} + \beta_{1,j} r_{mkt,t} + \beta'_{c,j} r_t + \delta_j max(r_{mkt,t} - k_j, 0) + \epsilon_{j,t} \quad t = 1, ..., T \tag{6}$$

- If δ is not significant, then a linear mimicking portfolio is sufficient ;

$$MP^1_{j,t} = \hat{\beta}_{0,j} + \hat{\beta}_{1,j} r_{mkt,t} + \hat{\beta}'_{c,j} r_t \quad t = 1, ..., T \tag{7}$$

- Else, we need a nonlinear mimicking portfolio.

$$MP^2_{j,t} = \hat{\beta}_{0,j} + \hat{\beta}_{1,j} r_{mkt,t} + \hat{\beta}'_{c,j} r_t + \hat{\delta}_j max(r_{mkt,t} - k_j, 0) \quad t = 1, ..., T \tag{8}$$

- The third mimicking portfolios we considered :

$$MP^3_{j,t} = \hat{\beta}_{0,j} + \hat{\beta}'_{c,j} r_t \quad t = 1, ..., T \tag{9}$$

Terminologies

Let f_{-k} be a set of 50-k linear PCs, excluding the first k linear PCs and NMP_k / NPC_k be a set of k nonlinear MPs/PCs. $k = 2, 3, ..., 6$.

- **Base case** : Price $[f_{-k}, NMP_k]$ using risk factors derived from $[f_{-k}, NMP_k]$. In formula : $\mu = \mathbb{E}([f_{-k}, NMP_k]), \Sigma = Cov([f_{-k}, NMP_k])$
- **Robustness check** : Price $[f_{-k}, NMP_k]$ using risk factors derived from $[f_{-k}, NPC_k]$. In formula : $\mu = \mathbb{E}([f_{-k}, NMP_k]), \Sigma = Cov([f_{-k}, NPC_k])$

Terminologies

- **Base case** : Let the LARS-EN algorithm adds the factors starting by the model with all the mimicking portfolios of the NLPCs.
- **Robustness check** : Let the LARS-EN algorithm adds the factors starting by the model with no risk factors ;

Results do not depend on the mimicking portfolios (MPs), so we present the figures only for MP^2

