CL202: Fluid Mechanics

Fluid Statics

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What has been covered till now?

Let us list down

- Introductory concepts
- Dimensional analysis
  - Buckingham Pi theorem
  - Classification of flows
    - Mach number
    - Boundary layer
- Similitude/ Similarity/ Scale-up

Fluid Statics: Motivation

- Also known as ‘Hydrostatics’ for incompressible fluids
- Statics: No motion
  - No momentum change
  - No shear stress are present
- Pressure distribution in the atmosphere and ocean
- Force on flat and curved surfaces: vessel / tank walls, dam gates
- Buoyancy
- Hydraulic actuation
- Design of manometers, mechanical and electronic pressure instruments

Pressure

- Thermodynamic property of fluid
- Defined as force per unit area
    - Force caused by fluid molecules bombarding the surface
- A scalar quantity
    - Pressure force acting on a surface is a vector
    - The direction of pressure force is determined by the surface on which it acts
- Always acts normal (perpendicular) to a surface
- Pressure is compressive i.e. pushes a surface (not pulls)

Pressure

- Pressure is isotropic i.e. acts equally is all directions
  - Has a single value at any point in a fluid
- Units:
  - kg/m^2; Pascal (Pa), Bar, psi.
- Gauge (or Gage) pressure: Pressure relative to the local ambient pressure (P_{atm}) when $P > P_{atm}$

$$P(gauge) = P_{abs} - P_{atm}$$

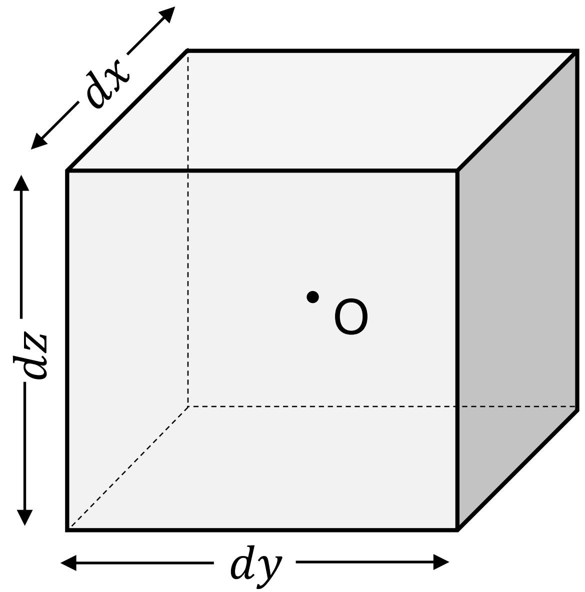
- Vacuum pressure: Pressure relative to the local ambient pressure (P_{atm}) when $P < P_{atm}$

$$P(vacuum) = P_{atm} - P_{abs}$$

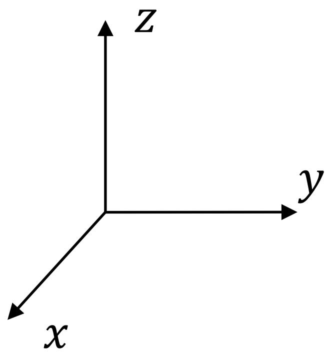
Pressure distribution in a static fluid

- We experience that the pressure increases with depth
- Let us derive an expression for pressure distribution in a fluid
- Consider a cuboid fluid element of mass dm
- Apply Newton's second law of motion on the fluid element

$$\boldsymbol{F} = \frac{d(m\boldsymbol{V})}{dt} = m\frac{d\boldsymbol{V}}{dt} \text{ (for constant } m\text{)}$$



- For a stationary fluid ($\frac{d\boldsymbol{V}}{dt}$), $\boldsymbol{F} = 0$
- Force can be body and surface force
- Body force is caused by gravity



$$\boldsymbol{dF}_{Body} = dm\,\boldsymbol{g} = \rho\, d\cancel{V}\,\boldsymbol{g} = \rho\, dxdydz\,\boldsymbol{g}$$

Pressure distribution in a static fluid

- Surface forces can be caused by pressure and viscous stresses
- Viscous stresses are zero- stationary fluid
- We need to find pressure force on each face
- Recall Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

