Chapter 5
American Options

Pricing American contracts requires, due to the early exercise feature of such contracts, the solution of optimal stopping problems for the price process. Similar to the pricing of European contracts, the solutions of these problems have a deterministic characterization. Unlike in the European case, the pricing function of an American option does not satisfy a partial differential equation, but a partial differential inequality, or to be more precise, a system of inequalities. We consider the discretization of this inequality both by the finite difference and the finite element method where the latter is approximating the solutions of variational inequalities. The discretization in both cases leads to a sequence of linear complementarity problems (LCPs). These LCPs are then solved iteratively by the PSOR algorithm. Thus, from an algorithmic point of view, the pricing of an American option differs from the pricing of a European option only as in the latter we have to solve linear systems, whereas in the former we need to solve linear complementarity problems. The calculation of the stiffness matrix is the same for both options since the matrix depends on the model and not on the contract.

We assume that the dynamics of the stock price is modeled by a geometric Brownian motion and that no dividends are paid. Under this assumption, the value of an American call contract is equal to the value of the corresponding European call option. Therefore, we focus on put options in the following.
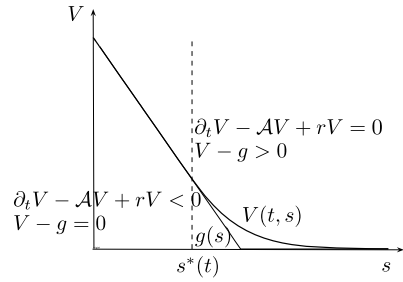
5.1 Optimal Stopping Problem

Recall that a *stopping time* τ for a given filtration $\mathcal{F}_t$ is a random variable taking values in $(0, \infty)$ and satisfying

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \forall t \geq 0.$$

Fig. 5.1 American put option

Denoting by $\mathcal{T}_{t,T}$ the set of all stopping times for S_t with values in the interval (t, T), the value of an American option is given by

$$V(t, s) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\big[e^{-r(\tau - t)} g(S_\tau) \mid S_t = s\big]. \tag{5.1}$$

As for the European vanilla style contracts, there is a close connection between the probabilistic representation (5.1) of the price and a deterministic, PDE based representation of the price. We have

Theorem 5.1.1 *Let $v(t, x)$ be a sufficiently smooth solution of the following system of inequalities*

$$\begin{aligned}
\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv &\geq 0 && in\ J \times \mathbb{R}, \\
v(t, x) &\geq g(e^x) && in\ J \times \mathbb{R}, \\
(\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv)(g - v) &= 0 && in\ J \times \mathbb{R}, \\
v(0, x) &= g(e^x) && in\ \mathbb{R}.
\end{aligned} \tag{5.2}$$

Then, $V(T - t, e^x) = v(t, x)$.

A proof can be found in [15], we also refer to [98] for further details. For each $t \in J$ there exists the so-called *optimal exercise price* $s^*(t) \in (0, K)$ such that for all $s \leq s^*(t)$ the value of the American put option is the value of immediate exercise, i.e. $V(t, s) = g(s)$, while for $s > s^*(t)$ the value exceeds the immediate exercise value, see Fig. 5.1. The region $\mathcal{C} := \{(t, s) \mid s > s^*(t)\}$ is called the *continuation region* and the complement $\mathcal{C}^c$ of $\mathcal{C}$ is the *exercise region*. Since the optimal exercise price is not known a priori, it is called a free boundary for the associated pricing PDE and the problem of determining the option price is then a *free boundary problem*. Note that the inequalities (5.2) do not involve the free boundary $s^*(t)$.

Remark 5.1.2 In the Black–Scholes model, the derivative of V at $x = s^*(t)$ is continuous which is known as the *smooth pasting* condition. This does not hold for pure jump models.

5.2 Variational Formulation

The set of admissible solutions for the variational form of (5.2) is the convex set $\mathcal{K}_g \subset H^1(\mathbb{R})$

$$\mathcal{K}_g := \{v \in H^1(\mathbb{R}) : v \geq g \text{ a.e. } x\}. \tag{5.3}$$

The variational form of (5.2) reads:

Find $u \in L^2(J; H^1(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R}))$ such that $u(t, \cdot) \in \mathcal{K}_g$ and

$$(\partial_t u, v - u) + a^{\text{BS}}(u, v - u) \geq 0, \ \forall v \in \mathcal{K}_g, \ \text{ a.e. in } J, \tag{5.4}$$

$$u(0) = g.$$

Since the bilinear form $a^{\text{BS}}(\cdot, \cdot)$ is continuous and satisfies a Gårding inequality in $H^1(\mathbb{R})$ by Proposition 4.2.1, problem (5.4) admits a unique solution for every payoff $g \in L^\infty(\mathbb{R})$ by Theorem B.2.2 of the Appendix B.

