Math 403 - Solutions for Problem Set 3

Problem 17.19 (a) Prove that R is an integral domain if and only if $\{0\}$ is a prime ideal of R.

SOLUTION: Suppose that R is an integral domain. Then R has at least two elements (since $1 \neq 0$) and hence the ideal $\{0\}$ is not the ring R itself. Furthermore, suppose that $a, b \in R$ and that $a \notin \{0\}$ and that $b \notin \{0\}$. Hence $a \neq 0$ and $b \neq 0$. Since R is an integral domain, it follows that $ab \neq 0$. Hence $ab \notin \{0\}$. We have shown that if $a \notin \{0\}$ and $b \notin \{0\}$, then $ab \notin \{0\}$. Since $\{0\}$ is not R itself. It follows that $\{0\}$ is indeed a prime ideal of R.

Conversely, suppose that R is a commutative ring with unity $1 \neq 0$ and that $\{0\}$ is a prime ideal of R. In order to show that R is an integral domain, we must just prove that, for $a, b \in R$, if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$. To prove this, assume that a and b are nonzero elements in R. Then $a \notin \{0\}$ and $b \notin \{0\}$. Since $\{0\}$ is a prime ideal in R, it follows that $ab \notin \{0\}$. Hence $ab \neq 0$. We have proved that R is an integral domain.

Problem 17.19 (b) Prove that R is a field if and only if $\{0\}$ is a maximal ideal of R.

SOLUTION: Suppose that R is a field and that J is an ideal of R such that $\{0\} \subseteq J \subseteq R$. Assume that $J \neq \{0\}$. Then J contains a nonzero element a of R. Since R is a field, the element a is a unit in R. It follows from problem 17.20 in problem set 2 that $aR = R$ Since J is an ideal of R and $a \in J$, it follows that $aR \subseteq J$. Hence $R \subseteq J$. We also have $J \subseteq R$. Therefore, if $J \neq \{0\}$, we have proved that $J = R$. Consequently, $\{0\}$ is indeed a maximal ideal of R.

Conversely, suppose that R is a commutative ring with unity $1 \neq 0$ and that $\{0\}$ is a maximal ideal of R. Suppose that $a \in R$ and that $a \neq 0$. Consider the principal ideal aR of the ring R. The ideal aR contains the nonzero element a and hence $aR \neq \{0\}$. Thus, aR is an ideal of R such that $\{0\} \subseteq aR \subseteq R$ and $aR \neq \{0\}$. Since $\{0\}$ is a maximal ideal of R, it follows that $aR = R$. In particular, we have $1 \in aR$. Thus, there exists an element $b \in R$ such that $1 = ab$. Since R is a commutative ring, we also have $ba = 1$. Thus, a is a unit in the R. We have proved that every nonzero element of R is a unit in R. It follows that R is a field.

Problem 17.25(a): Show that if I and J are ideals in a ring R, then $I \cap J$ is an ideal in R.

SOLUTION: Assume that I and J are ideals in a ring R. Let $K = I \cap J$. First, note that since I and J are subgroups of R under $+$, it follows that K is also a subgroup of R under $+$. This is a result from group theory (and very easy to prove). Furthermore, suppose that $k \in K$ and $r \in R$. Then $k \in I$ and so we have $rk \in I$ and $kr \in I$ since I is an ideal of R. Also, $k \in J$ and so we have $rk \in J$ and $kr \in J$ since J is an ideal of R. Hence, we have $rk \in I \cap J$ and $kr \in I \cap J$. That is, if $k \in K$ and $r \in R$, it follows that $rk \in K$ and $kr \in K$. We have proved that $K = I \cap J$ is indeed an ideal in the ring R.

Problem 17.27(b): Let I be the set of nilpotent elements in a commutative ring R. We proved in class that I is an ideal in the ring R. Show that R/I has no nonzero nilpotent elements.

