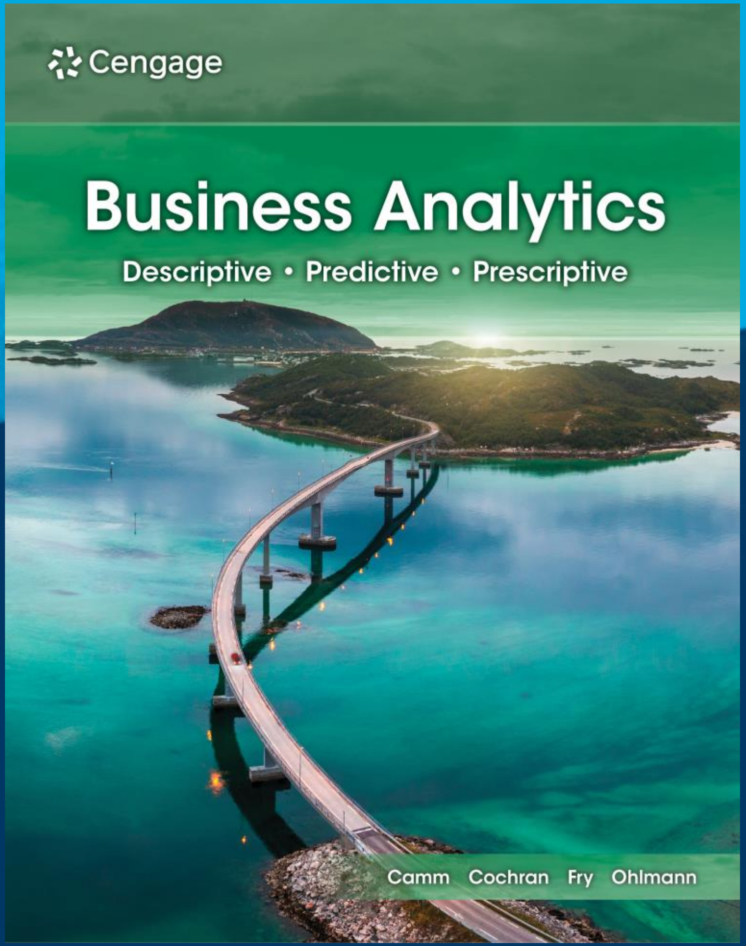


# Business Analytics, 5e

## Chapter 14 – Linear Optimization Models

# Chapter Contents

# Learning Objectives (1 of 2)

After completing this chapter, you will be able to:

LO 14-1 Create a linear programming model based on a problem description.

LO 14-2 Create a spreadsheet model of a linear program and solve it with Excel Solver.

LO 14-3 Interpret the solution of a linear program.

LO 14-4 Calculate the value of the slack or surplus of a constraint.

LO 14-5 Perform sensitivity analysis on an objective function coefficient and interpret an objective-function-coefficient range.

# Learning Objectives (2 of 2)

LO 14-6 Perform sensitivity analysis on the right-hand-side of a constraint and interpret a right-hand-side range.

LO 14-7 Interpret a shadow price for a constraint.

LO 14-8 Find an alternative optimal solution to a linear program when one exists.

# Introduction

**Optimization models**

- can be used to support and improve managerial decision-making,
- include an **objective function** to be
  - *maximized* (profit, return on investment, advertising effectiveness), or
  - *minimized* (cost, overtime, amount of fuel),
  - in the presence of a set of restrictions, called **constraints**, and
- can be linear or nonlinear.

In this chapter, we discuss *linear optimization models*, also known as *linear programs*.

- We also explain how to use Excel Solver to solve linear programs.

# 14.1 The Par, Inc. Problem

Management at Par, Inc., a small golf equipment and supplies manufacturer, has decided to move into the standard and deluxe golf bag market.

Management analyzed the manufacturing process of the golf bags and produced an estimate of the number of hours per bag needed by each department involved in the operations, as well as the available hours.

The accounting department also estimated the following profit contributions:

- Standard bag: $10/unit
- Deluxe bag: $9/unit

| | Production Time (hours) | | Available |
|---|---|---|---|
| **Department** | **Standard Bag** | **Deluxe Bag** | (hours) |
| Cutting and Dyeing | $^7/_{10}$ | 1 | 630 |
| Sewing | $^1/_2$ | $^5/_6$ | 600 |
| Finishing | 1 | $^2/_3$ | 708 |
| Inspection and Packaging | $^1/_{10}$ | $^1/_4$ | 135 |

# 14.1 Problem Formulation

**Problem formulation**, or **modeling**, is the process of translating the verbal statement of a problem into a mathematical statement.

The general guidelines for problem formulation include the following steps:

- **Understand the problem thoroughly**
- **Describe the objective**
- **Describe each constraint**
- **Define the decision variables**
- **Write the objective in terms of the decision variables**
- **Write the constraints in terms of the decision variables**

Let us now use these guidelines to develop a mathematical model for Par, Inc.

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# 14.1 Describe the Objective and Each Constraint

Par, Inc.'s objective is to maximize the total contribution to profit, subject to the following four constraints related to the available number of hours in each department.

1. $$\begin{bmatrix} \text{hours of cutting and} \\ \text{dyeing time used} \end{bmatrix} \leq \begin{bmatrix} \text{hours of cutting and} \\ \text{dyeing time available} \end{bmatrix}$$
2. $$\begin{bmatrix} \text{hours of sewing} \\ \text{time used} \end{bmatrix} \leq \begin{bmatrix} \text{hours of sewing} \\ \text{time available} \end{bmatrix}$$
3. $$\begin{bmatrix} \text{hours of finishing} \\ \text{time used} \end{bmatrix} \leq \begin{bmatrix} \text{hours of finishing} \\ \text{time available} \end{bmatrix}$$
4. $$\begin{bmatrix} \text{hours of inspection and} \\ \text{packaging time used} \end{bmatrix} \leq \begin{bmatrix} \text{hours of inspection and} \\ \text{packaging time available} \end{bmatrix}$$

# 14.1 Write Objective and Constraints in Terms of the Decision Variables

If we define the **decision variables** as

$S =$ number of standard bags $\qquad D =$ number of deluxe bags

we can use the unit profit contributions for standard and deluxe golf bags to write the objective function as the maximization of total profit contribution

$$\text{Max } 10S + 9D$$

and write the four department constraints as follows.

$$\tfrac{7}{10} S + 1D \le 630 \qquad 1S + \tfrac{2}{3} D \le 708$$

$$\tfrac{1}{2} S + \tfrac{5}{6} D \le 600 \qquad \tfrac{1}{10} S + \tfrac{1}{4} D \le 135$$

Finally, we add **non-negativity constraints** to prevent the decision variables $S$ and $D$ from having negative values: $S, D \ge 0$.

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# 14.1 Linear Programming Model for Par, Inc.

$$
\begin{aligned}
&\text{Max} && 10S + 9D \\
&\text{subject to (s.t.)} && \\
& && {}^{7}/_{10}\,S + 1D \le 630 && \text{Cutting and dyeing} \\
& && {}^{1}/_{2}\,S + {}^{5}/_{6}\,D \le 600 && \text{Sewing} \\
& && 1S + {}^{2}/_{3}\,D \le 708 && \text{Finishing} \\
& && {}^{1}/_{10}\,S + {}^{1}/_{4}\,D \le 135 && \text{Inspection and packaging} \\
& && S, D \ge 0 &&
\end{aligned}
$$

This mathematical model is a **linear programming model**, or **linear program**, because the objective function and all constraint functions (the left-hand sides of the constraint inequalities) are **linear functions** of the decision variables.

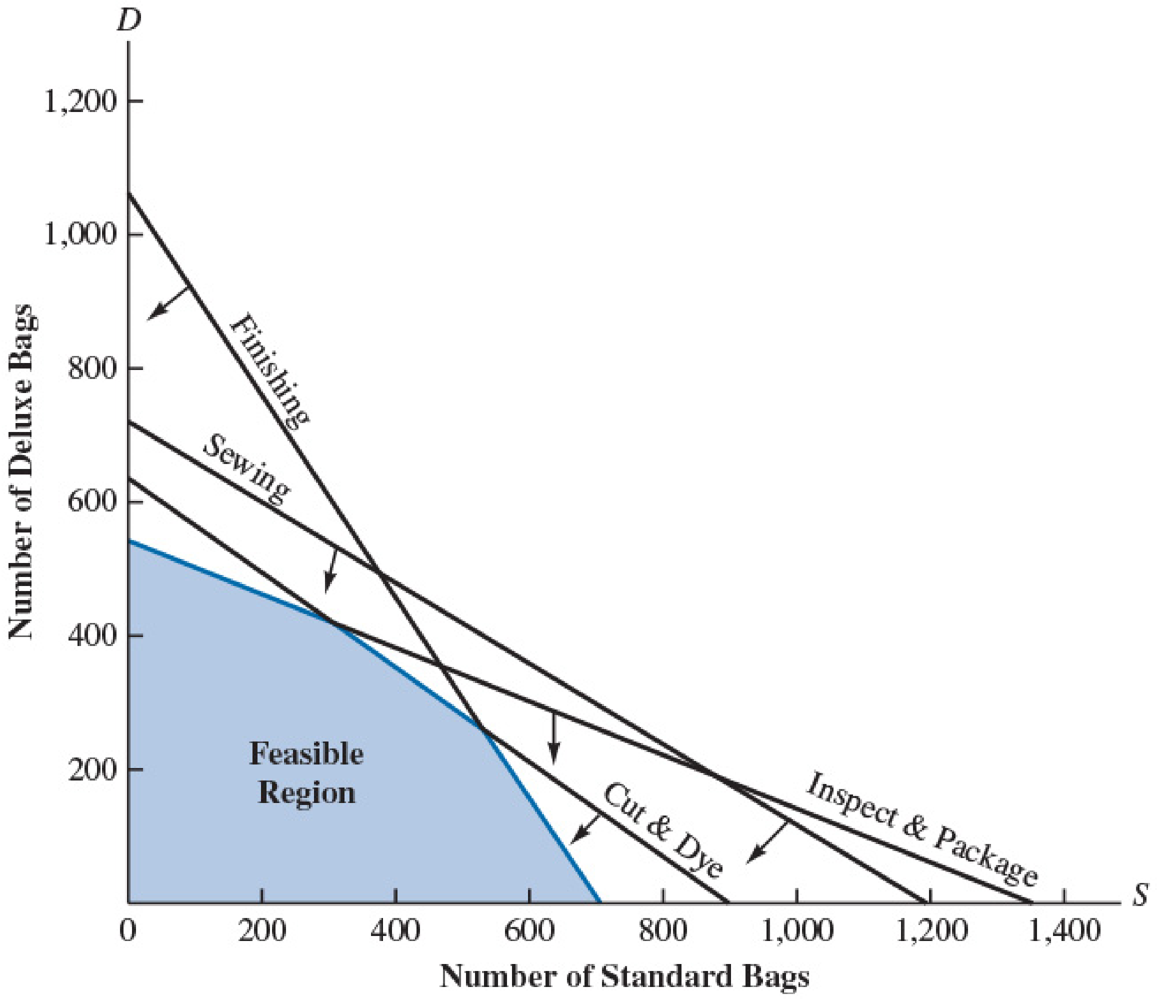
# 14.2 The Feasible Region of the Par, Inc. Problem

When we have only two decision variables, and the functions of these variables are linear, they form lines in a two-dimensional space.

