1. Find the D'Alembert's solution of one-dimensional wave equation, $u_{tt} = u_{xx}$, on the infinite string with the following initial conditions:

   (A) $u(x,0) = x^2,\ u_t(x,0) = \sin x.$

   (B) $u(x,0) = \sin x,\ u_t(x,0) = \cos x.$

Solution:

(A) The D'Alembert solution for the wave equation $u_{tt} = c^2 u_{xx}$ is given by:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(\eta)d\eta,$$

where $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Thus,

$$\begin{aligned} u(x,t) &= \frac{(x+t)^2 + (x-t)^2}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} \sin\eta d\eta, \\ &= x^2 + t^2 + \frac{1}{2}\left(\cos(x-t) - \cos(x+t)\right), \\ &= x^2 + t^2 + \sin x \sin t. \end{aligned}$$

(B) Similar to part (A), we obtain the solution,

$$u(x,t) = \frac{\sin(x+t) + \sin(x-t)}{2} + \frac{1}{2}\int\limits_{x-t}^{x+t} \cos\eta d\eta = \sin(x+t).$$

2. An infinite string stretching is given the initial displacement,

$$\phi(x) = \frac{1}{1+4x^2},$$

and released from rest. Find its subsequent motion as a function of x and t.

Solution:

Recall the D'Alembert's solution for the one-dimensional wave equation. Application of the initial displacement $u(x,0) = \phi(x) = \frac{1}{1+4x^2}$ and the initial velocity $u_t(x,0) = \psi(x) = 0$, results in the following expression for $u(x,t)$ as,

$$u(x,t) = \frac{1 + 4(x^2 + c^2t^2)}{\left[1 + 4(x+ct)^2\right]\left[1 + 4(x-ct)^2\right]}.$$

3. Determine the Fourier series of the following functions:

   (A) $f(x) = \begin{cases} -x, & -\pi \le x \le 0, \\ x, & 0 \le x \le \pi. \end{cases}$

(B) $f(x) = |\sin x|, \quad -\pi < x < \pi.$

Solution:

(A) This is an even periodic function with period 2π. Thus, $B_n = 0\ \forall n$. Hence,

$$\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \pi \\
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx dx \\
&= \frac{2}{\pi n^2}\Big|\cos nx\Big|_0^{\pi} = \frac{2}{\pi n^2}\{(-1)^n - 1\} \\
&= \begin{cases} 0, & \text{if } n \text{ is even,} \\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus, the Fourier series is given by,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right\}.$$

(B) Since $|\sin x|$ is an even function, we have $B_n = 0$ for $n = 1, 2, \dots$. Further,

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x] dx \\
&= \frac{2}{\pi}\frac{[1+(-1)^n]}{(1-n^2)}, \text{ for } n = 0, 2, 3, \dots.
\end{aligned}$$

For $n = 1$, $A_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x dx = 0$. Thus, the Fourier series is given by,

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(1-4n^2)}.$$

4. For the following functions, determine the Fourier cosine series and the Fourier sine series on the interval $0 < x < \pi$:

   (A) $f(x) = 1$.

   (B) $f(x) = \pi - x$.

   (C) $f(x) = x^2$.

Solution:

(A) Cosine series: $f(x) = 1$ and Sine series: $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin(2n-1)x}{2n-1}$.

(B) Cosine series: $f(x) = \dfrac{\pi}{2} + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos(2n-1)x}{(2n-1)^2}$ and Sine series: $f(x) = 2\displaystyle\sum_{n=1}^{\infty} \dfrac{\sin nx}{n}$.

(C) Cosine series: $f(x) = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n \cos nx}{n^2}$ and

Sine series: $f(x) = 2\pi^2 \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{(-1)^{n+1}}{n\pi} - 2\dfrac{(1-(-1)^n)}{(n\pi)^3}\right] \sin nx$.

5. Given the Fourier series for the function $f(t) = t^4, \ -\pi < t < \pi$, as,

$$\frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^4}(\pi^2 n^2 - 6)\cos nt,$$

determine the Fourier series for $f(t) = t^5, \ -\pi < t < \pi$.

