MTL 411: Functional Analysis

Lecture A: Adjoint operators

Recall, Riesz representation theorem: Suppose f is a continuous linear functional on a Hilbert space H. Then there exists a unique z in H such that $f(x) = \langle x, z \rangle$ for all $x \in H$.

As a consequence, for a given bounded linear operator, we can construct an associated bounded linear operator, which is called Hilbert-adjoint operator or simply adjoint operator.

1 Adjoint operator

Let H_1 and H_2 be (Real/complex) Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. Then an adjoint operator T^* of T is an operator $\underline{T^* : H_2 \to H_1}$ satisfies

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \forall x \in H_1, \forall y \in H_2.$$

Remark. In the above equation, the left hand side inner product is between elements in H_2, that is, the inner product is from H_2 and in the right hand side the inner product is from H_1.

Questions. Given $T \in \mathcal{B}(H_1, H_2)$, does an adjoint operator T^* exist? If it exists, will it be unique and bounded operator? The answers are affirmative.

Theorem 1.1. *Given $T \in \mathcal{B}(H_1, H_2)$, there exists a unique operator $T^* \in \mathcal{B}(H_2, H_1)$ such that*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \forall x \in H_1, \forall y \in H_2$$

and $||T|| = ||T^*||$.

Proof. Fix $y \in H_2$. Define $f_y(x) = \langle Tx, y \rangle$, $x \in H_1$.
Since T is bounded linear operator and using Cauchy-Schwarz inequality (CSI), we can show that f_y is a continuous linear functional on H_1, that is, f_y is linear and

$$|f_y(x)| \leq ||T|| \, ||y|| \, ||x||, \forall x \in H_1.$$

Then by Riesz representation theorem, there exists a unique $z \in H_1$ such that $f_y(x) = \langle x, z \rangle$.
Therefore for each $y \in H_2$, there exists a unique $z \in H_1$ such that $f_y(x) = \langle x, z \rangle = \langle Tx, y \rangle$.
Now, we define $T^* y = z$, for each $y \in H_2$. Thus

$$\langle x, T^* y \rangle = \langle Tx, y \rangle, \forall x \in H_1, \forall y \in H_2. \tag{1.1}$$

The uniqueness of T^* follows from the Riesz representation theorem or from a simple relation that $\langle (T_1 - T_2)y, x \rangle = 0$ for all $y \in H_2$, $x \in H_1$ implies $T_1 - T_2 = 0$.

Claim: T^* is linear.

Consider $y, z \in H_2$ and $\alpha \in \mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, we get

$$\begin{aligned} \langle x, T^*(\alpha y + z) \rangle &= \langle Tx, \alpha y + z \rangle \\ &= \overline{\alpha} \langle Tx, y \rangle + \langle Tx, z \rangle \\ &= \overline{\alpha} \langle x, T^* y \rangle + \langle x, T^* z \rangle \\ &= \langle x, \alpha T^* y + T^* z \rangle, \forall x \in H_1. \end{aligned}$$

Hence $T^*(\alpha y + z) = \alpha T^* y + T^* z, \forall y, z \in H_2, \forall \alpha \in \mathbb{K}$.

Claim: T^* is bounded and $||T|| = ||T^*||$.

Since the equation (1.1) is satisfied for all elements in H_1 and H_2, choose $x = T^*y$ and using CSI, we get

$$\begin{aligned} ||T^*y||^2 &\leq ||T||\ ||T^*y||\ ||y||, \forall y \in H_2. \\ \implies ||T^*y|| &\leq ||T||\ ||y||, \forall y \in H_2. \end{aligned}$$

Therefore, T^* is bounded and $||T^*|| \leq ||T||$. By a similar argument using (1.1), we can show that $||T|| \leq ||T^*||$. □

Remark. If we have an operator S such that $\langle x, Sy \rangle = \langle Tx, y \rangle, \forall x \in H_1, \forall y \in H_2$, then by uniqueness $T^* = S$.

