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## Abstract

The abstract of the report goes here. The abstract should state the topic(s) under investigation and the main results or conclusions. Methods or approaches should be stated if this is appropriate for the topic. The abstract should be self-contained, concise and clear. The typical length is one paragraph.

# Contents

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## 2.1 Preliminaries

A common problem in finance is to price financial derivatives, often referred just as derivatives. In essence, derivatives are contracts set between parties whose value in time derives from the price of their underlying assets. A notorious family of derivatives in financial markets are "options". Options are contracts set between two parties in which the holder has the right to sell or buy, commonly referred as exercise, an underlying stock at a preestablished price, also known as "strike price", in the future. Options are referred as "call options" or as "put options" if the exercise position is to buy or to sell, respectively. Similarly, options are classified depending on their exercise style. In that regard, the simplest of options are European options. European options give the right to exercise at the expiration date of the contract. Another well known type options are American options. American options work similar as European options with the difference that can be exercised at any point in time between the beginning and expiration date of the contract. Let us define the payoff as

$$H(S,t) = \max(S - K, 0) \tag{2.1a}$$

$$H(S,t) = \max(K - S, 0) \tag{2.1b}$$

where $K$ is the strike price, $S \in [0, \infty]$ is the stock price, and $t \in [0, T]$ is the current time. Note that $t$ is measure in years, $t = 0$ and $t = T$ denotes the beginning of the contract and the expiration date, and the region $[0, T]$ is the life span of the option. While an American option's payoff is defined for all $(S, t)$, European options' payoff is only defined at $t = T$.

Obviously, options give greater flexibility to holders by removing their exposure of a negative payoff which is why writers of the option charge a premium to the buyers at the time they enter the contract. The premium is often referred as the price or value of the option and the problem of finding this value is called option pricing. When pricing options, it is important to find the just price because otherwise the writer or buyer of the option could set some scheme in which option will always be profitable to them. In other words, options pricing must follow the principle of no-arbitrage. Therefore, we assume that the writer of the option uses the premium to construct a portfolio consisting of $\phi_0$ units of the stock and invests $\psi_0$ units of cash into a risk-free asset such as US treasury bill, certificate of deposit, or bank account. Then, the writer rebalanced the portfolio $(\phi_0, \psi_0)$ to hedge any possible claims from the buyer of the option at any future time $0 < t \leq T$. Therefore, at any time $t$, the writer holds a portfolio $(\phi(t), \psi(t))$ with value

$$\Pi(t) = \phi(t)S(t) + \psi(t)B(t) \tag{2.2}$$

Moreover, the portfolio is self-financing. In other words, the changes in the value of the portfolio $V(t)$ depend on the changes in $S(t)$ and $B(t)$, and the current portfolio $(\phi(t), \psi(t))$

$$d\Pi(t) = \phi(t)dS(t) + \psi(t)dB(t) \tag{2.3}$$

Finally, the value of an option must satisfy the following

$$\Pi(t) = V(t) \tag{2.4}$$

at any time $0 \leq t \leq T$.

The Black-Scholes model is built upon the self-financing portfolio hedging strategy and expresses a mathematical model for dynamics of option's price. The Black-Scholes model makes some assumptions about the market. For a complete list of these assumptions look at (reference). We enumerate the one we believe are important for our task. First, the stock price $S(t)$ is a log-normal random variable

$$dS = r(t)Sdt + \sigma(t)SdW \tag{2.5}$$

where the risk-free interest $r(t)$ and the price volatility $\sigma(t)$ are deterministic functions of time during the life of the option. Secondly, the bank account $B(t)$ is a deterministic function of time

$$dB = r(t)B(t)dt \tag{2.6}$$

Finally, the stock does not pay dividends. From now on, we will assume that the risk-free interest rate and stock price volatility are constant during the life of the option. Later on, we will address the assumption about dividends.

By applying the Black-Scholes model to price European options, the famous Black-Scholes PDE is obtained

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0 & \text{for } t \in [0, T) \text{ and } S \in [0, \infty) \\ V(S,T) = H(S,T) & \text{for } S \in [0, \infty) \end{cases} \tag{2.7}$$

where $V(S,t)$ is a deterministic function. For the derivation of (2.7), we reference to REFERENCE.

