SV model

Model:

$$y_t = \exp\left\{\frac{h_t}{2}\right\}\epsilon_t, \qquad \epsilon_t \sim i.i.d. N(0,1), \qquad 1 \le t \le n$$

$$h_1 = \mu + \eta_1, \qquad \eta_1 \sim i.i.d. N\left(0, \frac{\sigma^2}{1-\phi^2}\right)$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \eta_t, \quad \eta_t \sim i.i.d. N(0, \sigma^2), \qquad 2 \le t \le n$$

Likelihood ($\theta = (\mu, \phi, \sigma^2)$):

$$f(y_t|h_t) = \frac{1}{\sqrt{2\pi \exp\{h_t\}}} \exp\left\{-\frac{y_t^2}{2\exp\{h_t\}}\right\}$$

$$f(h_t|h_{t-1}, \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2}\right\}, 2 \le t \le n$$

$$f(h_1|\theta) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1-\phi^2}}} \exp\left\{-\frac{(h_1 - \mu)^2}{2\frac{\sigma^2}{1-\phi^2}}\right\}$$

$$\begin{aligned}
\therefore L(\theta) &= f(y|\theta) \\
&\propto \prod_{t=1}^{n} [f(y_t|h_t) f(h_t|h_{t-1}, \theta)] \\
&\propto \sigma^{2\times\left(-\frac{n}{2}\right)} (1-\phi^2)^{\frac{1}{2}} \exp\left\{\sum_{t=1}^{n}\left(-\frac{y_t^2}{2\,exp\{h_t\}} - \frac{h_t}{2}\right) - \frac{(h_1 - \mu)^2}{2\frac{\sigma^2}{1-\phi^2}} - \sum_{t=2}^{n} \frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2}\right\}
\end{aligned}$$

Prior distribution:

$$\mu \sim N(\mu_0, \sigma_0^2)$$

$$\sigma^2 \sim IG\left(\frac{v_0}{2}, \frac{\delta_0}{2}\right)$$

$\because |\phi| < 1$

$\therefore \phi\ has\ 3\ or\ more\ choices:$

$$\begin{cases} \phi \sim U(-1,1) \\ \phi \sim N_{|\phi|<1}(0,1) \\ \dfrac{1+\phi}{2} \sim Beta(a_0, b_0) \end{cases}$$

$\therefore Prior\ PDFs\ are:$

$$\begin{cases} p(\mu) \propto \exp\left\{-\dfrac{(\mu - \mu_0)^2}{2\sigma_0^2}\right\} \\ p(\sigma^2) \propto \sigma^{2\times\left(-\frac{v_0}{2}-1\right)} \exp\left\{-\dfrac{\delta_0}{2\sigma^2}\right\} \\ \begin{cases} p(\phi) \propto 1 \\ p(\phi) \propto \exp\left\{-\dfrac{\phi^2}{2}\right\} \\ p\left(\dfrac{1+\phi}{2}\right) \propto \left(\dfrac{1+\phi}{2}\right)^{a_0-1} \left(\dfrac{1-\phi}{2}\right)^{b_0-1} \end{cases}, -1 < \phi < 1 \end{cases}$$

Posterior distribution:

$$\pi(\mu, \sigma^2, \phi) \propto L(\theta) p(\sigma^2) p(\mu) p(\phi)$$

$$\pi(\mu|\sigma^2,\phi) \propto L(\theta)p(\mu)$$

$$\propto \exp\left\{-\frac{(h_1-\mu)^2}{2\frac{\sigma^2}{1-\phi^2}}+\sum_{t=2}^{n}-\frac{[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2}{2\sigma^2}\right\}\exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$$

$$\propto \exp\left\{-\frac{(1-\phi^2)(-2h_1\mu+\mu^2)+\sum_{t=2}^{n}[(\phi-1)^2\mu^2-2(\phi-1)(\phi h_{t-1}-h_t)\mu]}{2\sigma^2}-\frac{\mu^2-2\mu_0\mu}{2\sigma_0^2}\right\}$$

