Chapter 4
European Options in BS Markets

In the last chapters, we explained various methods to solve partial differential equations. These methods are now applied to obtain the price of a European option. We assume that the stock price follows a geometric Brownian motion and show that the option price satisfies a parabolic PDE. The unbounded log-price domain is localized to a bounded domain and the error incurred by the truncation is estimated. It is shown that the variational formulation has a unique solution and the discretization schemes for finite element and finite differences are derived. Furthermore, we describe extensions of the Black–Scholes model, like the constant elasticity of variance (CEV) and the local volatility model.

4.1 Black–Scholes Equation

Let X be the solution of the SDE (1.2), where we assume that the coefficients b, σ are independent of time t and satisfy the assumptions of Theorem 1.2.6. Further assume $r(x)$ to be a bounded and continuous function modeling the riskless interest rate. We want to compute the value of the option with payoff g which is the conditional expectation

$$V(t,x) = \mathbb{E}\left[e^{-\int_t^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_t = x\right]. \tag{4.1}$$

We show that $V(t,x)$ is a solution of a deterministic partial differential equation. Therefore, we first relate to the process X a differential operator $\mathcal{A}$, the so-called *infinitesimal generator* of the process X.

Proposition 4.1.1 *Let $\mathcal{A}$ denote the differential operator which is, for functions $f \in C^2(\mathbb{R})$ with bounded derivatives, given by*

$$(\mathcal{A}f)(x) = \frac{1}{2}\sigma^2(x)\partial_{xx} f(x) + b(x)\partial_x f(x). \tag{4.2}$$

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Then, the process $M_t := f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ is a martingale with respect to the filtration of W.

Proof We apply the Itô formula (1.7) to $f(X_t)$ and obtain, in integral form,

$$f(X_t) = f(X_0) + \int_0^t \partial_x f(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \partial_{xx} f(X_s)\sigma^2(X_s)\,\mathrm{d}s.$$

Using $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$, and the definition of $\mathcal{A}$, we have

$$f(X_t) = f(X_0) + \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s) f'(X_s)\,\mathrm{d}W_s.$$

The result follows if we can show that the stochastic integral $\int_0^t \sigma(X_s)\partial_x f(X_s)\,\mathrm{d}W_s$ is a martingale (with respect to the filtration of W). According to Proposition 1.2.7, it is sufficient to show that $\mathbb{E}[\int_0^t |\sigma(X_s)\partial_x f(X_s)|^2\,\mathrm{d}s] < \infty$. Since f has bounded derivatives and σ satisfies (1.4), we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^t |\sigma(X_s)|^2|\partial_x f(X_s)|^2\,\mathrm{d}s\right] &\le C^2 \sup_{x\in\mathbb{R}} |\partial_x f(x)|^2 \mathbb{E}\left[\int_0^t (1+|X_s|^2)\,\mathrm{d}s\right]\\ &\le TC^2 \sup_{x\in\mathbb{R}} |\partial_x f(x)|^2 \big(1+\mathbb{E}\big[\sup_{0\le s\le T}|X_s|^2\big]\big) \overset{(1.5)}{<} \infty.\end{aligned}$$

□

Remark 4.1.2 For $t > 0$ denote by X_t^x the solution of the SDE (1.2) starting from x at time 0. Then, since $M_t = f(X_t) - \int_0^t (\mathcal{A}f)(X_s)\,\mathrm{d}s$ is a martingale by Proposition 4.1.1, we know that $\mathbb{E}[M_0] = f(x) = \mathbb{E}[M_t]$. Therefore,

$$\mathbb{E}[f(X_t^x)] = f(x) + \mathbb{E}\left[\int_0^t (\mathcal{A}f)(X_s^x)\,\mathrm{d}s\right].$$

Since by assumption f has bounded derivatives and b, σ satisfy the global Lipschitz and linear growth condition (1.3)–(1.4)

$$\begin{aligned}\mathbb{E}\big[\sup_{0\le s\le T} |(\mathcal{A}f)(X_s^x)|\big] &\le C\mathbb{E}\big[\sup_{0\le s\le T} |\sigma^2(X_s^x)| + |b(X_s^x)|\big]\\ &\le C'\big(1+\mathbb{E}\big[\sup_{0\le s\le T}|X_s^x|^2\big]\big) < \infty.\end{aligned}$$

Thus, since $\mathcal{A}f$ and X_t^x are continuous, the dominated convergence theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(X_t^x)]|_{t=0} = \lim_{t\to 0}\mathbb{E}\left[\frac{1}{t}\int_0^t (\mathcal{A}f)(X_s^x)\,\mathrm{d}s\right] = (\mathcal{A}f)(x).$$

Therefore, $\mathcal{A}$ is called the *infinitesimal generator* of the process X_t^x.

For the purpose of option pricing, we need a discounted version of Proposition 4.1.1.

Proposition 4.1.3 *Let* $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ *with bounded derivatives in* x, *let* $\mathcal{A}$ *be as in* (4.2) *and assume that* $r \in C^0(\mathbb{R})$ *is bounded. Then, the process*

$$M_t := e^{-\int_0^t r(X_s)\,\mathrm{d}s} f(t, X_t) - \int_0^t e^{-\int_0^s r(X_\tau)\,\mathrm{d}\tau} (\partial_t f + \mathcal{A} f - r f)(s, X_s)\,\mathrm{d}s,$$

is a martingale with respect to the filtration of W.

Proof Denote by Z the process $Z_t := e^{-\int_0^t r(X_s)\,\mathrm{d}s}$. Then

$$\mathrm{d}(Z_t f(t, X_t)) = \mathrm{d}Z_t f(t, X_t) + Z_t\,\mathrm{d}f(t, X_t),$$

with $\mathrm{d}Z_t = -r(X_t) Z_t\,\mathrm{d}t$, and thus, by the Itô formula (1.7),

$$\begin{aligned}\mathrm{d}(Z_t f(t, X_t)) = & -r(X_t) Z_t f(t, X_t) + Z_t \big(\partial_t f(t, X_t)\,\mathrm{d}t + \partial_x f(t, X_t)\,\mathrm{d}X_t \\ & + \frac{1}{2}\sigma^2(X_t)\partial_{xx} f(t, X_t)\,\mathrm{d}t\big) \\ = & Z_t \big((-r f(t, X_t) + \partial_t f(t, X_t) + (\mathcal{A} f)(t, X_t))\,\mathrm{d}t \\ & + \sigma(X_t)\partial_x f(t, X_t)\,\mathrm{d}W_t\big).\end{aligned}$$

Thus, we need to show that $\int_0^t Z_s \sigma(X_s)\partial_x f(s, X_s)\,\mathrm{d}W_s$ is a martingale. But

$$\mathbb{E}\left[\int_0^t |Z_s \sigma(X_s)\partial_x f(s, X_s)|^2\,\mathrm{d}s\right] < \infty,$$

by the boundedness of r and by repeating the estimates in the proof of Proposition 4.1.1. □

We now are able to link the stochastic representation of the option price (4.1) with a parabolic partial differential equation.

