

## 3. Convex functions

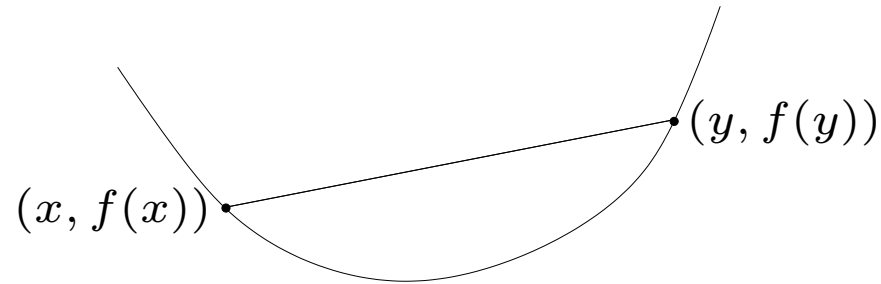
- basic properties and examples
- operations that preserve convexity
- the conjugate function
- quasiconvex functions
- log-concave and log-convex functions
- convexity with respect to generalized inequalities

# Definition

$f : \mathbf{R}^n \rightarrow \mathbf{R}$  is convex if  $\mathbf{dom} f$  is a convex set and

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

for all  $x, y \in \mathbf{dom} f$ ,  $0 \leq \theta \leq 1$



- $f$  is concave if  $-f$  is convex
- $f$  is strictly convex if  $\mathbf{dom} f$  is convex and

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$$

for  $x, y \in \mathbf{dom} f$ ,  $x \neq y$ ,  $0 < \theta < 1$

## Examples on $\mathbf{R}$

convex:

- affine:  $ax + b$  on  $\mathbf{R}$ , for any  $a, b \in \mathbf{R}$
- exponential:  $e^{ax}$ , for any  $a \in \mathbf{R}$
- powers:  $x^\alpha$  on  $\mathbf{R}_{++}$ , for  $\alpha \geq 1$  or  $\alpha \leq 0$
- powers of absolute value:  $|x|^p$  on  $\mathbf{R}$ , for  $p \geq 1$
- negative entropy:  $x \log x$  on  $\mathbf{R}_{++}$

concave:

- affine:  $ax + b$  on  $\mathbf{R}$ , for any  $a, b \in \mathbf{R}$
- powers:  $x^\alpha$  on  $\mathbf{R}_{++}$ , for  $0 \leq \alpha \leq 1$
- logarithm:  $\log x$  on  $\mathbf{R}_{++}$

## Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

### examples on $\mathbf{R}^n$

- affine function  $f(x) = a^T x + b$
- norms:  $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$  for  $p \geq 1$ ;  $\|x\|_\infty = \max_k |x_k|$

### examples on $\mathbf{R}^{m \times n}$ ( $m \times n$ matrices)

- affine function

$$A^T X = \langle A | X \rangle = A \circ X$$
$$f(X) = \text{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

## Restriction of a convex function to a line

$f : \mathbf{R}^n \rightarrow \mathbf{R}$  is convex if and only if the function  $g : \mathbf{R} \rightarrow \mathbf{R}$ ,

$$g(t) = f(x + tv), \quad \text{dom } g = \{t \mid x + tv \in \text{dom } f\}$$

is convex (in  $t$ ) for any  $x \in \text{dom } f$ ,  $v \in \mathbf{R}^n$

can check convexity of  $f$  by checking convexity of functions of one variable

**example.**  $f : \mathbf{S}^n \rightarrow \mathbf{R}$  with  $f(X) = \log \det X$ ,  $\text{dom } f = \mathbf{S}_{++}^n$

$$\begin{aligned} g(t) = \log \det(X + tV) &= \log \det X + \log \det(I + tX^{-1/2}VX^{-1/2}) \\ &= \log \det X + \sum_{i=1}^n \log(1 + t\lambda_i) \end{aligned}$$

where  $\lambda_i$  are the eigenvalues of  $X^{-1/2}VX^{-1/2}$

$g$  is concave in  $t$  (for any choice of  $X \succ 0$ ,  $V$ ); hence  $f$  is concave

