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Abstract

The abstract of the report goes here. The abstract should state the topic(s) under investigation and the main results or conclusions. Methods or approaches should be stated if this is appropriate for the topic. The abstract should be self-contained, concise and clear. The typical length is one paragraph.

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1 Introduction

2 Black-Scholes equation

2.1 Preliminaries

A common problem in finance is pricing financial derivatives, often referred to simply as derivatives. In essence, derivatives are contracts set between parties whose value over time derives from the price of their underlying assets. A notorious family of derivatives in financial markets are "options". Options are contracts set between two parties in which the holder has the right to sell or buy, commonly referred to as exercising, an underlying asset at a pre-established price, also known as the "strike price", in the future. Options are referred to as "call options" or "put options" if the exercise position is to buy or to sell, respectively. Similarly, options are classified depending on their exercise style. In that regard, the simplest options are European options. European options give the right to exercise on the expiration date of the contract. Another well-known type of option is the American option. American options work similarly to European options, with the difference that they can be exercised at any point in time between the beginning and the expiration date of the contract.

Let's define the payoff as

$$H(S,t) = \max(S-K,0) \tag{2.1a}$$

$$H(S,t) = \max(K-S,0) \tag{2.1b}$$

where K is the strike price, $S \in [0,\infty]$ is the asset price, and $t \in [0,T]$ is the current time. Note that t is measured in years, where $t=0$ and $t=T$ denote the beginning of the contract and the expiration date, respectively, and the interval $[0,T]$ represents the lifespan of the option. While an American option's payoff is defined for all (S,t), a European option's payoff is only defined at $t=T$.

Obviously, options provide greater flexibility to holders by eliminating their exposure to negative payoffs. This is why the writers charge a premium to the holders at the time they enter the contract. The premium is often referred to as the price or value of the option, and the problem of determining this value is called option pricing. When pricing options, it is crucial to find the fair price; otherwise, the writer or holder of the option could devise a scheme in which the option will always be profitable for them. In other words, options pricing must adhere to the principle of no-arbitrage. Therefore, we assume that the writer of the option uses the premium to construct a portfolio consisting of ϕ_0 units of the asset and invests ψ_0 units of cash in a risk-free asset, such as US Treasury bills, certificates of deposit, or a bank account. Then, the writer rebalances the portfolio (ϕ_0, ψ_0) to hedge against any potential claims from the holder of the option at any future time $0 < t \leq T$. Consequently, at any time t, the writer holds a portfolio (ϕ_t, ψ_t) with a value

$$\Pi_t = \phi_t S_t + \psi_t B_t \tag{2.2}$$

Moreover, the portfolio is self-financing. In other words, the changes in portfolio depend on the changes in S_t and B_t, and the rebalancing of portfolio (ϕ_t, ψ_t)

$$d\Pi_t = \phi_t dS_t + \psi_t dB_t$$
$$S_t d\phi_t + B_t d\psi_t = 0$$

Finally, the portfolio value matches the option value

$$\Pi_t = V_t \tag{2.3}$$

at any time $0 \leq t \leq T$. Using the self-financing portfolio hedging strategy, The Black-Scholes model presents a mathematical model for the dynamics of an option's price. The model makes certain assumptions about the market. A complete list of all the assumptions can be found in [4] and [5]. In the next part, we enumerate some them. First, the asset price S_t is distributed as a log-normally,

$$S_t = S_0 \exp\left\{ \int_0^t \Big(r(s) - \frac{1}{2}\sigma(s) \Big) ds + \sqrt{t}Z \right\} \tag{2.4}$$

where the risk-free interest $r(t)$ and the price volatility $\sigma(t)$ are deterministic functions of time during the life of the option. Secondly, the bank account $B(t)$ is a deterministic function

$$dB = r(t)B(t)dt \tag{2.5}$$

Finally, the asset does not pay dividends. Additionally, we will assume that the risk-free interest rate and asset price volatility are constant during the life of the option. Later on, we will address the assumption about dividends.

By applying the Black-Scholes model to price European options, the famous Black-Scholes PDE is obtained

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0 & \text{for } t \in [0, T) \text{ and } S \in [0, \infty) \\ V(S,T) = H(S,T) & \text{for } S \in [0, \infty) \end{cases} \tag{2.6}$$

where $V(S,t)$ is a deterministic function. A derivation of (2.6) can be found in [4] and [5].

We previously mention that the one of the assumptions of the Black-Scholes model is that the underlying asset does not pay dividends. In most cases, assets such as stocks pay out dividends just a few times at year. In this case, dividends are to be modelled discretely. However, there are certain assets that pay out a proportion of the current price during and interval of time. Thus, in such cases, it is useful to model dividends as a continuous yield. [5] shows as reasonable model of the asset price with volatility $\sigma(t)$, rate of return $r(t)$, and continuous dividend yield $\delta(S,t)$ paid at instant of time dt is modeled as

$$dS = (r(t) - \delta(S,t))Sdt + \sigma(t)SdW \tag{2.7}$$

Similarly to as we did for the risk-free interest rate and the volatility of the asset price, we will assume that continuous dividends yield is as constant from now on. In [5], it is shown by applying the Black-Scholes model under (2.7) to price European options, the slightly modified version of (2.6) is obtained

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 & \text{for } t \in [0,T) \text{ and } S \in [0,\infty) \\ V(S,T) = H(S,T) & \text{for } S \in [0,\infty) \end{cases} \tag{2.8}$$