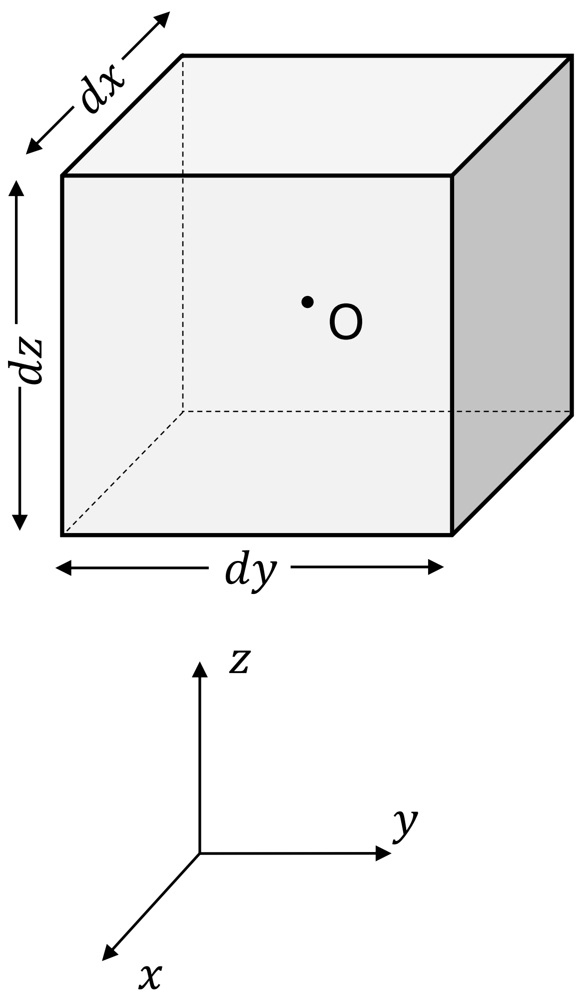
- Pressure on the right face on neglecting high order terms

$$p_{Right} = p + \frac{\partial p}{\partial y}\frac{dy}{2}$$

- Similarly, on the left face:

$$p_{Left} = p - \frac{\partial p}{\partial y}\frac{dy}{2}$$

Pressure at O: p



Pressure distribution in a static fluid

- Surface or pressure force along the *y*-direction

$$d\boldsymbol{F}_{P,y} = -\left(\cancel{p} + \frac{\partial p}{\partial y}\frac{dy}{2}\right)dxdz\,\hat{\boldsymbol{j}} + \left(\cancel{p} - \frac{\partial p}{\partial y}\frac{dy}{2}\right)dxdz\,\hat{\boldsymbol{j}}$$

$$dF_{P,y} = -\frac{\partial p}{\partial y}dydxdz\,\hat{\boldsymbol{j}}$$

- Similarly, we can obtain:

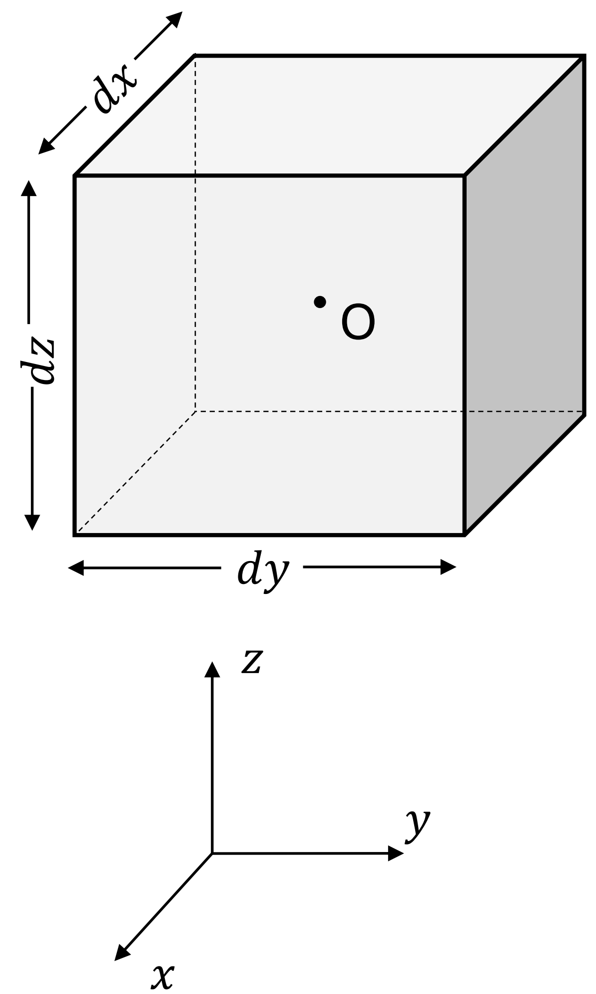
$$dF_{P,x} = -\frac{\partial p}{\partial x}dydxdz\,\hat{\boldsymbol{\imath}}$$

$$dF_{P,z} = -\frac{\partial p}{\partial z}dydxdz\,\widehat{\boldsymbol{k}}$$

- Total surface or pressure force will be

$$d\boldsymbol{F}_{Surface} = -\left(\frac{\partial p}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial y}\,\hat{\boldsymbol{j}} + \frac{\partial p}{\partial z}\,\widehat{\boldsymbol{k}}\right)dxdydz$$