As in the case of plain European vanilla contracts, we localize (5.4) to a bounded domain $G = (-R, R)$, $R > 0$, by approximating the value v in (5.1)

$$v(t, x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} g(e^{X_\tau}) \mid X_t = x \right],$$

by the value of a barrier option

$$v_R(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} g(e^{X_\tau}) 1_{\{\tau < \tau_G\}} \mid X_t = x \right]. \tag{5.5}$$

Repeating the arguments in the proof of Theorem 4.3.1, we obtain the estimate for the localization error: there exist constants $C(T, \sigma), \gamma_1, \gamma_2 > 0$ such that

$$|v(t, x) - v_R(t, x)| \leq C(T, \sigma) \, e^{-\gamma_1 R + \gamma_2 |x|},$$

valid for payoff functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the growth condition (4.10). Thus, we consider problem (5.1) restricted on G

$$\begin{aligned}
\partial_t v_R - \mathcal{A}^{\text{BS}} v_R + r v_R &\geq 0 & \text{in } J \times G, \\
v_R(t, x) &\geq g(e^x)|_G & \text{in } J \times G, \\
(\partial_t v_R - \mathcal{A}^{\text{BS}} v_R + r v_R)(g|_G - v_R) &= 0 & \text{in } J \times G, \\
v_R(0, x) &= g(e^x)|_G & \text{in } G, \\
v_R(t, \pm R) &= g(e^{\pm R}) & \text{in } J.
\end{aligned} \tag{5.6}$$

We cast the truncated problem into variational form. To get a simple convex set for admissible functions and to facilitate the numerical solution, we introduce the *time value* of the option (also called *excess to payoff*) $w_R := v_R - g|_G$ and consider

$$\mathcal{K}_{0,R} := \{v \in H_0^1(G) : v \geq 0 \text{ a.e. } x \in G\}.$$

The variational formulation of the truncated problem reads then

Find $u_R \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G))$ such that $u_R(t, \cdot) \in \mathcal{K}_{0,R}$ and

$$(\partial_t u_R, v - u_R) + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(g, v - u_R), \ \forall v \in \mathcal{K}_{0,R}, \qquad (5.7)$$

$$u_R(0) = 0.$$

We calculate the right hand side in (5.7) for a put option, i.e. $g(s) = \max\{0, K - s\}$. Let $\varphi \in H_0^1(G)$. Then, by the definition of $a^{\mathrm{BS}}(\cdot, \cdot)$ in (4.9) and integration by parts,

$$
\begin{aligned}
-a^{\mathrm{BS}}(g, \varphi) &= -\frac{1}{2}\sigma^2 \int_{-R}^{\ln K} (K - e^x)' \varphi' \, \mathrm{d}x + \left(r - \frac{1}{2}\sigma^2\right) \int_{-R}^{\ln K} (K - e^x)' \varphi \, \mathrm{d}x \\
&\quad - r \int_{-R}^{\ln K} (K - e^x) \varphi \, \mathrm{d}x \\
&= \frac{1}{2}\sigma^2 e^x \varphi \Big|_{-R}^{\ln K} - \frac{1}{2}\sigma^2 \int_{-R}^{\ln K} e^x \varphi \, \mathrm{d}x - \left(r - \frac{1}{2}\sigma^2\right) \int_{-R}^{\ln K} e^x \varphi \, \mathrm{d}x \\
&\quad - r \int_{-R}^{\ln K} (K - e^x) \varphi \, \mathrm{d}x \\
&= \frac{1}{2}K\sigma^2 \varphi(\ln K) - rK \int_{-R}^{\ln K} \varphi \, \mathrm{d}x.
\end{aligned}
$$

Since this holds for all φ, $-a^{\mathrm{BS}}(g, \varphi)$ defines the linear functional $f = \frac{1}{2}K\sigma^2 \delta_{\ln K} - rK \chi_{\{x \leq \ln K\}} \in H^{-1}(G)$, where $\chi_{\mathcal{B}}$ is the indicator function.

5.3 Discretization

As in the European case, we approximate the option price both by the finite difference and the finite element method. The finite difference method discretizes the partial differential inequalities whereas the finite element method approximates the solution of variational inequalities. In both cases, the discretization leads to a sequence of linear complementarity problems. These LCPs are then solved iteratively by the PSOR algorithm.

