1. Consider transient heat conduction in a circular region of radius a. Considering that heat conduction takes place only radially, find the solution of the transient heat conduction in the circular disk at any point r at any time $t > 0$
(i) When the boundary is kept at zero degrees and the initial temperature distribution is given by $u(r, 0) = 100$,
(ii) When the boundary is kept at zero degrees and the initial temperature distribution is given by $u(r, 0) = r$.
Solution:
Governing equation:
$$u_t = \alpha(u_{rr} + \frac{1}{r}u_r).$$
$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\nu_n}{a}r\right) e^{-\alpha \frac{\nu_n^2}{a^2}t},$$
where ν_n are zeros of $J_0(\lambda a)$, and
$$A_n = \frac{\int_0^a r f(r) J_0\left(\frac{\nu_n}{a}r\right)\, dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\, dr}.$$
(i) $f(r) = 100$:
$$A_n = 100\frac{\int_0^a r J_0\left(\frac{\nu_n}{a}r\right)\, dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\, dr}.$$
(ii) $f(r) = r$:
$$A_n = \frac{\int_0^a r^2 J_0\left(\frac{\nu_n}{a}r\right)\, dr}{\int_0^a r\left(J_0\left(\frac{\nu_n}{a}r\right)\right)^2\, dr}.$$

2. Solve the following boundary value problem in a circular disk:
$$\begin{aligned}\nabla^2 u &= 0, r < a,\ 0 \le \theta < 2\pi,\\ u(a,\theta) &= 4 + 3\sin\theta, 0 \le \theta < 2\pi.\end{aligned}$$
Solution:
$$u(r,\theta) = 4 + \frac{3}{a}r\sin\theta.$$
(All coefficients for $\cos n\theta, n \neq 0$ will vanish due to the given BC and $a_0 = 8$. Also for the coefficients of $\sin\theta$, all will be zero for $n \neq 1$ and

$b_1 = 3/a$.)
Recall the solution:

$$u(r,\theta) = a_0/2 + \sum_{n=1}^{\infty}[a_n \cos n\theta + b_n \sin n\theta]r^n,$$

with

$$\begin{aligned} a_n &= \frac{1}{\pi a^n}\int_0^{2\pi} u(a,\theta)\cos n\theta d\theta,\ n = 0,1,2,3,\ldots \\ b_n &= \frac{1}{\pi a^n}\int_0^{2\pi} u(a,\theta)\sin n\theta d\theta,\ n = 1,2,3,\ldots. \end{aligned}$$

3. Find the Fourier integral representation of the following non-periodic function:

$$f(t) = \begin{cases} \sin t, & t^2 < \pi^2, \\ 0, & t^2 > \pi^2. \end{cases}$$

Solution: The Fourier transform of $f(t)$:

$$\begin{aligned} \mathcal{F}\{f(t)\} &= g(\sigma) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\sigma t}\, dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \sin t\; e^{-i\sigma t}\, dt \\ &= -\frac{1}{\sqrt{2\pi}}\frac{2i\sin\sigma\pi}{1-\sigma^2}. \end{aligned}$$

Taking the inverse

$$\begin{aligned} f(t) = \mathcal{F}^{-1}\{g(\sigma)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\sigma)e^{i\sigma t}\, d\sigma \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2i\sin\sigma\pi}{1-\sigma^2}\right]e^{i\sigma t}\, d\sigma \\ &= \frac{2}{\pi}\int_0^{\infty}\frac{\sin\sigma\pi\sin\sigma t}{1-\sigma^2}\, d\sigma. \end{aligned}$$

4. Find the Fourier integral representation of the following non-periodic function

$$f(t) = \begin{cases} 0, & -\infty < t < -1, \\ -1, & -1 < t < 0, \\ 1, & 0 < t < 1, \\ 0, & 1 < t < \infty. \end{cases}$$

Solution: Realize that the given function is an odd function.
We can directly use the sine integral formula:

$$\begin{aligned}
\mathcal{F}_s\{f(t)\} &= g_s(\sigma) \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \sigma t \; dt \\
&= \sqrt{\frac{2}{\pi}} \;\; |(-\cos \sigma t)/\sigma|_0^1 \\
&= \sqrt{\frac{2}{\pi}} \; \frac{1 - \cos \sigma}{\sigma}.
\end{aligned}$$

Taking the inverse

$$f(t) = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos \sigma)}{\sigma} \sin \sigma t \; d\sigma.$$

5. Express

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi, \\ 0, & x > \pi, \end{cases}$$

as Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1 - \cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma.$$

Solution:
From the convergence of Fourier integral, we obtain

$$f_o(\pi) = \frac{1}{2}(f_o(\pi^+) + f_o(\pi^-)) = \frac{1}{2} = \int_0^\infty \sqrt{\frac{2}{\pi}} \sin(\sigma\pi) g_s(\sigma) d\sigma.$$