Theorem 4.1.4 *Let* $V \in C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ *with bounded derivatives in* x *be a solution of*

$$\partial_t V + \mathcal{A} V - r V = 0 \quad \text{in } J \times \mathbb{R}, \qquad V(T, x) = g(x) \quad \text{in } \mathbb{R}, \tag{4.3}$$

with $\mathcal{A}$ *as in* (4.2). *Then,* $V(t, x)$ *can also be represented as*

$$V(t, x) = \mathbb{E}\left[e^{-\int_t^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_t = x\right].$$

Proof We show the result only for $t = 0$. Since $\partial_t V + \mathcal{A}V - rV = 0$, we have, by Proposition 4.1.3, that the process $M_t := e^{-\int_0^t r(X_s)\,\mathrm{d}s} V(t, X_t)$ is a martingale. Thus,

$$\begin{aligned} V(0,x) &= \mathbb{E}[M_0 \mid X_0 = x] \\ &= \mathbb{E}[M_T \mid X_0 = x] \\ &= \mathbb{E}\Big[e^{-\int_0^T r(X_s)\,\mathrm{d}s} V(T, X_T) \mid X_0 = x\Big] \\ &= \mathbb{E}\Big[e^{-\int_0^T r(X_s)\,\mathrm{d}s} g(X_T) \mid X_0 = x\Big]. \end{aligned}$$

□

Remark 4.1.5 The converse of Theorem 4.1.4 is also true. Any $V(t,x)$ as in (4.1), which is $C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ with bounded derivatives in x, solves the PDE (4.3).

We apply Theorem 4.1.4 to the Black–Scholes model [21]. In the Black–Scholes market, the risky asset's spot-price is modeled by a geometric Brownian motion S_t, i.e. the SDE for this model is as in (1.2), with coefficients $b(t,s) = rs$, $\sigma(t,s) = \sigma s$, where $\sigma > 0$ and $r \geq 0$ denote the (constant) volatility and the (constant) interest rate, respectively. Therefore, the SDE is given by

$$\mathrm{d}S_t = r S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t.$$

We assume for simplicity that *no dividends are paid*. Based on Theorem 4.1.4, we get that the discounted price of a European contract with payoff $g(s)$, i.e. $V(0,s) = \mathbb{E}[e^{-rT} g(S_T) | S_0 = s]$, is equal to a regular solution $V(0,s)$ of the Black–Scholes equation

$$\partial_t V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V + rs\partial_s V - rV = 0 \quad \text{in } J \times \mathbb{R}_+. \tag{4.4}$$

The BS equation (4.4) needs to be completed by the *terminal condition*, $V(T,s) = g(s)$, depending on the type of option. Equation (4.4) is a parabolic PDE with the second order "spatial" differential operator

$$(\mathcal{A}f)(s) = \frac{1}{2}\sigma^2 s^2 \partial_{ss} f(s) + rs\partial_s f(s), \tag{4.5}$$

which degenerates at $s = 0$. To obtain a non-degenerate operator with constant coefficients, we switch to the price process $X_t = \log(S_t)$ which solves the SDE

$$\mathrm{d}X_t = \Big(r - \frac{1}{2}\sigma^2\Big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

The infinitesimal generator for this process has constant coefficients and is given by

$$(\mathcal{A}^{\mathrm{BS}} f)(x) = \frac{1}{2}\sigma^2 \partial_{xx} f(x) + \Big(r - \frac{1}{2}\sigma^2\Big)\partial_x f(x). \tag{4.6}$$

We furthermore change to *time-to-maturity* $t \to T - t$, to obtain a forward parabolic problem. Thus, by setting $V(t,s) =: v(T-t, \log s)$, the BS equation in real price (4.4) satisfied by $V(t,s)$ becomes the BS equation for $v(t,x)$ in *log-price*

$$\partial_t v - \mathcal{A}^{\mathrm{BS}} v + rv = 0 \quad \text{in } (0,T) \times \mathbb{R}, \tag{4.7}$$

with the initial condition $v(0,x) = g(e^x)$ in $\mathbb{R}$.

Remark 4.1.6 For put and call contracts with strike $K > 0$, it is convenient to introduce the so-called *log-moneyness* variable $x = \log(s/K)$ and setting the option price $V(t,s) := K w(T-t, \log(s/K))$. Then, the function $w(t,x)$ again solves (4.7), with the initial condition $w(0,x) = g(Ke^x)/K$. Thus, the initial condition becomes for a put $w(0,x) = \max\{0, 1 - e^x\}$ and for a call $w(0,x) = \max\{0, e^x - 1\}$ where both payoffs now do not depend on K.

4.2 Variational Formulation

We give the variational formulation of the Black–Scholes equation (4.7) which reads:

$$\begin{aligned}
&\text{Find } u \in L^2(J; H^1(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R})) \text{ such that} \\
&(\partial_t u, v) + a^{\mathrm{BS}}(u,v) = 0, \ \forall v \in H^1(\mathbb{R}), \ \text{a.e. in } J, \\
&u(0) = u_0,
\end{aligned} \tag{4.8}$$

where $u_0(x) := g(e^x)$ and the bilinear form $a^{\mathrm{BS}}(\cdot,\cdot) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ is given by

$$a^{\mathrm{BS}}(\varphi, \phi) := \frac{1}{2}\sigma^2(\varphi', \phi') + (\sigma^2/2 - r)(\varphi', \phi) + r(\varphi, \phi). \tag{4.9}$$

We show that $a^{\mathrm{BS}}(\cdot,\cdot)$ is continuous (3.8) and satisfies a Gårding inequality (3.9) on $\mathcal{V} = H^1(\mathbb{R})$.

Proposition 4.2.1 *There exist constants* $C_i = C_i(\sigma, r) > 0$, $i = 1,2,3$, *such that for all* $\varphi, \phi \in H^1(\mathbb{R})$

$$|a^{\mathrm{BS}}(\varphi,\phi)| \le C_1 \|\varphi\|_{H^1(\mathbb{R})} \|\phi\|_{H^1(\mathbb{R})}, \quad a^{\mathrm{BS}}(\varphi,\varphi) \ge C_2 \|\varphi\|^2_{H^1(\mathbb{R})} - C_3 \|\varphi\|^2_{L^2(\mathbb{R})}.$$

Proof We first show continuity. By Hölder's inequality,

$$\begin{aligned}
|a^{\mathrm{BS}}(\varphi,\phi)| &\le \frac{1}{2}\sigma^2 \|\varphi'\|_{L^2(\mathbb{R})} \|\phi'\|_{L^2(\mathbb{R})} + |\sigma^2/2 - r| \|\varphi'\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \\
&\quad + r \|\varphi\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \\
&\le C_1(\sigma, r) \|\varphi\|_{H^1(\mathbb{R})} \|\phi\|_{H^1(\mathbb{R})}.
\end{aligned}$$