Pressure at O is p



Pressure distribution in a static fluid

$$d\boldsymbol{F}_{Surface} = -\left(\frac{\partial p}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial p}{\partial z}\,\widehat{\boldsymbol{k}}\right) dxdydz$$

- Gradient operator

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\,\widehat{\boldsymbol{k}}\right)$$

- Pressure gradient

$$\boldsymbol{\nabla} p = \left(\frac{\partial p}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial p}{\partial z}\,\widehat{\boldsymbol{k}}\right)$$

Pressure gradient, not pressure, is important in calculation of surface force

Therefore:

$$d\boldsymbol{F}_{Surface} = -\boldsymbol{\nabla} p\; dxdydz$$

- Total force

$$d\boldsymbol{F} = d\boldsymbol{F}_{Surface} + \boldsymbol{dF}_{Body}$$

$$\mathrm{d}\boldsymbol{F} = -\boldsymbol{\nabla} p\; dxdydz + \rho\; dxdydz\; \boldsymbol{g}$$

$$\mathrm{d}\boldsymbol{F} = (-\boldsymbol{\nabla} p + \rho \boldsymbol{g})\; dxdydz = (-\boldsymbol{\nabla} p + \rho \boldsymbol{g})\; d\forall$$

Pressure distribution in a static fluid

$$\mathrm{d}\boldsymbol{F} = (-\boldsymbol{\nabla} p + \rho \boldsymbol{g})\, d\cancel{V}$$

Using Newton's second law $\mathrm{d}\boldsymbol{F} = (-\boldsymbol{\nabla} p + \rho \boldsymbol{g})\, dV = 0$

$$-\boldsymbol{\nabla} p \qquad + \qquad \rho \boldsymbol{g} = 0$$

Net pressure force per unit volume   Net body force per unit volume

- Pressure gradient, not pressure, is important in calculation of surface force
- The equations can also be written in terms of components in each direction

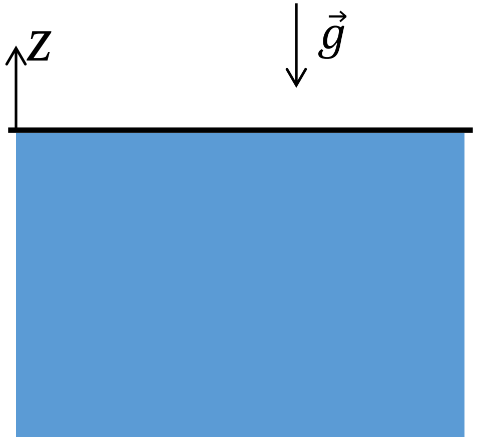
$$\boxed{\frac{\partial p}{\partial x} = \rho g_x} \qquad \boxed{\frac{\partial p}{\partial y} = \rho g_y} \qquad \boxed{\frac{\partial p}{\partial z} = \rho g_z}$$

Pressure distribution in a static fluid

If the z axis is chosen such that positive z axis points vertically upward,

$$\vec{g} = -g\,\hat{\boldsymbol{k}}$$

because gravity acts vertically downwards.



We have,

$$g_x = 0 \qquad g_y = 0 \qquad g_z = -g$$

So, $$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\rho g$$

As pressure is a function of z only in this case, we have

$$\boxed{\frac{dp}{dz} = -\rho g}$$

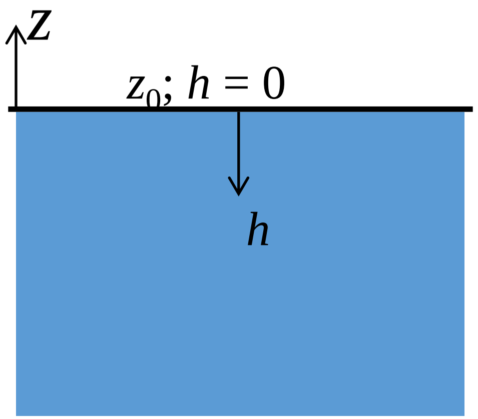
Pressure distribution in a static fluid

$$\boxed{\frac{dp}{dz} = -\rho g}$$

In general, both ρ and g can vary with z.

However, unless the change in elevation or depth is very large, the change in g can be neglected.

Further, if we are considering an **incompressible liquid**, ρ can also be taken as a constant.



As the z axis is pointing vertically upward, ***depth increases in the negative*** *z* ***direction***.

Let h be the depth of liquid from its surface and let the surface of the liquid be at $z = z_0$.

