SOLVED PROBLEMS: SECTION 1.3

13. Let P be a prime ideal of the commutative ring R. Prove that if P is a prime ideal of R, then $A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for all ideals A, B of R. Give an example to show that the converse is false.

    Solution: If P is a prime ideal of R, and $A \cap B \subseteq P$, then $AB \subseteq P$ since $AB \subseteq A \cap B$, and therefore $A \subseteq P$ or $B \subseteq P$.

    In the ring $\mathbf{Z}_4$, the zero ideal is not a prime ideal. On the other hand, since the ideals of $\mathbf{Z}_4$ form a chain, it is always true that either $A \subseteq A \cap B$ or $B \subseteq A \cap B$. It follows that if $A \cap B = (0)$, then either $A = (0)$ or $B = (0)$, so the zero ideal of $\mathbf{Z}_4$ provides the desired counterexample.

14. Show that in the polynomial ring $\mathbf{Z}[x]$, the ideal (n, x) generated by $n \in \mathbf{Z}$ and x is a prime ideal if and only if n is a prime number.

    Solution: Define $\phi : \mathbf{Z}[x] \to \mathbf{Z}_n$ by $\phi(a_0 + a_1 x + \cdots + a_n x^n) = a_0$, for all polynomials $a_0 + a_1 x + \cdots + a_n x^n \in \mathbf{Z}[x]$. It follows from Example 1.2.2 that ϕ is a ring homomorphism. (In the example, let θ be the projection of $\mathbf{Z}$ onto $\mathbf{Z}_n$, and let $\eta(x) = [0]_n$. Then $\phi = \widehat{\theta}$.) It is clear that ϕ is onto, and that $\ker(\phi) = (n, x)$. Thus $\mathbf{Z}[x]/\,(n, x) \cong \mathbf{Z}_n$, so (n, x) is a prime ideal iff $\mathbf{Z}_n$ is an integral domain, which occurs iff n is a prime number.

15. Let R be a Boolean ring (see Exercise 1.1.11 in the text) and let P be a prime ideal of R. Prove that P is maximal, and that $R/P \cong \mathbf{Z}_2$.

    Solution: The factor ring R/P is also a Boolean ring, since if $x^2 = x$ for all $x \in R$, then the same identity holds in the factor ring. Thus if $\overline{a} \neq \overline{0}$ in R/P, then $(\overline{a})^2 = \overline{a}$, so $\overline{a} = \overline{1}$ since R/P is an integral domain. We conclude that R/P is a field with only two elements, $\overline{0}$, and $\overline{1}$, so it must be isomorphic to $\mathbf{Z}_2$.

16. Let R be a commutative ring. Then R is called a *local ring* if it has a proper ideal P such that $P \supseteq I$, for all proper ideals I of R. Prove that the following conditions are equivalent for R.

    (1) R is a local ring;

    (2) the set of nonunits of R forms an ideal;

    (3) there exists a maximal ideal P of R such that $1 + x$ is a unit, for all $x \in P$.

    Solution: (1) $\Longrightarrow$ (3): If R has a proper ideal P such that $P \supseteq I$, for all proper ideals I of R, then P is certainly a maximal ideal of R. If $a \in P$, then $1 + a \notin P$, since otherwise we would have $1 = (1 + a) - a \in P$. From the assumption on P it follows that $1 + a$ does not generate a proper ideal, so $1 + a$ must be a unit.

    (3) $\Longrightarrow$ (2): Let $a \in R$. If $a \in P$, then a is certainly a nonunit of R. If $a \notin P$, then there exists $b \in R$ with $(a + P)(b + P) = 1 + P$, since R/P is a field and $a + P$ is a nonzero element of R/P. Therefore $ab = 1 + x$, for some $x \in P$, so by assumption, ab is a unit. It follows that a is a unit since, $a\left(b(ab)^{-1}\right) = 1$. Thus P is precisely the set of nonunits of R.

    (2) $\Longrightarrow$ (1): Suppose that the set of nonunits of R forms an ideal P of R. No proper ideal I of R can contain a unit, so we must have $P \supseteq I$ for all proper ideals I of R, and thus R is a local ring.

    Note: The usual definition states that a commutative ring is a local ring if it has a unique maximal ideal. We must wait until we have proved Lemma 2.1.13 to show that this is equivalent to the definition given here.

17. Prove that any nonzero homomorphic image of a local ring is again a local ring.

    Solution: Let R be a local ring, and let P be the proper ideal that contains all other proper ideals of R. By Theorem 1.2.7, any nonzero homomorphic image of R is isomorphic to R/K, for some nonzero ideal K of R. It follows from the definition of a local ring that $K \subseteq P$. Since there is a one-to-one correspondence between ideals of R/K and ideals of R that contain K (apply Proposition 1.2.9), it follows immediately that $P/K \supseteq I/K$ for all proper ideals I/K of R/K, and thus R/K is a local ring.

18. Show that the ring R defined in Exercise 1.2.9 of the text is a local ring.

    Solution: The ring R is defined to be the set of all rational numbers m/n such that p is not a factor of n. We can now see that this just the localization of $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$. Theorem 1.3.11 (c) shows that pR is the unique maximal ideal of R. In fact, pR is the set of nonunits of R, since a nonzero element m/n is invertible iff p is not a divisor of m.