2. **Modules.**

The concept of a module includes:

(1) any left ideal I of a ring R;

(2) any abelian group (which will become a $\mathbb{Z}$-module);

(3) an n-dimensional $\mathbb{C}$-vector space (which will become both a module over $\mathbb{C}$ and over $M_n(\mathbb{C})$).

The key point which these have in common is that one can both add elements of the module and multiply elements of the module from the left by elements of the ring. So, we generalize the idea:

Definition 2.1. *Let R be a ring. Then a* **(unital) left R-module** *M is an additive abelian group together with an operation*

$$R \times M \to M \qquad (r, m) \mapsto rm \tag{2.1}$$

satisfying

(i) $(r+s)m = rm + sm$ *and* $r(m+n) = rm + rn$ *(distributivity)*

(ii) $(rs)m = r(sm)$ *(associativity)*

(iii) $1_R\, m = m$ *(unitarity)*

for all $m, n \in M$ and $r, s \in R$.

Remark (a) In older books you may find that unitarity is not required in the definition.

(b) Similarly, one has the notion of a *right* R-module M, where one replaces (2.1) by an operation

$$M \times R \to M : (m, r) \mapsto mr \tag{2.2}$$

and adjusts (i), (ii), and (iii) accordingly.

(c) I will often write "Given ${}_RM$" for "Given a left R-module M".

Examples 2.2. (1) The set of n-dimensional column K-vectors, for a field K, is naturally a left module over $M_n(K)$ (or indeed over K itself).

(2) The set of n-dimensional row K-vectors, for a field K, is naturally a right module over $M_n(K)$.

(3) An abelian group A is a left (or right) $\mathbb{Z}$-module under the standard operation

$$n \cdot a \;=\; \underbrace{a + a + \cdots + a}_{n \text{ times}}.$$

(4) A ring R is always a left (or right) R-module and any left ideal of R is also a left R-module.

(5) Slightly more generally, given a subring S of a ring R then R is a (left or right) S-module.

(6) Let R be a ring. The **zero right R-module** is the right R-module $\{0\}$ with just one element 0 such that $0r = 0$ for all $r \in R$. The zero left R-module is defined similarly, and in both cases we just write it as 0.

(7) Over a *commutative* ring R there is no difference between left and right modules—given a left R-module M you get a right module by defining $m * r = rm$ for $m \in M$ and $r \in R$. However, over noncommutative rings, associativity is likely to fail when you do this. So they really are different.

(8) $\mathbb{C}[x]$ is a left module over the first Weyl algebra $A = A_1(\mathbb{C})$, where A acts on $\mathbb{C}[x]$ as differential operators. In this case the fact that we do get a module is just (a special case of) the assertion that, by definition, differential operators are linear operators.

We have the following basic properties of modules.

Lemma 2.3. *Given ${}_RM$, then for all $m \in M$, $r \in R$ we have*

(1) *(a) $0_R m = 0_M$, and (b) $r0_M = 0_M$, where 0_R stands for the zero element of R and 0_M stands for the zero element of M. So, from now on we will just write $0 = 0_M = 0_R$ without fear of confusion.*

(2) *$(-1) \cdot m = -m$.*

Proof. (1) $0_R m + 0_R m = (0_R + 0_R)m = 0_R m$. So, cancelling—which is allowed in an abelian group—gives $0_R m = 0_M$. The proof of (b) is similar starting with $r0_M + r0_M$.

(2) In this case

$$m + (-1)m = 1m + (-1)m = (1 + (-1))m = 0_R m = 0_M.$$

So $(-1)m$ is the additive inverse of m, i.e. $(-1)m = -m$. □

Most definitions from the theory of linear algebra or abelian groups ("subthings", "factor things", "thing" homomorphisms...) have analogues here. So, before reading the next few pages see if you can guess all the relevant definitions.

Definition 2.4. *Let R be a ring and M a left R-module. A* **submodule** *N of M is a subset of M which forms a left R-module under the operations inherited from M. Write $N \leq M$ for "N is a submodule of M"*

(I suppose more formally I should write "left submodule" here, but the second left is almost always superfluous.)

As you have seen with vector spaces and groups, we have the standard way of testing this:

Lemma 2.5. *A subset $N$of a left R-module M is a submodule $\Leftrightarrow$*

(i) $N \neq \varnothing$ (equivalently, $0_M \in N$)

(ii) $x, y \in N \implies x - y \in N$ (so N is a subgroup under addition)

(iii) $x \in N$, $r \in R \implies rx \in N$.

Proof. Use the proofs you have seen before. □

Examples 2.6. :

(1) The submodules of a vector space V over a field K are just the subspaces of V.

(2) The submodules of a $\mathbb{Z}$-module A are just the subgroups of A.

(3) As usual $\{0_M\}$ is a submodule of any module M and it will just be written 0. Similarly M is a submodule of M.

(4) For any ring R, the submodules of R_R are just the right ideals of R. Similarly the left ideals of R are just the (left) submodules of ${}_RR$.

In particular, for all $n \in \mathbb{Z}$, $n\mathbb{Z}$ is a submodule of $\mathbb{Z}_{\mathbb{Z}}$.

The module ${}_RM$ is **simple** if $M \neq 0$ and M has no submodules other than M and 0. For example, a vector space over a field is simple as a module if and only if it is 1-dimensional. An abelian group is simple (as a $\mathbb{Z}$-module) $\iff$ it is a simple abelian group.

Definition 2.7. *Given a left R-module M and elements $\{m_i \in M : i \in I\}$, we write $\sum_{i \in I} Rm_i$ for the set of elements*

$$\sum_{i\in I} Rm_i \;=\; \Big\{ m = \sum_{i\in I} r_i m_i, \text{ where the } r_i \in R \text{ and only finitely many } r_i \text{ are nonzero} \Big\}.$$

We say that M is **generated** *by $\{m_i \in M : i \in I\}$ if $M = \sum_{i \in I} Rm_i$. We say that M is* **cyclic** *if $M = Rm = \{rm : r \in R\}$ for some $m \in M$, and that M is* **finitely generated** *if $M = \sum_{i=0}^n Rm_i$ for some finite set $\{m_i\}_i$ of elements of M.*

Lemma 2.8. *Let $\{m_i : i \in I\}$ be elements of the left R-module M. Then*

(1) *The set $N = \sum_{i \in I} Rm_i$ is a submodule of M.*

(2) *$N = \sum_{i \in I} Rm_i$ is the unique smallest submodule of M containing $\{m_i : i \in I\}$.*

(3) *If M is a finitely generated module, then M has a maximal submodule (meaning a submodule maximal among the submodules $N \neq M$).*

Proof. Part (1) is left as an exercise.

