# Normally Distributed Errors

**Likelihood:**

$$y \sim N_n(X\beta, \sigma^2 I_n)$$

$$l = \left(\frac{1}{\sqrt{2\pi}}\right)^n \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right\}$$

$$\propto \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right\}$$

**Prior:**

$$\beta \sim N_k(\beta_0, B_0)$$

$$\sigma^2 \sim IG\left(\frac{\alpha_0}{2}, \frac{\delta_0}{2}\right)$$

$$p(\beta) = \left(\frac{1}{\sqrt{2\pi}}\right)^k |B_0|^{-1} \exp\left\{-\frac{1}{2}(\beta - \beta_0)' B_0^{-1}(\beta - \beta_0)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}(\beta - \beta_0)' B_0^{-1}(\beta - \beta_0)\right\}$$

$$p(\sigma^2) = \frac{\left(\frac{\delta_0}{2}\right)^{\frac{\alpha_0}{2}}}{\Gamma\left(\frac{\alpha_0}{2}\right)} \sigma^{2\left(-\frac{\alpha_0}{2} - 1\right)} e^{-\frac{\delta_0}{2\sigma^2}}$$

$$\propto \sigma^{2(-\frac{\alpha_0}{2} - 1)} e^{-\frac{\delta_0}{2\sigma^2}}$$

**Posterior:**

$$\pi(\beta, \sigma^2) \propto l \times p(\beta)\, p(\sigma^2)$$

$$= \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right\} \exp\{(\beta - \beta_0)' B_0^{-1}(\beta - \beta_0)\} \sigma^{2\left(-\frac{\alpha_0}{2} - 1\right)} e^{-\frac{\delta_0}{2\sigma^2}}$$

$$= \sigma^{2\left(-\frac{\alpha_0 + n}{2} - 1\right)} \exp\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta) - \frac{1}{2}(\beta - \beta_0)' B_0^{-1}(\beta - \beta_0) - \frac{\delta_0}{2\sigma^2}\}$$

$$\pi(\sigma^2|\beta) \propto \sigma^{2\left(-\frac{\alpha_0 + n}{2} - 1\right)} \exp\{-\frac{1}{2\sigma^2}[(y - X\beta)'(y - X\beta) + \delta_0]\}$$

$$\therefore \boldsymbol{\sigma^2 | \beta \sim IG\left(\frac{\alpha_1}{2}, \frac{\delta_1}{2}\right)}$$

Where

$$\alpha_1 = \alpha_0 + n$$
$$\delta_1 = (y - X\beta)'(y - X\beta) + \delta_0$$

$$\pi(\beta|\sigma^2) \propto \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta) - \frac{1}{2}(\beta - \beta_0)' B_0^{-1}(\beta - \beta_0)\right\}$$

$$= \exp\left\{-\frac{1}{2}[\sigma^{-2}(y'y - y'X\beta - \beta'X'y + \beta'X'X\beta) + \beta' B_0^{-1}\beta - \beta' B_0^{-1}\beta_0 - \beta_0' B_0^{-1}\beta + \beta_0' B_0^{-1}\beta_0]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}[\sigma^{-2}(-y'X\beta - \beta'X'y + \beta'X'X\beta) + \beta'B_0^{-1}\beta - \beta'B_0^{-1}\beta_0 - \beta_0'B_0^{-1}\beta]\right\}$$

$$\propto \exp\{-\frac{1}{2}[\beta'(\sigma^{-2}X'X + B_0^{-1})\beta - 2\beta'(\sigma^{-2}X'y + B_0^{-1}\beta_0)]\}$$

$$\therefore \boldsymbol{\beta}|\boldsymbol{\sigma}^{\mathbf{2}} \sim \boldsymbol{N}_{\boldsymbol{k}}(\boldsymbol{\beta}_{\mathbf{1}}, \boldsymbol{B}_{\mathbf{1}})$$

Where

$$B_1 = (\sigma^{-2}X'X + B_0^{-1})^{-1}$$
$$\beta_1 = B_1(\sigma^{-2}X'y + B_0^{-1}\beta_0)$$