Formas diferenciais

- Funcional em $X \to L : X \to \mathbb{R}$

$\uparrow$ espaço de objetos

K-forma diferencial $\to X = \{\text{objetos de dim } K\}$

$\hookrightarrow$ variedades, etc

0-formas : $p \in \mathbb{R}^n \to L(p) \in \mathbb{R}$

$\to L : \mathbb{R}^n \to \mathbb{R}$

$\mathbb{R}^n = \{\text{ptos}\}$

1- formas $X = \{\text{caminhos diferenciáveis}\}$

$\cong \{\alpha : [0,1] \to \mathbb{R}^n \text{ classe } C^1\}$

$\to \alpha \overset{\omega}{\longmapsto} \mathbb{R}$

dadas $f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}$ C^1

$\to \omega := \sum_{i=1}^{n} f_i \, dx_i$ / $L\omega : X \to \mathbb{R}$

$$L\omega(\alpha) = \int_0^1 \left(\sum_{i=1}^{n} f_i(\alpha(t)) \frac{dx_i}{dt} \right) dt$$

esse tipo de funcionais são as 1-formas

+

K – formas $K \geq 2$

$F : \mathbb{R}^n \to \mathbb{R}^n$ $\quad I = (i_1, \dots, i_K)$ $\quad 1 \leq i_r, j_r \leq n$

$J = (j_1, \dots, j_K)$

$$\to \frac{\partial F_I}{\partial X_J} = \frac{\partial(F_{i_1}, \dots, F_{i_K})}{\partial(x_{j_1}, \dots, x_{j_K})} = \det \begin{bmatrix} \frac{\partial F_{i_1}}{\partial x_{j_1}} & \cdots & \frac{\partial F_{i_1}}{\partial x_{j_K}} \\ \vdots & & \\ \frac{\partial F_{i_K}}{\partial x_{j_1}} & \cdots & \frac{\partial F_{i_K}}{\partial x_{j_K}} \end{bmatrix}$$

def Uma k-cela é um mapa C^1
$f: I^k = [0,1]^k \longrightarrow \mathbb{R}^m$

- $f: I^k \longrightarrow \mathbb{R}^k$ k-cela $I = (i_1, \dots i_k)$
$I^k = [0,1]^k$

- dx_I = x_I - area de k-celas

$$dx_I : f \longmapsto \int_{I^k} \frac{\partial f_I}{\partial u} du \doteq \int_0^1 \cdots \int_0^1 \frac{\partial(f_{i_1}, \dots f_{i_k})}{\partial(u_1, \dots u_k)} du_1 \cdots du_k$$

- Mais geralmente, dada $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ C^1 podemos considerar

$$f\,dx_I : f \longmapsto \int_{I_k} f(f(u)) \frac{\partial f_I}{\partial u} du$$

def Uma k-forma básica / simple é um funcional em $\{k\text{-celas}\}$ da forma dx_I / $f dx_I$

- Uma k-forma é um funcional em

$\{k\text{-celas}\}^*$ obtido como soma finita de formas simples

$$\sim \quad w = \sum_I f_I \, dx_I \; : \; \varphi \longmapsto \sum_I \int_I f_I(\varphi(u)) \frac{\partial \varphi_I}{\partial u} \, du$$

$f_I \; \mathbb{R}^n \longrightarrow \mathbb{R}$

Not $\quad w(\varphi) = L_w(\varphi) = \int_\varphi w$

Not $\quad k\text{-cela}(\mathbb{R}^m) = \{\varphi : \mathbb{R}^m \longrightarrow \mathbb{R} \quad C^1\}$

$k\text{-cela}^*(\mathbb{R}^m) = \{L : k\text{-cela}(\mathbb{R}^n) \longrightarrow \mathbb{R}\}$