To show coercivity, note that with $\int_{\mathbb{R}} \varphi' \varphi \, dx = \frac{1}{2} \int_{\mathbb{R}} (\varphi^2)' \, dx = 0$ we have

$$\begin{aligned} a^{\mathrm{BS}}(\varphi, \varphi) &= \frac{1}{2}\sigma^2 \|\varphi'\|^2_{L^2(\mathbb{R})} + r\|\varphi\|^2_{L^2(\mathbb{R})} = \frac{1}{2}\sigma^2 \|\varphi\|^2_{H^1(\mathbb{R})} + (r - \sigma^2/2)\|\varphi\|^2_{L^2(\mathbb{R})} \\ &\geq \frac{1}{2}\sigma^2 \|\varphi\|^2_{H^1(\mathbb{R})} - |r - \sigma^2/2| \|\varphi\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

□

Referring to the abstract existence result Theorem 3.2.2 in the spaces $\mathcal{V} = H^1(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$, we deduce that the variational problem (4.8) admits a unique weak solution $u \in L^2(J; H^1(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R}))$ for every $u_0 \in L^2(\mathbb{R})$. Since $u_0(x) = g(e^x)$, $u_0 \in L^2(\mathbb{R})$ implies an unrealistic growth condition on the payoff g. In the next section, we reformulate the problem on a bounded domain where this condition can be weakened. In particular, we require the following *polynomial growth condition* on the payoff function: There exist $C > 0$, $q \geq 1$ such that

$$g(s) \leq C(s+1)^q, \quad \text{for all } s \in \mathbb{R}_+. \tag{4.10}$$

This condition is satisfied by the payoff function of all standard contracts like, e.g. plain vanilla European call, put or power options.

4.3 Localization

The unbounded log-price domain $\mathbb{R}$ of the log price $x = \log s$ is truncated to a bounded domain G. In terms of financial modeling, this corresponds to approximating the option price by a knock-out barrier option. Let $G = (-R, R)$, $R > 0$ be an open subset and let $\tau_G := \inf\{t \geq 0 \mid X_t \in G^c\}$ be the first hitting time of the complement set $G^c = \mathbb{R} \backslash G$ by X. Then, the price of a knock-out barrier option in log-price with payoff $g(e^x)$ is given by

$$v_R(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(e^{X_T}) 1_{\{T < \tau_G\}} \mid X_t = x\right]. \tag{4.11}$$

We show that the barrier option price v_R converges to the option price

$$v(t, x) = \mathbb{E}\left[e^{-r(T-t)} g(e^{X_T}) \mid X_t = x\right],$$

exponentially fast in R.

Theorem 4.3.1 *Suppose the payoff function* $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *satisfies* (4.10). *Then, there exist* $C(T, \sigma), \gamma_1, \gamma_2 > 0$, *such that*

$$|v(t, x) - v_R(t, x)| \leq C(T, \sigma)\, e^{-\gamma_1 R + \gamma_2 |x|}.$$

Proof Let $M_T = \sup_{\tau \in [t,T]} |X_\tau|$. Then, with (4.10)

$$|v(t, x) - v_R(t, x)| \leq \mathbb{E}\big[g(e^{X_T}) 1_{\{T \geq \tau_G\}} \mid X_t = x\big] \leq C\mathbb{E}\big[e^{q M_T} 1_{\{M_T > R\}} \mid X_t = x\big].$$

Using [143, Theorem 25.18], it suffices to show that there exist a constant $C(T,\sigma) > 0$ such that

$$\mathbb{E}\left[e^{q|X_T|}1_{\{|X_T|>R\}} \mid X_t = x\right] \le C(T,\sigma)\, e^{-\gamma_1 R + \gamma_2 |x|}.$$

We have for $\mu = r - \sigma^2/2$, with the transition probability p_{T-t},

$$\begin{aligned}
\mathbb{E}\left[e^{q|X_T|}1_{\{|X_T|>R\}} \mid X_t = x\right] &= \int_{\mathbb{R}} e^{q|z+x|} 1_{\{|z+x|>R\}}\, p_{T-t}(z)\, \mathrm{d}z \\
&\le e^{q|x|} \int_{\mathbb{R}} e^{q|z|} 1_{\{|z+x|>R\}} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\, e^{-(z-\mu(T-t))^2/(2\sigma^2(T-t))}\, \mathrm{d}z \\
&\le C_1(T,\sigma)\, e^{q|x|} \int_{\mathbb{R}} e^{(q+\mu/\sigma^2)|z|} 1_{\{|z+x|>R\}} e^{-z^2/(2\sigma^2(T-t))}\, \mathrm{d}z \\
&\le C_1(T,\sigma)\, e^{q|x|} \int_{\mathbb{R}} e^{-(\eta-q-\mu/\sigma^2)(R-|x|)} e^{\eta|z|} e^{-z^2/(2\sigma^2(T-t))}\, \mathrm{d}z \\
&\le C_1(T,\sigma)\, e^{-\gamma_1 R + \gamma_2|x|} \int_{\mathbb{R}} e^{\eta|z|} e^{-z^2/(2\sigma^2(T-t))}\, \mathrm{d}z,
\end{aligned}$$

with $\gamma_1 = \eta - q - \mu/\sigma^2$, and $\gamma_2 = \gamma_1 + q$. Since $\int_{\mathbb{R}} e^{\eta|z|} e^{-z^2/(2\sigma^2(T-t))}\, \mathrm{d}z < \infty$ for any $\eta > 0$, we obtain the required result by choosing $\eta > q + \mu/\sigma^2$. □

Remark 4.3.2 We see from Theorem 4.3.1 that $v_R \to v$ exponentially for a fixed x as $R \to \infty$. The artificial zero Dirichlet barrier type conditions at $x = \pm R$ are *not* describing correctly the asymptotic behavior of the price $v(t,x)$ for large $|x|$. Since the barrier option price v_R is a good approximation to v for $|x| \ll R$, R should be selected substantially larger than the values of x of interest.

The barrier option price v_R can again be computed as the solution of a PDE provided some smoothness assumptions.

Theorem 4.3.3 *Let* $v_R(t,x) \in C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ *be a solution of*

$$\partial_t v_R + \mathcal{A}^{\mathrm{BS}} v_R - r v_R = 0 \tag{4.12}$$

on $(0,T) \times G$ *where the terminal and boundary conditions are given by*

$$v_R(T,x) = g(e^x), \quad \forall x \in G, \quad v_R(t,x) = 0 \quad \text{on } (0,T) \times G^c.$$

Then, $v_R(t,x)$ *can also be represented as in* (4.11).

Now we can restate the problem (4.8) on the bounded domain:

Find $u_R \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G))$ such that

$$
\begin{aligned}
&(\partial_t u_R, v) + a^{\mathrm{BS}}(u_R, v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } J, \\
&u_R(0) = u_0|_G.
\end{aligned} \tag{4.13}
$$

By Proposition 4.2.1 and Theorem 3.2.2, the problem (4.13) is well-posed, i.e. there exists a unique solution $u_R \in L^2(0, T; H_0^1(G)) \cap C^0([0, T]; L^2(G))$ which can be approximated by a finite element Galerkin scheme.

