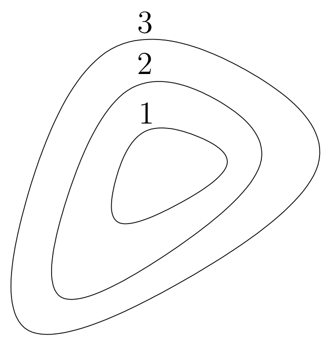
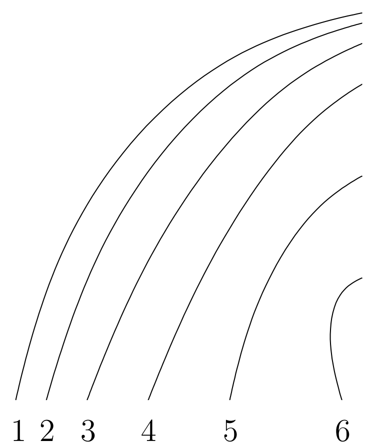
CVX101 Homework 2 solutions

3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.

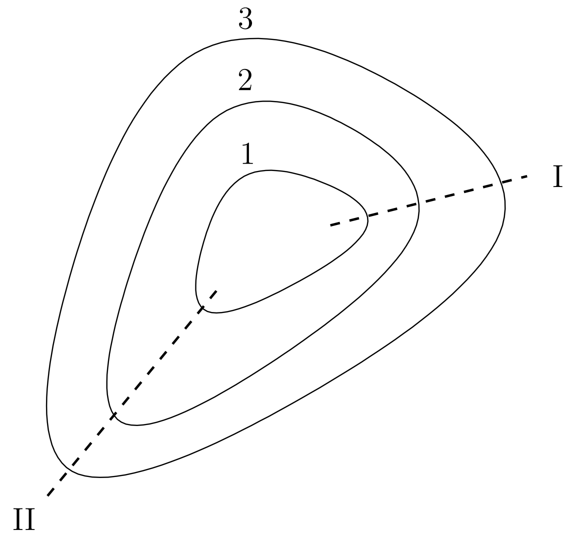


Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.

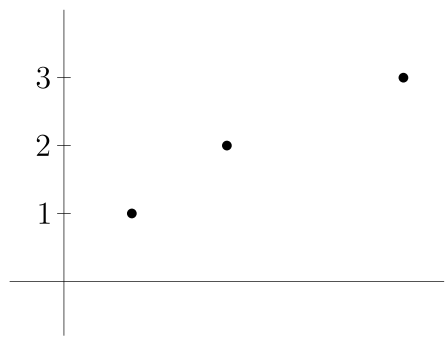


Solution. The first function could be quasiconvex because the sublevel sets appear to be convex. It is definitely not concave or quasiconcave because the superlevel sets are not convex.

It is also not convex, for the following reason. We plot the function values along the dashed line labeled I.



Along this line the function passes through the points marked as black dots in the figure below. Clearly along this line segment, the function is not convex.



If we repeat the same analysis for the second function, we see that it could be concave (and therefore it could be quasiconcave). It cannot be convex or quasiconvex, because the sublevel sets are not convex.

3.6 *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

Solution. The epigraph of f is a halfspace if and only if f is affine.

The epigraph of f is a convex cone if and only if f is convex and positively homogeneous, *i.e.*, $f(\alpha x) = \alpha f(x)$ for any x and any $\alpha \geq 0$.

The epigraph of f is a polyhedron if and only if f is convex and piecewise affine.

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

Solution. Strictly convex, and therefore quasiconvex. Also quasiconcave but not concave.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is neither positive semidefinite nor negative semidefinite. Therefore, f is neither convex nor concave. It is quasiconcave, since its superlevel sets

$$\{(x_1, x_2) \in \mathbf{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$$

are convex. It is not quasiconvex.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

Solution. The Hessian of f is

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/(x_1^2) & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix} \succeq 0$$

Therefore, f is convex and quasiconvex. It is not quasiconcave or concave.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

which is not positive or negative semidefinite. Therefore, f is not convex or concave.

It is quasiconvex and quasiconcave (*i.e.*, quasilinear), since the sublevel and superlevel sets are halfspaces.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

Solution. f is convex, as mentioned on page 72. (See also figure 3.3). This is easily verified by working out the Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix} = (2/x_2) \begin{bmatrix} 1 \\ -x_1/x_2 \end{bmatrix} \begin{bmatrix} 1 & -x_1/x_2 \end{bmatrix} \succeq 0.$$

Therefore, f is convex and quasiconvex. It is not concave or quasiconcave (see the figure).