Examples.

- $A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$. Then $\langle Ax, y \rangle = (Ax)^T y = (x^T A^T) y = x^T (A^T y) = \langle x, A^T y \rangle$. Therefore, the adjoint A^* is the transpose A^T of A.
- $B : \mathbb{C}^n \to \mathbb{C}^n$, $x \mapsto Bx$. Then $\langle Bx, y \rangle = (Bx)^T \overline{y} = (x^T B^T) \overline{y} = x^T (B^T \overline{y}) = \left\langle x, \overline{B^T} y \right\rangle$. Therefore, the adjoint B^* is the conjugate-transpose $\overline{B^T}$ of B.
- The integral operator $T : L^2[a,b] \to L^2[a,b]$ defined by

$$Tf(x) = \int_a^b k(x,t) f(t) dt, \ x \in [a,b]$$

where $k(x,t)$ is a real-valued continuous function on $[a,b] \times [a,b]$. Then it is easy to show that T is a bounded linear operator. Indeed,

$$|Tf(x)|^2 \leq \int_a^b |k(x,t)|^2 dt \int_a^b |f(t)|^2 dt \quad \text{(using CSI in continuous variable)}$$

$$\left(\int_a^b |Tf(x)|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_a^b \int_a^b |k(x,t)|^2 dt dx \right)^{\frac{1}{2}} \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Now we calculate T^* :

$$\begin{aligned} \langle Tf, g \rangle &= \int_a^b Tf(x) \overline{g(x)} dx \\ &= \int_a^b \left(\int_a^b k(x,t) f(t) dt \right) \overline{g(x)} dx \\ &= \int_a^b \left(\int_a^b k(x,t) \overline{g(x)} dx \right) f(t) dt \\ &= \int_a^b f(t) \overline{\left(\int_a^b \overline{k(x,t)} g(x) dx \right)} dt. \end{aligned}$$

Therefore, $T^* g(y) = \int_a^b \overline{k(s,y)} g(s) ds, \ y \in [a,b]$.

Exercises

1. Find the adjoint of right shift operator:

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^2.$$

2. Find the adjoint of multiplication operator:

$$Mf(x) = g(x)f(x), \; f \in L^2[a, b]$$

where $g(x)$ is a bounded function on $[a, b]$.

Next we will discuss a simple lemma which will be useful in several occasions.

Lemma 1.2. *(Zero operator) Let X and Y be inner product spaces and $T : X \to Y$ be a bounded linear operator. Then*

(a) $T = 0 \iff \langle Tx, y\rangle = 0$ *for all* $x \in X$ *and* $y \in Y$.

(b) If $T : X \to X$, *where* X *is complex, and* $\langle Tx, x\rangle = 0$ *for all* $x \in X$, *then* $T = 0$.

Proof. (a) The proof is trivial.
(b) Consider

$$\begin{aligned} \langle T(\alpha x + y), \alpha x + y\rangle &= 0 && (1.2)\\ \overline{\alpha}\,\langle Ty, x\rangle + \alpha\,\langle Tx, y\rangle &= 0, \; \forall \alpha \in \mathbb{C} \forall x, y \in X. && (1.3) \end{aligned}$$

Choose $\alpha = 1$ and $\alpha = i$, we get $\langle Tx, y\rangle = 0, \; \forall x, y \in X$. Then $T = 0$. $\square$

Remark. In the above lemma, if X is real inner product space then the statment (b) is not true. (Hint. Use rotation operator in $\mathbb{R}^2$.)

Properties of the adjoint operator

Theorem 1.3. *Let* $T : H_1 \to H_2$ *be a bounded linear operator. Then*

1. $(T^*)^* = T$
2. $||TT^*|| = ||T^*T|| = ||T||^2$
3. $\mathcal{N}(T) = \mathcal{R}(T^*)^\perp$
4. $\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}$
5. $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$
6. $\mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}$

Here $\mathcal{N}(T)$ *and* $\mathcal{R}(T)$ *denote the null space and range space of* T.