Similarly, the Black-Scholes model is applied to price American options. The work of [5] derives some important facts about the value surface of American options. Firstly, the value function is bounded from below by the payoff function:

$$V_{\mathsf{Am}}(S,t) \geq H(S,t) \qquad \text{for } t \in [0,T] \tag{2.9}$$

Moreover, the domain of $V(S,t)$ can be separated into the exercise region, where it is profitable for the holder to exercise the option,

$$\mathcal{S} := (S,t) : V(S,t) = H(S,t) \tag{2.10}$$

The continuation region, where it is preferable to continue holding the option because exercising is not profitable:

$$\mathcal{C} := (S,t) : V(S,t) > H(S,t) \tag{2.11}$$

The boundary of the continuation region, where it is most optimal for the holder to exercise the option:

$$\partial\mathcal{C} := (S,t) : S = \bar{S}(t) \tag{2.12}$$

where $\bar{S}(t)$ is the optimal exercise price. Lastly, the price dynamics of American options behaves as European options within the continuation region. Since we know $V(S,t)$ at the stopping region, we only need to solve $V(S,t)$ at continuation region and determine its boundary $\partial\mathcal{C}$ at the same time. Therefore, this is known as the free boundary problem formulation of the American option pricing problem and is equivalent to solve:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 & \text{for } (S,t) \in \mathcal{C} \\ V(S,t) = H(S,t) & \text{for } (S,t) \in \partial\mathcal{C} \end{cases} \tag{2.13}$$

at time T, the value of the option at exercise region will overlap with linear segment of the payoff function $H(S,T)$, in that case, the optimal exercise $S(T)$ will be equal to the strike price K payoff function. Therefore, we can use this information to define the terminal conditions of (2.13)

$$V(S,T) = H(S,T) \quad \bar{S}(T) = K \tag{2.14}$$

Next, we need to establish boundary conditions for the system (2.13). Generally, when pricing options, we would need two boundaries conditions for option pricing. However, the free boundary problem in (2.13) is expressed in terms on the moving boundary condition $\bar{S}(t)$. Therefore, we only need to determine one extra boundary condition. In case of American put options, the left boundary condition would be given by $\bar{S}(t)$, and the right boundary condition by $V(S,t) = 0$ for a sufficiently large S. Analogously, $\bar{S}(t)$ would be the right boundary condition for an American call option and its left boundary condition by $V(S,t) = 0$ for $S = 0$. Finally, $V(S,t)$ touches tangentially the exercise region S at $(\bar{S}(t), t)$. Since the exercise region is a linear segment, the derivative at that point is

$$\frac{\partial V}{\partial S}(\bar{S}(t), t) = 1 \tag{2.15a}$$

$$\frac{\partial V}{\partial S}(\bar{S}(t), t) = -1 \tag{2.15b}$$

which is called the smooth pasting condition in [4] and [5]. Grouping (2.13), (2.14) and (2.15) in one equation, we obtain the system.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta)S\dfrac{\partial V}{\partial S} - rV = 0 \quad \text{for } 0 \le S < \bar{S}(t) \text{ and } 0 \le t < T \\ V(S,T) = S - K \qquad \bar{S}(T) = K \\ V(0,t) = 0 \quad \dfrac{\partial V}{\partial S}(\bar{S}(t), t) = 1 \end{cases} \tag{2.16a}$$

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 & \text{for } \bar{S}(t) < S < \infty \text{ and } 0 \le t < T \\
V(S,T) = K - S \qquad \bar{S}(T) = K \\
\lim_{S\to\infty} V(S,t) = 0 \quad \dfrac{\partial V}{\partial S}(\bar{S}(t), t) = -1
\end{cases}
\tag{2.16b}
$$

2.2 Front-Fixing method

In the previous section, we presented the pricing of American options problem. By applying the Black-Scholes model, we derived the Black-Scholes PDE that describes the price dynamics in the continuation region $\mathcal{C}$ of call and put options. Moreover, we presented the moving boundary condition $\bar{S}(t)$ for this PDE. The moving boundary condition $\bar{S}(t)$ makes the Black-Scholes PDE more involved since we also need to determine this boundary as time changes. This type of problem are known as free boundary problems. The front fixing method was first introduced by [2] and is a strategy in which we define a map from the original domain to new domain where moving boundary remains constant as time changes. In this section, we explore two transformation based on the work of Nielsen and others [3], and the work of Company and others [1].

2.2.1 Inverse transformation

This method proposes the transformation

$$
x = \frac{S}{\bar{S}(t)} \tag{2.17}
$$

which maps the boundary of the continuation region ∂C defined in (2.10) to the fixed boundary

$$
\partial \mathcal{C}_x := \{(x,t) : x = 1\} \tag{2.18}
$$

which remain constant as t changes. Now, let us define the value $v(x,t)$ under this new map

$$
v(x,t) := V(S,t) \tag{2.19}
$$

which fixes the moving boundary $\bar{S}(t)$ at $x = 1$ when $S(t)$. Next, we compute the partial derivatives of V with respect of the partial derivatives of v which will allow us to rewrite the PDE in (2.16) with respect of (2.19)

$$\frac{\partial x}{\partial S} = \frac{1}{\bar{S}(t)}$$

$$\frac{\partial x}{\partial t} = -x\frac{\bar{S}'(t)}{\bar{S}(t)}$$

$$\frac{\partial V}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{\bar{S}(t)}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{\bar{S}(t)}\frac{\partial x}{\partial S}\frac{\partial^2 v}{\partial x^2} = \frac{1}{\bar{S}(t)^2}\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial V}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial v}{\partial t} - x\frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x}$$

Substituting these partial derivatives in the Black-Scholes PDE given by (2.16), we obtain the non-linear PDE

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \frac{\bar{S}'(t)}{\bar{S}(t)}\right]x\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x \in [0,1) \text{ and } t \in [0,T) \tag{2.20a}$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \frac{\bar{S}'(t)}{\bar{S}(t)}\right]x\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > 1 \text{ and } t \in (0,T] \tag{2.20b}$$

Likewise, we rewrite the terminal condition as

$$v(x,T) = \max(x\bar{S}(T) - K) = K\max(x-1,0) = 0 \tag{2.21a}$$

$$v(x,T) = \max(K - x\bar{S}(T)) = K\max(1-x,0) = 0 \tag{2.21b}$$

Note that x is always less than one for put options. Analogously, x is always greater than one for call options. Finally, we rewrite the boundary condition given by the optimal exercise price