Base case : 48 PCs + 2NMPs

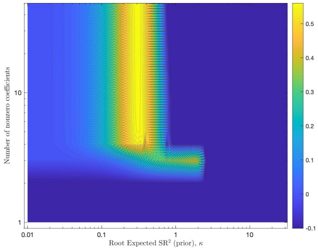


FIGURE – L1L2pen

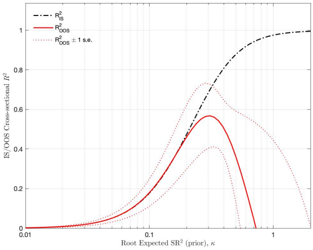


FIGURE – L2pen

Robustness check : 48 PCs + 2NPCs

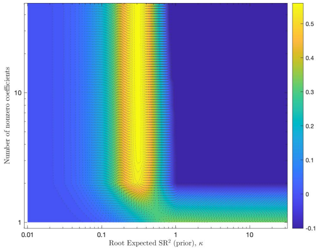


FIGURE – L1L2pen

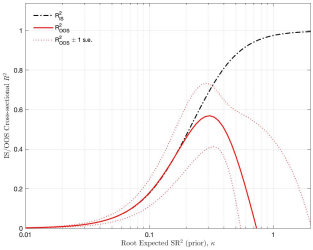


FIGURE – L2pen

Base case : 47 PCs + 3NMPs

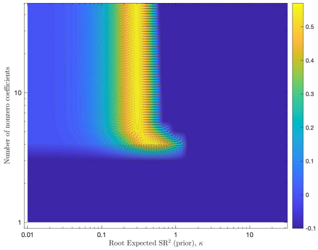


FIGURE – L1L2pen

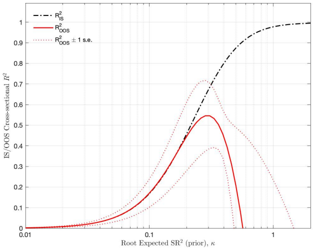


FIGURE – L2pen

Robustness check : 47 PCs + 3NPCs

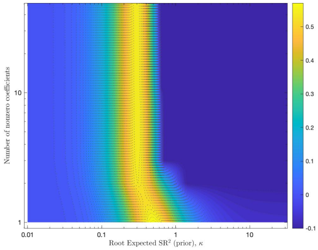


FIGURE – L1L2pen

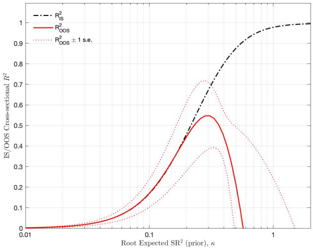


FIGURE – L2pen

Base case : 46 PCs + 4NMPs

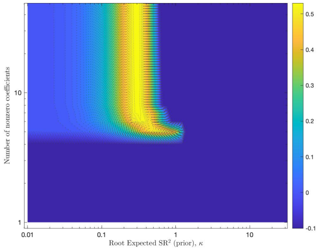


FIGURE – L1L2pen

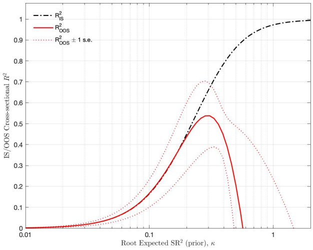


FIGURE – L2pen

Robustness check : 46 PCs + 4NPCs

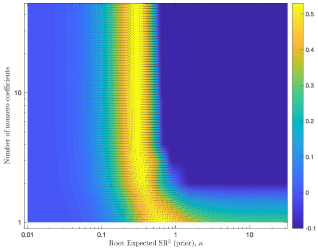


FIGURE – L1L2pen

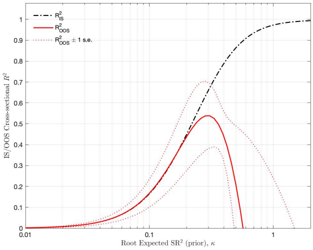


FIGURE – L2pen

Base case : 45 PCs + 5NMPs

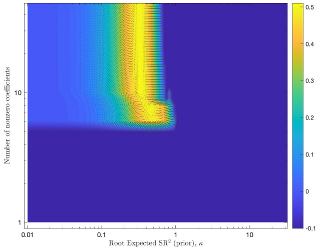


FIGURE – L1L2pen

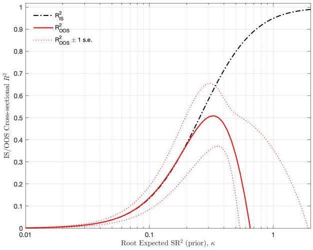


FIGURE – L2pen

Robustness check : 45 PCs + 5NPCs

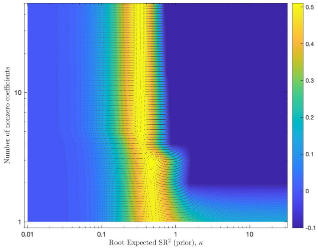


FIGURE – L1L2pen

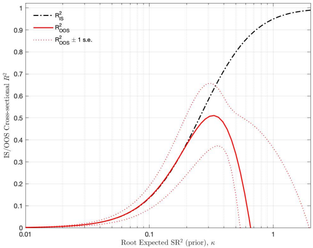


FIGURE – L2pen

Base case : 44 PCs + 6NMPs

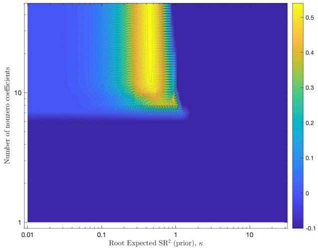


FIGURE – L1L2pen

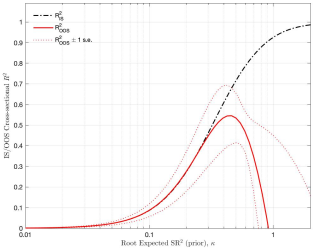


FIGURE – L2pen

Robustness check : 44 PCs + 6NPCs

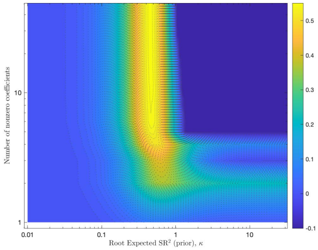


FIGURE – L1L2pen

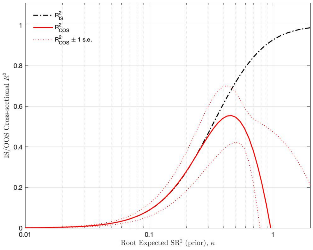


FIGURE – L2pen

Takeaway 2

- Results do not depend on whether one use the NLPCs or the NMPs ;
  - There is a difference but it is not that much as one can see from previous slides ;
  - One explanation is the quality of the NMPs which perfectly mimic the NLPCs ;
- Our results suggest that one should do supervised Elastic net instead of doing unsupervised Elastic net :
  - Benchmark analysis is much better than no benchmark analysis.

LF vs NLF

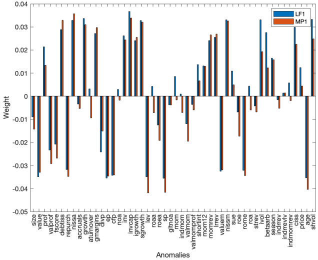


FIGURE – LF1 versus MP1

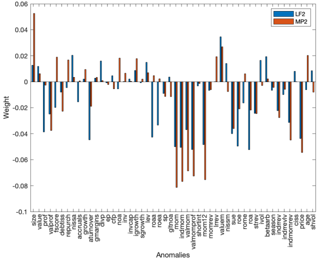


FIGURE – LF2 versus MP2

Conclusion

- The hybrid model requires less risk factors to achieve the highest out-of-sample performance
- Weight shifting on some anomalies. The mimicking portfolios (MPs) and the linear factors disagree on the anomalies that are marginal in terms of weights
- We believe that the nonlinear principal components have good prediction power.
- Thus, they should be taken into account for the development of future factor model.