(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbf{R}^2_{++}$.

Solution. Concave and quasiconcave. The Hessian is

$$\begin{array}{rcl} \nabla^2 f(x) & = & \left[\begin{array}{cc} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{array}\right] \\ & = & \alpha(1-\alpha)x_1^{\alpha}x_2^{1-\alpha}\left[\begin{array}{cc} -1/x_1^2 & 1/x_1x_2 \\ 1/x_1x_2 & -1/x_2^2 \end{array}\right] \\ & = & -\alpha(1-\alpha)x_1^{\alpha}x_2^{1-\alpha}\left[\begin{array}{c} 1/x_1 \\ -1/x_2 \end{array}\right]\left[\begin{array}{c} 1/x_1 \\ -1/x_2 \end{array}\right]^T \\ & \preceq & 0. \end{array}$$

f is not convex or quasiconvex.

A2.2 *A general vector composition rule.* Suppose

$$f(x) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

where $h : \mathbf{R}^k \to \mathbf{R}$ is convex, and $g_i : \mathbf{R}^n \to \mathbf{R}$. Suppose that for each i, one of the following holds:

- h is nondecreasing in the ith argument, and g_i is convex
- h is nonincreasing in the ith argument, and g_i is concave
- g_i is affine.

Show that f is convex. (This composition rule subsumes all the ones given in the book, and is the one used in software systems such as CVX.) You can assume that $\mathbf{dom}\, h = \mathbf{R}^k$; the result also holds in the general case when the monotonicity conditions listed above are imposed on $\tilde{h}$, the extended-valued extension of h.

Solution. Fix x, y, and $\theta \in [0, 1]$, and let $z = \theta x + (1-\theta)y$. Let's re-arrange the indexes so that g_i is affine for $i = 1, \ldots, p$, g_i is convex for $i = p+1, \ldots, q$, and g_i is concave for $i = q+1, \ldots, k$. Therefore we have

$$\begin{array}{ll} g_i(z) = \theta g_i(x) + (1-\theta)g_i(y), & i = 1, \ldots, p, \\ g_i(z) \le \theta g_i(x) + (1-\theta)g_i(y), & i = p+1, \ldots, q, \\ g_i(z) \ge \theta g_i(x) + (1-\theta)g_i(y), & i = q+1, \ldots, k. \end{array}$$

We then have

$$\begin{array}{rcl} f(z) & = & h(g_1(z), g_2(z), \ldots, g_k(z)) \\ & \le & h(\theta g_1(x) + (1-\theta)g_1(y), \ldots, \theta g_k(x) + (1-\theta)g_k(y)) \\ & \le & \theta h(g_1(x), \ldots, g_k(x)) + (1-\theta)h(g_1(y), \ldots, g_k(y)) \\ & = & \theta f(x) + (1-\theta)f(y). \end{array}$$

The second line holds since, for $i = p+1, \ldots, q$, we have increased the ith argument of h, which is (by assumption) nondecreasing in the ith argument, and for $i = q+1, \ldots, k$, we have decreased the ith argument, and h is nonincreasing in these arguments. The third line follows from convexity of h.

3.39 Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.
Solution. We will show that

$$f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0, \quad \mathbf{1}^T y = 1 \\ \infty & \text{otherwise.} \end{cases}$$

We first verify the domain of f^*. First suppose y has a negative component, say $y_k < 0$. If we choose a vector x with $x_k = -t$, $x_i = 0$ for $i \neq k$, and let t go to infinity, we see that

$$x^T y - \max_i x_i = -t y_k \to \infty,$$

so y is not in $\mathbf{dom}\, f^*$. Next, assume $y \succeq 0$ but $\mathbf{1}^T y > 1$. We choose $x = t\mathbf{1}$ and let t go to infinity, to show that

$$x^T y - \max_i x_i = t \mathbf{1}^T y - t$$

is unbounded above. Similarly, when $y \succeq 0$ and $\mathbf{1}^T y < 1$, we choose $x = -t\mathbf{1}$ and let t go to infinity.
The remaining case for y is $y \succeq 0$ and $\mathbf{1}^T y = 1$. In this case we have

$$x^T y \leq \max_i x_i$$

for all x, and therefore $x^T y - \max_i x_i \leq 0$ for all x, with equality for $x = 0$. Therefore $f^*(y) = 0$.