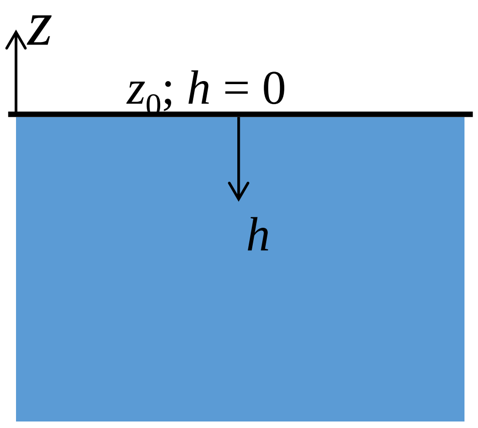
We have, $h = 0$ at the liquid surface and $h = z_0 - z$, as h is measured positive downward.

Pressure distribution in a static fluid

We have, $h = 0$ at the liquid surface and $h = z_0 - z$, as h is measured positive downward.

Using the basic equation of fluid statics, we have

$$\frac{dp}{dz} = \rho g_z = -\rho g$$



$$\Longrightarrow \quad p - p_0 = -\rho g(z - z_0) = \rho g h$$

where p_0 is the pressure at the liquid surface (i.e., $z = z_0$ or $h = 0$).

So,

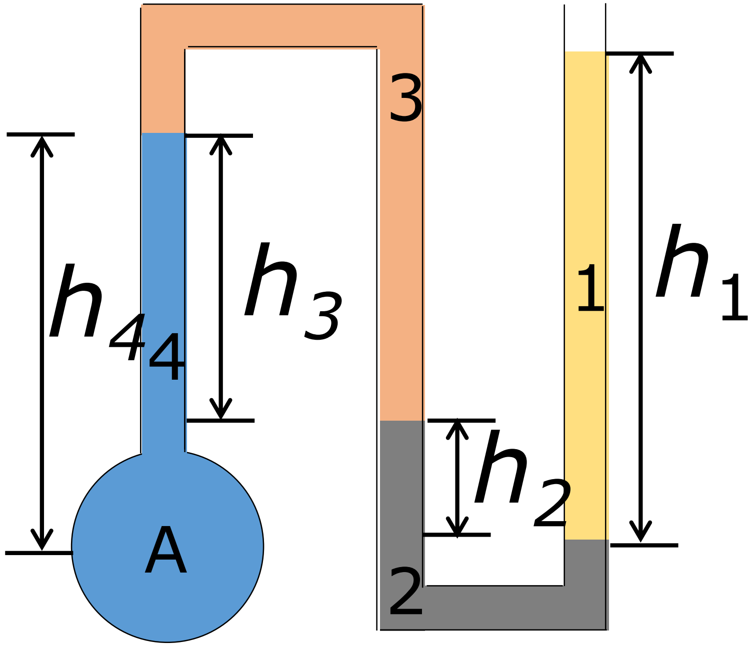
$$\boxed{p = p_0 + \rho g h}$$

Manometer

- A simple and inexpensive device to measure pressure
- Note that:
  - Two points at the same elevation in a continuous region of the same liquid are at the same pressure
  - Pressure increases with increasing depth

Manometer: Example

A manometer has four different fluids as shown in the Figure. Fluid 1 is open to atmosphere. Find the pressure at point A.



