Numerical solutions to the free boundary problem and the linear complementary problem arisen from American options.

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Abstract

The abstract of the report goes here. The abstract should state the topic(s) under investigation and the main results or conclusions. Methods or approaches should be stated if this is appropriate for the topic. The abstract should be self-contained, concise and clear. The typical length is one paragraph.

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1 Introduction

1.1 Overview

Options are equity-based derivates which are mainly used to mitigate risk. The option market is significantly big compared to other derivatives. In fact, options were the most traded derivates in 2019 with a volume of 18.55 billion contracts when combining index and individual equities contracts [6]. An enormous market, such as the option market, demands for reliable pricing mechanisms, so that arbitrage opportunities are minimized. Naturally, pricing American options is a wide research area because there is not a closed-form solution to PDE resulting from Black-Scholes model. In this work, we consider the free boundary and the linear complementary formulations of the pricing problem for American options. To solve the free boundary formulation, we apply the front fixing method along with two transformations that allow to fix the moving boundaries in the original problem. Alternatively, to solve the linear complementary problem, we apply the PSOR method. Moreover, for both the front-fixing and PSOR method, we present explicit and implicit central finite difference schemes. Finally, we conduct numerical experiments that allow us to compare the trade-offs and properties of each of the methods.

1.2 Brief history

Merton [9] was the first to consider the Black-Scholes model proposed by Black, and Scholes [1] to price European and American options. Although, in his work, Merton derived a nice formula to price European options, he stated that, in general, a closed-from solution was not attainable for American options. In 1977, Schwartz [11] and [2] proposed using finite difference schemes to solve the price problem for American options. The work of Merton and Schwartz served as foundation for the free boundary problem for American options pricing. Based on the work of Landau [8], Wu et al. [14] formulated the front fixing mention as an approach to solve the free boundary problem for options in which the Landau transformation is used to transform the moving boundary to fixed boundary. Since, multiple transformation has been proposed by Huang et al. [7], Nielsen et al. [10], and Company et al. [3]. During the same period, Dewynne et al. [5] proposed the linear complementary formulation of the pricing problem and introduced the PSOR method.

1.3 Motivation

In their work, Company et al. [3] and Nielsen et al. [10] proposed changes of variables for solving the free boundary problems for American put options with non-paying dividends underlies. As part of our work, we consider a price model in which the assets have continuous dividends yield. Such assets are found in index equity derivates. Consequently, we present methods for solving the price problem for both put and call options. Moreover, Company et al. [3] proposed an explicit finite difference scheme. Therefore, we formulate an implicit finite difference scheme based on the transformation proposed in [3]. Additionally, we solve the linear complementary formulation for the pricing problem proposed in [12] and [13]. Similarly, we consider a price model in which assets pay out dividends.

1.4 Aim

The goal of this work is to implement the numerical methods proposed by Nielsen, et al. [10], Company, et al. [3] and Seydel [12] to price American options for underlying assets that pay out with dividends. Additionally, we compare how well these methods approximate the price using as a reference the binomial model which is a widely used method for pricing American options in the industry. Finally, we explore the trade-off in terms of CPU time and memory, and do convergence analysis for each method.

1.5 Main results

1.6 Outline

The outline of this paper is the following. In section 2, we explore the Black-Scholes model for American options which results in the free boundary formulation of the pricing problem. Moreover, we explore the front fixing method as a strategy for fixing the moving boundary by applying a change of variable. We considered the changes of variable proposed by Nielsen, et al. [10] and Company, et al. [3]. In section 3, we explore an explicit and implicit schemes to solve partial differential equations that result from applying the front fixing method to the free boundary problem. In section 4, we discuss the results obtained from pricing American options using explicit and implicit methods given in section 3 and appendix 1. In section 5, we explore a reformulation of the pricing problem as the obstacle problem and the linear complementary system of equation resulted from it. Moreover, we explore the theta method that combines both explicit and implicit numerical scheme to solve obstacle problem. Finally, in the same section, we discuss the results of pricing American options using the theta method to solve the linear complementary problem.

2 Black-Scholes model

2.1 Preliminaries

A common problem in finance is pricing financial derivatives, often referred to simply as derivatives. In essence, derivatives are contracts set between parties whose value over time derives from the price of their underlying assets. A notorious family of derivatives in financial markets are "options". Options are contracts set between two parties in which the holder has the right to sell or buy, commonly referred to as exercising, an underlying asset at a pre-established price, also known as the "strike price", in the future. Options are referred to as "call options" or "put options" if the exercise position is to buy or to sell, respectively. Similarly, options are classified depending on their exercise style. In that regard, the simplest options are European options. European options give the right to exercise at the expiration date of the contract. Another well-known type of option is the American option. American options work similarly to European options, with the difference that they can be exercised at any point in time between the beginning and the expiration date of the contract. Obviously, American and European options are almost similar, and they only differ in at the times which the holder can exercise them. Therefore, we will start describing the pricing problem from the European options perspective and extended to the case of American options.
Let us define the payoff function of European option as

$$\textbf{Call:} \qquad H_{\text{Eur}}(S) = \max(S - K, 0) \tag{2.1a}$$

$$\textbf{Put:} \qquad H_{\text{Eur}}(S) = \max(K - S, 0) \tag{2.1b}$$

where K is the strike price and remains constant through lifespan of the option, $S \in [0, \infty]$ is the asset price at the maturity date. We can extend the European payoff to American options by introducing the time axis to equation above.

$$\textbf{Call:} \qquad H(S, t) = \max(S - K, 0) \tag{2.2a}$$

$$\textbf{Put:} \qquad H(S, t) = \max(K - S, 0) \tag{2.2b}$$

The interval $[0,T]$ represent the lifespan of option where T is the elapsed time, in years, since the starting and expiration date of the contract. For example, $T=1$ indicates that the contract expires one year after starting date. Moreover, $t \in [0,T]$ is the elapsed time, in years, since the starting date of the contract. While the payoff of European options is defined only at $t=T$, American option's payoff is defined for all $(S,t) \in [0,\infty] \times [0,T]$.
Options provide greater flexibility to holders by eliminating their exposure to negative payoffs. Therefore, writers charge premiums to the holders to acquire the contract. The premium is often referred to as the price or value of the option, and the problem of determining this value is called option pricing. Let V_t represent the value of the option at time t. For instance, V_0 represents the value of the option at the beginning of the contract. When pricing options, it is crucial to find the fair price; otherwise, the writer or holder of the option could devise a scheme in which the option will always be profitable for them. In other words, options pricing must adhere to the principle of no-arbitrage. Therefore, we assume that the writer of the option uses the premium to construct a portfolio consisting of ϕ_0 units of the asset and invests ψ_0 units of cash in a risk-free asset B_t, such as a US Treasury bills, certificates of deposit, or a bank account. Then, the writer rebalances the portfolio (ϕ_0, ψ_0) to hedge against any potential claims from the holder at any future time $0 < t \leq T$. Consequently, at any time t, the writer holds a portfolio (ϕ_t, ψ_t) with a value

$$\Pi_t = \phi_t S_t + \psi_t B_t$$

Moreover, the portfolio is self-financing. In other words, the changes in portfolio depend on the changes in S_t and B_t, and the rebalancing of portfolio (ϕ_t, ψ_t)

$$d\Pi_t = \phi_t dS_t + \psi_t dB_t$$

$$S_t d\phi_t + B_t d\psi_t = 0$$

Finally, the portfolio value matches the option value

$$\Pi_t = V_t$$

at any time $0 \leq t \leq T$. Using the self-financing portfolio hedging strategy, The Black-Scholes model presents a mathematical model for the dynamics of an option's price. The model makes certain assumptions about the market. A complete list of all the assumptions can be found in [9] and [13]. In the next part, we enumerate some them. Firstly, the asset price S_t is distributed as a log-normally

$$S_t = S_0 \exp\left\{ \int_0^t \Big(r(s) - \frac{1}{2}\sigma(s)\Big)ds + \sqrt{t}Z \right\} \tag{2.3}$$

where the risk-free interest rate $r(t)$ and the price volatility $\sigma(t)$ are deterministic functions of time during the life of the option. Secondly, the bank account $B(t)$ is a deterministic function

$$dB = r(t)B(t)dt$$

Finally, the asset does not pay dividends. Additionally, we will assume that the risk-free interest rate and asset price volatility are constant during the life of the option. Later on, we will address the assumption about dividends.

By applying the Black-Scholes model to price European options, the famous Black-Scholes PDE is obtained

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0 & \text{for } t \in [0, T) \text{ and } S \in [0, \infty) \\ V(S,T) = H(S,T) & \text{for } S \in [0, \infty) \end{cases}$$

where $V(S,t)$ is a deterministic function. A derivation of the Black-Scholes PDE for European options can be found in [9] and [13]. We previously mentioned that Black-Scholes model assumes that the underlying asset does not pay dividends. In most cases, assets such as stocks pay out dividends just a few times at year. In this case, dividends are to be modelled discretely. However, there are certain assets that pay out a proportion of the current price during and interval of time. For instance, indexes such as the SPX. Thus, in such cases, it is useful to model dividends as a continuous yield. [13] shows a slight adjusted asset price model that includes continuous yield dividends

$$S_t = S_0 \exp\left\{ \int_0^t \Big(r(s) - \delta(s) - \frac{1}{2}\sigma(s)\Big) ds + \sqrt{t}Z \right\} \tag{2.4}$$

Note that when the asset does not pay out dividends $\delta(t) = 0$, the asset price model will be exactly as in (2.3). Similarly to as we did for the risk-free interest rate and the volatility of the asset price, we will assume that continuous dividends yield is as constant from now on. As a consequence of the asset price mode (2.4), the Black-Scholes PDE changes to

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}_{\mathsf{BS}}(V) = 0 & \text{for } t \in [0, T) \text{ and } S \in [0, \infty) \\ V(S, T) = H(S, T) & \text{for } S \in [0, \infty) \end{cases} \tag{2.5}$$

where $\mathcal{L}_{\mathsf{BS}}(f)(x)$ is the linear parabolic operator applied to the function $f \in \mathcal{C}^2$

$$\mathcal{L}_{\mathsf{BS}}(f)(x) := \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + (r - \delta)x\frac{\partial f}{\partial x} - rf(x) \tag{2.6}$$