We previously mention that the one of the assumptions of the Black-Scholes model is that the underlying stock does not pay dividends. In most cases, assets such as stock pay out dividends just a few times at year. Therefore, dividends are to be modelled discretely. However, there are certain assets that pay out a proportion of the current asset price during and interval of time. Thus, in such cases, it is useful to model dividends as a continuous yield. By arbitrage arguments [REFERENCES], it can be shown that the asset price with volatility $\sigma(t)$, rate of return $r(t)$, and continuous dividend yield $\delta(S,t)$ paid at instant of time $dt$ is modeled as

$$dS = (r(t) - \delta(S,t))Sdt + \sigma(t)SdW \tag{2.8}$$

Similarly to as we did for the risk-free interest rate and the volatility of the asset price, we will assume that continuous dividends yield is as constant from now on. [REFERENCES] show that applying the Black-Scholes model under the price model (2.8) to price European options, we obtain the slightly modified version of (2.7)

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 & \text{for } t \in [0,T) \text{ and } S \in [0,\infty) \\
V(S,T) = H(S,T) & \text{for } S \in [0,\infty)
\end{cases}
\tag{2.9}
$$

Similarly, the Black-Scholes model is applied to price American options. One important result is that the value of an American option $V_{\text{Ame}}(t)$ is bounded from below by the payoff function

$$
V_{\text{Am}}(S,t) \geq H(S,t) \qquad \text{for } t \in [0,T] \tag{2.10}
$$

Moreover, the domain of $V(S,t)$ can be separated in the exercise region

$$
\mathcal{S} := \{(S,t) : V(S,t) = H(S,t)\} \tag{2.11}
$$

and the continuation region

$$
\mathcal{C} := \{(S,t) : V(S,t) > H(S,t)\} \tag{2.12}
$$

where the boundary of $\mathcal{C}$ is defined as

$$
\partial\mathcal{C} := \{(S,t) : S = \bar{S}(t)\} \tag{2.13}
$$

Lastly, the price dynamics of American options behaves as European options within the continuation region. Since we know $V(S,t)$ at the stopping region, we only need to solve $V(S,t)$ at continuation region and determine its boundary $\partial$ at the same time. Therefore, this is known as the free boundary problem formulation of the American option pricing problem and is equivalent to solve.

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 & \text{for } (S,t) \in \mathcal{C} \\
V(S,t) = H(S,t) & \text{for } (S,t) \in \partial\mathcal{C}
\end{cases}
\tag{2.14}
$$

Since the holder of an American option would always at the expiration date $T$ exercise the if it is profitable or would not make a profit otherwise. Therefore, American options value is equal to its payoff at the maturity date. With this information, we could establish terminal conditions for the free boundary problem in (2.14).

$$
\begin{aligned}
&V(S,T) = H(S,T)\\
&\bar{S}(T) = K
\end{aligned}
$$

Next, we need to establish boundary conditions for the system (2.14). Generally, when pricing options, we would need two boundaries conditions for option pricing. However, the free boundary problem in (2.14) is expressed in terms on the moving boundary condition $\bar{S}(t)$. Therefore, we only need to determine one extra boundary condition. In case of American put options, the left boundary condition would be given by $\bar{S}(t)$, and the right boundary condition by $V(S,t) = 0$ for a sufficiently large $S$. Analogously, $\bar{S}(t)$ would be the right boundary condition for an American call option and its left boundary condition by $V(S,t) = 0$ for $S = 0$. Finally, one extra information is that at the point $(\bar{S}(t), t)$, the derivative of the value with respect to the price is given by:

$$\frac{\partial V}{\partial S}(\bar{S}(t), t) = 1 \qquad \text{(call)} \tag{2.15a}$$

$$\frac{\partial V}{\partial S}(\bar{S}(t), t) = -1 \qquad \text{(put)} \tag{2.15b}$$

Grouping , boundary conditions and terminal condition in one equation, we obtain the system.

