1) Defina $f:\mathbb{R}^2 \to \mathbb{R}$ por:

$$f(x,y) = \begin{cases} xy\dfrac{(x^2-y^2)}{(x^2+y^2)}, & \text{se } (x,y) \neq (0,0) \\ 0, & \text{se } (x,y) = (0,0) \end{cases}$$

Prove que $\frac{\partial f}{\partial y}(x,0) = x$ para todo x e $\frac{\partial f}{\partial x}(0,y) = -y$ para todo y. Prove que $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Temos que: $f(x,y) = xy \cdot \dfrac{(x^2-y^2)}{(x^2+y^2)}$, se $(x,y) \neq 0$, e

$f(x,y) = 0$, se $(0,y) = (0,0)$

1) Devemos mostrar que: $\frac{\partial f}{\partial y}(x,0) = x$ e $\frac{\partial f}{\partial x}(0,y) = -y$, com isso temos:

i) $\frac{\partial f}{\partial y}(x,y)$ ou $\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(xy \cdot \left(\frac{x^2-y^2}{x^2+y^2}\right)\right) = x\frac{\partial}{\partial y} y \cdot \left(\frac{x^2-y^2}{x^2+y^2}\right)$,

Aplicando a regra do quociente: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{(g)^2} \longrightarrow$

$$\frac{x\left(\frac{\partial}{\partial y}\left(y(x^2-y^2)\right) \cdot (x^2+y^2) - \frac{\partial}{\partial y}(x^2+y^2) \cdot y(x^2-y^2)\right)}{(x^2+y^2)^2} =$$

$\dfrac{x \cdot (x^2-3y^2)(x^2+y^2) - 2y \cdot y(x^2-y^2)}{(x^2+y^2)^2}$, simplificando:

$\frac{\partial f}{\partial y} = \dfrac{x(-y^4-4x^2y^2+x^4)}{(x^2+y^2)^2}$, agora no ponto $(x,0)$ temos:

$\frac{\partial f}{\partial y}(x,0) = \dfrac{x(0-0+x^4)}{(x^2+0)^2} = x$, então mostramos que:

$\left(\frac{\partial f}{\partial y}\right)$ no ponto $(x,0) = x$. (A).

Agora a derivada parcial $\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(xy \cdot \frac{x^2-y^2}{x^2+y^2}\right) =$

$y \cdot \frac{\partial}{\partial x}\left(\frac{x(x^2-y^2)}{x^2+y^2}\right)$. Aplicando a regra do quociente:

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$, então:

$$\frac{y \cdot \frac{\partial}{\partial x}\left(x(x^2-y^2)\right) \cdot (x^2+y^2) - \frac{\partial}{\partial x}(x^2+y^2) \cdot x(x^2-y^2)}{(x^2+y^2)^2} =$$

$y \cdot \frac{(3x^2-y^2)(x^2+y^2) - 2x(x(x^2-y^2))}{(x^2+y^2)^2}$, simplificando.,

$\frac{\partial f(x,y)}{\partial x} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$, aplicando no ponto $(0,y)$

$\frac{\partial f(x,y)}{\partial x}(0,y) = y \cdot \frac{(-y^4)}{y^4} = -y$. Com isso provamos:

$\frac{\partial f(x,y)}{\partial x}$ no ponto $(0,y) = -y$. (B)

ii) Tomamos a derivada parcial (A) em x.

$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x(-y^4-4x^2y^2+x^4)}{(x^2+y^2)^2}\right)$, aplicando a regra

do quociente, $\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial/\partial x\left(x(-y^4-4x^2y^2+x^4)\right) \cdot (x^2+y^2)}{((x^2+y^2)^2)^2} -$

$\frac{\partial/\partial x(x^2+y^2) \cdot (x(-y^4-4x^2y^2+x^4))}{((x^2+y^2)^2)^2}$,

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{(5x^4-12x^2y^2-y^4)(x^2+y^2) - 4x(x^2+y^2)x(-y^4-4x^2y^2+x^4)}{((x^2+y^2)^2)^2}$$

Simplificando., $\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$, (C)

E agora tomando a derivada parcial de B em y.

$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial}{\partial y}\left(y \cdot \frac{(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}\right)$

Aplicando a regra do quociente,

$$\frac{\frac{\partial}{\partial y}\left[y(x^4+4x^2y^2-y^4)\right]((x^2+y^2)^2) - \frac{\partial}{\partial y}(x^2+y^2)\cdot y(x^4+4x^2y^2-y^4)}{((x^2+y^2)^2)^2} =$$

$$\frac{(-5y^4+12x^2y^2+x^4)(x^2+y^2)^2 - 4y(x^2+y^2)\cdot y(x^4+4x^2y^2-y^4)}{((x^2+y^2)^2)^2} =$$

Simplificando,

$$\frac{\partial^2}{\partial y \partial y} f(x,y) = \frac{-y^6-9y^4x^2+9y^2x^4+x^6}{(x^2+y^2)^3}$$, no ponto $(0,0)$,

$\frac{\partial^2 f}{\partial x \partial y} = 0$, e $\frac{\partial^2 f}{\partial y \partial x} = 0$, dado que $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Mas, $\frac{\partial f}{\partial y}(x,0) = x \longrightarrow \frac{\partial^2 f}{\partial x \partial y} = 1$, e

$\frac{\partial f}{\partial x}(0,y) = -y \longrightarrow \frac{\partial^2 f}{\partial y \partial x} = -1$

Portanto, conclui-se que: $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$