Note that for conciseness, we write $\mathcal{L}_{\mathsf{BS}}(V)(S)$ as $\mathcal{L}_{\mathsf{BS}}(V)$. Similarly, to the asset price model with dividends, the Black-Scholes PDE with dividends fall back to the original Black-Scholes PDE if the asset does not pay out dividends $\delta = 0$. So far, we have presented an equation that describes the dynamics of the value $V(S, t)$ of European options. So now, we will consider the pricing of American options. By applying the Black-Scholes model to price American options, Merton [9] derives some important facts. Firstly, the value $V(S, t)$ is bounded from below by the payoff function:

$$V(S, t) \geq H(S, t) \qquad \text{for } t \in [0, T] \tag{2.7}$$

Moreover, the domain of $V(S, t)$ can be separated into the exercise region in

$$\mathcal{S} := \{(S, t) : V(S, t) = H(S, t)\} \tag{2.8}$$

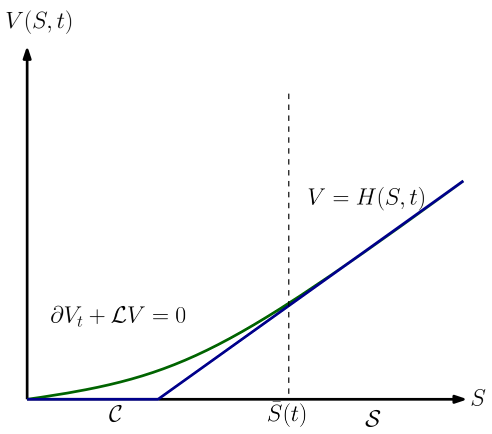
in which it is profitable for the holder to exercise the option, the continuation region

$$\mathcal{C} := \{(S,t) : V(S,t) > H(S,t)\} \tag{2.9}$$

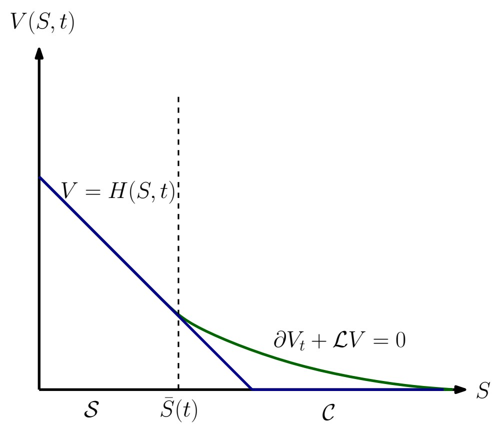
in which it is preferable to continue holding the option because exercising is not profitable, and the optimal exercise boundary that separates the continuation region and exercise region

$$\partial\mathcal{C} := \{(S,t) : S = \bar{S}(t)\} \tag{2.10}$$

where $\bar{S}(t)$ is the optimal exercise price.



(a) Call option



(b) Put option

Figure 2.1: Value $V(S,t)$ of American option value curve.

Lastly, the price dynamics of American options is governed by the same Black-Scholes PDE as European options in the continuation region. Pricing American options only requires solving $V(S,t)$ at continuation region and finding the optimal exercise price that serves as a boundary of continuation region.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta)S\dfrac{\partial V}{\partial S} - rV = 0 & \text{for } (S,t) \in \mathcal{C} \\ V(S,t) = H(S,t) & \text{for } (S,t) \in \partial\mathcal{C} \end{cases} \tag{2.11}$$

The problem above is also known as the free boundary formulation of the pricing problem because it requires to solve a PDE with a moving boundary. The terminal condition of the PDE will be given at time T. At the expiration date of the contract, the holder will either exercise or not the option. Therefore, the value of the option will be equal to the payoff function. Obviously, in that case, the optimal exercise price $S(T)$ will be equal to the strike price K. Hence,

$$V(S,T) = H(S,T), \qquad \bar{S}(T) = K \tag{2.12}$$

Next, we need to establish boundary conditions for the system (2.11). Generally, when pricing options, we need two boundaries conditions. As it can be observed in figure (2.1), for call options, the left boundary condition $V(0,t) = 0$ is given at $S = 0$ and the right boundary conditions $V(\bar{S}(t),t) = \bar{S}(t) - K$ is given at the optimal exercise price $\bar{S}(t)$. Analogously, for put options, the left boundary condition $V(\bar{S}(t),t) = K - \bar{S}(t)$ at $\bar{S}(t)$ and right boundary is $V(S,t) = 0$ for an arbitrary large S. Finally, $V(S,t)$ touches the payoff $H(S,t)$ tangentially at the optimal exercise price $\bar{S}(t)$

$$\textbf{Call:} \qquad \frac{\partial V}{\partial S}(\bar{S}(t),t) = 1 \tag{2.13a}$$

$$\textbf{Put:} \qquad \frac{\partial V}{\partial S}(\bar{S}(t),t) = -1 \tag{2.13b}$$

which is called the smooth pasting condition and later on will help us in obtaining $\bar{S}(t)$. By grouping (2.11), (2.12) and (2.13) in one equation, we obtain the system

Call:
$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 \quad \text{for } S \in (0, \bar{S}(t)) \text{ and } t \in [0, T) \\
V(S,T) = S - K \\
\bar{S}(T) = K \\
V(0,t) = 0 \\
\dfrac{\partial V}{\partial S}(\bar{S}(t), t) = 1
\end{cases}
\tag{2.14a}
$$

Put:
$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 \quad \text{for } S \in (\bar{S}(t), \infty), \text{ and } t \in [0, T) \\
V(S,T) = K - S \\
\bar{S}(T) = K \\
\lim_{S\to\infty} V(S,t) = 0 \\
\dfrac{\partial V}{\partial S}(\bar{S}(t), t) = -1
\end{cases}
\tag{2.14b}
$$

2.2 Front-Fixing method

In the previous section, we presented the pricing of American options problem. By applying the Black-Scholes model, we derived the Black-Scholes PDE that describes the price dynamics in the continuation region $\mathcal{C}$ of call and put options. Moreover, we presented the moving boundary condition $\bar{S}(t)$ for this PDE. The moving boundary condition $\bar{S}(t)$ makes the Black-Scholes PDE more involved since we also need to determine this boundary as time changes. This type of problems are known as free boundary problems. The front fixing method is a strategy in which a transformation is used to map the domain from the original problem to a new domain where moving boundary remains fixed as time changes. In this section, we explore two transformation based on the work of Nielsen et al. [10], and the work of Company and et al. [3].

2.2.1 Nielsen transformation

The Nielsen transformation suggests a really simple transformation in which the asset price S is divided by the optimal exercise price $\bar{S}$

$$x = \frac{S}{\bar{S}(t)} \tag{2.15}$$

Clearly, the moving boundary in the original problem will be fixed when $S = \bar{S}(t)$ at $x = 1$. Now, we define $v(x, t)$ as the value function of the option but under the front fixing domain given by x

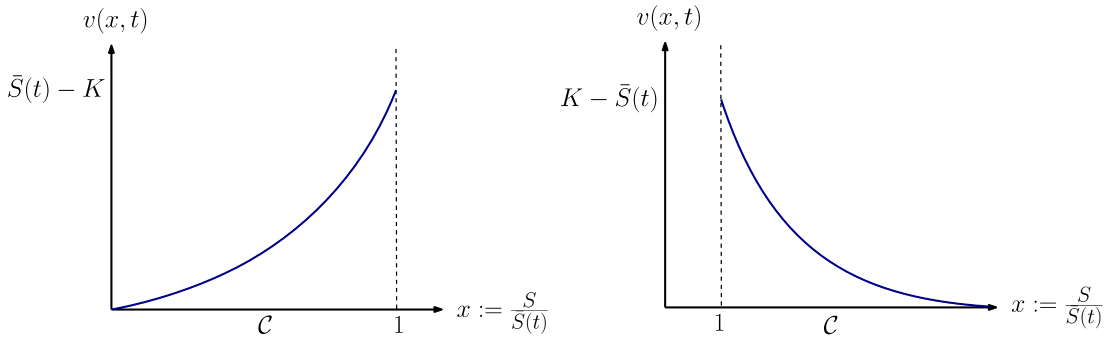
$$v(x, t) := V(S, t) \tag{2.16}$$

Moreover, we want to understand how this transformation affects the Black-Scholes PDE, the boundary, terminal and contact point conditions given in equation in (2.14).
Firstly, we start with the Black-Scholes PDE which is defined at the interval $S \in (0, \bar{S}(t))$ for call options or the open interval $S \in (\bar{S}(t), \infty)$ for put options. Under the front fixing domain, the transformed PDE will be defined in the interval $x \in (0, 1)$ for call and $x \in (1, \infty)$ for put. Moreover, we apply the chain rule to rewrite the Black-Scholes PDE in terms of v, so that, the new PDE is given as

$$\textbf{Call:} \qquad \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left[(r - \delta) - \frac{\bar{S}'(t)}{\bar{S}(t)} \right] x \frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x \in [0, 1) \text{ and } t \in [0, T) \tag{2.17a}$$

$$\textbf{Put:} \qquad \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \left[(r - \delta) - \frac{\bar{S}'(t)}{\bar{S}(t)} \right] x \frac{\partial v}{\partial x} - rv = 0 \quad \text{for } x > 1 \text{ and } t \in (0, T] \tag{2.17b}$$

Similarly, we express the boundary conditions in the front fixing domain. We already stated that the Nielsen transformation fixes the moving boundary $\bar{S}$ at $x = 1$. Additionally, x goes to infinity as S goes to infinity, and $x = 0$ for $S = 0$. Therefore, the boundary condition opposite to the optimal exercise price $\bar{S}(t)$ will remain as in the original problem. Hence, as it can be observed in figure (2.2), the call option has left boundary condition $v(0, t) = 0$ at $x = 0$ and right boundary condition $v(1, t) = \bar{S}(t) - K$ at $x = 1$. Alternatively, the put option has left boundary condition $v(1,t) = K - \bar{S}(t)$ at $x = 1$ and right boundary condition $v(x,t) = 0$ at a sufficiently large x.



(a) Call option (b) Put option

Figure 2.2: Value $v(x,t) := V(S,t)$ in the front fixing domain defined by Nielsen transformation.

