Algebra 2 Home Exercises 3

1. Let A be a Noetherian ring. Prove that an A-module M is a Noetherian A-module iff M is of finite type.

Hint. The left arrow was proved in the lectures. To prove the right arrow assume that M isn't of finite type, then there are infinitely many elements m_i in M such that m_{i+1} doesn't belong to $Am_1 + \cdots + Am_i$. Show that the infinite sequence of A-submodules

$$Am_1, Am_1 + Am_2, Am_1 + Am_2 + Am_3, \dots$$

is strictly increasing.

2. Define the set of *formal power series* over a ring A by $A[[X]] = \{\sum_{n=0}^{\infty} a_n X^n : a_n \in A\}$. Define

$$\sum_{n=0}^{\infty} a_n X^n + \sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} (a_n + b_n) X^n,$$

$$(\sum_{n=0}^{\infty} a_n X^n)(\sum_{n=0}^{\infty} b_n X^n) = \sum_{n=0}^{\infty} c_n X^n, \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then $A[[X]]$ is a commutative ring.

Prove that $A[[X]]$ is a Noetherian ring if A is a Noetherian ring.

Hint: Modify cleverly the proof of Hilbert's theorem and consider the sequence of ideals

$$J_n = \{a \in A : \quad \text{there is } f(X) = aX^n + \text{terms of higher order in } J\}.$$

Note: *higher* order, not smaller order. Instead of polynomials which generate the ideal in the proof of Hilbert's theorem consider power series $f_j^{(m)}(X) = a_j^{(m)} X^m +$ terms of higher order .

3. Let A be a Noetherian ring and let $f\colon A \to A$ be a ring homomorphism. Prove that f is an isomorphism iff f is surjective. [Remark: without the assumption of Noetherianity this is false].

Hint. Assume that f is surjective and denote by I_j the kernel of $f^{(j)} = f \circ \cdots \circ f\colon A \to A$. Show that I_j form an increasing sequence of ideals of A and therefore $I_j = I_{j+1}$ for some j. Deduce that $f(f^{(j)}(a)) = 0$ implies $f^{(j)}(a) = 0$ and use the surjectivity of f to complete the proof.

Algebra 2 Solutions of Home Exercises 3

1. Assume that M isn't of finite type. Then there are infinitely many elements m_i in M such that m_{i+1} doesn't belong to $Am_1 + \cdots + Am_i$. Then the infinite sequence of A-submodules

$$Am_1, Am_1 + Am_2, Am_1 + Am_2 + Am_3, \ldots$$

is strictly increasing, hence M is not a Noetherian A-module.

2. Let J be a non-zero ideal of $A[[X]]$. For $n \geq 0$ define

$$J_n = \{a \in A : \quad \text{there is } f(X) = aX^n + \text{terms of higher order } \text{ in } J\}.$$

Then J_n is an ideal of A and $J_1 \subset J_2 \subset \ldots$. Since A is Noetherian, we deduce that there is n such that $J_n = J_{n+1} = \ldots$. For $m \leq n$ the ideal J_m as an ideal of the Noetherian ring A is finitely generated, let $c_j^{(m)}$ for $1 \leq j \leq n_m$ be its generators. Pick up $f_j^{(m)}(X) = c_j^{(m)} X^m + \cdots \in J$.

Let $f = a_m X^m + \cdots \in J$. We shall pass from f to another series in J which starts with a *larger* power of X.

If $m \leq n$, then write $a_m = \sum_j a_j^{(m)} c_j^{(m)}$ with $a_j^{(m)} \in A$. Then $f - \sum_j a_j^{(m)} f_j^{(m)}(X) = b_{m+1}X^{m+1} + b_{m+2}X^{m+2} + \cdots \in J$. If $m > n$, then write $a_m = \sum_j a_j^{(m)} c_j^{(n)}$ with $a_j^{(m)} \in A$. Then $f - \sum_j a_j^{(m)} X^{m-n} f_j^{(n)}(X) = b_{m+1}X^{m+1} + b_{m+2}X^{m+2} + \cdots \in J$.

Continue this way (infinitely many times, keeping in mind that we work with power series that's fine). Then f is a linear combination with coefficients in $A[[X]]$ of $f_j^{(m)}(X)$, $0 \leq m \leq n$, $1 \leq j \leq n_m$. Thus, J is finitely generated.

3. The sequence of the ideals I_j stabilizes for sufficiently large $j > j_0$. Since f is surjective, $f^{(j)}$ is surjective for all j. Now, if $f(b) = 0$ then for a $j > j_0$ find a such that $b = f^{(j)}(a)$. Then $f(b) = f^{(j+1)}(a) = 0$, so $a \in I_{j+1} = I_j$, and so $b = 0$. Thus f is injective.