(2) If L is a submodule of M containing all the m_i then it also contains all finite sums $r_1 m_{i_1} + \cdots + r_n m_{i_n}$ and hence $L \subseteq N$. Since N is a submodule we are done.

(3) This is very similar to Corollary 1.27.

First, we can write $M = \sum_{i=1}^n Rm_i$ with n as small as possible. The advantage of this is that now $M \neq N = \sum_{i=1}^{n-1} Rm_i$. Let X be the set of all proper submodules of M that contain N and order X by inclusion. It is important to notice that a submodule $M \supseteq I \supseteq N$ is not equal to M if and only if $m_n \notin I$. Now suppose that we are given a chain $Y = \{I_\lambda\}$ of elements of X. As Y is a chain we claim that $I = \bigcup I_\lambda$ is a submodule of M. This is one of the few cases where addition is the subtle point. Indeed, $x, y \in I$ then $x \in I_\lambda$ and $y \in I_\mu$ for some λ, μ. Now either $I_\lambda \subseteq I_\mu$ or $I_\mu \subseteq I_\lambda$; assume the former. Then $x \pm y \in I_\mu \subseteq I$. If $m \in I$ and $r \in R$ then $m \in I_\lambda$ for some λ whence $rm \in I_\lambda \subseteq I$.

Finally as $m_n \notin I_\nu$ for any ν it follows that $m_n \notin I = \bigcup I_\nu$. Thus $I \neq M$.

Thus X is indeed inductive and any maximal element in X - and there is at least one by Zorn's Lemma - is a maximal submodule of M □

As was true of Corollary 1.27, part (3) fails if you do not assume Zorn's Lemma and it also fails if you do not assume that the module is finitely generated. The standard counterexample is $\mathbb{Q}$ as a $\mathbb{Z}$-module. Can you prove this? An easier example is

Exercise 2.9. Let $R = \{\frac{a}{b} \in \mathbb{Q} : b \text{ is odd}\}$.

(1) Prove that R is a ring.

(2) Prove that (apart from 0 and $\mathbb{Q}$) the R-submodules of $\mathbb{Q}$ are just the $\{R2^m = 2^m R : m \in \mathbb{Z}\}$; thus they form a chain:

$$0 \subset \cdots \subset R2^n \subset R2^{n-1} \cdots \subset R2 \subset R \subset \frac{1}{2}R \subset \cdots \subset \frac{1}{2^n}R \subset \cdots \subset \mathbb{Q} \qquad \text{for } n \in \mathbb{N}.$$

(3) Hence $\mathbb{Q}$ has no proper maximal R-submodule

The details of Exercise 2.9 are given in the Second Example Sheet. Instead, here, I will give the details of a variant:

Exercise *Let $\mathbb{Z}[2^{-1}] = \{a2^{-n} : n \geq 1, a \in \mathbb{Z}\} \subset \mathbb{Q}$. Then the $\mathbb{Z}$-submodules of $M = \mathbb{Z}[2^{-1}]/\mathbb{Z}$ are M itself and $[2^{-n}]\mathbb{Z}$, where I have used the short-hand $[x] = [x + \mathbb{Z}]$ for $x \in \mathbb{Z}[2^{-1}]$.*

In particular, N has no maximal submodule.

Proof. Suppose that $0 \neq [q] \in M$ and write $q = 2^{-n}b$ where $n \geq 1$ and b is odd. By Euclid's Algorithm, write $1 = 2^n x + by$ for some integers x and y. Then

$$[2^{-n}] = [x + 2^{-n}by] = [2^{-n}b]y = [q]y \in [q]\mathbb{Z}.$$

In particular, $[2^{-n}]\mathbb{Z} = [q]\mathbb{Z}$.

Now, suppose that N is some $\mathbb{Z}$-submodule of M. Then N is generated by all the elements q in N and so, by the last paragraph, is generated by a set of the form $\{2^{-m_i} : i \in I\}$ for some index set I. There are now two cases: It could be that the m_i are bounded above, in which case they are bounded above by some m_j and then $N = \mathbb{Z}2^{-m_j}$. Or, the m_j have no upper bound. But in this, case for any $n \geq 0$, there exists some $m_i > n$ and hence $2^{-n}\mathbb{Q} \subseteq 2^{m_i}\mathbb{Q} \subseteq N$. Thus $N = M$.

It follows from the last paragraph that the $\mathbb{Z}$-submodules of M form a chain:

$$0 \subset \cdots \subset 2^{-n}\mathbb{Z} \subset 2^{-n-1}\mathbb{Z} \cdots \cdots \subset M.$$

So, there is certainly no maximal R-submodule of $\mathbb{Q}$. □

Note that the definition and properties of modules depend upon the ring concerned—for example $\mathbb{Q}$ is cyclic as a $\mathbb{Q}$-module, but is not even finitely generated as a $\mathbb{Z}$-module. This follows from the last example, but a more direct proof is the following: Suppose that $\mathbb{Q}$ is finitely generated as a $\mathbb{Z}$-module, say by $x_1, ...x_n$.

Write the $x_i = a_i/b$ over a common denominator b (thus, for integers a_i, b). Then it is easy to see that $\mathbb{Q} = \sum \mathbb{Z}x_i \subseteq \mathbb{Z}.\frac{1}{b}$. But this is crazy since $\frac{1}{2b}$ is not contained in $\mathbb{Z}\frac{1}{b}$.

Definition 2.10. *Let R be a ring, and let M and N be left R-modules. An* R**-module homomorphism** *(or* R**-homomorphism***) from M to N is a map $\theta : M \to N$ which satisfies*

(i) $\theta(x+y) = \theta(x) + \theta(y)$ (thus θ is a homomorphism of abelian groups),

(ii) $r\theta(x) = \theta(rx)$

for all $x, y \in M$, $r \in R$.

We say that θ is a monomorphism/epimorphism/isomorphism when θ is (1-1), onto and bijective, respectively. If $M = N$ we say that θ is an endomorphism of M and write $\mathrm{End}_R(M)$ *for the set of all such endomorphisms. Finally an automorphism is a bijective endomorphism.*

Examples: (1) Let K be a field. Then K-module homomorphisms are just linear mappings between vector spaces.

(2) $\mathbb{Z}$-module homomorphisms are just homomorphisms of abelian groups. (Check this.)

(3) Given a homomorphism $\theta : M \to N$ then $\theta(0_M) = 0_N$, and $\theta(-x) = -\theta(x)$, for all $x \in M$.

The **kernel** $\ker\theta$ of an R-module homomorphism $\theta : M \to N$ is the subset of M defined by

$$\ker\theta = \{x \in M : \theta(x) = 0_N\}.$$

Lemma 2.11. *Given a homomorphism $\theta : M \to N$ of left (or right) R-modules then* $\ker(\theta)$ *is a submodule of M and $\theta(M) = Im(\theta)$ is a submodule of S*

Proof. This is an easy exercise, but for once let me give all the details.