The **feasible region** is the intersection of the *half spaces* described by each of the constraints' inequalities.

The feasible region for the Par, Inc. problem is shown to the right.

Notice that the horizontal axis corresponds to the value of $S$ and the vertical axis to the values of $D$.

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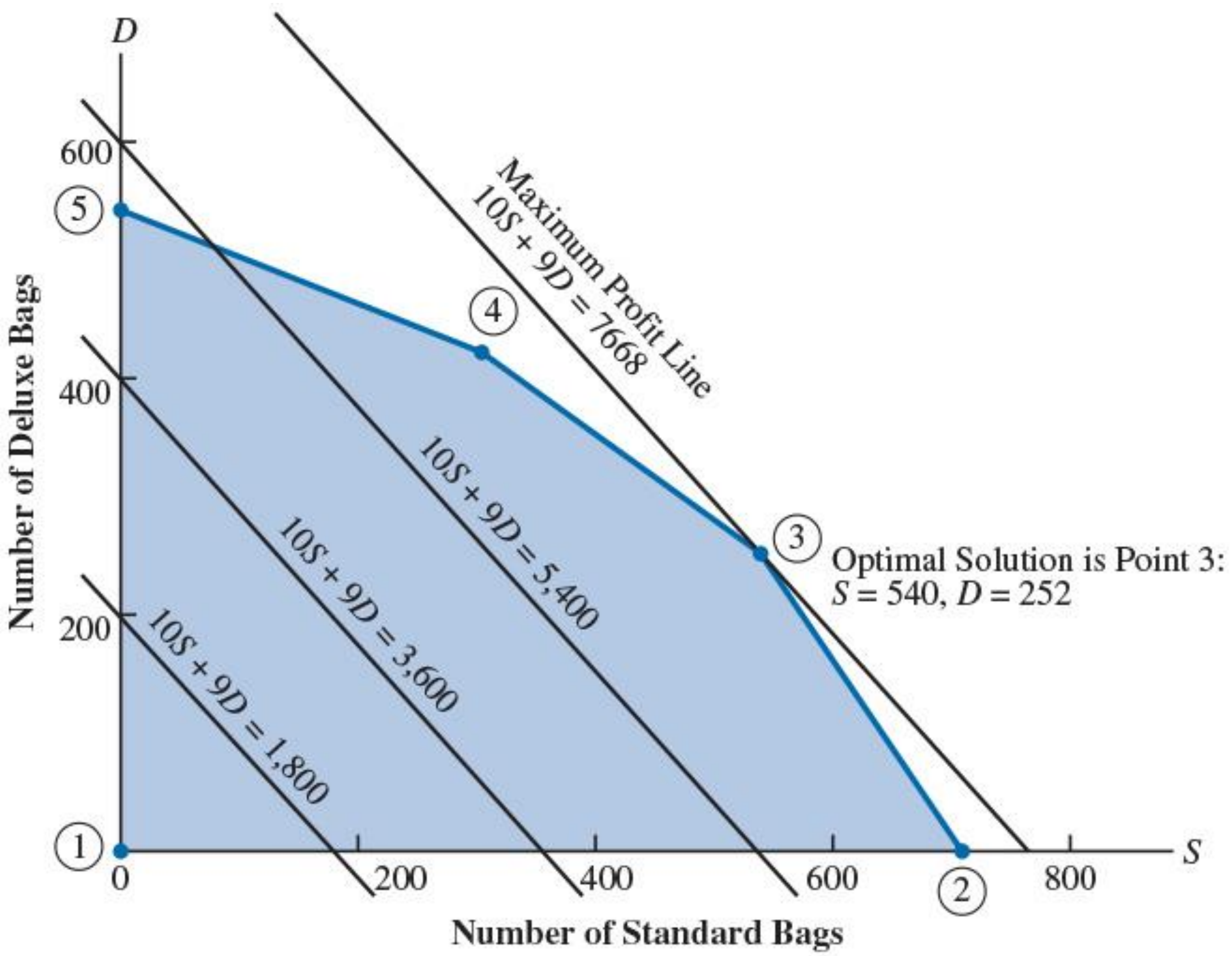
# 14.2 The Optimal Solution of the Par, Inc. Problem

The points where the constraints intersect on the boundary of the feasible region are called **extreme points**.

The optimal solution to the Par, Inc. problem is at ③, the extreme point that results in the highest possible objective function value.

The optimal solution is a **feasible solution** because extreme point ③ at $S = 540$ and $D = 252$ satisfies all of the problem's constraints.

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# 14.2 Extreme Points for the Par, Inc. Problem

The simplex algorithm, developed by George Dantzig, is quite effective at investigating extreme points in an intelligent way to find the optimal solution to even very large linear programs.

- Excel Solver utilizes Dantzig's simplex algorithm to solve linear programs by systematically finding which set of constraints forms the optimal extreme point of the feasible region.
- Once it finds an optimal solution, Solver reports the optimal values of the decision variables and the optimal objective function value.

| **Point** | ***S*** | ***D*** | **Profit = 10 *S* + 9 *D*** |
|---|---|---|---|
| ① | 0 | 0 | 10(0) + 9(0) = 0 |
| ② | 708 | 0 | 10(708) + 9(0) = 7,080 |
| ③ | 540 | 252 | 10(540) + 9(252) = 7,668 |
| ④ | 300 | 420 | 10(300) + 9(420) = 6,780 |
| ⑤ | 0 | 540 | 10(0) + 9(540) = 4,860 |

# 14.2 A What-If Model for the Par, Inc. Problem

A what-if model for optimization allows the user to try different values of the decision variables and see easily whether (a) that trial solution is feasible, and (b) the value of the objective function for that trial solution.

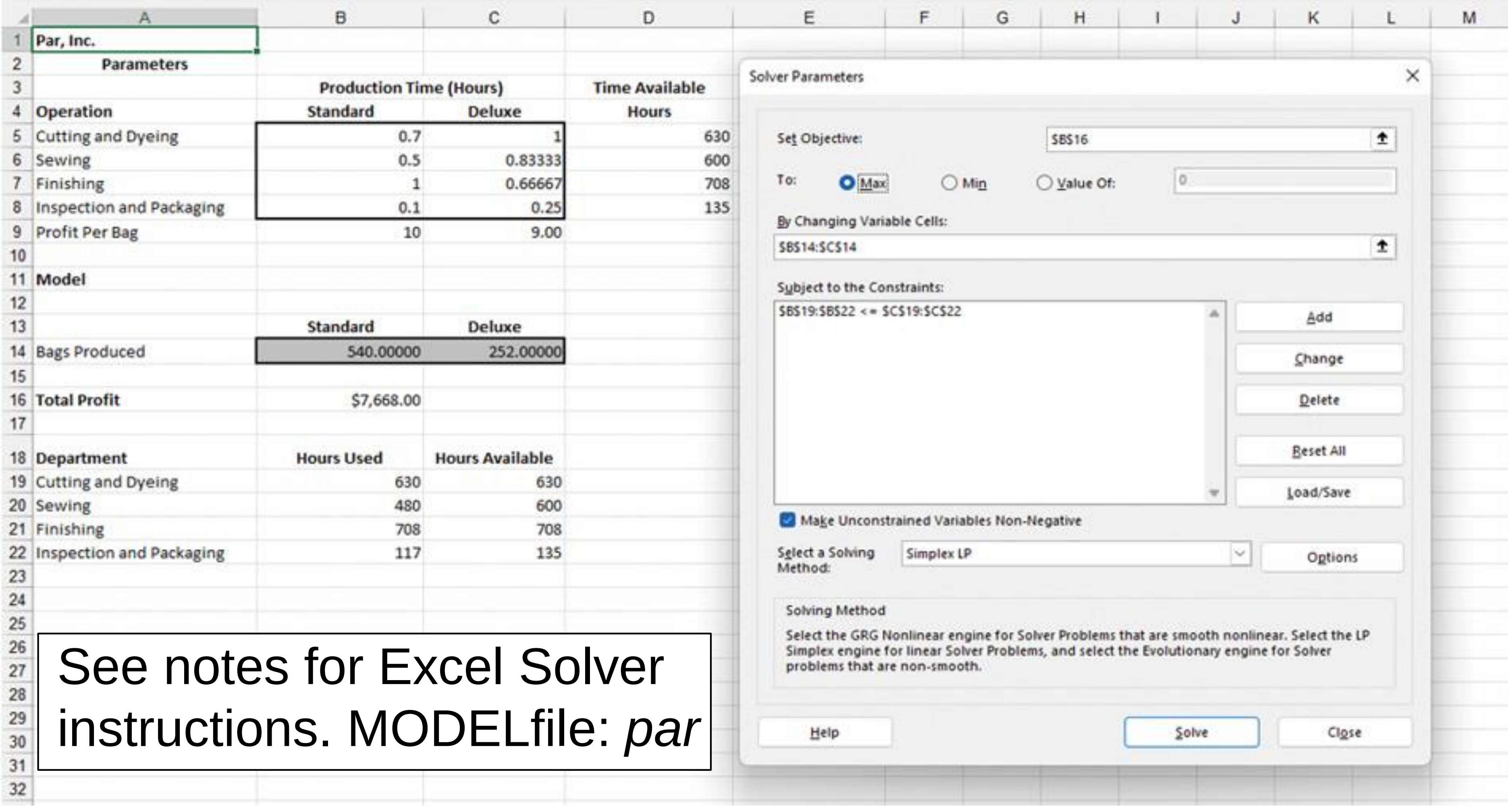
The next slide shows a spreadsheet model for the Par, Inc. problem with the optimal solution of 540 standard bags and 252 deluxe bags.

- Rows *1-9* contain the parameters for the problem.
- Cells *B14 and C14* contain the decision variable cells.
- Cell *B16* calculates the objective function using the Excel function:

$$= SUMPRODUCT(B9{:}C9, \$B\$14{:}\$C\$14)$$

Likewise, cells *B19:B22* use the *SUMPRODUCT* function to compute the left-hand side of the constraints. Cells *C19:C22* contain the right-hand side values.

# 14.2 Solving Linear Programs with Excel Solver




See notes for Excel Solver instructions. MODELfile: *par*

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# 14.2 Excel Solver Answer Report for Par, Inc.

The Answer Report has three sections:

- Objective Cell, Variable Cells, and Constraints (which include constraint's right-hand values, formulas, binding status, and slack)

A **binding constraint** holds equality at the optimal solution (status = binding.)