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 v}{\partial x^2} + (r-\delta)S\dfrac{\partial V}{\partial S} - rV = 0 \quad \text{for } 0 \le S < \bar{S}(t) \text{ and } 0 \le t < T\\
V(S,T) = S - K \qquad \bar{S}(T) = K\\
V(0,t) = 0 \quad \dfrac{\partial V}{\partial S}(\bar{S}(t), t) = 1
\end{cases} \tag{2.16a}
$$

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 v}{\partial x^2} + (r-\delta)S\dfrac{\partial V}{\partial S} - rV = 0 \quad \text{for } \bar{S}(t) < S < \infty \text{ and } 0 \le t < T \\ V(S,T) = K - S \qquad \bar{S}(T) = K \\ \lim_{S\to\infty} V(S,t) = 0 \quad \dfrac{\partial V}{\partial S}(\bar{S}(t),t) = -1 \end{cases} \tag{2.16b}$$

## 2.2 Front-Fixing method

In the previous section, we presented the pricing of American options problem. By applying the Black-Scholes model, we derived the Black-Scholes PDE that describes the price dynamics in the continuation region $\mathcal{C}$ of call and put options. Moreover, we presented the moving boundary condition $\bar{S}(t)$ for this PDE. The moving boundary condition $\bar{S}(t)$ makes the Black-Scholes PDE more involved since we also need to determine this boundary as time changes. This type of problem are known as free boundary problems. The front fixing method was first introduced by [2] and is a strategy in which we define a map from the original domain to new domain where moving boundary remains constant as time changes. In this section, we explore two transformation based on the work of Nielsen and others [3], and the work of Company and others [1].

### 2.2.1 Inverse transformation

This method proposes the transformation

$$x = \frac{S}{\bar{S}(t)} \tag{2.17}$$

which maps the boundary of the continuation region $\partial C$ defined in (2.13) to the fixed boundary

$$\partial\mathcal{C}_x := \{(x,t) : x = 1\} \tag{2.18}$$

which remain constant as t changes. Now, let us define the value $v(x,t)$ under this new map

$$v(x,t) := V(S,t) \tag{2.19}$$

which fixes the moving boundary $\bar{S}(t)$ at $x = 1$ when $S(t)$. Next, we compute the partial derivatives of $V$ with respect of the partial derivatives of $v$ which will allow us to rewrite the PDE in (2.16) with respect of (2.19)

$$\frac{\partial x}{\partial S} = \frac{1}{\bar{S}(t)}$$

$$\frac{\partial x}{\partial t} = -x\frac{\bar{S}'(t)}{\bar{S}(t)}$$

$$\frac{\partial V}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{\bar{S}(t)}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{\bar{S}(t)}\frac{\partial x}{\partial S}\frac{\partial^2 v}{\partial x^2} = \frac{1}{\bar{S}(t)^2}\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial V}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial v}{\partial t} - x\frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x}$$

Substituting these partial derivatives in the Black-Scholes PDE given by (2.16), we obtain the non-linear PDE

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \frac{\bar{S}'(t)}{\bar{S}(t)}\right]x\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x \in [0,1) \text{ and } t \in [0,T) \tag{2.20a}$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \frac{\bar{S}'(t)}{\bar{S}(t)}\right]x\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > 1 \text{ and } t \in (0,T] \tag{2.20b}$$

Likewise, we rewrite the terminal condition as

$$v(x,T) = \max(x\bar{S}(T) - K) = K\max(x-1,0) = 0 \tag{2.21a}$$

$$v(x,T) = \max(K - x\bar{S}(T)) = K\max(1-x,0) = 0 \tag{2.21b}$$

Note that $x$ is always less than one for put options. Analogously, $x$ is always greater than one for call options. Finally, we rewrite the boundary condition given by the optimal exercise price $\bar{S}(t)$ as

$$\frac{\partial v}{\partial x}(1,t) = 1 \tag{2.22a}$$

$$\frac{\partial v}{\partial x}(1,t) = -1 \tag{2.22b}$$

and the boundary condition opposite to the optimal exercise price as

$$v(0,t) = 0 \tag{2.23}$$

for both put and call options. In summary, by groping equations (2.20), (2.21), (2.22), and (2.22), we obtain the system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\right] x \dfrac{\partial v}{\partial x} - rv = 0 \quad \text{for } x \in (0,1) \text{ and } t \in [0,T] \\ v(x,T) = 0 \qquad \bar{S}(T) = K \\ v(0,t) = 0 \quad \dfrac{\partial v}{\partial x}(1,t) = \bar{S}(t) \end{cases} \tag{2.24a}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\right] x \dfrac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > 1 \text{ and } t \in [0,T] \\ v(x,T) = 0 \qquad \bar{S}(T) = K \\ \lim_{x\to\infty} v(x,t) = 0 \quad \dfrac{\partial v}{\partial x}(1,t) = -\bar{S}(t) \end{cases} \tag{2.24b}$$