Likewise, we express the contact point condition in terms of $v(x,t)$. Recall that at the contact point, the slope $V(S,t)$ with respect to S is the same as the slope of the linear segment in the payoff function. This can be seen clearly in figure (2.1). Hence, by the chain rule, the contact point condition of $v(x,t)$ is given by

$$\textbf{Call:} \qquad \frac{\partial v}{\partial x}(1,t) = \bar{S}(t) \tag{2.18a}$$

$$\textbf{Put:} \qquad \frac{\partial v}{\partial x}(1,t) = -\bar{S}(t) \tag{2.18b}$$

Finally, recall that the terminal condition of $\bar{S}(t)$ is given by (2.12). Moreover, $x >= 1$ for call options, and $x <= 1$ for put options. Hence, by simple substitution, we can rewrite the terminal conditions of $v(x,t)$ as

$$\textbf{Call:} \qquad v(x,T) = \max(x\bar{S}(T) - K) = K\max(x-1,0) = 0 \tag{2.19a}$$

$$\textbf{Put:} \qquad v(x,T) = \max(K - x\bar{S}(T)) = K\max(1-x,0) = 0 \tag{2.19b}$$

In summary, by groping equations (2.17), (2.19), and (2.18), we obtain the system

$$\textbf{Call:}\quad \begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\right] x \dfrac{\partial v}{\partial x} - rv = 0 & \text{for } x \in (0,1) \text{ and } t \in [0,T) \\ v(x,T) = 0 & \text{for } x \in [0,1] \\ \bar{S}(T) = K & \\ v(0,t) = 0 & \text{for } t \in [0,T) \\ v(1,t) = \bar{S}(t) - K & \text{for } t \in [0,T) \\ \dfrac{\partial v}{\partial x}(1,t) = \bar{S}(t) & \text{for } t \in [0,T) \end{cases} \tag{2.20a}$$

$$\textbf{Put:}\quad \begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 v}{\partial x^2} + \left[(r-\delta) - \dfrac{\bar{S}'(t)}{\bar{S}(t)}\right] x \dfrac{\partial v}{\partial x} - rv = 0 & \text{for } x > 1 \text{ and } t \in [0,T) \\ v(x,T) = 0 & \text{for } x \geq 1 \\ \bar{S}(T) = K & \\ v(1,t) = K - \bar{S}(t) & \text{for } t \in [0,T) \\ \lim_{x\to\infty} v(x,t) = 0 & \text{for } t \in [0,T) \\ \dfrac{\partial v}{\partial x}(1,t) = -\bar{S}(t) & \text{for } t \in [0,T) \end{cases} \tag{2.20b}$$

2.2.2 Company transformation

The Company transformation proposes set of change of variable for the asset price S, the time t, the value function $V(S,t)$ and the moving boundary

$$x := \log \frac{S}{\bar{S}_f(t)}, \quad \tau := T - t, \quad v(x,\tau) := \frac{V(S,t)}{K}, \quad \bar{S}_f(\tau) := \frac{\bar{S}(t)}{K} \tag{2.21}$$

Let us break down the transformations. Firstly, the transformation proposed are written forward in time. Therefore, $\tau = 0$ refers to the expiration date of the options $t = T$. Secondly, both the value function and the optimal exercise price is scaled by the strike price. Finally, the new moving boundary is fixed at $S = \bar{S}_f(t)$ or $x = 0$.

Similarly, as we did for the Nielsen method, we rewrite the Black-Scholes PDE in terms of $v(x,\tau)$. Note that as x goes to infinity S goes to infinity. Conversely, as x goes to negative infinity S goes to zero. Moreover, $S = \bar{S}(t)$ at $x = 0$. Using the previous information, we deduce that the Black-Scholes PDE is defined in the intervals $x \in (-\infty, 0)$ for call options and $x \in (0, \infty)$ for put options. Therefore, we have

$$\textbf{Call:} \quad \frac{\partial v}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} - \left((r-\delta) + \frac{\sigma^2}{2} - \frac{\bar{S}'(\tau)}{\bar{S}(\tau)}\right)\frac{\partial v}{\partial x} + rv = 0 \quad \text{for } x < 0 \text{ and } \tau \in (0,T] \tag{2.22a}$$

$$\textbf{Put:} \quad \frac{\partial v}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} - \left((r-\delta) - \frac{\sigma^2}{2} - \frac{\bar{S}'(\tau)}{\bar{S}(\tau)}\right)\frac{\partial v}{\partial x} + rv = 0 \quad \text{for } x > 0 \text{ and } \tau \in (0,T] \tag{2.22b}$$

Note that the term on the new PDE that correspond to the terms in the linear parabolic operator $\mathcal{L}V$ defined in (2.6) are negative because the Black-Scholes PDE was inverted in time.

Again, the boundary conditions for the call option in the original domain are $V(0,t)$ at $S = 0$ and $V(\bar{S},t) = \bar{S} - K$ at $S = \bar{S}(t)$, and when transforming those boundary conditions to the front fixing domain, they become $v(x,\tau) = 0$ for a sufficiently negative x and $v(0,\tau) := \bar{S}_f(\tau) - 1 = V(\bar{S},t)/K$ at $x = 0$. Similarly, the boundary conditions for the put option in the original domain are $V(\bar{S}(t),t) = K - \bar{S}$ at $S = \bar{S}(t)$ and $V(S,t) = 0$ for a sufficiently large S, and under the front fixing domain, they become $v(0,\tau) = 1 - \bar{S}_f(\tau) = V(\bar{S},t)/K$ at $x = 0$ and $v(x,\tau) = 0$ for a sufficiently large x. Similarly, to as we did for the Nielsen transformation, we also rewrite the contact point condition

$$\textbf{Call:} \qquad \frac{\partial v}{\partial x}(0,\tau) = \bar{S}_f(\tau) \tag{2.23a}$$

$$\textbf{Put:} \qquad \frac{\partial v}{\partial x}(0,\tau) = -\bar{S}_f(\tau) \tag{2.23b}$$

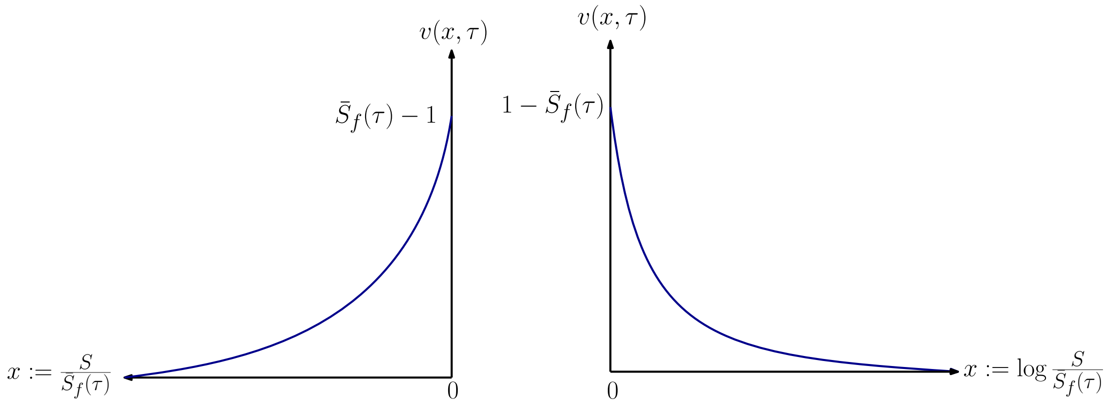


Figure 2.3: Value $v(x,t) := V(S,t)/K$ in the front fixing domain defined by Company transformation.

Since the transformed PDE is forward in time, we have to come up with initial conditions for $\bar{S}_f(\tau)$ and $v(x,\tau)$. For $\bar{S}_f(\tau)$, the initial condition is given by

$$\bar{S}_f(0) = \frac{\bar{S}(T)}{K} = 1 \tag{2.24}$$

Moreover, for call options, the initial condition is given by $v(x,0) = V(S,T)/K = \max\left(\bar{S}_f(0)e^x - 1, 0\right) = \max\left(e^x - 1, 0\right) = 0$ since x is always negative. Similarly, for put options, the initial condition is given by $v(x,0) = V(S,T)/K = \max\left(1 - \bar{S}_f(0)e^x, 0\right) = \max\left(1 - e^x, 0\right) = 0$ since x is always positive. Hence,

$$\textbf{Call:} \qquad v(x,0) = 0 \tag{2.25a}$$

$$\textbf{Put:} \qquad v(x,0) = 0 \tag{2.25b}$$

Finally, grouping the equations together, we have the system

$$
\textbf{Call:}\quad \begin{cases}
\dfrac{\partial v}{\partial \tau} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial x^2} - \left((r-\delta) + \dfrac{\sigma^2}{2} - \dfrac{\bar{S}'(\tau)}{\bar{S}(\tau)}\right)\dfrac{\partial v}{\partial x} + rv = 0 & \text{for } x < 0 \text{ and } \tau \in (0,T] \\
v(x,0) = 0 & \text{for } x < 0 \\
\bar{S}_f(0) = 1 & \\
\lim_{x\to-\infty} v(x,\tau) = 0 & \text{for } \tau \in (0,T] \\
\dfrac{\partial v}{\partial x}(0,\tau) = \bar{S}_f(\tau) & \text{for } \tau \in (0,T]
\end{cases}
\tag{2.26a}
$$

$$
\textbf{Put:}\quad \begin{cases}
\dfrac{\partial v}{\partial \tau} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial x^2} - \left((r-\delta) - \dfrac{\sigma^2}{2} - \dfrac{\bar{S}'(\tau)}{\bar{S}(\tau)}\right)\dfrac{\partial v}{\partial x} + rv = 0 & \text{for } x > 0 \text{ and } \tau \in (0,T] \\
v(x,0) = 0 & \text{for } x > 0 \\
\bar{S}_f(0) = 1 & \\
\lim_{x\to\infty} v(x,\tau) = 0 & \text{for } \tau \in (0,T] \\
\dfrac{\partial v}{\partial x}(0,\tau) = -\bar{S}_f(\tau) & \text{for } \tau \in (0,T]
\end{cases}
\tag{2.26b}
$$

3 Finite difference schemes

3.1 Overview

In this section, we present explicit and implicit central finite difference schemes for solving the PDE problem in (2.20). Previously, we considered the pricing problem of American options which requires solving the free boundary problem defined in (2.14). Then, we presented the front fixing method as a strategy to fix the moving boundary using a change of variable. Moreover, we derived the PDE problem for call and put options that resulted from applying the Nielsen transformation suggested by [10], and the Company transformation suggested by [3], resulting in the systems (2.20) and (2.26), respectively. In the following part, we present numerical methods for solving (2.20). But before we jump into that, we define what it means to compute a numerical solution to a PDE problem.

Recall that the solution $v(x,t)$ of (2.20) is defined in the continuous region

$$\textbf{Call:} \qquad \mathcal{T}:[0,T], \qquad \mathcal{X}:[0,1], \qquad \mathcal{F}:\mathcal{X}\times\mathcal{T} \tag{3.1a}$$

$$\textbf{Put:} \qquad \mathcal{T}:[0,T], \qquad \mathcal{X}:[1,\infty), \qquad \mathcal{F}:\mathcal{X}\times\mathcal{T}, \tag{3.1b}$$

Now, we want to discretize $\mathcal{F}$ using the grid $\mathcal{G}$ with $N+1$ and $M+1$ nodes

$$\mathcal{G} := \{(x_i,t_n) : (i,n) \in \{0,\dots,M+1\}\times\{0,\dots,N+1\}\} \tag{3.2}$$

where

$$x_i := x_{\mathsf{min}} + i\Delta x \qquad \text{for } i = 0,\dots,M+1 \tag{3.3}$$

$$t_n := t_{\mathsf{min}} + i\Delta t \qquad \text{for } i = 0,\dots,N+1 \tag{3.4}$$

$$\Delta x := \frac{x_{\mathsf{max}} - x_{\mathsf{min}}}{M+1} \tag{3.5}$$

$$\Delta t := \frac{t_{\mathsf{max}} - t_{\mathsf{min}}}{N+1} \tag{3.6}$$

Each contiguous node will be separated by Δx on the spatial axis and Δt on the temporal axis. As Δx and Δt decreases, the number of nodes in the grid will increase. Therefore, we refer to Δx and Δt as the resolution of the grid.

