Wave Digital Structures

SASP - SSSP

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Motivations

- A physical model is generally specified by
  - A set of «*constitutive equations*» (ODEs and/or PDEs), each describing a functional element and involving
    - extensive variables (pressure, voltage, force)
    - intensive variables (flow, current, velocity)
  - A set of «*continuity constraints*» that specify how such elements interact with each other
    - Kirchhoff laws, Laws of dynamics, conservation laws in fluid-dynamics, ...
- Physical models, in fact, can usually be seen as a collection of functional blocks that interact with each other through specific contact points
  - Blocks thought of as «black boxes» (even if they are inherently modeled in a distributed-parameter fashion) that we can interact with through specific interaction points
  - When modeled as PDEs, we can «lumpify» them (see slides on «source-based» modeling)
  - Interaction points correspond to ports of the equivalent electric circuit
- In the analog world, this is represented by a set of equations, that needs solving
  - Circuit simulation SW such as SPICE are specialized in solving such sets of equations
- Can we turn one such set of equation into a **computable signal flow**?

Getting inspiration from DWG theory

- A prototype of this idea was proposed in Digital WaveGuide (DWG) theory:
  - The fact that the general solution of the 1D d'Alambert equation is a pair of waves that are traveling undisturbed in opposite directions suggests some sort of «signal flow»

$$\ddot{v}(x,t) = c^2 v''(x,t) \Longrightarrow v(x,t) = v_r(ct-x) + v_l(ct+x)$$

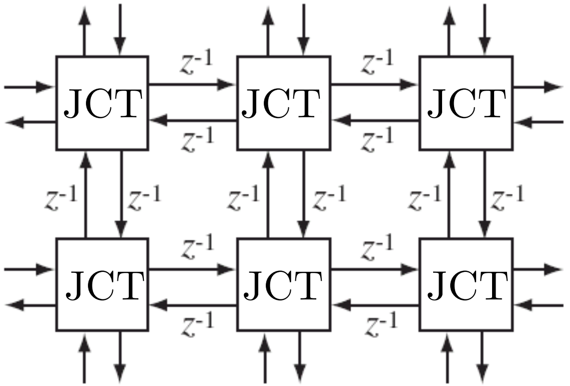
  - Pairs of (rightward-leftward) traveling waves turn out to be the result of a linear and invertible mapping from the corresponding Kirchooff (extensive-intensive) pair

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} \Longleftrightarrow \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{Z_0} & -\frac{1}{Z_0} \end{bmatrix}\begin{bmatrix} v_r \\ v_l \end{bmatrix}$$

  - The linear mapping depends on the characteristic impedance Z_0 of the physical structure that the PDE is modeling
  - We then discretize signals: $\begin{bmatrix} v_r \\ v_l \end{bmatrix} \xrightarrow{\text{discr}} \begin{bmatrix} v^+ \\ v^- \end{bmatrix}$
  - The resulting structure is a **signal flow**
    - Discontinuities in the characteristic impedance cause scattering, which is implemented as a 2-input 2-output block called Kelly-Lochbaum scattering cell
    - Interconnections btw multiple delay lines are modeled as multi-port junctions, which implement continuity equations (Kirchhoff laws) using a scattering matrix that depends on the characteristic impedances of the physical media that are connected together

Getting inspiration from DWG theory

- Junctions have a twofold function
  - Enforcing continuity equations
  - Implementing changes of the «reference frames» for the wave variables
- When building networks of delay lines the resulting signal flow is guaranteed to be computable (no loops without delay elements) because the scattering junctions are always connected together through at least one delay element
  - Delay elements express both temporal delay and spatial shifts, therefore their presence is guaranteed by the distributed-parameter nature of the physical model



Example of WGN modeling a 2D mesh

- Can we generalize this approach to define a computable signal flow in lumped-parameter model?
  - While in principle we can use similar definitions for waves in a circuit, we can no longer assume that scattering cell be separated from each other by a delay element (there is no spatial propagation), therefore computability problems arise
  - Unless we resort to other tricks, the signal flow that we construct through K2W mappings, will not be computable!

Wave Digital Filter principles

Wave Digital Filters (WDFs) are the “lumped” counterpart of DWGs. Originally devised to model a digital filter after an analog one, they actually offer an interesting approach to physical modeling, which is valid for any port-by-port interconnections of physical blocks

- We start from a reference electrical equivalent circuit
- For each port we formally define a linear K2W mapping, just as we did for DWGs
  - Each port mapping still depends on a Z_0, which can no longer to be interpreted as a characteristic impedance (there is no propagation to model). In fact, it now plays the role of a **free parameter**, sort of a «token» to be eventually spent for some specific purpose...
- We apply the K2W mappings to all variables of the circuit and discretize the individual blocks. In doing so, two main things happen:
  - Consitutive equations are replaced by input-output relationships
  - Topological interconnections (parallel, series, or other types...) are replaced by Junctions (scattering cells)
- The result is a generally non-computable «signal flow»
- We still have N «tokens» (free parameters) to spend (one per port) ➔ we can use such degrees of freedom in order to eliminate the loops with no delay elements, and render the signal flow computable

