Chapter 1
Notions of Mathematical Finance

The present notes deal with topics of computational finance, with focus on the analysis and implementation of numerical schemes for pricing derivative contracts. There are two broad groups of numerical schemes for pricing: stochastic (Monte Carlo) type methods and deterministic methods based on the numerical solution of the Fokker–Planck (or Kolmogorov) partial integro-differential equations for the price process. Here, we focus on the latter class of methods and address finite difference and finite element methods for the most basic types of contracts for a number of stochastic models for the log returns of risky assets. We cover both, models with (almost surely) continuous sample paths as well as models which are based on price processes with jumps. Even though emphasis will be placed on the (partial integro)differential equation approach, some background information on the market models and on the derivation of these models will be useful particularly for readers with a background in numerical analysis.

Accordingly, we collect synoptically terminology, definitions and facts about models in finance. We emphasise that this is a *collection* of terms, and it can, of course, in no sense claim to be even a short survey over mathematical modelling in finance. Readers who wish to obtain a perspective on mathematical modelling principles for finance are referred to the monographs of Mao [120], Øksendal [131], Gihman and Skorohod [71–73], Lamberton and Lapeyre [109], Shiryaev [152], as well as Jacod and Shiryaev [97].

1.1 Financial Modelling

Stocks Stocks are shares in a company which provide partial ownership in the company, proportional with the investment in the company. They are issued by a company to raise funds. Their value reflects both the company's real assets as well as the estimated or imagined company's earning power. *Stock* is the generic term for assets held in the form of shares. For publicly quoted companies, stocks are quoted and traded on a stock exchange. An *index* tracks the value of a *basket of stocks*.

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Assets for which future prices are not known with certainty are called *risky assets*, while assets for which the future prices are known are called *risk free*.

Price Process The price at which a stock can be bought or sold at any given time t on a stock exchange is called *spot price* and we shall denote it by S_t. All possible future prices S_t as functions of t (together with probabilistic information on the likelihood of a particular price history) constitute the *price process* $S = \{S_t : t \geq 0\}$ of the asset. It is mathematically modelled by a *stochastic process* to be defined below.

Derivative Securities A *derivative security*, derivative for short, is a security whose value depends on the value of one or several *underlying assets* and the decisions of the investor. It is also called *contingent claim*. It is a financial contract whose value at *expiration time* (or *time of maturity*) T is determined by the price process of the underlying assets up to time T. After choosing a price process for the asset(s) under consideration, the task is to determine a price for the derivative security on the asset. There are several types of derivatives: options, forwards, futures and swaps. We focus exemplary on the pricing of options, since pricing other assets leads to closely related problems.

Options An *option* is a derivative which gives its holder the *right, but not the obligation* to make a specified transaction at or by a specified date at a specified price. Options are sold by one party, the *writer* of the option, to another, the *holder*, of the option. If the holder chooses to make the transaction, he *exercises* the option. There are many conditions under which an option can be exercised, giving rise to different types of options. We list the main ones: *Call options* give the right (but not the obligation) to buy, *put options* give the right (but not an obligation) to sell the underlying at a specified price, the so-called *strike price* K. The simplest options are the *European call and put* options. They give the holder the right to buy (resp., sell) exactly at *maturity* T. Since they are described by very simple rules, they are also called *plain vanilla* options. Options with more sophisticated rules than those for plain vanillas are called *exotic options*. A particular type of exotic options are *American options* which give the holder the right (but not the obligation) to buy (resp., sell) the underlying at any time t at or before maturity T. For European options the price does not depend on the path of the underlying, but only on the realisation at maturity T. There are also so-called *path dependent* options, like Asian, lookback or barrier contracts. The value of *Asian options* depends on the average price of the option's underlying over a period, *lookback options* depend on the maximum or minimum asset price over a period, and *barrier options* depend on particular price level(s) being attained over a period.