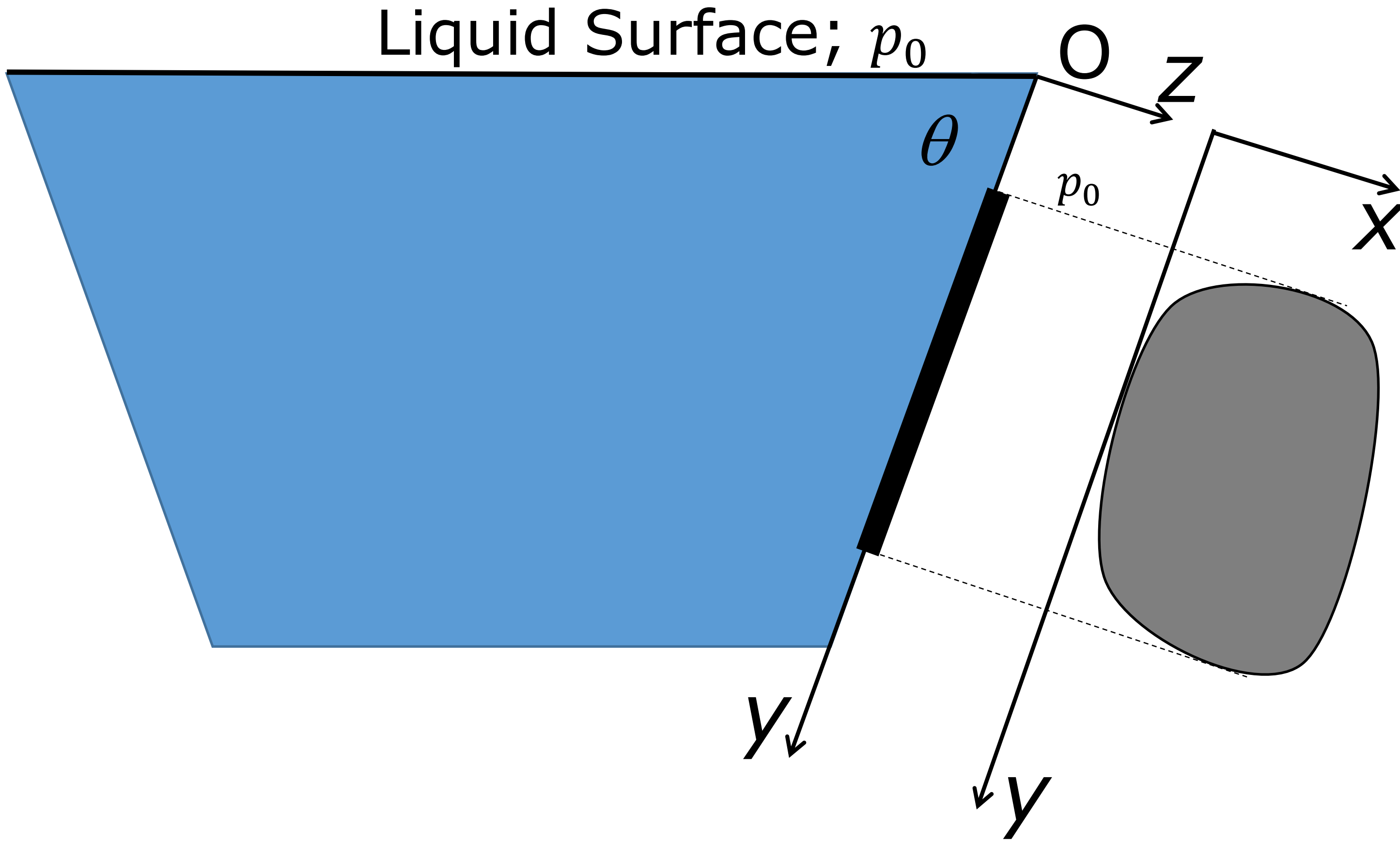
$$p_A = p_{atm} + \rho_1 g h_1 - \rho_2 g h_2 - \rho_3 g h_3 + \rho_4 g h_4$$

Hydrostatic Force on a Plane Submerged Surface

- Consider a plane submerged surface

- Co-ordinates are chosen such that the surface lies in *xy* plane

- We would obtain:
  - The magnitude and direction of pressure force on the surface
  - The line of action of force



