4 Pricing and hedging financial derivatives

In this section we consider pricing of derivatives when prices of underliers are modelled by SDEs. Before we consider pricing and hedging of derivatives within a (more or less) general setting using Girsanov's theorem and the martingale representation theorem, we first look (Section 4.1) how this approach works in the one-dimensional case and, in particular, when an underlier's price is modeled by GBM. This was already considered in FIM, where the Black-Scholes formula was obtained via a different route. Then (Section 4.6) we generalise this approach to a market model with multiple assets. The key results in this section are the two fundamental pricing theorems as well as the derivative pricing formula and hedging strategies. We start with related quant job interview questions on which you should be able to answer after this section (you should be able to answer on some of them even using FIM):

- Write down the Black-Scholes PDE and briefly explain its derivation.
- What are the assumptions used in derivation of the Black-Scholes pricing formula?
- What is the Black-Scholes formula?
- How to hedge an option?
- What is implied volatility and a volatility smile?
- What would you use to forecast volatility: implied standard deviation or historical standard deviation?
- What is a drawback of local volatility models?
- Compare local and stochastic volatility models.
- What is Gamma of an option?
- Why is it preferable to have small Gamma of an option?
- Why is the Gamma of plain vanilla options positive?

4.1 The case of one risky asset

In this section the market consists of one risky asset and one money market account. We need in some preliminaries before we can discuss option pricing as such.

Let $W(t)$, $0 \le t \le T$, be a one-dimensional standard Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Here $\{\mathcal{F}_t\}$ is the natural filtration for $W(t)$. Assume that the stock price $S(t)$ is modelled as

$$dS = \mu(t)\, Sdt + \sigma(t)\, SdW(t), \quad S(0) = S_0, \tag{4.1}$$

where the (instantaneous) rate of return $\mu(t)$ and the volatility $\sigma(t)$ are $\mathcal{F}_t$-adapted stochastic processes with 'good' properties and $\sigma(t)$ is almost surely not zero. The solution to (4.1) is a generalised GBM and it can be written explicitly:

$$S(t) = S_0 \exp\left(\int_0^t \sigma(s)\, dW(s) + \int_0^t \left[\mu(s) - \frac{\sigma^2(s)}{2}\right] ds\right). \tag{4.2}$$

Exercise 24 *Show that $S(t)$ from (4.2) is indeed a solution of (4.1).*

Exercise 25 *What is the main reason for modelling prices so that we can write them in the form (4.2)?*

Exercise 26 *Give an example of negative prices.*

Further, assume that the model for the money market account (the current account) is the following differential equation

$$dB = r(t)Bdt, \tag{4.3}$$

where $r(t)$ is a short rate (or in other words an instantaneous spot rate; it was defined in FIM and we recall the basics of fixed-income later in this course, see Section 6.1), which is assumed to be $\mathcal{F}_t$-adapted.

The solution of (4.3) is

$$B(t) = B(0)\exp\left(\int_0^t r(s)\,ds\right). \tag{4.4}$$

Remark 4.1 *Note that $B(t)$ has a derivative in the usual sense while $S(t)$ – not, i.e. $B(t)$ is smoother than $S(t)$; $B(t)$ has zero quadratic variation while $S(t)$ has non zero quadratic variation. On the physical level of rigour, we can say that $S(t)$ is 'more random' than $B(t)$. In the case of the money market account, we have a high degree of certainty about short-term returns but we have less certainty over longer period of time. However, returns on stock are highly uncertain both on short term and long term.*

Remark 4.2 *Obviously, the case of $r = 0$ corresponds to the market model with free money borrowing.*

Remark 4.3 *The model of this section is simpler than the generic multi-asset model of Section 4.6 in two major aspects: it is always arbitrage-free and complete. The simplicity of this section's model allows us to understand pricing and hedging in complete markets in a more transparent way.*

Note that the discounted factor $D(t) = 1/B(t)$ satisfies the differential equation

$$dD = -r(t)Ddt \tag{4.5}$$

and

$$D(t) = \frac{1}{B(t)} = \frac{1}{B(0)}\exp\left(-\int_0^t r(s)\,ds\right). \tag{4.6}$$

Suppose an investor holds a portfolio $\phi(t) = (\Delta(t), \psi(t))$, where $\Delta(t)$ is the amount of units of stock held at time t and $\psi(t)$ is the amount of units on the current account at time t. The value $V(t)$ of this portfolio at time t is

$$V(t) = \Delta(t)S(t) + \psi(t)B(t). \tag{4.7}$$

As in FIM, we assume that

- both the stock and money on the current account can be traded in arbitrary amounts;
- no transaction costs or taxes are charged;
- short positions are allowed[16];
- borrowing and lending are at the same rate of interest;
- trading in assets is continuous in time.

We will also, for simplicity, ignore that stock may pay dividends.

Further, we require that the processes $\Delta(t)$ and $\psi(t)$ are $\mathcal{F}_t$-adapted and that

$$E\int_0^T \left[\Delta^2(t)S^2(t) + \psi^2(t)B^2(t)\right] dt < \infty, \tag{4.8}$$

[16] I.e. $\Delta(t)$ and $\psi(t)$ are allowed to be negative.

which is our class of **admissible** strategies[17] $\phi(t) = (\Delta(t), \psi(t))$.

We say that the portfolio $\phi(t)$ is **self-financing** if

$$dV = \Delta(t)dS + \psi(t)dB \tag{4.9}$$

or equivalently

$$V(t) = V(s) + \int_s^t \Delta(s')dS(s') + \int_s^t \psi(s')dB(s')$$

and

$$\Delta(t)S(t) + \psi(t)B(t) - \Delta(s)S(s) - \psi(s)B(s) = \int_s^t \Delta(s')dS(s') + \int_s^t \psi(s')dB(s'),$$

i.e. the change in the portfolio's value is only due to gains/losses in trade and never needs to be topped up with extra cash nor can ever afford withdrawals.

Requiring from $\Delta(t)$ and $\psi(t)$ to be $\mathcal{F}_t$-adapted and satisfying (4.8) are needed (together with the assumption that the volatility $\sigma(t)$ is a 'good' function) for the SDE (4.9) to be well defined. The first term $\Delta(t)dS$ in (4.9) is the capital gain on the stock position and the second term

$$\psi(t)dB = (V(t) - \Delta(t)S(t))r(t)dt \tag{4.10}$$

is the interest earnings on the cash position.

Exercise 27 *Show that (4.10) is correct.*

Example 21 *A simple example of an admissible strategy is a keep-only-bonds strategy, i.e. when* $\Delta(t) = 0$ *for all* $t \geq 0$. *In this case the portfolio value is*

$$V(t) = \psi(t)B(t) = \psi(0)B(0)\exp\left(\int_0^t r(s)\,ds\right). \tag{4.11}$$

Exercise 28 *Show that (4.11) is correct.*

To progress towards pricing and hedging, let us first manipulate with the equation (4.9). We have from (4.9), (4.7), (4.1) and (4.3):

$$\begin{aligned} dV &= \Delta(t)dS + \psi(t)dB = \Delta(t)dS + \frac{V(t) - \Delta(t)S(t)}{B(t)}dB \\ &= \Delta(t)\left[\mu(t)\,S(t)dt + \sigma(t)\,S(t)dW(t)\right] + r(t)\left(V(t) - \Delta(t)S(t)\right)dt \\ &= \Delta(t)\sigma(t)\,S(t)\left[\gamma(t)dt + dW(t)\right] + r(t)V(t)dt, \end{aligned} \tag{4.12}$$

where $\gamma(t)$ is the *market price of risk*

$$\gamma(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \tag{4.13}$$

which is the excess rate (i.e. above the 'low risk' rate $r(t)$ on the rolling money market account) of return per unit of risk, which is associated with the stock and measured via the volatility $\sigma(t)$. As we will see in Section 4.6, when we simultaneously consider several risky assets, the market price of risk is the same for all of these assets which is one of the important consequences of the risk-neutral pricing.

Introduce the discounted stock price process

$$\tilde{S}(t) = \frac{S(t)}{B(t)} = S(t)D(t) \tag{4.14}$$

[17] You might notice that admissible strategies are slightly different in different books. E.g., some require for the risk to be bounded by assuming that there is a constant C such that $V(t) \geq C$ for all $t \geq 0$ a.s. But we will not go for more complicated versions in this module.

and the discounted portfolio value

$$\tilde{V}(t) = \frac{V(t)}{B(t)}. \tag{4.15}$$

From (4.14), (4.2) and (4.4), we get

$$\tilde{S}(t) = \tilde{S}(0) \exp\left(\int_0^t \sigma(s)\, dW(s) + \int_0^t \left[\mu(s) - r(s) - \frac{\sigma^2(s)}{2} \right] ds \right), \tag{4.16}$$

the SDE for which has the form

$$\begin{aligned} d\tilde{S} &= [\mu(t) - r(t)]\, \tilde{S}dt + \sigma(t)\tilde{S}dW \\ &= \sigma(t)\, \tilde{S}\, [\gamma(t)dt + dW(t)]\,. \end{aligned} \tag{4.17}$$

Note that the volatility of the discounted stock $\tilde{S}$ is the same as of the undiscounted stock S while the rate of return has changed to $\mu(t) - r(t)$.

We also obtain form (4.15), (4.12) and (4.3):

$$d\tilde{V}(t) = \Delta(t)\sigma(t)\, \tilde{S}\, [\gamma(t)dt + dW(t)]\,. \tag{4.18}$$

Exercise 29 *Show that (4.17) and (4.18) is correct.*

Using Girsanov's theorem, we find a unique measure Q under which $\tilde{S}$ is a martingale:

$$d\tilde{S} = \sigma(t)\tilde{S}dW^Q. \tag{4.19}$$

As in FIM, we will call this measure Q an *equivalent martingale measure* (*EMM*). Let us emphasise that in the case of the one-dimensional model with single Wiener process (4.1) there is only one $\gamma(t)$ using which (4.1) can be rearranged into (4.18) and hence there is a unique Q. We will see in Section 4.6 that in a general case it is not always that an EMM exists and if it exists, it might be not unique.

Exercise 30 *Show that (4.19) implies*

$$dS = r(t)Sdt + \sigma(t)SdW^Q. \tag{4.20}$$

Further, under this EMM Q the discounted discounted portfolio value process is also a martingale:

$$d\tilde{V}(t) = \Delta(t)\sigma(t)\, \tilde{S}dW^Q. \tag{4.21}$$

Exercise 31 *Show that (4.21) is true.*

We emphasise that, according to (4.21), an arbitrary discounted portfolio value process $\tilde{V}(t)$ is a martingale under an EMM Q.

