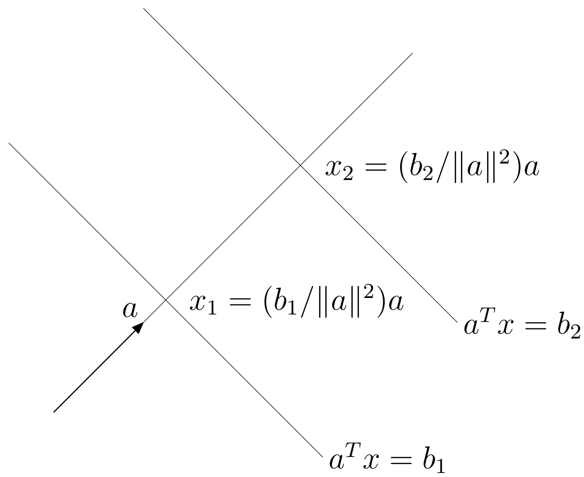
CVX101 Homework 1 solutions

2.5 What is the distance between two parallel hyperplanes $\{x \in \mathbf{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbf{R}^n \mid a^T x = b_2\}$?

Solution. The distance between the two hyperplanes is $|b_1 - b_2|/\|a\|_2$. To see this, consider the construction in the figure below.



The distance between the two hyperplanes is also the distance between the two points x_1 and x_2 where the hyperplane intersects the line through the origin and parallel to the normal vector a. These points are given by

$$x_1 = (b_1/\|a\|_2^2)a, \qquad x_2 = (b_2/\|a\|_2^2)a,$$

and the distance is

$$\|x_1 - x_2\|_2 = |b_1 - b_2|/\|a\|_2.$$

2.7 *Voronoi description of halfspace.* Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, *i.e.*, $\{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Solution. Since a norm is always nonnegative, we have $\|x-a\|_2 \leq \|x-b\|_2$ if and only if $\|x-a\|_2^2 \leq \|x-b\|_2^2$, so

$$\begin{aligned} \|x-a\|_2^2 \leq \|x-b\|_2^2 &\iff (x-a)^T(x-a) \leq (x-b)^T(x-b) \\ &\iff x^T x - 2a^T x + a^T a \leq x^T x - 2b^T x + b^T b \\ &\iff 2(b-a)^T x \leq b^T b - a^T a. \end{aligned}$$

Therefore, the set is indeed a halfspace. We can take $c = 2(b - a)$ and $d = b^T b - a^T a$. This makes good geometric sense: the points that are equidistant to a and b are given by a hyperplane whose normal is in the direction $b - a$.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \le a^T x \le \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \dots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

X (e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \le \theta \|x - b\|_2\}$. You can assume $a \ne b$ and $0 \le \theta \le 1$.

Solution.

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) This set is convex because it can be expressed as

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\},$$

i.e., an intersection of halfspaces. (For fixed y, the set

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$$

is a halfspace; see exercise 2.9).

(e) In general this set is not convex, as the following example in $\mathbf{R}$ shows. With $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\} = \{x \in \mathbf{R} \mid x \leq -1/2 \text{ or } x \geq 1/2\}$$

which clearly is not convex.

(f) This set is convex. $x + S_2 \subseteq S_1$ if $x + y \in S_1$ for all $y \in S_2$. Therefore

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y),$$

the intersection of convex sets $S_1 - y$.

(g) The set is convex, in fact a ball.

$$\begin{aligned} &\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\} \\ &= \{x \mid \|x - a\|_2^2 \leq \theta^2 \|x - b\|_2^2\} \\ &= \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\} \end{aligned}$$

If $\theta = 1$, this is a halfspace. If $\theta < 1$, it is a ball

$$\{x \mid (x - x_0)^T (x - x_0) \leq R^2\},$$

with center x_0 and radius R given by

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, \qquad R = \left(\frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1 - \theta^2} + \|x_0\|_2^2 \right)^{1/2}.$$

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

(a) $\alpha \leq \mathbf{E} f(x) \leq \beta$, where $\mathbf{E} f(x)$ is the expected value of $f(x)$, *i.e.*, $\mathbf{E} f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbf{R} \to \mathbf{R}$ is given.)

(b) $\mathbf{prob}(x > \alpha) \leq \beta$.

(c) $\mathbf{E} |x^3| \leq \alpha \mathbf{E} |x|$.

(d) $\mathbf{E} x^2 \leq \alpha$.

(e) $\mathbf{E} x^2 \geq \alpha$.

(f) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E} x)^2$ is the variance of x.

(g) $\mathbf{var}(x) \geq \alpha$.

(h) $\mathbf{quartile}(x) \geq \alpha$, where $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(i) $\mathbf{quartile}(x) \leq \alpha$.

Solution. We first note that the constraints $p_i \geq 0$, $i = 1, \ldots, n$, define halfspaces, and $\sum_{i=1}^n p_i = 1$ defines a hyperplane, so P is a polyhedron.

The first five constraints are, in fact, linear inequalities in the probabilities p_i.