$\bar{S}(t)$ as

$$\frac{\partial v}{\partial x}(1,t) = 1 \tag{2.22a}$$

$$\frac{\partial v}{\partial x}(1,t) = -1 \tag{2.22b}$$

and the boundary condition opposite to the optimal exercise price as

$$v(0,t) = 0 \tag{2.23}$$

for both put and call options. In summary, by groping equations (2.20), (2.21), (2.22), and (2.22), we obtain the system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\right] x \dfrac{\partial v}{\partial x} - rv = 0 & \text{for } x \in (0,1) \text{ and } t \in [0,T) \\ v(x,T) = 0 \qquad \bar{S}(T) = K & \text{for } x \in [0,1] \\ v(0,t) = 0 \quad v(1,t) = \bar{S}(t) - K \quad \dfrac{\partial v}{\partial x}(1,t) = \bar{S}(t) & \text{for } t \in [0,T) \end{cases} \tag{2.24a}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\right] x \dfrac{\partial v}{\partial x} - rv = 0 & \text{for } x > 1 \text{ and } t \in [0,T) \\ v(x,T) = 0 \qquad \bar{S}(T) = K & \text{for } x \geq 1 \\ v(1,t) = K - \bar{S}(t) \quad \lim_{x\to\infty} v(x,t) = 0 \quad \dfrac{\partial v}{\partial x}(1,t) = -\bar{S}(t) & \text{for } t \in [0,T) \end{cases} \tag{2.24b}$$

2.2.2 Log transformation

In this section, we define the transformations

$$x := \log \frac{KS}{\bar{S}(t)} \qquad v(x,t) := \frac{V(S,t)}{K} \tag{2.25}$$

which maps the boundary $\partial\mathcal{C}$ to the region

$$\partial\mathcal{C}_x := \{(x,t) : x = \log K\} \tag{2.26}$$

Similarly to the previous section, we compute the partial derivatives of $v(x,t)$,

$$\frac{\partial x}{\partial t} = -\frac{\bar{S}'(t)}{\bar{S}(t)}$$

$$\frac{\partial x}{\partial S} = \frac{1}{S}$$

$$\frac{\partial V}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial^2 S} = \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2} - \frac{K}{S^2}\frac{\partial v}{\partial x}$$

$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial t} - K\frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x}$$

Using the partial derivates, we rewrite the Black-Scholes PDE as follows

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \left((r-\delta) - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x} - \frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x < \log K \text{ and } t \in [0,T) \tag{2.27a}$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \left((r-\delta) - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x} - \frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > \log K \text{ and } t \in [0,T) \tag{2.27b}$$

with terminal condition

$$v(x,T) = \max\left(\frac{e^x}{K} - 1, 0\right) = 0 \tag{2.28a}$$

$$v(x,T) = \max\left(1 - e^x, 0\right) = 0 \tag{2.28b}$$

Next, we rewrite the boundary condition given by the optimal exercise price

$$\frac{\partial v}{\partial x}(\log K, t) = \frac{\bar{S}(t)}{K} \tag{2.29a}$$

$$\frac{\partial v}{\partial x}(\log K, t) = -\frac{\bar{S}(t)}{K} \tag{2.29b}$$

and the boundary condition opposite to the optimal exercise price

$$\lim_{x \to -\infty} v(x,t) = 0 \tag{2.30a}$$

$$\lim_{x \to \infty} v(x,t) = 0 \tag{2.30b}$$

Finally, grouping equations together, we have the system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial x^2} + \left((r-\delta) - \dfrac{\sigma^2}{2}\right)\dfrac{\partial v}{\partial x} - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\dfrac{\partial v}{\partial x} - rv = 0 \quad \text{for } x < \log K \text{ and } t \in [0,T) \\ v(x,T) = 0 \qquad \bar{S}(T) = K \\ \lim_{x\to-\infty} v(x,t) = 0 \quad \dfrac{\partial v}{\partial x}(\log K, t) = \dfrac{\bar{S}(t)}{K} \end{cases} \tag{2.31a}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial x^2} + \left((r-\delta) - \dfrac{\sigma^2}{2}\right)\dfrac{\partial v}{\partial x} - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\dfrac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > \log K \text{ and } t \in [0,T) \\ v(x,T) = 0 \qquad \bar{S}(T) = K \\ \lim_{x\to\infty} v(x,t) = 0 \quad \dfrac{\partial v}{\partial x}(\log K, t) = -\dfrac{\bar{S}(t)}{K} \end{cases} \tag{2.31b}$$

3 Finite difference schemes

3.1 Overview

In this section, we present explicit and implicit finite difference schemes for solving the non-linear PDE problem in (2.24). The explicit and implicit schemes of (2.31) are provided in appendix A.

Previously, we considered the pricing problem of American options which requires solving the free boundary problem defined in (2.16). Then, we presented the front fixing method used to rewrite the PDE problem using a fixed boundary by applying the inverse transformation presented in [3] and the log transformation presented in [1] resulting in the systems (2.24) and (2.31).