First, $\ker\theta \neq \varnothing$ since $0_M \in \ker\theta$. Suppose that $x, y \in \ker\theta$. Then

$$\theta(x-y) = \theta(x) - \theta(y) = 0_N - 0_N = 0_N.$$

So $x - y \in \ker\theta$.

Now suppose that $x \in \ker\theta$, and $r \in R$. Then

$$\begin{aligned}\theta(rx) &= r\theta(x) \qquad \text{[by homomorphism condition (ii)]}\\ &= r0_N \qquad [\text{as } x \in \ker\theta]\\ &= 0_N.\end{aligned}$$

So $rx \in \ker\theta$. Hence $\ker\theta$ is a submodule of M.

Now $\operatorname{im}\theta \neq \varnothing$ since $0_N = \theta(0_M) \in \operatorname{im}\theta$. Suppose that $u, v \in \operatorname{im}\theta$. Then $u = \theta(x)$, $v = \theta(y)$ for some $x, y \in M$. Then

$$u - v = \theta(x) - \theta(y) = \theta(x-y) \in \operatorname{im}\theta.$$

Suppose further that $r \in R$. Then

$$ru = r\theta(x) = \theta(rx) \in \operatorname{im}\theta.$$

Hence $\operatorname{im}\theta$ is a submodule of N. $\square$

Factor Modules. Recall that, if I is a left ideal of a ring R that is not a two-sided ideal, then one cannot make the factor abelian group R/I into a factor ring. (See the comments after Lemma 1.19.) However, we *can* make it into a factor module, as we next describe.

For completeness, let us recall the construction of the factor group M/N when $N \subseteq M$ are abelian groups. (As M is an abelian group, N is a normal subgroup of M.) The **cosets** of N in M are the subsets $x+N$ of M with $x \in M$, where $x+N = \{x+u : u \in N\}$. Two cosets $x+N$ and $x'+N$ of N in M are either identical or disjoint, i.e. $(x+N)\cap(x'+N) = \varnothing$. Furthermore, $x+N = x'+N \iff x-x' \in N$. Every element $y \in M$ belongs to some coset of N in M. In particular, $y \in y+N$ since $y = y+0_N$. The set of cosets of N in M, which is denoted by M/N, forms a partition of M. We define $(x+N)+(y+N) = (x+y)+N$ for $x, y \in M$. This consistently defines an operation of addition on M/N, which makes M/N into an additive abelian group.

Now suppose that N is a left R-submodule of a module M. Assume that

$$x+N = x'+N$$

for some $x, x' \in M$. Then $x-x' \in N$. Let $r \in R$. Then $rx - rx' = r(x-x') \in N$ since N is a submodule of M. Hence

$$rx + N = rx' + N.$$

This means we can consistently define an operation $R \times M/N \to M/N$ by putting

$$r(x+N) = rx+N \tag{2.3}$$

for all $x \in M$, $r \in R$. We have:

Theorem 2.12. *Let $N \subset M$ be left R-modules. The rule* (2.3) *turns M/N into a left R-module called the* **factor module** *of M by N.*

Proof. As we started by defining M/N as the factor of abelian groups, certainly M/N is an abelian group, and we have explained why we have a consistent multiplication map. To check module condition (i) from Definition 2.1 for M/N, let $x \in M$, $r, s \in R$. Then

$$\begin{aligned} (r+s)(x+N) &= (r+s)x + N && \text{by (2.3)} \\ &= (rx+sx)+N && \text{as } M \text{ is an } R\text{-module} \\ &= (rx+N)+(sx+N) && \\ &= r(x+N)+s(x+N) && \text{by (2.3).} \end{aligned}$$

You should check that module conditions (ii), and (iii) also hold. $\square$

As usual, all the results we have proved above for left R-modules also have analogues for right modules. It is a good way to check that you understand these concepts by writing out the analogous results on the right!

THE HOMOMORPHISM THEOREMS FOR MODULES:

You will have seen homomorphism theorems for factor groups and for factor rings. As we see next, almost exactly the same theorems apply for factor modules.

To begin, note that for any R-module homomorphism θ, because it is a homomorphism of abelian groups,

$$\theta \text{ is a monomorphism if and only if } \ker\theta = \{0\}.$$

Let M and N be left (or right) R-modules. If there is an R-module isomorphism $\theta : M \to N$ then M and N are said to be **isomorphic**. We indicate that this is the case by writing $M \cong N$. The inverse mapping $\theta^{-1} : N \to M$ is also an R-module homomorphism. To see this, let $u, v \in N$, $r \in R$, and just for a change, we will prove it for *right* modules. Thus $u = \theta(x)$, $v = \theta(y)$ for some $x, y \in M$. So

$$\begin{aligned} \theta^{-1}(u+v) = \theta^{-1}(\theta(x)+\theta(y)) \; &= \; \theta^{-1}(\theta(x+y)) \\ = x + y \; &= \; \theta^{-1}(u) + \theta^{-1}(v) \end{aligned}$$

and

$$\begin{aligned} \theta^{-1}(ur) = \; \theta^{-1}(\theta(x)r) \; &= \; \theta^{-1}(\theta(xr)) \\ = xr \; &= \; \theta^{-1}(u)r. \end{aligned}$$

Hence, being bijective, θ^{-1} is an isomorphism.

If $\psi : L \to M$ and $\theta : M \to N$ are R-module isomorphisms then so is $\theta \circ \psi : L \to N$. Hence $\cong$ defines an equivalence relation on the collection of all right R-modules. We often use the notation $\theta\psi$ for the composition $\theta \circ \psi$.

Theorem 2.13. (The First Isomorphism Theorem for Modules) *Let R be a ring, M and N right R-modules and $\theta : M \to N$ an R-module homomorphism. Then*

$$M/\ker\theta \cong \operatorname{im}\theta.$$

Proof. Suppose that $x, x' \in M$ and that $x + \ker\theta = x' + \ker\theta$. Then $x - x' \in \ker\theta$. So

$$\theta(x) - \theta(x') = \theta(x - x') = 0_N, \qquad \text{i.e. } \theta(x) = \theta(x').$$

Therefore, we may consistently define a mapping

$$\overline{\theta} : M/\ker\theta \to \operatorname{im}\theta$$

by

$$\overline{\theta}(x + \ker\theta) = \theta(x) \qquad \text{for } x \in M.$$

It is easy to check that $\overline{\theta}$ is an R-module homomorphism. Indeed, from the First isomorphism Theorem for Groups, we know that it is a well-defined group homomorphism, so we need only check multiplication. Thus, suppose that $x + \ker\theta \in M/\ker(\theta)$ and $r \in R$. Then

$$r\overline{\theta}(x + \ker\theta) = r\theta(x) = \theta(rx) = \overline{\theta}(rx + \ker\theta) = \overline{\theta}\left(r(x + \ker\theta)\right),$$

as required.