Now consider a European-type option and assume that $H(T)$ is an $\mathcal{F}_T$-measurable variable which represents the (random) value of the payoff of this option at its maturity T. We assume that $EH^2(T) < \infty$. Note that we allow here that this payoff can be not only of the form $f(S(T))$ (like e.g., in the cases of plain-vanilla European calls and puts) but also to correspond to path dependent options (like e.g. European Asian options), i.e. when the payoff depends on anything that occurs during the time interval $[0, T]$, which is what $\mathcal{F}_T$-measurability means. Our aim now is to find an initial value $V(0)$ of this option and a hedging strategy $\Delta(t)$ so that the corresponding wealth process $V(t)$ at the maturity time T matches the claim:

$$V(T) = H(T) \quad \text{almost surely.} \tag{4.22}$$

Introduce the discounted payoff

$$\tilde{H}(T) = \frac{H(T)}{B(T)}. \tag{4.23}$$

By the martingale representation theorem (Theorem 3.1 and see Example 11), there is an $\mathcal{F}_s$-measurable integrand $\theta(s)$ such that

$$\tilde{H}(T) = E_Q\tilde{H}(T) + \int_0^T \theta(s)\, dW^Q(s). \tag{4.24}$$

It follows from (4.21), (4.22), (4.15), (4.23) and (4.24) that we should have

$$\tilde{V}(0) + \int_0^T \Delta(s)\sigma(s)\,\tilde{S}(s)dW^Q(s) = E_Q\tilde{H}(T) + \int_0^T \theta(s)\, dW^Q(s) \quad \text{a.s.}, \tag{4.25}$$

which implies that the discounted initial portfolio value is

$$\tilde{V}(0) = E_Q\tilde{H}(T) \tag{4.26}$$

and the hedging strategy is

$$\Delta(t) = \frac{\theta(t)}{\sigma(t)\,\tilde{S}(t)} = \frac{\theta(t)B(t)}{\sigma(t)\,S(t)}. \tag{4.27}$$

To summarise, if the writer sells the option at time $t = 0$ for

$$\begin{aligned} V(0) &= B(0)E_Q\left[\frac{H(T)}{B(T)}\right] = E_Q\left[\frac{B(0)}{B(T)}H(T)\right] \\ &= E_Q\left[\exp\left(-\int_0^T r(s)\, ds\right) H(T)\right] \end{aligned} \tag{4.28}$$

and follows the **hedging** or **replicating** strategy $\Delta(t)$ from (4.27) for $0 \le t \le T$ then at the maturity time T he has the required amount of funds $H(T)$ so that he can pay the holder the required amount of money according to the contingent claim almost surely (in other words, with probability 1), i.e. he can perfectly replicate[18] (hedge) the pay-off $H(T)$.

Exercise 32 *Explain why (4.26) follows from (4.25).*

We emphasise that in the case of the one-dimensional model with single Wiener process (4.1) the hedging strategy always exists as we see from (4.27). Consequently, any contingent claim H with maturity at a finite $T > 0$ is attainable (recall Definition 2.11): for any H, if we start with the initial capital $V(0)$ and follow the hedging strategy $\Delta(t)$ in adjusting the portfolio with the wealth process $V(t)$ then (4.22) holds. Hence, the market considered in this section is complete (recall Definition 2.12). We will see in Section 4.6 that in a general case a hedging strategy does not always exists for every claim (see also Example 19, where we proved that there is a unique EMM in this market and related this to the second fundamental theorem of asset pricing).

Remark 4.4 *The above result of this section is more general than the pricing formula and delta hedging you considered in FIM, where the payoff was only a function of the stock price at the maturity time T: $H(T) = f(S(T))$. This result covers essentially arbitrary payoffs including path-dependent ones. However, there is a price to pay for this generality: we only showed that a replicating portfolio exists and we did not come up with an explicit formula for $\Delta(t)$. In many particular, interesting cases one can easily find hedging strategies as we will see, e.g. in Example 22.*

Exercise 33 *Show that within the market model of this section, the price of an option at time $0 \le t \le T$ is given by*

$$V(t) = E_Q\left[\exp\left(-\int_t^T r(s)\, ds\right) H(T)\middle| \mathcal{F}_t\right]. \tag{4.29}$$

[18] That is why this strategy (which includes the initial value $V(0)$ and $\Delta(t)$) is also called replicating.

Example 22 *Within the market model of this section, let us find an explicit formula for the hedging strategy in the case when the claim $H(T)$ can be expressed as $f(S(T))$, i.e. when the contingent claim depends on the final price of the asset only (this is e.g. the case for plain-vanilla calls and puts). Assume for simplicity that the short rate $r(s)$ is constant: $r(s) \equiv r$. We also assume that the volatility $\sigma(t)$ is deterministic.*

According to (4.29), we have

$$V(t) = E_Q\left[\mathrm{e}^{-r(T-t)} f(S(T))|\mathcal{F}_t\right]. \tag{4.30}$$

Since $S(t)$, $t \geq 0$, is a Markov process[19]*, the conditional expectation in (4.30) can be written as a function of time t and price $S(t)$ at the time t : $V(t) = v(t, S(t))$, i.e.*

$$V(t) = E_Q\left[\mathrm{e}^{-r(T-t)} f(S(T))|\mathcal{F}_t\right] = E_Q\left[\mathrm{e}^{-r(T-t)} f(S(T))|S(t)\right] := v(t, S(t)). \tag{4.31}$$

We know from the earlier consideration that the discounted portfolio value $\tilde{V}(t)$ is a martingale under the EMM Q. Hence $\mathrm{e}^{-rt}v(t, S(t))$ is a Q-martingale. Applying the Ito formula to $\mathrm{e}^{-rt}v(t, S(t))$, we obtain[20]

$$d\left(\mathrm{e}^{-rt}v(t, S(t))\right) = \mathrm{e}^{-rt}\left[\frac{\partial v}{\partial t} - rv + rS\frac{\partial v}{\partial x}(t, S) + \frac{\sigma^2(t)}{2}S^2\frac{\partial^2 v}{\partial x^2}(t, S)\right]dt + \mathrm{e}^{-rt}\sigma(t)S\frac{\partial v}{\partial x}(t, S)dW^Q. \tag{4.32}$$

Since $\mathrm{e}^{-rt}v(t, S(t))$ is a Q-martingale, the expression at dt must be equal zero:

$$\frac{\partial v}{\partial t} - rv + rS(t)\frac{\partial v}{\partial x}(t, S) + \frac{\sigma^2(t)}{2}S^2(t)\frac{\partial^2 v}{\partial x^2}(t, S) = 0, \tag{4.33}$$

which should be valid for any $S > 0$. Consequently, the price $v(t, x)$ solves the PDE problem

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv &= 0, \quad t \in [0, T), \quad x \in (0, \infty), \\ v(T, x) &= f(x). \end{aligned} \tag{4.34}$$

The problem (4.34) is just a probabilistic representation (the Feynman-Kac formula) for $v(t, S(t))$ from (4.31). As you know from FIM, (4.34) is the Black-Scholes equation[21]*. Furthermore, from (4.32)-(4.33), we get*

$$d\left(\mathrm{e}^{-rt}v(t, S(t))\right) = d\tilde{V}(t) = \sigma(t)\tilde{S}\frac{\partial v}{\partial x}(t, S)dW^Q(t). \tag{4.35}$$

Comparing (4.35) with (4.21), we conclude that the strategy Δ has the form

$$\Delta(t) = \frac{\partial v}{\partial x}(t, S(t)). \tag{4.36}$$

This hedging strategy (4.36) (delta hedging) is already known to you from FIM and you know that it is constructive: we can compute $v(t, x)$ and its first spatial derivative at least numerically (by e.g. solving the PDE problem using finite differences) or, on some occasions, analytically. The way of finding hedging strategies shown in this example works in more general cases than we considered here, e.g. for some path-dependent options.

The price V from (4.28) and (4.29) of the considered option is an arbitrage price (see Definition 2.10). Let us reformulate Definition 2.8 (on which Definitions 2.9 and 2.10 rest) in a mathematical way.

[19] Recall that SDE solutions are Markov processes and $S(t)$ is an SDE solution.
[20] Recall that $S(t)$ under the measure Q satisfies the SDE (4.20).
[21] In FIM you saw it in the case of constant volatility σ and it was derived differently.

Definition 4.1 *An* ***arbitrage*** *means existence of a self-financing portfolio value process $V(t)$ such that $V(0) = 0$ and under the 'market' measure P for some $t > 0$*

$$P(V(t) \geq 0) = 1 \tag{4.37}$$

and

$$P(V(t) > 0) > 0. \tag{4.38}$$

The equality (4.37) means that probability of a loss is zero and (4.38) means that probability of a profit is non-zero. This is a typical 'free lunch' situation. It is useful to re-read Definition 2.8 now.

In Section 4.6, where we generalise ideas of this section to multi-asset models, we will revise the fundamental asset pricing theorems. Here we have a preliminary example which is, in a sense, gives us the first fundamental theorem in the single-stock setting.

Example 23 *In this example we, in particular, show that the price V from (4.28) and (4.29) of the considered option is an arbitrage price indeed. More generically, we prove in this example that the market model considered in this section is free of arbitrage.*

We showed earlier in this section that an arbitrary discounted admissible portfolio value process $\tilde{V}(t)$ is a martingale under an EMM Q which, in particular, implies

$$E_Q\tilde{V}(t) = E_Q\left[\frac{V(t)}{B(t)}\right] = \frac{V(0)}{B(0)}.$$

And if $V(0) = 0$ then

$$E_Q\left[\frac{V(t)}{B(t)}\right] = 0. \tag{4.39}$$

Suppose $V(t)$ satisfies[22] *(4.37) and hence $P(V(t) < 0) = 0$. Since $Q \sim P$, we also have*

$$Q(V(t) < 0) = 0. \tag{4.40}$$

It follows from (4.39) and (4.40) that[23]

$$Q(V(t) > 0) = 0.$$

Since $Q \sim P$, we also have $P(V(t) > 0) = 0$. Consequently, $V(t)$ does not produce an arbitrage. Since we considered an arbitrary admissible strategies with the wealth value $V(t)$ so that $V(0) = 0$, there is no arbitrage in the considered model. Therefore, in particular, the price V from (4.28) and (4.29) is an arbitrage price.

