Math 432
Exam 2
November 8, 2011
Key

1. Mark each of the following statements as "True" or "False". Justify any "False" answers with either a counterexample or result from the text.

a. Let $G =< a >$ be a cyclic group of order n. If r divides n, then G has a subgroup of order r.
True

b. Every abelian group is cyclic.
False, the Klein 4 group is abelian but not cyclic.

c. Every cyclic group is abelian.
True

d. The relation of being isomorphic is an equivalence relation on a collection of groups.
True

e. If a is an element of order m in a group G, and $a^k = e$, then m divides k.
True

2.

a. Suppose $G = <a>$ is a cyclic group of order 30. List all the generators of G.
$a, a^7, a^{11}, a^{13}, a^{17}, a^{19}, a^{23}, a^{29}$

b. Recall U_{11} denotes the set of all invertible elements of $\mathbb{Z}_{11}$. U_{11} is a cyclic group with respect to multiplication. Given that 2 is a generator of U_{11}, find all other generators of U_{11}.
$2^3, 2^7, 2^9$

c. List the distinct subgroups of $\mathbb{Z}_{35}$ under addition.
$< [1] >= \mathbb{Z}_{35}$
$< [5] >= \{[0], [5], [10], [15], [20], [25], [30]\}$
$< [7] >= \{[0], [7], [14], [21], [28]\}$
$< [0] >= \{[0]\}$

d. Let $G = <a>$ be a cyclic group of order 96. Find the order of a^{24}.
The order of $a^{24} = 96/(24, 96) = 96/24 = 4$

3. Let f=(1,7,2,5,4)(1,4,2,3)(1,5,4,6,3,2).

   a. Express f as a product of disjoint cycles.
   (1,4,6,7,2)

   b. Compute f^2 and f^{-1}.
   $f^2 = (1, 6, 2, 4, 7)$
   $f^{-1} = (2, 7, 6, 4, 1)$

   c. Determine the order of f.
   f has order 5

   d. Is f even or odd? Justify your answer.
   $f = (1, 2)(1, 7)(1, 6)(1, 4)$, so f is even since it can be written as a product of an even number of transpositions.

4. Let G and G' be groups and H be a subgroup of G. Prove that if there exists an isomorphism from G to G',then $\phi(H) = \{x \in G' | x = \phi(h) \text{ for some } h \in H\}$ is a subgroup of G'.
Since H is a subgroup of G, e, the identity in G is in H. Since ϕ is an isomorphism from G to G', $\phi(e) = e'$ where e' is the identity in G', so $\phi(e) \in \phi(H)$, thus $\phi(H)$ is nonempty.
Let $x, y \in \phi(H)$, then there exist elements $h_1, h_2 \in H$, such that $\phi(h_1) = x, \phi(h_2) = y$. Now since H is a group, $h_1h_2 \in H$ and thus $\phi(h_1h_2) \in \phi(H)$. Now since ϕ is an isomorphism from G to G', $\phi(h_1h_2) = \phi(h_1)\phi(h_2) = xy$, so $xy = \phi(h_1h_2) \in \phi(H)$. Thus $\phi(H)$ is closed.
Now since $x = \phi(h_1) \in \phi(H)$, we must show $x^{-1} \in \phi(H)$. $x^{-1} = [\phi(h_1)]^{-1} = \phi(h_1^{-1}) \in \phi(H)$, since $h_1 \in H$, H a group implies $h_1^{-1} \in H$.

5. If G and H are finite groups and $\phi : G- > H$ is an isomorphism, prove that a and $\phi(a)$ have the same order, for any $a \in G$.
Let H and G be groups. Let $a \in G$ have order n, and ϕ be an isomorphism from G to H. We will prove by induction that $\phi(a^n) = \phi(a)^n$ for every natural number n.
When $n = 1$, the statement is clearly true.
Assume that for some natural number k, $\phi(a^k) = \phi(a)^k$.
Now $\phi(a^{k+1}) = \phi(a^k a) = \phi(a^k)\phi(a) = \phi(a)^k\phi(a) = \phi(a)^{k+1}$.
We will show that $\phi(a)$ has order n. Assume to the contrary, that there exists some natural number j, $j < n$ such that $\phi(a)^j = e'$, where e' is the identity in H, Then $\phi(a^j) = e' = \phi(e)$ which implies $a^j = e$, since ϕ is one to one. This contradicts the fact that the order of a is n.

6. Prove that U_5 is isomorphic to U_{10} but not U_{12}. (You must come up with a mapping and prove it is an isomorphism, just drawing group tables does not constitute a proof. $U_5 =< 2 >$, $U_{10} =< 3 >$. We will show $\phi : U_5- > U_{10}$ defined by $\phi(2^n) = 3^n$ is an isomorphism.
Let $3^j \in U_{10}, j \in \mathbb{Z}$. The preimage of 3^j under ϕ is 2^j. Thus ϕ is onto.
Suppose $\phi(2^j) = \phi(2^k)$, for some $j, k \in \mathbb{Z}$. Then $3^j = 3^k$ and $j \equiv k \mod 4$. Thus $2^j = 2^k$. Hence ϕ is one to one.
Let $2^j, 2^k \in U_5$. $\phi(2^j 2^k) = \phi(2^{j+k} = 3^{j+k} = 3^j 3^k = \phi(2^j)\phi(2^k)$. Thus ϕ is operation preserving.
Thus ϕ is an isomorphism.
$U_{12} = \{1, 5, 7, 11\}$ is not cyclic, so it is not isomorphic to U_5.

7. Show that the mapping ϕ from the additive group $\mathbb{Z}_{12}$ to the additive group $\mathbb{Z}_{12}$ defined by $\phi([x]) = [3x]$ is a homomorphism and find kerϕ.
Let $x, y \in \mathbb{Z}_{12}$, $\phi(x+y) = [3(x+y)] = [3x+3y] = [3x]+[3y] = \phi(x)+\phi(y)$, so ϕ is a homomorphism.
$ker\phi = \{0, 4, 8\}$