Here, $g_s(\sigma)$ is the Fourier sine transform of f_o (odd extension of f). By definition, $g_s(\sigma)$ is given by

$$\begin{aligned}
g_s(\sigma) &= \int_0^\infty \sqrt{\frac{2}{\pi}} f_0(\tau) \sin \sigma\tau d\tau \\
&= \int_0^\pi \sqrt{\frac{2}{\pi}} \sin \sigma\tau d\tau = -\sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \cos \sigma\tau|_0^\pi \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} (1 - \cos \sigma\pi)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\frac{1}{2} &= \int_0^\infty \sqrt{\frac{2}{\pi}} \sin(\sigma\pi) g_s(\sigma) d\sigma \\
&= \int_0^\infty \sqrt{\frac{2}{\pi}} \sin(\sigma\pi) \sqrt{\frac{2}{\pi}} \frac{1}{\sigma}(1-\cos\sigma\pi) d\sigma \\
&= \frac{2}{\pi} \int_0^\infty \frac{1-\cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma.
\end{aligned}$$

Hence,

$$\int_0^\infty \frac{1-\cos(\pi\sigma)}{\sigma} \sin(\pi\sigma) d\sigma = \frac{\pi}{4}.$$

6. If $U(x,t)$ is the temperature at time t and α the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time by solving the following boundary value problem:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \alpha \frac{\partial^2 U}{\partial x^2}, \quad x > 0,\ t > 0 \\
\frac{\partial U}{\partial x}(0,t) &= 0, \quad U(x,0) = f(x).
\end{aligned}$$

Solution:
Taking Fourier cosine transform

$$\frac{d}{dt}\overline{U_c}(\sigma,t) + \alpha\sigma^2 \overline{U_c} = 0,$$

which gives us

$$\overline{U_c}(\sigma,t) = Ae^{-\alpha\sigma^2 t}$$

Using the initial condition,

$$\overline{U_c}(\sigma,t) = \overline{f_c}(\sigma) e^{-\alpha\sigma^2 t},$$

where $\overline{f_c}(\sigma)$ is the Fourier cosine transform of $f(x)$. Taking inverse, we get the solution

$$U(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{f_c}(\sigma) e^{-\alpha\sigma^2 t} \cos\sigma x \, d\sigma.$$

7. Find the Laplace transforms of

$$\text{(i) } te^{3t}\cos 4t, \text{(ii) } t\int_0^t e^{-3t}\sin 2t\ dt, \text{(iii) } \int_0^t \frac{e^{-3t}\sin 2t}{t}\ dt.$$

Solution:
(i)

$$\begin{aligned}
\mathcal{L}\{te^{3t}\cos 4t\} &= -\frac{d}{ds}\mathcal{L}\{e^{3t}\cos 4t\} \\
&= -\frac{d}{ds}\mathcal{L}\{\cos 4t\}_{s\to\ s-3} \\
&= -\frac{d}{ds}\left\{\frac{s-3}{(s-3)^2+4^2}\right\} \\
&= \frac{(s-3)^2-16}{((s-3)^2+16)^2}
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{L}\{t\int_0^t e^{-3t}\sin 2t\} &= -\frac{d}{ds}\frac{\mathcal{L}\{e^{-3t}\sin 2t\}}{s} \\
&= -\frac{d}{ds}\frac{\mathcal{L}\{\sin 2t\}_{s\to\ s+3}}{s} \\
&= -\frac{d}{ds}\frac{2}{s((s+3)^2+4)} \\
&= \frac{2(3s^2+12s+13)}{(s^3+6s^2+13s)^2}
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}\{\int_0^t \frac{e^{-3t}\sin 2t}{t}\ dt\} &= \frac{1}{s}\int_s^\infty \mathcal{L}\{e^{-3t}\sin 2t\}ds \\
&= \frac{1}{s}\int_s^\infty \frac{2}{(s+3)^2+4}ds \\
&= \frac{1}{s}\Big|\tan^{-1}(\frac{s+3}{2})\Big|_s^\infty \\
&= \frac{1}{s}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{s+3}{2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)\right] \\
&= \frac{1}{s}\,\cot^{-1}\left[\frac{s+3}{2}\right]
\end{aligned}
$$

(Using various properties.)

8. Find the Laplace transform of the following step functions:
(i) $2H(\sin \pi t) - 1$, (ii) $H(t^3 - 6t^2 + 11t - 6)$.
Solution: (i) From definition of unit step function

$$
H(\sin \pi t) = \begin{cases} 1, & \sin \pi t > 0, \\ 0, & \sin \pi t < 0. \end{cases}
$$

This will give the given function as $+1$ between 0 and 1, 2 and 3, and so on whereas it will be -1 between 1 and 2, 3 and 4, and so on. That is

$$
\begin{aligned}
\mathcal{L}\{2H(\sin \pi t) - 1\} &= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt + \int_2^3 e^{-st}dt - \int_3^4 e^{-st}dt + \cdots \\
&= \left|\frac{e^{-st}}{-s}\right|_0^1 - \left|\frac{e^{-st}}{-s}\right|_1^2 + \left|\frac{e^{-st}}{-s}\right|_2^3 - \left|\frac{e^{-st}}{-s}\right|_3^4 + \cdots \\
&= \frac{1}{s}\left[(1 - e^{-s}) + (e^{-2s} - e^{-s}) + (e^{-2s} - e^{-3s}) + (e^{-4s} - e^{-3s}) + \cdots\right] \\
&= \frac{1}{s}[1 - 2e^{-s}(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots)] \\
&= \frac{1}{s}\left[1 - 2e^{-s}\frac{1}{1+e^{-s}}\right] \\
&= \frac{1}{s}\frac{1 - e^{-s}}{1 + e^{-s}} \\
&= \frac{1}{s}\tanh\frac{s}{2}
\end{aligned}
$$

