Progress report

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1 June 2023

1 Project Overview

My dissertation project is about Partial Differential Equations PDEs: Modelling and Computation in Finance. Specifically, I am exploring how to price American options by solving the system of PDE resulting from applying to black-schole-merton model to hedge the risk of the option.

1.1 Background

An American option is a contract that gives the holder the right to buy, or sell an underly stock S at strike price K at any point in time between the start ($t = 0$) and expiration date ($t = T$) of the contract inclusively. At any point in time $t \in [0, T]$, the payoff function of an american call/put option can be defined as deterministc of the price at that time $S = S(t)$:

$$\begin{aligned} C(S) &= \max(S - K, 0) \quad \text{call option} \\ P(S) &= \max(K - S, 0) \quad \text{put option} \end{aligned} \tag{1}$$

Assuming that stock does not have dividends, pricing an American option is similar to price an European option. Therefore, we focus on the pricing American put options. The problem of pricing of American put options could be reformulated as a free boundary. For the free boundary formulation, we are task to to find the value of the option $V(S, t)$ at time 0 while at

same time finding the optimal exercise price $\bar{S}(t)$, the exercise region $\mathcal{C} := \{(S,t) : S > \bar{S}(t)\}$, and the optimal exercise boundary $\partial\mathcal{C} := \{(S,t) : S = \bar{S}(t)\}$.

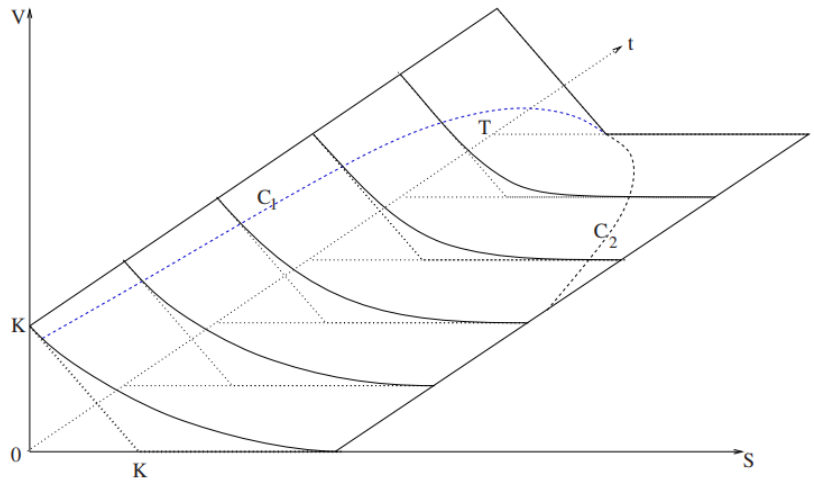


Figure 1: Value surface of an American option by Seidel [3].

The problem stated is described by the well known system of equation:

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 && \text{for } S > \bar{S}(t) \text{ and } 0 \le t < T \\
&V(T,S) = \max(K - S, 0) && \text{for } S \ge 0 \\
&\frac{\partial V}{\partial S}(t, \bar{S}(t)) = -1 \\
&V(t, \bar{S}(t)) = K - \bar{S}(t) \\
&\lim_{S\to\infty} V(t, \bar{S}(t)) = 0 \\
&\bar{S}(T) = K \\
&V(t,S) = K - S && \text{for } 0 \le S < \bar{S}(t)
\end{aligned}
\tag{2}
$$

For the given parameters σ (price volatility) and r (risk-free interest rate).

2 Progress

In the week from the 12-18 of June, I read some of the materials provided to me, specifically, the chapters 1-4 [1]; section 1-3 of [2];.