$$\propto \exp\left\{-\frac{[(n-1)(1-\phi)^2+1-\phi^2]\mu^2-2[(1-\phi^2)h_1+(1-\phi)\sum_{t=2}^{n}(h_t-\phi h_{t-1})]\mu}{2\sigma^2}-\frac{\mu^2-2\mu_0\mu}{2\sigma_0^2}\right\}$$

$$\propto \exp\{-\frac{\{\sigma_0^2[(n-1)(1-\phi)^2+1-\phi^2]+\sigma^2\}\mu^2-2\{\sigma_0^2[(1-\phi^2)h_1+(1-\phi)\sum_{t=2}^{n}(h_t-\phi h_{t-1})]+\sigma^2\mu_0\}\mu}{2\sigma^2\sigma_0^2}\}$$

$$\therefore \boldsymbol{\mu|\sigma^2,\phi \sim N(\mu_1,\sigma_1^2)}$$

Where:

$$\sigma_1^2=\frac{\sigma^2\sigma_0^2}{\sigma_0^2[(n-1)(1-\phi)^2+1-\phi^2]+\sigma^2}$$

$$\mu_1=\frac{\sigma_1^2\{\sigma_0^2[(1-\phi^2)h_1+(1-\phi)\sum_{t=2}^{n}(h_t-\phi h_{t-1})]+\sigma^2\mu_0\}}{\sigma^2\sigma_0^2}$$

$$=\sigma_1^2\left\{\frac{1}{\sigma^2}\left[(1-\phi^2)h_1+(1-\phi)\sum_{t=2}^{n}(h_t-\phi h_{t-1})\right]+\frac{1}{\sigma_0^2}\mu_0\right\}$$

$$\pi(\sigma^2|\mu,\phi) \propto L(\theta)p(\sigma^2)$$

$$\propto \sigma^{2\times\left(-\frac{n}{2}\right)}\exp\left\{-\frac{(h_1-\mu)^2}{2\frac{\sigma^2}{1-\phi^2}}+\sum_{t=2}^{n}-\frac{[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2}{2\sigma^2}\right\}\sigma^{2\times\left(-\frac{\nu_0}{2}-1\right)}\exp\left\{-\frac{\delta_0}{2\sigma^2}\right\}$$

$$\propto \sigma^{2\times\left(-\frac{\nu_0+n}{2}-1\right)}\exp\{-\frac{(1-\phi^2)(h_1-\mu)^2+\sum_{t=2}^{n}[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2+\delta_0}{2\sigma^2}\}$$

$$\therefore \boldsymbol{\sigma^2|\mu,\phi \sim IG\left(\frac{\nu_1}{2},\frac{\delta_1}{2}\right)}$$

Where:

$$\boldsymbol{\nu_1}=\nu_0+n$$

$$\boldsymbol{\delta_1}=(1-\phi^2)(h_1-\mu)^2+\sum_{t=2}^{n}[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2+\delta_0$$

$$\pi(\phi|\mu,\sigma^2) \propto L(\theta)p(\phi)$$

$$\propto (1-\phi^2)^{\frac{1}{2}}\exp\left\{-\frac{(h_1-\mu)^2}{2\frac{\sigma^2}{1-\phi^2}}+\sum_{t=2}^{n}-\frac{[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2}{2\sigma^2}\right\}p(\phi)$$

$$log\pi(\phi|\mu,\sigma^2)=const\ C-\frac{1}{2\sigma^2}\sum_{t=2}^{n}[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2+\ln[\varphi(\phi)],\qquad -1<\phi<1$$

$$\boldsymbol{where}\quad \ln[\varphi(\phi)]=\frac{1}{2}\log(1-\phi^2)-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma^2}+logp(\phi),\qquad -1<\phi<1$$

This is a special distribution, but can be sampled by using the AR (Acceptance-Rejection) algorithm or the MH (Metropolis-Hastings) algorithm.