(ii) $\frac{1}{s}[e^{-s} - e^{-2s} + e^{-3s}]$.
From definition of unit step function

$$
H(t^3 - 6t^2 + 11t - 6) = \begin{cases} 1, & (t-1)(t-2)(t-3) > 0, \\ 0, & (t-1)(t-2)(t-3) < 0. \end{cases}
$$

This will give the given function as +1 between 1 and 2, and 3 and ∞.

$$\begin{aligned}\mathcal{L}\{H(t^3-6t^2+11t-6)\} &= \int_1^2 e^{-st}dt + \int_3^\infty e^{-st}dt \\ &= \left|\frac{e^{-st}}{-s}\right|_1^2 + \left|\frac{e^{-st}}{-s}\right|_3^\infty \\ &= \frac{1}{s}[e^{-s} - e^{-2s} + e^{-3s}]\end{aligned}$$

9. Find the inverse Laplace transforms:
(i) $\dfrac{2s+3}{s^2+4s+6}$, (ii) $\dfrac{2s^2-3s+5}{s^2(s^2+1)}$.
Solution: (i)

$$\begin{aligned}f(s) &= \frac{2s+3}{s^2+4s+6} \\ &= \frac{2(s+2)-1}{(s+2)^2+(\sqrt{2})^2}\end{aligned}$$

Taking inverse

$$F(t) = e^{-2t}(2\cos\sqrt{2}t - \frac{1}{\sqrt{2}}\ \sin\sqrt{2}t)$$

(ii) Take $\dfrac{2s^2-3s+5}{s^2(s^2+1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+1}$. Find $A=-3, B=5, C=3, D=-3$ so that

$$\frac{2s^2-3s+5}{s^2(s^2+1)} = -\frac{3}{s} + \frac{5}{s^2} + \frac{3s}{s^2+1} - \frac{3}{s^2+1}.$$

By inverting,

$$F(t) = 5t - 3 - 3(\sin t - \cos t)$$

10. Solve the following Initial Value Problems:
(i) $\ddot{y} + 2\dot{y} + 5y = e^{-t}\sin t,\ y(0) = 0,\ \dot{y}(0) = 1,$ (ii) $t\ddot{y} + 2\dot{y} + ty = 0,\ y(0) = 1.$
Solution:
(i) After getting the Laplace transform of y as

$$\mathcal{L}\{y(t)\} = \frac{1}{3}\frac{1}{(s+1)^2+1} + \frac{2}{3}\frac{1}{(s+1)^2+2^2},$$

and then inverting

$$y(t) = \frac{e^{-t}}{3}[\sin t + \sin 2t]$$

(ii)

Taking Laplace transform on the equation,

$$-\frac{d}{ds}\mathcal{L}\{\ddot{y}\} + 2\mathcal{L}\{\dot{y}\} - \frac{d}{ds}\mathcal{L}\{y\} = 0$$
$$\Rightarrow -\frac{d}{ds}[s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0)] + 2[s\mathcal{L}\{y\} - y(0)] - \frac{d}{ds}\mathcal{L}\{y\} = 0,$$

which gives the ODE

$$\frac{d}{ds}\mathcal{L}\{y\} = -\frac{1}{s^2+1},$$

$\Rightarrow \mathcal{L}\{y\} = -\tan^{-1}(s) + C.$

Inverting

$$y(t) = \frac{1}{t}\mathcal{L}^{-1}\{\frac{d}{ds}(\tan^{-1}(s) + C)\} = \sin t/t.$$

11. Solve the following IBVP for one dimensional heat conduction equation for a rod of unit length and with unit diffusivity:

$$\begin{aligned} U_t &= U_{xx},\ 0 < x < 1, t > 0, \\ U(x,0) &= 3\sin(2\pi x),\ 0 < x < 1, \\ U(0,t) &= 0 = U(1,t),\ t > 0. \end{aligned}$$

Solution:

Using $\mathcal{L}\{U(x,t)\} = \overline{u}(x,s)$ and taking transform on the equation, we get

$$\frac{d^2\overline{u}(x,s)}{dx^2} - s\overline{u}(x,s) = -3\sin(2\pi x).$$

Solving this non-homogeneous ODE:

$$\overline{u}(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x} + \frac{3\sin(2\pi x)}{s+4\pi^2}.$$

Converting and using the boundary conditions, $A(s) = 0 = B(s)$ thereby getting

$$\overline{u}(x,s) = \frac{3\sin(2\pi x)}{s + 4\pi^2}.$$

Inverting,

$$u(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x).$$