SOLUTION: An element α of R/I has the form $\alpha = a + I$, where $a \in R$. Suppose that α is nilpotent. We then have $\alpha^n = 0_{R/I}$ for some positive integer n. The additive identity in R/I is $0_{R/I} = 0 + I = I$. Multiplication in R/I is defined by $(a+I)(b+I) = ab + I$ for all $a, b \in R$. In particular,

$$(a+I)^2 = (a+I)(a+I) \;=\; aa + I = a^2 + I$$

$$(a+I)^3 = (a+I)(a+I)^2 = (a+I)(a^2+I) = aa^2 + I \;=\; a^3 + I, \; . \;\; . \;\; . \;\; .$$

and we can show by a simple mathematical induction argument that $(a+I)^n = a^n + I$ for all positive integers n. Thus, $\alpha^n = a^n + I$. Since $\alpha^n = 0_{R/I}$, it follows that $a^n + I = 0 + I = I$. Therefore, $a^n \in I$. This implies that a^n is a nilpotent element in R. Therefore, there exists a positive integer m such that

$$(a^n)^m \;=\; 0_R \quad .$$

It follows that $a^{nm} = 0_R$. Since nm is a positive integer, it follows that a is a nilpotent element in the ring R. Therefore, $a \in I$. Hence $\alpha = a + I = 0 + I = 0_{R/I}$. We have proved that if α is a nilpotent element in R, then $\alpha = 0_{R/I}$. Therefore, the ring R/I has non nonzero nilpotent elements.

Problem 17.33(a): Suppose that I and J are ideals in a ring R. Prove that $I + J = \{\, i + j \;\big|\; i \in I, \; j \in J \,\}$ is an ideal in the ring R.

SOLUTION: Let $K = I + J$. By group theory, we know that K is a subgroup of R under the operation $+$. This is true because R is an abelian group and both I and J are subgroups of R under the operation $+$. Suppose that $r \in R$ and $k \in K$. We can write k in the form $k = i + j$, where $i \in I$ and $j \in J$. It follows that $ri \in I$ and $rj \in J$ since I and J are ideals in R. It also follows that $ir \in I$ and $jr \in J$. Therefore, $K = I + J$ contains $ri + rj$ and also contains $ir + jr$. By the distributive laws, we have

$$ri + rj \;=\; r(i+j) \;=\; rk \qquad\qquad ir + jr \;=\; (i+j)r \;=\; kr \quad .$$

Therefore, if $r \in R$ and $k \in K$, it follows that $rk \in K$ and $kr \in K$. We have proved that $K = I + J$ is indeed an ideal in the ring R.

ADDITIONAL PROBLEMS:

A: Let R be the ring of continuous real-valued functions on the interval $(0, 1)$. Let

$$I \;=\; \{\ f \in R \bigm| f(1/2) = 0 \ \ \textit{and} \ \ f(1/3) = 0 \ \} \ \ .$$

Prove that I is an ideal of R. Prove that I is not a prime ideal of R.

SOLUTION: The fact that R is a ring was discussed in class. We first prove that I is a subgroup of R under addition. The element 0_R is just the constant function 0 on the interval $(0, 1)$. That element is clearly in I. Suppose that f and g are in I. Then $f(1/2) = f(1/3) = 0$ and $g(1/2) = g(1/3) = 0$. Then

$$(f+g)(1/2) = f(1/2)+g(1/2) = 0+0 = 0 \quad \textit{and} \quad (f+g)(1/3) = f(1/3)+g(1/3) = 0+0 = 0$$

and hence $f + g$ is in I. Also, the additive inverse of f is $-f$ and we have

$$(-f)(1/2) = -f(1/2) = -0 = 0 \quad \textit{and} \quad (-f)(1/3) = -f(1/3) = -0 = 0$$

and hence $-f$ is in I. Finally, suppose that $h \in R$. Then we have

$$(hf)(1/2) = h(1/2){\cdot}f(1/2) = h(1/2){\cdot}0 = 0 \quad \textit{and} \quad (hf)(1/3) = h(1/3){\cdot}f(1/3) = h(1/3){\cdot}0 = 0$$

and hence hf is in the set I. Since R is a commutative ring, we have $fh = hf$ and hence fh is in I. We have proved that I is an ideal in the ring R.