Payoff The *payoff* of an option is its value at the time of exercise T. For a European call with strike price K, the payoff g is

$$g(S_T) = (S_T - K)_+ = \begin{cases} S_T - K & \text{if } S_T > K, \\ 0 & \text{else.} \end{cases}$$

At time $t \leq T$ the option is said to be *in the money*, if $S_t > K$, the option is *out of the money*, if $S_t < K$, and the option is said to be *at the money*, if $S_t \approx K$.

Modelling Assumptions Most market models for stocks assume the existence of a *riskless bank account* with *riskless interest rate* $r \geq 0$. We will also consider stochastic interest rate models where this is not the case. However, unless explicitly stated otherwise, we assume that money can be deposited and borrowed from this bank account with continuously compounded, known interest rate r. Therefore, 1 currency unit in this account at $t = 0$ will give e^{rt} currency units at time t, and if 1 currency unit is borrowed at time $t = 0$, we will have to pay back e^{rt} currency units at time t. We also assume a *frictionless market*, i.e. there are no transaction costs, and we assume further that there is no default risk, all market participants are rational, and the market is efficient, i.e. there is no arbitrage.

1.2 Stochastic Processes

We refer to the texts Mao [120] and Øksendal [131] for an introduction to stochastic processes and stochastic differential equations. Much more general stochastic processes in the Markovian and non-Markovian setup are treated in the monographs Gihman and Skorohod [71–73] as well as Jacod and Shiryaev [97].

Prices of the so-called risky assets can be modelled by stochastic processes in continuous time $t \in [0, T]$ where the maturity $T > 0$ is the *time horizon*. To describe stochastic price processes, we require a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, Ω is the set of elementary events, $\mathcal{F}$ is a σ-algebra which contains all events (i.e. subsets of Ω) of interest and $\mathbb{P} : \mathcal{F} \to [0, 1]$ assigns a probability of any event $A \in \mathcal{F}$.

We shall always assume the probability space to be complete, i.e. if $B \subset A$ with $A \in \mathcal{F}$ and $\mathbb{P}[A] = 0$, then $B \in \mathcal{F}$. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with a *filtration*, i.e. a family $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ of σ-algebras which are monotonic with respect to t in the sense that for $0 \leq s \leq t \leq T$ holds that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T \subset \mathcal{F}$. In financial modelling, the σ-algebra $\mathcal{F}_t \in \mathbb{F}$ represents the information available in the model up to time t. We assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ satisfies the *usual assumptions*, i.e.

(i) $\mathcal{F}$ is $\mathbb{P}$-complete,
(ii) $\mathcal{F}_0$ contains all $\mathbb{P}$-null subsets of Ω and
(iii) The filtration $\mathbb{F}$ is right-continuous: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

Definition 1.2.1 (Stochastic processes) A *stochastic process* $X = \{X_t : 0 \leq t \leq T\}$ is a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, parametrised by the time variable t. For $\omega \in \Omega$, the function $X_t(\omega)$ of t is called *a sample path* of X. The process is $\mathbb{F}$-*adapted* if X_t is $\mathcal{F}_t$ measurable (denoted by $X_t \in \mathcal{F}_t$) for each t.

To model asset prices by stochastic processes, knowledge about past events up to time t should be incorporated into the model. This is done by the concept of *filtration*.

Definition 1.2.2 (Natural filtration) We call $\mathbb{F}^X = \{\mathcal{F}_t^X : 0 \le t \le T\}$ the *natural filtration for* X if it is the completion with respect to $\mathbb{P}$ of the filtration $\widetilde{\mathbb{F}}^X = \{\widetilde{\mathcal{F}}_t^X : 0 \le t \le T\}$, where for each $0 \le t \le T$, $\widetilde{\mathcal{F}}_t^X = \sigma(X_r : r \le s)$.