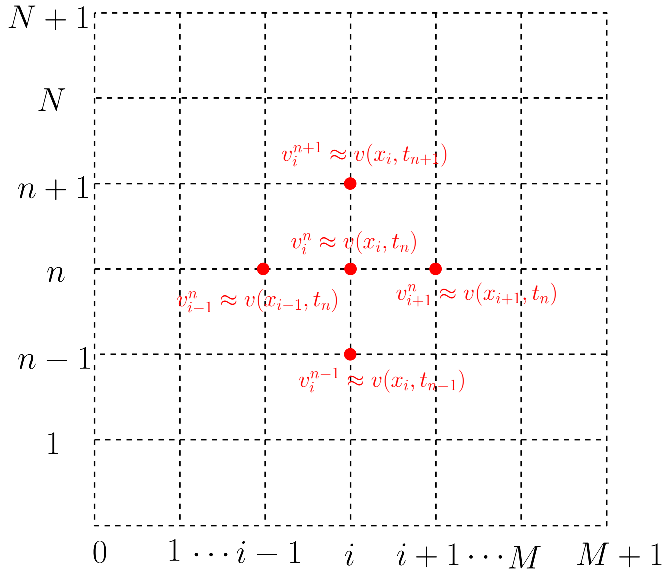


Figure 3.1: The grid $\mathcal{G}$ and the approximation $v_i^n \approx v(x_i, t_n)$ in each node.

From (3.1), it is clear that $t_{\text{min}} = 0$ and $t_{\text{max}} = T$. Moreover, for call options, $x_{\text{min}} = 0$ and $x_{\text{max}} = 1$. Likewise, for put options, $x_{\text{min}} = 1$ and $x_{\text{max}} = x_\infty$ where x_∞ is arbitrary large value. Now that we defined our grid, our goal is to approximate the value function $v(x, t)$ and the optimal exercise price $\bar{S}(t)$ at each node of the grid $\mathcal{G}$

$$v_i^n \approx v(x_i, t_n), \quad \bar{S}^n \approx \bar{S}(t_n)$$

Moreover, we want that the error of the approximation converges to zero value. Specifically, we want that the approximation error at each node

$$e_i^n := v_i^n - v(x_i, t_n) \tag{3.7}$$

goes to zero as Δx and Δt decrease. (3.7) is the local truncation error, and it measures the approximation error at time t_n. It is important to state if a single node has inferior order than the rest of the nodes, it might degrade the order of the truncation error in overall.

Finally, we need ways to approximate derivatives. Here is where finite differences schemes come into play. The idea of finite differences is trivial which is approximating derivatives as the difference of contiguous nodes in the grid. Let us say we are at point x, then the forward differences approximate the derivative as

$$\frac{f(x+h)-f(x)}{h}=\frac{df}{dx}+O(h)$$

Conversely, the backward difference approximate the derivative as

$$\frac{f(x)-f(x-h)}{h}=\frac{df}{dx}+O(h)$$

As you can observe forward and backward difference approximation yield a local truncation error of $O(h)$. Moreover, the central finite difference approximate the first order derivative as

$$\frac{f(x+h)-f(x-h)}{2h}=\frac{df}{dx}+O(h^2)$$

and for second order derivatives as

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=\frac{d^2f}{dx^2}+O(h^2)$$

Note that both approximation offers a better order of convergence that forward and backward difference, but you are required to come up with strategies for approximating the derivative at the boundary of your grid where $x+h$ or $x-h$ is not defined.

3.2 Explicit scheme

Generally, explicit schemes use forward finite difference to approximate the temporal partial derivative and central finite difference to approximate the spatial derivative at time t_{n+1} and position x_i. However, since the problem (2.20) is written backward in time, we use backward finite difference at t_{n+1}, and a central finite difference at x_i.

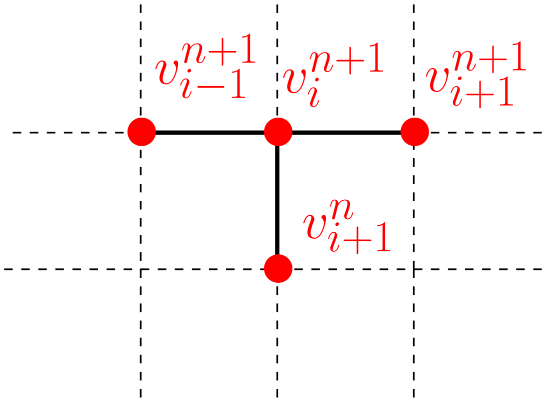


Figure 3.2: Stencil diagram of the explicit scheme.

The central finite difference for the first order and second order spatial partial derivative is given by

$$\frac{v_{i+1}^{n+1}-v_{i-1}^{n+1}}{2\Delta x} = \frac{\partial v}{\partial x} + O(\Delta x^2) \qquad \text{for } i = 1, \dots, M \tag{3.8}$$
$$\frac{v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}}{\Delta x^2} = \frac{\partial^2 v}{\partial x^2} + O(\Delta x^2) \qquad \text{for } i = 1, \dots, M \tag{3.9}$$

As it can be observed in figure (3.2), the first and second order central finite difference approximations at node (x_i, t_{n+1}) require to compute the difference at the nodes (x_{i-1}, t_{n+1}) and (x_{i+1}, t_{n+1}). Hence, we can only approximate the spatial partial derivative at the internal region of the grid $\mathcal{G}$ given by the nodes (x_i, t_n) for $i = 1, \dots, M$. Also note, that the central finite difference has second order convergence in space. In other words, as we decrease Δx by one decimal place, the approximation error will decrease by two decimal places.

Analogously, the backward difference approximation at t_{n+1} for $v(x,t)$ and the optimal exercise price $\bar{S}(t)$ is given by

$$\frac{v_i^{n+1}-v_i^n}{\Delta t} = \frac{\partial v}{\partial t} + O(\Delta t) \qquad \text{for } n = N, \dots, 0 \tag{3.10}$$
$$\frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t} = \bar{S}'(t) + O(\Delta t) \qquad \text{for } n = N, \dots, 0 \tag{3.11}$$

Contrary to the central finite difference, the backward finite difference approximations have first order convergence in time. While it would be desirable to have second order convergence for the temporal partial derivative approximation, it is not possible use central finite difference because we would be required to have two boundary conditions in the time axis. By combining the finite difference approximations (3.8), (3.9), (3.10), and (3.10),the approximation of the PDE in (2.20) is given by

$$\begin{aligned}\frac{v_i^{n+1}-v_i^n}{\Delta t}&+\frac{1}{2}\sigma^2 x_i^2\frac{v_{i-1}^{n+1}-2v_i^{n+1}+v_{i+1}^{n+1}}{(\Delta x)^2}\\&+x_i\left((r-\delta)-\frac{1}{\bar{S}^{n+1}}\frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t}\right)\frac{v_{i+1}^{n+1}-v_{i-1}^{n+1}}{2\Delta x}-rv_i^{n+1}=0\end{aligned}$$

for $i=1,\dots,M$ and $n=N,\dots,0$. To simplify the expression above, we introduce the terms

$$\begin{aligned}\lambda &:= \frac{\Delta t}{(\Delta x)^2}\\ A_i &:= \frac{\lambda}{2}\sigma^2 x_i^2-\frac{\lambda}{2}\left((r-\delta)-\frac{1}{\Delta t}\right)x_i\Delta x && \text{for } i=1,\dots,M\\ B_i &:= 1-\lambda\sigma^2 x_i^2-r\Delta t && \text{for } i=1,\dots,M\\ C_i &:= \frac{\lambda}{2}\sigma^2 x_i^2+\frac{\lambda}{2}\left((r-\delta)-\frac{1}{\Delta t}\right)x_i\Delta x && \text{for } i=1,\dots,M\\ D_i^{n+1} &:= \frac{x_i}{2\Delta x}\frac{v_{i+1}^{n+1}-v_{i-1}^{n+1}}{\bar{S}^{n+1}} && \text{for } i=1,\dots,M\end{aligned}$$

Then, we rearrange the finite difference approximation of the PDE as

$$v_i^n-D_i^{n+1}\bar{S}^n=A_i v_{i-1}^{n+1}+B_i v_i^{n+1}+C_i v_{i+1}^{n+1} \tag{3.12}$$

for $i=1,\dots,M$ and $t=N,\dots,0$. Moreover, the PDE problem in (2.20) have well-defined spatial boundary conditions. For call options, the boundary conditions are located at $x=0$ and $x=1$. Similarly, for put options, the boundary conditions are located at $x=1$ and at a sufficient large x. However, since the $\mathcal{G}$ is defined in terms of x_{min} and x_{max}, regardless of the option type, the boundary conditions will be always at x_0 and x_{M+1}.

$$\textbf{Call:}\qquad v_0^n=0,\qquad v_{M+1}^n=\bar{S}^n-K \tag{3.13a}$$

$$\textbf{Put:}\qquad v_0^n=K-\bar{S}^n,\qquad v_{M+1}^n=0 \tag{3.13b}$$

Likewise, the terminal conditions are located at t_{N+1} $i = 0, \dots, M+1$

$$v_i^{N+1} = 0, \qquad \bar{S}^{N+1} = K \tag{3.14a}$$

Moreover, for the problem (2.20), we have contact point condition (2.18). The contact point condition gives the slope at $x = 1$. When the option is a call option, $x = 1$ correspond to x_{M+1} in the grid $\mathcal{G}$. Reciprocally, for a put option, $x = 1$ correspond to x_0. Therefore, by using backward difference at x_{M+1} and forward difference at x_0, the contact point approximation for call and put options, respectively.

$$\begin{aligned}
&\textbf{Call:} \qquad \frac{v_{M+1}^n - v_M^n}{\Delta x} = \frac{\partial v}{\partial x}(1,t) + O(\Delta x) \\
&\textbf{Put:} \qquad \frac{v_1^n - v_0^n}{\Delta x} = \frac{\partial v}{\partial x}(1,t) + O(\Delta x)
\end{aligned}$$