5.3.1 Finite Difference Discretization

Discretizing (5.6) with finite differences and the backward Euler scheme, i.e. the θ-scheme with $\theta = 1$, we obtain a sequence of *matrix linear complementarity prob-*

lems

$$\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \dots, M-1,$$

$$\big(\mathbf{I} + k\mathbf{G}^{\text{BS}}\big)\underline{u}^{m+1} \geq \underline{u}^m + k\underline{f},$$

$$\underline{u}^{m+1} \geq \underline{u}_0, \tag{5.8}$$

$$(\underline{u}^{m+1} - \underline{u}_0)^\top \big((\mathbf{I} + k\mathbf{G}^{\text{BS}})\underline{u}^{m+1} - \underline{u}^m - k\underline{f}\big) = 0,$$

$$\underline{u}^0 = \underline{u}_0,$$

where $\mathbf{G}^{\text{BS}}$ and $\underline{u}_0$ are as in (4.14). The vector $\underline{f}$ accounts for the (non-homogeneous) Dirichlet boundary conditions and is given by $\underline{f} = k(f^-, 0, \dots, 0, f^+)^\top \in \mathbb{R}^N$, with $f^\pm := g(e^{\pm R})(\sigma^2/(2h^2) \mp (\sigma^2/2 - r)/(2h))$. Note that we cannot approximate the time value of the option $w_R = v_R - g|_G$ by the FDM, since the corresponding inequality (5.6) satisfied by w_R involves functionals $f \in H^{-1}(G)$ which cannot be approximated by finite difference quotients.

5.3.2 Finite Element Discretization

We discretize (5.7) using the backward Euler scheme and the finite element space $S_{\mathcal{T},0}^1$ given in (3.21). We obtain

$$\text{Find } \underline{u}_N^{m+1} \in \mathbb{R}_{\geq 0}^N \text{ such that for } m = 0, \dots, M-1,$$

$$(\underline{v} - \underline{u}_N^{m+1})^\top \big(\mathbf{M} + k\mathbf{A}^{\text{BS}}\big)\underline{u}_N^{m+1} \geq (\underline{v} - \underline{u}_N^{m+1})^\top \big(k\underline{f} + \mathbf{M}\underline{u}_N^m\big), \quad \forall \underline{v} \in \mathbb{R}_{\geq 0}^N, \tag{5.9}$$

$$\underline{u}_N^0 = \underline{0},$$

where $\underline{u}_N^m$ is the coefficient vector of $u_N(t_m) \in S_{\mathcal{T},0}^1 \cap \mathcal{K}_{0,R}$, and $\mathbf{M}$ and $\mathbf{A}^{\text{BS}}$ are as in the European case, see (4.15). The vector $\underline{f}$ is given by $\underline{f}_i = -a^{\text{BS}}(g, b_i)$. The sequence of inequalities (5.9) can be rewritten as sequence of LCPs.

Lemma 5.3.1 *Denote by* $\mathbf{B} := \mathbf{M} + k\mathbf{A}^{\text{BS}}$, $\underline{F}^m := k\underline{f} + \mathbf{M}\underline{u}_N^m$. *Then, problem* (5.9) *is equivalent to*: *given* $\underline{u}_N^0 = \underline{0}$, *find* $\underline{u}_N^{m+1} \in \mathbb{R}^N$ *such that for* $m = 0, \dots, M-1$,

$$\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m,$$

$$\underline{u}_N^{m+1} \geq \underline{0}, \tag{5.10}$$

$$(\underline{u}_N^{m+1})^\top \big(\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m\big) = 0.$$

Proof '$\Leftarrow$': Clearly, from $\mathbb{R}^N \ni \underline{u}_N^{m+1} \geq \underline{0}$ follows $\underline{u}_N^{m+1} \in \mathbb{R}_{\geq 0}^N$. From $\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m$, by the definition of $\mathbf{B}$ and $\underline{F}^m$, it follows

$$\underline{w}^{m+1} := \mathbf{M}(\underline{u}_N^{m+1} - \underline{u}_N^m) + k\mathbf{A}^{\text{BS}}\underline{u}_N^{m+1} - k\underline{f} \geq \underline{0}.$$

Now,

$$(\underline{v} - \underline{u}_N^{m+1})^\top \underline{w}^{m+1} = \underline{v}^\top \underbrace{\underline{w}^{m+1}}_{\geq \underline{0}} - \underbrace{(\underline{u}_N^{m+1})^\top \underline{w}^{m+1}}_{=0},$$

hence, $(\underline{v} - \underline{u}_N^{m+1})^\top \underline{w}^{m+1} = (\underline{v} - \underline{u}_N^{m+1})^\top (\mathbf{M}(\underline{u}_N^{m+1} - \underline{u}_N^m) + k\mathbf{A}^{\mathrm{BS}}\underline{u}_N^{m+1} - k\underline{f}) \geq 0$
for all $\underline{v} \in \mathbb{R}_{\geq 0}^N$, which is the second line of (5.9).

