MATH 403: Homework Chapter 21 Proof Solutions

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1. Let E be a field extension of F and suppose α is algebraic over F. Prove that α is a root of a unique irreducible monic polynomial in $F[x]$.

   Proof: We know that α is a root of some irreducible $p(x)$ of minimal degree. We can multiply by the multiplicative inverse of the leading coefficient to get an irreducible monic polynomial.

   To show it's unique assume $p(x)$ and $q(x)$ are both irreducible monic polynomials with α as a root. We know they have the same degree and so we see that $(p-q)(x)$ is a polynomial of smaller degree for which α is a root. Since $p(x)$ and $q(x)$ have minimal degree we have $(p-q)(x) = 0$ and so $p(x) = q(x)$.

3. Find the degree and a basis for $\mathbb{Q}\left(\sqrt{3}+\sqrt{5}\right)$ over $\mathbb{Q}\left(\sqrt{15}\right)$.

   Answer: Since $\mathbb{Q}\left(\sqrt{3}+\sqrt{5}\right) = \mathbb{Q}(\sqrt{3},\sqrt{5})$ we see a basis for $\mathbb{Q}(\sqrt{3},\sqrt{5})$ over $\mathbb{Q}(\sqrt{15})$ is $\{1,\sqrt{3}\}$ and the degree is 2.

4. Let E be a field extension of F and let $\alpha \in E$. Show that $\left[F(\alpha) : F(\alpha^3)\right] \leq 3$. Find examples to illustrate that it could be 1, 2 or 3.

   Proof: Observe that α is a root of $p(x) = x^3 - \alpha^3 \in F(\alpha^3)[x]$ and so α is a root of an irreducible polynomial of degree at most 3 ($p(x)$ may or may not be irreducible but it definitely factors to irreducibles since $F[x]$ is a PID.)

   Example: $[\mathbb{Q}(1) : \mathbb{Q}(1^3)] = [\mathbb{Q} : \mathbb{Q}] = 1$ with basis $\{1\}$.

   Example: $\left[\mathbb{Q}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) : \mathbb{Q}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3\right] = \left[\mathbb{Q}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) : \mathbb{Q}(-1)\right] = \left[\mathbb{Q}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) : \mathbb{Q}\right] =$ 2 since $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a root of $x^2 - x + 1$ which is irreducible over $\mathbb{Q}$ since it has no rational roots.

   Example: $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}(2)] = [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ with basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$.

5. Find the minimal polynomial for $\sqrt[3]{2}+\sqrt[3]{4}$ over $\mathbb{Q}$. Justify why it's minimal.

   Proof: If we let $x = 2^{1/3} + 4^{1/3} = 2^{1/3} + 2^{2/3}$ then

$$
\begin{aligned}
x^3 &= \left(2^{1/3} + 2^{2/3}\right)^3 \\
x^3 &= \left(2^{2/3} + 4 + 2^{4/3}\right)\left(2^{1/3} + 2^{2/3}\right) \\
x^3 &= 2 + 4\cdot 2^{1/3} + 2^{5/3} + 2^{4/3} + 4\cdot 2^{2/3} + 4 \\
x^3 &= 6 + 4\cdot 2^{1/3} + 2\cdot 2^{2/3} + 2\cdot 2^{1/3} + 4\cdot 2^{2/3} \\
x^3 &= 6 + 6\cdot 2^{1/3} + 6\cdot 2^{2/3} \\
x^3 - 6 &= 6\left(2^{1/3} + 2^{2/3}\right) \\
x^3 - 6 &= 6x \\
x^3 - 6x - 6 &= 0
\end{aligned}
$$

   Which is irreducible by Eisenstein, hence is minimal degree.