The method proposed in Chib and Greenberg (1994) is presented here. They neglected $\mathbf{ln}\,[\boldsymbol{\varphi(\phi)}]$ on the right-hand side in the above equation, which is a $\boldsymbol{const\ C + log[h(\phi)]}$, i.e. :.

$$log\pi(\phi|\mu,\sigma^2) \approx const\ C - \frac{1}{2\sigma^2}\sum_{t=2}^{n}[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2$$

$$= const\ C - \frac{\sum_{t=2}^{n}(h_{t-1}-\mu)^2}{2\sigma^2}\left\{\phi - \sum_{t=2}^{n}\frac{(h_t-\mu)(h_{t-1}-\mu)}{\sum_{t=2}^{n}(h_{t-1}-\mu)^2}\right\}^2$$

$$= const\ C + \log[h(\phi)], \qquad -1<\phi<1$$

Then $h(\phi)$ is a truncated normal distribution with mean $\sum_{t=2}^{n}\frac{(h_t-\mu)(h_{t-1}-\mu)}{\sum_{t=1}^{n}(h_t-\mu)^2}$ variance $\frac{\sigma^2}{\sum_{t=1}^{n}(h_t-\mu)^2}$, leaving a range of (-1, 1).

Sampling ϕ and σ^2 together

For ϕ, let its prior distribution be:

$$\frac{1+\phi}{2} \sim Beta(a_0, b_0)$$

Hence

$$p(\phi) \propto \left(\frac{1+\phi}{2}\right)^{a_0-1}\left(\frac{1-\phi}{2}\right)^{b_0-1}$$

$$\propto (1+\phi)^{a_0-1}(1-\phi)^{b_0-1}, a_0, b_0 > 0.5$$

∴ The posterior distribution of ϕ, σ^2 is:

$$\pi(\phi) \propto L(\theta)p(\phi)p(\sigma^2)$$

$$\propto \sigma^{2\times\left(-\frac{n}{2}\right)}(1-\phi^2)^{\frac{1}{2}}\exp\left\{-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma^2} - \sum_{t=2}^{n}\frac{[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2}{2\sigma^2}\right\}\exp\left\{-\frac{\delta_0}{2\sigma^2}\right\}\left(\frac{1+\phi}{2}\right)^{a_0-1}\left(\frac{1-\phi}{2}\right)^{b_0-1}$$

$$\propto \sigma^{2\times\left(-\frac{v_0+n}{2}-1\right)}(1+\phi)^{a_0-\frac{1}{2}}(1-\phi)^{b_0-\frac{1}{2}}\exp\left\{-\frac{1}{2\sigma^2}\left\{(1-\phi^2)(h_1-\mu)^2 + \sum_{t=2}^{n}[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2 + \delta_0\right\}\right\}$$

$$\log[\pi(\phi)] = const\ C - \left(\frac{v_0+n}{2}+1\right)\log(\sigma^2) + \left(a_0-\frac{1}{2}\right)\log(1+\phi) + (b_0-\frac{1}{2})\log(1-\phi)$$

$$-\frac{1}{2\sigma^2}\left\{(1-\phi^2)(h_1-\mu)^2 + \sum_{t=2}^{n}[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2 + \delta_0\right\}$$

Sampling volatility:

The conditional posterior distribution of the parameters (μ,σ^2,ϕ) of the SV model can be obtained by adding latent variables to the conditions and sampling from them. Therefore, to apply the Gibbs sampler, the latent variables should be treated as parameters in the same way as (μ,σ^2,ϕ).

There are three sampling methods for h proposed so far: single-move sampler, multi-move sampler and mixture sampler.

single-move sampler:

$$f(h|\theta) \propto f(h_1|\theta)\prod_{t=2}^{n}f(h_t|h_{t-1},\theta)$$

$$\propto \exp\left\{-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma^2} - \sum_{t=2}^{n}\frac{[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2}{2\sigma^2}\right\}$$

When $t=1$:

$$f^*(h_t|h_{-t},\theta) = f^*(h_1|h_2,\theta)$$

$$\propto f(h_1|\theta)f(h_2|h_1,\theta)$$

$$\propto \exp\left\{-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma^2} - \frac{[(h_2-\mu)-\phi(h_1-\mu)]^2}{2\sigma^2}\right\}$$

$$\propto \exp\left\{-\frac{(1-\phi^2)(h_1^2-2\mu h_1)+\phi^2h_1^2-2\phi[(\phi-1)\mu+h_2]h_1}{2\sigma^2}\right\}$$

$$\propto \exp\left\{-\frac{h_1^2-2\phi[(1-\phi)\mu+h_2]h_1}{2\sigma^2}\right\}$$