### 2.2.2 Log transformation

In this section, we define the transformations

$$x := \log \frac{KS}{\bar{S}(t)} \qquad v(x,t) := \frac{V(S,t)}{K} \tag{2.25}$$

which maps the boundary $\partial\mathcal{C}$ to the region

$$\partial\mathcal{C}_x := \{(x,t) : x = \log K\} \tag{2.26}$$

Similarly to the previous section, we compute the partial derivatives of $v(x,t)$,

$$\frac{\partial x}{\partial t} = -\frac{\bar{S}'(t)}{\bar{S}(t)}$$

$$\frac{\partial x}{\partial S} = \frac{1}{S}$$

$$\frac{\partial V}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial^2 S} = \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2} - \frac{K}{S^2}\frac{\partial v}{\partial x}$$

$$\frac{\partial V}{\partial t} = K\frac{\partial v}{\partial t} - K\frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x}$$

Using the partial derivates, we rewrite the Black-Scholes PDE as follows

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \left((r-\delta) - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x} - \frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x < \log K \text{ and } t \in [0,T) \tag{2.27a}$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} + \left((r-\delta) - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x} - \frac{\bar{S}'(t)}{\bar{S}(t)}\frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > \log K \text{ and } t \in [0,T) \tag{2.27b}$$

with terminal condition

$$v(x,T) = \max\left(\frac{e^x}{K} - 1, 0\right) = 0 \tag{2.28a}$$

$$v(x,T) = \max\left(1 - e^x, 0\right) = 0 \tag{2.28b}$$

Next, we rewrite the boundary condition given by the optimal exercise price

$$\frac{\partial v}{\partial x}(\log K, t) = \frac{\bar{S}(t)}{K} \tag{2.29a}$$

$$\frac{\partial v}{\partial x}(\log K, t) = -\frac{\bar{S}(t)}{K} \tag{2.29b}$$

and the boundary condition opposite to the optimal exercise price

$$\lim_{x \to -\infty} v(x, t) = 0 \tag{2.30a}$$

$$\lim_{x \to \infty} v(x, t) = 0 \tag{2.30b}$$

Finally, grouping equations together, we have the system

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial x^2} + \left((r - \delta) - \dfrac{\sigma^2}{2}\right)\dfrac{\partial v}{\partial x} - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\dfrac{\partial v}{\partial x} - rv = 0 \quad \text{for } x < \log K \text{ and } t \in [0, T) \\ v(x, T) = 0 \qquad \bar{S}(T) = K \\ \lim_{x \to -\infty} v(x, t) = 0 \quad \dfrac{\partial v}{\partial x}(\log K, t) = \dfrac{\bar{S}(t)}{K} \end{cases} \tag{2.31a}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial x^2} + \left((r - \delta) - \dfrac{\sigma^2}{2}\right)\dfrac{\partial v}{\partial x} - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\dfrac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > \log K \text{ and } t \in [0, T) \\ v(x, T) = 0 \qquad \bar{S}(T) = K \\ \lim_{x \to \infty} v(x, t) = 0 \quad \dfrac{\partial v}{\partial x}(\log K, t) = -\dfrac{\bar{S}(t)}{K} \end{cases} \tag{2.31b}$$

# 3 Finite Difference schemes

The Black-Scholes PDE can be transformed to heat diffusion PDE using the following change of variables

$$
\begin{aligned}
S &= Ke^x \\
t &= T - \frac{2\tau}{\sigma^2} \\
q &:= \frac{2r}{\sigma^2} \\
q_\delta &:= \frac{2(r-\delta)}{\sigma^2} \\
\alpha &:= \frac{1}{2}(q_\delta - 1) \\
\beta &:= \frac{1}{4}(q_\delta - 1)^2 + q \\
v(x,\tau) &:= e^{-(\alpha x + \beta\tau)} y(x,\tau) = V(S,t)
\end{aligned}
$$

The system (**??**) is the free boundary formulation for the pricing problem for American options. A detailed derivation of (**??**) can be found at [REFERENCES].