Note that in this proof we only used the fact that there is an EMM Q, which leads us to the first fundamental theorem of asset pricing in the continuous setting that if there is an EMM then the market is arbitrage free (see Theorem 4.1 in Section 4.6).

4.2 Volatility smile and other models

As it was briefly mentioned in FIM, in practice, for pricing derivatives one uses *implied volatility.*

Recall that Vega $\mathcal{V}$ measures sensitivity of option prices to volatility, it is the partial derivative of the option price with respect to volatility. The higher the volatility, the more risky the underlying is and it is more risky to write an option on this underlying. Hence, the writer charges a higher premium for options on an underlier with larger volatility. Hence, Vega should be positive for any option before its maturity. Further, it is easy to mathematically show that $\mathcal{V} > 0$ before maturity for the Black-Scholes prices of plain-vanilla put and call. It immediately follows from the put-call parity relationship (2.1) that $\mathcal{V}$ for call is always equal to $\mathcal{V}$ for put.

[22] That is, we prove here by contradiction.

[23] Otherwise, we would have $Q(V(t)/B(t) > 0) > 0$ and $Q(V(t)/B(t) < 0) = 0$, which imply $E_Q\left[V(t)/B(t)\right] > 0$.

Recall from FIM that in the case of the European call with $f(x) = (x - K)_+$ and maturity T the Black-Scholes formula for this option's price is

$$C(t, x; \sigma, T, r) = x\Phi\left(\frac{\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\ln\frac{x}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right), \tag{4.41}$$

where σ is the volatility and r is the short rate. Then, for Vega of a European call, we have (see also FIM):

$$\begin{aligned}
\mathcal{V} = \frac{\partial}{\partial\sigma}C(t, x; \sigma, T, r) &= x\frac{\partial}{\partial\sigma}\Phi\left(\frac{\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&- Ke^{-r(T-t)}\frac{\partial}{\partial\sigma}\Phi\left(\frac{\ln\frac{S_0}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= x(-\frac{\ln\frac{x}{K} + r(T-t)}{\sigma^2\sqrt{T-t}} + \frac{\sqrt{T-t}}{2})\Phi'\left(\frac{\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&- Ke^{-r(T-t)}(-\frac{\ln\frac{x}{K} + r(T-t)}{\sigma^2\sqrt{T-t}} - \frac{\sqrt{T-t}}{2})\Phi'\left(\frac{\ln\frac{x}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{x\sqrt{T-t}}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)^2\right].
\end{aligned}$$

Because in the Black–Scholes world prices are positive and the exponent is always positive, $\mathcal{V} > 0$ before maturity.

Since $\mathcal{V} > 0$, there is a one-to-one correspondence between the Black-Scholes option prices for put or call and the volatilities σ if all the other parameters are fixed.

Discussion on why models are needed.

Observables	**Questions of interest**

Let us recall how implied volatility is evaluated. Suppose there is a European-type derivative traded on the market, which the underlier is the stock whose volatility we need to find. Then we can observe the current option price x and invert the Black–Scholes formula (4.41) to find the implied volatility σ_{imp} (recall that all the other parameters in (4.41) are known: the interest rate r and the stock's spot price are observable and the payoff of the derivative is given), which can be viewed as the value of volatility that was implicitly used by the market for pricing this benchmark option. This procedure of finding volatility is called *calibration* to distinguish it from the estimation of σ based on historical data. Usually, if we need to price an OTC derivative with maturity T (say, one year), then, as a 'benchmark' used in calibration, one takes an observable derivative on the same underlier and with the same maturity.

The reasoning for using implied volatility rather than historical volatility is as follows. The real volatility is obviously changing with time. Historical volatility reflects market behaviour and expectations in the past whilst in pricing a derivative we need the best information possible about future volatility (e.g., over the next year as in the example in the previous paragraph). Current prices on the corresponding tradable options reflect traders' anticipation of what will happen with this particular underlying stock in the future and thus incorporate current and past information on which such a prognosis can be made. So, in other words, implied volatility is a better estimate for future volatility than its historical counterpart. From another angle, we want to price our OTC derivative in a consistent way with how the market prices other derivatives on the same underlier, which obviously requires the use of implied volatility.

There is another important use of implied volatilities. As Emanuel Derman says[24] in (Derman, 2004), "One of the things you learn repeatedly in a career in financial modeling is the importance of units". How to tell which option is better, say for speculation? This cannot be answered looking just on an option price – it is an insufficient measure of its value. E.g., it is impossible to tell whether £100 asked for an ATM put is more attractive than £50 for a deep OTM put. A better measure of option's value is its implied volatility. Emanuel Derman says in (Derman, 2004): "The Black-Scholes model views a stock option as a kind of bet on the future volatility of a stock's returns. The more volatile the stock, the more likely the bet will pay off, and therefore the more you should pay for it. ... Even today, when no one believes that the Black-Scholes model is absolutely the best way to estimate option value, ... the Black-Scholes model's implied volatilities are still the market convention for quoting prices."

As we have seen here and will see again and again in this course, financial engineering operates not in the real world but in the 'implied world'.

In reality there is a problem with implied volatilities due to a deficiency in the Black–Scholes world. On a market we can usually observe a number of calls and puts written on the same stock but having different maturities and different strikes. According to the Black–Scholes formula, we should be able to price all of them with the same volatility σ or, in other words, all these options should have the same implied volatility σ_{imp}.

However, the reality is different. Empirical data suggest that σ_{imp} depends on values of strikes K: usually options, which are far OTM or deep ITM, are traded at higher implied volatilities than ATM options. Tomas Björk says (see p. 104 in Björk, 2004): 'The graph of the observed implied volatility function thus often looks like the smile of the Cheshire cat, and for this reason the implied volatility curve is termed the *volatility smile*'. Sometimes it is also called *volatility skew*. Implied volatilities also depend on the maturity time: σ_{imp} is usually larger for larger T, which reflects the intuitively obvious fact that the uncertainty of the market is higher over a larger time horizon. The function $\sigma_{imp}(K, T)$ is called the *volatility surface*.

To fix this deficiency in the Black–Scholes formula, a number of more complicated models for stock were introduced, which include (i) deterministic models, when the volatility $\sigma(t)$ is modelled as a deterministic function of time; (ii) local volatility models, when $\sigma(t)$ is modelled as $\sigma(t, S(t))$ with $\sigma(t, x)$ being a deterministic function of time t and the underlier's price x (see Section 4.3 below); (iii) stochastic volatility models, when $\sigma(t)$ is modelled as a stochastic process containing 'additional randomness' to 'randomness' present in the SDE for the underlier with this volatility (see Section 4.4 below); and (iv) models with jumps considered in Chapter 5 later in the course.

[24] It is an interesting book. Among others, it is useful for preparing being a quant. As one of former graduates from the FCM MSc wrote after starting a job in the financial sector as a feedback: "If there's any advice I'd give to students that want to move into the financial industry, I guess it would be to 'read around' the topic."

4.3 Local volatility models

A popular example of local volatility models is the *constant elasticity of variance* (CEV) model for asset prices, which has the form under the EMM:

$$dS = rSdt + \sigma S^{\delta} dW^{Q}, \tag{4.42}$$

where $\sigma \geq 0$ and $\delta \geq 0$ are constants. The local volatility is equal to

$$\sigma(t) = \sigma(S(t)) = \sigma S^{\delta - 1}. \tag{4.43}$$

This model was first considered in Cox, 1975. The parameter δ determines the relationship between the volatility and the price. When $\delta = 1$, (4.42) coincides with GBM. There is an empirical evidence and economic justification that there is an inverse relationship (so-called leverage effect) between the stock's price level and its volatility, which implies that $0 < \delta < 1$. The economics arguments are as follows[25]. First, as a rule, every firm has fixed costs, which have to be covered irrespective of its income. A decrease of the firm's income decreases its value (and hence stock price) and at the same time its riskiness increases. Second, in can be the other way round: a downturn in the overall economy or in a particular sector can lead to an increase in stock volatility and to stock price decrease. Third, if a firms stock price decreases, the market value of its equity tend to decrease faster than the market value of its debt. Consequently, the firm's debt-to-equity ratio increases which leads to growth of riskiness of the stock, i.e. to increase of the volatility.

However, for commodities, one often observes[26] the opposite effect (so-called inverse leverage): the volatility of the price of a commodity tends to increase as its price increases, which implies that $\delta > 1$ in (4.42).

In terms of the volatility smile effect, the choice (4.43) of the local volatility can be used to fit the modelled option prices to the observed option prices with different strikes in the case of a fixed maturity. Fitting options with different strikes and different maturity times usually require to consider local volatilities dependent not only on $S(t)$ but also on time t explicitly.

The main benefit of local volatility models includes that they can perfectly fit plain-vanilla call and put options and, it is very cheap to calibrate them thanks to the Dupire formula (Dupire, 1994; Derman and Kani, 1994) as we show in the next example.

Example 24 *(Dupire's formula) For simplicity, let us assume that the interest rate $r = 0$. Consider a generic local volatility model written under an EMM Q :*

$$dS = \sigma(t, S) S dW^{Q}, \tag{4.44}$$

where $\sigma(t, x)$ being a deterministic function of time t and the underlier's price x. The forward Kolmogorov (Fokker-Planck) equation for the density $\rho(t, x)$ of the random variable $S(t)$ (see Example 15) is

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(\sigma^2(t, x) x^2 \rho \right). \tag{4.45}$$

We assume that $\rho(t, x)$ and its derivatives go to 0 sufficiently fast as $x \to \pm\infty$, which is a natural assumption.

We can then write the price of a call option with strike K and maturity T written on $S(t)$ as

$$C(K, T) = E_Q(S(T) - K)_+ = \int_K^{\infty} (x - K) \rho(T, x) dx. \tag{4.46}$$

It is not difficult to get (see Exercise 34 below)

$$\frac{\partial^2}{\partial K^2} C(K, T) = \rho(T, K). \tag{4.47}$$

[25] See, e.g. Black, 1975, 1976; Beckers, 1980.