Recall that the solution $v(x, t)$ of (2.17) is defined in the continuous region

$$\mathcal{F} : \mathcal{X} \times \mathcal{T}$$

where

$$\mathcal{T} : [0, T]$$

$$\mathcal{X} : [0, \infty)$$

Now, we want to discretize $\mathcal{F}$ using a grid with resolution Δx and Δt. Let's define the bound of the grid as:

$$x_{\text{min}} := 0$$

$$x_{\text{max}} := x_\infty$$

$$t_{\text{min}} := 0$$

$$t_{\text{max}} := T$$

where x_∞ means an arbitrary large value in this context. Moreover, the grid dimension will be given by

$$M := \frac{x_{\text{max}} - x_{\text{min}}}{\Delta x} \tag{3.1}$$

$$N := \frac{t_{\text{max}} - t_{\text{min}}}{\Delta t} \tag{3.2}$$

$$x_i := x_{\text{min}} + i\Delta x \qquad \text{for } i = 0, \ldots, M \tag{3.3}$$

$$t_i := t_{\text{min}} + i\Delta t \qquad \text{for } i = 0, \ldots, N \tag{3.4}$$

Hence, the grid $\mathcal{G}$ is defined as

$$\mathcal{G} := \{(x_i, t_n) : (i, n) \in \{0, \ldots, M\} \times \{0, \ldots, N\}\} \tag{3.5}$$

Thus, solving equation (2.24) numerically is the find approximations

$$v_i^n \approx v(x_i, t_n)$$

and

$$\bar{S}^n \approx \bar{S}(t_n)$$

Since (2.24) and (2.31) are written backward in time, all information of the previous time step is contained at t_{n+1}. Therefore, as an explicit scheme, we will use backward central difference to approximate the (partial) derivates of $v(x, t)$ and $\bar{S}(t)$. Conversely, we will use forward central difference as an implicit scheme.

3.2 Explicit scheme

In the explicit scheme, we use the information available in the previous time step t_{n+1} when computing the central finite difference. Therefore, we approximate the spatial partial derivatives of $v(x, t)$ at the nodes (x_i, t_n) as

$$\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{2\Delta x} = \frac{\partial v}{\partial x} + O((\Delta x)^2) \qquad \text{for } i = 1, \ldots, M-1 \tag{3.6}$$

$$\frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{\Delta x^2} = \frac{\partial^2 v}{\partial x^2} + O(h^2) \qquad \text{for } i = 1, \ldots, M-1 \tag{3.7}$$

Analogously, we approximate the time partial derivative of $v(x,t)$ and the time derivative of $\bar{S}(t)$ as

$$\frac{v_i^{n+1} - v_{i-1}^n}{\Delta t} = \frac{\partial v}{\partial t} + O(\Delta t) \qquad \text{for } n = N-1, \ldots, 0 \tag{3.8}$$

$$\frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t} = \bar{S}'(t) + O(\Delta t) \qquad \text{for } n = N-1, \ldots, 0 \tag{3.9}$$

Using the finite difference approximations above, and approximation of the PDE in (2.24) is given by

$$\begin{aligned}\frac{v_i^{n+1} - v_i^n}{\Delta t} &+ \frac{1}{2}\sigma^2 x_i^2 \frac{v_{i-1}^{n+1} - 2v_i^{n+1} + v_{i+1}^{n+1}}{(\Delta x)^2} \\ &+ x_i\left((r-\delta) - \frac{1}{\bar{S}^{n+1}}\frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t}\right)\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{2\Delta x} - r v_i^{n+1} = 0\end{aligned}$$

for $i = 1, \ldots, M-1$ and $t = N-1, \ldots, 0$. To simplify the expression above, we introduce the terms

$$\begin{aligned}
\lambda &:= \frac{\Delta t}{(\Delta x)^2} \\
A_i &:= \frac{\lambda}{2}\sigma^2 x_i^2 - \frac{\lambda}{2}\left((r-\delta) - \frac{1}{\Delta t}\right)x_i \Delta x && \text{for } i = 1, \ldots, M-1 \\
B_i &:= 1 - \lambda\sigma^2 x_i^2 - r\Delta t && \text{for } i = 1, \ldots, M-1 \\
C_i &:= \frac{\lambda}{2}\sigma^2 x_i^2 + \frac{\lambda}{2}\left((r-\delta) - \frac{1}{\Delta t}\right)x_i \Delta x && \text{for } i = 1, \ldots, M-1 \\
D_i^{n+1} &:= \frac{x_i}{2\Delta x}\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}} && \text{for } i = 1, \ldots, M-1
\end{aligned}$$

Then, we rearrange the finite difference approximation of the PDE as

$$v_i^n - D_i^{n+1}\bar{S}^n = A_i^{n+1} v_{i-1}^{n+1} + B_i^{n+1} v_i^{n+1} + C_i^{n+1} v_i^{n+1}$$

for $i = 1, \ldots, M-1$ and $t = N-1, \ldots, 0$. Moreover, we have well-defined boundary conditions

$$v_0^n = 0 \qquad v_M^n = \bar{S}^n - K \qquad \text{for } n = N-1, \ldots, 0$$

$$v_0^n = K - \bar{S}^n \qquad v_M^n = 0 \qquad \text{for } n = N-1, \ldots, 0$$

and terminal condition

$$v_i^N = 0 \qquad \text{for } i = 0, \ldots, M$$

Moreover, using the contact point condition,

$$v_{M-1}^n = v_M^n - \Delta x \bar{S}^n = (1 - \Delta x)\bar{S}^n - K \qquad \text{for } n = N-1, \ldots, 0 \tag{3.12a}$$

$$v_1^n = v_0^n - \Delta x \bar{S}^n = K - (1 + \Delta x)\bar{S}^n \qquad \text{for } n = N-1, \ldots, 0 \tag{3.12b}$$

we obtain an explicit expression for $\bar{S}^n$

$$\bar{S}^n = \frac{K + A_{M-2}v_{M-2}^{n+1} + B_{M-1}v_{M-1}^{n+1} + C_{M-1}v_M^{n+1}}{(1 - \Delta x) - D_{M-1}^{n+1}} \qquad \text{for } n = N-1, \ldots, 0 \tag{3.13a}$$

$$\bar{S}^n = \frac{K - (A_1 v_0^{n+1} + B_1 v_1^{n+1} + C_1 v_2^{n+1})}{D_1^{n+1} + (1 + \Delta x)} \qquad \text{for } n = N-1, \ldots, 0 \tag{3.13b}$$