The **slack** is the difference between a Constraint left- and right-hand values.

Objective Cell (Max)

| Cell | Name | Original Value | Final Value |
|---|---|---|---|
| $B$16 | Total Profit | $19.00 | $7,668.00 |

Variable Cells

| Cell | Name | Original Value | Final Value | Integer |
|---|---|---|---|---|
| $B$14 | Bags Produced Standard | 1.00000 | 540.00000 | Contin |
| $C$14 | Bags Produced Deluxe | 1.00000 | 252.00000 | Contin |

Constraints

| Cell | Name | Cell Value | Formula | Status | Slack |
|---|---|---|---|---|---|
| $B$19 | Cutting and Dyeing Hours Used | 630 | $B$19<=$C$19 | Binding | 0 |
| $B$20 | Sewing Hours Used | 480 | $B$20<=$C$20 | Not Binding | 120 |
| $B$21 | Finishing Hours Used | 708 | $B$21<=$C$21 | Binding | 0 |
| $B$22 | Inspection and Packaging Hours Used | 117 | $B$22<=$C$22 | Not Binding | 18 |

By adding a nonnegative **slack variable**, we can make the constraint equality.

Sewing constraint: $^1/_2\,S + {}^5/_6\,D \le 600 \;\Rightarrow\; {}^1/_2\,S + {}^5/_6\,D + \mathrm{slack}_{sewing} = 600$

# 14.3 M&D Chemicals Cost Minimization

M&D Chemicals wants to satisfy the following requirements for the production of two raw products (A and B) at a minimum cost:

- A customer's order for 125 gallons of product A must be satisfied.
- The combined production for A and B must total at least 350 gallons.

Management at M&D Chemicals has also identified the following information on product processing time, cost, and available hours for processing.

- Product A: 2 hours per gallon at a cost of $2 per gallon
- Product B: 1 hour per gallon at a cost of $3 per gallon
- For the coming month, 600 hours of processing time are available.

Let us begin by formulating M&D Chemicals' problem using a linear program.

# 14.3 Problem Formulation for M&D Chemicals

Let the decision variables for the amount of each raw product be

$A =$ number of gallons of product A to produce

$B =$ number of gallons of product B to produce

The objective function for the minimization of total production cost.

$$\text{Min } 2A + 3B$$

Subjected to the following constraints.

| | |
|---|---|
| Customer's demand for 125 gallons of product A: | $1A \geq 125$ |
| Minimum combined production of 350 gallons: | $1A + 1B \geq 350$ |
| Limitation of 600 hours on available processing time: | $2A + 1B \leq 600$ |
| Non-negativity on the produced amounts of A and B: | $A \geq 0,\ B \geq 0$ |

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# 14.3 Linear Programming Model for M&D Chemicals

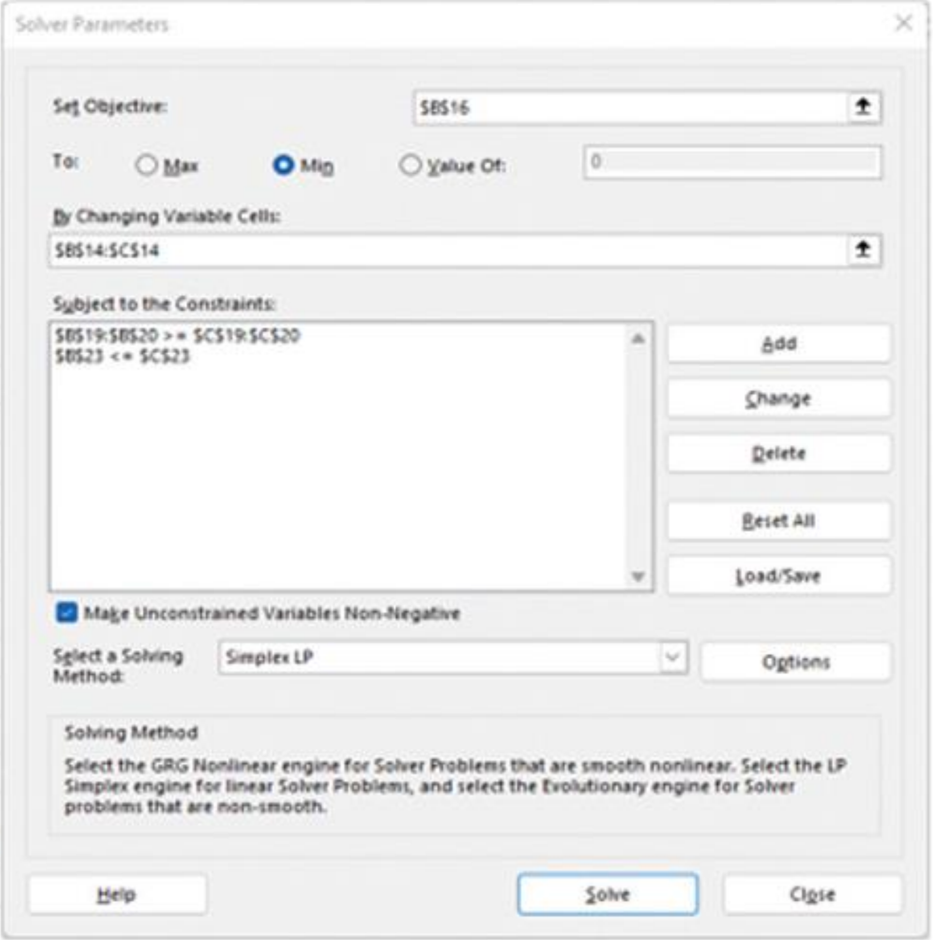
$$
\begin{array}{lll}
\text{Min} & 2A + 3B & \\
\text{subject to (s.t.)} & & \\
& 1A \qquad\quad \geq 125 & \text{Demand for product A} \\
& 1A + 1B \geq 350 & \text{Total production} \\
& 2A + 1B \leq 600 & \text{Processing time} \\
& A, B \qquad \geq 0 &
\end{array}
$$

In the next slide, we show a spreadsheet model for the M&D Chemicals problem along with the Solver dialog box.

See MODELfile: *mdmodel* for the complete linear programming model.

# 14.3 Solution for the M&D Chemicals Problem

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **M&D Chemicals** | | | |
| 2 | **Parameters** | | | |
| 3 | | Product A | Product B | Time Available |
| 4 | Processing Time (hours) | 2 | 1 | 600 |
| 5 | Production Cost | $2.00 | $3.00 | |
| 6 | | | | |
| 7 | Minimum Total Production | 350 | | |
| 8 | Product A Minimum | 125 | | |
| 9 | | | | |
| 10 | | | | |
| 11 | **Model** | | | |
| 12 | | | | |
| 13 | | Product A | Product B | |
| 14 | Gallons Produced | 250 | 100 | |
| 15 | | | | |
| 16 | Minimize Total Cost | $800.00 | | |
| 17 | | | | |
| 18 | | Provided | Required | |
| 19 | Product A | 250 | 125 | |
| 20 | Total Production | 350 | 350 | |
| 21 | | | | |
| 22 | | Hours Used | Hours Available | Unused Hours |
| 23 | Processing Time | 600 | 600 | 0 |
| 24 | | | | |
| 25 | | | | |



# 14.3 Excel Solver Answer Report for M&D Chemicals

The optimal solution is to produce 250 gallons of product A and 100 gallons of product B at a total cost of $800.

The constraints section shows that the production of product A is not binding.

We can add a non-negative **surplus variable** to turn a not-binding greater-than-or-equal-to constraint into equality.

Objective Cell (Min)

| Cell | Name | Original Value | Final Value |
|---|---|---|---|
| $B$16 | Minimize Total Cost | $0.00 | $800.00 |

Variable Cells

| Cell | Name | Original Value | Final Value | Integer |
|---|---|---|---|---|
| $B$14 | Gallons Produced Product A | 0 | 250 | Contin |
| $C$14 | Gallons Produced Product B | 0 | 100 | Contin |

Constraints

| Cell | Name | Cell Value | Formula | Status | Slack |
|---|---|---|---|---|---|
| $B$19 | Product A Provided | 250 | $B$19>=$C$19 | Not Binding | 125 |
| $B$20 | Total Production Provided | 350 | $B$20>=$C$20 | Binding | 0 |
| $B$23 | Processing Time Hours Used | 600 | $B$23<=$C$23 | Binding | 0 |

A surplus variable is subtracted from the left-hand side of the constraint:

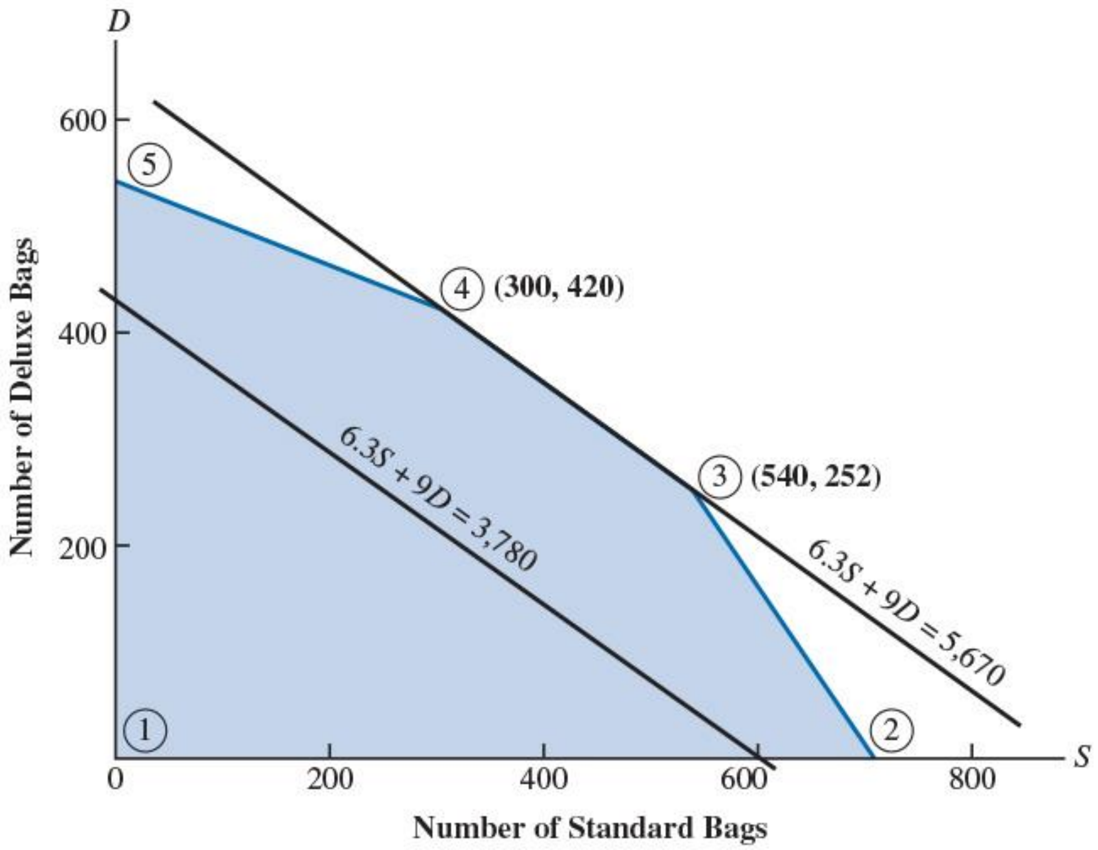
Demand for Product A: $1A \geq 125$ ➔ $1A - \text{surplus}_A = 125$

# 14.4 Alternative Optimal Solutions

**Alternative optimal solutions** occur when the optimal objective function contour line coincides with one of the binding constraint lines on the boundary of the feasible region.