Hydrostatic Force on a Plane Submerged Surface

Consider an element of surface of area

dA at a distance y from O

Pressure force on the surface element

caused by the liquid

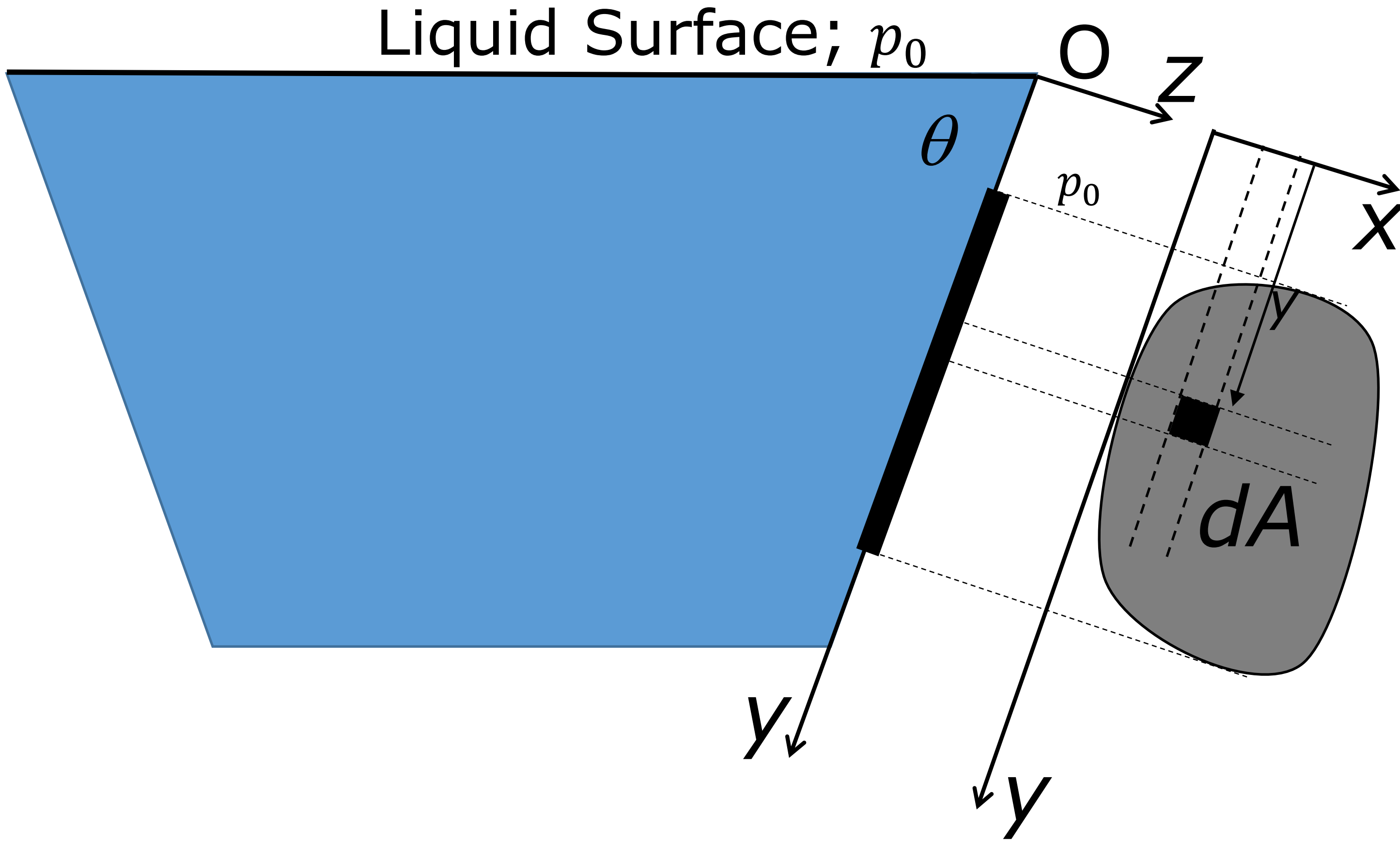
$$dF = p\, dA$$

Net pressure force on the surface due to liquid

$$\boldsymbol{F}_R = \int_{Area\ of\ plane} p\, dA$$

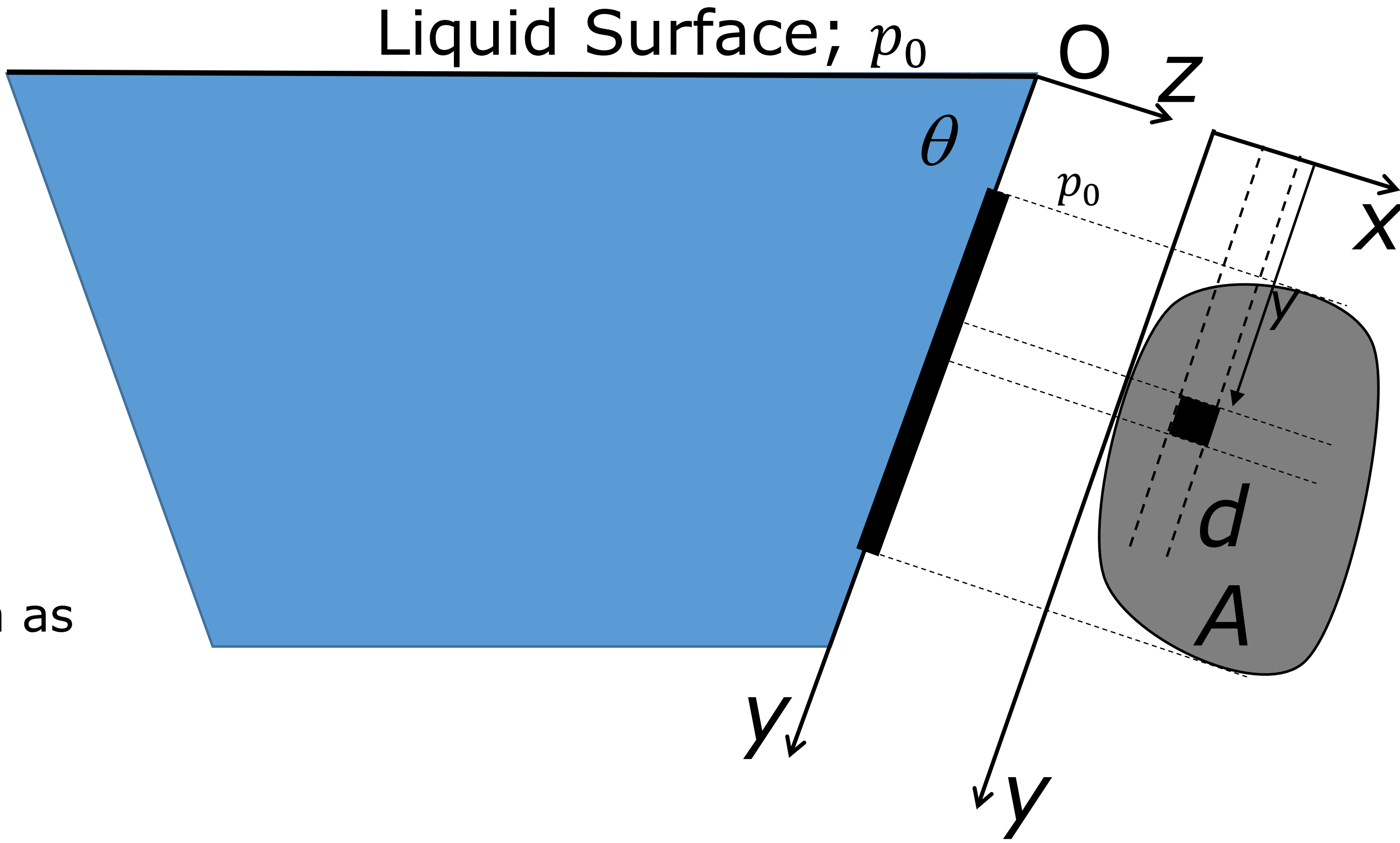
From hydrostatics,

$$p = p_0 + \rho g y\, sin\theta$$



$$\boldsymbol{F}_R = \int_{Area\ of\ plane} (p_0 + \rho g y\, sin\theta) dA$$

Hydrostatic Force on a Plane Submerged Surface



$$\boldsymbol{F}_R = \int_{Area\ of\ plane} (p_0 + \rho g y\ sin\theta) dA$$