## Extended-value extension

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extended-value extension  $\tilde{f}$  of  $f$  is

$$\tilde{f}(x) = f(x), \quad x \in \mathbf{dom} f, \quad \tilde{f}(x) = \infty, \quad x \notin \mathbf{dom} f$$

often simplifies notation; for example, the condition

$$0 \leq \theta \leq 1 \implies \tilde{f}(\theta x + (1 - \theta)y) \leq \theta \tilde{f}(x) + (1 - \theta)\tilde{f}(y)$$

(as an inequality in  $\mathbf{R} \cup \{\infty\}$ ), means the same as the two conditions

- $\mathbf{dom} f$  is convex
- for  $x, y \in \mathbf{dom} f$ ,

$$0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

# First-order condition

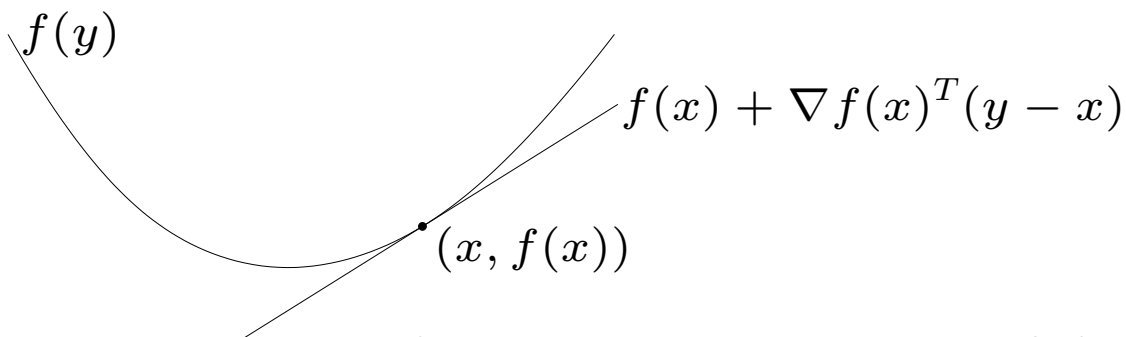
$f$  is **differentiable** if  $\text{dom } f$  is open and the gradient

$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each  $x \in \text{dom } f$

**1st-order condition:** differentiable  $f$  with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \text{dom } f$$



first-order approximation of  $f$  is global underestimator

## Second-order conditions

$f$  is **twice differentiable** if  $\text{dom } f$  is open and the Hessian  $\nabla^2 f(x) \in \mathbf{S}^n$ ,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n,$$

exists at each  $x \in \text{dom } f$

**2nd-order conditions:** for twice differentiable  $f$  with convex domain

- $f$  is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \text{dom } f$$

- if  $\nabla^2 f(x) \succ 0$  for all  $x \in \text{dom } f$ , then  $f$  is strictly convex





## Examples

**quadratic function:**  $f(x) = (1/2)x^T P x + q^T x + r$  (with  $P \in \mathbf{S}^n$ )

$$\nabla f(x) = Px + q, \quad \nabla^2 f(x) = P$$

convex if  $P \succeq 0$

**least-squares objective:**  $f(x) = \|Ax - b\|_2^2$

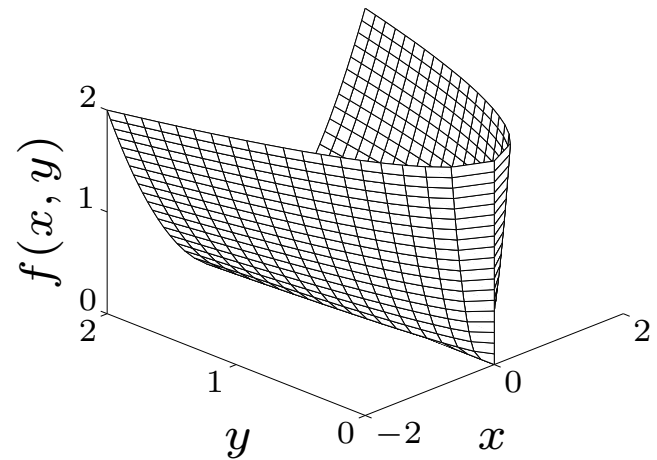
$$\nabla f(x) = 2A^T(Ax - b), \quad \nabla^2 f(x) = 2A^T A$$

convex (for any  $A$ )

