Seja $f: A \to \mathbb{R}$ e $g: B \to \mathbb{R}$ funções limitadas não negativas nos retângulos A e B. Defina $\varphi: A \times B \to \mathbb{R}$ pondo $\varphi(x,y) = f(x) \cdot g(y)$. Prove que:

$$\overline{\int}_{A\times B} \varphi(z)\,dz = \overline{\int}_A f(x)\,dx \cdot \overline{\int}_B g(y)\,dy$$

. E que vale um resultado similar para integrais inferiores

Resposta: Seja $(P, \overline{P})$ uma partição de $A \times B$, então para todo bloco $(B, \overline{B})$ da partição $(P, \overline{P})$, temos que:

$$0 \leq m_B(f) \leq f(x), \ \forall x \in B \quad e \quad 0 \leq m_{\overline{B}}(g) \leq g(y), \ \forall y \in \overline{B}$$

Assim, temos que:

$$0 \leq m_B(f) \cdot m_{\overline{B}}(g) \leq f(x) \cdot g(y), \ \forall (x,y) \in (B, \overline{B})$$

$$0 \leq m_B(f) \cdot m_{\overline{B}}(g) \leq \inf_{(x,y) \in (B,\overline{B})} (f(x) \cdot g(y))$$

$$0 \leq m_B(f) \cdot m_{\overline{B}}(g) \leq m_{(B,\overline{B})}(\varphi)$$

Por outro lado,

$$m_{(B,\overline{B})}(\varphi) = \inf_{(x,y) \in (B,\overline{B})} (f(x) \cdot g(y)) \leq f(x) \cdot g(y), \ \forall (x,y) \in (B, \overline{B})$$

Daí,

$$m_{(B,\overline{B})}(\varphi) \leq \inf_{x \in B} (f(x) \cdot g(y)) = g(y) \inf_{x \in B} (f(x)) = g(y) \cdot m_B(f)$$

$$m_{(B,\overline{B})}(\varphi) \leq \inf_{g \in \overline{B}} (g(y) \cdot m_B(f)) = m_B(f) \cdot \inf_{g \in \overline{B}} (g(y)) = m_B(f) \cdot m_B(\overline{g})$$

Portanto: $m_{(B,\overline{B})}(\varphi) = m_B(f) \cdot m_B(\overline{g}) \to s(\varphi, (P, \overline{P})) = \sum_{(B,\overline{B}) \in (P,\overline{P})} m_{(B,\overline{B})}(\varphi) \operatorname{Vol}(B, \overline{B}) =$

$$\sum_{(B,\overline{B}) \in (P,\overline{P})} m_{(B,\overline{B})}(\varphi) \operatorname{Vol}(B) \operatorname{Vol}(\overline{B}) = \sum_{(B,\overline{B}) \in (P,\overline{P})} m_B(f) \cdot m_B(\overline{g}) \operatorname{Vol}(B) \cdot \operatorname{Vol}(\overline{B}) =$$

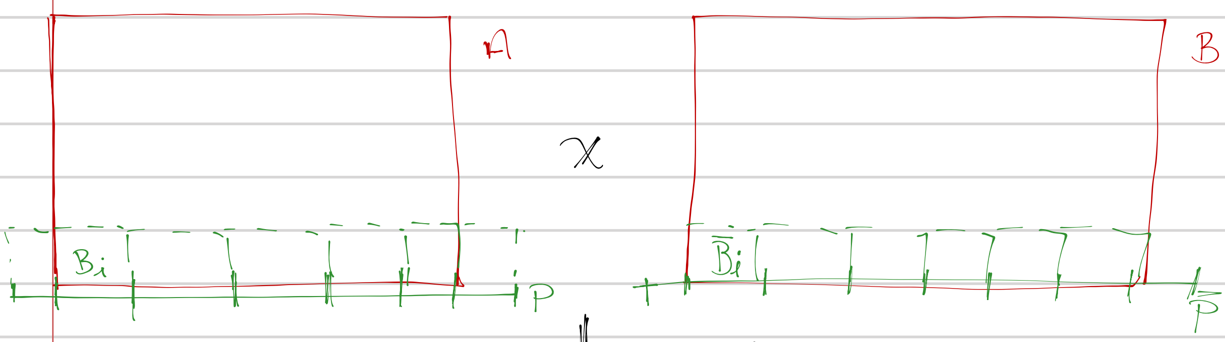
$$\sum_{B \in P, \overline{B} \in \overline{P}} (m_B(f) \cdot \operatorname{Vol}(B))(m_B(\overline{g}) \cdot \operatorname{Vol}(\overline{B})) = \sum_{B \in P} m_B(f) \cdot \operatorname{Vol}(B) \cdot \sum_{\overline{B} \in \overline{P}} m_B(\overline{g}) \cdot \operatorname{Vol}(\overline{B}) =$$

$s(f, P) \cdot s(g, \overline{P})$, e com isso temos que:

$$\underline{\int} \varphi(z)\,dz = \sup_{(P,\overline{P})} s(\varphi, (P, \overline{P})) = \sup_{(P,\overline{P})} s(f, P) \cdot s(g, \overline{P}) =$$

$$\sup_{P} s(f,P) \cdot \sup s(g,\bar{P}) = \underline{\int_A} f(x)\,dx \cdot \underline{\int_B} g(y)\,dy$$

O mesmo argumento vale para a integral superior,



$$\Downarrow \quad \varphi : (A \times B) \longrightarrow \mathbb{R}$$