A stochastic process is called *càdlàg* (from French 'continue à droite avec des limités à gauche') if it has càdlàg sample paths, and a mapping $f : [0, T] \to \mathbb{R}$ is said to be càdlàg if for all $t \in [0, T]$ it has a left limit at t and is right-continuous at t. A stochastic process is called *predictable* if it is measurable with respect to the σ-algebra $\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is the smallest σ-algebra generated by all adapted càdlàg processes on $[0, T] \times \Omega$.

Asset prices are often modelled by *Markov processes*. In this class of stochastic processes, the stochastic behaviour of X after time t depends on the past only through the current state X_t.

Definition 1.2.3 (Markov property) A stochastic process $X = \{X_t : 0 \le t \le T\}$ is *Markov* with respect to $\mathbb{F}$ if

$$\mathbb{E}[f(X_s)|\mathcal{F}_t] = \mathbb{E}[f(X_s)|X_t],$$

for any bounded Borel function f and $s \ge t$.

No arbitrage considerations require discounted log price processes to be martingales, i.e. the best prediction of X_s based on the information at time t contained in $\mathcal{F}_t$ is the value X_t. In particular, the expected value of a martingale at any finite time T based on the information at time 0 equals the initial value X_0, $\mathbb{E}[X_T|\mathcal{F}_0] = X_0$.

Definition 1.2.4 (Martingale) A stochastic process $X = \{X_t : 0 \le t \le T\}$ is a *martingale* with respect to $(\mathbb{P}, \mathbb{F})$ if

(i) X is $\mathbb{F}$ adapted,
(ii) $\mathbb{E}[|X_t|] < \infty$ for all $t \ge 0$,
(iii) $\mathbb{E}[X_s|\mathcal{F}_t] = X_t$ $\mathbb{P}$-a.s. for $s \ge t \ge 0$.

There is a one-to-one correspondence between models that satisfy the *no free lunch with vanishing risk* condition and the existence of a so-called *equivalent local martingale measure* (ELMM). We refer to [54, 55] for details. The most widely used price process is a *Brownian motion* or *Wiener process*. Its use in modelling log returns in prices of risky assets goes back to Bachelier [4]. Recall that the *normal distribution* $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu \in \mathbb{R}$ and variance σ^2 with $\sigma > 0$ has the density

$$f_{\mathcal{N}}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)},$$

and it is symmetric around μ. Normality assumptions in models of log returns of risky assets' prices imply the assumption that upward and downward moves of prices occur symmetrically.

Definition 1.2.5 (Wiener process) A stochastic process $X = \{X_t : t \geq 0\}$ is a *Wiener process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if (i) $X_0 = 0$ $\mathbb{P}$-a.s., (ii) X has independent increments, i.e. for $s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$, (iii) $X_{t+s} - X_t$ is normally distributed with mean 0 and variance $s > 0$, i.e. $X_{t+s} - X_t \sim \mathcal{N}(0, s)$, and (iv) X has $\mathbb{P}$-a.s. continuous sample paths. We shall denote this process by W for N. Wiener.

In the Black–Scholes stock price model, the price process S of the risky asset is modelled by assuming that the return due to price change in the time interval $\Delta t > 0$ is

$$\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S_t}{S_t} = r\Delta t + \sigma \Delta W_t,$$

in the limit $\Delta t \to 0$, i.e. that it consists of a deterministic part $r\Delta t$ and a random part $\sigma(W_{t+\Delta t} - W_t)$. In the limit $\Delta t \to 0$, we obtain the stochastic differential equation (SDE)

$$\mathrm{d}S_t = r S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \quad S_0 > 0. \tag{1.1}$$

The above SDE admits the unique solution

$$S_t = S_0 e^{(r-\sigma^2/2)t + \sigma W_t}.$$

This exponential of a Brownian motion is called the *geometric Brownian motion*. The stochastic differential equation (1.1) for the geometric Brownian motion is a special case of the more general SDE

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \quad X_0 = Z, \tag{1.2}$$

for which we give an existence and uniqueness result.