Thus, the non-linear PDE problem (2.24) can be rewritten as:

$$\begin{cases} v_i^n - D_i^{n+1}\bar{S}^n = A_i v_{i-1}^{n+1} + B_i v_i^{n+1} + C_i v_{i+1}^{n+1} & \text{for } i = 1, \ldots, M-2 \text{ and } n = N-1, \ldots, 0 \\ v_i^N = 0 \qquad \bar{S}^N = K & \text{for } i = 0, \ldots, M \\ v_0^n = 0 \quad v_{M-1}^n = (1 - \Delta x)\bar{S}^n - K \quad v_M^n = \bar{S}^n - K & \text{for } n = N-1, \ldots 0 \\ x_{\text{min}} := 0 \quad x_{\text{max}} := 1 \\ t_{\text{min}} := 0 \quad t_{\text{max}} := T \end{cases} \tag{3.14a}$$

$$
\begin{cases}
v_i^n - D_i^{n+1}\bar{S}^n = A_i v_{i-1}^{n+1} + B_i v_i^{n+1} + C_i v_{i+1}^{n+1} & \text{for } i = 2, \dots, M-1 \text{ and } n = N-1, \dots, 0 \\
v_i^N = 0 \qquad \bar{S}^N = K & \text{for i = 0, \dots, M} \\
v_0^n = K - \bar{S}^n \quad v_1^n = K - (1+\Delta x)\bar{S}^n \quad v_M^n = 0 & \text{for } n = N-1, \dots 0 \\
x_{\min} := 1 \quad x_{\max} := x_\infty \\
t_{\min} := 0 \quad t_{\max} := T
\end{cases}
\tag{3.14b}
$$

Therefore, we can formulate the following algorithm

Algorithm 1 Explicit method for call options

Ensure: $\lambda \leq 0.5$

- **for** $i = 0, \dots, M$ **do**
  - $v_i^N = 0$
- $\bar{S}^N = K$
- **for** $i = 1, \dots, M-2$ **do**
  - $A_i = \dfrac{\lambda}{2}\sigma^2 x_i^2 - \dfrac{\lambda}{2}\left((r-\delta) - \dfrac{1}{\Delta t}\right)x_i \Delta x$
  - $B_i = 1 - \lambda\sigma^2 x_i^2 - r\Delta t$
  - $C_i = \dfrac{\lambda}{2}\sigma^2 x_i^2 + \dfrac{\lambda}{2}\left((r-\delta) - \dfrac{1}{\Delta t}\right)x_i \Delta x$
- **for** $n = N-1, \dots, 0$ **do**
  - **for** $i = 1, \dots, M-1$ **do**
    - $D_i^{n+1} = \dfrac{x_i}{2\Delta x}\dfrac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}}$
  - $\bar{S}^n = \dfrac{K - (A_{M-2}v_{M-2}^{n+1} + B_{M-1}v_{M-1}^{n+1} + C_{M-1}v_M^{n+1})}{D_{M-1}^{n+1} + (1-\Delta x)}$
  - $v_0^n = 0$
  - $v_{M-1}^n = (1-\Delta x)\bar{S}^n - K$
  - $v_M^n = \bar{S}^n - K$
  - **for** $i = 1, \dots, M-2$ **do**
    - $v_i^n = A_i^{n+1}v_{i-1}^{n+1} + B_i^{n+1}v_i^{n+1} + C_i^{n+1}v_i^{n+1} + D_i^{n+1}\bar{S}^n$

Algorithm 2 Explicit method for put options

Require: $n \geq 0$

Ensure: $\lambda \leq 0.5$

for $i = 0, \ldots, M$ **do**

  $v_i^N = 0$

$\bar{S}^N = K$

for $i = 2, \ldots, M-1$ **do**

  $A_i = \dfrac{\lambda}{2}\sigma^2 x_i^2 - \dfrac{\lambda}{2}\left((r-\delta) - \dfrac{1}{\Delta t}\right)x_i \Delta x$

  $B_i = 1 - \lambda\sigma^2 x_i^2 - r\Delta t$

  $C_i = \dfrac{\lambda}{2}\sigma^2 x_i^2 + \dfrac{\lambda}{2}\left((r-\delta) - \dfrac{1}{\Delta t}\right)x_i \Delta x$

for $n = N-1, \ldots, 0$ **do**

  for $i = 1, \ldots, M-1$ **do**

    $D_i^{n+1} = \dfrac{x_i}{2\Delta x}\dfrac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}}$

  $\bar{S}^n = \dfrac{K - (A_1 v_0^{n+1} + B_1 v_1^{n+1} + C_1 v_2^{n+1})}{D_1^{n+1} + (1+\Delta x)}$

  $v_0^n = K - \bar{S}^n$

  $v_1^n = K - (1+\Delta x)\bar{S}^n$

  $v_M^n = 0$

  for $i = 2, \ldots, M-1$ **do**

    $v_i^n = A_i^{n+1} v_{i-1}^{n+1} + B_i^{n+1} v_i^{n+1} + C_i^{n+1} v_i^{n+1} + D_i^{n+1} \bar{S}^n$

3.3 Implicit scheme

Analogously to the previous section, the PDE in (2.24) is approximated using forward central difference

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{1}{2}\sigma^2 x_i^2 \frac{v_{i-1}^n - 2v_i^n + v_{i+1}^n}{(\Delta x)^2} + x_i\left((r-\delta) - \frac{1}{\bar{S}^n}\frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t}\right)\frac{v_{i+1}^n - v_{i-1}^n}{2\Delta x} - r v_i^n = 0$$