**quadratic-over-linear:**  $f(x, y) = x^2/y$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for  $y > 0$



**log-sum-exp:**  $f(x) = \log \sum_{k=1}^n \exp x_k$  is convex

$$\nabla^2 f(x) = \frac{\mathbf{1}^T \mathbf{z} \cdot \text{diag}(\mathbf{z}) - \mathbf{z} \mathbf{z}^T}{(\mathbf{1}^T \mathbf{z})^2}$$

$$\nabla^2 f(x) = \frac{1}{\mathbf{1}^T \mathbf{z}} \mathbf{diag}(\mathbf{z}) - \frac{1}{(\mathbf{1}^T \mathbf{z})^2} \mathbf{z} \mathbf{z}^T \quad (z_k = \exp x_k)$$

to show  $\nabla^2 f(x) \succeq 0$ , we must verify that  $v^T \nabla^2 f(x) v \geq 0$  for all  $v$ :

$$v^T \nabla^2 f(x) v = \frac{(\sum_k z_k v_k^2)(\sum_k z_k) - (\sum_k v_k z_k)^2}{(\sum_k z_k)^2} \geq 0$$

since  $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$  (from Cauchy-Schwarz inequality)

**geometric mean:**  $f(x) = (\prod_{k=1}^n x_k)^{1/n}$  on  $\mathbf{R}_{++}^n$  is concave

$$z_k = \frac{1}{x_k}$$

(similar proof as for log-sum-exp)

$$\begin{aligned} \nabla^2 f(x) &= \frac{\sqrt[n]{\prod x_i}}{n^2} \cdot \left( \mathbf{z} \mathbf{z}^T - n \cdot \text{diag}(\mathbf{z}^2) \right) \\ &= \frac{\sqrt[n]{\prod x_i}}{n^2} \left( (\sum v_k z_k)^2 - n \cdot \sum (v_k z_k)^2 \right) \leq 0 \end{aligned}$$

# Epigraph and sublevel set

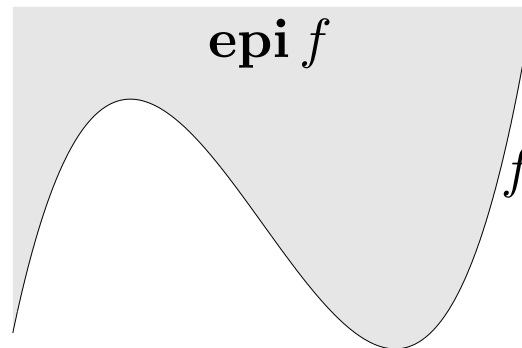
$\alpha$ -sublevel set of  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ :

$$C_\alpha = \{x \in \mathbf{dom} f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

**epigraph** of  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ :

$$\mathbf{epi} f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom} f, f(x) \leq t\}$$



$f$  is convex if and only if  $\mathbf{epi} f$  is a convex set

# Jensen's inequality

**basic inequality:** if  $f$  is convex, then for  $0 \leq \theta \leq 1$ ,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

**extension:** if  $f$  is convex, then

$$f(\mathbf{E} z) \leq \mathbf{E} f(z)$$

for any random variable  $z$

basic inequality is special case with discrete distribution

$$\mathbf{prob}(z = x) = \theta, \quad \mathbf{prob}(z = y) = 1 - \theta$$

# Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show  $\nabla^2 f(x) \succeq 0$
3. show that  $f$  is obtained from simple convex functions by operations that preserve convexity
  - nonnegative weighted sum
  - composition with affine function
  - pointwise maximum and supremum
  - composition
  - minimization
  - perspective

# Positive weighted sum & composition with affine function

**nonnegative multiple:**  $\alpha f$  is convex if  $f$  is convex,  $\alpha \geq 0$

**sum:**  $f_1 + f_2$  convex if  $f_1, f_2$  convex (extends to infinite sums, integrals)

**composition with affine function:**  $f(Ax + b)$  is convex if  $f$  is convex

## examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^m \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i, i = 1, \dots, m\}$$

- (any) norm of affine function:  $f(x) = \|Ax + b\|$

## Pointwise maximum

if  $f_1, \dots, f_m$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex

### examples

- piecewise-linear function:  $f(x) = \max_{i=1,\dots,m}(a_i^T x + b_i)$  is convex
- sum of  $r$  largest components of  $x \in \mathbf{R}^n$ :

$$f(x) = x_{[1]} + x_{[2]} + \dots + x_{[r]}$$

is convex ( $x_{[i]}$  is  $i$ th largest component of  $x$ )