Let $x \in M$ such that $\overline{\theta}(x + \ker\theta) = 0_N$. By the definition of $\overline{\theta}$, $\theta(x) = 0_N$, i.e. $x \in \ker\theta$. Therefore $x + \ker\theta = \ker\theta = 0_{M/\ker\theta}$. Hence $\ker\overline{\theta} = \{0_{M/\ker\theta}\}$, i.e. $\overline{\theta}$ is a monomorphism.

Now let $u \in \operatorname{im}\theta$. Then $u = \theta(x)$ for some $x \in M$ and so

$$u = \overline{\theta}(x + \ker\theta).$$

Therefore $\overline{\theta}$ is also surjective and hence an isomorphism. $\square$

Theorem 2.14. (The Correspondence Theorem for Modules) *Let M be a left module over a ring R and let N be a submodule of M.*

(i) Every submodule of M/N has the form K/N, where K is some submodule of M with $N \subseteq K$.

(ii) There is a 1–1 correspondence

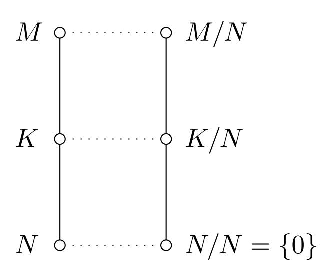
$$K \mapsto K/N$$

between the submodules K of M which contain N and the submodules of M/N. This correspondence preserves inclusions.

(iii) If $M \to N$ is an isomorphism of left R-modules then there is a (1-1) correspondence between submodules of M and N.

Proof. If K is a submodule of M with $N \subseteq K$ then, clearly, N is a submodule of K and K/N is a submodule of M/N since

$$K/N = \{x + N : x \in K\} \subseteq \{x + N : x \in M\} = M/N.$$



[Recall $0_{M/N} = N$]

(i) Let T be any submodule of M/N. Let

$$K = \{x \in M : x + N \in T\}.$$

It is easy to check that K is a submodule of M. For $u \in N$,

$$u + N = N = 0_{M/N} \in T$$

and so $u \in K$. Hence $N \subseteq K$. Furthermore $T = K/N$.

(ii) In part (i), we saw that the mapping from the set of submodules K of M such that $N \subseteq K$ to the submodules of M/N defined by $K \mapsto K/N$ is surjective.

Now suppose that J, K are submodules of M that contain N and $J/N = K/N$. Let $j \in J$. Then $j + N = k + N$ for some $k \in K$. But then $j \in j + N = k + N \subseteq K$. So $J \subseteq K$. Similarly $K \subseteq J$. Hence $J = K$. This shows the mapping is injective. It clearly preserves inclusion.

(iii) Let $\theta : M \to N$ be the isomorphism and recall that $\theta^{-1} : N \to M$ is also an isomorphism from the chat before Theorem 2.13. Now, given a submodule $K \subset M$ then certainly $\theta(M)$ is a submodule of N with $\theta^{-1}(\theta(K)) = K$. Hence the mapping $K \mapsto \theta(K)$ gives a (1-1) correspondence. $\square$

Theorem 2.15. (1) **(The Second Isomorphism Theorem for Modules)** *If A and B are submodules of a left R-module M then $A + B$ and $A \cap B$ are submodules of M and*

$$\frac{A+B}{B} \cong \frac{A}{A \cap B}.$$

(2) **(The Third Isomorphism Theorem for Modules)** *Let N, K, M be right (or left) modules over a ring R such that $N \subseteq K \subseteq M$. Then*

$$\frac{M/N}{K/N} \cong M/K.$$

Proof. (1) Let us first check that $A \cap B$ is a submodule. As usual, it is a sub-abelian group, so it only remains to check that $rx \in A \cap B$ for all $r \in R$ and $x \in A \cap B$. But as $x \in A$ certainly $rx \in A$ and as $x \in B$ similarly $rx \in B$. Thus $rx \in A \cap B$, as required. The proof for $A + B$ is similar. We now define

$$\psi : A \to \frac{A+B}{B} \quad \text{by} \quad \psi(x) = [x + B] \text{ for } x \in A.$$

This is a surjective homomorphism of abelian groups and trivially $\psi(rx) = [rx + B] = r[x + B] = r\psi(x)$, for any $r \in R$. Hence it is also an R-homomorphism. As abelian groups it has kernel $A \cap B$ and so by Theorem 2.13 we get the desired isomorphism

$$\frac{A}{A \cap B} = \frac{A}{\ker(\psi)} \cong Im(\psi) = \frac{A+B}{B}.$$

By chasing the maps you see that this isomorphism is also given by $[a + A \cap B] \mapsto [a + B]$ fior all $a \in A$.

(2) We can define a mapping $\theta : M/N \to M/K$ by putting $\theta(x + N) = x + K \quad (x \in M)$. By the corresponding result for abelian groups, this is an isomorphism of abelian groups. Thus, to prove the stated result, all we need to check is that it is also a homomorphism of R-modules. This is (almost) obvious. $\square$

As usual, everything we have stated for left modules has an analogue on the right.

Remark: One feature of this proof works for many results about modules, especially for factor modules: What we have really done is to observe that the result *does* hold for the factor abelian group. So then really all that is left to do is to check that the given map of groups also preserves the natural R-module structures. This is a valid approach in proving many such results.

Generalizing the observation at the beginning of the Second Isomorphism Theorem, we note that one can consider arbitrary sums and intersections of submodules. To be precise, suppose that N_λ $(\lambda \in \Lambda)$ is a collection of submodules of a right module M over a ring R. (The nonempty index set Λ may be finite or infinite.) Then

$$\bigcap_{\lambda \in \Lambda} N_\lambda$$

(the intersection of all the submodules) is also a submodule of M (see Example Sheet 2).

The **sum** $\sum_{\lambda \in \Lambda} N_\lambda$ of the submodules is defined by

$$\sum_{\lambda \in \Lambda} N_\lambda = \{x_1 + x_2 + \cdots + x_m : m \in \mathbb{N},\ x_i \in N_{\lambda_i},\ \lambda_i \in \Lambda\ (i = 1, 2, \ldots, m)\}.$$

This is also a submodule of M (see Example Sheet 2). It is the smallest submodule of M that contains all N_λ $(\lambda \in \Lambda)$.