$\Omega_k(\mathbb{R}^n) = \{w \;\; k\text{-forma em } \mathbb{R}^m\}$

$\subset k\text{-cela}^*(\mathbb{R}^m)$

Obs

i) π permutação de k-símbolos

$\Rightarrow dx_{\pi I} = sg(\pi)\, dx_I$

ii) se I tem uma entrada repetida

$\Rightarrow dx_I = 0$

exemplos $n=3$

$\Rightarrow dx\,dy = -\,dy\,dx$

$dz\,dx\,dx = 0$

Teorema

$T: I^k \to I^k$ difeomorfismo

$\varphi: I^k \to \mathbb{R}^m$ k-cela

ω k-forma

$\Rightarrow \displaystyle\int_{\varphi \circ T} \omega = \pm \int_{\varphi} \omega$
$+$ se T pres orientação
$-$ se não

dem:

$w = f\,dx_I$

$$\Rightarrow \int_{\varphi \circ T} w = \int_{I^k} f(\varphi \circ T) \frac{\partial(\varphi \circ T)_I}{\partial u} du$$

$$= \int_{I^k} f(\varphi(Tu)) \cdot \left(\frac{\partial \varphi_I}{\partial v}\right)_{v = T(u)} \cdot \frac{\partial T}{\partial u} du$$

$$= \pm \int_{I^k} f(\varphi(v)) \frac{\partial \varphi_I}{\partial v} dv = \int_{\varphi} w$$

Teo de mudança de variáveis

- Como $w = \sum_I f\,dx_I$ o resultado se deduz pela linearidade da integral

$\times$

Obs Podemos extender a noção de k-cela a $\varphi : C \to \mathbb{R}^m$ onde C é difeomorfo a I^k

com isto o teorema anterior fala que $w(\varphi)$ não depende (módulo ± 1) da parametrização

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Representação

$I = (i_1, i_2, \dots i_k)$ é crescente se

$i_1 < i_2 < \dots < i_k$

- w k-forma $\Rightarrow$ pode se escrever como

$$w = \sum_A f_A \, dx_A \qquad A \text{ crescente}$$

(pelo teorema anterior)

- Fixemos A crescente e $p \in \mathbb{R}^n$

$\varphi_r: u \longmapsto p + r L(u)$ $\qquad L: \mathbb{R}^k \to$ plano X_A em $\mathbb{R}^m$

$\Rightarrow \varphi_r$ k-cela

seja I crescente

$$\rightarrow \quad \frac{\partial z_I}{\partial u} = \begin{cases} r^k & I = A \\ 0 & I \neq A \end{cases}$$

$$\therefore \ w(i) = f_A \, dx_A(i) = r^k \int_{I_k} f_A(z(u)) \, du$$

$\rightarrow$

$$f_A(p) = \lim_{r \to 0} \frac{1}{r^k} w(i_r)$$

Em particular a representação $w = \sum_A f_A \, dx_A$ é única

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* de agora em diante escreveremos as formas em sua presentação crescente

Corolário: $k > m \rightarrow \Omega^k(\mathbb{R}^m) = 0$

Produto Exterior

$\alpha = \sum_{I} f_I \, dx_I$ $\quad$ k-forma

$\beta = \sum_{J} g_J \, dx_J$ $\quad$ ℓ- "

$\Rightarrow \alpha \wedge \beta := \sum_{I,J} f_I g_J \, dx_{IJ}$

$I = (i_1, \dots, i_k)$ $\quad \rightarrow \quad IJ = (i_1, \dots, i_k, j_1, \dots j_\ell)$
$J = (j_1, \dots, j_\ell)$

ex: $dx\,dy = dx \wedge dy$ em $\mathbb{R}^3$

Propriedades

$\wedge : \Omega^k \times \Omega^\ell \longrightarrow \Omega^{k+\ell}$

a) $(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$
$\gamma \wedge (\alpha + \beta) = \gamma \wedge \alpha + \gamma \wedge \beta$

b) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

c) $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$

dem (Pugh)