Using the contact point condition, we obtain an explicit expression for v_M^n

$$\textbf{Call:} \qquad v_M^n = v_{M+1}^n - \Delta x \bar{S}^n = (1 - \Delta x)\bar{S}^n - K \qquad \text{for } n = N, \dots, 0 \tag{3.15a}$$

$$\textbf{Put:} \qquad v_1^n = v_0^n - \Delta x \bar{S}^n = K - (1 + \Delta x)\bar{S}^n \qquad \text{for } n = N, \dots, 0 \tag{3.15b}$$

Note that the approximation for v_M^n has first order convergence in space which could degrade the global convergence of the explicit method to first order in space even if we are using central finite difference to approximate the spatial partial derivatives of $v(x,t)$. Similarly, we can obtain explicit expression for $\bar{S}^n$ by computing (3.12) at x_M and at x_1 for call and put options, respectively. Then, rearranging the resulting expression in terms of $\bar{S}^n$

$$\textbf{Call:} \qquad \bar{S}^n = \frac{K + A_M v_{M-1}^{n+1} + B_M v_M^{n+1} + C_M v_{M+1}^{n+1}}{(1 - \Delta x) - D_M^{n+1}} \tag{3.16a}$$

$$\textbf{Put:} \qquad \bar{S}^n = \frac{K - (A_1 v_0^{n+1} + B_1 v_1^{n+1} + C_1 v_2^{n+1})}{D_1^{n+1} + (1 + \Delta x)} \tag{3.16b}$$

for $n = N, \dots, 0$. Thus, combining (3.12), (3.13), (3.14), (3.15), and (3.16), the explicit scheme of PDE problem (2.20) is given by

$$\textbf{Call:}\quad \begin{cases} v_i^n - D_i^{n+1}\bar{S}^n = A_i v_{i-1}^{n+1} + B_i v_i^{n+1} + C_i v_{i+1}^{n+1} & \text{for } i = 1, \dots, M-1 \text{ and } n = N, \dots, 0 \\ v_i^{N+1} = 0 & \text{for } i = 0, \dots, M+1 \\ \bar{S}^{N+1} = K & \\ v_0^n = 0 & \text{for } n = N, \dots, 0 \\ v_M^n = (1-\Delta x)\bar{S}^n - K & \text{for } n = N, \dots, 0 \\ v_{M+1}^n = \bar{S}^n - K & \text{for } n = N, \dots, 0 \end{cases} \tag{3.17a}$$

$$\textbf{Put:}\quad \begin{cases} v_i^n - D_i^{n+1}\bar{S}^n = A_i v_{i-1}^{n+1} + B_i v_i^{n+1} + C_i v_{i+1}^{n+1} & \text{for } i = 2, \dots, M \text{ and } n = N, \dots, 0 \\ v_i^{N+1} = 0 & \text{for } i = 0, \dots, M+1 \\ \bar{S}^{N+1} = K & \\ v_0^n = K - \bar{S}^n & \text{for } n = N, \dots, 0 \\ v_1^n = K - (1+\Delta x)\bar{S}^n & \text{for } n = N, \dots, 0 \\ v_M^n = 0 & \text{for } n = N, \dots 0 \end{cases} \tag{3.17b}$$

Finally, we formulate an algorithm for solving the system (3.17)

Algorithm 3.1 Explicit method for call options

Ensure: $\lambda \leq 0.5$

for $i = 0, \dots, M+1$ **do**

$\quad v_i^{N+1} = 0$

$\bar{S}^{N+1} = K$

for $i = 1, \dots, M$ **do**

$$A_i = \frac{\lambda}{2}\sigma^2 x_i^2 - \frac{\lambda}{2}\left((r-\delta) - \frac{1}{\Delta t}\right)x_i \Delta x$$

$$B_i = 1 - \lambda\sigma^2 x_i^2 - r\Delta t$$

$$C_i = \frac{\lambda}{2}\sigma^2 x_i^2 + \frac{\lambda}{2}\left((r-\delta) - \frac{1}{\Delta t}\right)x_i \Delta x$$

for $n = N, \dots, 0$ **do**

$\quad$ **for** $i = 1, \dots, M$ **do**

$$D_i^{n+1} = \frac{x_i}{2\Delta x}\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}}$$

$$\bar{S}^n = \frac{K + A_M v_{M-1}^{n+1} + B_M v_M^{n+1} + C_M v_{M+1}^{n+1}}{(1-\Delta x) - D_M^{n+1}}$$

$\quad v_0^n = 0$

$\quad v_M^n = (1-\Delta x)\bar{S}^n - K$

$\quad v_{M+1}^n = \bar{S}^n - K$

$\quad$ **for** $i = 1, \dots, M-1$ **do**

$$v_i^n = A_i v_{i-1}^{n+1} + B_i v_i^{n+1} + C_i v_{i+1}^{n+1} + D_i^{n+1}\bar{S}^n$$

Algorithm 3.2 Explicit method for put options

for $i = 0, \dots, M+1$ **do**

$\quad v_i^{N+1} = 0$

$\bar{S}^{N+1} = K$

for $i = 1, \dots, M$ **do**

$$A_i = \frac{\lambda}{2}\sigma^2 x_i^2 - \frac{\lambda}{2}\left((r-\delta) - \frac{1}{\Delta t}\right)x_i \Delta x$$

$$B_i = 1 - \lambda\sigma^2 x_i^2 - r\Delta t$$

$$C_i = \frac{\lambda}{2}\sigma^2 x_i^2 + \frac{\lambda}{2}\left((r-\delta) - \frac{1}{\Delta t}\right)x_i \Delta x$$

for $n = N, \dots, 0$ **do**

$\quad$ **for** $i = 1, \dots, M$ **do**

$$D_i^{n+1} = \frac{x_i}{2\Delta x}\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{\bar{S}^{n+1}}$$

$$\bar{S}^n = \frac{K - (A_1 v_0^{n+1} + B_1 v_1^{n+1} + C_1 v_2^{n+1})}{D_1^{n+1} + (1+\Delta x)}$$

$$v_0^n = K - \bar{S}^n$$

$$v_1^n = K - (1+\Delta x)\bar{S}^n$$

$$v_{M+1}^n = 0$$

$\quad$ **for** $i = 2, \dots, M$ **do**

$$v_i^n = A_i v_{i-1}^{n+1} + B_i v_i^{n+1} + C_i v_{i+1}^{n+1} + D_i^{n+1}\bar{S}^n$$

3.3 Implicit scheme

Analogously to the previous section, implicit methods approximate the temporal partial derivative using backward difference and the spatial partial derivative using a central difference at time t_n and position x_i. Since the PDE in (2.20) is written backward in time, we use a forward difference instead.

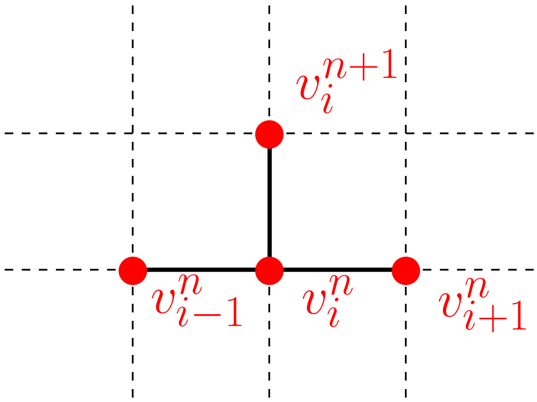


Figure 3.3: Stencil diagram of the implicit scheme.

Therefore, the central difference for the first and second order spatial partial derivative at time t_n is

$$\frac{v_{i+1}^n - v_{i-1}^n}{2\Delta x} = \frac{\partial v}{\partial x} + O(\Delta x^2) \tag{3.18}$$

$$\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2} = \frac{\partial^2 v}{\partial x^2} + O(\Delta x^2) \tag{3.19}$$

for $i = 1, \ldots, M$. Likewise, the forward difference of $v(x,t)$ and $\bar{S}(t)$ at position x_i is

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = \frac{\partial v}{\partial t} + O(\Delta t) \tag{3.20}$$

$$\frac{\bar{S}^{n+1} - \bar{S}^n}{\Delta t} = \bar{S}'(t) + O(\Delta t) \tag{3.21}$$

for $n = N, \ldots, 0$. Hence, combining (3.18), (3.19), (3.20) and (3.21),we obtain the implicit approximation of the PDE (2.20) as

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}+\frac{1}{2}\sigma^2 x_i^2\frac{v_{i-1}^n-2v_i^n+v_{i+1}^n}{(\Delta x)^2}$$
$$+x_i\left((r-\delta)-\frac{1}{\bar{S}^n}\frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t}\right)\frac{v_{i+1}^n-v_{i-1}^n}{2\Delta x}-rv_i^n=0$$

for $i=1,\dots,M$ and $n=N,\dots,0$. Similar to the explicit method, the approximation error is second order in space and first order in time. Again, to make the implicit approximation more manageable, we introduce the following terms

$$\alpha_i^n:=-\frac{\lambda}{2}\sigma^2x_i^2+\frac{\lambda\Delta x}{2}x_i\left(r-\delta+\frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t\bar{S}^n}\right) \tag{3.22}$$

$$\beta_i^n:=1+\lambda\sigma^2x_i^2+r\Delta t \tag{3.23}$$

$$\gamma_i^n:=-\frac{\lambda}{2}\sigma^2x_i^2+\frac{\lambda\Delta x}{2}x_i\left(r-\delta+\frac{\bar{S}^{n+1}-\bar{S}^n}{\Delta t\bar{S}^n}\right) \tag{3.24}$$

and rearrange the PDE as

$$\alpha_i^n v_{i-1}^n+\beta_i^n v_i^n+\gamma_i^n v_{i+1}^n=v_i^{n+1} \tag{3.25}$$