Similarly to the previous section, the terms are introduced

$$\alpha_i^n := -\frac{\lambda}{2}\sigma^2 x_i^2 + \frac{\lambda\Delta x}{2} x_i\left(r - \delta + \frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t \bar{S}^n}\right) \tag{3.15}$$

$$\beta_i^n := 1 + \lambda\sigma^2 x_i^2 + r\Delta t \tag{3.16}$$

$$\gamma_i^n := -\frac{\lambda}{2}\sigma^2 x_i^2 + \frac{\lambda\Delta x}{2} x_i\left(r - \delta + \frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t \bar{S}^n}\right) \tag{3.17}$$

to make the expression above more manageable. Next, the PDE approximation is rearranged as

$$\alpha_i^n v_{i-1}^n + \beta_i^n v_i^n + \gamma_i^n v_{i+1}^n = v_i^{n+1}$$

Contrary to the explicit method, there is not an explicit expression for $\bar{S}^n$. Therefore, the PDE problem in (2.24) is rewritten as

$$\begin{cases} \alpha_i^n v_{i-1}^n + \beta_i^n v_i^n + \gamma_i^n v_{i+1}^n = v_i^{n+1} & \text{for } i = 1, \dots, M-2 \text{ and } n = N-1, \dots, 0 \\ v_i^N = 0 \qquad \bar{S}^N = K & \text{for } i = 0, \dots, M \\ v_0^n = 0 \quad v_{M-1}^n = (1-\Delta x)\bar{S}^n - K \quad v_M^n = \bar{S}^n - K & \text{for } n = N-1, \dots 0 \\ x_{\text{min}} := 0 \quad x_{\text{max}} := 1 & \\ t_{\text{min}} := 0 \quad t_{\text{max}} := T & \end{cases} \tag{3.18a}$$

$$\begin{cases} \alpha_i^n v_{i-1}^n + \beta_i^n v_i^n + \gamma_i^n v_{i+1}^n = v_i^{n+1} & \text{for } i = 2, \dots, M-1 \text{ and } n = N-1, \dots, 0 \\ v_i^N = 0 \qquad \bar{S}^N = K & \text{for i = 0, \dots, M} \\ v_0^n = K - \bar{S}^n \quad v_1^n = K - (1+\Delta x)\bar{S}^n \quad v_M^n = 0 & \text{for } n = N-1, \dots 0 \\ x_{\text{min}} := 1 \quad x_{\text{max}} := x_\infty & \\ t_{\text{min}} := 0 \quad t_{\text{max}} := T & \end{cases} \tag{3.18b}$$

Since there is not an explicit formula for v_i^n and $\bar{S}^n$, we will have to solve a non-linear system of equation. Let's define the vector $\mathbf{v}^n \in \mathbb{R}^{M-2}$

$$\mathbf{v}^n := \begin{bmatrix} v_1^n, & v_2^n, & \cdots, & v_{M-2}^n \end{bmatrix}^\mathsf{T} \tag{3.19a}$$

$$\mathbf{v}^n := \begin{bmatrix} v_2^n, & v_3^n, & \cdots, & v_{M-1}^n \end{bmatrix}^\mathsf{T} \tag{3.19b}$$

the matrix $\Lambda^n \in \mathbb{R}^{M-1,M-2}$

$$\Lambda^n = \begin{bmatrix} \beta_1^n & \gamma_1^n & & & & & \\ \alpha_2^n & \beta_2^n & \gamma_2^n & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \alpha_{M-3}^n & \beta_{M-3}^n & \gamma_{M-3}^n \\ & & & & \alpha_{M-2}^n & \beta_{M-2}^n \\ & & & & & \alpha_{M-2}^n \end{bmatrix} \tag{3.20a}$$

$$\Lambda^n = \begin{bmatrix} \gamma_1^n & & & & & \\ \beta_2^n & \gamma_2^n & & & & \\ \alpha_3^n & \beta_3^n & \gamma_3^n & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{M-2}^n & \beta_{M-2}^n & \gamma_{M-2}^n \\ & & & & \alpha_{M-1}^n & \beta_{M-1}^n \end{bmatrix} \tag{3.20b}$$

and the vector $\mathbf{f}^n \in \mathbb{R}^{M-1}$

$$\mathbf{f}^n = \begin{bmatrix} v_1^{n+1} \\ \vdots \\ v_{M-2}^{n+1} - \gamma_{M-2}^n[(1-\Delta x)\bar{S}^n - K] \\ v_{M-1}^{n+1} - \gamma_{M-1}^n(\bar{S}^n - K) - \beta_{M-1}^n[(1-\Delta x)\bar{S}^n - K] \end{bmatrix} \tag{3.21a}$$

$$\mathbf{f}^n = \begin{bmatrix} v_1^{n+1} - \alpha_1^n(K - \bar{S}^n) - \beta_1^n[K - (1+\Delta x)\bar{S}^n] \\ v_2^{n+1} - \beta_2^n[K - (1+\Delta x)\bar{S}^n] \\ v_3^{n+1} \\ \vdots \\ v_{M-1}^{n+1} \end{bmatrix} \tag{3.21b}$$

Thus, the non-linear system of equations that we need to solve is

$$F(\mathbf{v}^n, \bar{S}^n) = \Lambda^n \mathbf{v}^n - \mathbf{f}^n = 0 \tag{3.22}$$

By computing the Jacobian of the system, we con solve the non-linear system using the newton's method

$$\mathbf{y}_{k+1} = \mathbf{y}_k - J^{-1}(\mathbf{y}_k)F(\mathbf{y}_k) \tag{3.23}$$

where y_k is some approximation of the solution

$$\mathbf{y} = \left[\mathbf{v}^n | \bar{S}^n\right]^{\mathsf{T}} \tag{3.24}$$

4 Numerical results

5 Linear complementary problem

5.1 Overview

In section 2, we defined some

$$
\begin{aligned}
V(S,t) - H(S,t) > 0 \qquad & \text{for all } (S,t) \in \mathcal{C} \\
V(S,t) - H(S,t) = 0 \qquad & \text{for all } (S,t) \in \mathcal{S}
\end{aligned}
$$