In particular, if N_1, N_2, ..., N_n are submodules of M, then $\bigcap_{i=1}^{n} N_i$ and $N_1 + N_2 + \cdots + N_n$ are submodules of M, where

$$N_1 + N_2 + \cdots + N_n = \{x_1 + x_2 + \cdots + x_n : x_i \in N_i\ (i = 1, 2, \ldots, n)\}.$$

Let us apply these results to answer the question:

What are the simple modules over a commutative ring R?

Some of the lemmas will also be used for other results.

Lemma 2.16. (1) *If M is a finitely generated left R-module, then M has a simple factor module.*

(2) *If M is a simple left module over any ring R then $M = Rm$ for any $0 \neq m \in M$.*

Proof. (i) By Lemma 2.8(3), M has a maximal submodule, say N. We claim that M/N is a simple module. Indeed, if it has a nonzero submodule $\overline{K}$, then by the Correspondence Theorem, we can write $\overline{K} = K/N$ for some module $N \subseteq K \subseteq M$. The maximality of N then ensures that either $N = K$ or $K = M$. Equivalently, either $\overline{K} = N/N = 0$ or $K = M/N$, as desired.

(ii) Rm is a submodule by Lemma 2.8 and is nonzero as it contains $m = 1 \cdot m$. By simplicity of M, Rm must therefore equal M. □

Lemma 2.17. *If M is a left module over any ring R and $a \in M$ then:*

(1) *There is an R-module homomorphism $\theta : R \to Ra$ given by $\theta(r) = ra$ for $r \in R$.*

(2) *Moreover, $Ra \cong R/I$ where $I = \{r \in R : ra = 0\}$.*

Proof. (1) is routine. For (2) just note that, by the first isomorphism theorem for modules, $Ra \cong R/\ker(\theta)$ and $\ker(\theta) = I$ by definition. □

Corollary 2.18. *If R is a (nontrivial) commutative ring, then simple R-modules are just the factors R/P where P runs through the maximal ideals of R. Moreover, $R/P \not\cong R/Q$ for distinct maximal ideals P, Q.*

Proof. As we remarked in Examples 2.2 it does not matter if we work with right or left R-modules in this case, but let's work with left modules for concreteness.

If N is a simple (left) R-module, then: $N = Ra$ for some $a \in N$, by Lemma 2.16 and then $N \cong R/I$ for some ideal I by Lemma 2.17 (as R is commutative, ideals, left ideals and submodules of R are all the same thing). Note that, by definition, $M \neq 0$ and so $I \neq R$. If I is *not* maximal, say $I \subsetneq J \subsetneq R$, then the Correspondence Theorem 2.14 says that J/I is a proper submodule of R/I, in the sense that J/I is a submodule of R/I that is neither zero nor R/I. Hence by Theorem 2.15(iii) $N \cong R/I$ also has a proper submodule.

In order to prove the last part we digress a little. Suppose that M is a left R-module over a possibly noncommutative ring R. Given a subset S of M we write

$$ann_R S = \{r \in R : rm = 0 \text{ for all } m \in S\}$$

for the **annihilator** of S.

Suppose that M is cyclic; say $M = Ra \cong R/I$ where $I = \{r \in R : ra = 0\}$ as in Lemma 2.17. There are several observations to make here. First, the definition of I just says that $I = ann_R(a)$. Secondly, if there is an isomorphism $\theta : M \to N$ then for any $r \in ann_R M$ and $n = \theta(m) \in N$ we have $rn = r\theta(m) = \theta(rm) = \theta(0) = 0$ and so $ann(M) \subseteq ann(N)$. Applying the same argument to $\theta^{-1} : N \to M$ we obtain

(2.4) If M and N are isomorphic modules over (any) ring R then $ann_R M = ann_R N$.

Now return to the special case of a cyclic module $M = Rm$ over a commutative ring R. Then we claim that in fact $ann_R M = ann_R(m)$. The inclusion $\subseteq$ is obvious, so suppose that $r \in ann_R(a)$ and that $m \in M$. Then $m = sa$ for some $s \in R$ and so

$$rm = rsa = sra = s \cdot 0 = 0,$$

as claimed.

Note that the final assertion of the corollary is a special case of these observations: We are given a module $M \cong R/P \cong R/Q$. From (2.4) we see that $ann_R M = ann_R(R/P) = ann_R(R/Q)$. But from the last paragraph we see that $ann_R(R/P) = P$. Thus $P = Q$, as required. $\square$

We proved rather more than was necessary in the last part of the proof of the corollary, but it does show that the concept of annihilators is useful; indeed the concept will return several times in this course.

Exercise 2.19. (i) Show that, if R is a noncommutative ring then the simple left R-modules are the same as the modules R/I where I runs through the maximal left ideals of R. However, the left ideal I will *not* be unique (see Example 2.21 below).

(ii) If M is a left module over a ring R and $m \in R$ show that $ann_R M$ is a *two-sided* ideal of R and that $ann(m)$ is a left ideal of R.

(iii) Show that $ann_R m$ is usually not an ideal of R when M is a module over a non-commutative ring R and $m \in M$. (Consider, for example, one column of $M_2(\mathbb{C})$ as a left $M_2(\mathbb{C})$-module.)

In the noncommutative case, it is very difficult to say much more in general about simple R-modules—except that they are complicated. Let us illustrate this with the Weyl algebra $A_1 = A_1(\mathbb{C})$.

Example 2.20. (A simple module over the Weyl Algebra)

(1) $\mathbb{C}[x]$ *is a left* A_1*-module where* $\alpha = \sum f_i \partial^i \in A_1$ *acts on* $g(x) \in \mathbb{C}[x]$ *as a differential operator*

$$\alpha \cdot g(x) \;=\; \alpha * g(x) \;=\; \sum f_i(x) \frac{d^i g}{dx^i}.$$

The proof of this is almost obvious—A_1 was defined as the set of all differential operators and the fact that they act linearly on functions is really the same as saying that those functions form an A_1-module.

Of course, this same argument means that other spaces of differentiable functions, like $\mathbb{C}(x)$, are left A_1-modules.

(2) $\mathbb{C}[x] = A_1 \cdot 1$ *simply because* $g(x) = g(x) * 1$ *for all* $g(x) \in \mathbb{C}[x]$.

(3) $\mathbb{C}[x] \cong A_1/A_1\partial$.

To see this, take the map $\chi : A_1 \to M = \mathbb{C}[x]$ given by $\chi(\alpha) = \alpha * 1$. Then Lemma 2.17 shows that $\mathbb{C}[x] \cong A_1/\ker(\chi)$. Now, clearly $A_1\partial \in \ker(\chi)$, so what about the other inclusion? The fact that A_1 has $\mathbb{C}$-basis $\{x^i\partial^j\}$ (see Corollary 1.13) means that any element $\alpha \in A_1$ can be written as $\alpha = \beta\partial + h(x)$, where $\beta \in A_1$ but $h(x) \in \mathbb{C}[x]$. Now

$$\alpha * 1 = \beta * (\partial * 1) + h(x) * 1 = 0 + h(x).$$

Thus, $\alpha \in \ker(\chi) \iff h(x) = 0$, as required.