$$\boldsymbol{F}_R = p_0 A + \rho g\ sin\theta \int_{Area\ of\ plane} y\ dA$$

What is $\int_{Area\ of\ plane} y\ dA$?

We know that centroid of an area is given as

$$x_C = \frac{\int_{Area} x\ dA}{\int_{Area} dA};\ y_C = \frac{\int_{Area} y\ dA}{\int_{Area} dA}$$

$$\boldsymbol{F}_R = p_0 A + \rho g y_C sin\theta\ A$$

$$\boldsymbol{F}_R = (p_0 + \rho g y_C sin\theta) A$$

$$\boxed{\boldsymbol{F}_R = (p_0 + \rho g h_C) A}$$

where $h_C = y_C sin\theta$ i.e. depth of centroid

To find net force on the surface, the force on other side should also be accounted for

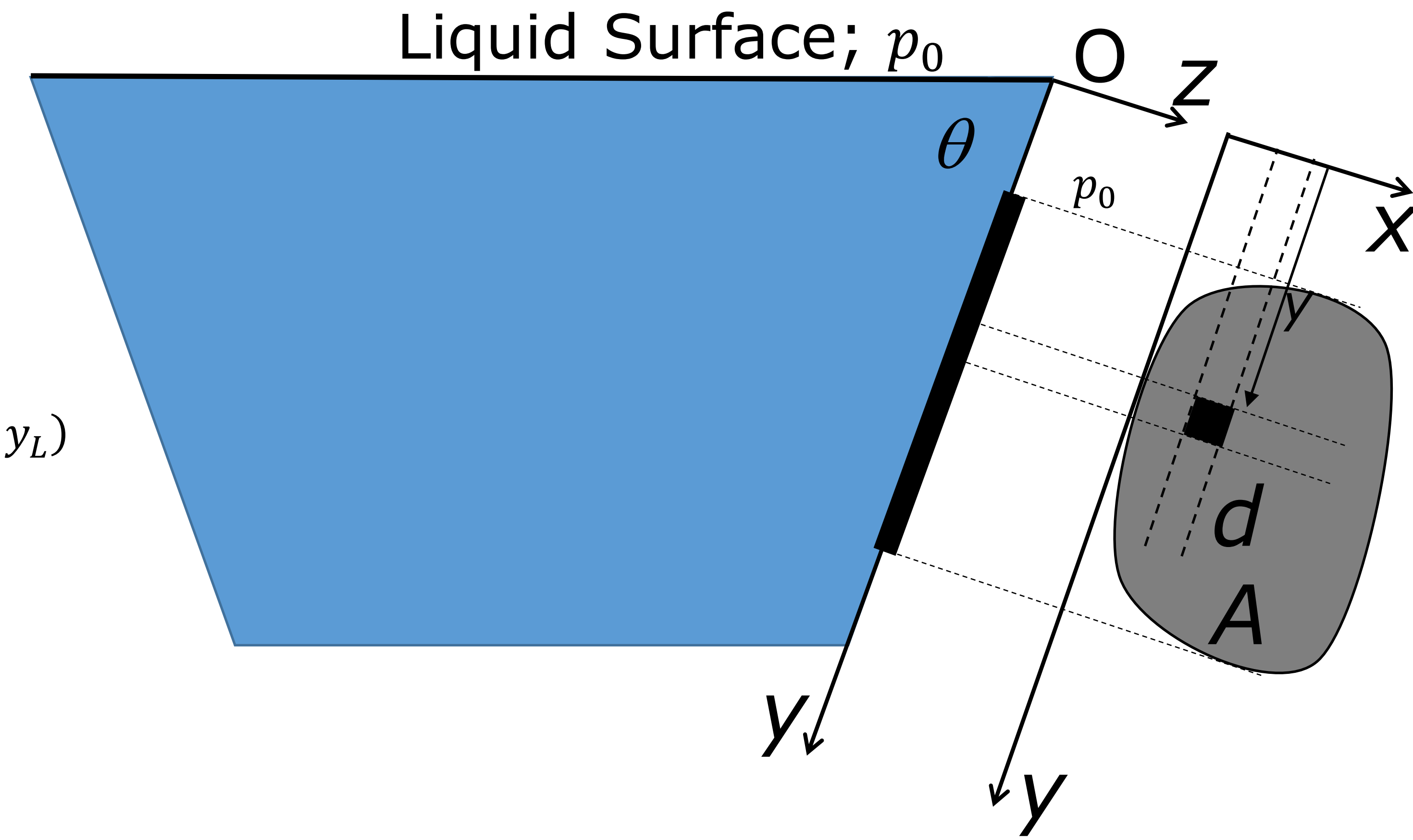
$$\boldsymbol{F}_{net} = (p_0 + \rho g h_C) A - p_0 A$$