[26] See e.g. Emanuel and MacBeth, 1982; Geman and Shih, 2009

Then we have

$$\begin{aligned} \frac{\partial C}{\partial T} &= \int_K^\infty (x-K)\frac{\partial}{\partial T}\rho(T,x)dx = \frac{1}{2}\int_K^\infty (x-K)\frac{\partial^2}{\partial x^2}\left(\sigma^2(T,x)x^2\rho\right)dx \\ &= \frac{1}{2}\sigma^2(T,K)K^2\rho(T,K) = \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2}{\partial K^2}C(K,T) \end{aligned} \tag{4.48}$$

and hence

$$\sigma^2(T,K) = \frac{2\frac{\partial C}{\partial T}(K,T)}{K^2\frac{\partial^2}{\partial K^2}C(K,T)}. \tag{4.49}$$

Exercise 34 *Derive (4.47).*

Exercise 35 *Explain all steps in the derivation (4.48).*

In practice the Dupire formula is used in the following way:

- Collect the option data, consisting of a matrix of quoted call prices $C(K_i,T_j)$, $i=1,\dots,L$, $j=1,\dots,N$, for the corresponding liquid options;
- Convert the prices to the implied vol values;
- Interpolate and extrapolate these values to create a volatility surface;
- Calculate $\sigma(t,x)$ using (4.49) with the right-hand side computed using the reconstructed vol surface;
- Substitute the obtained $\sigma(t,x)$ in (4.44) and use it for finding prices of other options.

Remark 4.5 *It is not difficult to adjust the argument used in deriving (4.49) in the case when $r \neq 0$.*

4.4 Stochastic volatility models

Despite local volatility models being able to perfectly fit European-style path-independent options, they do not capture the true dynamics of volatility and, as a result, hedging exotics under local volatility models is problematic. The authors of the famous SABR model (Hagan *et al.*, 2002) say: "We discover that the dynamics of the market smile predicted by local vol models is opposite of observed market behavior: when the price of the underlying decreases, local vol models predict that the smile shifts to higher prices; when the price increases, these models predict that the smile shifts to lower prices. Due to this contradiction between model and market, delta and vega hedges derived from the model can be unstable and may perform worse than naive Black-Scholes' hedges." Statistical studies showed that volatility has an additional source of randomness independent of randomness present in the spot price processes. These problems can be addressed by using stochastic volatility models such as Heston model (Heston (1993)), which we saw already in Example 16 and which we discuss below, and SABR (Hagan *et al.* (2002); Antonov *et al.* (2019)). For further reading on stochastic volatility models, you can use e.g. Musiela and Rutkowski (2005); Gatheral and Taleb (2006); Henry-Labordère (2008).

Consider the Heston stochastic volatility model written under the 'market' measure P (Heston (1993)):

$$\begin{aligned} dS &= \mu S dt + \sqrt{v} S dW_1(t), \\ dv &= \kappa(\theta - v)dt + \delta\sqrt{v}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right), \end{aligned} \tag{4.50}$$

where $S(t)$ is an asset price, $v(t)$ is a stochastic volatility, $\mu \geq 0$ is the rate of return of the asset, $\kappa \geq 0$ is the rate at which $v(t)$ reverts to the long run average volatility $\theta > 0$, $\delta > 0$ is a volatility of the volatility (vol of vol), and $W_1(t)$ and $W_2(t)$ are independent Wiener processes with the correlation coefficient $\rho \in [-1,1]$. The condition

$$2\kappa\theta \geq \delta^2; \tag{4.51}$$

guarantees that zero is unattainable by $v(t)$ in finite time.

What is an EMM Q in this case? As usual, the discounted price process $\tilde{S}(t)$ should be a Q-martingale. If we choose the market price of risk

$$\gamma_1(t) = \frac{\mu - r}{\sqrt{v(t)}} \tag{4.52}$$

then, according to Girsanov's theorem, there is a measure Q under which $W_1^Q(t)$ is a Wiener process so that $dW_1^Q(t) = \gamma_1(t)dt + dW_1(t)$. Hence the first equation in (4.50) becomes under Q :

$$dS = rSdt + \sqrt{v}SdW_1^Q(t). \tag{4.53}$$

Exercise 36 *Confirm that $\tilde{S}(t)$ is a Q-martingale in this case and that S under Q satisfies (4.53).*

What should we do with the other equation? Since volatility is typically considered as a non-tradable asset, we do not put a requirement that it should (one way or another) become a martingale and we have a flexibility how $W_2(t)$ can be changed under Q, which leads to the observation that we have infinitely many EMMs here and hence the market is not complete. To specify a particular Q under which pricing within the Heston model is done, it is standard for the Heston model to choose

$$\gamma_2(t) = \lambda\sqrt{v(t)}. \tag{4.54}$$

Then under the corresponding Q the model (4.50) takes the form

$$\begin{aligned} dS &= rSdt + \sqrt{v}SdW_1^Q(t), \\ dv &= \kappa(\theta - v)dt + \delta\sqrt{v}\left(\rho dW_1^Q(t) + \sqrt{1-\rho^2}dW_2^Q(t)\right). \end{aligned} \tag{4.55}$$

with new κ and θ.

Exercise 37 *Confirm (4.55) and find the new κ and θ.*

The Heston model is popular with practitioners because there are explicit formulas for call and other option prices in the form of some complicated integrals (Heston (1993)). A fast and accurate numerical approximation of these integrals is not easy, but there are efficient methods for them, see e.g. Heston (1993); Lewis (2000); Rouah (2013).

The drawbacks of stochastic volatility models are that they are often harder to calibrate (i.e., find their parameters so that the smile is reproduced) and they can suffer from calibration errors. To read more on this topic, see e.g. http://www.math.ku.dk/˜rolf/teaching/ctff03/Gatheral.1.pdf and Gatheral and Taleb (2006). Please also keep in mind that in practice a quant should choose a model which is best for a particular task and, hence, it is useful to have, understand and be able to use a large arsenal of models (this is true for any modelling, not just in financial engineering).

4.5 American options

The most popular type of options on financial markets are American options. Although the labels (European and American options) apparently have some historical justification, both types of options are now traded everywhere in the world.

Remark 4.6 (about revision) *You already studied the topic 'American options' in FIM in the discrete setting. Main ideas concerning pricing American options remains the same in the continuous case as they are in the discrete setting. At the same time, in the continuous case pricing American options is a harder problem and in this section our aim is to introduce related notions and transfer ideas from the discrete case but without going too much into technical details. Before proceeding to consideration of American option in the continuous case, it is very useful if you could look through and refresh your corresponding knowledge from FIM or/and read Chapter 12 in (Tretyakov, 2013).*

Let us recall the definition of American options.

Definition 4.2 ***American options*** *are options which can be exercised at any time prior to or at the expiration date, i.e., the choice of exercise time is left to the contract's holder.*

Example 25 *For example, an American call (put) option with strike price K and expiry time T gives the holder the right, but not the obligation, to buy (sell) an asset for price K, at any time up to T.*

Let us further discuss Definition 4.2. Let the maturity (in other words, the final exercise date) of an American option be T and its payoff at time $0 \le t \le T$ be $H(t)$, which can depend on the past and present price $S(s)$, $0 \le s \le t$, of the underlier.

Exercise 38 (Revision) *Give examples of options which $H(t)$ depends only on $S(t)$ and which depends on both the past and present price $S(s)$, $0 \le s \le t$.*

A holder of a European-type option can exercise it at the maturity T only and get $H(T)$. In contrast, a holder of an American option can exercise it at any time t before and at the maturity T and get $H(t)$. Moreover, the exercise time t does not need to be chosen a priori (i.e., at time $t = 0$); it can be chosen using the information generated by the underlier's price $S(s)$, $0 \le s \le t$, which is contained in a filtration $\mathcal{F}_t$ to which this process $S(t)$ is adapted. Consequently, the holder of an American option can exercise her right at a random time $\tau(\omega)$. A decision to exercise an option or not can depend only on the information available at the decision time (we do not allow insider trading which is a criminal offence). Hence, admissible random times here are stopping times as you know from FIM. In the continuous case the definition of stopping times is essentially the same as Definition 2.28 given in the discrete setting in FIM.

Definition 4.3 *A random variable $\tau = \tau(\omega)$ taking values in $[0, \infty]$ is called a* ***stopping time*** *or a* ***Markov time*** *with respect to a filtration $\mathcal{F}_t$ if the event*

$$\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t \tag{4.56}$$

for all $t \ge 0$.

The intuitive understanding of this definition is also the same as in the discrete case. The expression (4.56) means that we can determine, standing at time t whether the stopping event has occurred or not. The random moment $\tau(\omega)$ is the instant when one must make a certain decision to take an action like to exercise an option, buy a stock, stop waiting at a bus stop, stop gambling, etc. The condition (4.56) is equivalent to $\{\omega : \tau(\omega) > t\} \in \mathcal{F}_t$ (see Definition 2.15 of a σ-algebra), which can be interpreted as follows. If we decide at time t to postpone a certain action, the only information we could use for making this decision is $\mathcal{F}_t$, i.e. the information available to us up to time t only, as we cannot take into account the 'future', i.e. what might happen after time t (because we do not know it yet!). In a sense the concept of stopping time explains why we cannot catch the best time to, e.g., sell a stock at the maximal price over a given time interval $[0, T]$ or to stop gambling: this time, of course, exists in $[0, T]$ but we can only determine it by observing data over the whole period $[0, T]$ and so we can find this best time only *a posteriori*. Note that the notion of stopping time has no connection with probability P on Ω, it is related only to the sample space Ω and a filtration $\mathcal{F}_t$ on it. For further understanding, please attempt the following exercise.

Exercise 39 (Stopping time) *Which of the following random moments are stopping times and which are not?*

(1) $\tau(\omega) = T$, where $T > 0$ is a constant;

(2) the last exit time of a standard Wiener process from the interval $(-1, 1)$ during the time interval $[0, T]$;

(3) $\tau(\omega)$ is the first exit time of a standard Wiener process from the interval $(-1, 1)$ (to justify, use your common sense rather than full rigour);

(4) let τ and θ be two stopping times, then what can be said about [27]*:*

$$\tau \wedge \theta, \quad \tau \vee \theta, \quad (\tau + \theta) \wedge T;$$

(5) the first time the process $S(t)$ reaches its minimum over $[0, T]$.

Let $\mathcal{T}_t$, $0 \le t \le T$, be the set of all possible stopping times τ_t, i.e. all possible random variables with values in $[t, T] \cup \infty$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ and independent of the 'future'. In particular, the set $\mathcal{T}_0$ contains every stopping time $\tau = \tau_0$. A stopping time τ_T in $\mathcal{T}_T$ can take the value T on some paths and the value ∞ on others; it cannot take any other value.

In what follows in this section we will be assuming that the market, on which an American option considered here is traded, is arbitrage free as usual and hence there is an EMM Q under which we price options.

