

SV model

Model:

$$y_t = \exp\left\{\frac{h_t}{2}\right\} \epsilon_t, \quad \epsilon_t \sim i.i.d. N(0, 1), \quad 1 \leq t \leq n$$

$$h_1 = \mu + \eta_1, \quad \eta_1 \sim i.i.d. N\left(0, \frac{\sigma^2}{1 - \phi^2}\right)$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \eta_t, \quad \eta_t \sim i.i.d. N(0, \sigma^2), \quad 2 \leq t \leq n$$

Likelihood ($\theta = (\mu, \phi, \sigma^2)$):

$$f(y_t|h_t) = \frac{1}{\sqrt{2\pi \exp\{h_t\}}} \exp\left\{-\frac{y_t^2}{2 \exp\{h_t\}}\right\}$$

$$f(h_t|h_{t-1}, \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2}\right\}, 2 \leq t \leq n$$

$$f(h_1|\theta) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1 - \phi^2}}} \exp\left\{-\frac{(h_1 - \mu)^2}{2 \frac{\sigma^2}{1 - \phi^2}}\right\}$$

$$\therefore L(\theta) = f(y|\theta)$$

$$\propto \prod_{t=1}^n [f(y_t|h_t)f(h_t|h_{t-1}, \theta)]$$

$$\propto \sigma^{2 \times \left(-\frac{n}{2}\right)} (1 - \phi^2)^{\frac{1}{2}} \exp\left\{\sum_{t=1}^n \left(-\frac{y_t^2}{2 \exp\{h_t\}} - \frac{h_t}{2}\right) - \frac{(h_1 - \mu)^2}{2 \frac{\sigma^2}{1 - \phi^2}} + \sum_{t=2}^n -\frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2}\right\}$$

Prior distribution:

$$\mu \sim N(\mu_0, \sigma_0^2)$$

$$\sigma^2 \sim IG\left(\frac{\nu_0}{2}, \frac{\delta_0}{2}\right)$$

$$\therefore |\phi| < 1$$

$\therefore \phi$ has 3 or more choices:

$$\begin{cases} \phi \sim U(-1, 1) \\ \phi \sim N_{|\phi| < 1}(0, 1) \\ \frac{1 + \phi}{2} \sim \text{Beta}(a_0, b_0) \end{cases}$$

\therefore Prior PDFs are:

$$p(\mu) \propto \exp\left\{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2}\right\}$$

$$p(\sigma^2) \propto \sigma^{2 \times \left(-\frac{\nu_0}{2} - 1\right)} \exp\left\{-\frac{\delta_0}{2\sigma^2}\right\}$$

$$p(\phi) \propto 1$$

$$\text{or } p(\phi) \propto \exp\left\{-\frac{\phi^2}{2}\right\}$$

$$\text{or } p\left(\frac{1 + \phi}{2}\right) \propto \left(\frac{1 + \phi}{2}\right)^{a_0 - 1} \left(\frac{1 - \phi}{2}\right)^{b_0 - 1}$$

Posterior distribution:

$$\pi(\mu, \sigma^2, \phi) \propto L(\theta) p(\sigma^2) p(\mu) p(\phi)$$

$$\pi(\mu | \sigma^2, \phi) \propto L(\theta) p(\mu)$$

$$\begin{aligned} & \propto \exp \left\{ -\frac{(h_1 - \mu)^2}{2 \frac{\sigma^2}{1 - \phi^2}} + \sum_{t=2}^n -\frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2} \right\} \exp \left\{ -\frac{(\mu - \mu_0)^2}{2\sigma_0^2} \right\} \\ & \propto \exp \left\{ -\frac{(1 - \phi^2)(-2h_1\mu + \mu^2) + \sum_{t=2}^n [(\phi - 1)^2\mu^2 - 2(\phi - 1)(\phi h_{t-1} - h_t)\mu] - \frac{\mu^2 - 2\mu_0\mu}{2\sigma_0^2}}{2\sigma^2} \right\} \\ & \propto \exp \left\{ -\frac{[(n-1)(1 - \phi)^2 + 1 - \phi^2]\mu^2 - 2[(1 - \phi^2)h_1 + (1 - \phi) \sum_{t=2}^n (h_t - \phi h_{t-1})]\mu - \frac{\mu^2 - 2\mu_0\mu}{2\sigma_0^2}}{2\sigma^2} \right\} \\ & \propto \exp \left\{ -\frac{\{\sigma_0^2[(n-1)(1 - \phi)^2 + 1 - \phi^2] + \sigma^2\}\mu^2 - 2\{\sigma_0^2[(1 - \phi^2)h_1 + (1 - \phi) \sum_{t=2}^n (h_t - \phi h_{t-1})] + \sigma^2\mu_0\}\mu}{2\sigma^2\sigma_0^2} \right\} \\ & \therefore \mu | \sigma^2, \phi \sim N(\mu_1, \sigma_1^2) \end{aligned}$$