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \dots + x_{i_r} \mid 1 \leq i_1 < i_2 < \dots < i_r \leq n\}$$

# Pointwise supremum

if  $f(x, y)$  is convex in  $x$  for each  $y \in \mathcal{A}$ , then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

## examples

- support function of a set  $C$ :  $S_C(x) = \sup_{y \in C} y^T x$  is convex
- distance to farthest point in a set  $C$ :

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for  $X \in \mathbf{S}^n$ ,

$$\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$$

$$X(y_1, \dots, y_n) = (\lambda_1 y_1, \dots, \lambda_n y_n)$$



# Composition with scalar functions

composition of  $g : \mathbf{R}^n \rightarrow \mathbf{R}$  and  $h : \mathbf{R} \rightarrow \mathbf{R}$ :

$$f(x) = h(g(x))$$

$f$  is convex if  $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

- proof (for  $n = 1$ , differentiable  $g, h$ )

$$f''(x) = \underbrace{h''(g(x))}_{>0} \underbrace{g'(x)^2}_{>0} + \underbrace{h'(g(x))}_{>0} \underbrace{g''(x)}_{>0}$$

- note: monotonicity must hold for extended-value extension  $\tilde{h}$

## examples

- $\exp g(x)$  is convex if  $g$  is convex
- $1/g(x)$  is convex if  $g$  is concave and positive

# Vector composition

composition of  $g : \mathbf{R}^n \rightarrow \mathbf{R}^k$  and  $h : \mathbf{R}^k \rightarrow \mathbf{R}$ :

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \dots, g_k(x))$$

$f$  is convex if  $\begin{array}{l} g_i \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing in each argument} \\ g_i \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing in each argument} \end{array}$

proof (for  $n = 1$ , differentiable  $g, h$ )

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \underbrace{\nabla h(g(x))}_{k \times 1}^T \underbrace{g''(x)}_{1 \times k}$$

## examples

- $\sum_{i=1}^m \log g_i(x)$  is concave if  $g_i$  are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$  is convex if  $g_i$  are convex

# Minimization

if  $f(x, y)$  is convex in  $(x, y)$  and  $C$  is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

## examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$  with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \quad C \succ 0 \in \mathcal{S}_{++}^n$$

minimizing over  $y$  gives  $g(x) = \inf_y f(x, y) = x^T (A - B C^{-1} B^T) x$

$g$  is convex, hence Schur complement  $A - B C^{-1} B^T \succeq 0$

- distance to a set:  $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$  is convex if  $S$  is convex

# Perspective

the **perspective** of a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is the function  $g : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$ ,

$$g(x, t) = tf(x/t), \quad \text{dom } g = \{(x, t) \mid x/t \in \text{dom } f, t > 0\}$$

$g$  is convex if  $f$  is convex

## examples

- $f(x) = x^T x$  is convex; hence  $g(x, t) = x^T x/t$  is convex for  $t > 0$
- negative logarithm  $f(x) = -\log x$  is convex; hence relative entropy  $g(x, t) = t \log t - t \log x$  is convex on  $\mathbf{R}_{++}^2$
- if  $f$  is convex, then

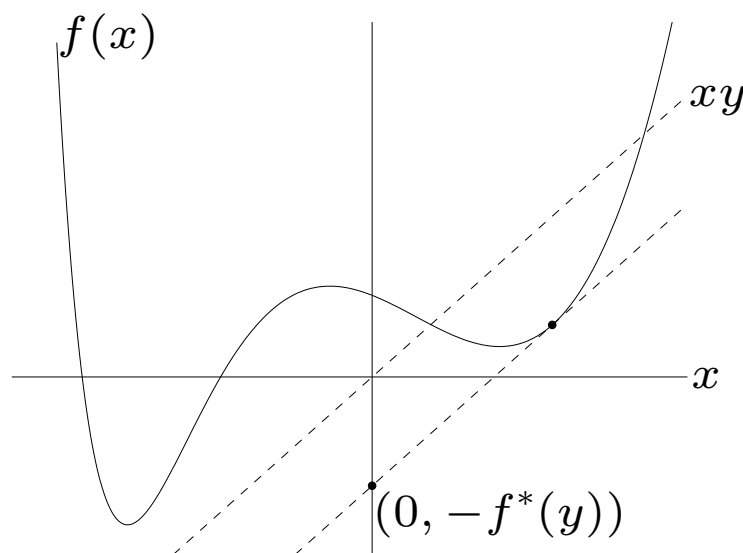
$$g(x) = (c^T x + d)f((Ax + b)/(c^T x + d))$$

is convex on  $\{x \mid c^T x + d > 0, (Ax + b)/(c^T x + d) \in \text{dom } f\}$

# The conjugate function

the **conjugate** of a function  $f$  is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^T x - f(x))$$