Theorem 1.2.6 *We consider a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with filtration* $\mathbb{F}$ *and a Brownian motion* W *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *adapted to* $\mathbb{F}$. *Assume there exists* $C > 0$ *such that* $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *in* (1.2) *satisfy*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x - y|, \quad x, y \in \mathbb{R},\ t \in \mathbb{R}_+, \tag{1.3}$$

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \quad x \in \mathbb{R},\ t \in \mathbb{R}_+. \tag{1.4}$$

Assume further $X_0 = Z$ *for a random variable which is* $\mathcal{F}_0$*-measurable and satisfies* $\mathbb{E}[|Z|^2] < \infty$. *Then, for any* $T \geq 0$, (1.2) *admits a* $\mathbb{P}$*-a.s. unique solution in* $[0, T]$ *satisfying*

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |X_t|^2\Big] < \infty. \tag{1.5}$$

We refer to [131, Theorem 5.2.1] or [120, Theorem 2.3.1] for a proof of this statement. Note that the Lipschitz continuity (1.3) implies the linear growth condition (1.4) for time-independent coefficients $\sigma(x)$ and $b(x)$. For any $t \geq 0$ one has $\int_0^t |b(s, X_s)| \,\mathrm{d}s < \infty$, $\int_0^t |\sigma(s, X_s)|^2 \,\mathrm{d}s < \infty$, $\mathbb{P}$-a.s., i.e. the solution process X is a particular case of a so-called *Itô process*. Equation (1.2) is formally the differential form of the equation

$$X_t = Z + \int_0^t b(s, X_s) \,\mathrm{d}s + \int_0^t \sigma(s, X_s) \,\mathrm{d}W_s,$$

for $t \in [0, T]$. In the derivation of pricing equations, it will become important to check under which conditions the integrals with respect to W, i.e. $\int_0^t \phi_s \,\mathrm{d}W_s$, are martingales. The notion of stochastic integrals is discussed in detail in [120, Sect. 1.5].

Proposition 1.2.7 *Let the process ϕ be predictable and let ϕ satisfy, for $T \geq 0$,*

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 \,\mathrm{d}t\right] < \infty. \tag{1.6}$$

Then, the process $M = \{M_t : t \geq 0\}$, $M_t := \int_0^t \phi_s \,\mathrm{d}W_s$ is a martingale.

For a proof of this statement, we refer to [131, Theorem 3.2.1]. In mathematical finance, we are interested in the dynamics of $f(t, X_t)$, e.g. where $f(t, X_t)$ denotes the option price process. Here, the Itô formula plays an important role.

Theorem 1.2.8 (Itô formula) *Let X be given by the Itô process* (1.2), *and let $f(t,x) \in C^2([0,\infty) \times \mathbb{R})$, i.e. f is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$. Then, for $Y_t = f(t, X_t)$ we obtain*

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t}(t, X_t) \,\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t) \,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t) \cdot (\mathrm{d}X_t)^2, \tag{1.7}$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t) \cdot (\mathrm{d}X_t)$ is computed according to the rules

$$\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}W_t = \mathrm{d}W_t \cdot \mathrm{d}t = 0, \quad \mathrm{d}W_t \cdot \mathrm{d}W_t = \mathrm{d}t.$$

We refer to [120, Theorem 1.6.2] for a proof of the Itô formula. A sketch of the proof is given in [131, Theorem 4.1.2]. We note in passing that the smoothness requirements on the function f in Theorem 1.2.8 can be substantially weakened. We refer to [132, Sects. II.7 and II.8] and [40, Sect. 8.3] for general versions of the Itô formula for Lévy processes and semimartingales.

1.3 Further Reading

An introduction to financial modelling and option pricing can be found in Wilmott et al. [161] and the corresponding student version [162]. More details on risk-neutral pricing, absence of arbitrage and equivalent martingale measures are given in Delbaen and Schachermayer [53]. For a general introduction to stochastic differential equations, see Øksendal [131] and Mao [120], Protter [132] and the references therein.