Wave variables

Let us consider an arbitrary circuit port, described by the Kirchhoff pair (v,i),

The corresponding wave pair is obtained by applying the linear K2W mapping

$$\begin{bmatrix} v^+ \\ v^- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & R \\ 1 & -R \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix}$$

Where «+» and «–» denote «incident» and «reflected» wave (*), respectively, and R is a free parameter called «reference resistance»

The inverse mapping is $$\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{R} & -\frac{1}{R} \end{bmatrix}\begin{bmatrix} v^+ \\ v^- \end{bmatrix}$$

In this context we interpret such mappings as a simple parametric change of reference frame

(*) the meaning of “incident” and “reflected” must be specified with reference to a specific object (a circuit element, a junction port, ...)

Wave variables

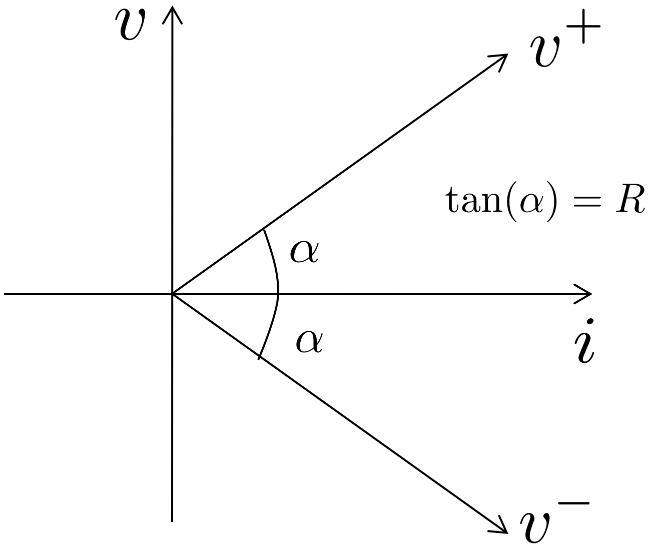
Let us consider an arbitrary circuit port, described by the Kirchhoff pair (v,i),

The corresponding wave pair is obtained by applying the linear K2W mapping

$$\begin{bmatrix} v^+ \\ v^- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & R \\ 1 & -R \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$$

$$\begin{aligned} v^+ \,\text{axis} &= \{(v,i) : v^- = 0\} \\ &= \{(v,i) : v = Ri\} \end{aligned}$$

$$\begin{aligned} v^- \,\text{axis} &= \{(v,i) : v^+ = 0\} \\ &= \{(v,i) : v = -Ri\} \end{aligned}$$



Resistors in the Wave Domain

- The resistor is an element described by a simple algebraic constitutive equation $v = R_0 i$

- Its representation in the wave domain is obtained by simply applying the K2W mappings to the constitutive equation

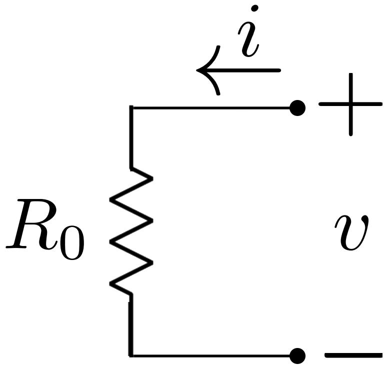
$$v = R_0 i \Longleftrightarrow v^+ + v^- = R_0 \frac{v^+ - v^-}{R}$$

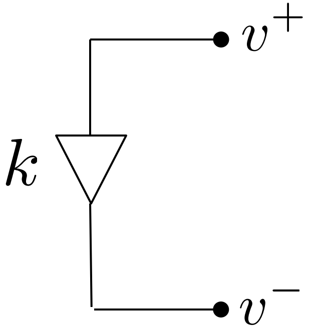
$$\Downarrow$$

$$v^- = k v^+ \qquad k = \frac{R_0 - R}{R_0 + R}$$

- We obtain a reflection coefficient that becomes zero when the reference resistance matches the resistor's impedance

$$R = R_0 \Rightarrow k = 0$$





Generic Impedances in the Wave Domain

- A generic impedance is described by an ODE, which can be compactly expressed in the Laplace domain as