For example, in the Par, Inc. problem, reducing the unit profit contribution for the standard bags to $6.30 would make the total profit contour line parallel to the cutting and dyeing constraint line.

All points on the segment between extreme points ③ and ④ are optimal.

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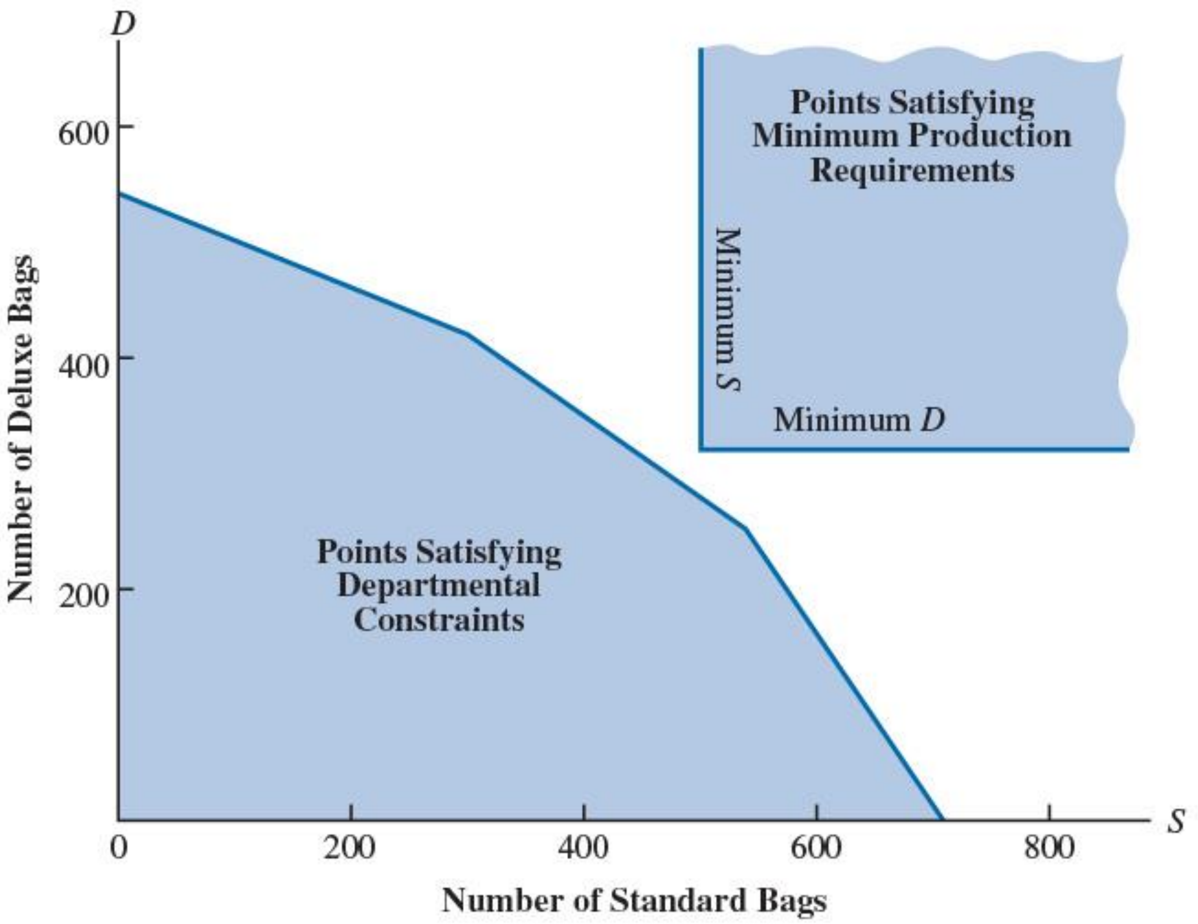
# 14.4 Infeasibility

**Infeasibility** means that no solution to the linear programming problem satisfies all the constraints, including non-negativity conditions.

- Graphically, a feasible region does not exist.

Infeasibility occurs because too many restrictions have been placed on the problem.

On the Par, Inc. problem, requiring production of $S = 500$ and $D = 360$ results in no feasible solutions.



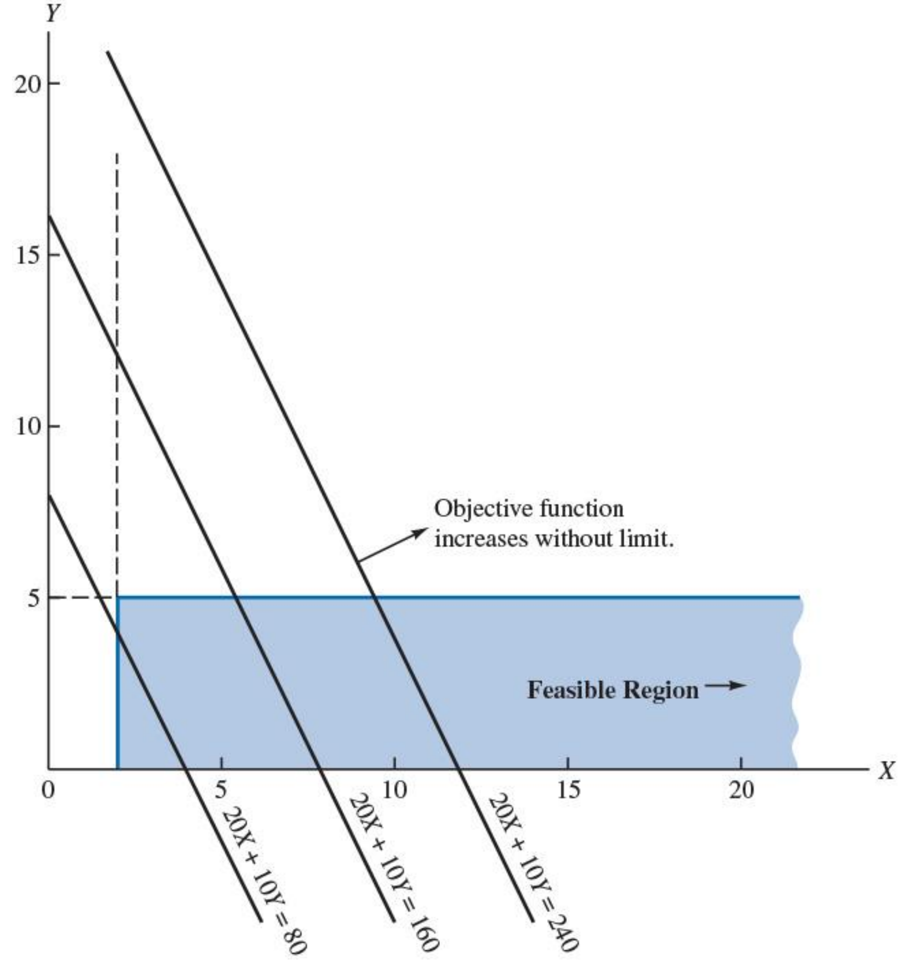
# 14.4 Unbounded

A linear program solution is **unbounded** if, without violating any of the constraints, the value of the solution may be made

- infinitely large for a maximization problem,
- infinitely small for a minimization problem.

As an illustration, consider the following linear program with two decision variables, *X* and *Y*:

$$\begin{aligned} \text{Max} \quad & 20X + 10Y \\ \text{s.t.} \quad & 1X \geq 2 \\ & 1Y \leq 5 \\ & X, Y \geq 0 \end{aligned}$$

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# 14.5 Sensitivity Analysis

**Sensitivity analysis**, the study of how the changes in the input parameters of an optimization model affect the optimal solution, helps in answering the following questions:

1. How will a change in a *coefficient of the objective function* affect the optimal solution?
2. How will a change in the *right-hand-side value for a constraint* affect the optimal solution?

Because sensitivity analysis is concerned with how these changes affect the optimal solution, the analysis does not begin until the optimal solution to the linear programming problem has been obtained.

For this reason, sensitivity analysis is sometimes called *postoptimality analysis*.

# 14.5 Shadow Price

The **shadow price** for a constraint is the *change* in the optimal objective function value if the right-hand side of that constraint is *increased* by one.

- Making a binding constraint more restrictive degrades or leaves unchanged the optimal objective function value.
- Making a binding constraint less restrictive improves or leaves unchanged the optimal objective function.
- *Nonbinding constraints will always have a shadow price of zero*.

Shadow prices are symmetric.

- The negative of the shadow price is the *change* in the objective function if the right-hand side of that constraint is *decreased* by one.

# 14.5 Reduced Cost and Allowable Increase/Decrease

The **reduced cost** for a decision variable is the shadow price of the nonnegativity constraint for that variable.

- In other words, the reduced cost indicates the change in the optimal objective function value that results from changing the right-hand side of the nonnegativity constraint from 0 to 1

The **allowable increase** and **allowable decrease** indicate the change in the objective function coefficient for which the current optimal solution will remain optimal.

The Excel Solver Sensitivity Report can provide useful information about the sensitivity of the optimal solution to changes in the model input data.

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# 14.5 Solver Sensitivity Report for M&D Chemicals

The shadow price for $1A \geq 125$ is zero because the constraint is non-binding.

The shadow price of \$4 for the $1A + 1B \geq 350$ constraints can be interpreted as the *increase* in total cost when the constraint is changed to $1A + 1B \geq 351$.

The shadow price of −\$1 for the $2A + 1B \leq 600$ constraints can be interpreted as the *decrease* in total cost when the constraint is changed to $2A + 1B \leq 601$.

The allowable increase and decrease for each shadow price indicate the range of right-hand side values in which the solution will remain optimal.