To repeat, the holder of an American option may exercise it at the maturity time T, as in the case of its European counterpart. Hence, the value of an American option should be at least not less than that of the corresponding European one. The possibility to get a payoff at an earlier time tends to make American options more expensive than their European counterparts: the better (and here better because more flexible) the product the higher the price is. At any stopping time $0 \le \tau \le T$, the holder of an American option can exercise it and receive the payment $H(\tau)$, which is called the *intrinsic value* of the option. Thus, the value $V(t)$ should always satisfy

$$V(t) \ge H(t), \; 0 \le t \le T, \tag{4.57}$$

(obviously for any possible realisation of $S(t)$), i.e. the value of an American derivative security is always greater than or equal to its intrinsic value.

When is it optimal for the holder of an American option to exercise it? We provide a heuristic argument to answer on this question assuming that the market is complete. Obviously, if $V(t) > H(t)$, the replicating portfolio worth more than the intrinsic value and it would be irrational for the holder to exercise her option. Now assume that there is the smallest time τ^* in $[0, T]$ so that $V(\tau^*) = H(\tau^*)$, i.e., this τ^* is the first time the equality occurs and it is a stopping time. Denote the time immediately after τ^* as $\tau^* +$. Since $V(t)$ is the value of a replicating portfolio for this option, $V(\tau^*)$ is not only sufficient to cover the cost of the exercise at τ^* but also the amount $V(\tau^*+)$ needed to hedge the American option over the time period $(\tau^*, T]$ if the option is not exercised at time τ^*. Clearly, $V(\tau^*+) \le V(\tau^*)$ which implies that τ^* is the optimal time to exercise this option (if we exercise, we get $V(\tau^*)$ and can use it to purchase the remaining American option keeping the remaining non-negative balance $V(\tau^*) - V(\tau^*+)$; if we do not exercise at τ^*, we either lose money or gain nothing; see an analogous argument in the discrete case, e.g. in Example 12.2 on pp. 128-129 of (Tretyakov, 2013)). We can formally define this optimal time as

$$\tau^* = \inf\{0 \le t \le T : V(t) = H(t)\}.$$

Hence the optimal strategy is to hold the option until τ^* when $V(\tau^*) = H(\tau^*)$, which also implies that $V(t) > H(t), \; 0 \le t < \tau^*$. Since it is optimal for the holder to exercise the American option at time τ^*, the option price should be

$$V(0) = E_Q\left[\tilde{H}(\tau^*) I_{\{\tau^* < \infty\}}\right], \tag{4.58}$$

where Q is an EMM (unique as we assumed that the market is complete) and $\tilde{H}(\tau)$ is the discounted payoff $H(\tau)$. Note that τ^* in (4.58) is unknown and should be found together with the price as a result of solving the pricing problem (see the same argument in FIM for the discrete case).

Further, because it is optimal for the holder to have a strategy so that she exercises this option at a stopping time τ when $\tilde{H}(\tau)$ takes the maximum value, the price of an American option can be found by solving the optimisation problem[28]:

$$V(0) = \sup_{\tau \in \mathcal{T}_0} E_Q\left[\tilde{H}(\tau)\right]. \tag{4.59}$$

[27] Here $a \wedge b := \min(a, b), \; a \vee b := \max(a, b)$.

[28] Assuming that the maximum in (4.59) exists, we can write $V(0) = \max_{\tau \in \mathcal{T}_0} E_Q\left[\tilde{H}(\tau)\right]$.

This is an optimisation problem which solution is the pair: the optimal time τ^* and the price $V(0)$.

As we see (and you already know from FIM), pricing American options is a more difficult problem than pricing European options. However, as you again know from FIM, there is a particular case when pricing an American option reduces to pricing the corresponding European option.

Exercise 40 (Revision) *Using arbitrage arguments, show that the arbitrage price of an American call option on an underlier which does not pay dividends coincides with the price of the corresponding European call option. (What does the word 'corresponding' exactly mean here?). Hint: first prove that for a European call option with value $C(t)$ and exercise price K on a stock with price $S(t)$ the inequality $C(t) \geq (S(t) - K)_+$ holds for all $t \leq T$.*[29]

In the discrete case you were able to price American options using replication and the backward induction. In the continuous case this problem is even more complicated. There is no closed form solution for the price of an American option even when the underlier is modelled using GBM (recall that in this case for plain vanilla European options we have the Black-Scholes formula), with very rare exceptions (as an example see Exercise 40).

Let us now consider a slightly simpler setting[30]. Consider the market model as in Section 4.1, i.e., that the stock price $S(t)$ is modelled under a 'market' measure as (4.1) and the money market account (the current account) is modelled by (4.3), and for simplicity we assume that the short rate $r(t)$ is constant: $r(t) = r$. As we established in Section 4.1, there is a a unique EMM Q under which the discounted price process $\tilde{S}(t) = e^{-rt}S(t)$ is a martingale (cf. (4.19)):

$$d\tilde{S} = \sigma(t)\tilde{S}dW^Q. \tag{4.60}$$

Further, we will assume from now (in this section) that the payoff $H(t)$ depends only on the price at time t (like it is for plain-vanilla put and call options) and we will then write $H(t, S(t))$, where $H(t, x)$ is a deterministic function. Hence (see Example 22), we can also write $V(t) = v(t, S(t))$, where $v(t, x)$ is a deterministic function which is assumed to be sufficiently smooth. By the Ito formula, we get (cf. (4.32))

$$d\left(\mathrm{e}^{-rt}v(t, S(t))\right) = \mathrm{e}^{-rt}\left[\frac{\partial v}{\partial t} - rv + rS\frac{\partial v}{\partial x}(t, S) + \frac{\sigma^2(t)}{2}S^2\frac{\partial^2 v}{\partial x^2}(t, S)\right]dt + \mathrm{e}^{-rt}\sigma(t)S\frac{\partial v}{\partial x}(t, S)dW^Q. \tag{4.61}$$

Hence for $h \geq 0$

$$\begin{aligned} e^{-rh}E_Q v(t+h, S_{t,x}(t+h)) \;=\; & v(t, x) \\ & + \int_t^{t+h} \mathrm{e}^{-r(s-t)}E_Q\left[\frac{\partial v}{\partial t} - rv(s, S_{t,x}(s)) + rS\frac{\partial v}{\partial x}(s, S_{t,x}(s))\right. \\ & \left. + \frac{\sigma^2(s)}{2}S^2\frac{\partial^2 v}{\partial x^2}(s, S_{t,x}(s))\right]ds. \end{aligned} \tag{4.62}$$

Exercise 41 *Explain how to obtain (4.61) from (4.62).*

If standing at (t, x) it is not optimal to stop holding the option, i.e., $v(t, x) > H(t, x)$, then the optimal value (or in other words, the optimal stopping time) is in future and instead of exercising the option at t the holder keeps it. Assume she holds the option until $t + h$ then, by the usual pricing formula, we have

$$v(t, x) = e^{-rh}E_Q v(t+h, S_{t,x}(t+h)),$$

which together with (4.62) gives

$$\int_t^{t+h} \mathrm{e}^{-r(s-t)}E_Q\left[\frac{\partial v}{\partial t} - rv(s, S_{t,x}(s)) + rS\frac{\partial v}{\partial x}(s, S_{t,x}(s)) + \frac{\sigma^2(s)}{2}S^2\frac{\partial^2 v}{\partial x^2}(s, S_{t,x}(s))\right]ds = 0. \tag{4.63}$$

[29] This exercise was examined in FIM, hence it is not examinable in this module. But it is useful for fin-math motivated graduates to know this fact.

[30] As usual, we will not follow a full rigour but concentrate on ideas.

To minimize loss of information between t and $t+h$, we tend now h to zero. We divide (4.63) by h and tend $h \to 0$ to obtain[31]

$$\frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) = 0. \tag{4.64}$$

Alternatively, if we had $v(t,x) = H(t,x)$, it would not be optimal to continue to hold the option any further. This leads to the strict inequality

$$v(t,x) > e^{-rh}E_Q v(t+h, S_{t,x}(t+h)),$$

which together with (4.62) gives (similarly to how (4.64) has been obtained)

$$\frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) < 0. \tag{4.65}$$

Note that (4.65) is a *partial differential inequality.*

Let us summarise what we have obtained. Define the region $\mathcal{C} := \{(t,x) : v(t,x) > H(t,x)\}$, which is called the **continuation domain**. The set $\mathcal{S} = \{(t,x) : v(t,x) = H(t,x)\}$ is called **stopping domain**. Then the optimal value function $v(t,x)$ satisfies

$$\begin{gathered} v(T,x) = H(T,x), \\ v(t,x) \geq H(t,x) \text{ for all } (t,x) \\ \frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) = 0 \text{ for } (t,x) \in \mathcal{C}, \\ \frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) < 0 \text{ for } (t,x) \in \mathcal{S}. \end{gathered} \tag{4.66}$$

Given (t,x) the optimal stopping time $\tau^*_{t,x} \in \mathcal{T}_t$ is

$$\tau^*_{t,x} = \inf\{s \geq t : \ (s, S(s)) \in \mathcal{S}\}.$$

It is rather clear that finding an analytical solution of (4.66) is not realistic. There are two reformulations of (4.66) which are useful for solving this problem numerically. First, we re-write (4.66) as a set of *variational inequalities*:

$$\begin{gathered} v(T,x) = H(T,x), \\ v(t,x) \geq H(t,x) \text{ for all } (t,x) \\ \frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) \leq 0 \text{ for for all } (t,x), \\ [v(t,x) - H(t,x)] \cdot \left[\frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x)\right] = 0 \text{ for all } (t,x). \end{gathered} \tag{4.67}$$

The benefit of the variational inequalities formulation is that the continuation region $\mathcal{C}$ is not present in it. The second re-writing takes the form of a *free boundary problem:*

$$\begin{gathered} \frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) = 0 \text{ for } (t,x) \in \mathcal{C}, \\ v(t,x) = H(t,x) \text{ for } (t,x) \in \partial\mathcal{C}. \end{gathered} \tag{4.68}$$

This is a free boundary problem because the domain $\mathcal{C}$ and its boundary $\partial\mathcal{C}$ are unknown, they need to be found as a part of the solution. The boundary $\partial\mathcal{C}$ is the *optimal exercise boundary.*

When we considered European options, the discounted value of a portfolio was a martingale (see Section 4.1). To have a similar probabilistic characterisation of American options, we need to introduce the new notion.

[31] Think how you would apply L'Hospital's rule to the limit $\lim_{h\to 0}\left[\frac{\int_0^h f(x)dx}{h}\right]$.