$$V(s) = Z(s)I(s)$$

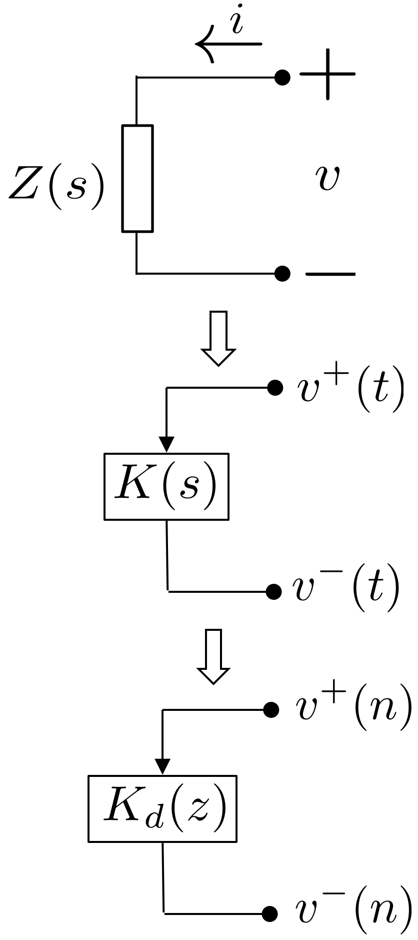
- Its representation in the wave domain is obtained by simply applying in the Laplace domain the K2W mappings to the constitutive equation

$$V(s) = Z(s)I(s) \Longleftrightarrow V^{+}(s) + V^{-}(s) = Z(s)\frac{V^{+}(s) - V^{-}(s)}{R}$$

$$\Downarrow$$

$$V(s)^{-} = K(s)V^{+}(s) \qquad K(s) = \frac{Z(s) - R}{Z(s) + R}$$

- We obtain a reflection filter that can be readily discretized to $K_\mathrm{d}(z)$
- Assuming that $K_\mathrm{d}(z)$ is causal and $k_\mathrm{d}(n)$ the corresponding impulse response, we can easily eliminate the instantaneous input-output dependency of the reflection filter ($k_\mathrm{d}(0)=0$) by setting $R=z_d(0)$, where $z_d(n)$ is the impulse response associated to $Z_d(s)$, discretization of $Z(s)$



Capacitor in the Wave Domain

- The constitutive equation of a capacitor is an ODE of the form

$$v = \frac{q}{C} \iff \dot{v} = \frac{i}{C} \iff sV(s) = \frac{1}{C} I(s)$$

$$\Downarrow$$

$$V(s) = Z(s) I(s)\,, \quad Z(s) = \frac{1}{sC}$$

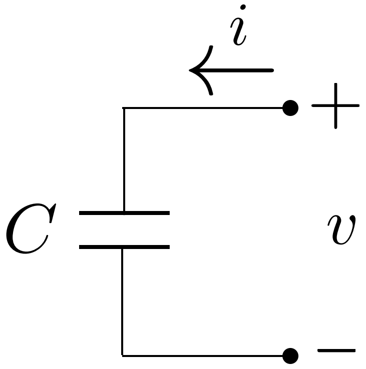
$$\Downarrow$$

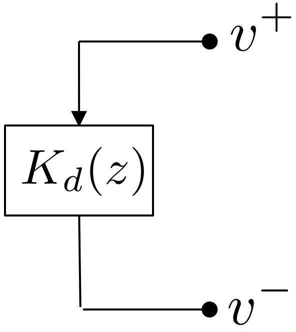
$$K(s) = \frac{Z(s) - R}{Z(s) + R} = \frac{\frac{1}{sC} - R}{\frac{1}{sC} + R}$$

$$\Downarrow$$

$$s = \frac{2}{T_s} \frac{1 - z^{-1}}{1 + z^{-1}} \Longrightarrow \boxed{K_d(z) = \frac{p + z^{-1}}{1 + p z^{-1}}\,, \quad p = \frac{T - 2RC}{T + 2RC} = \frac{T - 2\tau}{T + 2\tau}}$$

Allpass filter!

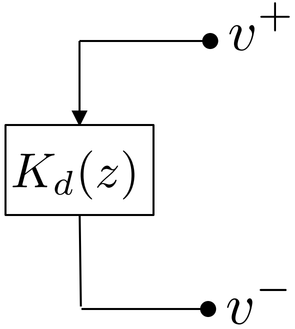




Capacitor in the Wave Domain

- The allpass filter representing the capacitor in the wave domain has an instantaneous I/O connection

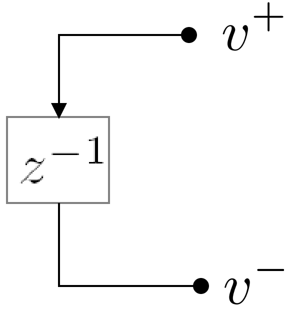
$$K_d(z) = \frac{p + z^{-1}}{1 + pz^{-1}}, \quad p = \frac{T - 2RC}{T + 2RC}$$



- Which can be eliminated by properly selecting the parameter *R*

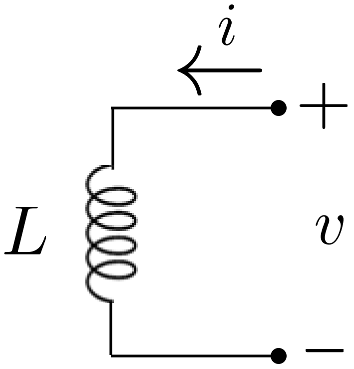
$$\Downarrow$$

$$2RC = T \Rightarrow \boxed{R = \frac{T}{2C}} \Longrightarrow \boxed{K_d(z) = z^{-1}}$$