4.4 Discretization

We use the finite element and the finite difference method to discretize the Black–Scholes equation. As for the heat equation in Chap. 2, we use the variational formulation of the differential equations for FEM and determine approximate solutions that are piecewise linear. For FDM we replace the derivatives in the differential equation by difference quotients.

4.4.1 Finite Difference Discretization

We discretize the PDE (4.12) directly using finite differences on a bounded domain with homogeneous Dirichlet boundary conditions. Proceeding as in Sect. 2.3.1, we obtain the matrix problem:

$$
\begin{aligned}
&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \ldots, M-1, \\
&\big(\mathbf{I} + \theta k \mathbf{G}^{\mathrm{BS}}\big)\underline{u}^{m+1} = \big(\mathbf{I} - (1-\theta)k\mathbf{G}^{\mathrm{BS}}\big)\underline{u}^{m}, \\
&\underline{u}^0 = \underline{u}_0,
\end{aligned} \tag{4.14}
$$

where $\mathbf{G}^{\mathrm{BS}} = \sigma^2/2\,\mathbf{R} + (\sigma^2/2 - r)\mathbf{C} + r\mathbf{I}$, is given explicitly with

$$
\mathbf{R} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}, \quad \mathbf{C} = \frac{1}{2h}\begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}.
$$

4.4.2 Finite Element Discretization

We discretize (4.13) using the θ-scheme and the finite element space $V_N = S_{\mathcal{T}}^1 \cap H_0^1(G)$ with $S_{\mathcal{T}}^1$ given as in (3.17). Setting $u_0(x) := g(e^x)$ and proceeding exactly as in Sect. 3.3 with uniform mesh width h and uniform time steps k, we obtain the matrix problem:

$$\begin{aligned}
&\text{Find } \underline{u}^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \ldots, M-1 \\
&(\mathbf{M} + k\theta \mathbf{A}^{\mathrm{BS}})\underline{u}^{m+1} = (\mathbf{M} - k(1-\theta)\mathbf{A}^{\mathrm{BS}})\underline{u}^{m}, \\
&\underline{u}^0 = \underline{u}_0,
\end{aligned} \tag{4.15}$$

where $\mathbf{M}_{ij} = (b_j, b_i)_{L^2(G)}$ and $\mathbf{A}^{\mathrm{BS}}_{ij} = a^{\mathrm{BS}}(b_j, b_i)$. Let $\mathbf{M}$ be given as in (2.25). Using (3.22), we can compute $\mathbf{A}^{\mathrm{BS}} = \sigma^2/2\mathbf{S} + (\sigma^2/2 - r)\mathbf{B} + r\mathbf{M}$ explicitly with

$$\mathbf{S} = \frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{2}\begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}. \tag{4.16}$$

4.4.3 Non-smooth Initial Data

As already mentioned, the advantage of finite elements is that we have low smoothness assumptions on the initial data u_0, and therefore on the payoff function g. In particular, as shown in Theorem 3.2.2, we have a unique solution for every $u_0 \in L^2(G)$. However, according to Theorem 3.6.5, we need $u_0 \in H^2(G)$ to obtain the optimal convergence rate $\|u - u_N\|_{L^2(J;L^2(G))} = \mathcal{O}(h^2 + k^r)$ where $r = 1$ for $\theta \in [0,1] \setminus \{1/2\}$ and $r = 2$ for $\theta = 1/2$. This is due to the time discretization since uniform time steps are used. To recover the optimal convergence rate for $u_0 \in H^s(G)$, $0 < s < 2$ we need to use *graded meshes* in time or space. We assume for simplicity $T = 1$.

Let $\lambda : [0,1] \to [0,1]$ be a grading function which is strictly increasing and satisfies

$$\lambda \in C^0([0,1]) \cap C^1((0,1)), \quad \lambda(0) = 0, \quad \lambda(1) = 1.$$

We define for $M \in \mathbb{N}$ the algebraically graded mesh by the time points,

$$t_m = \lambda\Big(\frac{m}{M}\Big), \quad m = 0, 1, \ldots, M.$$

It can be shown [146, Remark 3.11] that we obtain again the optimal convergence rate if $\lambda(t) = \mathcal{O}(t^\beta)$ where β depends on r and s, $\beta = \beta(r, s)$ (algebraic grading).

Example 4.4.1 Consider the payoff functions

$$g_c(s) = \begin{cases} s - K & \text{if } s > K, \\ 0 & \text{else,} \end{cases} \qquad g_d(s) = \begin{cases} 1 & \text{if } s > K, \\ 0 & \text{else.} \end{cases}$$

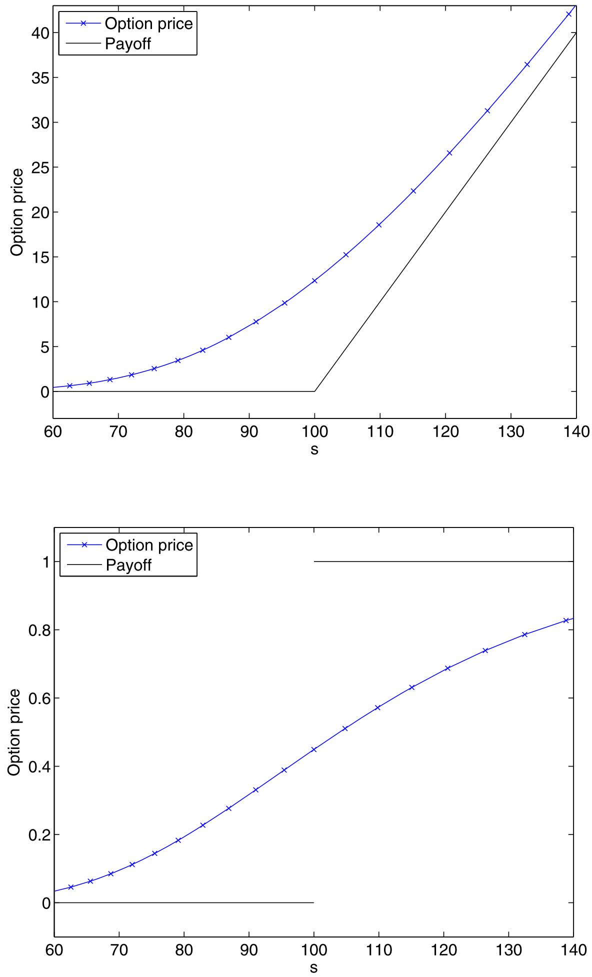


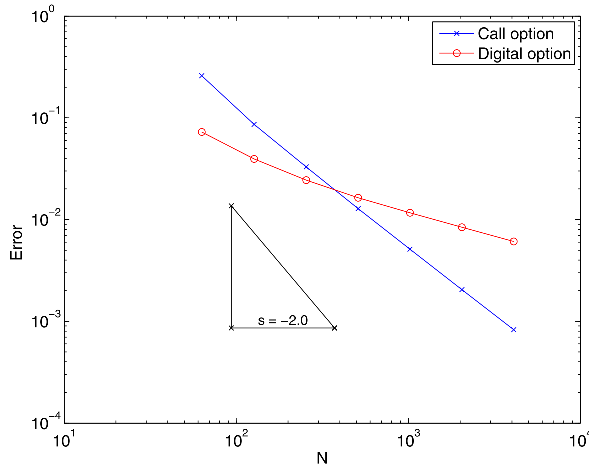
Fig. 4.1 Option price of European call (*top*) and digital (*bottom*) option

