Math 0430
Homework #10

Sec. 6.2: #4: (a) Let $[a]_n$ denote the congruence class of the integer a modulo n. Show that the map f: $\mathbf{Z_{12}} \to \mathbf{Z_4}$ that sends $[a]_{12}$ to $[a]_4$ is a well-defined surjective homomorphism.

Proof: Let $[a]_{12} = [b]_{12}$. Then $12|(a - b)$, so $4|(a - b)$ and $f([a]_{12}) = f([b]_{12})$. Thus f is a well-defined mapping. $f([a]_{12} + [b]_{12}) = f([a + b]_{12}) = [a]_4 + [b]_{12} = f([a]_{12}) + f([b]_{12})$ and $f([a]_{12} [b]_{12}) = f([ab]_{12}) =[a]_4 [b]_4 = f([a]_{12}) f([b]_{12})$, so f is a homomorphism. Clearly, if $[a]_4 \in \mathbf{Z_4}$, a is an integer, $[a]_{12} \in \mathbf{Z_{12}}$ and $f([a]_{12}) = [a]_4$, so f is surjective. ♦

(b) Find the kernel of f.

Proof: $\text{Ker}(f) = \{[a]_{12} \in \mathbf{Z_{12}} \mid f([a]_{12}) = [a]_4 =[0]_4\} = \{[a]_{12} \in \mathbf{Z_{12}} : 4|a\} = ([4]_{12}) = \{[0]_{12}, [4]_{12} , [8]_{12}\}$. ♦

Sec. 6.2: #8: (a) Let I = {0,3} in $\mathbf{Z_6}$. Verify that I is an ideal in $\mathbf{Z_6}$ and show that $Z_6/I \cong Z_3$.

Proof: Define f: $Z_6 \to Z_3$ by $f([a]_6 =[a]_3$. Then f is clearly surjective. For $[a]_6 ,[b]_6 \in Z_6$, $f([a]_6 + [b]_6) = f([a + b]_6) = [a + b]_3 = [a]_3 + [b]_3 = f([a]_6) + f([b]_6)$ and $f([a]_6 [b]_6) = f([ab]_6) = [ab]_3 = [a]_3 [b]_3 = f([a]_6) f([b]_6)$; so f is also a homorphism. The kernel of f is I. So I is an ideal in $\mathbf{Z_6}$ and by the First Isomorphism Theorem, $Z_6/I \cong Z_3$. ♦

Sec. 6.3: #7: Let R be a commutative ring with identity. Prove that R is a field if and only if (0_R) is a maximal ideal.

Proof: Assume that R is a field. Let $(0_R) \subseteq M \subseteq R$. If $M \neq (0_R)$, then there is $a \in M$ with $a \neq 0_R$. This means that a is a unit, $1_R \in M$ and M = R. Therefore, M is a maximal ideal of R.

Conversely, assume that (0_R) is a maximal ideal. Let $a \neq 0_R$ and consider (a). $(0_R) \subset (a) \subseteq R$, so (a) = R. It follows that $1_R \in (a)$ and there is $b \in R$ such that $1_R = ba$, and a is a unit. Therefore, R is a field.

Sec. 6.3: #10: Let p be a fixed prime and let J be the set of polynomials in Z[x] whose constant terms are divisible by p. Prove that J is a maximal ideal in Z[x].

Proof: Since p is prime, Z_p is a field, Define Φ: Z[x] $\rightarrow$ Z_p by Φ(f(x)) = $[a_0]$ where f(x) = $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Let g(x) = $b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0$. Then $\Phi(f(x) + g(x)) = [a_0 + b_0] = [a_0] + [b_0] = \Phi(f(x)) + \Phi(g(x))$ and $\Phi(f(x)g(x)) = [a_0 b_0] = [a_0][b_0] = \Phi(f(x))\Phi(g(x))$, so Φ is a homomorphism. If $[a] \in Z_p$, then $f(x) = a \in Z[x]$, and $\Phi(a) = [a]$; so Φ is surjective. Furthermore, the kernel of Φ is $\{f(x) \in Z[x] \mid \Phi(f(x)) = [a_0] = [0]\} = \{f(x) \in Z[x] \mid p|a_0\}$ = J. By the First Isomorphism Theorem, $\frac{Z[x]}{J} \cong Z_p$. Since Z_p is a field, J is a maximal ideal of Z[x]. ■

Sec. 6.3: 11: Show that the principal ideal (x-1) in Z[x] is prime but not maximal.

Proof: Define Φ: Z[x] $\rightarrow$ Z by Φ(f(x)) = f(1) where f(x) = $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Let g(x) = $b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0$. Then $\Phi(f(x) + g(x)) = f(1) + g(1) = \Phi(f(x)) + \Phi(g(x))$ and $\Phi(f(x)g(x)) = f(1)g(1) = \Phi(f(x))\Phi(g(x))$, so Φ is a homomorphism. If $a \in Z$, then $f(x) = a \in Z[x]$, and $\Phi(f(x)) = f(1) = a$; so Φ is surjective. Furthermore, the kernel of Φ is $\{f(x) \in Z[x] \mid \Phi(f(x)) = f(1) = 0\} = \{f(x) \in Z[x] \mid x-1|f(x)\}$ = (x-1). By the First Isomorphism Theorem, $\frac{Z[x]}{(x-1)} \cong Z$. Since Z is an integral domain and not a field, (x-1) is a principal ideal, but not a maximal ideal. ■

Sec. 6.3: #13: Find an ideal in $\mathbf{Z} \times \mathbf{Z}$ that is prime, but not maximal.

Proof: Define $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by $f((z_1,z_2)) = z_1$. F is a surjective homomorphism and the kernel of f is I = $\mathbf{(0)} \times \mathbf{Z}$. So the First Isomorphism Theorem implies that $\frac{\mathbf{Z} \times \mathbf{Z}}{\mathbf{(0)} \times \mathbf{Z}} \cong \mathbf{Z}$. Since **Z** is an integral domain but not a field, $\mathbf{(0)} \times \mathbf{Z}$ is a prime ideal that is not maximal. ■