Because $A > 0$ and $B > 0$, their reduced costs are both zero.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 6 | Variable Cells | | | | | | | |
| 7 | | | | Final | Reduced | Objective | Allowable | Allowable |
| 8 | | Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| 9 | | \$B\$14 | Gallons Produced Product A | 250 | 0 | 2 | 1 | 1E+30 |
| 10 | | \$C\$14 | Gallons Produced Product B | 100 | 0 | 3 | 1E+30 | 1 |
| 11 | | | | | | | | |
| 12 | Constraints | | | | | | | |
| 13 | | | | Final | Shadow | Constraint | Allowable | Allowable |
| 14 | | Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| 15 | | \$B\$19 | Product A Provided | 250 | 0 | 125 | 125 | 1E+30 |
| 16 | | \$B\$20 | Total Production Provided | 350 | 4 | 350 | 125 | 50 |
| 17 | | \$B\$23 | Processing Time Hours Used | 600 | -1 | 600 | 100 | 125 |
| 18 | | | | | | | | |

# 14.6 General Linear Programming Notation

The general notation for linear programs uses the letter $x$ with a subscript.

In the Par, Inc. problem, the decision variables could be denoted as $x_1 =$ number of standard bags, and $x_2 =$ number of deluxe bags, so that:

$$
\begin{array}{lrll}
\text{Max} & 10x_1 + 9x_2 & & \\
\text{s.t.} & & & \\
& \tfrac{7}{10}x_1 + 1x_2 \le 630 & & \text{Cutting and dyeing} \\
& \tfrac{1}{2}x_1 + \tfrac{5}{6}x_2 \le 600 & & \text{Sewing} \\
& 1x_1 + \tfrac{2}{3}x_2 \le 708 & & \text{Finishing} \\
& \tfrac{1}{10}x_1 + \tfrac{1}{4}x_2 \le 135 & & \text{Inspection and packaging} \\
& x_1, x_2 \ge 0 & & 
\end{array}
$$

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# 14.6 Investment Portfolio Selection

Welte Mutual Funds, Inc., based in New York City, is looking to maximize the return on the investment of $100,000 in

- five investment opportunities with given projected annual rates of return, and
- subject to the following investment guidelines:
  - Neither oil nor steel investments should receive more than $50,000.
  - The amount invested in government bonds should be at least 25% of the steel industry investments.
  - The investment in Pacific Oil cannot be more than 60% of the total oil industry investment.

| **Investment** | **Projected Rate of Return (%)** |
|---|---|
| Atlantic Oil | 7.3 |
| Pacific Oil | 10.3 |
| Midwest Steel | 6.4 |
| Huber Steel | 7.5 |
| Government bonds | 4.5 |

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# 14.6 Problem Formulation for Welte Mutual Funds

Using general linear programming notation, we define the following five decision variables:

$x_1$ = dollars invested in Atlantic Oil

$x_2$ = dollars invested in Pacific Oil

$x_3$ = dollars invested in Midwest Steel

$x_4$ = dollars invested in Huber Steel

$x_5$ = dollars invested in government bonds

Using the projected rate of return, we write the objective function for maximizing the total return for the portfolio as

$$\text{Max } 0.073x_1 + 0.103x_2 + 0.064x_3 + 0.075x_4 + 0.045x_5$$

# 14.6 Constraints for Welte Mutual Funds

The constraint specifying investment of the available $100,000 is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100{,}000$$

Neither the oil nor steel industry should receive more than $50,000

$$x_1 + x_2 \leq 50{,}000$$

$$x_3 + x_4 \leq 50{,}000$$

Amount invested in gov.t bonds be at least 25% of the steel industry investment

$$x_5 \geq 0.25(x_3 + x_4)$$

Pacific Oil cannot be more than 60% of the total oil industry investment

$$x_2 \leq 0.60(x_1 + x_2)$$

Finally, we have the nonnegativity constraints: $x_1, x_2, x_3, x_4, x_5 \geq 0$

# 14.6 Linear Programming Model for Welte Mutual Funds

$$\text{Max } 0.073x_1 + 0.103x_2 + 0.064x_3 + 0.075x_4 + 0.045x_5$$

s.t.

| | |
|---|---|
| $x_1 + x_2 + x_3 + x_4 + x_5 = 100{,}000$ | Available funds |
| $x_1 + x_2 \leq 50{,}000$ | Oil industry maximum |
| $x_3 + x_4 \leq 50{,}000$ | Steel industry maximum |
| $x_5 \geq 0.25(x_3 + x_4)$ | Government bonds minimum |
| $x_2 \leq 0.60(x_1 + x_2)$ | Pacific Oil restriction |
| $x_1, x_2, x_3, x_4, x_5 \geq 0$ | |

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# 14.6 The Solution for the Welte Mutual Funds Problem

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Welte Mutual Funds Problem** | | | | | |
| 2 | | | | | | |
| 3 | **Parameters** | | | | | |
| 4 | Investment | Projected Rate of Return | | | | |
| 5 | Atlantic Oil | 0.073 | | Available Funds | 100000 | |
| 6 | Pacific Oil | 0.103 | | Oil Max | 50000 | |
| 7 | Midwest Steel | 0.064 | | Steel Max | 50000 | |
| 8 | Huber Steel | 0.075 | | Pacific Oil Max | 0.6 | |
| 9 | Gov't Bonds | 0.045 | | Gov't Bonds Min | 0.25 | |
| 10 | | | | | | |
| 11 | **Model** | | | | | |
| 12 | | | | | | |
| 13 | Investment | Amount Invested | | | | |
| 14 | Atlantic Oil | 20000 | | | | |
| 15 | Pacific Oil | 30000 | | | | |
| 16 | Midwest Steel | 0 | | | | |
| 17 | Huber Steel | 40000 | | | | |
| 18 | Gov't Bonds | 10000 | | | | |
| 19 | | | | | | |
| 20 | Max Total Return | =SUMPRODUCT(B5:B9,B14:B18) | | | | |
| 21 | | | | | | |
| 22 | | Funds Invested | Funds Available | Unused Funds | | |
| 23 | Total | =SUM(B14:B18) | =E5 | =C23-B23 | | |
| 24 | | | | | | |
| 25 | | Funds Invested | Max Allowed | | | |
| 26 | Oil | =SUM(B14:B15) | =E6 | | | |
| 27 | Steel | =SUM(B16:B17) | =E7 | | | |
| 28 | Pacific Oil | =B15 | =E8*(B14+B15) | | | |
| 29 | | | | | | |
| 30 | | Funds Invested | Min Required | | | |
| 31 | Gov't Bonds | =B18 | =E9*(B16+B17) | | | |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Welte Mutual Funds Problem** | | | | |
| 2 | | | | | |
| 3 | **Parameters** | | | | |
| 4 | Investment | Projected Rate of Return | | | |
| 5 | Atlantic Oil | 0.073 | | Available Funds | $100,000.00 |
| 6 | Pacific Oil | 0.103 | | Oil Max | $50,000.00 |
| 7 | Midwest Steel | 0.064 | | Steel Max | $50,000.00 |
| 8 | Huber Steel | 0.075 | | Pacific Oil Max | 0.6 |
| 9 | Gov't Bonds | 0.045 | | Gov't Bonds Min | 0.25 |
| 10 | | | | | |
| 11 | **Model** | | | | |
| 12 | | | | | |
| 13 | Investment | Amount Invested | | | |
| 14 | Atlantic Oil | $20,000.00 | | | |
| 15 | Pacific Oil | $30,000.00 | | | |
| 16 | Midwest Steel | $0.00 | | | |
| 17 | Huber Steel | $40,000.00 | | | |
| 18 | Gov't Bonds | $10,000.00 | | | |
| 19 | | | | | |
| 20 | Max Total Return | $8,000.00 | | | |
| 21 | | | | | |
| 22 | | Funds Invested | Funds Available | Unused Funds | |
| 23 | Total | $100,000.00 | $100,000.00 | $0.00 | |
| 24 | | | | | |
| 25 | | Funds Invested | Max Allowed | | |
| 26 | Oil | $50,000.00 | $50,000.00 | | |
| 27 | Steel | $40,000.00 | $50,000.00 | | |
| 28 | Pacific Oil | $30,000.00 | $30,000.00 | | |
| 29 | | | | | |
| 30 | | Funds Invested | Min Required | | |
| 31 | Gov't Bonds | $10,000.00 | $10,000.00 | | |

MODELfile: *welte.* See notes for details.

CENGAGE

# 14.6 Transportation Planning

In a **transportation problem**, the quantity of goods available at each supply location (origin) is limited, and the quantity needed at several demand locations (destinations) is known.

- The usual objective in a transportation problem is to minimize the cost of shipping goods from origin to destination.

Let us revisit Foster Generators' transportation problem with the shipping of a product from three plants in Cleveland, Bedford, and York to four distribution centers located in Boston, Chicago, St. Louis, and Lexington.

- For convenience, the what-if model we developed for Foster Generators in Chapter 12 is restated in the next slide.