Definition 4.4 *Let $\zeta(t)$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and on some time interval $[t_0, T]$. It is called a* ***supermartingale*** *with respect to a measure P and a filtration $\{\mathcal{F}_t\}$ if*

$$E_P|\zeta(t)| < \infty \text{ and } E_P(\zeta(t)|\mathcal{F}_s) \leq \zeta(s) \quad \text{for all } t_0 \leq s \leq t \leq T. \tag{4.69}$$

Example 26 *Since for American options we have from (4.64) and (4.65):*

$$\frac{\partial v}{\partial t}(t,x) - rv(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2(t)}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x) \leq 0,$$

we obtain as in (4.61) for $s \leq t$:

$$e^{-rt}v(t, S_{0,S_0}(t)) - e^{-rs}v(s, S_{0,S_0}(s)) = \tag{4.70}$$
$$+\int_s^t \mathrm{e}^{-rs'}\left[\frac{\partial v}{\partial t} - rv + rS\frac{\partial v}{\partial x} + \frac{\sigma^2(s')}{2}S^2\frac{\partial^2 v}{\partial x^2}\right]ds' + \int_s^t \mathrm{e}^{-rs'}\sigma(s')S\frac{\partial v}{\partial x}dW^Q(s')$$

whence

$$\mathrm{e}^{-rt}E_Q(v(t,S(t))|\mathcal{F}_s) - \mathrm{e}^{-rs}E_Q(v(s,S(s))|\mathcal{F}_s) \leq 0$$

and hence $\mathrm{e}^{-rt}E_Q(v(t,S(t))|\mathcal{F}_s) \leq \mathrm{e}^{-rs}v(s,S(s))$ and the discounted value of a portfolio is a supermartingale here (assuming that $E_Q v(t,S(t)) < \infty$).

The following notion is often used in analysis and approximation of American option prices.

Definition 4.5 *Consider a process $Y(t)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ with $E|Y(t)| < \infty$. We say that a process $X(t)$* ***dominates*** *the process $Y(t)$ if $X(t) \geq Y(t)$ P-a.s. for all $t \geq 0$. The* ***Snell envelope*** *$Z(t)$ of the process $Y(t)$ is defined as the smallest supermartingale dominating $Y(t)$, i.e., $Z(t)$ is a supermartingale dominating $Y(t)$, and if $\tilde{X}(t)$ is another supermartingale dominating $Y(t)$ then $Z(t) \leq \tilde{X}(t)$ P-a.s. for all $t \geq 0$.*

Consider the optimal value process

$$V(t) = \sup_{\tau\in\mathcal{T}_t} E_Q\left[\mathrm{e}^{-r(\tau-t)}H(\tau)|\mathcal{F}_t\right]$$

and the corresponding discounted process $\tilde{V}(t) = \mathrm{e}^{-rt}V(t)$. One can prove (Björk (2004); Karatzas and Shreve (1998); Lamberton and Lapeyre (2007)) that $\tilde{V}(t)$ is the Snell envelope for the discounted payoff process $\tilde{H}(t)$. The Snell envelope property guarantees that $V(t)$ is the best possible price.

But nobody would buy a supermartingale as it tends to go down. However, the stopped process $\tilde{V}(s \wedge \tau^*_{t,x})$, $t \leq s \leq T$, is a martingale (see Example 28 below), which gives the buyer the guarantee that the price is fair if she exercises her American option optimally. To this end, we need the following proposition ensuring that the martingale property is valid for stopped processes (Karatzas and Shreve (1991); Björk (2004); Karatzas and Shreve (1998)).

Proposition 4.1 *Let $\zeta(t)$ be a martingale with respect to the measure Q and the filtration $\{\mathcal{F}_t\}$ and let τ be a stopping time. Then the stopped process $X(t) := \zeta(t \wedge \tau)$ is also a Q-martingale.*

Example 27 *Let τ be a stopping time and $h(t)$ be a 'good' process. According to the property that Ito integrals are martingales and Proposition 4.1, the stopped Ito integral*

$$X(t) = \int_0^{t\wedge\tau} h(s')dW^Q(s')$$

is a martingale.

Example 28 *Continue Example 26. For $t < \tau^*_{s,x}$, we have $(t, S(t)) \in \mathcal{C}$ and hence*

$$\frac{\partial v}{\partial s}(s,x) - rv(s,x) + rx\frac{\partial v}{\partial x}(s,x) + \frac{\sigma^2(s)}{2}x^2\frac{\partial^2 v}{\partial x^2}(s,x) = 0,$$

whence from (4.70) for $s \le t < \tau^*_{s,x}$

$$\mathrm{e}^{-rt}v(t,S(t)) = \mathrm{e}^{-rs}v(s,S(s)) + \int_s^t \mathrm{e}^{-rs'}\sigma(s')S\frac{\partial v}{\partial x}(s',S)dW^Q(s')$$

and, since $\mathrm{e}^{-rt}v(t,S(t))$ is continuous[32] *in t, the above equality is also true for $t = \tau^*_{s,x}$. For $t \ge \tau^*_{s,x}$, we have for the stopped process*

$$\mathrm{e}^{-r(t\wedge\tau^*_{s,x})}v(t\wedge\tau^*_{s,x}, S(t\wedge\tau^*_{s,x})) = \mathrm{e}^{-r\tau^*_{s,x}}v(\tau^*_{s,x}, S(\tau^*_{s,x})).$$

Hence

$$\mathrm{e}^{-r(t\wedge\tau^*_{s,x})}v(t\wedge\tau^*_{s,x}, S(t\wedge\tau^*_{s,x})) = \mathrm{e}^{-rs}v(s,S(s)) + \int_s^{t\wedge\tau^*_{s,x}} \mathrm{e}^{-rs'}\sigma(s')S\frac{\partial v}{\partial x}(s',S)dW^Q(s').$$

which together with Proposition 4.1 (see Example 27) implies that indeed the stopped process $\tilde{V}(t\wedge\tau^*_{s,x}) = \mathrm{e}^{-r(t\wedge\tau^*_{s,x})}v(t\wedge\tau^*_{s,x}, S(t\wedge\tau^*_{s,x}))$ is a Q-martingale.

For further reading on pricing and hedging American options, see e.g. Björk (2004); Karatzas and Shreve (1998); Lamberton and Lapeyre (2007); Shreve (2004).

4.6 Multi-asset models

In this section the market consists of d risky asset and one money market account. Let $W_i(t)$, $i = 1,\dots,q$, $0 \le t \le T$, be independent one-dimensional standard Wiener processes on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Here $\{\mathcal{F}_t\}$ is the natural filtration for the q-dimensional standard Wiener process $W(t) = (W_1(t),\dots,W_q(t))^\top$. Assume that the stock prices $S_i(t)$, $i = 1,\dots,d$, $0 \le t \le T$, are modelled by

$$dS_i = \mu_i(t)\,S_i dt + S_i\sum_{j=1}^q \sigma_{ij}(t)\,dW_j(t),\ \ i = 1,\dots,d, \tag{4.71}$$

where the rate of return vector $\mu(t) = (\mu_1(t),\dots,\mu_d(t))^\top$ and the volatility matrix $\sigma(t) = \{\sigma_{ij}(t), i = 1,\dots,d,\ j = 1,\dots,q\}$ are $\mathcal{F}_t$-adapted stochastic processes with 'good' properties.

Exercise 42 *Show that if $S_i(0) > 0$ then $S_i(t)$ from (4.71) are strictly positive for all $t > 0$.*

We use the same model for the money market account (the current account) as in the case of the single-asset market (cf. (4.3) and (4.4)):

$$B(t) = B(0)\exp\left(\int_0^t r(s)\,ds\right), \tag{4.72}$$

where $r(t)$ is a short rate, which is assumed to be $\mathcal{F}_t$-adapted.

Introduce the discounted stock price processes

$$\tilde{S}_i(t) = \frac{S_i(t)}{B(t)}, \tag{4.73}$$

which satisfy the system of SDEs

$$d\tilde{S}_i = \tilde{S}_i\left[(\mu_i(t) - r(t))\,dt + \sum_{j=1}^q \sigma_{ij}(t)\,dW_j(t)\right],\ \ i = 1,\dots,d. \tag{4.74}$$

[32] It follows from the natural fact (which we do not prove) that $v(t,x)$ is a smooth function.

Exercise 43 *Derive (4.74).*

What is an EMM in this case? It is defined analogously to the one asset case.

Definition 4.6 *A probability measure Q is* ***EMM*** *if Q and P are equivalent and under Q all the discounted prices $\tilde{S}_i$, $i = 1, \dots, d$, are martingales.*

We will now try to figure out when there is an EMM for the model (4.71)-(4.72) and when there could be no EMM.

Suppose there is 'good' $\mathcal{F}_t$-adapted q-dimensional process $\gamma(t) = (\gamma_1(t), \dots, \gamma_q(t))^\top$ such that we can re-write (4.74) in the form

$$d\tilde{S}_i = \tilde{S}_i \sum_{j=1}^{q} \sigma_{ij}(t) \, \left[\gamma_j(t)dt + dW_j(t)\right], \; i = 1, \dots, d. \tag{4.75}$$

Then by the Girsanov theorem (i.e., Theorem 3.3), there is an equivalent probability measure Q under which

$$W^Q(t) = W(t) + \int_0^t \gamma(s)ds \tag{4.76}$$

is a q-dimensional standard Wiener process. It is easy to see from (4.75)-(4.76) that under this measure Q the discounted price processes $\tilde{S}_i$, $i = 1, \dots, d$, satisfy the SDEs

$$d\tilde{S}_i = \tilde{S}_i \sum_{j=1}^{q} \sigma_{ij}(t) \, dW_j^Q(t), \; i = 1, \dots, d, \tag{4.77}$$

and $\tilde{S}_i(t)$ are Q-martingales. From the previous considerations, we know why it is important to look for martingales.

Now the question is when we can find such $\gamma(t)$. Comparing (4.74) and (4.75), we obtain that $\gamma(t)$ must satisfy the relationships

$$\mu_i(t) - r(t) = \sum_{j=1}^{q} \sigma_{ij}(t) \, \gamma_j(t), \; i = 1, \dots, d, \tag{4.78}$$

which are called *the market price of risk equations* and $\gamma(t)$ is called the *market price of risk vector*. On the left-hand side of (4.78) we have the excess of return over the risk free rate r for asset i. On the right-hand side we have a linear combination of the volatilities $\sigma_{ij}(t)$ of the asset i with respect to the individual risk factors W_j with factor loadings $\gamma_j(t)$, which explains why $\gamma(t)$ is called the market price of risk vector. Let us emphasise that $\gamma(t)$ is the same for all of the d risky assets.

