Lab 6 – Bayesian Inference

Ex. 1 (Bayes' Theorem)

We know:

$$P(B) = 0.01, \quad P(+|B) = 0.95, \quad P(-|\neg B) = 0.90$$

Thus:

$$P(+|\neg B) = 1 - 0.90 = 0.10, \quad P(\neg B) = 0.99$$

a) Probability of having the disease given a positive test

Using Bayes' Theorem:

$$P(B|+) = \frac{P(+|B) \cdot P(B)}{P(+|B) \cdot P(B) + P(+|\neg B) \cdot P(\neg B)}$$

$$P(B|+) = \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.10 \cdot 0.99} \approx 0.0876$$

So the probability is approximately **8.76%**.

b) Minimum specificity s.t. $P(B|+) = 0.5$

Let x be the specificity. Then:

$$0.5 = \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + (1 - x) \cdot 0.99}$$

Solving for x:

$$x \approx 0.9899$$

So the specificity must be at least **98.99%**.

Ex. 2 (Poisson – Gamma conjugacy)

We observe 180 calls in 10 hours. Let λ be the call rate per hour.

a) Posterior distribution

Likelihood:

$$P(k|\lambda) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k = 180,\ t = 10$$

Choose a Gamma prior:

$$\lambda \sim \text{Gamma}(\alpha, \beta), \quad \alpha = 1,\ \beta = 1$$

Posterior is:

$$\lambda|\text{data} \sim \text{Gamma}(\alpha + 180,\ \beta + 10) = \text{Gamma}(181, 11)$$

b) 94% Highest Density Interval (HDI)

Using the 3rd and 97th percentiles of the posterior:

$$\text{HDI}_{94\%} \approx (16.15,\ 20.06)$$

c) Mode of posterior

Mode of a Gamma distribution:

$$\text{mode} = \frac{\alpha' - 1}{\beta'} = \frac{181 - 1}{11} = \frac{180}{11} \approx 16.36$$

Final Results

- Ex 1 (a): $P(B|+) \approx 8.76\%$
- Ex 1 (b): Min. specificity $\approx 98.99\%$
- Ex 2 Posterior: $\lambda \sim \text{Gamma}(181, 11)$
- Ex 2 HDI (94%): $\approx (16.15,\ 20.06)$
- Ex 2 Mode: ≈ 16.36