Where:

$$\begin{aligned} \sigma_1^2 &= \frac{\sigma^2 \sigma_0^2}{\sigma_0^2[(n-1)(1 - \phi)^2 + 1 - \phi^2] + \sigma^2} \\ \mu_1 &= \frac{\sigma_1^2 \{\sigma_0^2[(1 - \phi^2)h_1 + (1 - \phi) \sum_{t=2}^n (h_t - \phi h_{t-1})] + \sigma^2\mu_0\}}{\sigma^2 \sigma_0^2} \\ &= \sigma_1^2 \left\{ \frac{1}{\sigma^2} \left[(1 - \phi^2)h_1 + (1 - \phi) \sum_{t=2}^n (h_t - \phi h_{t-1}) \right] + \frac{1}{\sigma_0^2} \mu_0 \right\} \end{aligned}$$

$$\pi(\sigma^2 | \mu, \phi) \propto L(\theta) p(\sigma^2)$$

$$\begin{aligned} & \propto \sigma^{2 \times \left(-\frac{n}{2}\right)} \exp \left\{ -\frac{(h_1 - \mu)^2}{2 \frac{\sigma^2}{1 - \phi^2}} + \sum_{t=2}^n -\frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2} \right\} \sigma^{2 \times \left(-\frac{\nu_0 - 1}{2}\right)} \exp \left\{ -\frac{\delta_0}{2\sigma^2} \right\} \\ & \propto \sigma^{2 \times \left(-\frac{\nu_0 + n}{2} - 1\right)} \exp \left\{ -\frac{(1 - \phi^2)(h_1 - \mu)^2 + \sum_{t=2}^n [(h_t - \mu) - \phi(h_{t-1} - \mu)]^2 + \delta_0}{2\sigma^2} \right\} \\ & \therefore \sigma^2 | \mu, \phi \sim IG \left(\frac{\nu_1}{2}, \frac{\delta_1}{2} \right) \end{aligned}$$

Where:

$$\nu_1 = \nu_0 + n$$

$$\delta_1 = (1 - \phi^2)(h_1 - \mu)^2 + \sum_{t=2}^n [(h_t - \mu) - \phi(h_{t-1} - \mu)]^2 + \delta_0$$

$$\pi(\phi | \mu, \sigma^2) \propto L(\theta) p(\phi)$$

$$\propto (1 - \phi^2)^{\frac{1}{2}} \exp \left\{ -\frac{(h_1 - \mu)^2}{2 \frac{\sigma^2}{1 - \phi^2}} + \sum_{t=2}^n -\frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2} \right\} p(\phi)$$

$$\log \pi(\phi | \mu, \sigma^2) = \text{const } C - \frac{1}{2\sigma^2} \sum_{t=2}^n [(h_t - \mu) - \phi(h_{t-1} - \mu)]^2 + \ln [\varphi(\phi)], \quad -1 < \phi < 1$$

$$\text{where } \ln [\varphi(\phi)] = \frac{1}{2} \log(1 - \phi^2) - \frac{(1 - \phi^2)(h_1 - \mu)^2}{2\sigma^2} + \log p(\phi), \quad -1 < \phi < 1$$

This is a special distribution, but can be sampled by using the AR (Acceptance-Rejection) algorithm or the MH (Metropolis-Hastings) algorithm. The method proposed in Chib and Greenberg (1994) is presented here. They neglected $\ln [\varphi(\phi)]$ on

the right-hand side in the above equation, which is a $\text{const } C + \log[h(\phi)]$, i.e. ∴

$$\begin{aligned} \log \pi(\phi | \mu, \sigma^2) &\approx \text{const } C - \frac{1}{2\sigma^2} \sum_{t=2}^n [(h_t - \mu) - \phi(h_{t-1} - \mu)]^2 \\ &= \text{const } C - \frac{\sum_{t=2}^n (h_{t-1} - \mu)^2}{2\sigma^2} \left\{ \phi - \sum_{t=2}^n \frac{(h_t - \mu)(h_{t-1} - \mu)}{\sum_{t=2}^n (h_{t-1} - \mu)^2} \right\}^2 \\ &= \text{const } C + \log [h(\phi)], \quad -1 < \phi < 1 \end{aligned}$$

Then $h(\phi)$ is a truncated normal distribution with mean $\sum_{t=2}^n \frac{(h_t - \mu)(h_{t-1} - \mu)}{\sum_{t=2}^n (h_{t-1} - \mu)^2}$ variance $\frac{\sigma^2}{\sum_{t=2}^n (h_{t-1} - \mu)^2}$, leaving a range of (-1, 1).

Sampling of volatility:

The conditional posterior distribution of the parameters (μ, σ^2, ϕ) of the SV model can be obtained by adding latent variables to the conditions and sampling from them. Therefore, to apply the Gibbs sampler, the latent variables should be treated as parameters in the same way as (μ, σ^2, ϕ) .