In the week from the 19-26 of June, I used the tranformation

$$x = \frac{S(t)}{\bar{S}(t)} \tag{3}$$

proposed by Nielsen [2], to fix the free boundary in equation (2). Therefore, the value function depends on $x \in [0, \infty]$ that is fixed in time.

$$v(t, x) = V(t, x\bar{S}(t)) \tag{4}$$

Using (3) in (2), the system of equation (2) is reformulated as a system of non-linear PDEs:

$$\begin{aligned}
&\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + x\left(r - \frac{\bar{S}'(t)}{\bar{S}'(t)}\right)\frac{\partial v}{\partial x} - rv = 0 && \text{for } x > 1 \text{ and } 0 \leq t < T \\
&v(T, x) = 0 && \text{for } x \geq 1 \\
&\frac{\partial v}{\partial x}(t, 1) = -S(t) \\
&v(t, 1) = K - \bar{S}(t) \\
&\lim_{S \to \infty} v(t, x) = 0 \\
&\bar{S}(T) = K
\end{aligned} \tag{5}$$

Then I proceeded to solve the system in (4) by applying both the explicit and implicit finite difference schemes. The code was written in python and to test the implementation, I used the same parameters listed in Nielsen [2]

$$r = 0.1$$

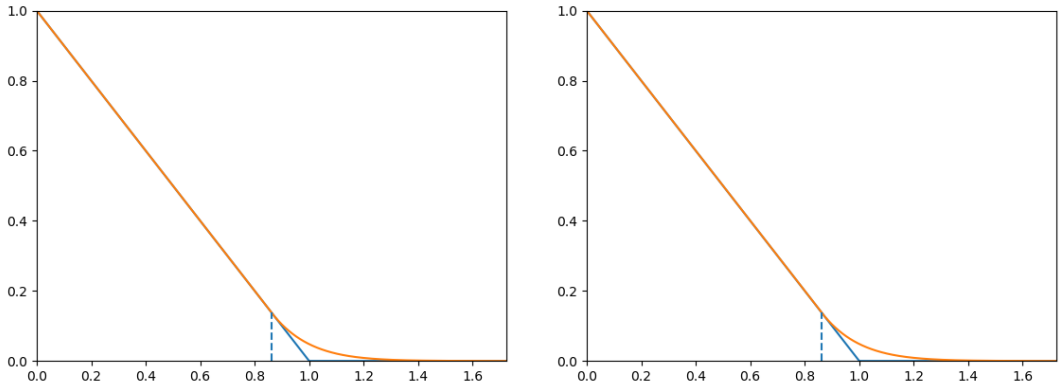
$$\sigma = 0.2$$

$$K = 1$$

$$T = 1$$

$$x_\infty = 2$$

using a grid resolution of $\Delta t = \Delta x = 0.0001$ for the explicit scheme, and $\Delta x = 5.0^{-6}$ and $\Delta x = 0.001$. Below, there are the produced by my implementation of both scheme in python. As you can see both plots looks quite similar to figure (1) where the solid orange line is the vale $V(S,t)$ of the option a time 0, the solid blue is the payoff function of the american put option and the dashed blue line is the optimal exercise price $\bar{S}(t)$ at time 0.



(a) $\bar{S}(0) = 8.6232 \times 10^{-1}$ (b) $\bar{S}(0) = 8.6178 \times 10^{-1}$

Figure 2: American put option value at time 0. The left and right plots are from apply an explicit and implicit scheme respectevely.

3 Future work

Another reformulation of the pricing problem or an American put option is the linear complmentary problem. My plan for the week from 3-9 of July is to solve the Linear Complementary problem for an american option using the PSOR method.

During the week 10-17 of July, I will explore directions my dissertation could take. The original plan is to do a comparative numerical analysis of each the methods. However, I would like to explore other possibilities. For instance, pricing american call options with dividends, solving the problem of pricing an american put option using jump diffusion models instead of the black scholes model by solving a Partial Integro-Differential equation, or pricing american swaptions using numerical PDEs.

References

[1] Norbert Hilber et al. "Computational Methods for Quantitative Finance". In: *SpringerLink* (2013). DOI: https://doi.org/10.1007-978-3-642-35401-4. URL: https://link.springer.com/book/10.1007/978-3-642-35401-4.

[2] Bjrn Nielsen, Ola Skavhaug, and Aslak Tveito. "Penalty and front-fixing methods for the numerical solution of American option problems". In: *Journal of Computational Finance* 5 (Sept. 2001). DOI: 10.21314/JCF.2002.084.

[3] Rüdiger U. Seydel. *Tools for Computational Finance*. Springer Berlin, Heidelberg, 2009. DOI: 10.1007/978-3-540-92929-1.