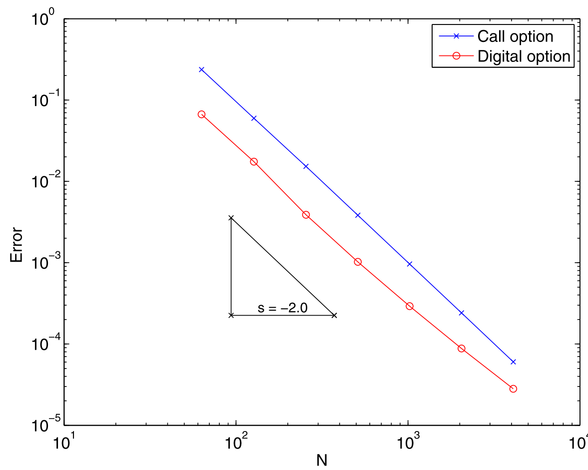
We set strike $K = 100$, volatility $\sigma = 0.3$, interest rate $r = 0.01$, maturity $T = 1$. For the discretization we use $N = 300$, $\theta = 1/2$, $R = 3$ and apply the L^2-projection for u_0. As in Remark 4.1.6, we use $K = 1$ in the calculations, for which $R = 3$ is a sufficiently large localization parameter. Using time steps $M = \mathcal{O}(N)$, we obtain the option prices shown in Fig. 4.1.

For $G_0 = (K/2, 3/2K)$ we measure the discrete $L^2(J; L^2(G_0))$-error defined by

$$\sqrt{\sum_{m=1}^{M} k_m h \|\underline{\varepsilon}^m\|_{\ell_2}^2}, \quad \text{with} \quad \|\underline{\varepsilon}^m\|_{\ell_2}^2 := \sum_{i=1}^{N} |u(t_m, x_i) - u_N(t_m, x_i)|^2,$$

Fig. 4.2 $L^2(J; L^2(G))$ convergence rate for $\theta = 1/2$ using uniform time steps (*top*) and graded time steps (*bottom*)





both with uniform time steps and with graded time steps. The exact values are obtained using the analytic formulas [21]. We use the algebraic grading factor $\beta = 3$ for the call option and the extreme grading factor $\beta = 25$ for the digital option.

It can be seen in Fig. 4.2 that for the call and the digital option we only obtain the optimal convergence rate using a graded time mesh.

Remark 4.4.2 Note that the theory from [146, Chap. 3.3] suggests that choosing $\beta = 10$ is sufficient in the case of a digital option in the Black–Scholes market to obtain full convergence order of the given discretization. But the analysis in [146, Chap. 3.3] is performed in a semi-discrete setting, i.e. there is no discretization error in the space domain. This explains why in our situation, discretizing in space and time, a larger grading factor has to be chosen.

4.5 Extensions of the Black–Scholes Model

We end this chapter by considering two extensions of the Black–Scholes model

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t, \quad S_0 = s \geq 0.$$

In the *constant elasticity of variance* (CEV) model, σS_t is replaced by σS_t^ρ for some $0 < \rho < 1$. Another possible extension is to replace the constant volatility σ by a deterministic function $\sigma(s)$ which leads to the so-called *local volatility* models.

4.5.1 CEV Model

Under a unique equivalent martingale measure, the stock price dynamics are given by

$$dS_t = rS_t\,dt + \sigma S_t^\rho\,dW_t, \qquad S_0 = s \geq 0, \tag{4.17}$$

where ρ is the *elasticity of variance*. We assume $0 < \rho < 1$. Under this condition, the point 0 is an attainable state. As soon as $S = 0$, we keep S equal to zero, with the resulting process still satisfying (4.17). Note that (4.17) is of the form (1.2) but with $\sigma(t,s) = \sigma s^\rho$, non-Lipschitz. The transformation to log-price, $x = \log s$, will not allow removing the factor s^ρ. A formal application of Theorem 4.1.4 yields that the value $V(t,s)$ of a European vanilla with payoff g is the solution of

$$\partial_t V + \mathcal{A}_\rho^{\mathrm{CEV}} V - rV = 0 \quad \text{in } J \times \mathbb{R}_{\geq 0}, \tag{4.18}$$

with the terminal condition $V(T,s) = g(s)$ and generator

$$(\mathcal{A}_\rho^{\mathrm{CEV}} f)(s) = \frac{1}{2}\sigma^2 s^{2\rho}\partial_{ss} f(s) + rs\partial_s f(s).$$

Note that for $\rho = 1$, the CEV generator is the same as the BS generator. In (4.18), we change to time-to-maturity $t \to T - t$ and localize to a bounded domain $G := (0, R)$, $R > 0$. Thus, we consider

$$\begin{aligned} \partial_t v - \mathcal{A}^{\mathrm{CEV}} v + rv &= 0 && \text{in } J \times G, \\ v &= 0 && \text{on } J \times \{R\}, \\ v(0,s) &= g(s) && \text{in } G. \end{aligned} \tag{4.19}$$

For the variational formulation of (4.19), we multiply the first equation in (4.19) by a test function w and integrate from $s = 0$ to $s = R$. Using $s^{2\rho}\partial_{ss}^2 v = \partial_s(s^{2\rho}\partial_s v) - 2\rho s^{2\rho-1}\partial_s v$, we find, upon integration by parts,

$$\int_0^R s^{2\rho}\partial_{ss} v w\,ds = -\int_0^R s^{2\rho}\partial_s v \partial_s w\,ds - 2\rho\int_0^R s^{2\rho-1}\partial_s v w\,ds + s^{2\rho}\partial_s v w\Big|_{s=0}^{s=R}.$$

The boundary terms vanish for $w \in C_0^\infty(G)$.