Solution:

Integrating the given series will result in a sine series, along with another t with the constant $\dfrac{\pi^4}{5}$. We know that the Fourier series for $\dfrac{t}{2}$ is determined as,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

Now, substituting this back into the series for t^5, we obtain,

$$t^5 = 2\pi^4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt + 40\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}(n^2\pi^2 - 6)\sin nt.$$

6. Deduce the Fourier series for the function $f(x) = e^{ax}, \ -\pi < x < \pi, \ a \in \mathbb{R}$. Hence find the values of the following four series:

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}$

(B) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a^2+n^2}$

(D) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2}$.

Solution:

The Fourier series of given function $f(x)$ is:

$$\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos(nx)-n\sin(nx)]\right\}.$$

(A) Substituting $x=0$, which is a point of continuity, we obtain,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{a\pi-\sinh a\pi}{2a^2\sinh a\pi}.$$

(B) We write

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\sum_{n=-\infty}^{-1}\frac{(-1)^n}{a^2+n^2}+(\text{the term for } n=0)+\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}.$$

Changing n to $-n$ in the first sum on RHS, simplifying and using the result in part (A), we obtain,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{1}{a^2}+\frac{2(a\pi-\sinh a\pi)}{2a^2\sinh a\pi}.$$

(C) We have to put $x=\pi$ in the above Fourier series. Observe that we can not directly put $x=\pi$ in the above Fourier series, due to convergence problem. Accordingly, we define $g:[-\pi,\pi]\to\mathbb{R}$ by,

$$g(-\pi)=g(\pi)=\frac{1}{2}\left[f(-\pi^+)+f(\pi^-)\right]=\frac{e^{-a\pi}+e^{a\pi}}{2}=\cosh(a\pi),\ g(x)=f(x),x\in(-\pi,\pi).$$

Clearly g is a piece-wise C^1 function on $[-\pi,\pi]$ and $g(-\pi)=g(\pi)$. Thus the Fourier series of g converges to g in $[-\pi,\pi]$,

$$g(x)=\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}[a\cos(nx)-n\sin(nx)]\right\}.$$

Now substituting $x=\pi$, we obtain,

$$g(\pi)=\cosh(a\pi)=\frac{\sinh(\pi a)}{\pi}\left\{\frac{1}{a}+2a\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos(n\pi)\right\}.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}=\frac{a\pi\cosh(a\pi)-\sinh(\pi a)}{2a^2\sinh(a\pi)}.$$

(D) Similar to part(B), from part (C), we obtain the result for part (D) as,

$$\sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2}=\frac{1}{a^2}+\frac{a\pi\cosh(a\pi)-\sinh(\pi a)}{a^2\sinh(a\pi)}.$$

7. Consider $f(x)=\sqrt{1-\cos x},\ 0<x<2\pi$.

(A) Determine Fourier series expansion of f in $(0, 2\pi)$.

(B) Does the limit of Fourier series exist at $x = 0$?

(C) Use part (B) to find the series

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots.$$

Solution:

(A) We have $f(x) = \sqrt{2}\sin\frac{x}{2}$. Then the Fourier series expansion is given by,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\},\ x \in (0, 2\pi),$$

with $L = \pi$ and

$$\begin{aligned} a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx,\ n = 0, 1, 2, \cdots \\ b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx,\ n = 1, 2, \cdots. \end{aligned}$$

We now calculate each of these coefficients to be,

$$\begin{aligned} a_0 &= \frac{4\sqrt{2}}{\pi} \\ a_n &= -\frac{4\sqrt{2}}{\pi(4n^2-1)},\ n = 1, 2, \cdots, \\ b_n &= 0,\ n = 1, 2, \cdots. \end{aligned}$$

Therefore the desired Fourier series is given by,

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)}\cos nx.$$

(B) For this part, we define $g : [0, 2\pi] \to \mathbb{R}$ by

$$g(0) = g(2\pi) = \frac{1}{2}\left[f(0^+) + f(2\pi^-)\right] = 0,\ g(x) = f(x),\ x \in (0, 2\pi).$$

Clearly g is a piece-wise C^1 function in $[0, 2\pi]$ and $g(0) = g(2\pi)$. Thus Fourier series of g converges to g in $[0, 2\pi]$. Observe that Fourier series of g is given by

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)}\cos nx.$$

Thus, the Fourier series of f in part (A) converges to $g(0) = 0$.

(C) Using part (B), we obtain,

$$\frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(4n^2-1)}\cos 0 = 0,$$

which gives,

$$\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n-1)} = \frac{2\sqrt{2}}{\pi}.$$