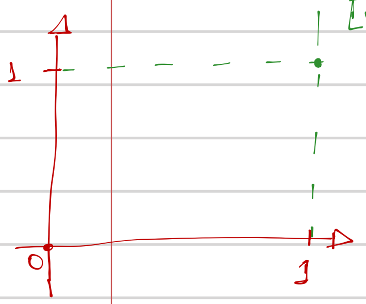
Sejam $A=[0,1]\times[0,1]$, $A_0 = A-\{(0,0)\}$ e $f: A_0 \to \mathbb{R}$ a função contínua (ilimitada) definida por:

$$f(x,y)=\frac{x-y}{(x+y)^3}.$$ Mostre que se tem:

$$\int_0^1 dx\int_0^1 f(x,y)\,dy=\frac{1}{2} \quad e \quad \int_0^1 dy\int_0^1 f(x,y)\,dx=-\frac{1}{2}$$




$[0,1]\times[0,1]=A$

$A_0=A-\{(0,0)\}$

$f: A_0\to\mathbb{R}$, contínua ilimitada

$f(x,y)=\frac{x-y}{(x+y)^3}$, não está definida em $(0,0)$.

Seja $\int_0^1 dx\int_0^1 f(x,y)\,dy = 1/2 \to \int_0^1 dx \int_0^1 \frac{(x-y)}{(x+y)^3}dy = \int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}dy\,dx$

Seja $I=\int_0^1\frac{(x-y)}{(x+y)^3}dy$, podemos reescrever o numerador então:

$$I=\int_0^1 \frac{(x-y)}{(x+y)^3}dy = \int\frac{2x-(x+y)}{(x+y)^3}dy = \int\frac{2x}{(x+y)^3}dy - \int\frac{\cancel{(x+y)}}{(x+y)^{\cancel{3}2}}dy = \int\frac{2x}{(x+y)^3}dy-\int\frac{1}{(x+y)^2}dy$$

Tomamos $u=x+y \to du=dy$ (diferenciando com respeito a $y$).

$$\boxed{\int x^n = \frac{x^{n+1}}{n+1}}$$

$$2x\int\frac{1}{u^3}du - \int\frac{1}{u^2}du = 2x\left[\frac{u^{-2}}{-2}\right]-\left[\frac{u^{-1}}{-1}\right]$$

$$\cancel{2}x\left[\frac{1}{-\cancel{2}(x+y)^2}\right]-\left[-\frac{1}{(x+y)}\right]=\left[-\frac{x}{(x+y)^2}+\frac{1}{(x+y)}\right]=\left[\frac{-x+x+y}{(x+y)^2}\right]=\frac{y}{(x+y)^2}$$

Agora tomamos a variável $x$, então:

$$\int_0^1\left[\frac{y}{(x+y)^2}\right]\Big|_0^1 dx = \int_0^1\left[\frac{1}{(x+1)^2}\right]dx$$, se $w=(x+1)\to dw=dx$, com respeito a $x$.

Os intervalos será: $w=x+1$, se $x=0\to w=1$, se $x=1\to w=2$, logo:

$$\int_1^2\frac{1}{w^2}dw=\int_1^2 w^{-2}dw \Rightarrow [-w^{-1}]\Big|_1^2 = \left[-\frac{1}{w}\right]\Big|_1^2=\left[-\frac{1}{2}-\left(-\frac{1}{1}\right)\right]=$$

$$\left[-\frac{1}{2}+1\right]=\frac{-1+2}{2}=\frac{1}{2}\to\int_0^1\int_0^1\frac{x-y}{(x+y)^3}dy\,dx=\frac{1}{2}.$$

Agora tomamos $\int_0^1 dy \int_0^1 f(x,y)\,dx = \frac{-1}{2} = \int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}\,dx\,dy$, assumimos que $I = \int \frac{(x-y)}{(x+y)^3}\,dx$, agora reescrevemos o numerador da integral como:

$$\int \frac{(x-y)}{(x+y)^3}\,dx = \int \frac{(x+y)-2y}{(x+y)^3}\,dx = \int \frac{\cancel{(x+y)}}{(x+y)^{\cancel{3}2}}\,dx - \int \frac{2y}{(x+y)^3}\,dx = \int \frac{1}{(x+y)^2}\,dx - \int \frac{2y}{(x+y)^3}\,dx$$

, se $t = x+y \rightarrow dt = dx$, assim:

$$\int \frac{1}{t^2}\,dt - 2y\int \frac{1}{t^3}\,dt = -\frac{1}{t} + \cancel{2}y\frac{1}{\cancel{2}t^2} = \frac{y}{t^2} - \frac{1}{t}$$

, mas $t = x+y$, então:

$$\frac{1}{(x+y)^2} - \frac{1}{(x+y)} = \frac{y-(x+y)}{(x+y)^2} = \left(\frac{-x}{(x+y^2)}\right)$$

, agora:

$$\int_0^1 \left[\frac{-x}{(x+y)^2}\right]\Big|_0^1 dy = \int_0^1 \frac{-1}{(1+y)^2}\,dy$$

, se $s = y+1 \rightarrow ds = dy$, os intervalos de integração:

se $y=0 \rightarrow s=1$
$y=1 \rightarrow s=2$

Então:

$$\int_1^2 -\frac{1}{s^2}\,ds = -\left[-\frac{1}{s}\right]\Big|_1^2 = \left[\frac{1}{s}\right]\Big|_1^2 = \left[\frac{1}{2} - 1\right] = -\frac{1}{2}$$

, então:

$$\int_0^1\int_0^1 \frac{(x-y)}{(x+y)^3}\,dx\,dy = -\frac{1}{2}.$$

$\square$