Proof. 1. Consider $\langle (T^*)^*x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle$, for all $x \in H_1$, $y \in H_2$.
Then $\langle ((T^*)^* - T)x, y\rangle = 0$, for all $x \in H_1$, $y \in H_2$.

2. Recall that the norm of composition operators $||SH|| \leq ||S|| \; ||H||$. Henceforth, we get $||T^*T|| \leq ||T^*|| \; ||T|| = ||T||^2$.
For the other inequality, consider

$$\begin{aligned}
||T||^2 &= \left(\sup_{0 \neq x \in H_1} \frac{||Tx||}{||x||} \right) \left(\sup_{0 \neq x \in H_1} \frac{||Tx||}{||x||} \right) \\
&= \sup_{0 \neq x \in H_1} \frac{||Tx||^2}{||x||^2} = \sup_{0 \neq x \in H_1} \frac{\langle Tx, Tx \rangle}{||x||^2} \\
&= \sup_{0 \neq x \in H_1} \frac{\langle T^*Tx, x \rangle}{||x||^2} \\
&\leq \sup_{0 \neq x \in H_1} \frac{||T^*T|| ||x|| ||x||}{||x||^2} = ||T^*T||.
\end{aligned}$$

3. Suppose $x \in \mathcal{N}(T)$, then $Tx = 0$. Thus $\langle T^*y, x \rangle = \langle y, Tx \rangle = 0$, for all $y \in H_2$.
Therefore, $x \in \mathcal{R}(T^*)^\perp$. Similarly, it is easy to show that $\mathcal{R}(T^*)^\perp \subset \mathcal{N}(T)$.

4. For $z \in \mathcal{R}(T^*)$, there exists a $y \in H_2$ such that $T^*y = z$.
Then $\langle z, x \rangle = \langle T^*y, x \rangle = \langle y, Tx \rangle = 0$ for all $x \in \mathcal{N}(T)$. Therefore, $z \perp \mathcal{N}(T)$. This implies that $\mathcal{R}(T^*) \subset \mathcal{N}(T)^\perp$, and hence $\overline{\mathcal{R}(T^*)} \subset \mathcal{N}(T)^\perp$ (why?).

Claim. $\overline{\mathcal{R}(T^*)} = \mathcal{N}(T)^\perp$.
Suppose this is not true, that is, $\overline{\mathcal{R}(T^*)}$ is a proper closed subspace of $\mathcal{N}(T)^\perp$. Then by projection theorem, there exists a $0 \neq x_0 \in \mathcal{N}(T)^\perp$ and $x_0 \perp \overline{\mathcal{R}(T^*)}$. In particular,

$$\begin{aligned}
\langle x_0, T^*y \rangle &= 0, \forall y \in H_2 \\
\implies \langle Tx_0, y \rangle &= 0, \forall y \in H_2 \\
\implies Tx_0 &= 0, \\
\implies x_0 \in \mathcal{N}(T) \cap \mathcal{N}(T)^\perp &
\end{aligned}$$

Therefore, $x_0 = 0$ which is contradiction to $x_0 \neq 0$, hence the claim is proved.