# 14.6 Transportation Model for Foster Generators

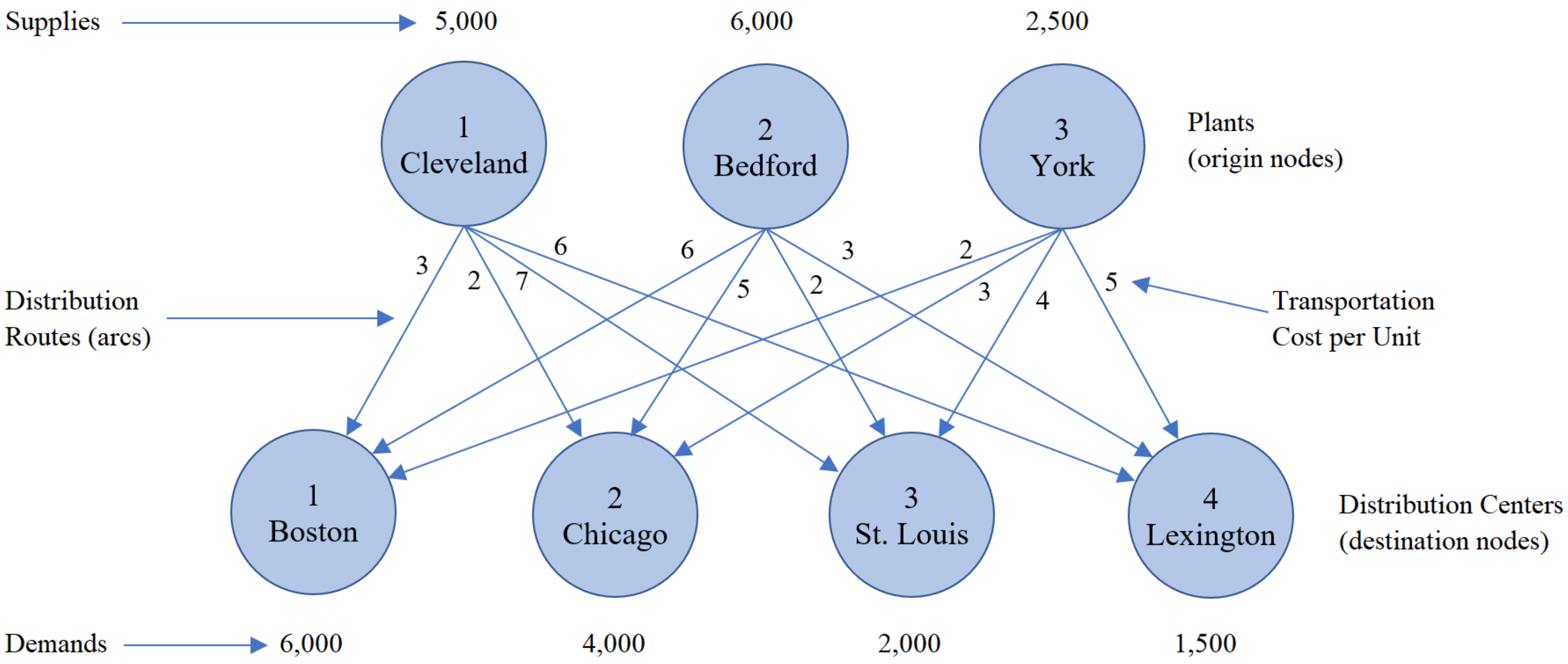
| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Foster Generators** | | | | | | |
| 2 | **Parameters** | | | | | | |
| 3 | Shipping Cost/Unit | | Destination | | | | |
| 4 | Origin | Boston | Chcago | St. Louis | Lexington | Supply | |
| 5 | Cleveland | 3 | 2 | 7 | 6 | 5000 | |
| 6 | Bedford | 6 | 5 | 2 | 3 | 6000 | |
| 7 | York | 2 | 5 | 4 | 5 | 2500 | |
| 8 | Demand | 6000 | 4000 | 2000 | 1500 | | |
| 9 | | | | | | | |
| 10 | | | | | | | |
| 11 | **Model** | | | | | | |
| 12 | | | | | | | |
| 13 | Total Cost | =SUMPRODUCT(B5:E7,B17:E19) | | | | | |
| 14 | | | | | | | |
| 15 | | | Destination | | | | |
| 16 | Origin | Boston | Chcago | St. Louis | Lexington | **Total** | |
| 17 | Cleveland | 5000 | 0 | 0 | 0 | =SUM(B17:E17) | |
| 18 | Bedford | 1000 | 4000 | 1000 | 0 | =SUM(B18:E18) | |
| 19 | York | 0 | 0 | 1000 | 1500 | =SUM(B19:E19) | |
| 20 | Total | =SUM(B17:B19) | =SUM(C17:C19) | =SUM(D17:D19) | =SUM(E17:E19) | | |
| 21 | | | | | | | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Foster Generators** | | | | | |
| 2 | **Parameters** | | | | | |
| 3 | Shipping Cost/Unit | | Destination | | | |
| 4 | Origin | Boston | Chcago | St. Louis | exingtoı | Supply |
| 5 | Cleveland | $3.00 | $2.00 | $7.00 | $6.00 | 5000 |
| 6 | Bedford | $6.00 | $5.00 | $2.00 | $3.00 | 6000 |
| 7 | York | $2.00 | $5.00 | $4.00 | $5.00 | 2500 |
| 8 | Demand | 6000 | 4000 | 2000 | 1500 | |
| 9 | | | | | | |
| 10 | | | | | | |
| 11 | **Model** | | | | | |
| 12 | | | | | | |
| 13 | Total Cost | $54,500.00 | | | | |
| 14 | | | | | | |
| 15 | | | Destination | | | |
| 16 | Origin | Boston | Chcago | St. Louis | exingtoı | **Total** |
| 17 | Cleveland | 5000 | 0 | 0 | 0 | 5000 |
| 18 | Bedford | 1000 | 4000 | 1000 | 0 | 6000 |
| 19 | York | 0 | 0 | 1000 | 1500 | 2500 |
| 20 | Total | 6000 | 4000 | 2000 | 1500 | |

MODELfile: *foster*. See notes for details about the spreadsheet model.

CENGAGE

# 14.6 Network Representation for Foster Generators



# 14.6 Problem Formulation for Foster Generators

The objective function is the sum of the following three cost expressions:

1. Transportation cost from Cleveland $= 3x_{11} + 2x_{12} + 7x_{13} + 6x_{14}$
2. Transportation cost from Bedford $= 6x_{21} + 5x_{22} + 2x_{23} + 3x_{24}$
3. Transportation cost from York $= 2x_{31} + 3x_{32} + 4x_{33} + 5x_{34}$

The supply constraints limit product flow to the capacity of each plant:

1. Cleveland supply $= x_{11} + x_{12} + x_{13} + x_{14} \leq 5{,}000$
2. Bedford supply $= x_{21} + x_{22} + x_{23} + x_{24} \leq 6{,}000$
3. York supply $= x_{31} + x_{32} + x_{33} + x_{34} \leq 2{,}500$

The destination constraints meet the demand at the four distribution centers:

1. Boston $= x_{11} + x_{21} + x_{31} = 6{,}000$
2. Chicago $= x_{12} + x_{22} + x_{32} = 4{,}000$
3. St. Louis $= x_{13} + x_{23} + x_{33} = 2{,}000$
4. Lexington $= x_{14} + x_{24} + x_{34} = 1{,}500$

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# 14.6 Linear Programming Model for Foster Generators

Min $3x_{11} + 2x_{12} + 7x_{13} + 6x_{14} + 6x_{21} + 5x_{22} + 2x_{23} + 3x_{24} + 2x_{31} + 3x_{32} + 4x_{33} + 5x_{34}$

s.t.

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &\leq 5{,}000 \\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 6{,}000 \\
x_{31} + x_{32} + x_{33} + x_{34} &\leq 2{,}500 \\
x_{11} + x_{21} + x_{31} &= 6{,}000 \\
x_{12} + x_{22} + x_{32} &= 4{,}000 \\
x_{13} + x_{23} + x_{33} &= 2{,}000 \\
x_{14} + x_{24} + x_{34} &= 1{,}500
\end{aligned}
$$

$x_{ij} \geq 0 \quad$ for $i = 1,2,3$ and $j = 1,2,3,4$

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# 14.6 Spreadsheet Solution for Foster Generators

The minimum total transportation cost is $39,500.

The Cleveland plant should ship 1,000 units to the Boston distribution center and 4,000 units to the Chicago distribution center.

The Bedford plant should ship 2,500 units to the Boston distribution center, 2,000 units to the S, Louis distribution center, and 1,500 units to the Lexington distribution center.

The York distribution center should ship all its 2,500 units to the Boston distribution center.

The distribution network is balanced, with all the plants operating at capacity and matching the demand of all the distribution centers.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Foster Generators** | | | | | |
| 2 | **Parameters** | | | | | |
| 3 | Shipping Cost/Unit | | Destination | | | |
| 4 | Origin | Boston | Chcago | St. Louis | Lexington | Supply |
| 5 | Cleveland | $3.00 | $2.00 | $7.00 | $6.00 | 5000 |
| 6 | Bedford | $6.00 | $5.00 | $2.00 | $3.00 | 6000 |
| 7 | York | $2.00 | $5.00 | $4.00 | $5.00 | 2500 |
| 8 | Demand | 6000 | 4000 | 2000 | 1500 | |
| 9 | | | | | | |
| 10 | | | | | | |
| 11 | **Model** | | | | | |
| 12 | | | | | | |
| 13 | Total Cost | $39,500.00 | | | | |
| 14 | | | | | | |
| 15 | | | Destination | | | |
| 16 | Origin | Boston | Chcago | St. Louis | Lexington | **Total** |
| 17 | Cleveland | 1000 | 4000 | 0 | 0 | 5000 |
| 18 | Bedford | 2500 | 0 | 2000 | 1500 | 6000 |
| 19 | York | 2500 | 0 | 0 | 0 | 2500 |
| 20 | Total | 6000 | 4000 | 2000 | 1500 | |
| 21 | | | | | | |

CENGAGE

# 14.6 Maximizing Banner Ad Revenue

In this example, we consider the Business, Science, and Sports sections of MHT, a large content publisher.

MHT has contracts for banner ads with five companies:

- Nile, Zstart, Cheetah, Stride, and Stove.

The expected click-through rates of MHT web content, the number of guaranteed impressions by section, and the contract impression limits for each advertiser are shown in the next slide.

MHT wants to use linear programming to allocate impressions for each of the five advertisers across the three sections of the MHT website so to maximize revenue, knowing that MHT receives $0.30 for each click-through achieved.

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# 14.6 Data for the MHT Banner Ad Allocation Problem

| | Click-Through Rates | | | | | |
|---|---|---|---|---|---|---|
| **Section** | **Nile** | **Zstar** | **Cheetah** | **Stride** | **Stove** | **Available Impressions** |
| Business | 0.0155 | 0.0265 | 0.0100 | 0.0170 | 0.0105 | 2,550,000 |
| Science | 0.0165 | 0.0110 | 0.0125 | 0.0265 | 0.0125 | 2,150,000 |
| Sports | 0.0145 | 0.0235 | 0.0190 | 0.0225 | 0.0160 | 2,500,000 |

| | Contract Impression Limits | | | | |
|---|---|---|---|---|---|
| **Section** | **Nile** | **Zstar** | **Cheetah** | **Stride** | **Stove** |
| Ads Lower Limit | 1,500,000 | 1,000,000 | 1,100,000 | 1,000,000 | 1,500,000 |
| Ads Upper Limit | 2,000,000 | 2,000,000 | 1,800,000 | 2,000,000 | 2,000,000 |

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## 14.6 Decision Variables and Objective Function for MHT

Let us define decision variables as follows:

$$x_{ij} = \text{the number of impressions in section } i \text{ allocated to company } j$$

where, $i = 1$ is Business, $i = 2$ is Science, and $i = 3$ is Sports; $j = 1$ is Nile, $j = 2$ is Zstar, $j = 3$ is Cheetah, $j = 4$ is Stride, and $j = 5$ is Stove.

The objective function is the maximized expected revenue, calculated as:

$$\text{Max} \quad \sum_{i=1}^{3}\sum_{j=1}^{5}(\$0.30)(\text{click-through rate})_{ij}(x_{ij})$$

That is, Max

$$\begin{aligned}&0.0047x_{11} + 0.0080x_{12} + 0.0030x_{13} + 0.0051x_{14} + 0.0032x_{15} + \\ &0.0050x_{21} + 0.0033x_{22} + 0.0038x_{23} + 0.0008x_{24} + 0.0038x_{25} + \\ &0.0044x_{31} + 0.0071x_{32} + 0.0057x_{33} + 0.0068x_{34} + 0.0048x_{35}\end{aligned}$$

# 14.6 Constraints for the MHT Problem

We have three impression availability constraints, one for each section:

| | |
|---|---|
| Business | $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 2{,}550{,}000$ |
| Science | $x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 2{,}150{,}000$ |
| Sports | $x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 2{,}500{,}000$ |

Lower and upper limit constraints for each company:

| | |
|---|---|
| Nile | $1{,}500{,}000 \leq x_{11} + x_{21} + x_{31} \leq 2{,}000{,}000$ |
| Zstar | $1{,}000{,}000 \leq x_{12} + x_{22} + x_{32} \leq 2{,}000{,}000$ |
| Cheetah | $1{,}100{,}000 \leq x_{13} + x_{23} + x_{33} \leq 1{,}800{,}000$ |
| Stride | $1{,}000{,}000 \leq x_{14} + x_{24} + x_{34} \leq 2{,}000{,}000$ |
| Stove | $1{,}500{,}000 \leq x_{15} + x_{25} + x_{35} \leq 2{,}000{,}000$ |

And the non-negativity constraints: $x_{ij} \geq 0$ for $i = 1{,}2{,}3{,}4{,}5$ and $j = 1{,}2{,}3$.