Inductor in the Wave Domain

- The constitutive equation of an inductor is an ODE of the form



$$v = L\frac{\partial i}{\partial t}$$

$$\Downarrow$$

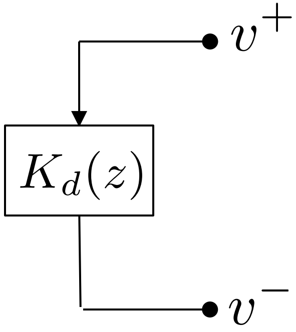
$$V(s) = Z(s)I(s)\,,\ \ Z(s) = sL$$

$$\Downarrow$$

$$K(s) = \frac{Z(s)-R}{Z(s)+R} = \frac{sL-R}{sL+R}$$

$$\Downarrow$$

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \Longrightarrow \boxed{K_d(z) = \frac{p-z^{-1}}{1-pz^{-1}}\,, \quad p = \frac{\frac{2L}{R}-T}{\frac{2L}{R}+T} = \frac{2\tau-T}{2\tau+T}}$$

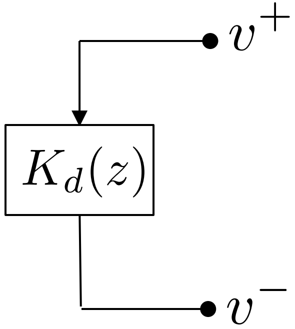


Allpass filter!

Inductor in the Wave Domain

- The allpass filter representing the capacitor in the wave domain has an instantaneous I/O connection

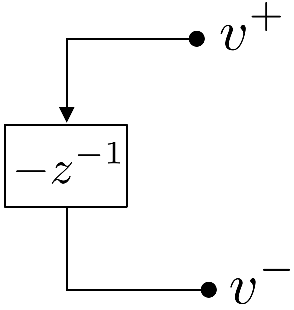
$$K_d(z) = \frac{p - z^{-1}}{1 - pz^{-1}}\,, \qquad p = \frac{\frac{2L}{R} - T}{\frac{2L}{R} + T}$$



- Which can be eliminated by properly selecting the parameter *R*

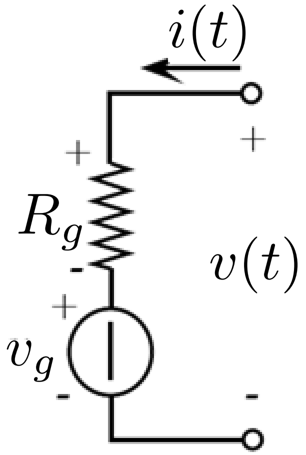
$$\Downarrow$$

$$\frac{2L}{R} = T \Rightarrow \boxed{R = \frac{2L}{T}} \Longrightarrow \boxed{K_d(z) = -z^{-1}}$$



Voltage Generator in the Wave Domain

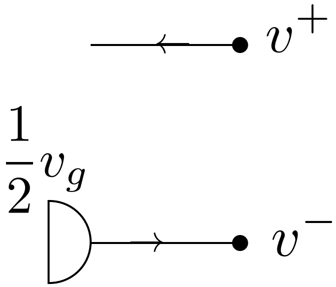
$$
\begin{aligned}
v^+ &= \frac{1}{2}(v + Ri) = \frac{1}{2}(v_g + R_g i + Ri) \\
v^- &= \frac{1}{2}(v - Ri) = \frac{1}{2}(v_g + R_g i - Ri)
\end{aligned}
$$



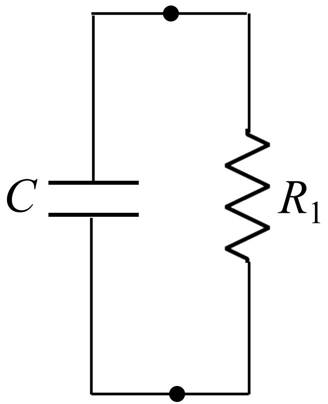
Adaptation:

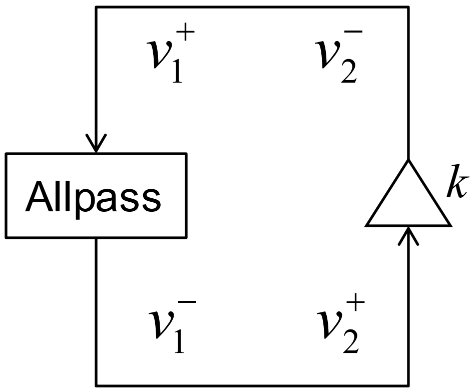
$$\boxed{R = R_g}$$

$$
\boxed{
\begin{aligned}
v^+ &= \frac{1}{2}v_g + \frac{1}{2}(R_g + R)i = \frac{1}{2}v_g + R_g i \\
v^- &= \frac{1}{2}v_g + \frac{1}{2}(R_g - R)i = \frac{1}{2}v_g \ ,
\end{aligned}
}
$$



