Bayesian Inference – Worked Solutions

Exercise 1 (Bayes' Theorem)

Data:
$$P(B) = 0.01, \quad P(+ \mid B) = 0.95, \quad P(- \mid \neg B) = 0.90.$$

Hence
$$P(+ \mid \neg B) = 1 - 0.90 = 0.10, \qquad P(\neg B) = 0.99.$$

(a) Posterior probability $P(B \mid +)$

By Bayes' theorem,

$$P(B \mid +) = \frac{P(+ \mid B)\,P(B)}{P(+ \mid B)\,P(B) + P(+ \mid \neg B)\,P(\neg B)} = \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.10 \cdot 0.99} \approx 0.0876.$$

Thus, the probability is approximately **8.76**%.

(b) Minimum specificity such that $P(B \mid +) = 0.5$

Let x denote the specificity, i.e., $x = P(- \mid \neg B)$. Then $P(+ \mid \neg B) = 1 - x$. Solve

$$0.5 = \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + (1 - x) \cdot 0.99}.$$

Equivalently,
$$0.5\big(0.95 \cdot 0.01 + (1 - x) \cdot 0.99\big) = 0.95 \cdot 0.01,$$
$$0.5 \cdot 0.0095 + 0.5 \cdot 0.99\,(1 - x) = 0.0095,$$
$$0.00475 + 0.495\,(1 - x) = 0.0095 \quad \implies \quad 0.495\,(1 - x) = 0.00475.$$

Hence
$$1 - x = \frac{0.00475}{0.495} \approx 0.00959596 \quad \implies \quad x \approx 0.9899.$$

Therefore, the specificity must be at least **98.99**%.

Exercise 2 (Poisson–Gamma Conjugacy)

We observe $k = 180$ calls over $t = 10$ hours. Let λ be the call rate (per hour).

Model and Likelihood

For the aggregated count over t hours, $X \sim \text{Poisson}(\lambda t)$. The likelihood is

$$\mathcal{L}(\lambda \mid k, t) = P(X = k \mid \lambda) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \ \propto \ \lambda^k e^{-\lambda t}.$$

Prior

Choose the conjugate prior $\lambda \sim \text{Gamma}(\alpha, \beta)$ with density

$$p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \qquad \alpha, \beta > 0.$$

Posterior

By Bayes' rule,

$$p(\lambda \mid k, t) \ \propto \ \mathcal{L}(\lambda \mid k, t)\, p(\lambda) \ \propto \ \lambda^k e^{-\lambda t}\, \lambda^{\alpha-1} e^{-\beta\lambda} = \lambda^{(\alpha+k)-1}\, e^{-(\beta+t)\lambda},$$

which is the kernel of a Gamma distribution. Therefore,

$$\lambda \mid k, t \ \sim \ \text{Gamma}\big(\alpha', \beta'\big), \qquad \alpha' = \alpha + k, \ \ \beta' = \beta + t.$$

Numerical Posterior (weakly informative prior)

Using $\alpha = 1$, $\beta = 1$ (weak prior),

$$\alpha' = 1 + 180 = 181, \qquad \beta' = 1 + 10 = 11,$$

so

$$\lambda \mid \text{data} \ \sim \ \text{Gamma}(181, 11).$$

94% Highest Density Interval (HDI)

For a unimodal Gamma, the HDI is (practically) the same as the equal–tail credible interval. Using the Gamma quantile function $F^{-1}_{\text{Gamma}(\alpha',\beta')}$ with $\gamma = 0.06$ (i.e., 94% mass in the middle),

$$\text{HDI}_{94\%} = \left[F^{-1}_{\text{Gamma}(\alpha',\beta')}\big(0.03\big), \ F^{-1}_{\text{Gamma}(\alpha',\beta')}\big(0.97\big) \right] \approx (16.15, \ 20.06).$$

Mode

For $\text{Gamma}(\alpha', \beta')$ with $\alpha' > 1$,

$$\text{mode} = \frac{\alpha' - 1}{\beta'} = \frac{181 - 1}{11} = \frac{180}{11} \approx 16.36 \text{ calls/hour.}$$

Summary

$$\boxed{\begin{array}{l} \text{Ex. 1(a)}: \ P(B \mid +) \approx 8.76\%, \quad \text{Ex. 1(b)}: \ \text{specificity} \approx 98.99\%; \\[4pt] \lambda \mid \text{data} \sim \text{Gamma}(181, 11), \quad \text{HDI}_{94\%} \approx (16.15, 20.06), \quad \text{mode} \approx 16.36. \end{array}}$$