# 14.6 Spreadsheet Model for the MHT Problem

| | | | D | E | F | G |
|---|---|---|---|---|---|---|
| **MHT** | | | | | | |
| **Parameters** | | | | | | |
| Section | Nile | Zstar | Cheetah | Stride | Stove | Impressions Avai |
| Business | 0.0155 | 0.0265 | 0.01 | 0.017 | 0.0105 | 2550000 |
| Science | 0.0165 | 0.011 | 0.0125 | 0.0265 | 0.0125 | 2150000 |
| Sports | 0.0145 | 0.0235 | 0.019 | 0.0225 | 0.016 | 2500000 |
| Contracted Ads Lower Limit | 1500000 | 1000000 | 1100000 | 1000000 | 1500000 | |
| Contracted Ads Upper Limit | 2000000 | 2000000 | 1800000 | 2000000 | 2000000 | |
| | | | | | | |
| Revenue per click-through | 0.3 | | | | | |
| | | | | | | |
| Revenue per view | Nile | Zstar | Cheetah | Stride | Stove | |
| Business | =B4*$B$10 | =C4*$B$10 | =D4*$B$10 | =E4*$B$10 | =F4*$B$10 | |
| Science | =B5*$B$10 | =C5*$B$10 | =D5*$B$10 | =E5*$B$10 | =F5*$B$10 | |
| Sports | =B6*$B$10 | =C6*$B$10 | =D6*$B$10 | =E6*$B$10 | =F6*$B$10 | |
| | | | | | | |
| **Model** | | | | | | |
| | | Number of Banner Ads | | | | |
| | Nile | Zstar | Cheetah | Stride | Stove | |
| Business | 550000 | 2000000 | 0 | 0 | 0 | =SUM(B20:F20) |
| Science | 950000 | 0 | 0 | 1100000 | 100000 | =SUM(B21:F21) |
| Sports | 0 | 0 | 1100000 | 0 | 1400000 | =SUM(B22:F22) |
| | =SUM(B20:B22) | =SUM(C20:C22) | =SUM(D20:D22) | =SUM(E20:E22) | =SUM(F20:F22) | |
| | | | | | | |
| Click-Through Revenue | =SUMPRODUCT(B13:F15,B20:F22) | | | | | |

MODELfile: *mht*

CENGAGE

# 14.6 Spreadsheet Solution for the MHT Problem

All available impressions should be used, resulting in $45,270 total revenue.

Nile impressions are split between the Business and Science sections.

Stove impressions are split between the Science and Sports sections.

Zstar, Cheetah and Stride are allocated exclusively to the Business, Sports and Science sections, respectively.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **MHT** | | | | | | |
| 2 | **Parameters** | | | | | | |
| 3 | Section | Nile | Zstar | Cheetah | Stride | Stove | Impressions Available |
| 4 | Business | 0.0155 | 0.0265 | 0.0100 | 0.0170 | 0.0105 | 2,550,000 |
| 5 | Science | 0.0165 | 0.0110 | 0.0125 | 0.0265 | 0.0125 | 2,150,000 |
| 6 | Sports | 0.0145 | 0.0235 | 0.0190 | 0.0225 | 0.0160 | 2,500,000 |
| 7 | Contracted Ads Lower Limit | 1,500,000 | 1,000,000 | 1,100,000 | 1,000,000 | 1,500,000 | |
| 8 | Contracted Ads Upper Limit | 2,000,000 | 2,000,000 | 1,800,000 | 2,000,000 | 2,000,000 | |
| 9 | | | | | | | |
| 10 | Revenue per click-through | $0.30 | | | | | |
| 11 | | | | | | | |
| 12 | Revenue per view | Nile | Zstar | Cheetah | Stride | Stove | |
| 13 | Business | $0.0047 | $0.0080 | $0.0030 | $0.0051 | $0.0032 | |
| 14 | Science | $0.0050 | $0.0033 | $0.0038 | $0.0080 | $0.0038 | |
| 15 | Sports | $0.0044 | $0.0071 | $0.0057 | $0.0068 | $0.0048 | |
| 16 | | | | | | | |
| 17 | **Model** | | | | | | |
| 18 | | | Number of Banner Ads | | | | |
| 19 | | Nile | Zstar | Cheetah | Stride | Stove | |
| 20 | Business | 550000 | 2000000 | 0 | 0 | 0 | 2550000 |
| 21 | Science | 950000 | 0 | 0 | 1100000 | 100000 | 2150000 |
| 22 | Sports | 0 | 0 | 1100000 | 0 | 1400000 | 2500000 |
| 23 | | 1500000 | 2000000 | 1100000 | 1100000 | 1500000 | |
| 24 | | | | | | | |
| 25 | Click-Through Revenue | $45,270.00 | | | | | |

# 14.6 Reduced Cost Analysis for the MHT Problem

The Variable Cells section of the Solver sensitivity report provides information on the model variables' reduced cost.

The negative reduced cost for un-allocated impressions ($\text{Final Value} = 0$) indicates the amount to be subtracted from the click-through rates (i.e., increased) for them to be allocated in the optimal solution.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 6 | Variable Cells | | | | | | | |
| 7 | | | | Final | Reduced | Objective | Allowable | Allowable |
| 8 | | Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| 9 | | $B$20 | Business Nile | 550000 | 0 | 0.00465 | 0.0003 | 0.0003 |
| 10 | | $C$20 | Business Zstar | 2E+06 | 0 | 0.00795 | 1E+30 | 0.0003 |
| 11 | | $D$20 | Business Cheetah | 0 | -0.0014 | 0.003 | 0.00135 | 1E+30 |
| 12 | | $E$20 | Business Stride | 0 | -0.0026 | 0.0051 | 0.00255 | 1E+30 |
| 13 | | $F$20 | Business Stove | 0 | -0.0003 | 0.00315 | 0.0003 | 1E+30 |
| 14 | | $B$21 | Science Nile | 950000 | 0 | 0.00495 | 0.0003 | 0.0003 |
| 15 | | $C$21 | Science Zstar | 0 | -0.005 | 0.0033 | 0.00495 | 1E+30 |
| 16 | | $D$21 | Science Cheetah | 0 | -0.0009 | 0.00375 | 0.0009 | 1E+30 |
| 17 | | $E$21 | Science Stride | 1E+06 | 0 | 0.00795 | 0.0003 | 0.00225 |
| 18 | | $F$21 | Science Stove | 100000 | 0 | 0.00375 | 0.00165 | 0.0003 |
| 19 | | $B$22 | Sports Nile | 0 | -0.0017 | 0.00435 | 0.00165 | 1E+30 |
| 20 | | $C$22 | Sports Zstar | 0 | -0.0022 | 0.00705 | 0.00225 | 1E+30 |
| 21 | | $D$22 | Sports Cheetah | 1E+06 | 0 | 0.0057 | 0.0033 | 0.0009 |
| 22 | | $E$22 | Sports Stride | 0 | -0.0022 | 0.00675 | 0.00225 | 1E+30 |
| 23 | | $F$22 | Sports Stove | 1E+06 | 0 | 0.0048 | 0.0009 | 0.00165 |

CENGAGE

# 14.6 Shadow Price Analysis for the MHT Problem

The Constraints section of the sensitivity report provides information on the constraints' shadow price.

The only non-zero shadow prices are for binding constraints:

- Negative shadow prices for constraints bound to an allocation upper limit.
- Positive shadow prices for constraints bound to an allocation lower limit.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 25 | Constraints | | | | | | | |
| 26 | | | | Final | Shadow | Constraint | Allowable | Allowable |
| 27 | | Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| 28 | | $B$23 | Nile | 2E+06 | 0 | 2000000 | 1E+30 | 500000 |
| 29 | | $C$23 | Zstar | 2E+06 | 0.0003 | 2000000 | 100000 | 900000 |
| 30 | | $D$23 | Cheetah | 1E+06 | 0 | 1800000 | 1E+30 | 700000 |
| 31 | | $E$23 | Stride | 1E+06 | 0 | 2000000 | 1E+30 | 900000 |
| 32 | | $F$23 | Stove | 2E+06 | 0 | 2000000 | 1E+30 | 500000 |
| 33 | | $B$23 | Nile | 2E+06 | -0.003 | 1500000 | 100000 | 900000 |
| 34 | | $C$23 | Zstar | 2E+06 | 0 | 1000000 | 1000000 | 1E+30 |
| 35 | | $D$23 | Cheetah | 1E+06 | -0.0033 | 1100000 | 100000 | 100000 |
| 36 | | $E$23 | Stride | 1E+06 | 0 | 1000000 | 100000 | 1E+30 |
| 37 | | $F$23 | Stove | 2E+06 | -0.0042 | 1500000 | 100000 | 100000 |
| 38 | | $G$20 | Business Impressions Available | 3E+06 | 0.00765 | 2550000 | 900000 | 100000 |
| 39 | | $G$21 | Science Impressions Available | 2E+06 | 0.00795 | 2150000 | 900000 | 100000 |
| 40 | | $G$22 | Sports Impressions Available | 3E+06 | 0.009 | 2500000 | 100000 | 100000 |

CENGAGE

# 14.6 Assigning Project Leaders to Clients

An **assignment problem** involves assigning "agents" to "tasks" so to

- optimize a stated objective, with
- each agent assigned to at most one task.

As an example of an assignment problem, consider the case of Fowle Marketing Research, which needs to assign three project leaders (agents) to one of three clients (tasks) while minimizing the total time required.

The table provides all the project leader-client combinations and the estimated project completion times in days.

| | Client | | |
|---|---|---|---|
| **Project Leader** | **1** | **2** | **3** |
| 1. Terry | 10 | 15 | 9 |
| 2. Carle | 9 | 18 | 5 |
| 3. McClymonds | 6 | 11 | 3 |

CENGAGE

# 14.6 Problem Formulation for the Fowle Problem

Let us define $x_{ij}$ decision variables such that:

$$x_{ij} = \begin{cases} 1 \text{ if project leader } i \text{ is assigned to client } j \\ 0 \text{ otherwise} \end{cases}$$

where $i = 1,2,3$ and $j = 1,2,3$.