'$\Rightarrow$': From the second line of (5.9), we have $\forall \underline{v} \in \mathbb{R}_{\geq 0}^N$

$$\underline{v}^\top \underline{w}^{m+1} \geq (\underline{u}_N^{m+1})^\top \underline{w}^{m+1}.$$

Now suppose $(\underline{w}^{m+1})_k < 0$ for some $k \in \{1, \dots, N\}$, and let $(\underline{v})_k \gg 1$. Then the left hand side becomes arbitrarily small, which is a contradiction. Hence $\underline{w}^{m+1} \geq \underline{0}$, which is the inequality $\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m$. Now

$$\underline{u}_N^{m+1} \in \mathbb{R}_{\geq 0}^N \Rightarrow (\underline{u}_N^{m+1})^\top \underline{w}^{m+1} \geq 0.$$

Set in the second line of (5.9) $\underline{v} = \underline{0}$, i.e. $(-\underline{u}_N^{m+1})^\top \underline{w}^{m+1} \geq 0$, hence $(\underline{u}_N^{m+1})^\top \underline{w}^{m+1} \leq 0$. We conclude $(\underline{u}_N^{m+1})^\top \underline{w}^{m+1} = 0$, which is $(\underline{u}_N^{m+1})^\top (\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m) = 0$. $\qquad\square$

We give a convergence result for the price approximated by FEM. Let $\{u_N^m\}_{m=1}^M$ be the solution of (5.9) and let $u^m := u(t_m, x)$ be the solution of (5.4) at time level t_m. The proof of the next convergence result is shown in [121, Theorem 22].

Theorem 5.3.2 *Assume $u \in C^0((0, T]; H^2(G)) \cap C^{1,1}(\overline{J}; L^2(G))$. Then, there exists a constant $C = C(u) > 0$ such that the following error bound holds*

$$\max_m \|u^m - u_N^m\|_{L^2(G)} + \left(k \sum_{m=1}^M \|u^m - u_N^m\|_{H^1(G)}^2 \right)^{1/2} \leq C(k + h).$$

Thus, as in the European case (compare with Theorem 3.6.5), we obtain first order convergence in the energy norm $\|u^M - u_N^M\|_{H^1(G)} = \mathcal{O}(k + h)$, provided that $u(t, x)$ is sufficiently smooth.

5.4 Numerical Solution of Linear Complementarity Problems

In the following, two methods for solving the derived LCPs are described. Both methods are iterative approaches.

Table 5.1 Description of the PSOR algorithm

Choose an initial guess $\underline{x}^0 \geq \underline{c}$.
Choose $\omega \in (0, 1]$ and $\varepsilon > 0$.
For $k = 0, 1, 2, \ldots,$
  For $i = 1, \ldots, N$,

$$\widetilde{x}_i^{k+1} := \frac{1}{\mathbf{A}_{ii}} \Big(b_i - \sum_{j=1}^{i-1} \mathbf{A}_{ij} x_j^{k+1} - \sum_{j=i+1}^{N} \mathbf{A}_{ij} x_j^k \Big)$$

$$x_i^{k+1} := \max \big\{ c_i, x_i^k + \omega(\widetilde{x}_i^{k+1} - x_i^k) \big\}$$

  Next i
  If $\|\underline{x}^{k+1} - \underline{x}^k\|_2 < \varepsilon$ stop else
Next k

5.4.1 Projected Successive Overrelaxation Method

As shown in Eqs. (5.9) and (5.10), the numerical pricing by both the FDM and FEM of an American option reduces to solving (in each time step) an LCP. Both can be written in the abstract form: find $\underline{x} \in \mathbb{R}^N$ such that

$$\mathbf{A}\underline{x} \geq \underline{b},$$
$$\underline{x} \geq \underline{c}, \tag{5.11}$$
$$(\underline{x} - \underline{c})^\top (\mathbf{A}\underline{x} - \underline{b}) = 0.$$

Problem (5.11) was solved by Cryer [47] using the *projected successive overrelaxation* (PSOR) method for matrices $\mathbf{A}$ being symmetric and positive definite. In applications in finance, however, $\mathbf{A}$ is not symmetric due the presence of a drift term in the infinitesimal generator of the price process. It can be shown that PSOR works also for matrices which are not symmetric, but diagonally dominant.