(a) $\mathbf{E}\, f(x) = \sum_{i=1}^n p_i f(a_i)$, so the constraint is equivalent to two linear inequalities

$$\alpha \leq \sum_{i=1}^n p_i f(a_i) \leq \beta.$$

(b) $\mathbf{prob}(x \geq \alpha) = \sum_{i:\, a_i \geq \alpha} p_i$, so the constraint is equivalent to a linear inequality

$$\sum_{i:\, a_i \geq \alpha} p_i \leq \beta.$$

(c) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i(|a_i^3| - \alpha|a_i|) \leq 0.$$

(d) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i a_i^2 \leq \alpha.$$

(e) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i a_i^2 \geq \alpha.$$

The first five constraints therefore define convex sets.

(f) The constraint

$$\mathbf{var}(x) = \mathbf{E}\, x^2 - (\mathbf{E}\, x)^2 = \sum_{i=1}^n p_i a_i^2 - (\sum_{i=1}^n p_i a_i)^2 \leq \alpha$$

is not convex in general. As a counterexample, we can take $n = 2$, $a_1 = 0$, $a_2 = 1$, and $\alpha = 1/5$. $p = (1, 0)$ and $p = (0, 1)$ are two points that satisfy $\mathbf{var}(x) \leq \alpha$, but the convex combination $p = (1/2, 1/2)$ does not.

(g) This constraint is equivalent to

$$\sum_{i=1}^n p_i a_i^2 - (\sum_{i=1}^n p_i a_i)^2 = b^T p - p^T A p \geq \alpha,$$

where $b_i = a_i^2$ and $A = aa^T$. We write this as

$$p^T A p - b^T p + \alpha \leq 0.$$

This defines a convex set, since the matrix aa^T is positive semidefinite. (See exercise 2.10.)

Let us denote $\mathbf{quartile}(x) = f(p)$ to emphasize it is a function of p. The figure illustrates the definition. It shows the cumulative distribution for a distribution p with $f(p) = a_2$.

(h) The constraint $f(p) \geq \alpha$ is equivalent to

$$\mathbf{prob}(x \leq \beta) < 0.25 \text{ for all } \beta < \alpha.$$

If $\alpha \leq a_1$, this is always true. Otherwise, define $k = \max\{i \mid a_i < \alpha\}$. This is a fixed integer, independent of p. The constraint $f(p) \geq \alpha$ holds if and only if

$$\mathbf{prob}(x \leq a_k) = \sum_{i=1}^{k} p_i < 0.25.$$

This is a strict linear inequality in p, which defines an open halfspace.

(i) The constraint $f(p) \leq \alpha$ is equivalent to

$$\mathbf{prob}(x \leq \beta) \geq 0.25 \text{ for all } \beta \geq \alpha.$$

Here, let us define $k = \max\{i \mid a_i \leq \alpha\}$. Again, this is a fixed integer, independent of p. The constraint $f(p) \leq \alpha$ holds if and only if

$$\mathbf{prob}(x \leq a_k) = \sum_{i=1}^{k} p_i \geq 0.25.$$

If $\alpha < a_1$, then no p satisfies $f(p) \leq \alpha$, which means that the set is empty. Thus, the constraint $f(p) \leq \alpha$ is a linear inequality on p.

2.28 *Positive semidefinite cone for $n = 1,\ 2,\ 3$.* Give an explicit description of the positive semidefinite cone $\mathbf{S}_+^n$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1,\ 2,\ 3$. To describe a general element of $\mathbf{S}^n$, for $n = 1,\ 2,\ 3$, use the notation

$$x_1, \qquad \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

Solution. A symmetric matrix X is positive semidefinite if and only if all principal minors (determinants of symmetric submatrices) are nonnegative. For $n = 1$ the condition is just $x_1 \geq 0$. For $n = 2$ the condition is

$$x_1 \geq 0, \qquad x_3 \geq 0, \qquad x_1 x_3 - x_2^2 \geq 0.$$

For $n = 3$ the condition is

$$x_1 \geq 0, \qquad x_4 \geq 0, \qquad x_6 \geq 0, \qquad x_1x_4 - x_2^2 \geq 0, \qquad x_4x_6 - x_5^2 \geq 0, \qquad x_1x_6 - x_3^2 \geq 0$$

and

$$x_1x_4x_6 + 2x_2x_3x_5 - x_1x_5^2 - x_6x_2^2 - x_4x_3^2 \geq 0.$$

A1.7 *Dual cones in* $\mathbf{R}^2$. Describe the dual cone for each of the following cones.

(a) $K = \{0\}$.

(b) $K = \mathbf{R}^2$.

(c) $K = \{(x_1, x_2) \mid |x_1| \leq x_2\}$.

(d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

Solution.

(a) $K^* = \mathbf{R}^2$. To see this:

$$\begin{aligned} K^* &= \{y \mid y^T x \geq 0 \text{ for all } x \in K\} \\ &= \{y \mid y^T 0 \geq 0\} \\ &= \mathbf{R}^2. \end{aligned}$$

(b) $K^* = \{0\}$. To see this, we need to identify the values of $y \in \mathbf{R}^2$ for which $y^T x \geq 0$ for all $x \in \mathbf{R}^2$. But given any $y \neq 0$, consider the choice $x = -y$, for which we have $y^T x = -\|y\|_2^2 < 0$. So the only possible choice is $y = 0$ (which indeed satisfies $y^T x \geq 0$ for all $x \in \mathbf{R}^2$).

(c) $K^* = K$. (This cone is self-dual.)

(d) $K^* = \{(x_1, x_2) \mid x_1 - x_2 = 0\}$. Here K is a line, and K^* is the line orthogonal to it.