$+$

lema

$$dx_I \wedge dx_J = dx_{IJ}$$

dem

seja π, ρ permutações | πI, ρJ crescentes

$$\Rightarrow dx_I \wedge dx_J = sg(\pi)\, sg(\rho)\, dx_{\pi I} \wedge dx_{\rho J}$$

$$= sg(\pi)\, sg(\rho)\, dx_{\pi I \cdot \rho J}$$

$$= dx_{IJ}$$

$+$

consequência

$$\alpha = \sum_I f_I dx_I$$ e as presentações não são crescentes

$$\beta = \sum_J g_J dx_J$$

$$\Rightarrow \alpha \wedge \beta = \sum_{I,J} f_I g_J dx_{IJ}.$$

Derivada exterior

ω k-forma $\to$ $\omega = \sum_I f_I \, dx_I$ (crescente)

$$\Rightarrow d\omega := \sum_I df_I \wedge dx_I$$

$$\hookrightarrow df_I = \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i$$

ex: $\omega = f dx + g dy$ em $\mathbb{R}^2$

$$\Rightarrow d\omega = (g_x - f_y) \, dx \, dy$$

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claramente $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ é linear

obs: se $\omega = \sum_I f_I \, dx_I$ não crescente

seja π / πI é crescente

$$\to d(f_I dx_I) = \operatorname{sg}\pi \, d(f_{\pi I} dx_{\pi I})$$
$$= \operatorname{sg}\pi \underbrace{df_{\pi I}}_{= df_I} \wedge dx_{\pi I} = df_I \wedge dx_I$$

$\therefore\ dw = \sum_I df_I \wedge dx_I$

Proposisão

1) $\alpha \in \Omega^k$, $\beta \in \Omega^p$

$\Rightarrow d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$

2) $d^2 = 0$ ie $\forall \alpha$ $d(d\alpha) = 0$

dem:

1) $\alpha = f dx_I$ $\beta = g dx_J$

$\Rightarrow d(\alpha \wedge \beta) = d(f \cdot g\, dx_{IJ})$

$= (g df + f dg) \wedge dx_{IJ}$

$= (df \wedge dx_I) \wedge (g dx_J) + (-1)^k (f dx_I) \wedge (dg \wedge dx_J)$

$= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

Pela prop distributiva queda

$\checkmark$

2)

<u>0-formas</u> : $f : \mathbb{R}^n \to \mathbb{R}$

$\to df = \sum_{i=1}^{n} f_{x_i}\, dx_i$ $\Big\}$ $d^2 f = 0$

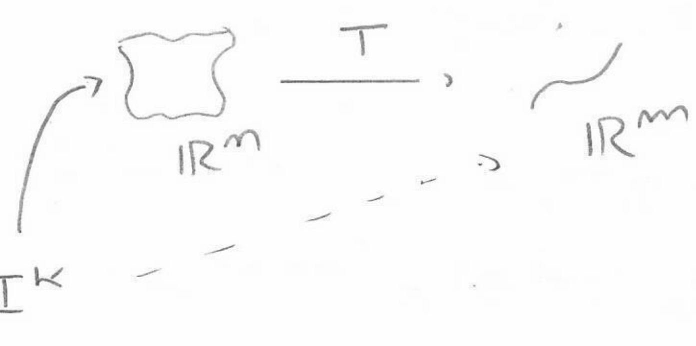
$d(dx_i) = 0$

<u>k-formas</u> : $d^2 x_I = 0$ ✓

$$d(f\, dx_I) = df \wedge dx_I \to d^2(f\, dx_I) = d(df \wedge dx_I) \overset{\text{produto}}{=} 0$$

<u>Push Forward / Pull back</u>

$T : \mathbb{R}^n \to \mathbb{R}^m$ $\quad C^1$



$$T_* : k\text{-celas}(\mathbb{R}^n) \to k\text{-celas}(\mathbb{R}^m)$$
$$\varphi \mapsto T \circ \varphi$$

$$T^* : k\text{-celas}^*(\mathbb{R}^m) \longrightarrow k\text{-celas}^*(\mathbb{R}^n)$$

$$k\text{-celas}(\mathbb{R}^m) \xrightarrow{\ \omega\ } \mathbb{R}$$

$$\begin{tikzcd}
k\text{-cela}(\mathbb{R}^n) \ni \varphi \arrow[r] & T_* \varphi \in k\text{-cela}(\mathbb{R}^m) \arrow[d, "\omega"] \\
& \mathbb{R}
\end{tikzcd}$$

$$\longrightarrow T^*\omega(\varphi) = \omega(T_* \varphi) = \omega(T \circ \varphi)$$