Additionally,

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0 \qquad \text{for } (S,t) \in \mathcal{C}
$$

Moreover

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV < 0 \qquad \text{for } (S,t) \in \mathcal{S}
$$

Therefore, grouping the bounds above we form a linear complementary system of equations

$$
\begin{cases}
\left[\frac{\partial V}{\partial t} - \mathcal{L}_{\text{BS}}(V)\right] \cdot [V(S,t) - H(S,t)] = 0 & \text{for all } (S,t) \\
V(S,t) - H(S,t) \geq 0 & \text{for all } (S,t) \\
\frac{\partial V}{\partial t} - \mathcal{L}_{\text{BS}}(V) \leq 0 & \text{for all } (S,t) \\
V(S,T) = H(S,T)
\end{cases}
\tag{5.1}
$$

The benefit of the reformulation (5.1) is that there is no dependence on the unknown boundary of the continuation region. Later in section (XXX) and section (XXX), we will explore numerical methods for solving both type of problems.

The Black-Scholes PDE can be transformed to heat diffusion PDE using the following change of variables

$$
\begin{aligned}
S &= Ke^x \\
t &= T - \frac{2\tau}{\sigma^2} \\
q &:= \frac{2r}{\sigma^2} \\
q_\delta &:= \frac{2(r-\delta)}{\sigma^2} \\
\alpha &:= \frac{1}{2}(q_\delta - 1) \\
\beta &:= \frac{1}{4}(q_\delta - 1)^2 + q \\
v(x,\tau) &:= e^{-(\alpha x + \beta \tau)} y(x,\tau) = V(S,t)
\end{aligned}
$$

The system (**??**) is the free boundary formulation for the pricing problem for American options. A detailed derivation of (**??**) can be found at [REFERENCES].

The equation (**??**) is a paraboblic PDE. Moreover, by applying the transformation, the equation (**??**) converts to the heat diffusion PDE.

$$
h(x,\tau) := \frac{H(S,t)}{K} = \begin{cases} \max(e^x - 1, 0) \\ \max(1 - e^x, 0) \end{cases} \tag{5.2}
$$

$$
\bar{x}(\tau) := \log \bar{S}(t) - \log K \tag{5.3}
$$

$$
\begin{cases}
\dfrac{\partial y}{\partial \tau} = \dfrac{\partial^2 y}{\partial x^2} & \text{for } \tau \in [0, \dfrac{\sigma^2}{2}T) \text{ and } x \in (\bar{x}(t), \infty) \\
y(x,\tau) = e^{(\alpha x + \beta\tau)} h(x,\tau) & \text{for } \tau \in [0, \dfrac{\sigma^2}{2}T] \text{ and } x \in (-\infty, \bar{x}(\tau)] \\
\bar{x}(0) = 0
\end{cases} \tag{5.4}
$$

We can reformulate equation (5.4) as:

$$g := e^{\alpha x + \beta \tau} h(x, \tau) \tag{5.5}$$

$$\begin{cases} \Big(\dfrac{\partial y}{\partial \tau} - \dfrac{\partial^2 y}{\partial x^2}\Big)(y - g) = 0 \\ \dfrac{\partial y}{\partial \tau} - \dfrac{\partial^2 y}{\partial x^2} \geq 0 \quad y - g \geq 0 \\ y(x, 0) = g(x, 0) \end{cases} \tag{5.6}$$

By exploring the geometric properties of the value function $V(S, t)$, we can determine useful conditions that will later help on in solving the equation (**??**). Firstly, at any given time $0 \leq t \leq T$, American options match the linear segment of the payoff function within the stopping region. Therefore, we could say that

$$\frac{\partial V}{\partial S}(S, t) = \begin{cases} -1 & \text{(put)} \\ 1 & \text{(call)} \end{cases} \tag{5.7}$$

Moreover, as the price goes to infinity the value of the option tends to zero

$$\lim_{S \to \infty} V(S, t) = 0 \tag{5.8}$$

Pricing American options requires using numerical methods. The Black-Scholes PDE in (XXX) can be converted to the heat diffusion equation
To obtain such approximation, we rely on central difference approximations (REFERENCE).

5.2 Explicit scheme

An explicit scheme is one where we approximate the time partial derivative using a forward difference approximation. Hence, the PDE in (XXX) is approximated as

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} = \frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{(\Delta x)^2} \tag{5.9}$$

By rearranging the terms,

$$\lambda := \frac{\Delta\tau}{(\Delta x)^2} \tag{5.10}$$

$$y_i^{n+1} = \lambda y_{i-1}^n + (1 - 2\lambda)y_i^n + \lambda y_{i+1}^n \tag{5.11}$$

It is shown by reference [REFERENCE] that method (XXX) is stable and consistent under the following condition

$$0 < \Delta\tau \leq \frac{(\Delta x)^2}{2} \tag{5.12}$$

Moreover, the method has order of convergence $O(\Delta\tau, (\Delta x)^2)$.

