15) Seja $f:[0,+\infty) \to \mathbb{R}$ limitada em cada intervalo limitado. Demonstre que se

$$\lim_{x \to \pm\infty} [f(x+1) - f(x)] = L,$$ então:

$$\lim_{x \to \pm\infty} f(x)/x = L.$$

Resposta: Temos que f é limitada em todos os intervalos de $[0,+\infty)$, para todo $c \in \mathbb{Z}_+$ existe $M_c > 0$ tal que:

$$|f(\lambda)| < M_c, \text{ para todo } \lambda \in [c-1, c).$$

Para $c \in \mathbb{Z}_+$, $\lambda \in [c-1, c)$ e $k \in \mathbb{Z}_+$, assim temos que:

$$\left| \frac{f(\lambda+k)}{\lambda+k} - L \right| \leq \left| \frac{f(\lambda+k)}{\lambda+k} - \frac{f(\lambda+k)}{k} + \frac{f(\lambda)}{k} \right| +$$

$$\left| \frac{f(\lambda+k)}{k} - \frac{f(\lambda)}{k} - L \right|$$

$$\left| \frac{f(\lambda+k)}{\lambda+k} - L \right| \leq \left| \frac{f(\lambda+k)}{\lambda+k} - \frac{f(\lambda+k)}{k} \right| + \left| \frac{f(\lambda)}{k} \right| +$$

$$\left| \frac{f(\lambda+k)}{k} - \frac{f(\lambda)}{k} - L \right|$$

$$\left|\frac{f(\lambda+k)}{\lambda+k}-L\right| \leq \left|\frac{\lambda}{\lambda+k}\right| \cdot \left|\frac{f(\lambda+k)}{k}\right| + \left|\frac{f(\lambda+k)}{k}\right| +$$

$$\left|\frac{f(\lambda+k)}{k}-\frac{f(\lambda)}{k}-L\right|$$

$$\left|\frac{f(\lambda+k)-L}{\lambda+k}\right| \leq \left|\frac{\lambda}{\lambda+k}\right| \left(\left|\frac{f(\lambda+k)}{k}-\frac{f(\lambda)}{k}-L\right| + \left|\frac{f(\lambda+k)}{k}\right| + |L|\right) + \left|\frac{f(\lambda)}{k}\right| + \left|\frac{f(\lambda+k)}{k}-\frac{f(\lambda)}{k}-L\right|$$

$$\left|\frac{f(\lambda+k)-L}{\lambda+k}\right| \leq \frac{C}{k}\left(\left|\frac{f(\lambda+k)}{k}-\frac{f(\lambda)}{k}-L\right| + \left|\frac{f(\lambda)}{k}\right| + |L|\right)$$

$$+\left|\frac{f(\lambda)}{k}\right| + \left|\frac{f(\lambda+k)}{k}-\frac{f(\lambda)}{k}-L\right|$$

$$\left|\frac{f(\lambda+k)}{k}-L\right| \leq \frac{C}{k}|L| + \left(1+\frac{C}{k}\right)\left|\frac{f(\lambda)}{k}\right| + \left(1+\frac{C}{k}\right)\left|\frac{f(\lambda+k)}{k}\right.$$

$$\left.-\frac{f(\lambda)}{k}-L\right|$$

$$\left|\frac{f(\lambda+k)}{\lambda+k}-L\right| < \frac{1}{k}\left(C|L| + \left(1+\frac{C}{k}\right)M_C\right) + \left(1+\frac{C}{k}\right)\Big|$$

$$\left.\frac{f(\lambda+k)}{k}-\frac{f(\lambda)}{k}-L\right|$$

Seja $\varepsilon>0$, devemos provar que existe C e $N_0 \in \mathbb{Z}$ tal que para todo $k \geq N_0$ e $x \in [C-1, C]$ valem as desigualdades;

$$\frac{1}{K}\left(C|L| + \left(1+\frac{C}{K}\right)M\right) < \frac{\varepsilon}{2} \text{, e}$$

$$\left(1 + \frac{C}{K}\right)\left|\frac{f(\lambda+K)}{K} - \frac{f(\lambda)}{K} - L\right| < \frac{\varepsilon}{2}$$

Com isso, se $x \geq (C-1) + N_0$, existem $\lambda \in [C-1, C)$ e $K \in \mathbb{Z}_+$ tais que $x = \lambda + K$, $K \geq N_0$ e consequentemente,

$$\left|\frac{f(x)}{x} - L\right| = \left|\frac{f(\lambda+K)}{\lambda+K} - L\right|$$

$$\left|\frac{f(x)}{x} - L\right| < \frac{1}{K}\left(C|L| + \left(1+\frac{C}{K}\right)M_C\right) + \left(1+\frac{C}{K}\right)\cdot\left|\frac{f(\lambda+K)}{K} - \frac{f(\lambda)}{K} - L\right|$$

$$\left|\frac{f(x)}{x} - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\left|\frac{f(x)}{x} - L\right| = \varepsilon$$

Com isso, pode-se concluir que $\lim_{N \to \infty} \frac{f(x)}{x} = L$.

Por hipótese, $\lim_{x \to +\infty} (f(x+1) - f(x)) = L$.

Fixemos $C \in \mathbb{Z}_+$ tal que para todo número real $x \geq C-1$, vale a desigualdade:

Temos portanto, que para todo $l \in [c-1, c]$ e $k \in \mathbb{Z}_+$, temos que:

$$\left| \frac{f(l+k)}{k} - \frac{f(l)}{k} - L \right| = \left| \sum_{i=0}^{k-1} \left(\frac{f(l+i+1)}{k} - \frac{f(l+i)}{k} - \frac{L}{k} \right) \right|$$

$$\left| \frac{f(l+k)}{k} - \frac{f(l)}{k} - L \right| \leq \sum_{i=0}^{k-1} \left| \frac{f(l+i+1) - f(l+i) - L}{k} \right|$$

$$\left| \frac{f(l+k)}{k} - \frac{f(l)}{k} - L \right| < \sum_{i=0}^{k-1} \frac{\varepsilon/3}{k}$$

$$\left| \frac{f(l+k)}{k} - \frac{f(l)}{k} - L \right| = \frac{\varepsilon}{3}$$

Como: $\lim\limits_{k \to \infty} \frac{1}{k} = 0$, e:

$$\lim_{k \to +\infty} \left(c|L| + \left(1 + \frac{c}{k}\right) M_c \right) = c|L| + M$$

Segue que: $\lim\limits_{k \to +\infty} \frac{1}{k} \left(c|L| + \left(1 + \frac{c}{k}\right) M_c \right) = 0$

Além disso: $\lim\limits_{k \to +\infty} \left(1 + \frac{c}{k}\right) = 1$

Ao fixaremos $n_0 \in \mathbb{Z}_+$ tal que para todo $k \geq n_0$ valem as desigualdades:

$$\left|\frac{1}{k}\left(c|L| + \left(1+\frac{c}{k}\right)M_c\right)\right| < \frac{\varepsilon}{2}, \text{ e}$$

$$1 + \frac{c}{k} < 3/2$$

Desta forma, para $l \in [c-1, c]$ e $k \geq n_0$, temos que:

$$\left|\frac{1}{k}\left(c|L| + \left(1+\frac{c}{k}\right)M_c\right| < \frac{\varepsilon}{2}$$

$$\left(1+\frac{c}{k}\right)\left|\frac{f(l+k)}{k} - \frac{f(l)}{k} - L\right| < \frac{3}{2}\frac{\varepsilon}{3} = \frac{\varepsilon}{2}$$