(4) $\mathbb{C}[x]$ *is a* **simple** A_1*-module.*

To see this, suppose that $f(x), g(x)$ are nonzero polynomials in $\mathbb{C}[x]$ with $\deg f = d$, say $f = \lambda x^d + \cdots$ where $\lambda \neq 0$. Then

$$\frac{1}{\lambda d!} g(x)\partial^d * f = \frac{1}{\lambda d!} g(x) \frac{d^d f}{dx^d} = g(x).$$

So certainly $\mathbb{C}[x]$ is simple as a left A_1-module.

Exercise. We should note that there are lots of other A_1-modules. For example prove that, as A_1-modules,

$$\mathbb{C}(x) \supset \mathbb{C}[x, x^{-1}] = A_1 \cdot x^{-1}$$

and that

$$A_1 x^{-1}/\mathbb{C}[x] \;\cong\; A_1/A_1 x$$

is also a simple left A_1-module. (Why is this? It will be explained in more detail in an exercise set.)

Example 2.21. *One complicating feature of noncommutative rings is that lots of different-looking modules can be isomorphic. To see this we again take $A_1 = A_1(\mathbb{C})$ and the simple module $N = \mathbb{C}[x] \cong A_1/A_1\partial$. As $\mathbb{C}[x]$ is a simple A_1-module we can, by Lemmas 2.16, and 2.17 write*

$$N \;=\; A_1x \;\cong\; A_1/ann_{A_1}(x).$$

So, what is $ann_{A_1}(x)$? In fact

$$ann_{A_1}(x) = A_1\partial^2 + A_1(x\partial - 1)$$

and hence

$$A_1/A_1\partial \;\cong\; N \;\cong\; A_1/(A_1\partial^2 + A_1(x\partial - 1)).$$

Proof. First of all it is easy to check that $\partial^2 * x = 0 = (x\partial - 1) * x$ and hence that $ann_{A_1}(x) \supseteq A_1\partial^2 + A_1(x\partial - 1)$.

In order to prove the other direction, recall from Corollary 1.13 that any element $\alpha \in A_1$ can be written as $\alpha = \sum_{i=0}^{n} f_i(x)\partial^i$ and hence as $\alpha = \beta\partial^2 + f(x)\partial + g(x)$, for $\beta \in A_1$ but $f, g \in \mathbb{C}[x]$. Now suppose that $\alpha \in ann_{A_1}(x)$ and write $\alpha = \beta\partial^2 + f(x)\partial + g(x)$ as above. Rearranging slightly we see that $\alpha = \beta\partial^2 + f_1(x)(x\partial - 1) + \lambda\partial + g_1(x)$ for some $\lambda \in \mathbb{C}$ and $f_1(x), g_1(x) \in \mathbb{C}[x]$.

But $\alpha * x = 0$ and hence $(\lambda\partial + g_1(x)) * x = 0$. This in turn forces

$$0 = (\lambda\partial + g_1(x)) * x = \lambda + g_1(x)x \qquad \textit{and so} \qquad \lambda = 0 = g_1(x).$$

Thus, $\alpha = \beta\partial^2 + f_1(x)(x\partial - 1) \in A_1\partial^2 + A_1(x\partial - 1)$. $\square$

DIRECT SUMS: Just as for groups and rings we have direct sums of modules. We begin by reminding you of the definitions in the first two cases.

Abelian groups: Let A_1 and A_2 be additive abelian groups (with zeros 0_1 and 0_2, respectively). The **(external) direct sum** of A_1 and A_2 denoted by $A_1 \oplus A_2$, is the set of ordered pairs

$$\{(a_1, a_2) : a_1 \in A_1,\; a_2 \in A_2\}$$

made into an additive abelian group by defining

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

for all $a_1, b_1 \in A_1,\; a_2, b_2 \in A_2$. It is straightforward to check that the conditions for an additive abelian group hold. The zero is $(0_1, 0_2)$ and $-(a_1, a_2) = (-a_1, -a_2)$.

Rings: Let R_1 and R_2 be rings (with identities 1_1 and 1_2, respectively). The additive abelian group $R_1 \oplus R_2$ can be made into a ring, also denoted $R_1 \times R_2$, by defining

$$(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$$

for all $a_1, b_1 \in R_1$, $a_2, b_2 \in R_2$. The identity is $(1_1, 1_2)$. Note that $R_1 \oplus R_2$ and $R_1 \times R_2$ are different notations for the same set. The second is the standard notation for the product of two rings but, in this course, it is convenient to use the first notation (which is more typical when we forget the multiplication and consider just the underlying abelian group).

Modules: Let R be a ring and M_1, M_2 left R-modules. The additive abelian group $M_1 \oplus M_2$ can be made into a left R-module by defining

$$r(x_1, x_2) = (rx_1, rx_2)$$

for all $x_1 \in M_1$, $x_2 \in M_2$, $r \in R$.

These constructions can be generalised to more than 2 summands.

Definition 2.22. *Given additive abelian groups A_1, A_2, $\ldots$, A_t, the* **(external) direct sum** $A_1 \oplus A_2 \oplus \cdots \oplus A_t$ *is the set*

$$\{(a_1, a_2, \ldots, a_t) : a_i \in A_i \ (i = 1, 2, \ldots, t)\}$$

made into an additive abelian group in the obvious way.

If the A_i happen to be (left) modules over a ring R then $A_1 \oplus A_2 \oplus \cdots \oplus A_t$ becomes a left R-module under the natural map

$$r(a_1, a_2, \ldots, a_t) = (ra_1, ra_2, \ldots, ra_t), \qquad \textit{for } a_i \in A_i \textit{ and } r \in R.$$

Remark: As remarked already, if R_1 and R_2 are rings, then either term direct product or direct sum can be used for the ring defined above. Both are fine, being equivalent for two, or any finitely many, rings R_i. (If there are infinitely many rings R_i to be combined then the direct product is the one which gives a ring with 1, whereas direct sum would give a non-unital ring.)

Exercise 2.23. (i) Check that $A_1 \oplus A_2 \oplus \cdots \oplus A_t$ really is a module in the last definition.

(ii) Suppose that M_i are left modules over a ring R. Then

$$M_1 \oplus M_2 \oplus M_3 \cong (M_1 \oplus M_2) \oplus M_3 \cong M_1 \oplus (M_2 \oplus M_3)$$

under the mappings

$$(x_1, x_2, x_3) \mapsto ((x_1, x_2), x_3) \mapsto (x_1, (x_2, x_3)).$$

Similarly, $M_1 \oplus M_2 \cong M_2 \oplus M_1$ since $(x_1, x_2) \mapsto (x_2, x_1)$ defines an isomorphism.