<u>obs</u> $\quad T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow$

$\quad S : \mathbb{R}^m \longrightarrow \mathbb{R}^p$

$$(S \circ T)^* : k\text{-celas}^*(\mathbb{R}^p) \longrightarrow k\text{-celas}^*(\mathbb{R}^n)$$

e $\quad (S\circ T)^*\omega(\varphi) = \omega((S\circ T)_* \varphi) = \omega(S \circ T \circ \varphi)$

$\quad = (S^*\omega)(T \circ \varphi) = T^*(S^*\omega)(\varphi)$

ie $\quad (S \circ T)^* = T^* \circ S^*$

Lema (Cauchy - Binet)

$k \leq m$, $A \in Mat_{k\times m}$, $B \in Mat_{m\times k}$ $\to$ $C = AB \in Mat_{k\times k}$

$$\Rightarrow \det C = \sum_{J} \det A^J \cdot \det B_J$$

$J = (j_1, \dots, j_k)$ crescente $1 \leq j_\ell \leq m$

- A^J = matrix $k\times k$ obtida escolhendo as colunas em J

  B_J = matrix $k\times k$ " " " linhas em J

$+$

Corolário

$T : \mathbb{R}^n \to \mathbb{R}^m$

$f dy_I$ forma básica em $\mathbb{R}^m$

$$\Rightarrow T^*(f dy_I) = T^*f \cdot T^* dy_I = T^*f \, dT_I$$

$$dT_I = dT_{i_1} \wedge \dots \wedge dT_{i_k}$$

Em particular $T^*(f dy_I)$ é uma k-forma dem:

$$T^*(f\,dy_I) : \varphi \longmapsto \int_{I^k} f(T\circ\varphi(u)) \frac{\partial (T\circ\varphi)_I}{\partial u}\,du$$

$$\underset{\substack{\text{cauchy}\\ \text{Binet}}}{=} \sum_J \int_{I^k} f(T\circ\varphi(u)) \left.\frac{\partial T_I}{\partial x_J}\right|_{x=\varphi(u)} \cdot \frac{\partial \varphi_J}{\partial u}\,du$$

$$\rightarrow T^*(f\,dy_I) = \sum_J T^*f \,\frac{\partial T_I}{\partial x_J}\,dx_J$$

$$\underset{\substack{\uparrow\\ \text{ex}}}{=} T^*f \cdot dT_I$$

$+$

como consequência

$$\rightarrow T^* : \Omega^k(\mathbb{R}^m) \rightarrow \Omega^k(\mathbb{R}^n)$$

Corolário

1) $T^*(\alpha \wedge \beta) = T^*\alpha \wedge T^*\beta$

2) $dT^* = T^*d$

dem:

1) consideramos $\alpha = f\,dx_I$
$\beta = g\,dx_J$

$\Rightarrow T^*(\alpha \wedge \beta) = T^*(fg\,dx_{IJ}) = T^*(f \cdot g)\,dT_{IJ}$

$= T^*f \cdot T^*g\,dT_I \wedge dT_J$

$= T^*\alpha \wedge T^*\beta$

Pela distributiva queda

2) f 0-forma $\to$ $dT^*f = d(f \circ T) = T^*df$

regra da cadeia

$\omega = f_I\,dx_I \to d(T^*\omega) = d(T^*f_I\,dT_I)$

$= d(T^*f_I) \wedge dT_I + (-1)^0 T^*f_I\,d^2T_I$

$= T^*df_I \wedge dT_I = T^*d(f_I\,dx_I)$ ✓