5.3 Implicit scheme

The implicit scheme approximates the time derivative using a backward difference

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} = \frac{y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}}{(\Delta x)^2} \tag{5.13}$$

$$y_i^{n+1} - \lambda(y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}) = y_i^n \tag{5.14}$$

$$K := \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \tag{5.15}$$

$$(I + \lambda K)\boldsymbol{y}^{n+1} = \boldsymbol{y}^n \tag{5.16}$$

5.4 Theta method

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} = (1-\theta)\frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{(\Delta x)^2} + \theta\frac{y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}}{(\Delta x)^2} \tag{5.17}$$

$$y_i^{n+1} - \lambda\theta(y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}) = y_i^n + (1-\theta)\lambda(y_{i-1}^n - 2y_i^n + y_{i+1}^n) \tag{5.18}$$

$$(1 + \lambda\theta K)\boldsymbol{y}^{n+1} = (1 - \lambda\theta K)\boldsymbol{y}^n \tag{5.19}$$

5.4.1 Theta method

We discretize the system of equation containing (3.17) and (3.18) to solve it. Firstly, we define an uniform meshgrid within the region $[1, x_{\mathsf{max}}] \times [0, T]$ and with resolution Δx and Δt.

$$M := \frac{x_{\mathsf{max}} - 1}{\Delta x} \qquad N := \frac{T}{\Delta t}$$

$$\begin{aligned} x_i &:= 1 + i\Delta x && \text{for } i = 0, \dots, M \\ t_n &:= n\Delta t && \text{for } n = 0, \dots, N \end{aligned}$$

Now we define the approximations

$$v_i^n \approx v(x_i, t_n) \qquad \text{for } (x_i, t_n) \in \{x_k\}_0^M \times \{t_k\}_0^N \tag{5.20}$$

$$\bar{S}^n \approx \bar{S}(t_n) \qquad \text{for } t_n \in \{t_k\}_0^N \tag{5.21}$$

By the boundary conditions, we can derive an expression for

$$v_0^n = K - \bar{S}^n \qquad \text{for } n = 0, \dots, N-1, N \tag{5.22}$$

$$v_{M+1}^n = 0 \qquad \text{for } n = 0, \dots, N-1, N \tag{5.23}$$

Additionally by using the smothness condition, we get:

$$\frac{v_1^n - v_0^n}{\Delta x} = -\bar{S}^n \tag{5.24}$$

or

$$v_1^n = K - (1 + \Delta x)\bar{S}^n \tag{5.25}$$

Next, we equation (3.XX) discretize using centered finite difference. The discretization method, we use is the theta method which a interpolation between an implicit and explicit scheme.

$$\begin{aligned} v_i^{t+1} - v_i^t + \theta\Bigg\{ &\frac{1}{2}\sigma^2 x_i^2 \frac{\Delta t}{(\Delta x)^2}(v_{i-1}^t - 2v_i^t + v_{i+1}^t) + \left[(r-\delta) - \frac{1}{\bar{S}^t}\frac{\bar{S}^{t+1} - \bar{S}^t}{\Delta t}\right]\frac{\Delta t}{2\Delta x}(v_{i+1}^t - v_{i-1}^t) - r v_i^t \Delta t\Bigg\} \\ &+ (1-\theta)\Bigg\{\frac{1}{2}\sigma^2 x_i^2 \frac{\Delta t}{(\Delta x)^2}(v_{i-1}^{t+1} - 2v_i^{t+1} + v_{i+1}^{t+1}) \\ &+ \left[(r-\delta) - \frac{1}{\bar{S}^{t+1}}\frac{\bar{S}^{t+1} - \bar{S}^t}{\Delta t}\right]\frac{\Delta t}{2\Delta x}(v_{i+1}^{t+1} - v_{i-1}^{t+1}) - r v_i^{t+1}\Delta t\Bigg\} = 0 \end{aligned} \tag{5.26}$$

To simplify the expression above, we introduce the following terms

$$\lambda := \frac{\Delta t}{(\Delta x)^2} \tag{5.27}$$

$$\alpha_i := 1 + \theta(\lambda\sigma^2 x_i^2 + r\Delta t) \tag{5.28}$$

$$\beta_i := -\frac{1}{2}\lambda\theta\left[\sigma^2 x_i^2 - x_i\Delta x(r-\delta)\right] - \frac{1}{2}\lambda\theta x_i \Delta x \frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t \bar{S}^n} \tag{5.29}$$

$$\gamma_i := -\frac{1}{2}\lambda\theta\left[\sigma^2 x_i^2 + x_i\Delta x(r-\delta)\right] + \frac{1}{2}\lambda\theta x_i \Delta x \frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t \bar{S}^n} \tag{5.30}$$

$$a_i := 1 - (1-\theta)(\lambda\sigma^2 x_i^2 + r\Delta t) \tag{5.31}$$

$$b_i := \frac{1}{2}(1-\theta)\lambda\left[\sigma^2 x_i^2 - x_i\Delta x\left((r-\delta) - \frac{1}{\Delta t}\right)\right] \tag{5.32}$$

$$c_i := \frac{1}{2}(1-\theta)\lambda\left[\sigma^2 x_i^2 + x_i\Delta x\left((r-\delta) - \frac{1}{\Delta t}\right)\right] \tag{5.33}$$

$$d_i^{n+1} := (1-\theta)\frac{x_i}{2\Delta x}\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}} \tag{5.34}$$

Now, the expression above becomes

$$\beta_i^n v_{i-1}^n + \alpha_i^n v_i^n + \gamma_i^n v_{i+1}^n = b_i v_{i-1}^{n+1} + a_i v_i^{n+1} + c_i v_{i+1}^{n+1} + d_i^{n+1}\bar{S}^n \tag{5.35}$$

$$\gamma_1^n v_2^n = b_1 v_0^{n+1} + a_1 v_1^{n+1} + c_1 v_2^{n+1} + d_1^{n+1}\bar{S}^n - \beta_1^n(K - \bar{S}^n) - \alpha_1^n(K - (1+\Delta x)\bar{S}^n) \tag{5.36}$$

$$\alpha_2^n v_2^n + \gamma_2^n v_3^n = b_2 v_1^{n+1} + a_2 v_2^{n+1} + c_2 v_3^{n+1} + d_2^{n+1}\bar{S}^n - \beta_2^n(K - (1+\Delta x)\bar{S}^n) \tag{5.37}$$

$$\beta_M^n v_{M-1}^n + \alpha_M^n v_M^n = b_i v_{i-1}^{n+1} + a_i v_i^{n+1} + c_i v_{i+1}^{n+1} + d_i^{n+1}\bar{S}^n \tag{5.38}$$

6 Conclusion

7 Conclusions

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