The boundary and terminal conditions are given by (3.13) and (3.14). Likewise, the approximation of v_M^n or v_1^n for put and call, respectively, is given by (3.15). Similar to the explicit method, the approximation v_M^n and v_0^n given by the contact point condition is first order in space. Hence, the global approximation error of implicit scheme might be degraded to first order in space. Contrary to the explicit method, there is not an explicit expression for $\bar{S}^n$. Now, we formulate the system of equations of the problem (2.20) using (3.13), (3.14), (3.15) and (3.25)

$$\textbf{Call:} \quad \begin{cases} \alpha_i^n v_{i-1}^n + \beta_i^n v_i^n + \gamma_i^n v_{i+1}^n = v_i^{n+1} & \text{for } i = 1, \dots, M-1 \text{ and } n = N, \dots, 0 \\ v_i^{N+1} = 0 & \text{for } i = 0, \dots, M+1 \\ \bar{S}^{N+1} = K & \\ v_0^n = 0 & \text{for } n = N, \dots 0 \\ v_M^n = (1 - \Delta x)\bar{S}^n - K & \text{for } n = N, \dots 0 \\ v_{M+1}^n = \bar{S}^n - K & \text{for } n = N, \dots 0 \end{cases} \tag{3.26a}$$

$$\textbf{Put:} \quad \begin{cases} \alpha_i^n v_{i-1}^n + \beta_i^n v_i^n + \gamma_i^n v_{i+1}^n = v_i^{n+1} & \text{for } i = 2, \dots, M \text{ and } n = N, \dots, 0 \\ v_i^{N+1} = 0 & \text{for } i = 0, \dots, M+1 \\ \bar{S}^{N+1} = K & \\ v_0^n = K - \bar{S}^n & \text{for } n = N, \dots 0 \\ v_1^n = K - (1 + \Delta x)\bar{S}^n & \text{for } n = N, \dots 0 \\ v_M^n = 0 & \text{for } n = N, \dots 0 \end{cases} \tag{3.26b}$$

Since there is not an explicit formula for v_i^n and $\bar{S}^n$, we will have to solve a non-linear system of equation. Let's define the vector $\mathbf{v}^n \in \mathbb{R}^{M-1}$

$$\textbf{Call:} \qquad \mathbf{v}^n := \begin{bmatrix} v_1^n, & v_2^n, & \cdots, & v_{M-1}^n \end{bmatrix}^\mathsf{T} \tag{3.27a}$$

$$\textbf{Put:} \qquad \mathbf{v}^n := \begin{bmatrix} v_2^n, & v_3^n, & \cdots, & v_M^n \end{bmatrix}^\mathsf{T} \tag{3.27b}$$

the matrix $\Lambda^n \in \mathbb{R}^{M-1,M-2}$

$$\textbf{Call:} \qquad \Lambda^n = \begin{bmatrix} \beta_1^n & \gamma_1^n & & & & \\ \alpha_2^n & \beta_2^n & \gamma_2^n & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{M-2}^n & \beta_{M-2}^n & \gamma_{M-2}^n \\ & & & & \alpha_{M-1}^n & \beta_{M-1}^n \\ & & & & & \alpha_M^n \end{bmatrix} \tag{3.28a}$$

$$\textbf{Put:} \qquad \Lambda^n := \begin{bmatrix} \gamma_1^n & & & & & \\ \beta_2^n & \gamma_2^n & & & & \\ \alpha_3^n & \beta_3^n & \gamma_3^n & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{M-1}^n & \beta_{M-1}^n & \gamma_{M-1}^n \\ & & & & \alpha_M^n & \beta_M^n \end{bmatrix} \tag{3.28b}$$

and the vector $\mathbf{f}^n \in \mathbb{R}^{M-1}$

$$\textbf{Call:} \qquad \mathbf{f}^n := \begin{bmatrix} v_1^{n+1} \\ \vdots \\ v_{M-1}^{n+1} - \gamma_{M-1}^n[(1-\Delta x)\bar{S}^n - K] \\ v_M^{n+1} - \gamma_M^n(\bar{S}^n - K) - \beta_M^n[(1-\Delta x)\bar{S}^n - K] \end{bmatrix} \tag{3.29a}$$

$$\textbf{Put:} \qquad \mathbf{f}^n := \begin{bmatrix} v_1^{n+1} - \alpha_1^n(K - \bar{S}^n) - \beta_1^n[K - (1+\Delta x)\bar{S}^n] \\ v_2^{n+1} - \beta_2^n[K - (1+\Delta x)\bar{S}^n] \\ v_3^{n+1} \\ \vdots \\ v_{M-1}^{n+1} \end{bmatrix} \tag{3.29b}$$

Thus, the non-linear system of equations that we need to solve is

$$F(\mathbf{v}^n, \bar{S}^n) = \Lambda^n \mathbf{v}^n - \mathbf{f}^n = 0 \tag{3.30}$$

By computing the Jacobian of the system, we con solve the non-linear system using the newton's method

$$\mathbf{y}_{k+1} = \mathbf{y}_k - J^{-1}(\mathbf{y}_k) F(\mathbf{y}_k) \tag{3.31}$$

where y_k is some approximation of the solution

$$\mathbf{y} = \left[\mathbf{v}^n | \bar{S}^n \right]^{\mathsf{T}} \tag{3.32}$$

3.4 Numerical results

Generally, Datasets for American options are hard to get and often require paying substantial amount of money. Therefore, to validate our implementation, we mainly relied on the data available in Company, et al. [3], Nielsen, et al. [10], Seydel [12], and Wilmott, et al. [13]. Moreover, we used the approximations produced by the binomial model introduced by Cox et al. [4] as benchmark for assessing the consistency of our method. We chose the binomial model because it uses a completely different approach to price options than the one considered in our work, is widely used in the industry, and is simple to implement. First, we want to assess the correct functionality of our implementation. With that objective, we define the set of parameters taken from [10]

$$K = 1, \quad T = 1, \quad r = 0.2, \quad \sigma = 0.2, \quad \delta = 0 \tag{3.33}$$

Moreover, Nielsen [10] suggests that the optimal exercise boundary $\bar{S}(t) \approx 0.86$ for parameters (3.33). Then, we validated that the binomial model produces a similar approximation for $\bar{S}(t)$. Finally, we priced a put option using our implementation of the explicit (3.17) and implicit (3.26) method for the Nielsen transformation, and the explicit method for the Company transformation (A.1). In tables (3.1) and (3.2), you can find the RMSE error produced by the explicit method for Nielsen and Company transformation.

Asset price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000
1.0	0.048167	0.046048	0.047748	0.048176	0.048160	0.048155
1.2	0.008666	0.008661	0.008638	0.008705	0.008674	0.008662
1.4	0.001285	0.001467	0.001369	0.001313	0.001292	0.001286
1.6	0.000167	0.000237	0.000196	0.000176	0.000170	0.000168
1.8	0.000020	0.000038	0.000027	0.000022	0.000021	0.000020
2.0	0.000002	0.000006	0.000004	0.000003	0.000002	0.000002
	RMSE	0.000080	0.000016	0.000002	0.00000	0.000000

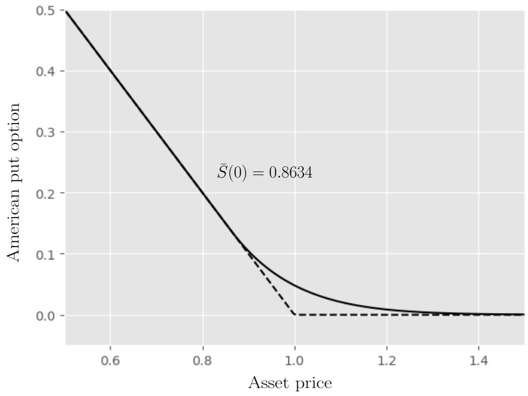
Table 3.1: RSME error produced by the explicit scheme for the Nielsen transformation for $\Delta t = 1/8, 1/16, \ldots, 1/128$ and $\Delta t = \Delta x^2/2$.

Asset price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000
1.0	0.048167	0.049286	0.049274	0.048465	0.048256	0.048174
1.2	0.008666	0.010736	0.009108	0.008829	0.008686	0.008667
1.4	0.001285	0.001950	0.001501	0.001349	0.001295	0.001287
1.6	0.000167	0.000354	0.000224	0.000185	0.000172	0.000168
1.8	0.000020	0.000076	0.000035	0.000024	0.000021	0.000020
2.0	0.000002	0.000017	0.000006	0.000003	0.000003	0.000002
	RSME	0.000092	0.000045	0.000013	0.000003	0.00000

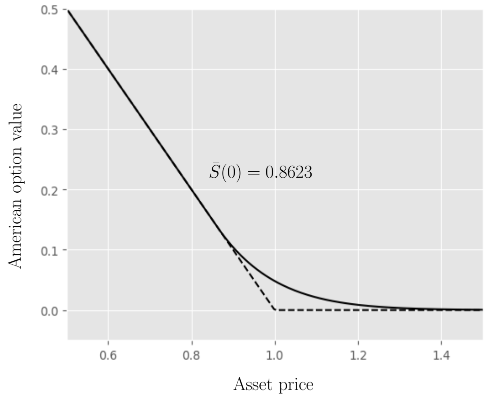
Table 3.2: RSME error produced by the explicit scheme for Company transformation for $\Delta t = 1/8, 1/16, \ldots, 1/128$ and $\Delta t = \Delta x^2/2$.

Moreover, figure (3.4) show the $V(S,0)$ curve obtained by the Nielsen and Company explicit method. In each plot, we have listed the optimal exercise boundary $\bar{S}(0)$. Also, we have listed the correspondent value curve produced using the binary option pricing model using 2^500 nodes. As you see, Nielsen and Company approximated optimal exercise boundary within

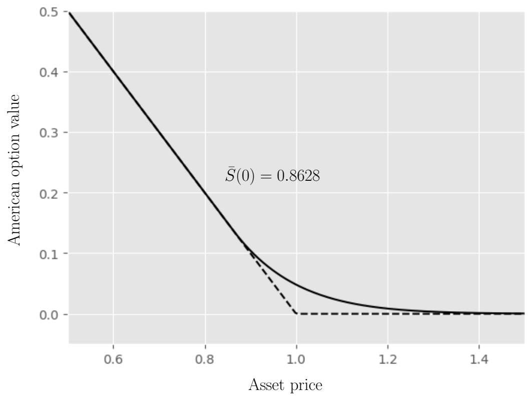
2 decimal places.



(a) Binary model of 2^{500} nodes.



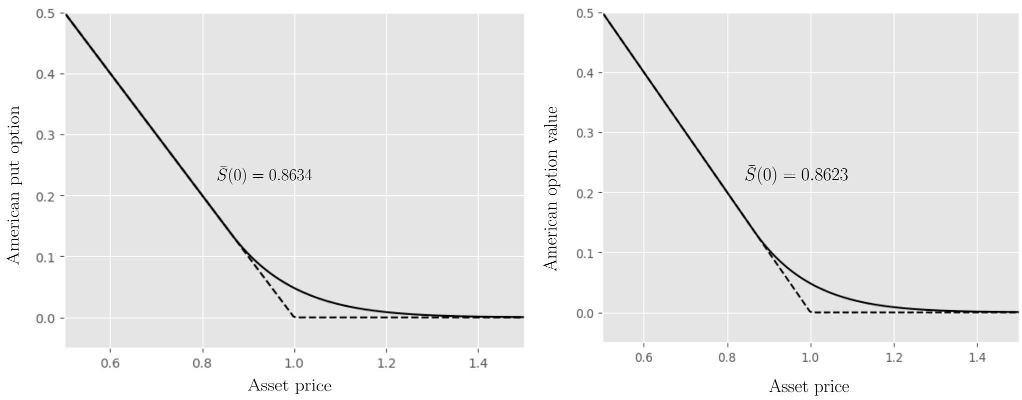
(b) Nielsen transformation with $\Delta x = 1\text{e}{-3}$ and $\Delta t = 0.5 \times 1\text{e}{-6}$



(c) Company transformation $\Delta x = 1\text{e}{-3}$ and $\Delta t = 0.5 \times 1\text{e}{-6}$

Figure 3.4: American put option value $V(S, 0)$ curve.