The algorithm is described in Table 5.1 where we denote by $\mathbf{A}_{ij}$ the entries of $\mathbf{A}$, i.e. $\mathbf{A} = (\mathbf{A}_{ij})_{1 \leq i, j \leq N}$, and we assume that $\mathbf{A}_{ii} \neq 0$, $\forall i$. If the step $x_i^{k+1} := \max\{c_i, x_i^k + \omega(\widetilde{x}_i^{k+1} - x_i^k)\}$ is replaced by

$$x_i^{k+1} = x_i^k + \omega(\widetilde{x}_i^{k+1} - x_i^k),$$

then we get the classical SOR for the solution of $\mathbf{A}\underline{x} = \underline{b}$. This step ensures the inequality condition $\underline{x} \geq \underline{c}$. The parameter ω is a relaxation parameter. Since the PSOR is an iterative algorithm, we have to discuss its convergence. The following result is proven in [1, Chap. 6].

Proposition 5.4.1 *Assume* $\mathbf{A}$ *satisfies*

(i) *There exist constants* $C_1, C_2 > 0$ *such that* $C_1 \underline{v}^\top \underline{v} \leq \underline{v}^\top \mathbf{A}\underline{v} \leq C_2 \underline{v}^\top \underline{v}$, $\forall \underline{v} \in \mathbb{R}^N$;
(ii) *It is diagonally dominant, i.e.* $|\mathbf{A}_{ii}| > \sum_{j \neq i} |\mathbf{A}_{ij}|$, $\forall i$.

Furthermore, assume $\omega \in (0, 1]$. *Then, the sequence* $\{\underline{x}^k\}$ *generated by PSOR converges, as* $k \to \infty$, *to the unique solution* $\underline{x}$ *of* (5.11).

Table 5.2 Description of the primal–dual active set algorithm

Choose an initial guess $\underline{x}^0 \geq \underline{c}$, $\underline{\lambda}^0 \geq 0$.
Choose $\varepsilon > 0$.
For $k = 0, 1, 2, \ldots,$
  Set $\mathcal{I}_k = \{i : \lambda_i^k + k_1(c_i^k - x_i^k) \leq 0\}$,
    $\mathcal{A}_k = \{i : \lambda_i^k + k_1(c_i^k - x_i^k) > 0\}$
  Solve $\mathbf{A}\underline{x}^{k+1} + \underline{\lambda}^{k+1} = \underline{b}$, $\underline{x}^{k+1} = \underline{c}$ on $\mathcal{A}_k$, $\underline{\lambda}^{k+1} = \underline{0}$ on $\mathcal{I}_k$.
  If $\|\underline{x}^{k+1} - \underline{x}^k\|_2 < \varepsilon$ stop else
Next k

Note that assumption (i) of Proposition 5.4.1 ensures the existence of a unique solution of (5.11).

5.4.2 Primal–Dual Active Set Algorithm

Problem (5.11) can be formulated as follows: find $\underline{x}, \underline{\lambda} \in \mathbb{R}^N$ such that

$$
\begin{aligned}
\mathbf{A}\underline{x} + \underline{\lambda} &= \underline{b}, \\
\underline{x} &\geq \underline{c}, \\
\underline{\lambda} &\geq 0, \\
(\underline{x} - \underline{c})^\top \underline{\lambda} &= 0.
\end{aligned}
\tag{5.12}
$$

Problem (5.12) was solved by Ito and Kunisch [84] using the primal–dual active set algorithm that can also be formulated as a semi-smooth Newton method for P-matrices $\mathbf{A}$. A matrix is called a P-matrix if all its principal minors are positive. This can be shown to hold in our setup for the FEM discretization as well as for the FDM discretization for sufficiently small time-step k. The complementarity system in (5.12) can equivalently be expressed as

$$\mathcal{C}(\underline{x}, \underline{\lambda}) = 0, \ \text{where} \ \mathcal{C}(\underline{x}, \underline{\lambda}) := \underline{\lambda} - \max(0, \underline{\lambda} + k_1(\underline{c} - \underline{x})),$$