Define the weighted Sobolev space

$$W_\rho = \overline{C_0^\infty(G)}^{\|\cdot\|_\rho}, \tag{4.20}$$

where the weighted Sobolev norm $\|\cdot\|_\rho$ is defined by

$$\|\varphi\|_\rho^2 := \int_0^R (s^{2\rho}|\partial_s\varphi|^2 + |\varphi|^2)\,\mathrm{d}s, \quad 0 \le \rho \le 1. \tag{4.21}$$

For $\varphi, \phi \in C_0^\infty(G)$, we define the bilinear form

$$\begin{aligned} a_\rho^{\mathrm{CEV}}(\varphi,\phi) := \frac{1}{2}\sigma^2 \int_0^R s^{2\rho}\partial_s\varphi\partial_s\phi\,\mathrm{d}s + \rho\sigma^2 \int_0^R s^{2\rho-1}\partial_s\varphi\phi\,\mathrm{d}s \\ - r\int_0^R s\partial_s\varphi\phi\,\mathrm{d}s + r\int_0^R \varphi\phi\,\mathrm{d}s. \end{aligned} \tag{4.22}$$

The variational formulation of (4.19) is based on the triple of spaces $\mathcal{V} = W_\rho \hookrightarrow \mathcal{H} = L^2(G) = \mathcal{H}^* \hookrightarrow W_\rho^* = \mathcal{V}^*$, and reads:

$$\begin{aligned} &\text{Find } v \in L^2(J; W_\rho) \cap H^1(J; L^2(G)) \text{ such that} \\ &(\partial_t v, w) + a_\rho^{\mathrm{CEV}}(v, w) = 0, \ \forall w \in W_\rho, \ \text{a.e. in } J, \\ &v(0) = g. \end{aligned} \tag{4.23}$$

To establish well-posedness of (4.23), we show continuity and coercivity of the bilinear form $a_\rho^{\mathrm{CEV}}(\cdot,\cdot)$ on W_ρ.

Proposition 4.5.1 *Assume $r > 0$. There exist $C_1, C_2 > 0$ such that for $\varphi, \phi \in W_\rho$*

$$|a_\rho^{\mathrm{CEV}}(\varphi,\phi)| \le C_1\|\varphi\|_\rho\|\phi\|_\rho, \quad \rho \in [0,1]\backslash\{1/2\}, \tag{4.24}$$

$$a_\rho^{\mathrm{CEV}}(\varphi,\varphi) \ge C_2\|\varphi\|_\rho^2, \quad 0 \le \rho \le \frac{1}{2}. \tag{4.25}$$

Proof Let $\varphi \in C_0^\infty(G)$. By Hardy's inequality, for $\varepsilon \ne 1$, $\varepsilon > 0$, and any $R > 0$

$$\left(\int_0^R s^{\varepsilon-2}|\varphi|^2\,\mathrm{d}s\right)^{\frac{1}{2}} \le \frac{2}{|\varepsilon-1|}\left(\int_0^R s^{\varepsilon}|\partial_s\varphi|^2\,\mathrm{d}s\right)^{\frac{1}{2}}, \tag{4.26}$$

we find with $\varepsilon = 2\rho \ne 1$, that

$$\begin{aligned} \left|\int_0^R s^{2\rho-1}\partial_s\varphi\phi\,\mathrm{d}s\right| &\le \left(\int_0^R s^{2\rho}(\partial_s\varphi)^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^R s^{2\rho-2}\phi^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\ &\le \|\varphi\|_\rho \frac{2}{|2\rho-1|}\|\phi\|_\rho \end{aligned}$$

and, by the Cauchy–Schwarz inequality,

$$\left|\int_0^R s\partial_s\varphi\phi\,\mathrm{d}s\right| \le \|\varphi\|_\rho\Big(\int_0^R s^{2-2\rho}\phi^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \le \|\varphi\|_\rho R^{1-\rho}\,\|\phi\|_{L^2(G)}.$$

Thus, for $\varphi, \phi \in C_0^\infty(G)$, $\rho \neq \frac{1}{2}$, one has

$$|a_\rho^{\mathrm{CEV}}(\varphi,\phi)| \le C(\rho,\sigma,r)\|\varphi\|_\rho\|\phi\|_\rho.$$

Hence, we may extend the bilinear form $a_\rho^{\mathrm{CEV}}(\cdot,\cdot)$ from $C_0^\infty(G)$ to W_ρ by continuity for $\rho \in [0,1]\backslash\{\frac{1}{2}\}$. Furthermore, we have

$$\begin{aligned} a_\rho^{\mathrm{CEV}}(\varphi,\varphi) &= \frac{1}{2}\sigma^2\|s^\rho\partial_s\varphi\|^2_{L^2(G)} + \frac{1}{2}\rho\sigma^2\int_0^R s^{2\rho-1}\partial_s(\varphi^2)\,\mathrm{d}s \\ &\quad - \frac{1}{2}r\int_0^R s\partial_s(\varphi^2)\,\mathrm{d}s + r\int_0^R \varphi^2\,\mathrm{d}s. \end{aligned}$$

Integrating by parts, we get, for $0 \le \rho \le \frac{1}{2}$,

$$\int_0^R s^{2\rho-1}\partial_s(\varphi^2)\,\mathrm{d}s = -(2\rho-1)\int_0^R s^{2\rho-2}\varphi^2\,\mathrm{d}s \ge 0.$$

Analogously, $\frac{1}{2}\int_0^R s\partial_s(\varphi^2)\,\mathrm{d}s = -\frac{1}{2}\int_0^R \varphi^2\,\mathrm{d}s$, hence we get for $0 \le \rho \le \frac{1}{2}$

$$a_\rho^{\mathrm{CEV}}(\varphi,\varphi) \ge \frac{1}{2}\sigma^2\|s^\rho\partial_s\varphi\|^2_{L^2(G)} + \frac{3}{2}r\|\varphi\|_{L^2(G)} \ge \frac{1}{2}\min\{\sigma^2, 3r\}\|\varphi\|_\rho. \qquad \square$$

By Theorem 3.2.2, we deduce

Corollary 4.5.2 *Problem* (4.23) *admits a unique solution* $V \in L^2(J;W_\rho) \cap H^1(J;L^2(G))$ *for* $0 \le \rho < 1/2$.

The previous result addressed only the case $0 \le \rho < \frac{1}{2}$. The case $\frac{1}{2} \le \rho < 1$ (which includes, for $\rho = \frac{1}{2}$, the Heston model and the CIR process) requires a modified variational framework due to the failure of the Hardy inequality (4.26) for $\varepsilon = 1$. Let us develop this framework. We multiply the first equation in (4.19) by an $s^{2\mu}w$, where μ is a parameter to be selected and $w \in C_0^\infty(G)$ is a test function, and integrate from $s = 0$ to $s = R$. We get from (4.19)

$$(\partial_t v, s^{2\mu}w) + a_{\rho,\mu}^{\mathrm{CEV}}(v,w) = 0, \quad \forall w \in C_0^\infty(G), \tag{4.27}$$

where the bilinear form $a_{\rho,\mu}^{\mathrm{CEV}}(\cdot,\cdot)$ is defined by

$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\phi) := \frac{1}{2}\sigma^2 \int_0^R s^{2\rho+2\mu}\partial_s\varphi\partial_s\phi\,\mathrm{d}s + \sigma^2(\rho+\mu)\int_0^R s^{2\rho+2\mu-1}\partial_s\varphi\phi\,\mathrm{d}s$$
$$- r\int_0^R s^{1+2\mu}\partial_s\varphi\phi\,\mathrm{d}s + r\int_0^R s^{2\mu}\varphi\phi\,\mathrm{d}s. \tag{4.28}$$

Note that $a^{\mathrm{CEV}}_{\rho,0} = a^{\mathrm{CEV}}_{\rho}$ and $a^{\mathrm{CEV}}_{1} = a^{\mathrm{BS}}$. We introduce the spaces $W_{\rho,\mu}$ as closures of $C_0^\infty(G)$ with respect to the norm

$$\|\varphi\|^2_{\rho,\mu} := \int_0^R \left(s^{2\rho+2\mu}|\partial_s\varphi|^2 + s^{2\mu}|\varphi|^2\right)\mathrm{d}s\,, \tag{4.29}$$

compare with (4.20). Note that $\|\varphi\|_\rho = \|\varphi\|_{\rho,0}$. We now show the analog to Proposition 4.5.1, for $\rho \in [1/2, 1]$.