$$\boldsymbol{F}_{net} = \rho g h_C A$$

Hydrostatic Force on a Plane Submerged Surface

- Next, we need to find line of action of $\boldsymbol{F}_R$
- Remember that the force does not necessarily act at the centroid
  - Variation of force with *y*
- Let us say the resultant force acts at (x_L, y_L)

Moment of $\boldsymbol{F}_R$ about *x*-axis = moment of distributed force about *x*-axis



$$y_L \boldsymbol{F}_R = \int_{Area\ of\ plane} y\ p\ dA$$

$$y_L = \frac{\int_{Area\ of\ plane} y(p_0 + \rho g y sin\theta)\ dA}{\boldsymbol{F}_R}$$

$$y_L = \frac{p_0 \int_{Area\ of\ plane} y\ dA + \rho g sin\theta \int_{Area\ of\ plane} (y^2)\ dA}{\boldsymbol{F}_R}$$

Hydrostatic Force on a Plane Submerged Surface

Second moment of area:

About x-axis: $I_{xx} = \int_A y^2 \, dA$

Product moment of area

$$I_{xy} = \int_A xy \, dA$$

Parallel-axis theorem:

$$I_{xx} = I_{xx,centroid} + A\, y_c^2$$

$$I_{xy} = I_{xy,centroid} + A x_C y_C$$

Hydrostatic Force on a Plane Submerged Surface

$$y_L = \frac{p_0 \int_{Area\ of\ plane} y\ dA + \rho g sin\theta \int_{Area\ of\ plane} (y^2)\ dA}{\boldsymbol{F}_R}$$

$$\boxed{\boldsymbol{F}_R = (p_0 + \rho g h_C)A}$$

$$y_L = \frac{p_0 y_C A + \rho g sin\theta\ I_{xx}}{\boldsymbol{F}_R} = \frac{p_0 y_C A + \rho g sin\theta\ \left(I_{xx,centroid} + A\ y_c^2\right)}{(p_0 + \rho g y_C sin\theta)A}$$

$$y_L = \frac{p_0 y_C A + \rho g sin\theta A\ y_c^2 + \rho g sin\theta I_{xx,centroid}}{(p_0 + \rho g y_C sin\theta)A}$$

$$\boxed{y_L = y_C + \frac{\rho g sin\theta\ I_{xx,centroid}}{\boldsymbol{F}_R}}$$

To find x_L:

Moment of $\boldsymbol{F}_R$ about y-axis = Moment of distributed force about y-axis

$$x_L \boldsymbol{F}_R = \int_{Area\ of\ plane} x\, p\, dA$$

$$x_L \boldsymbol{F}_R = \int_{Area\ of\ plane} x(p_0 + \rho g y sin\theta)\, dA$$

$$= p_0 \int_A x\, dA + \rho g\, sin\theta \int_A xy\ \ dA$$

$$= p_0 x_C A + \rho g\, sin\theta I_{xy}$$

$$= p_0 x_C A + \rho g\, sin\theta \left(I_{xy,centroid} + A x_C y_C\right)$$

$$\boxed{x_L = x_C + \frac{\rho g\, sin\theta I_{xy,centroid}}{\boldsymbol{F}_R}}$$