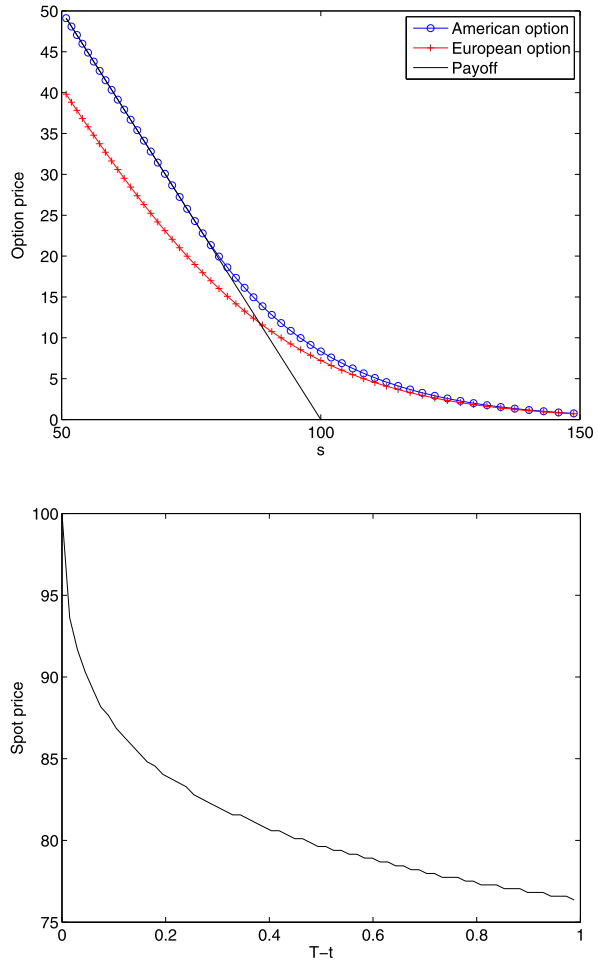for each $k_1 > 0$. Therefore, (5.12) is equivalent to

$$
\begin{aligned}
\mathbf{A}\underline{x} + \underline{\lambda} &= \underline{b}, \\
\mathcal{C}(\underline{x}, \underline{\lambda}) &= 0.
\end{aligned}
\tag{5.13}
$$

The primal–dual active set algorithm is based on using (5.13) for a prediction strategy, i.e. given a pair $(\underline{x}, \underline{\lambda})$, the active and inactive sets are given as

$$\mathcal{I} = \{i : \lambda_i + k_1(c_i - x_i) \leq 0\}, \quad \mathcal{A} = \{i : \lambda_i + k_1(c_i - x_i) > 0\}.$$

This leads to the algorithm given in Table 5.2.

Fig. 5.2 American option price (*top*) and corresponding exercise boundary (*bottom*)

Proposition 5.4.2 *Let* **A** *satisfy* (i) *and* (ii) *from Proposition* 5.4.1 *and let* $(\underline{x}^*, \underline{\lambda}^*)$ *be the unique solution of* (5.13), *then the primal dual active set method converges superlinearly to* $(\underline{x}^*, \underline{\lambda}^*)$, *provided that* $\|\underline{x}^0 - \underline{x}^*\|_2^2 + \|\underline{\lambda}^0 - \underline{\lambda}^*\|_2^2$ *is sufficiently small.*

A proof of this result can be found in [84, Theorem 3.1].

Example 5.4.3 Consider an American put with strike $K = 100$ and maturity $T = 1$. For $\sigma = 0.3$, $r = 0.1$, we compute the price of the option (at $t = T$) as well as the free boundary $s^*(t)$. The results are shown in Fig. 5.2, where we also plot the option price of the corresponding European option, which is, due to the single exercise right, lower than the price of the American contract.

5.5 Further Reading

Approximate, semi-analytic solutions of American option pricing problems can, for example, be found in Barone-Adesi and Whaley [12], Carr [33] or Geske and Johnson [70]. An overview of various methods for pricing an American put in a Black–Scholes setting is given in Barone-Adesi [11]. A rigorous treatment is provided by Jaillet et al. [98] where the Brennan and Schwartz algorithm [25] is used for the discretization. A semi-smooth Newton approach is analyzed in Ito et al. [84, 92] and Hager and Wohlmuth [77]. Holtz and Kunoth [87] provide high-order B-spline approximations for American puts in the Black–Scholes model and the corresponding Greeks.

Computable a posteriori error bounds on the discretization error in the numerical solution of American put contracts in a Black–Scholes setting have been obtained in [126].