Proposition 4.5.3 *Assume* $0 \le \rho \le 1$ *and select*

$$\begin{cases} \mu = 0 & \text{if } 0 \le \rho < \frac{1}{2},\ \rho = 1, \\ -\frac{1}{2} < \mu < \frac{1}{2} - \rho & \text{if } \frac{1}{2} \le \rho < 1. \end{cases} \tag{4.30}$$

Assume also $r > 0$. *Then there exist* $C_1, C_2 > 0$ *such that* $\forall \varphi, \phi \in W_{\rho,\mu}$ *the following holds*:

$$|a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\phi)| \le C_1\|\varphi\|_{\rho,\mu}\|\phi\|_{\rho,\mu}, \tag{4.31}$$
$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\varphi) \ge C_2\|\varphi\|^2_{\rho,\mu}. \tag{4.32}$$

Proof The continuity (4.31) of $a^{\mathrm{CEV}}_{\rho,\mu}$ in $W_{\rho,\mu}(0,R)\times W_{\rho,\mu}(0,R)$ follows from the Cauchy–Schwarz inequality and by Hardy's inequality (4.26) with $\varepsilon = 2(\rho+\mu) \neq 1$

$$|a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\phi)| \le \frac{1}{2}\sigma^2\|\varphi\|_{\rho,\mu}\|\phi\|_{\rho,\mu} + \sigma^2(\rho+\mu)\|\varphi\|_{\rho,\mu}\left(\int_0^R s^{2\rho+2\mu-2}\phi^2\,\mathrm{d}s\right)^{1/2}$$
$$+ r\|\varphi\|_{\rho,\mu}\left(\int_0^R s^{2+2\mu-2\rho}\phi^2\,\mathrm{d}s\right)^{1/2}$$
$$\le \left(\frac{\sigma^2}{2} + \frac{2\sigma^2(\rho+\mu)}{|2\rho+2\mu-1|} + rR^{1-\rho}\right)\|\varphi\|_{\rho,\mu}\|\phi\|_{\rho,\mu}.$$

Let $\varphi \in C_0^\infty(G)$. We calculate

$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\varphi) = \frac{1}{2}\sigma^2\|s^{\rho+\mu}\partial_s\varphi\|^2_{L^2(G)} + \frac{1}{2}\sigma^2(\rho+\mu)\int_0^R s^{2\rho+2\mu-1}\partial_s(\varphi^2)\,\mathrm{d}s$$
$$- \frac{1}{2}r\int_0^R s^{1+2\mu}\partial_s(\varphi^2)\,\mathrm{d}s + r\|s^\mu\varphi\|^2_{L^2(G)}$$

$$= \frac{1}{2}\sigma^2 \|s^{\rho+\mu}\varphi_s\|^2_{L^2(G)} - \frac{1}{2}\sigma^2(\rho+\mu)(2\rho+2\mu-1)\int_0^R s^{2\rho+2\mu-2}\varphi^2\,\mathrm{d}s$$
$$+ \frac{1}{2}r(1+2\mu)\int_0^R s^{2\mu}\varphi^2\,\mathrm{d}s + r\|s^\mu\varphi\|^2_{L^2(G)}.$$

Given $1/2 \le \rho < 1$, we now choose μ such that $-1/2 \le \mu < 1/2-\rho$. Then, $2\rho + 2\mu - 1 < 0$, $1+2\mu \ge 0$, $\rho+\mu \ge 0$, and we get

$$a^{\mathrm{CEV}}_{\rho,\mu}(\varphi,\varphi) \ge \frac{1}{2}\sigma^2\|s^{\rho+\mu}\partial_s\varphi\|^2_{L^2(G)} + r\|s^\mu\varphi\|^2_{L^2(G)} \ge \frac{1}{2}\min\{\sigma^2, 2r\}\|\varphi\|^2_{\rho,\mu}.$$

By density of $C_0^\infty(0,R)$ in $W_{\rho,\mu}$, we have shown (4.32). □

We are now ready to cast Eq. (4.19) into the abstract parabolic framework. We choose μ as in (4.30) and observe that $\mu \le 0$ then. Hence, if $\mathcal{H}_\mu := L^2(0,R; s^{2\mu}\,\mathrm{d}s)$ denotes the weighted L^2-space corresponding to $W_{\rho,\mu}$, we have the dense inclusions

$$W_{\rho,\mu} \hookrightarrow \mathcal{H}_\mu \cong (\mathcal{H}_\mu)^* \hookrightarrow (W_{\rho,\mu})^*. \tag{4.33}$$

Denote by $(\cdot,\cdot)_\mu$ the inner product in $\mathcal{H}_\mu$. The weak formulation then reads:

$$\begin{aligned}
&\text{Find } v \in L^2(J; W_{\rho,\mu}) \cap H^1(J;\mathcal{H}_\mu) \text{ such that}\\
&(\partial_t v, w)_\mu + a^{\mathrm{CEV}}_{\rho,\mu}(v,w) = 0,\ \forall w \in W_{\rho,\mu},\quad \text{a.e. in } J,\\
&v(0) = g.
\end{aligned} \tag{4.34}$$

Applying Theorem 3.2.2, we have shown

Theorem 4.5.4 *Let* $\rho \in [0,1]$, *and assume that* μ *satisfies* (4.30). *Then, the problem* (4.34) *admits a unique solution.*

4.5.2 Local Volatility Models

We replace the constant volatility σ in the Black–Scholes model (1.1) by a deterministic function $\sigma(s)$, i.e. the Black–Scholes model is extended to

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma(S_t)S_t\,\mathrm{d}W_t, \tag{4.35}$$

with $r \in \mathbb{R}_{\ge 0}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that the function $s \mapsto s\sigma(s)$ satisfies (1.3)–(1.4), such that the SDE (4.35) admits a unique solution. Thus, the infinitesimal generator $\mathcal{A}$ of the process S is

$$(\mathcal{A}f)(s) = \frac{1}{2}s^2\sigma^2(s)\partial_{ss}f(s) + rs\partial_s f(s).$$