Internal Direct Sums: Suppose that

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_t,$$

where M_1, M_2, $\ldots$, M_t are modules over a ring R. For each $i = 1, 2, \ldots, t$, let

$$N_i = \{(0, 0, \ldots, \underset{\nwarrow\, i\text{th entry}}{x_i}, \ldots, 0) : x_i \in M_i\}. \tag{2.5}$$

Then N_i is a submodule of M and $M = N_1 + N_2 + \cdots + N_t$. Furthermore $M_i \cong N_i$ (as R-modules). The isomorphism is given by

$$x_i \mapsto (0, 0, \ldots, x_i, \ldots, 0).$$

Every element $x \in M$ can be expressed *uniquely* in the form

$$x = \widehat{x_1} + \widehat{x_2} + \cdots + \widehat{x}_t,$$

where $\widehat{x_i} \in N_i$ $(i = 1, 2, \ldots, t)$. (In fact, $\widehat{x_i} = (0, 0, \ldots, x_i, \ldots, 0)$.)

We can now turn these observations around and make:

Definition 2.24. *Given any submodules N_1, N_2, ..., N_t of an R-module M, we say that M is the* **(internal) direct sum** *of N_1, N_2, ..., N_t if*

(i) $M = N_1 + N_2 + \cdots + N_t$,

(ii) every element x of M can be expressed uniquely in the form

$$x = x_1 + x_2 + \cdots + x_t, \tag{2.6}$$

where $x_i \in N_i$ $(i = 1, 2, \ldots, t)$.

We immediately get

Lemma 2.25. *(i) Let N_1, N_2, ..., N_t be submodules of an R-module M. If M is the internal direct sum of the N_i then $N_1 \oplus \cdots \oplus N_t \cong M$ under the map $\phi : (n_1, \ldots, n_t) \mapsto n_1 + \cdots n_t$.*

(ii) Conversely if $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t$, is an external direct sum of modules M_i then M is the internal direct sum of the submodules N_i defined by (2.5)

Proof. (i) The definitions of internal and external direct sums ensure that ϕ is an isomorphism of sets and it is then routine to check that it is a module homomorphism.

(ii) See the discussion before Definition 2.24. □

If N_1, N_2, ..., N_t are submodules of an R-module M then we can form their external direct sum (which has underlying set the cartesian product $N_1 \times \cdots \times N_t$ of the sets $N_1, \ldots, N_t$) and their "internal" sum $N_1 + \cdots + N_t$ (which is a subset of M and usually not "direct"). In the special case that M *is* the internal *direct* sum of $N_1, \ldots, N_t$ then, as we have just seen, these modules (M and the module based on the cartesian product) are isomorphic, so we usually omit the words "internal" and "external" and depend on the context to make it clear which we mean (if it matters). We also use the notation

$$M = N_1 \oplus N_2 \oplus \cdots \oplus N_t$$

for the internal direct sum (that is, to emphasise that an internal sum is direct). The condition for an internal sum to be direct is given in the next remark.

Remark 2.26. Since
$$x_1 + x_2 + \cdots + x_t = 0 \iff x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{t} x_j,$$
the uniqueness of (2.6) is also equivalent to the statement:

(ii)$'$ for $i = 1, 2, \ldots, t$,
$$N_i \cap \sum_{\substack{j=1 \\ j\neq i}}^{t} N_j = 0.$$

In the case $t = 2$, the condition (ii)$'$ is simply the assertion that $N_1 \cap N_2 = 0$ (but pairwise intersections being 0 is *not* enough when $t \geq 3$). Since we use the special case $t = 2$ so often I will call it:

Corollary 2.27. *A module M is a direct sum of submodules N_1 and N_2 if and only if*

(i) $M = N_1 + N_2$ *and*

(ii) $N_1 \cap N_2 = 0$.

You have to be a bit more careful when relating the direct sums of rings to direct sums of modules.

Lemma 2.28. *Let R and S be rings.*

(i) If I is a left ideal of R and J is a left ideal of S then $I \oplus J$ is a left ideal of $R \oplus S$.

(ii) If K is a left ideal of $R \oplus S$ then $K = K_1 \oplus K_2$ for some left ideals K_1 of R and K_2 of S.

Proof. (i) This is an easy exercise, but let us give the proof for once.

Assume that $I \underset{l}{\trianglelefteq} R$, $J \underset{l}{\trianglelefteq} S$. Then $I \oplus J$ is an abelian subgroup of $R \oplus S$ by standard results for groups. Let $a, b \in I \oplus J$, $x \in R \oplus S$. Then
$$a = (a_1, a_2), \quad b = (b_1, b_2), \quad x = (r, s)$$
for some $a_1, b_1 \in I$, $a_2, b_2 \in J$, $r \in R$, $s \in S$. Then
$$xa = (ra_1, sa_2) \in I \oplus J$$
since $ra_1 \in I$, $sa_2 \in J$. Hence $I \oplus J \underset{l}{\trianglelefteq} R \oplus S$.

(ii) Assume that $K \underset{l}{\trianglelefteq} R \oplus S$. Let
$$K_1 = \{a_1 \in R : (a_1, 0_S) \in K\}, \quad K_2 = \{a_2 \in S : (0_R, a_2) \in K\}.$$
It is easy to check that $K_1 \underset{l}{\trianglelefteq} R$, $K_2 \underset{l}{\trianglelefteq} S$.

Let $a \in K_1 \oplus K_2$. Then $a = (a_1, a_2)$ for some $a_1 \in K_1$, $a_2 \in K_2$, and so $(a_1, 0) \in K$, and $(0, a_2) \in K$. So $a = (a_1, 0) + (0, a_2) \in K$. Therefore $K_1 \oplus K_2 \subseteq K$.

Conversely let $a \in K$. Then $a = (a_1, a_2)$ for some $a_1 \in R$, $a_2 \in S$, and
$$(a_1, 0) = (1, 0)(a_1, a_2) \in K, \quad (0, a_2) = (0, 1)(a_1, a_2) \in K.$$
So $a_1 \in K_1$, $a_2 \in K_2$. Consequently $a \in K_1 \oplus K_2$ and $K \subseteq K_1 \oplus K_2$. Hence $K = K_1 \oplus K_2$. $\square$

Remark. So, the situation for ideals of a direct sum of rings is very different to that of submodules of a direct sum of modules. For example consider the 2D vector space $M = \mathbb{C} \oplus \mathbb{C}$ as a $\mathbb{C}$-module. Then, for any $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, the module $\mathbb{C}(\lambda, \mu)$ is a one dimensional submodule that certainly does not split as a direct sum of its components.