The equation (**??**) is a paraboblic PDE. Moreover, by applying the transformation,

the equation (**??**) converts to the heat diffusion PDE.

$$
h(x,\tau) := \frac{H(S,t)}{K} = \begin{cases} \max(e^x - 1, 0) \\ \max(1 - e^x, 0) \end{cases} \tag{3.1}
$$

$$
\bar{x}(\tau) := \log \bar{S}(t) - \log K \tag{3.2}
$$

$$
\begin{cases}
\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} & \text{for } \tau \in [0, \frac{\sigma^2}{2}T) \text{ and } x \in (\bar{x}(t), \infty) \\
y(x,\tau) = e^{(\alpha x + \beta\tau)} h(x,\tau) & \text{for } \tau \in [0, \frac{\sigma^2}{2}T] \text{ and } x \in (-\infty, \bar{x}(\tau)] \\
\bar{x}(0) = 0
\end{cases} \tag{3.3}
$$

We can reformulate equation (3.3) as:

$$g := e^{\alpha x+\beta\tau}h(x,\tau) \tag{3.4}$$

$$\begin{cases} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g) = 0 \\ \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0 \quad y-g \geq 0 \\ y(x,0) = g(x,0) \end{cases} \tag{3.5}$$

By exploring the geometric properties of the value function $V(S,t)$, we can determine useful conditions that will later help on in solving the equation (**??**). Firstly, at any given time $0 \leq t \leq T$, American options match the linear segment of the payoff function within the stopping region. Therefore, we could say that

$$\frac{\partial V}{\partial S}(S,t) = \begin{cases} -1 & \text{(put)} \\ 1 & \text{(call)} \end{cases} \tag{3.6}$$

Moreover, as the price goes to infinity the value of the option tends to zero

$$\lim_{S\to\infty} V(S,t) = 0 \tag{3.7}$$

Pricing American options requires using numerical methods. The Black-Scholes PDE in (XXX) can be converted to the heat diffusion equation

Therefore, we focus on analyzing the numerical solution of the heat diffusion equation. Suppose the equation (XXX) is defined within the rectangular region $[x_{\text{min}}, x_{\text{max}}] \times [\tau_{\text{min}}, \tau_{\text{max}}]$. By discretizing uniformly along the spatial direction $x$ and temporal direction $\tau$,

$$M := \frac{x_{\text{max}} - x_{\text{min}}}{\Delta x} \tag{3.8}$$

$$N := \frac{\tau_{\text{max}} - \tau_{\text{min}}}{\Delta \tau} \tag{3.9}$$

$$x_i := x_{\text{min}} + i\Delta x \qquad \text{for } i = 0, \ldots, M \tag{3.10}$$

$$\tau_i := \tau_{\text{min}} + i\Delta \tau \qquad \text{for } i = 0, \ldots, N \tag{3.11}$$

where $\Delta x$ and $\Delta \tau$ are the distance between two consecutive points, then, the grid is defined as the discrete region.

$$\mathcal{G} := \{(x_i, \tau_j) : (i, j) \in \{0, \ldots, M\} \times \{0, \ldots, N\}\} \tag{3.12}$$

Thus, solving numerically the equation (XXX) means finding an approximation for $y(x_i, \tau_n)$ at every $(x_i, \tau_n)$ within the grid $\mathcal{G}$,

$$y_i^n \approx y(x_i, \tau_n) \tag{3.13}$$

To obtain such approximation, we rely on central difference approximations (REFERENCE).

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} + O(h) \tag{3.14}$$

$$f'(x_i) = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2) \tag{3.15}$$

$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O(h^2) \tag{3.16}$$

## 3.1 Explicit scheme

An explicit scheme is one where we approximate the time partial derivative using a forward difference approximation. Hence, the PDE in (XXX) is approximated as

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} = \frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{(\Delta x)^2} \tag{3.17}$$

By rearranging the terms,

$$\lambda := \frac{\Delta\tau}{(\Delta x)^2} \tag{3.18}$$

$$y_i^{n+1} = \lambda y_{i-1}^n + (1 - 2\lambda) y_i^n + \lambda y_{i+1}^n \tag{3.19}$$

It is shown by reference [REFERENCE] that method (XXX) is stable and consistent under the following condition

$$0 < \Delta\tau \leq \frac{(\Delta x)^2}{2} \tag{3.20}$$

Moreover, the method has order of convergence $O(\Delta\tau, (\Delta x)^2)$.