Connecting two bipoles





$$v_1^+ = \frac{1}{2}(v_1 + Ri_1) = \frac{1}{2}(v_1 + Ri_2)$$
$$v_1^- = \frac{1}{2}(v_1 - Ri_1) = \frac{1}{2}(v_1 - Ri_2)$$

$$v_2^+ = \frac{1}{2}(v_2 + Ri_2)$$
$$v_2^- = \frac{1}{2}(v_2 - Ri_2)$$

$$v_2^- = v_1^+ \,, \quad v_1^- = v_2^+ \,, \quad i_1 = i_2$$

$$\text{Case 1)} \quad R = \frac{T}{2C} \Rightarrow p = 0 \,, \quad k = \frac{R_1 - \frac{T}{2C}}{R_1 + \frac{T}{2C}}$$

$$\text{Case 2)} \quad R = R_1 \Rightarrow p = \frac{R_1 - \frac{T}{2C}}{R_1 + \frac{T}{2C}} \,, \quad k = 0$$

Linking blocks

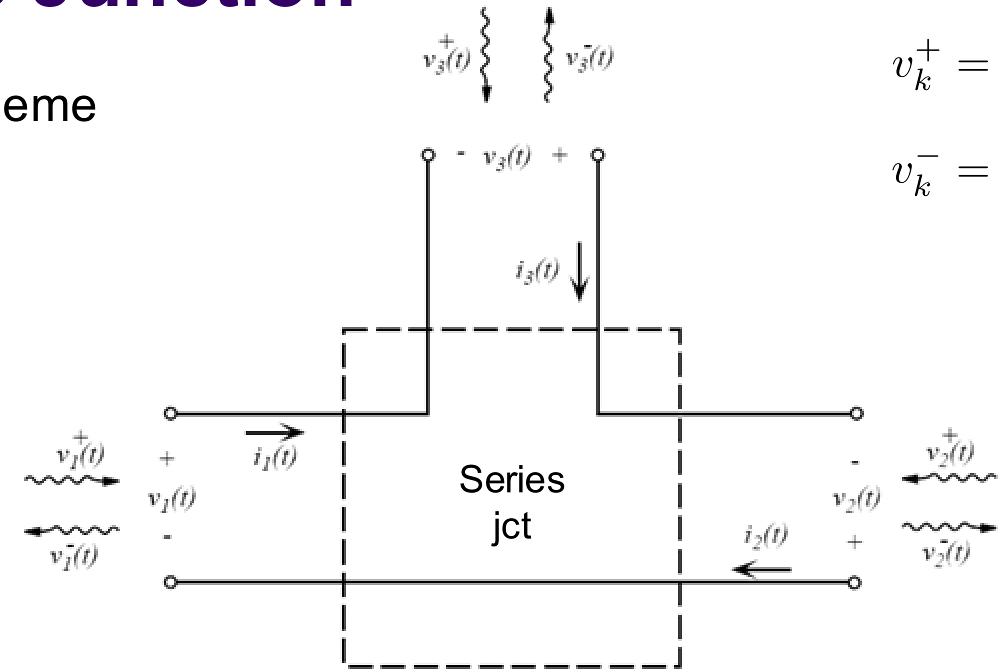
- What we know:
  - description of elementary blocks in the WD domain
  - how to choose the wave parameter R that makes a WD block (or a port) reflection-free (adaptation = no instantaneous reflection)
  - when a block is reflection free, we can connect it to a port that is NOT reflection-free (no computability problems)
  - blocks that exhibit an instantaneous reflection can only be connected to a reflection-free port
- What we don’t know:
  - How to connect more than two elementary blocks together...

Adaptors

- Junctions in DWGs are meant to implement topological interconnections (series or parallel) between different pairs of digital waves, but we cannot connect junctions together due to computability problems
- Can we exploit specific adaptation conditions that modify the junctions in such a way to allow us to connect them together without creating computability problems?
- Adaptors are special types of Junctions that exhibit one «adapted port», so that they can be connected with each other and form circuits. Just like Junctions, they:
  - Implement a change of reference impedance while preserving the numerical energy
  - Make sure that the interconnection btw wave blocks is computable
- We begin by deriving the scattering matrices of multiport junctions in a simplified form (with respect to the DWG case), and we will then turn them into adaptors

Series Junction

- Analog scheme



$$v_k^+ = \frac{1}{2}(v_k + R_k i_k)$$
$$v_k^- = \frac{1}{2}(v_k - R_k i_k)$$

Wave definitions

Continuity equations

$$v_1 + v_2 + v_3 = 0 \Longrightarrow (v_1^+ + v_1^-) + (v_2^+ + v_2^-) + (v_3^+ + v_3^-) = 0$$

$$i_1 = i_2 = i_3 \Longrightarrow \frac{v_1^+ - v_1^-}{R_1} = \frac{v_2^+ - v_2^-}{R_2} = \frac{v_3^+ - v_3^-}{R_3}$$

Series Junction

- If we write reflected waves (outputs) as a function of the incident waves (inputs) we obtain the scattering equation