Using the above notation and the completion time data, we can write

Days required for Terry's assignment $= 10x_{11} + 15x_{12} + 9x_{13}$

Days required for Carle's assignment $= 9x_{21} + 18x_{22} + 5x_{23}$

Days required for McClymond's assignment $= 6x_{31} + 11x_{32} + 3x_{33}$

The objective function is the minimized sum of the completion times:

$$\text{Min } 10x_{11} + 15x_{12} + 9x_{13} + 9x_{21} + 18x_{22} + 5x_{23} + 6x_{31} + 11x_{32} + 3x_{33}$$

# 14.6 Network Representation of the Fowle Problem

The constraints for the assignment problem reflect the conditions that each project leader can be assigned to at most one client and that each client must have one assigned project leader.

$x_{11} + x_{12} + x_{13} \leq 1$ Terry's assignment

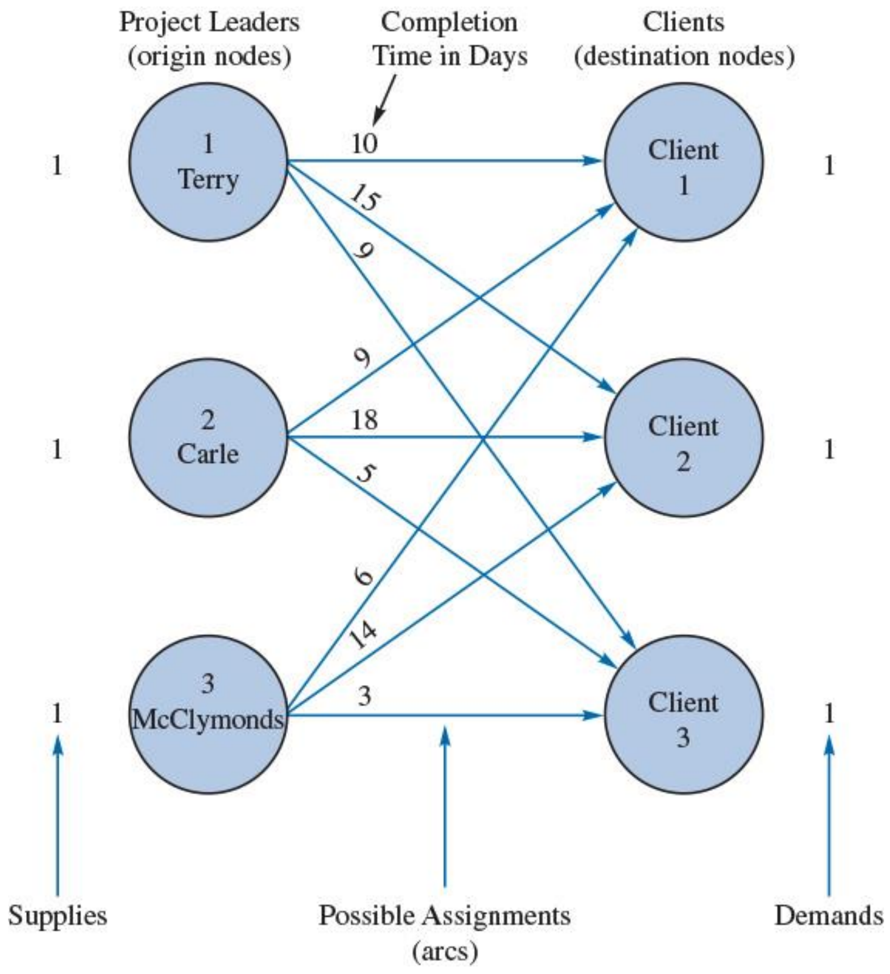
$x_{21} + x_{22} + x_{23} \leq 1$ Carle's assignment

$x_{31} + x_{32} + x_{33} \leq 1$ McClymond's assignment

$x_{11} + x_{21} + x_{31} = 1$ Client 1

$x_{12} + x_{22} + x_{32} = 1$ Client 2

$x_{13} + x_{23} + x_{33} = 1$ Client 3




CENGAGE

# 14.6 Linear Programming Model for the Fowle Problem

Min $\quad 10x_{11} + 15x_{12} + 9x_{13} + 9x_{21} + 18x_{22} + 5x_{23} + 6x_{31} + 11x_{32} + 3x_{33}$

s.t.

$$
\begin{array}{lllllllllll}
x_{11} + & x_{12} + & x_{13} & & & & & & & \le 1 \\
& & & x_{21} + & x_{22} + & x_{23} & & & & \le 1 \\
& & & & & & x_{31} + & x_{32} + & x_{33} & \le 1 \\
x_{11} + & & & x_{21} + & & & x_{31} & & & = 1 \\
& x_{12} + & & & x_{22} + & & & x_{32} & & = 1 \\
& & x_{13} + & & & x_{23} + & & & x_{33} & = 1
\end{array}
$$

$$x_{ij} \ge 0 \quad \text{for } i = 1,2,3 \text{ and } j = 1,2,3$$

The next slide shows the optimal solution for this model (MODELfile: *fowle*.)

# 14.6 Spreadsheet Model & Solution for the Fowle Problem

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Fowle Marketing Research** | | | | |
| 2 | **Parameters** | | | | |
| 3 | | | **Client** | | |
| 4 | **Project Leader** | 1 | 2 | 3 | |
| 5 | Terry | 10 | 15 | 9 | |
| 6 | Carle | 9 | 18 | 5 | |
| 7 | McClymonds | 6 | 14 | 3 | |
| 8 | | | | | |
| 9 | **Model** | | | | |
| 10 | | | | | |
| 11 | **Total Completion Time** | =SUMPRODUCT(B5:D7,B15:D17) | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | **Project Leader** | to Client 1 | to Client 2 | to Client 3 | Total |
| 15 | Terry | 0 | 1 | 0 | =SUM(B15:D15) |
| 16 | Carle | 0 | 0 | 1 | =SUM(B16:D16) |
| 17 | McClymonds | 1 | 0 | 0 | =SUM(B17:D17) |
| 18 | Total | =SUM(B15:B17) | =SUM(C15:C17) | =SUM(D15:D17) | |
| 19 | | | | | |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Fowle Marketing Research** | | | | |
| 2 | **Parameters** | | | | |
| 3 | | | **Client** | | |
| 4 | **Project Leader** | 1 | 2 | 3 | |
| 5 | Terry | 10 | 15 | 9 | |
| 6 | Carle | 9 | 18 | 5 | |
| 7 | McClymonds | 6 | 14 | 3 | |
| 8 | | | | | |
| 9 | **Model** | | | | |
| 10 | | | | | |
| 11 | **Total Completion Time** | 26 | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | **Project Leader** | to Client 1 | to Client 2 | to Client 3 | Total |
| 15 | Terry | 0 | 1 | 0 | 1 |
| 16 | Carle | 0 | 0 | 1 | 1 |
| 17 | McClymonds | 1 | 0 | 0 | 1 |
| 18 | Total | 1 | 1 | 1 | |
| 19 | | | | | |

Terry is assigned to Client 2, Carle to Client 1, and McClymond to Client 3 for a total completion time of 26 days.

# 14.7 Generating an Alternative Optimal Solution

Consider the solution to the Foster Generators transportation problem with an optimal total transportation cost of $39,500.

| | | | |
|---|---|---|---|
| $x_{11} = 1{,}000$ | $x_{12} = 4{,}000$ | $x_{13} = 0$ | $x_{14} = 0$ |
| $x_{21} = 2{,}500$ | $x_{22} = 0$ | $x_{23} = 2{,}000$ | $x_{24} = 1{,}500$ |
| $x_{31} = 2{,}500$ | $x_{32} = 0$ | $x_{33} = 0$ | $x_{34} = 0$ |

We can solve for an alternative optimal solution by adding the constraint for the original objective function set equal to the total transportation cost.

$$3x_{11} + 2x_{12} + 7x_{13} + 6x_{14} + 6x_{21} + 5x_{22} + 2x_{23} + 3x_{24} + 2x_{31} + 3x_{32} + 4x_{33} + 5x_{34} = 39{,}500$$

And set an objective function that maximizes the un-allocated shipping routes.

$$\text{Max } x_{13} + x_{14} + x_{22} + x_{32} + x_{33} + x_{34}$$

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# 14.7 Alternative Linear Program for Foster Generators

$$
\begin{array}{ll}
\text{Max} & x_{13} + x_{14} + x_{22} + x_{32} + x_{33} + x_{34} \\
\text{s.t.} & \\
& x_{11} + x_{12} + x_{13} + x_{14} \le 5{,}000 \\
& x_{21} + x_{22} + x_{23} + x_{24} \le 6{,}000 \\
& x_{31} + x_{32} + x_{33} + x_{34} \le 2{,}500 \\
& x_{11} + x_{21} + x_{31} = 6{,}000 \\
& x_{12} + x_{22} + x_{32} = 4{,}000 \\
& x_{13} + x_{23} + x_{33} = 2{,}000 \\
& x_{14} + x_{24} + x_{34} = 1{,}500 \\
& 3x_{11} + 2x_{12} + 7x_{13} + 6x_{14} + 6x_{21} + 5x_{22} + 2x_{23} + 3x_{24} + 2x_{31} + 3x_{32} + 4x_{33} + 5x_{34} = 39{,}500 \\
& x_{ij} \ge 0 \quad \text{for } i = 1,2,3 \text{ and } j = 1,2,3,4
\end{array}
$$

# 14.7 Alternative Optimal Solution to the Foster Problem

In the alternative solution, Bedford ships 2,500 units to Chicago instead of Boston.

To find an alternative optimal solution to a linear program:

| | Amount Shipped To: | | | | |
|---|---|---|---|---|---|
| **From:** | Boston | Chicago | St. Louis | Lexington | Total |
| Cleveland | 3,500 | 1,500 | 0 | 0 | 5,000 |
| Bedford | 0 | 2,500 | 2,000 | 1,500 | 6,000 |
| York | 2,500 | 0 | 0 | 0 | 2,500 |
| Total | 6,000 | 4,000 | 2,000 | 1,500 | |

**Step 1.** Solve the linear program

**Step 2.** Make a new objective function to be maximized. It is the sum of those variables that were equal to zero in the solution from Step 1

**Step 3.** Add a constraint to the original set that forces the original objective function to be equal to the optimal objective function value from Step 1

**Step 4.** Solve the problem created in Steps 2 and 3. If the objective function value is positive, you have found an alternative optimal solution

# Summary

- We showed how the graphical solution of a two-variable linear programming problem helps us understand how the computer solves large linear programs.
- We discussed how Excel Solver can be used to solve linear optimization problems.
- Slack and surplus variables are used to represent inequality constraints as equality.
- If a linear program solution is infeasible, no optimal solution can be found.
- In the case of an unbounded solution, the objective function diverges.
- Two or more optimal extreme points exist in the case of alternative optimal solutions.
- The Excel Solver Sensitivity Report allows us to analyze the impact of changes in the objective function coefficients and right-hand side values of constraints.
- We showed how to write a mathematical model using general linear programming notation and presented additional examples of linear programming applications.