In (4.78) we have d equations and q unknown processes $\gamma_j(t)$. If we cannot solve these equations then, based on our previous experience, we expect that there is an arbitrage in our market model and hence it is a wrong model. We will not prove this in the general case but will illustrate in a simplified case of two risky assets.

Example 29 *(when an EMM does not exist) Suppose in our market model (4.71)-(4.72) the number of risky assets $d = 2$ while $q = 1$. For simplicity assume that μ_i, σ_{i1}, $i = 1, 2$, and r are constant. It is natural to assume $\sigma_{i1} > 0$. Then the market price of risk equations (4.78) become*

$$\begin{aligned} \mu_1 - r &= \sigma_{11}\gamma_1, \\ \mu_2 - r &= \sigma_{21}\gamma_1, \end{aligned} \tag{4.79}$$

which has a solution only if

$$\frac{\mu_1 - r}{\sigma_{11}} = \frac{\mu_2 - r}{\sigma_{21}}.$$

Otherwise, the market price of risk equations do not have a solution and, as we show below, there is an arbitrage.

Indeed, assume that the market price of risk equations (4.79) do not have a solution and, for definiteness, let us consider the case

$$\frac{\mu_1 - r}{\sigma_{11}} > \frac{\mu_2 - r}{\sigma_{21}}$$

and introduce

$$\alpha = \frac{\mu_1 - r}{\sigma_{11}} - \frac{\mu_2 - r}{\sigma_{21}} > 0.$$

Consider a self-financing portfolio $\phi(t) = (\Delta_1(t), \Delta_2(t), \psi(t))$ consisting of $\Delta_i(t)$ amount of the corresponding assets and $\psi(t)$ is the amount of units on the current account at time t. The value $V(t)$ of this portfolio at time t is

$$V(t) = \Delta_1(t)S_1(t) + \Delta_2(t)S_2(t) + \psi(t)B(t). \tag{4.80}$$

Let us choose $\Delta_1(t) = 1/[S_1(t)\sigma_{11}]$ and $\Delta_2(t) = -1/[S_2(t)\sigma_{21}]$ and $\psi(0)$ so that $V(0) = 0$. We have

$$\begin{aligned} dV &= \Delta_1(t)dS_1 + \Delta_2(t)dS_2 + \psi(t)dB \\ &= \Delta_1(t)dS_1 + \Delta_2(t)dS_2 + \frac{V(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t)}{B(t)}dB \\ &= \Delta_1(t)dS_1 + \Delta_2(t)dS_2 + r\left[V(t) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t)\right]dt \\ &= \frac{\mu_1 - r}{\sigma_{11}}dt + dW - \frac{\mu_2 - r}{\sigma_{21}}dt - dW + rVdt \\ &= \alpha dt + rVdt. \end{aligned}$$

Then for the discounted portfolio $\tilde{V}(t) = V(t)/B(t) = D(t)V(t)$ (see (4.6) for $D(t)$), we have

$$d\tilde{V}(t) = D(t)\left[\alpha dt + rVdt - rVdt\right] = D(t)\alpha dt, \quad \tilde{V}(0) = 0. \tag{4.81}$$

Hence

$$\tilde{V}(t) = \frac{\alpha}{B(0)}\int_0^t \exp(-rs)\,ds = \frac{\alpha}{rB(0)}\left[1 - \exp(-rt)\right]$$

and

$$V(t) = \frac{\alpha}{r}\left[\exp(rt) - 1\right] \tag{4.82}$$

which is always positive for all $t > 0$. Therefore, there is a possibility of arbitrage in this market (see Definition 4.1).

If the market price of risk equations (4.78) do have a solution, then our market model has no arbitrage opportunities by essentially the same arguments as in Section 4.1. Consider (again) a self-financing $\mathcal{F}_t$-adapted portfolio $\phi(t) = (\Delta_i(t), i = 1, \dots, d; \psi(t))$ consisting of $\Delta_i(t)$ amount of the corresponding assets at time t and $\psi(t)$ is the amount of units on the current account at time t. Further, we require that the processes $\Delta_i(t)$ and $\psi(t)$ satisfy the following condition (cf. (4.8)):

$$E\int_0^T \left[\sum_{i=1}^d \Delta_i^2(t)S_i^2(t) + \psi^2(t)B^2(t)\right]dt < \infty. \tag{4.83}$$

As in Section 4.1, we consider such strategies $\phi(t)$ to be **admissible**[33].

It is not difficult to obtain for the discounted value of this portfolio

$$d\tilde{V}(t) = \sum_{i=1}^d \Delta_i(t)d\tilde{S}_i. \tag{4.84}$$

[33] You might notice that admissible strategies are defined in different books slightly differently. E.g., some require for the risk to be bounded by assuming that there is a constant C such that $V(t) \geq C$ for all $t \geq 0$ a.s. But we will not go for more complicated versions in this course.

Exercise 44 *Derive (4.84).*

Exercise 45 *Verify that the discounted portfolio value $\tilde{V}(t)$ is a Q-martingale.*

To conclude: if we assume that the market price of risk equations (4.78) have a solution, there is an EMM Q under which $\tilde{S}_i$ are martingales (see (4.77)), and, consequently, $\tilde{V}(t)$ is also a Q-martingale.

Since $\tilde{V}(t)$ is a Q-martingale, we can prove the following theorem by the same arguments as used in Example 23.

Theorem 4.1 ***(The first fundamental theorem of asset pricing)*** *If a market model has an EMM, then it is arbitrage free.*

Remark 4.7 *Theorem 4.1 gives the sufficient condition for absence of arbitrage in a market model. In the discrete setting (see FIM and the reminder in Section 2.1) existence of an EMM was also the necessary condition. In the continuous setting the necessary condition*[34] *is more complex and we do not include it here.*

Now, within our market model for which we assume that an EMM Q exists, we consider pricing and hedging of a derivative with payoff $H(T)$ which is an $\mathcal{F}_T$-measurable random variable. To this end, we generalise the arguments from Section 4.1. As before, our first aim is to find an initial value $V(0)$ of the writer of this option and his hedging strategy $\Delta(t)$ so that

$$V(T) = H(T) \text{ almost surely} \tag{4.85}$$

if it is possible, i.e., when the claim is attainable. Note that an important part of this exercise is to establish when any claim is attainable (recall that in the case of the simplified model of Section 4.1 every claim was always attainable).

Recall the definition of the discounted payoff

$$\tilde{H}(T) = \frac{H(T)}{B(T)}. \tag{4.86}$$

By the (multidimensional) martingale representation theorem (Theorem 3.1 and Remark 3.3), there is an $\mathcal{F}_s$-measurable q-dimensional random process $\theta(s)$ such that

$$\tilde{H}(T) = E_Q\tilde{H}(T) + \int_0^T \theta^\top(s)\, dW^Q(s). \tag{4.87}$$

It follows from (4.77), (4.84), (4.85), (4.86) and (4.87) that we should have

$$\tilde{V}(0) + \int_0^T \sum_{j=1}^q \sum_{i=1}^d \Delta_i(t)\tilde{S}_i\sigma_{ij}(t)\, dW_j^Q(t) = E_Q\tilde{H}(T) + \int_0^T \theta^\top(s)\, dW^Q(s) \quad \text{a.s.}, \tag{4.88}$$

which implies that the discounted initial portfolio value is

$$\tilde{V}(0) = E_Q\tilde{H}(T), \tag{4.89}$$

and hence the option's price at time $t = 0$ is

$$V(0) = E_Q\left[\exp\left(-\int_0^T r(s)\, ds\right) H(T)\right]. \tag{4.90}$$

Exercise 46 *Do the complete derivation of (4.88).*

[34] It was first proved in the landmark paper Delbaen and Schachermayer, 1994.

Further, since $\tilde{V}(t)$ is a Q-martingale, the option's price at any time $t \le T$:

$$V(t) = E_Q \left[\exp\left(-\int_t^T r(s)\, ds \right) H(T) \middle| \mathcal{F}_t \right]. \tag{4.91}$$

This is the ***derivative pricing formula***.

Exercise 47 *Derive (4.91).*

However, when do the above arguments work? It follows from (4.88) that for a hedging strategy to exist we should require (cf. (4.27)):

$$\theta_j(t)B(t) = \sum_{i=1}^{d} \Delta_i(t)S_i(t)\sigma_{ij}(t), \tag{4.92}$$

which are sometimes called *hedging equations*. We note that (4.92) has q equations in d unknown random processes $\Delta_i(t)$.

Suppose that (4.92) has a solution. In this case we can conclude that if the writer sells the option at time $t = 0$ for $V(0)$ from (4.90) and follows the **hedging** or **replicating** strategy $\Delta_i(t)$, $i = 1, \ldots, d$, from (4.92) for $0 \le t \le T$, then at the maturity time T he has the required amount of funds $H(T)$ so that he can pay the holder the required amount of money according to the contingent claim almost surely (in other words, with probability 1), i.e. he can perfectly replicate (hedge) the pay-off $H(T)$. Therefore, we can conclude that if (4.92) has a solution for any $\theta_j(t)$, then any claim is attainable (i.e., indeed, there is an initial capital $V(0)$ and a strategy $\Delta_i(t)$, $i = 1, \ldots, d$, so that the corresponding wealth process $V(t)$ takes the value at the maturity $V(T)$ so that (4.85) holds). So, if (4.92) has a solution for any $\theta_j(t)$ then the market is complete; otherwise it is incomplete.

Of course, we need to know when there are $\Delta_i(t)$, $i = 1, \ldots, d$, satisfying (4.92). The answer is given by the second fundamental theorem.

Theorem 4.2 ***(The second fundamental theorem of asset pricing)*** *Assume that the market model has an EMM and it is driven by Wiener processes only*[35]*. Then the market model is complete if and only if the EMM is unique.*

Proof. Let us first prove[36] that the EMM Q is unique on $\mathcal{F}_T = \mathcal{F}$ under the theorem's assumptions+that the market is complete. Take any event $A \in \mathcal{F}$ and a claim $f := B(T)I_A$. Assume that $B(0) = 1$. Since the market is assumed to be complete, there is a portfolio with its value process $V(t)$, which at time $t = 0$ is equal to $V(0)$, so that this claim is replicatable, i.e., $V(T) = f$. Suppose there are two EMMs, Q_1 and Q_2. Then, by (4.90) (also recall that a discounted portfolio value process $\tilde{V}(t)$ is a martingale under any EMM), this initial wealth $V(0)$ is calculated as

$$\begin{aligned} V(0) &= E_{Q_1} \left[\exp\left(-\int_0^T r(s)\, ds \right) f \right] = E_{Q_1} I_A = Q_1(A), \\ V(0) &= E_{Q_2} \left[\exp\left(-\int_0^T r(s)\, ds \right) f \right] = E_{Q_2} I_A = Q_2(A). \end{aligned}$$

Since A was an arbitrary event from $\mathcal{F}_T$, Q is uniquely defined on $\mathcal{F}_T$.