Additionally, table (3.3) present the RSME error produced by the implicit method for the Nielsen transformation and figure (3.5) show the value curve obtained by using the implicit method for the Nielsen approximation.



(a) Binary model of 2^{500} nodes.

(b) Implicit method for Nielsen transformation with $\Delta x = \Delta t = 1\mathrm{e}{-3}$.

Figure 3.5: American put option value $V(S, 0)$ curve.

Asset price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.200000	0.224810	0.227045	0.200000	0.200000	0.200000
1.0	0.048167	0.156141	0.159308	0.080886	0.053942	0.048944
1.2	0.008666	0.114815	0.117140	0.041319	0.013783	0.009290
1.4	0.001285	0.080638	0.081485	0.023604	0.003694	0.001519
1.6	0.000167	0.047914	0.048026	0.014269	0.001085	0.000229
1.8	0.000020	0.019130	0.020665	0.008591	0.000353	0.000033
2.0	0.000002	0.000022	0.003408	0.004695	0.000125	0.000005
	Asset price	0.06812077	0.06970396	0.02045601	0.00307761	0.00038755

Table 3.3: RSME error produced by the implicit scheme for the Nielsen transformation for $\Delta x = \Delta t = 1/8, 1/16, \ldots, 1/128$.

In the figures (3.4) and (3.5), we observe that the approximation follows the geometry of American options. Specifically, within the continuation region $S > \bar{S}(0)$, the value is larger than the payoff function and as S gets larger, the value goes to zero. In addition, the value curve is exactly the payoff. Also notice that the RSME of the implicit method for Nielsen transformation is larger than the explicit method for both Company and Nielsen transformation. The explanation we have for this is that for the implicit method we have to solve a non-linear system of equation. Therefore, the error produced by the implicit version of Nielsen will be affected by the error produced by non-linear solver.

We realized numerous testings to validate the correct functionality of our implementation. As Merton [9] showed, pricing American call options without dividends is equivalent to solve the price of European options. Therefore, the most natural test to conduct is to price American call options without dividends ($\delta = 0$) using our implementation and compare the results to the one obtain using the Black-Scholes formula for [9].

For the following experiments, we use the following set of parameters.

Moreover, (3.33), we tested each scheme for the case of put options. For the setup of the experiments, we priced a put option using the binomial model [4] with parameters (3.33). Then, we proceed to compute the RSME error

where V_i^{BM} is the price produced by the binomial model, and $\hat{V}_i$ is the price produced by our numerical method. In Tables (3.1) and (3.2) contain the results obtained for Nielsen and Company transformation, respectively, for $\Delta x = 1/8, 1/16, \ldots, 1/128$ and $\Delta t = \Delta x^2$

Asset price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000
1.0	0.048155	0.046048	0.047748	0.048176	0.048160	0.048155
1.2	0.008662	0.008661	0.008638	0.008705	0.008674	0.008662
1.4	0.001286	0.001467	0.001369	0.001313	0.001292	0.001286
1.6	0.000168	0.000237	0.000196	0.000176	0.000170	0.000168
1.8	0.000020	0.000038	0.000027	0.000022	0.000021	0.000020
2.0	0.000002	0.000006	0.000004	0.000003	0.000002	0.000002
	RMSE	0.000081	0.000017	0.000002	0.000000	0.000000

Asset price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.200000	0.224810	0.227045	0.200000	0.200000	0.200000
1.0	0.048180	0.156141	0.159308	0.080886	0.053942	0.048944
1.2	0.008661	0.114815	0.117140	0.041319	0.013783	0.009290
1.4	0.001283	0.080638	0.081485	0.023604	0.003694	0.001519
1.6	0.000167	0.047914	0.048026	0.014269	0.001085	0.000229
1.8	0.000020	0.019130	0.020665	0.008591	0.000353	0.000033
2.0	0.000002	0.000022	0.003408	0.004695	0.000125	0.000005
	RMSE	0.068119	0.069703	0.020455	0.003076	0.000385

Asset price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.200000	0.200000	0.200000	0.200000	0.200000	0.200000
1.0	0.048180	0.054188	0.053469	0.052440	0.052340	0.052259
1.2	0.008661	0.012658	0.010936	0.010331	0.010294	0.010271
1.4	0.001283	0.002423	0.001855	0.001694	0.001650	0.001640
1.6	0.000167	0.000462	0.000292	0.000250	0.000234	0.000229
1.8	0.000020	0.000107	0.000048	0.000034	0.000030	0.000029
2.0	0.000002	0.000028	0.000009	0.000005	0.000004	0.000004
	RMSE	0.002763	0.002187	0.001737	0.001695	0.001663

Asset Price	**BOPM**	0.125	0.0625	0.03125	0.015625	0.0078125
0.8	0.170000	0.170000	0.170000	0.170000	0.170000	0.170000
1.0	0.037785	0.038365	0.038260	0.037619	0.037754	0.037773
1.2	0.006576	0.007278	0.006645	0.006602	0.006573	0.006573
1.4	0.000953	0.001132	0.001013	0.000969	0.000953	0.000952
1.6	0.000122	0.000185	0.000142	0.000128	0.000124	0.000123
1.8	0.000015	0.000038	0.000021	0.000016	0.000015	0.000015
2.0	0.000002	0.000008	0.000003	0.000002	0.000002	0.000002
	RMSE	0.000035	0.000018	0.000006	0.000001	0.000000

4 Linear complementary problem

4.1 Overview

A linear complementary problem (LCP) is an optimization problem in which goal is to solve a system of linear equations subject to a complementary condition. In other words, given the matrix $\mathbf{A} \in \mathbb{R}^{d\times d}$ and vector $\mathbf{r} \in \mathbb{R}^d$, we want to find the vectors $\mathbf{v} \in \mathbb{R}^d$ and $\mathbf{w} \in \mathbb{R}^d$ such as

$$\mathbf{A}\mathbf{v} - \mathbf{w} = \mathbf{b}$$

subject to the complementary conditions

$$\begin{aligned} v_i &\geq 0 \qquad \text{for } i = 1, \dots, d \\ w_i &\geq 0 \qquad \text{for } i = 1, \dots, d \\ \mathbf{v}^T\mathbf{w} &= 0 \end{aligned}$$

Note that the complementary conditions require that either $v_i = 0$ or $w_i = 0$ for each component. As you will see later, the pricing problem for American options can be reformulated as a linear complementary problem. But first, we need to reconsider the results obtained from applying the Black-Scholes to pricing problem in section 2. Recall that the value curve of American options $V(S,t)$ is bounded from below by the payoff function

$$V(S,t) - H(S,t) \geq 0 \qquad \text{for all } (S,t)$$

Moreover, the value function can be divided in two complementary regions: the exercise region (2.8) where

$$V(S,t) - H(S,t) > 0 \qquad \text{for } (S,t) \in \mathcal{C} \tag{4.1}$$

$$\tag{4.2}$$

and continuation region (2.9) where

$$V(S,t) - H(S,t) = 0 \qquad \text{for } (S,t) \in \mathcal{S} \tag{4.3}$$

and the value function $V(S,t)$ is the solution to the Black-Scholes PDE within the continuation region

$$\frac{\partial V}{\partial t} + \mathcal{L}_{\mathsf{BS}}(V) = 0 \qquad \text{for } (S,t) \in \mathcal{C}$$

where $\mathcal{L}_{\mathsf{BS}}(V)$ is the linear parabolic operator (2.6) applied to the function $V(S,t)$. Finally, by plugin $V(S,t) = H(S,t)$ into the Black-Scholes PDE, we obtain the bound for PDE in the exercise region

$$\frac{\partial V}{\partial t} + \mathcal{L}_{\mathsf{BS}} < 0 \qquad \text{for } (S,t) \in \mathcal{S} \tag{4.4}$$

By grouping (4.1), (4.3), (4.4), we reformulate the problem of pricing American options as system of variational inequalities

$$\begin{cases} \left[\frac{\partial V}{\partial \tau} - \mathcal{L}_{\mathsf{BS}}(V)\right] \cdot [V(S,T-\tau) - H(S,T-\tau)] = 0 & \text{for all } (S,t) \\ V(S,T-\tau) - H(S,T-\tau) \geq 0 & \text{for } (S,t) \\ \frac{\partial V}{\partial \tau} - \mathcal{L}_{\mathsf{BS}}(V) \geq 0 & \text{for all } (S,t) \\ V(S,T) = H(S,T) & \end{cases} \tag{4.5}$$

The benefit of the variational inequalities is that there is no explicit dependence on the optimal exercise boundary defined in (2.10). Also, note that we rewrote the Black-Scholes PDE forward in time by introducing the transformation $\tau = T - t$ deliberately so that the variational inequalities' formulation looks similar in structure to LCP problem. But before elaborating more about the relationship between the variational inequalities and LCP problems, we introduce the transformations

$$S = Ke^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad h(x,\tau) := e^{(\alpha x + \beta \tau)} \frac{H(S,t)}{K}, \quad v(x,\tau) := V(S,t) =: Ke^{-(\alpha x+\beta\tau)} y(x,\tau) \tag{4.6}$$

where

$$q := \frac{2r}{\sigma^2}, \quad q_\delta := \frac{2(r-\delta)}{\sigma^2}, \quad \alpha := \frac{1}{2}(q_\delta - 1) \quad \beta := \frac{1}{4}(q_\delta - 1)^2 + q \tag{4.7}$$

so that the Black-Scholes PDE (2.5) to transform to the heat diffusion equation

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} = 0 \tag{4.8}$$

with boundary conditions, its solution $y(x,\tau)$ lies within the continuous region

$$\mathcal{F} : \mathbb{R} \times [0, \sigma^2 T/2] \tag{4.9}$$

Although we might solve (4.5) without transforming the problem, Dewynne et al. [5] and Seydel [12] suggest that transformation to the heat diffusion equation (4.8) introduces desirable numerical properties when applying finite difference schemes. Hence, under given transformation, the system of variational inequalities (4.5) becomes

$$\begin{cases} \left[\dfrac{\partial y}{\partial \tau} - \dfrac{\partial^2 y}{\partial x^2}\right] \cdot [y(x,\tau) - h(x,\tau)] = 0 & \text{for all } (x,\tau) \\ y(x,\tau) - h(x,\tau) \geq 0 & \text{for all } (x,\tau) \\ \dfrac{\partial y}{\partial \tau} - \dfrac{\partial^2 y}{\partial x^2} \geq 0 & \text{for all } (x,\tau) \\ y(x,0) = h(x,0) \end{cases} \tag{4.10}$$