Fenchel's Inequality  
 $f(x) + f^*(y) \geq x^T y$

- $f^*$  is convex (even if  $f$  is not)
- will be useful in chapter 5

## examples

- negative logarithm  $f(x) = -\log x$

$$\begin{aligned} f^*(y) &= \sup_{x>0} (xy + \log x) \\ &= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

- strictly convex quadratic  $f(x) = (1/2)x^T Qx$  with  $Q \in \mathbf{S}_{++}^n$

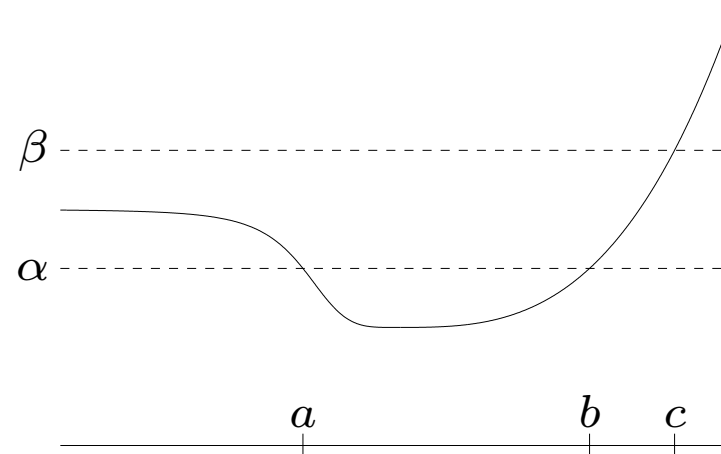
$$\begin{aligned} f^*(y) &= \sup_x (y^T x - (1/2)x^T Qx) \\ &= \frac{1}{2} y^T Q^{-1} y \end{aligned}$$

# Quasiconvex functions

$f : \mathbf{R}^n \rightarrow \mathbf{R}$  is quasiconvex if  $\mathbf{dom} f$  is convex and the sublevel sets

$$S_\alpha = \{x \in \mathbf{dom} f \mid f(x) \leq \alpha\}$$

are convex for all  $\alpha$



- $f$  is quasiconcave if  $-f$  is quasiconvex
- $f$  is quasilinear if it is quasiconvex and quasiconcave

## Examples

- $\sqrt{|x|}$  is quasiconvex on  $\mathbf{R}$
- $\text{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$  is quasilinear
- $\log x$  is quasilinear on  $\mathbf{R}_{++}$
- $f(x_1, x_2) = x_1 x_2$  is quasiconcave on  $\mathbf{R}_{++}^2$
- linear-fractional function

$$f(x) = \frac{a^T x + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}$$

is quasilinear

- distance ratio

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2}, \quad \text{dom } f = \{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$$

is quasiconvex



## internal rate of return

- cash flow  $x = (x_0, \dots, x_n)$ ;  $x_i$  is payment in period  $i$  (to us if  $x_i > 0$ )
- we assume  $x_0 < 0$  and  $x_0 + x_1 + \dots + x_n > 0$
- present value of cash flow  $x$ , for interest rate  $r$ :

$$\text{PV}(x, r) = \sum_{i=0}^n (1+r)^{-i} x_i$$

- internal rate of return is smallest interest rate for which  $\text{PV}(x, r) = 0$ :

$$\text{IRR}(x) = \inf\{r \geq 0 \mid \text{PV}(x, r) = 0\}$$

IRR is quasiconcave: superlevel set is intersection of open halfspaces

$$\text{IRR}(x) \geq R \iff \sum_{i=0}^n (1+r)^{-i} x_i > 0 \text{ for } 0 \leq r < R$$