Now let us prove the opposite statement, i.e., that an arbitrage-free market with the unique EMM Q on $\mathcal{F}_T = \mathcal{F}$ is complete. Uniqueness of the EMM implies that the market price of risk equations (4.78) have a unique solution $\gamma_i(t)$, $i = 1, \ldots, q$. The market price of risk equations (4.78) can be written in the matrix form as

$$\sigma(t)\gamma(t) = \mu(t) - r(t)e, \tag{4.93}$$

[35] I.e. the filtration $\mathcal{F}_t$ is the natural filtration for the Wiener processes present in the model.

[36] The proof is similar to the one in the discrete case considered in FIM.

where $e = (1, 1, \dots, 1)^\top$. Since this system of linear equations has a unique solution for every t and ω, the $d \times q$-dimensional matrix $\sigma(t)$ must have[37] rank q. Consequently, $d \geq q$[38]. Next, the hedging equations (4.92) can be written in the matrix form as

$$\sigma^\top(t)y(t) = \theta(t)B(t), \tag{4.94}$$

where the unknown $y(t)$ is a random vector with components $\Delta_i(t)S_i(t)$. For the market to be complete, this system (4.94) must have a solution $y(t)$ for any vector $\theta(t)B(t)$ on its right hand side. If this is the case then there are deltas $\Delta_i(t) = y_i(t)/S_i(t)$ for any derivative. To see that (4.94) has a solution, we first observe that ranks of a matrix and its transpose are equal[39], hence $rank(\sigma) = rank(\sigma^\top) = q$. Then, by the Kronecker–Capelli theorem[40], the system (4.94) has a solution for any $\theta(t)B(t)$. Theorem 4.2 is proved.

Remark 4.8 *We saw in the proof of Theorem 4.2 that we should require $q \leq d$ for a market to be complete. Hence in a complete arbitrage-free market model the number of Wiener processes q cannot exceed the number of risky assets d.*

Exercise 48 *Confirm (4.94).*

4.7 Incomplete markets

Real markets are generically incomplete, the completeness is just an approximation of the reality. Pricing and hedging in incomplete markets is a difficult (and interesting!) problem. It follows from the second fundamental theorem of asset pricing that in incomplete market models there are many EMMs. However, as we show below, there is still a consistent arbitrage pricing rule.

Suppose the market model from the previous section is arbitrage free and incomplete. We are interested in finding a value $\pi(t)$ of an option with the claim $H(T) \in \mathcal{F}_T$ at the maturity time T. Observe that by the definition of $\pi(t)$:

$$\pi(T) = H(T).$$

Consider the enlarged market consisting of

- the stock prices $S_i(t)$, $i = 1, \dots, d$;
- money market account $B(t)$;
- $\pi(t)$, $t \leq T$, which is a price process for $H(T)$.

For this enlarged market to be arbitrage free, the discounted process

$$\tilde{\pi}(t) := \frac{\pi(t)}{B(t)}$$

should be a martingale under Q which is an EMM for $S_i(t)$, $i = 1, \dots, d$. Hence,

$$\tilde{\pi}(t) = E_Q\left[\tilde{H}(T)\middle|\mathcal{F}_t\right]$$

[37] It is a fact from Linear Algebra often called the Kronecker–Capelli or Rouché–Capelli Theorem (compatibility criterion). It says the following.

The system of linear equations $Ax = b$ with an $d \times q$-dimensional matrix A has a solution (i.e., it is consistent) if and only if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|b]$ obtained by appending the column of free terms b to the right of A. And if the system is consistent, then the number of degrees of freedom is equal to $q - rank(A)$, where q is the number of unknowns.

Note that when a solution is unique then the degree of freedom is zero.

[38] We know from Linear Algebra that the rank of an $d \times q$-dimensional matrix A is less than or equal to $\min(d, q)$.

[39] It is an obvious fact from Linear Algebra.

[40] Note that $\sigma^\top$ is a $q \times d$-dimensional matrix and the argumented matrix $[\sigma^\top|\theta(t)B(t)]$ is a $q \times (d+1)$-dimensional matrix.

and

$$\pi(t) = E_Q \left[\exp\left(-\int_t^T r(s)\, ds \right) H(T) \middle| \mathcal{F}_t \right]. \tag{4.95}$$

Note that the price $\pi(t)$ depends on Q and, when the market is incomplete, there are many EMMs and each of the EMMs may give a different price (4.95), and choosing the 'right' price (or what is the same, the 'right' measure Q) is the main difficulty in modelling incomplete markets. For further reading on incomplete markets, see e.g. Bingham and Kiesel (2004); Björk (2004).

Also, compare (4.95) with the pricing formula (4.91) we had in the complete market case. It is useful to revisit the Heston model (Section 4.4) to see how we dealt with the issue of incompleteness of the market there.

4.8 Currency options

Since the end of Bretton Woods agreement in 1970s central banks of Western countries and many development countries have allowed market forces to determine exchange rates for their currencies. In FIM you had an example of a forward on foreign currency and you observed the difference in dealing with foreign currencies and usual stock. Here we briefly look at the idea of pricing options on foreign currency (so called *Quanto options*). For further reading on this topic, see e.g. Björk (2004); Lipton (2001); Clark (2011); Musiela and Rutkowski (2005); Shreve (2004).

Consider a UK firm which buys some parts from US for its production cycle. Then the firm has to deal with two currencies: domestic (GBP) and foreign (USD). Let us denote by $X(t)$ the spot USD-GBP exchange rate at time t, which is quoted as

$$\frac{\text{units of domestic currency}}{\text{one unit of foreign currency}},$$

i.e., how much GBP one needs to pay for one USD in our example. We assume that our market has two riskless assets: one domestic money market account with short rate r_d and one foreign money market account with short rate r_f, which dynamics and which exchange rate X are described by the following differential equations

$$\begin{aligned} dB_d &= r_d B_d dt, \\ dB_f &= r_f B_f dt, \\ dX &= \mu X dt + \sigma X dW, \end{aligned} \tag{4.96}$$

where $r_d \geq 0$, $r_f \geq 0$, $\sigma > 0$ and μ are some constants and $W(t)$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ with P being the 'real' probability measure and $\mathcal{F}_t$ being the natural filtration for $W(t)$.

Consider now a European option written on the foreign currency. Payoffs of currency (path independent) options at the maturity time T can be written as $f(X(T))$, where $f(x)$ is a deterministic function. For instance, to protect its business against raise of USD against GBP, the UK firm can buy the corresponding call option with payoff $f(X(T)) = (X(T) - K)_+$, which gives the firm the right to buy one unit of the foreign currency (i.e. one USD in our example) at the price K in the domestic currency (i.e. GBP in our example).

The risky asset here is USD (the foreign currency) paid for by GBP (the domestic currency), i.e.,

$$Y(t) = X(t) B_f(t). \tag{4.97}$$

To price such options, we need to find an EMM Q, i.e. a measure under which the discounted price of our risky asset

$$\tilde{Y}(t) = \frac{Y(t)}{B_d(t)} \tag{4.98}$$

is a martingale.

By Ito's formula we have

$$d\tilde{Y} = (r_f - r_d + \mu)\,\tilde{Y}dt + \sigma\tilde{Y}dW. \tag{4.99}$$

Exercise 49 *Confirm (4.99).*

Using Girsanov's theorem, we find a unique EMM Q under which $\tilde{Y}$ is a martingale:

$$d\tilde{Y} = \sigma\tilde{Y}dW^Q. \tag{4.100}$$

So, the considered market is arbitrage free and complete.

Under this measure Q the equation for X from (4.96) takes the form

$$dX = (r_d - r_f)\,Xdt + \sigma XdW^Q. \tag{4.101}$$

Exercise 50 *Derive (4.101).*

Using the derivative pricing formula (4.91), we get the price of the quanto option

$$V(t) = e^{-r_d(T-t)}E_Q\left[f(X(T))|\mathcal{F}_t\right]. \tag{4.102}$$

Exercise 51 *Write the PDE problem which solutions gives us the price $V(t)$ of the currency option (4.102).*

Example 30 *(Garman-Kohlhagen formula) Consider a call option, i.e., $f(x) = (x-K)_+$ on \$1. To derive its price assuming the market model (4.96), we first recall that we know from FIM (see also (4.41)) that if*

$$d\mathbb{S} = r\mathbb{S}dt + \sigma\mathbb{S}dW^Q \tag{4.103}$$

and the price of a European option is given by

$$u(t,x) = e^{-r(T-t)}E_Q f(\mathbb{S}_{t,x}(T)), \tag{4.104}$$

then we have the analytical expression for this European call option's price:

$$C(t,x;\sigma,T,r) = x\Phi\left(\frac{\ln\frac{x}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\ln\frac{x}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right). \tag{4.105}$$

Comparing (4.102), (4.101) and (4.104), (4.103), we notice that the only difference is that we have $r_d - r_f$ in (4.101) instead of r in (4.103) and r_d in (4.102) instead of r in (4.104). Then substituting $r_d - r_f$ in (4.105) instead of r and multiplying the resulted formula by $e^{-r_f(T-t)}$, we arrive at the required Garman-Kohlhagen formula

$$\begin{aligned} v(t,x) &= e^{-r_d(T-t)}E_Q(X_{t,x}(T)-K)_+ \\ &= e^{-r_f(T-t)}x\Phi\left(\frac{\log(x/K) + \left(r_d - r_f + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad -e^{-r_d(T-t)}K\Phi\left(\frac{\log(x/K) + \left(r_d - r_f - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{4.106}$$

Exercise 52 *Let $Z(t) = 1/X(t)$, i.e. USD/GBP exchange rate. If $X(t)$ follows the SDE from (4.96), what is the SDE for $Z(t)$?*

Exercise 53 *The measure $Q = Q_d$ in (4.100) is the EMM for the domestic (i.e., GBP) market. Using Exercise 52 or otherwise, find an EMM Q_f for the foreign (i.e., USD) market. Do the measures Q_d and Q_f coincide? Discuss a consequence of your observation from the option pricing prospective.*