## 3.2 Implicit scheme

The implicit scheme approximates the time derivative using a backward difference

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} = \frac{y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}}{(\Delta x)^2} \tag{3.21}$$

$$y_i^{n+1} - \lambda(y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}) = y_i^n \tag{3.22}$$

$$K := \begin{bmatrix} 2 & -1 & & 0 & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & \end{bmatrix} \tag{3.23}$$

$$(I + \lambda K)\boldsymbol{y}^{n+1} = \boldsymbol{y}^n \tag{3.24}$$

## 3.3 Theta method

$$\frac{y_i^{n+1} - y_i^n}{\Delta\tau} = (1-\theta)\frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{(\Delta x)^2} + \theta\frac{y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}}{(\Delta x)^2} \tag{3.25}$$

$$y_i^{n+1} - \lambda\theta(y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}) = y_i^n + (1-\theta)\lambda(y_{i-1}^n - 2y_i^n + y_{i+1}^n) \tag{3.26}$$

$$(1 + \lambda\theta K)\boldsymbol{y}^{n+1} = (1 - \lambda\theta K)\boldsymbol{y}^n \tag{3.27}$$

# 4 Numerical results

# 5 Linear complementary problem

Now, note that the bound of $V(S,t)$ in each region is

$$
\begin{aligned}
V(S,t) - H(S,t) &> 0 \qquad \text{for all } (S,t) \in \mathcal{C} \\
V(S,t) - H(S,t) &= 0 \qquad \text{for all } (S,t) \in \mathcal{S}
\end{aligned}
$$

Similarly, the bound of $\mathcal{L}_{\mathsf{BS}}(V)$ is

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV &= 0 \qquad \text{for } (S,t) \in \mathcal{C} \\
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV &< 0 \qquad \text{for } (S,t) \in \mathcal{S}
\end{aligned}
$$

Therefore, grouping the bounds above we form a linear complementary system of equations

$$
\begin{cases}
\left[\frac{\partial V}{\partial t} - \mathcal{L}_{\mathsf{BS}}(V)\right] \cdot [V(S,t) - H(S,t)] = 0 & \text{for all } (S,t) \\
V(S,t) - H(S,t) \geq 0 & \text{for all } (S,t) \\
\frac{\partial V}{\partial t} - \mathcal{L}_{\mathsf{BS}}(V) \leq 0 & \text{for all } (S,t) \\
V(S,T) = H(S,T)
\end{cases}
\tag{5.1}
$$

The benefit of the reformulation (5.1) is that there is no dependence on the unknown boundary of the continuation region. Later in section (XXX) and section (XXX), we will explore numerical methods for solving both type of problems.

### 5.0.1 Theta method

We discretize the system of equation containing (3.17) and (3.18) to solve it. Firstly, we define an uniform meshgrid within the region $[1, x_{\mathsf{max}}] \times [0, T]$ and with resolution $\Delta x$ and $\Delta t$.

$$M := \frac{x_{\mathsf{max}} - 1}{\Delta x} \qquad N := \frac{T}{\Delta t}$$

$$x_i := 1 + i\Delta x \qquad \text{for } i = 0, \ldots, M$$
$$t_n := n\Delta t \qquad \text{for } n = 0, \ldots, N$$

Now we define the approximations

$$v_i^n \approx v(x_i, t_n) \qquad \text{for } (x_i, t_n) \in \{x_k\}_0^M \times \{t_k\}_0^N \tag{5.2}$$

$$\bar{S}^n \approx \bar{S}(t_n) \qquad \text{for } t_n \in \{t_k\}_0^N \tag{5.3}$$

By the boundary conditions, we can derive an expression for

$$v_0^n = K - \bar{S}^n \qquad \text{for } n = 0, \ldots, N-1, N \tag{5.4}$$

$$v_{M+1}^n = 0 \qquad \text{for } n = 0, \ldots, N-1, N \tag{5.5}$$

Additionally by using the smothness condition, we get:

$$\frac{v_1^n - v_0^n}{\Delta x} = -\bar{S}^n \tag{5.6}$$

or

$$v_1^n = K - (1 + \Delta x)\bar{S}^n \tag{5.7}$$

Next, we equation (3.XX) discretize using centered finite difference. The discretization method, we use is the theta method which a interpolation between an implicit and explicit scheme.