However, I is not a prime ideal. To see this, suppose that f and g are defined by the formulas

$$f(x) \;=\; x - 1/2 \qquad\qquad \textit{and} \qquad\qquad g(x) = x - 1/3$$

for all x in the interval $(0, 1)$. Both f and g are continuous real-valued functions on that interval and hence they are elements of R. Note that $f(1/3) = -1/6 \neq 0$ and hence $f \notin I$. Also, $g(1/2) = 1/6 \neq 0$ and hence $g \notin I$. However,

$$(fg)(1/2) \;=\; f(1/2)g(1/2) \;=\; 0 \cdot (1/6) \;=\; 0$$

and

$$(fg)(1/3) \;=\; f(1/3)g(1/3) \;=\; (-1/6)\cdot 0 \;=\; 0$$

and therefore fg is in I. Thus, f and g are elements in the ring R, $f \notin I$, and $g \notin I$, but $fg \in I$. It follows that I is not a prime ideal of R.

B: This question concerns idempotents in a ring R. Suppose that R is a commutative ring with unity. As usual, let 1_R denote the unity in R. Suppose that e is an idempotent in R. Thus, $e \in R$ and $e^2 = e$.

(a) Let $f = 1_R - e$. Show that f is an idempotent in R. Furthermore, show that $ef = 0_R$ and $fe = 0_R$.

SOLUTION: We will use elementary facts about rings. We will just write 1 instead of 1_R. We have

$$f^2 \;=\; ff \;=\; (1-e)(1-e) \;=\; (1-e)1-(1-e)e \;=\; 1-e-(e-e^2)$$

$$=\; 1-e-e+e^2 \;=\; 1-e-e+e \;=\; 1-e \;=\; f \quad .$$

We have used the fact that $e^2 = e$ in this calculation. Thus, we indeed have $f^2 = f$ and so f is an idempotent in R. Finally, note that

$$ef = e(1-e) = e-e^2 = e-e = 0_R \qquad \textit{and} \qquad fe = (1-e)e = e-e^2 = e-e = 0_R \quad ,$$

exactly as stated in the problem.

(b) Let $S = Re$ and $T = Rf$. Thus, S is the principal ideal of R generated by e and T is the principal ideal of R generated by f. In particular, S and T are subrings of R. Prove that S and T are commutative rings with unity.

SOLUTION: As pointed out in the problem, S and T are subrings of the commutative ring R. Therefore, it is clear that multiplication in S and T is commutative. We must just show that S and T have a unity element. Both arguments are the same and so we just give the argument for S.

Note that S contains re for all $r \in R$. In particular, S contains $1e = e$. We will show that e is a unity for S. Suppose that $s \in S$. Thus, $s = re$ for some $r \in R$. We have

$$se \;=\; (re)e \;=\; r(ee) \;=\; r(e^2) \;=\; re \;=\; s$$

Since S is a commutative ring, we also have $es = s$. Thus, for all $s \in S$, we have $se = s$ and $es = s$. This shows that e is indeed a unity for S. Of course, as we know, the unity for a ring (if it exists) is unique and so e is the unity for the ring S. A similar argument works for T. The unity in T is f.

(c) Prove that $S \cap T \ = \ \{ \ 0_R \ \}$.

SOLUTION: Clearly, $S \cap T$ contains 0_R. Now suppose that $a \in S \cap T$. Since $a \in S$, we have $ae = a$. Since $a \in T$, we have $af = a$. Therefore,

$$a \ = \ af \ = \ (ae)f \ = \ a(ef) \ = \ a(0_R) \ = \ 0_R \ .$$

Therefore, $S \cap T = \{0_R\}$, as stated.

C: Let $R = M_2(\mathbb{R})$. Consider the following subset of R:

$$I \ = \ \left\{ \ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \ \middle| \ a, b \in \mathbb{R} \ \right\} \ .$$

Is I an ideal in the ring R? Justify your answer carefully.

SOLUTION: Actually, I is not an ideal in the ring R. To see this, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A \in I$ and $B \in R$. However,

$$AB \ = \ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ = \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and hence $AB \notin I$. It follows that I is not an ideal in R.