There are three sampling methods for h proposed so far: single-move sampler, multi-move sampler and mixture sampler.

single-move sampler:

$$\begin{aligned} f(h|\theta) &\propto f(h_1|\theta) \prod_{t=2}^n f(h_t|h_{t-1}, \theta) \\ &\propto \exp \left\{ -\frac{(1-\phi^2)(h_1 - \mu)^2}{2\sigma^2} - \sum_{t=2}^n \frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2} \right\} \end{aligned}$$

When $t = 1$:

$$\begin{aligned} f^*(h_t|h_{-t}, \theta) &= f^*(h_1|h_2, \theta) \\ &\propto f(h_1|\theta) f(h_2|h_1, \theta) \\ &\propto \exp \left\{ -\frac{(1-\phi^2)(h_1 - \mu)^2}{2\sigma^2} - \frac{[(h_2 - \mu) - \phi(h_1 - \mu)]^2}{2\sigma^2} \right\} \\ &\propto \exp \left\{ -\frac{(1-\phi^2)(h_1^2 - 2\mu h_1) + \phi^2 h_2^2 - 2\phi[(\phi - 1)\mu + h_2]h_1}{2\sigma^2} \right\} \\ &\propto \exp \left\{ -\frac{h_1^2 - 2\phi[(1-\phi)\mu + h_2]h_1}{2\sigma^2} \right\} \end{aligned}$$

When $2 \leq t \leq n-1$:

$$\begin{aligned} f^*(h_t|h_{-t}, \theta) &\propto f^*(h_t|h_{t-1}, h_{t+1}, \theta) \\ &\propto f(h_{t+1}|h_t, \theta) f(h_t|h_{t-1}, \theta) \\ &\propto \exp \left\{ -\frac{[(h_{t+1} - \mu) - \phi(h_t - \mu)]^2}{2\sigma^2} - \frac{[(h_t - \mu) - \phi(h_{t-1} - \mu)]^2}{2\sigma^2} \right\} \\ &\propto \exp \left\{ -\frac{(1+\phi^2)h_t^2 - 2\phi(h_{t-1} + h_{t+1})h_t}{2\sigma^2} \right\} \end{aligned}$$

When $t = n$:

$$\begin{aligned} f^*(h_t|h_{-t}, \theta) &= f^*(h_n|h_{n-1}, \theta) \\ &\propto f(h_n|h_{n-1}, \theta) \\ &\propto \exp \left\{ -\frac{[(h_n - \mu) - \phi(h_{n-1} - \mu)]^2}{2\sigma^2} \right\} \\ &\propto \exp \left\{ -\frac{\{h_n - [\phi h_{n-1} + (1-\phi)\mu]\}^2}{2\sigma^2} \right\} \end{aligned}$$

$$\therefore h_t|h_{-t}, \theta \sim N(h_t^*, \sigma_t^{2*})$$

Where

$$\sigma_t^{2*} = \begin{cases} \sigma^2, & t = 1 \\ \frac{\sigma^2}{1 + \phi^2}, & 2 \leq t \leq n - 1 \\ \sigma^2, & t = n \end{cases}$$

$$h_t^* = \begin{cases} \phi[(1 - \phi)\mu + h_2], & t = 1 \\ \frac{\phi(h_{t-1} + h_{t+1})}{1 + \phi^2}, & 2 \leq t \leq n - 1 \\ \phi h_{n-1} + (1 - \phi)\mu, & t = n \end{cases}$$

Then

$$\begin{aligned} \pi(h_t|h_{-t}, \theta) &\propto f(y_t|h_t, \theta) f^*(h_t|h_{-t}, \theta) \\ &\propto f(y_t|h_t, \theta) f_N(h_t; h_t^*, \sigma_t^{2*}) \\ &\propto \exp\left\{-\frac{y_t^2}{2\exp(h_t)} - \frac{h_t}{2} - \frac{(h_t - h_t^*)^2}{2\sigma_t^{2*}}\right\} \\ \log \pi(h_t|h_{-t}, \theta) &= \text{const } C - \frac{y_t^2}{2\exp(h_t)} - \frac{h_t}{2} - \frac{(h_t - h_t^*)^2}{2\sigma_t^{2*}} \end{aligned}$$

Note that $\exp(-h_t)$ is a convex function and can be bounded by a function linear in h_t . Let $\log f(y_t|h_t, \theta) = \text{const } C + \log f^*(y_t, h_t, \theta)$. Then

$$\begin{aligned} \log f^*(y_t, h_t, \theta) &= -\frac{1}{2}h_t - \frac{y_t^2}{2}\exp\{-h_t\} \\ &\leq -\frac{1}{2}h_t - \frac{y_t^2}{2}[\exp\{-h_t^*\}(1 + h_t^*) - \mu_t^* \exp\{-h_t^*\}] \\ &= \log g^*(y_t, h_t, \theta, h_t^*) \end{aligned}$$

Hence

$$\begin{aligned} f(y_t|h_t, \theta) f_N(h_t; h_t^*, \sigma_t^{2*}) &\leq g^*(y_t, h_t, \theta, \mu_t^*) f_N(h_t; h_t^*, \sigma_t^{2*}) \propto f_N(h_t; \mu_t^*, \sigma_t^{2*}) \\ \mu_t^* &= h_t^* + \frac{\sigma_t^{2*}}{2}[y_t^2 \exp\{-h_t^*\} - 1] \end{aligned}$$