$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \underbrace{\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{bmatrix} \right\}}_{\text{Scattering matrix}} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix}$$

- Where $\alpha_i = \dfrac{2R_i}{R_1 + R_2 + R_3}\,, \quad i = 1, 2, 3$

- Reflection coefficients: $1 - \alpha_i\,,\ i = 1, 2, 3$

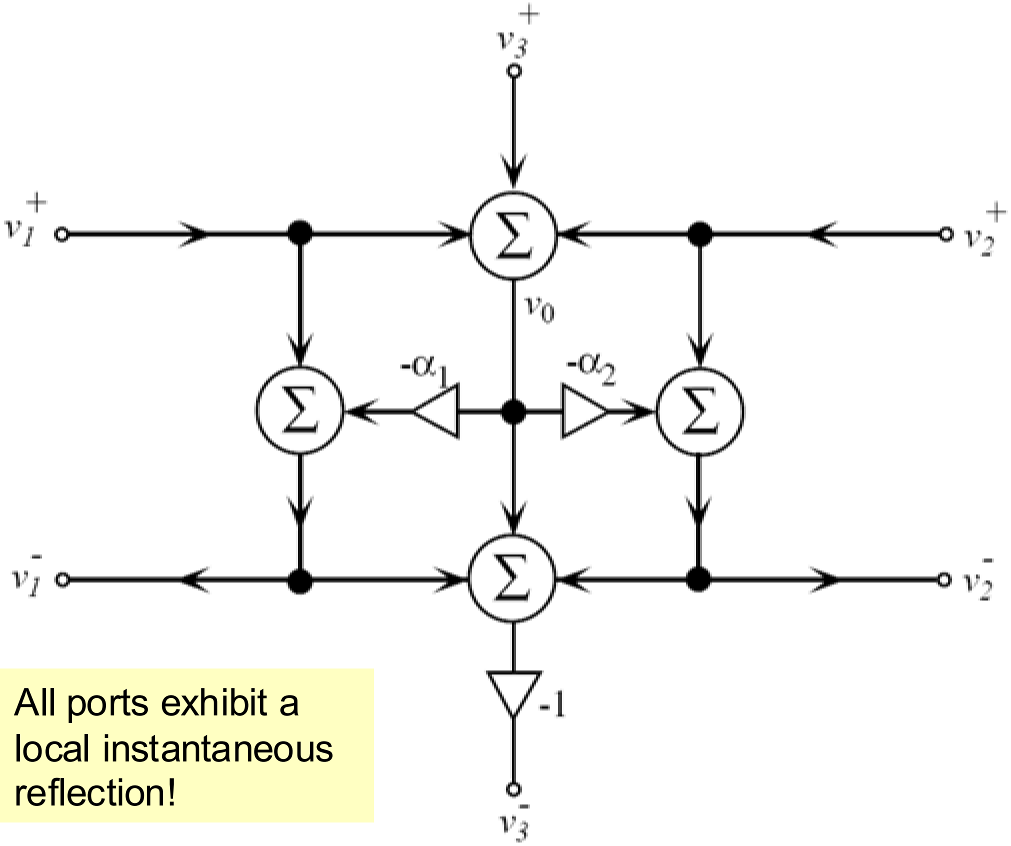
Series Junction

$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{bmatrix} \right\} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix} \qquad \alpha_i = \frac{2R_i}{R_1 + R_2 + R_3}, \quad i = 1, 2, 3$$

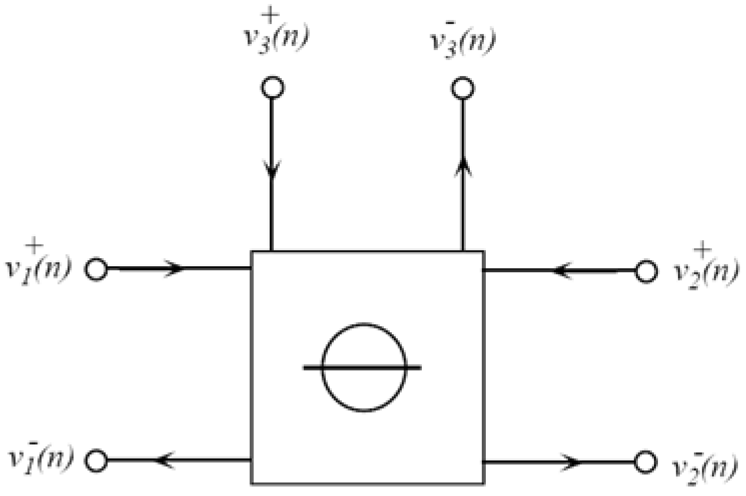
- The three ports are not independent of each other, as $\alpha_1 + \alpha_2 + \alpha_3 = 2$ therefore:

$$\begin{aligned} v_1^- &= v_1^+ - \alpha_1 v_0^+ , \quad v_0^+ = v_1^+ + v_2^+ + v_3^+ \\ v_2^- &= v_2^+ - \alpha_2 v_0^+ \\ v_3^- &= v_3^+ - \alpha_3 v_0^+ \\ &= v_3^+ - (2 - \alpha_1 - \alpha_2) v_0^+ \\ &= v_3^+ - 2v_0^+ + \alpha_1 v_0^+ + \alpha_2 v_0^+ \\ &= v_3^+ - 2v_0^+ + (v_1^+ - v_1^-) + (v_2^+ - v_2^-) \\ &= (v_1^+ + v_2^+ + v_3^+) - 2v_0^+ - v_1^- - v_2^- \\ &= v_0^+ - 2v_0^+ - v_1^- - v_2^- \\ &= -(v_0^+ + v_1^- + v_2^-) \end{aligned}$$

Series Junction



All ports exhibit a local instantaneous reflection!



$$v_1^- = v_1^+ - \alpha_1 v_0^+$$
$$v_2^- = v_2^+ - \alpha_2 v_0^+$$
$$v_3^- = -(v_0^+ + v_1^- + v_2^-)$$

$$v_0^+ = v_1^+ + v_2^+ + v_3^+$$

Turning a Series Junction Into an Adaptor

- The 3-port series junction is fully equivalent to the multiport series junction described in the case of DWGs for N=3
- As mentioned before, all ports exhibit an instantaneous reflection unless we add a constraint on the three reference port resistances
- For example, in order to have a zero reflection coefficient at port 3 we need

$$1-\alpha_3=0\,,\qquad \alpha_1+\alpha_2+\alpha_3=2 \Longleftrightarrow \alpha_2=1-\alpha_1$$