It follows from Theorem 4.1.4 that the option price V in (4.1) solves $\partial_t V + \mathcal{A}V - rV = 0$ in $\overline{J} \times \mathbb{R}_+$, $V(T,s) = g(s)$ in $\mathbb{R}_+$. As in the Black–Scholes model, we switch to time-to-maturity $t \to T - t$, to log-price $x = \ln(s)$ and localize the PDE to a bounded domain $G = (-R, R)$, $R > 0$. Thus, we consider the parabolic problem for $v(t,x) = V(T-t, e^x)$

$$\begin{aligned} \partial_t v - \mathcal{A}^{\mathrm{LV}} v + rv &= 0 && \text{in } J \times G, \\ v &= 0 && \text{on } J \times \partial G, \\ v(0,x) &= g(e^x) && \text{in } G, \end{aligned} \tag{4.36}$$

where, for $\widetilde{\sigma}(x) := \sigma(e^x)$, we denote by $\mathcal{A}^{\mathrm{LV}}$ the operator

$$(\mathcal{A}^{\mathrm{LV}} f)(x) = \frac{1}{2}\widetilde{\sigma}^2(x)\partial_{xx} f(x) + \Big(r - \frac{1}{2}\widetilde{\sigma}^2(x)\Big)\partial_x f(x).$$

The weak formulation to (4.36) reads:

$$\begin{aligned} &\text{Find } u \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G)) \text{ such that} \\ &(\partial_t u, v) + a^{\mathrm{LV}}(u, v) = 0, \ \forall v \in H_0^1(G), \ \text{a.e. in } J, \\ &u(0) = g, \end{aligned} \tag{4.37}$$

where the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot) : H_0^1(G) \times H_0^1(G) \to \mathbb{R}$ is given by

$$a^{\mathrm{LV}}(\varphi, \phi) := \frac{1}{2}\int_G \widetilde{\sigma}^2 \varphi' \phi' \,\mathrm{d}x + \int_G \big(\widetilde{\sigma}\widetilde{\sigma}' + \widetilde{\sigma}^2/2 - r\big)\varphi' \phi \,\mathrm{d}x + r\int_G \varphi\phi \,\mathrm{d}x.$$

Note that the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$ is a particular case of the bilinear form $a(\cdot,\cdot)$ defined in (3.16) by identifying the coefficients $\alpha(x) = \widetilde{\sigma}^2(x)/2$, $\beta(x) = (\widetilde{\sigma}\widetilde{\sigma}')(x) + \widetilde{\sigma}^2(x)/2 - r$ and $\gamma(x) = r$. Hence, the stiffness matrix $\mathbf{A}^{\mathrm{LV}}$ corresponding to the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$ can be implemented using the method described in Sect. 3.4.

Proposition 4.5.5 *Assume that $\widetilde{\sigma} : \mathbb{R} \to \mathbb{R}_+$ satisfies $\widetilde{\sigma} \in W^{1,\infty}(G)$ and $\exists \sigma_0 > 0$ such that $\widetilde{\sigma}(x) \geq \sigma_0 > 0$, $\forall x \in G$. Then, there exist constants $C_i > 0$, $i = 1,2,3$, such that for all $\varphi, \phi \in H_0^1(G)$ there holds*

$$\begin{aligned} |a^{\mathrm{LV}}(\varphi, \phi)| &\leq C_1 \|\varphi\|_{H^1(G)} \|\phi\|_{H^1(G)}, \\ |a^{\mathrm{LV}}(\varphi, \varphi)| &\geq C_2 \|\varphi\|_{H^1(G)}^2 - C_3 \|\varphi\|_{L^2(G)}^2. \end{aligned}$$

Proof We have

$$\begin{aligned} |a^{\mathrm{LV}}(\varphi, \phi)| \leq{}& \frac{1}{2}\|\widetilde{\sigma}\|_{L^\infty(G)}^2 \|\varphi'\|_{L^2(G)} \|\phi'\|_{L^2(G)} + r\|\varphi\|_{L^2(G)} \|\phi\|_{L^2(G)} \\ &+ \big(\|\widetilde{\sigma}\|_{L^\infty(G)} \|\widetilde{\sigma}'\|_{L^\infty(G)} + 1/2\|\widetilde{\sigma}\|_{L^\infty(G)}^2 + r\big)\|\varphi'\|_{L^2(G)} \|\phi\|_{L^2(G)} \\ \leq{}& C_1 \|\varphi\|_{H^1(G)} \|\phi\|_{H^1(G)}. \end{aligned}$$

Denote by $\overline{\sigma}$ the constant $\overline{\sigma} := \|\widetilde{\sigma}\|_{L^\infty(G)}\|\widetilde{\sigma}'\|_{L^\infty(G)} + 1/2\|\widetilde{\sigma}\|^2_{L^\infty(G)}$. Then, for an arbitrary $\varepsilon > 0$,

$$\begin{aligned} a^{\mathrm{LV}}(\varphi,\varphi) &\geq \frac{1}{2}\sigma_0^2\|\varphi'\|^2_{L^2(G)} + r\|\varphi\|^2_{L^2(G)} - \overline{\sigma}\|\varphi'\|_{L^2(G)}\|\varphi\|_{L^2(G)} \\ &\geq \big(\sigma_0^2/2 - \varepsilon\overline{\sigma}\big)\|\varphi'\|^2_{L^2(G)} + \big(r - \overline{\sigma}/(4\varepsilon)\big)\|\varphi\|^2_{L^2(G)}. \end{aligned}$$

Choosing $\varepsilon = \sigma_0^2/(4\overline{\sigma})$, we have

$$a^{\mathrm{LV}}(\varphi,\varphi) \geq \sigma_0^2/4\|\varphi'\|^2_{L^2(G)} + (r - (\overline{\sigma}/\sigma_0)^2)\|\varphi\|^2_{L^2(G)},$$

which gives, by the Poincaré inequality (3.4),

$$\begin{aligned} a^{\mathrm{LV}}(\varphi,\varphi) &\geq \sigma_0^2/8\|\varphi'\|^2_{L^2(G)} + \sigma_0^2/(8C)\|\varphi\|^2_{L^2(G)} - |r - (\overline{\sigma}/\sigma_0)^2|\|\varphi\|^2_{L^2(G)} \\ &\geq \sigma_0^2/8\min\{1, C^{-1}\}\|\varphi\|^2_{H^1(G)} - C_3\|\varphi\|^2_{L^2(G)}. \end{aligned}$$

□

By Proposition 4.5.5, the bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$ is continuous and satisfies a Gårding inequality. Hence, for $g \in L^2(G)$, the weak formulation (4.37) admits a unique solution $u \in L^2(J; H_0^1(G)) \cap H^1(J; L^2(G))$.

4.6 Further Reading

To derive the partial differential equations, we followed the line of Lamberton and Lapeyre [109]. Using finite differences to price options was first described in Brennan and Schwartz [26]. A rigorous treatment can be found in Achdou and Pironneau [1]. Finite elements were first applied to finance in Wilmott et al. [161]. Error estimates for non-smooth initial data are given in Thomée [154]. The CEV model was introduced by Cox and Ross [45, 46], and analytic formulas can be found, for example, in Hsu et al. [88]. See also [56]. The probabilistic argument to estimate the localization error is due to Cont and Voltchkova in [41], even in a more general setting of Lévy processes. The local volatility model is used to recover the volatility smile observed in the stock market as shown in Derman and Kani [57] or Dupire [59].