The other parts are exercise. $\square$

Exercises Let H_1, H_2 and H_3 be Hilbert spaces. Then show that

1. $(TS)^* = S^*T^*$ for $S \in \mathcal{B}(H_1, H_2), T \in \mathcal{B}(H_2, H_3)$
2. $(\alpha T + \beta S)^* = \overline{\alpha}T^* + \overline{\beta}S^*$ for $S, T \in \mathcal{B}(H_1, H_2)$, and $\alpha, \beta \in \mathbb{K}$.
3. $T^*T = 0 \iff T = 0$.
4. Construct a bounded linear operator on ℓ^2 whose range is not closed.
   Hint: $T(x_1, x_2, \ldots) = (x_1, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \ldots)$, $(x_n) \in \ell^2$.
5. Let H be a Hilbert space and $T : H \to H$ be a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and $(T^*)^{-1} = [T^{-1}]^*$.
6. Let T_1 and T_2 be bounded linear operators on a complex Hilbert space H into itself. If $\langle T_1x, x \rangle = \langle T_2x, x \rangle$ for all $x \in H$, show that $T_1 = T_2$.
7. Let $S = I + T^*T : H \to H$, where T is linear and bounded. Show that $S^{-1} : S(H) \to H$ exists.

Self-adjoint, Normal and Unitary operators

Definition 1.4. *A bounded linear operator $T : H \to H$ on a Hilbert space H is said to be*

self-adjoint *if*	$T = T^*$,
unitary *if T is bijective and*	$T^* = T^{-1}$,
normal *if*	$T^*T = TT^*$.

Remarks.

- T is unitary iff $T^*T = TT^* = I$.
- Unitary, Self-adjoint $\implies$ Normal. The converse is not true. (Hint. $T = 2iI$, where I is the identity operator on a complex Hilbert space).

Let us discuss a simple criterion for self-adjointness.

Theorem 1.5. *(Self-adjointness) Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Then:*

(a) If T is self-adjoint, then $\langle Tx, x\rangle$ is real for all $x \in H$.

(b) If H is complex Hilber space and $\langle Tx, x\rangle$ is real for all $x \in H$, then T is self-adjoint.

Proof. (a) Given $T = T^*$, we get $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$, for all $x \in H$. Hence the imaginary part of $\langle Tx, x\rangle = 0$ for all $x \in H$.
(b) Consider $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = \langle T^*x, x\rangle$ for all $x \in H$.
Thus $\langle (T - T^*)x, x\rangle = 0$, for all $x \in H$. Hence by zero operator lemma, we get $T = T^*$. $\square$

Remarks.

1. Suppose S and T are self-adjoint operators. Then $(ST)^* = T^*S^* = ST$. Therefore, ST is self-adjoint iff S and T commutes.
2. The set of self-adjoint operators in $\mathcal{B}(H)$ is a closed set with respect to operator norm.

Now we discuss some basic properties of unitary operators.

Theorem 1.6. *(Unitary operator) Let H be a Hilbert space and the operators $U : H \to H$ and $V : H \to H$ be unitary. Then:*

(a) U is isometric, i.e., $||Ux|| = ||x||$, for all $x \in H$.

(b) $||U|| = 1$, provided $H \neq \{0\}$.

(c) U^{-1}, UV are unitary.

(d) A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and surjective.

Proof. (a) Since $UU^* = U^*U = I$, we get $\langle U^*Ux, x\rangle = \langle Ix, x\rangle = ||x||^2$. Thus $||Ux|| = ||x||$.
(b) If $H \neq \{0\}$, choose $0 \neq x \in H$, then $||U(\frac{x}{||x||})|| = 1$.
(c) Since $U^{-1} = U^*$, we have $(U^{-1})^{-1} = (U^*)^{-1} = (U^{-1})^*$. Next, $(UV)^*UV = V^*U^*UV = I$. Then the result follows.

(d) Suppose T is a unitary operator on a complex Hilbert space H, then it is clear that T is isometric and surjective. Conversely, suppose T is isometric and surjective on a complex Hilbert space H. Then, consider

$$\langle (U^*U - I)x, x\rangle = 0 \text{ for all } x \in H,$$

then by zero operator lemma, we get $U^*U = I$. Since U is injective and surjective, U is invertible. Therefore, $U^{-1} = U^*$. $\square$

Exercises Problems 4, 5, 6, 14, 15 from E. Kreyzig (Pages 207-208).

Reference

E. Kreyzig, Introductory Functional Analysis with Applications, John Wiley & Sons. Inc, 1978.