There are results similar to Lemma 2.28 for right ideals, and hence for ideals. The results can be extended to direct sums of t rings, where $t \in \mathbb{N}$. (See Example Sheet 2.)

Similar remarks apply to direct sums of rings, but there is more to say in connection with multiplication since ideals are not the same as subrings.

Definition 2.29. *Let A, B be nonempty subsets of a ring R and let $x \in R$. Then AB is defined by*

$$AB = \left\{ \sum_{i=1}^{m} a_i b_i : m \in \mathbb{N},\ a_i \in A,\ b_i \in B\ (i = 1, 2, \ldots, m) \right\},$$

i.e. AB is the set of all finite sums of elements of the form ab with $a \in A$, $b \in B$.

If A is closed under addition then we find that $\{x\}A$ is the same as xA and $A\{x\}$ is the same as Ax, where

$$xA = \{xa : a \in A\} \qquad \textit{and} \qquad Ax = \{ax : a \in A\}.$$

Internal Direct Sums of Ideals: We make a few more observations about direct sums of rings.

Suppose that $R_1, \ldots, R_t$ are rings and write

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_t$$

for their (external) direct sum. Let

$$S_i = \{(0, 0, \ldots, \underset{\nwarrow\, i\text{th entry}}{r_i}, \ldots, 0) : r_i \in R_i\} \qquad (i = 1, 2, \ldots, t).$$

Then S_i is a ring in its own right and $R_i \cong S_i$ (I do not like calling it a subring of R since the identity of S_i, $(0, 0, \ldots, 1_i, \ldots, 0)$, is not the same as the identity of R, $(1_1, 1_2, \ldots, 1_t)$ when $t > 1$). The isomorphism is given by

$$r_i \mapsto (0, 0, \ldots, r_i, \ldots, 0).$$

In addition, S_i is an ideal of R and $S_i S_j = 0$ if $i \neq j$.

We have seen above that every ideal I of R has the form

$$I = I_1 \oplus I_2 \oplus \cdots \oplus I_t,$$

where $I_i \trianglelefteq R_i$ $(i = 1, 2, \ldots, t)$, and every set of this form is an ideal of R. Let

$$J_i = \{(0, 0, \ldots, \underset{\nwarrow\, i\text{th entry}}{a_i}, \ldots, 0) : a_i \in I_i\} \qquad (i = 1, 2, \ldots, t).$$

Then $J_i \trianglelefteq S_i$, and I_i corresponds to J_i under the above isomorphism from R_i to S_i. Also $J_i \trianglelefteq R$ and $J_i J_j = 0$ if $i \neq j$.

Since

$$R = S_1 + S_2 + \cdots + S_t$$

and every element $a \in R$ can be expressed uniquely in the form

$$a = a_1 + a_2 + \cdots + a_t,$$

where $a_i \in S_i$ $(i = 1, 2, \ldots, t)$, we also have

$$R = S_1 \oplus S_2 \oplus \cdots \oplus S_t,$$

an internal direct sum of *ideals* of R (each of which has its own 1), and

$$I = J_1 \oplus J_2 \oplus \cdots \oplus J_t,$$

an internal direct sum of ideals of R.

Conversely, given such ways of representing R and I as internal direct sums (with $J_i \trianglelefteq S_i$ $(i = 1, 2, \ldots, t)$), we can form the external direct sums and

$$J_1 \oplus J_2 \oplus \cdots \oplus J_t \trianglelefteq S_1 \oplus S_2 \oplus \cdots \oplus S_t \cong R$$

just as for modules. Furthermore, $J_1 \oplus J_2 \oplus \cdots \oplus J_t$ corresponds to I under the isomorphism.

Lemma 2.30. (The Modular Law) *Let R be a ring, M be a left R-module and A, B, C be submodules of M with $B \subseteq A$. Then*

$$A \cap (B + C) = B + (A \cap C).$$

Proof. Let $x \in A \cap (B + C)$. Then $x \in A$ and $x = b + c$ for some $b \in B$, $c \in C$. As $B \subseteq A$, we have $c = x - b \in A \cap C$. So $x \in B + (A \cap C)$ and hence $A \cap (B + C) \subseteq B + (A \cap C)$.

Also $B + (A \cap C) \subseteq A$ since $B \subseteq A$, and $B + (A \cap C) \subseteq B + C$ because $A \cap C \subseteq C$. Hence $B + (A \cap C) \subseteq A \cap (B + C)$. Therefore $A \cap (B + C) = B + (A \cap C)$. □

Lemma 2.31. *Let R be a ring and M a right R-module with submodules K, L and N. Suppose that $M = L \oplus K$ and $L \subseteq N$. Then $N = L \oplus (N \cap K)$.*

Proof. By the Modular Law,

$$N = N \cap M = N \cap (L + K) = L + (N \cap K)$$

since $L \subseteq N$. But $L \cap (N \cap K) = 0$ since $L \cap K = 0$. Therefore $N = L \oplus (N \cap K)$ by Corollary 2.27. □

Remark 2.32. There are versions of this lemma where M, K, L, N are either:

(a) left R-modules,

(b) right (or left) ideals of R,

(c) ideals of R.

Version (b) (for left ideals) is just a special case of Lemma 2.31. Version (c) requires further comment. We require M, K, L, N to be ideals of R such that $L \subseteq N \subseteq M$.

Lemma 2.33. *Let I be an ideal of a ring R which is a direct sum*

$$I = J_1 \oplus J_2 \oplus \cdots \oplus J_t \oplus K$$

of ideals $J_1, J_2, \ldots, J_t, K$ of R. Suppose that K is a direct sum

$$K = J_{t+1} \oplus J_{t+2}$$

of ideals J_{t+1}, J_{t+2} of R. Then

$$I = J_1 \oplus J_2 \oplus \cdots \oplus J_t \oplus J_{t+1} \oplus J_{t+2}.$$

Proof. Clearly,

$$I = J_1 + J_2 + \cdots + J_t + (J_{t+1} + J_{t+2}) = J_1 + J_2 + \cdots + J_{t+2}.$$

Let $a \in I$. Then

$$a = a_1 + a_2 + \cdots + a_t + b$$

for some uniquely determined $a_i \in J_i$ $(i = 1, 2, \ldots, t)$, $b \in K$. Also

$$b = a_{t+1} + a_{t+2}$$

for some uniquely determined $a_{t+1} \in J_{t+1}$, $a_{t+2} \in J_{t+2}$. So

$$a = a_1 + a_2 + \ldots + a_t + a_{t+1} + a_{t+2}.$$

The elements $a_i \in J_i$ $(i = 1, 2, \ldots, t+2)$ are uniquely determined by a. This establishes the result. $\square$