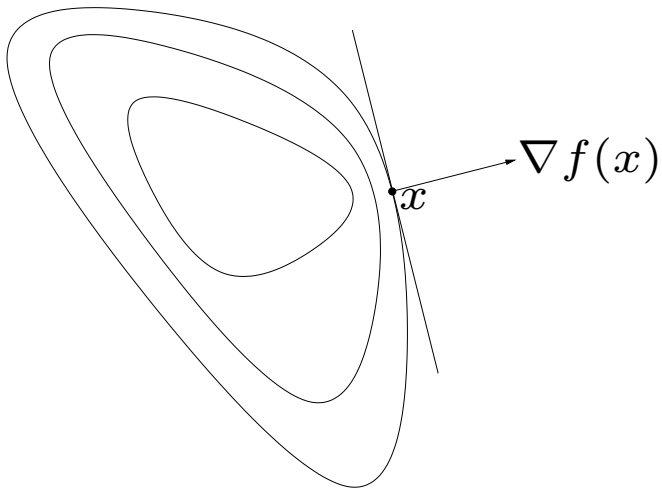
# Properties

**modified Jensen inequality:** for quasiconvex  $f$

$$0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\}$$

**first-order condition:** differentiable  $f$  with cvx domain is quasiconvex iff

$$f(y) \leq f(x) \implies \nabla f(x)^T (y - x) \leq 0$$



**sums** of quasiconvex functions are not necessarily quasiconvex

# Log-concave and log-convex functions

a positive function  $f$  is log-concave if  $\log f$  is concave:

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

$f$  is log-convex if  $\log f$  is convex

- powers:  $x^a$  on  $\mathbf{R}_{++}$  is log-convex for  $a \leq 0$ , log-concave for  $a \geq 0$
- many common probability densities are log-concave, *e.g.*, normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$

- cumulative Gaussian distribution function  $\Phi$  is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

To prove  
 $\Phi(x) \cdot \Phi'(x) \leq \Phi'(x)^2$

# Properties of log-concave functions

- twice differentiable  $f$  with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T$$

for all  $x \in \text{dom } f$

- product of log-concave functions is log-concave
- sum of log-concave functions is not always log-concave
- integration: if  $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}$  is log-concave, then

sum of log-concave  
is log-concave

$$g(x) = \int f(x, y) dy$$

is log-concave (not easy to show)

## consequences of integration property

- convolution  $f * g$  of log-concave functions  $f, g$  is log-concave

$$(f * g)(x) = \int f(x - y)g(y)dy$$

- if  $C \subseteq \mathbf{R}^n$  convex and  $y$  is a random variable with log-concave pdf then

$$f(x) = \mathbf{prob}(x + y \in C)$$

is log-concave

proof: write  $f(x)$  as integral of product of log-concave functions

$$f(x) = \int g(x + y)p(y) dy, \quad g(u) = \begin{cases} 1 & u \in C \\ 0 & u \notin C, \end{cases}$$

$p$  is pdf of  $y$

## example: yield function

$$Y(x) = \mathbf{prob}(x + w \in S)$$

- $x \in \mathbf{R}^n$ : nominal parameter values for product
- $w \in \mathbf{R}^n$ : random variations of parameters in manufactured product
- $S$ : set of acceptable values

if  $S$  is convex and  $w$  has a log-concave pdf, then

- $Y$  is log-concave
- yield regions  $\{x \mid Y(x) \geq \alpha\}$  are convex

# Convexity with respect to generalized inequalities

$f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is  $K$ -convex if  $\text{dom } f$  is convex and

$$f(\theta x + (1 - \theta)y) \preceq_K \theta f(x) + (1 - \theta)f(y)$$

for  $x, y \in \text{dom } f$ ,  $0 \leq \theta \leq 1$

**example**  $f : \mathbf{S}^m \rightarrow \mathbf{S}^m$ ,  $f(X) = X^2$  is  $\mathbf{S}_+^m$ -convex

proof: for fixed  $z \in \mathbf{R}^m$ ,  $z^T X^2 z = \|Xz\|_2^2$  is convex in  $X$ , i.e.,

$$z^T (\theta X + (1 - \theta)Y)^2 z \leq \theta z^T X^2 z + (1 - \theta) z^T Y^2 z$$

for  $X, Y \in \mathbf{S}^m$ ,  $0 \leq \theta \leq 1$

therefore  $(\theta X + (1 - \theta)Y)^2 \preceq \theta X^2 + (1 - \theta)Y^2$