$$
\begin{aligned}
&v_i^{t+1} - v_i^t + \theta\Bigg\{\frac{1}{2}\sigma^2 x_i^2 \frac{\Delta t}{(\Delta x)^2}(v_{i-1}^t - 2v_i^t + v_{i+1}^t) + \Bigg[(r-\delta) - \frac{1}{\bar{S}^t}\frac{\bar{S}^{t+1}-\bar{S}^t}{\Delta t}\Bigg]\frac{\Delta t}{2\Delta x}(v_{i+1}^t - v_{i-1}^t) - r v_i^t \Delta t\Bigg\} \\
&\quad + (1-\theta)\Bigg\{\frac{1}{2}\sigma^2 x_i^2 \frac{\Delta t}{(\Delta x)^2}(v_{i-1}^{t+1} - 2v_i^{t+1} + v_{i+1}^{t+1}) \\
&\quad + \Bigg[(r-\delta) - \frac{1}{\bar{S}^{t+1}}\frac{\bar{S}^{t+1}-\bar{S}^t}{\Delta t}\Bigg]\frac{\Delta t}{2\Delta x}(v_{i+1}^{t+1} - v_{i-1}^{t+1}) - r v_i^{t+1}\Delta t\Bigg\} = 0
\end{aligned}
\tag{5.8}
$$

To simplify the expression above, we introduce the following terms

$$\lambda := \frac{\Delta t}{(\Delta x)^2} \tag{5.9}$$

$$\alpha_i := 1 + \theta(\lambda\sigma^2 x_i^2 + r\Delta t) \tag{5.10}$$

$$\beta_i := -\frac{1}{2}\lambda\theta\Big[\sigma^2 x_i^2 - x_i\Delta x(r-\delta)\Big] - \frac{1}{2}\lambda\theta x_i \Delta x \frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t \bar{S}^n} \tag{5.11}$$

$$\gamma_i := -\frac{1}{2}\lambda\theta\Big[\sigma^2 x_i^2 + x_i\Delta x(r-\delta)\Big] + \frac{1}{2}\lambda\theta x_i \Delta x \frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t \bar{S}^n} \tag{5.12}$$

$$a_i := 1 - (1-\theta)(\lambda\sigma^2 x_i^2 + r\Delta t) \tag{5.13}$$

$$b_i := \frac{1}{2}(1-\theta)\lambda\Bigg[\sigma^2 x_i^2 - x_i\Delta x\Bigg((r-\delta) - \frac{1}{\Delta t}\Bigg)\Bigg] \tag{5.14}$$

$$c_i := \frac{1}{2}(1-\theta)\lambda\Bigg[\sigma^2 x_i^2 + x_i\Delta x\Bigg((r-\delta) - \frac{1}{\Delta t}\Bigg)\Bigg] \tag{5.15}$$

$$d_i^{n+1} := (1-\theta)\frac{x_i}{2\Delta x}\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}} \tag{5.16}$$

Now, the expression above becomes

$$\beta_i^n v_{i-1}^n + \alpha_i^n v_i^n + \gamma_i^n v_{i+1}^n = b_i v_{i-1}^{n+1} + a_i v_i^{n+1} + c_i v_{i+1}^{n+1} + d_i^{n+1}\bar{S}^n \tag{5.17}$$

$$\gamma_1^n v_2^n = b_1 v_0^{n+1} + a_1 v_1^{n+1} + c_1 v_2^{n+1} + d_1^{n+1}\bar{S}^n - \beta_1^n(K - \bar{S}^n) - \alpha_1^n(K - (1+\Delta x)\bar{S}^n) \tag{5.18}$$

$$\alpha_2^n v_2^n + \gamma_2^n v_3^n = b_2 v_1^{n+1} + a_2 v_2^{n+1} + c_2 v_3^{n+1} + d_2^{n+1}\bar{S}^n - \beta_2^n(K - (1+\Delta x)\bar{S}^n) \tag{5.19}$$

$$\beta_M^n v_{M-1}^n + \alpha_M^n v_M^n = b_i v_{i-1}^{n+1} + a_i v_i^{n+1} + c_i v_{i+1}^{n+1} + d_i^{n+1}\bar{S}^n \tag{5.20}$$

# 6 Conclusion

# 7 Conclusions

## References

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