When $2 \leq t \leq n-1$:

$$
\begin{aligned}
f^*(h_t|h_{-t},\theta) &\propto f^*(h_t|h_{t-1},h_{t+1},\theta) \\
&\propto f(h_{t+1}|h_t,\theta)f(h_t|h_{t-1},\theta) \\
&\propto \exp\left\{-\frac{[(h_{t+1}-\mu)-\phi(h_t-\mu)]^2}{2\sigma^2}-\frac{[(h_t-\mu)-\phi(h_{t-1}-\mu)]^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\frac{(1+\phi^2)h_t^2-2\phi(h_{t-1}+h_{t+1})h_t}{2\sigma^2}\right\}
\end{aligned}
$$

When $t = n$:

$$
\begin{aligned}
f^*(h_t|h_{-t},\theta) &= f^*(h_n|h_{n-1},\theta) \\
&\propto f(h_n|h_{n-1},\theta) \\
&\propto \exp\left\{-\frac{[(h_n-\mu)-\phi(h_{n-1}-\mu)]^2}{2\sigma^2}\right\} \\
&\propto \exp\left\{-\frac{\{h_n-[\phi h_{n-1}+(1-\phi)\mu]\}^2}{2\sigma^2}\right\}
\end{aligned}
$$

$$\therefore h_t|h_{-t},\theta \sim N(h_t^*,\sigma_t^{2^*})$$

Where

$$
\sigma_t^{2^*} = \begin{cases} \sigma^2 & ,t=1 \\ \dfrac{\sigma^2}{1+\phi^2} & ,2 \leq t \leq n-1 \\ \sigma^2 & ,t=n \end{cases}
$$

$$
h_t^* = \begin{cases} \phi[(1-\phi)\mu+h_2] & ,t=1 \\ \dfrac{\phi(h_{t-1}+h_{t+1})}{1+\phi^2} & ,2 \leq t \leq n-1 \\ \phi h_{n-1}+(1-\phi)\mu & ,t=n \end{cases}
$$

Then

$$
\begin{aligned}
\pi(h_t|h_{-t},\theta) &\propto f(y_t|h_t,\theta)f^*(h_t|h_{-t},\theta) \\
&\propto f(y_t|h_t,\theta)f_N\left(h_t;h_t^*,\sigma_t^{2^*}\right) \\
&\propto \exp\left\{-\frac{y_t^2}{2\exp(h_t)}-\frac{h_t}{2}-\frac{(h_t-h_t^*)^2}{2\sigma_t^{2^*}}\right\}
\end{aligned}
$$

$$log\pi(h_t|h_{-t},\theta) = const\ C - \frac{y_t^2}{2\exp(h_t)}-\frac{h_t}{2}-\frac{(h_t-h_t^*)^2}{2\sigma_t^{2^*}}$$

Note that $exp(-h_t)$ is a convex function and can be bounded by a function linear in h_t. Let $logf(y_t|h_t,\theta) = const\ C + logf^*(y_t,h_t,\theta)$. Then

$$
\begin{aligned}
logf^*(y_t,h_t,\theta) &= -\frac{1}{2}h_t-\frac{y_t^2}{2}\exp\{-h_t\} \\
&\leq -\frac{1}{2}h_t-\frac{y_t^2}{2}[\exp\{-h_t^*\}(1+h_t^*)-\mu_t^*\exp\{-h_t^*\}] \\
&= logg^*(y_t,h_t,\theta,h_t^*)
\end{aligned}
$$

Hence

$$f(y_t|h_t,\theta)f_N\left(h_t;h_t^*,\sigma_t^{2^*}\right) \leq g^*(y_t,h_t,\theta,\mu_t^*)f_N\left(h_t;h_t^*,\sigma_t^{2^*}\right) \propto f_N\left(h_t;\mu_t^*,\sigma_t^{2^*}\right)$$

$$\mu_t^* = h_t^*+\frac{\sigma_t^{2^*}}{2}[y_t^2\exp\{-h_t^*\}-1]$$