Now that we have formulated the linear complementary problem in terms of the heat equation, we proceed to apply the finite difference scheme framework presented in section 3. Specifically, we want to approximate $y(x, \tau)$ at each of the nodes within the grid

$$\mathcal{G} := \{(x_i, \tau_n) : (i, n) \in \{0, \ldots, M+1\} \times \{0, \ldots, N+1\}\} \tag{4.11}$$

where

$$x_i := x_{\text{min}} + i\Delta x \qquad \text{for } i = 0, \ldots, M+1 \tag{4.12}$$

$$\tau_n := t_{\text{min}} + i\Delta \tau \qquad \text{for } i = 0, \ldots, N+1 \tag{4.13}$$

$$\Delta x := \frac{x_{\text{max}} - x_{\text{min}}}{M+1} \tag{4.14}$$

$$\Delta t := \frac{t_{\text{max}} - t_{\text{min}}}{N+1} \tag{4.15}$$

where x_{min} is set to a sufficiently small value, x_{max} to a sufficiently large value, τ_{min} to 0 and τ_{max} to $\frac{\sigma^2}{2}T$

We dedicate the Crank-Nicholson scheme. Morover, the Crank-Nicholson approximation is given by

$$\frac{y_i^{n+1} - y_i^n}{\Delta \tau} = \frac{1}{2}\left(\frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{(\Delta x)^2} + \frac{y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}}{(\Delta x)^2}\right) \tag{4.16}$$

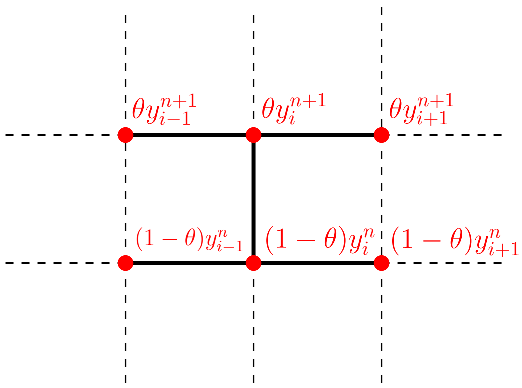


Figure 4.1: Stencil diagram for the Crank-Nicholson scheme.

4.2 Theta method

The theta method is a combination of the explicit and implicit method controlled by parameter $\theta \in [0, 1]$

$$\frac{y_i^{n+1} - y_i^n}{\Delta \tau} = (1-\theta)\frac{y_{i-1}^n - 2y_i^n + y_{i+1}^n}{(\Delta x)^2} + \theta\frac{y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}}{(\Delta x)^2} \tag{4.17}$$

for $i = 1, \ldots, M$, and $n = 0, \ldots, N$. The PDE approximation is fully explicit for $\theta = 0$, fully implicit for $\theta = 1$, and Crank-Nicholson for $\theta = 1/2$. Introducing $\lambda = \Delta t / \Delta x^2$, and rearranging (4.17), we have

$$y_i^{n+1} - \lambda\theta(y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^{n+1}) = y_i^n + (1-\theta)\lambda(y_{i-1}^n - 2y_i^n + y_{i+1}^n) \tag{4.18}$$

for $i = 1, \ldots, M$, and $n = 0, \ldots, N$. Also, let us define b_i^n

$$b_i^n := y_i^n + (1-\theta)\lambda(y_{i-1}^n - 2y_i^n + y_{i+1}^n) \tag{4.19}$$

for $i = 1, \ldots, M$, and $n = 0, \ldots, N$. Then, we define the vectors $\mathbf{b}^n \in \mathbb{R}^M$, $\mathbf{y}^{n+1} \in \mathbb{R}^M$ and $\mathbf{h}^n \in \mathbb{R}^M$ ad

$$\mathbf{b}^n := \left[b_1^n, \ldots, b_M^n\right]^T \tag{4.20}$$

$$\mathbf{y}^{n+1} := \left[y_1^{n+1}, \ldots, y_M^{n+1}\right]^T \tag{4.21}$$

$$\mathbf{h}^{n+1} := \left[h(x_1, \tau_{n+1}), \ldots, h(x_M, \tau_{n+1})\right]^T \tag{4.22}$$

and the matrix $\mathbf{A} \in \mathbb{R}^{M \times M}$ as

$$A := \begin{bmatrix} 1+2\theta & -\lambda\theta & & 0 & \\ -\lambda\theta & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & \end{bmatrix} \tag{4.23}$$

Hence, we can write the equations in (4.24) as

$$\begin{cases} (\mathbf{A}\mathbf{y}^{n+1} + \mathbf{b}^n)^T(\mathbf{y}^{n+1} - \mathbf{h}^{n+1}) = 0 \\ \mathbf{A}\mathbf{y}^{n+1} + \mathbf{b}^n \geq 0 \\ \mathbf{y}^{n+1} - \mathbf{h}^{n+1} \geq 0 \\ \mathbf{y}^0 = \mathbf{h}^0 \end{cases} \tag{4.24}$$

which is the LCP formulation. Note that if we define the vectors

$$\mathbf{v} := \mathbf{y}^{n+1} - \mathbf{h}^{n+1}$$

$$\mathbf{w} := \mathbf{A}\mathbf{y}^{n+1} - \mathbf{b}^{n}$$

$$\mathbf{r} := \mathbf{b}^{n} - \mathbf{A}\mathbf{h}^{n+1}$$

$$b_i^n := y_i^n + (1-\theta)\lambda(y_{i-1}^n - 2y_i^n + y_{i+1}^n) \tag{4.25}$$

$$h_i^n := h(x_i, \tau_n) \tag{4.26}$$

4.3 PSOR method

5 Conclusion

A FD schemes for Company transformation

A.1 Explicit scheme

The explicit scheme is given by

$$\begin{aligned}\frac{v_i^{n+1}-v_i^n}{\Delta t} &- \frac{1}{2}\sigma^2\frac{v_{i-1}^n-2v_i^n+v_{i+1}^n}{(\Delta x)^2}\\ &- \left((r-\delta)-\frac{\sigma^2}{2}-\frac{\bar{S}_f^{n+1}-\bar{S}_f^n}{\Delta t\bar{S}_f^n}\right)\frac{v_{i+1}^n-v_{i-1}^n}{2\Delta x}+rv_i^n=0\end{aligned}$$

for $i=1\ldots,M$ and $n=0,\ldots,N$.

$$v_i^{n+1}=av_{i-1}^n+bv_i^n+cv_{i+1}^n+\frac{\bar{S}_f^{n+1}-\bar{S}_f^n}{2\Delta x\bar{S}_f^n}(v_{i+1}^n-v_{i-1}^n)$$

where

$$\begin{aligned}\lambda=&\frac{\Delta t}{\Delta x^2}\\ a=&\frac{\lambda}{2}\left(\sigma^2-\left(r-\delta-\frac{\sigma^2}{2}\right)\Delta x\right)\\ b=&1-\sigma^2\lambda-r\Delta t\\ c=&\frac{\lambda}{2}\left(\sigma^2+\left(r-\delta-\frac{\sigma^2}{2}\right)\Delta x\right)\end{aligned}$$

Moreover, the boundary conditions

$$\begin{aligned}&\textbf{Call:}\qquad v_0^n=0,\quad v_{M+1}^n=\bar{S}_f^n-1\\ &\textbf{Put:}\qquad v_0^n=1-\bar{S}_f^n,\quad v_{M+1}^n=0\end{aligned}$$

for $n=0,\ldots,N$. Moreover, the contact point condition is approximated using central finite difference

$$\begin{aligned}&\textbf{Call:}\qquad \frac{v_{M+2}^n-v_M^n}{2\Delta x^2}=\frac{\partial v}{\partial x}(0,t)+O(\Delta x^2)\\ &\textbf{Put:}\qquad \frac{v_1^n-v_{-1}^n}{2\Delta x}=\frac{\partial v}{\partial x}(0,t)+O(\Delta x^2)\end{aligned}$$

for $n = 0, \ldots, N$. Moreover, the contact point condition is approximated using central finite difference

$$\textbf{Call:} \quad \frac{v^n_{M+2} - v^n_M}{2\Delta x^2} = \bar{S}^n_f$$
$$\textbf{Put:} \quad \frac{v^n_1 - v^n_{-1}}{2\Delta x} = -\bar{S}^n_f$$

By combining the central difference approximation for the PDE, the boundary conditions and the contact point, it is obtained

$$\textbf{Call:} \quad v^n_M =$$
$$\textbf{Put:} \quad v^n_1 = \alpha - \beta \bar{S}^n_f$$

Algorithm A.1 Explicit method for put options

for $i = 0, \ldots, M+1$ **do**

  $v_i^0 = 0$

$\bar{S}_f^0 = K$

$a = \frac{\lambda}{2}\left(\sigma^2 - \left(r - \delta - \frac{\sigma^2}{2}\right)\Delta x\right)$

$b = 1 - \sigma^2\lambda - r\Delta t$

$c = \frac{\lambda}{2}\left(\sigma^2 + \left(r - \delta - \frac{\sigma^2}{2}\right)\Delta x\right)$

$\alpha = 1 + \frac{r\Delta x^2}{\sigma^2}$

$\beta = 1 + \Delta x + \frac{1}{2}\Delta x^2$

for $n = 0, \ldots, N$ **do**

  $d^n = \frac{\alpha - (av_0^n + bv_1^0 + cv_2^n - (v_2^n - v_0^n)/(2\Delta x))}{(v_2^n - v_0^n)/(2\Delta x) + \beta\bar{S}_f^n}$

  $\bar{S}_f^{n+1} = d^n\bar{S}_f^n$

  $a^n = a - \frac{\bar{S}_f^{n+1} - \bar{S}_f^n}{2\Delta x\bar{S}_f^n}$

  $c^n = c - \frac{\bar{S}^{n+1} - \bar{S}^n}{2\Delta x\bar{S}_f^n}$

  $v_0^{n+1} = 1 - \bar{S}_f^{n+1}$

  $v_1^{n+1} = \alpha - \beta\bar{S}_f^{n+1}$

  $v_{M+1}^{n+1} = 0$

  for $i = 2, \ldots, M$ **do**

    $v_i^{n+1} = a^n v_{i-1}^n + bv_i^n + c^n v_{i+1}^n$

A.2 Implicit Scheme

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