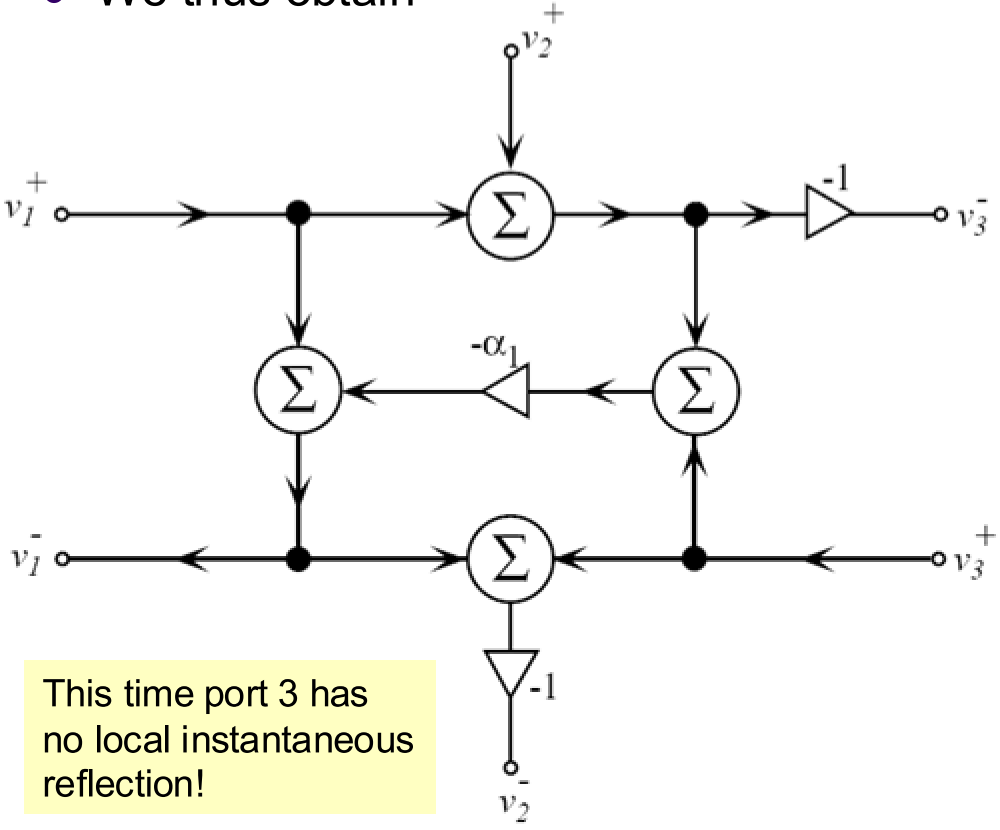
$$\Updownarrow$$

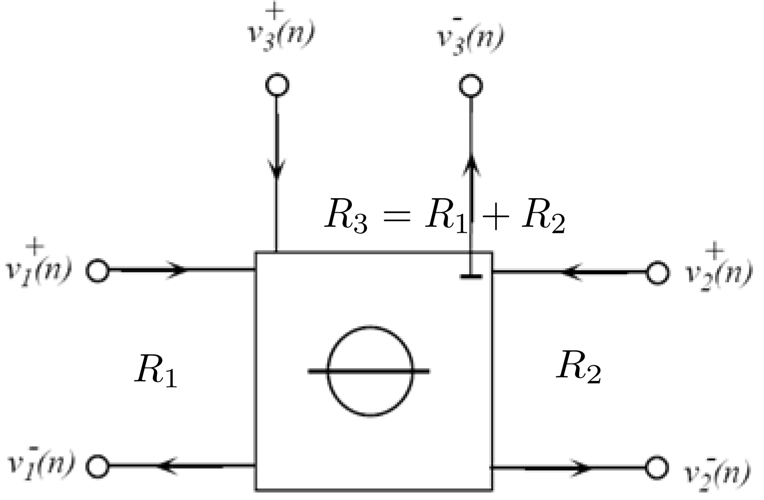
$$\boxed{R_3=R_1+R_2}$$

➔ In a series junction, choosing one of the port resistences to be equal to the series of the other two port resistances makes that port reflection free (adapted)!

Series Adaptor

- We thus obtain





$$\alpha_3 = 1 \quad \alpha_2 = 1 - \alpha_1$$

$$\Downarrow$$

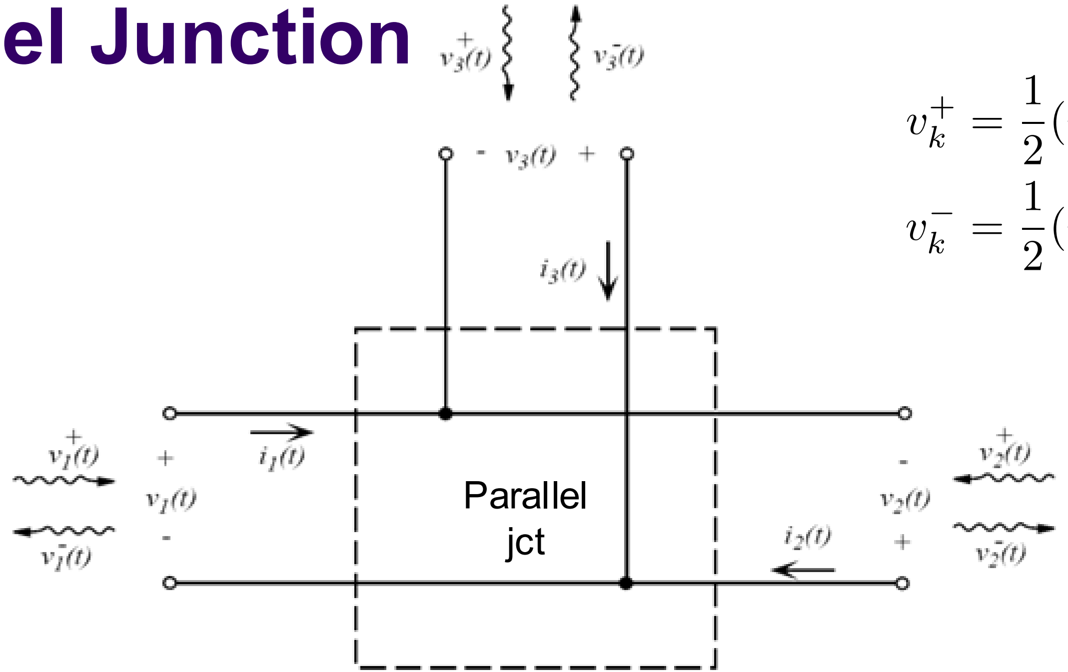
$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 \\ -\alpha_2 & 1-\alpha_2 & -\alpha_2 \\ -\alpha_3 & -\alpha_3 & 1-\alpha_3 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix}$$

$$= \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 \\ -1+\alpha_1 & \alpha_1 & -1+\alpha_1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix}$$

$$R_3 = R_1 + R_2$$

This time port 3 has no local instantaneous reflection!

Parallel Junction



$$v_k^+ = \frac{1}{2}(v_k + R_k i_k)$$
$$v_k^- = \frac{1}{2}(v_k - R_k i_k)$$

Wave definitions

$$G_k = 1/R_k$$

Continuity equations

$$i_1 + i_2 + i_3 = 0 \implies G_1(v_1^+ - v_1^-) + G_2(v_2^+ - v_2^-) + G_3(v_3^+ - v_3^-) = 0$$
$$v_1 = v_2 = v_3 \implies v_1^+ + v_1^- = v_2^+ + v_2^- = v_3^+ + v_3^-$$

Parallel Junction

- If we write reflected waves (outputs) as a function of the incident waves (inputs) we obtain the scattering equation

$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \underbrace{\left\{ \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}}_{\text{Scattering matrix}} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix}$$

- Where $\alpha_i = \dfrac{2G_i}{G_1 + G_2 + G_3}\,, \quad i = 1, 2, 3$

- Reflection coefficients: $\alpha_i - 1\,, \quad i = 1, 2, 3$

Parallel Junction

$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \left\{ \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix} \qquad \alpha_i = \frac{2G_i}{G_1 + G_2 + G_3}, \qquad i = 1, 2, 3$$

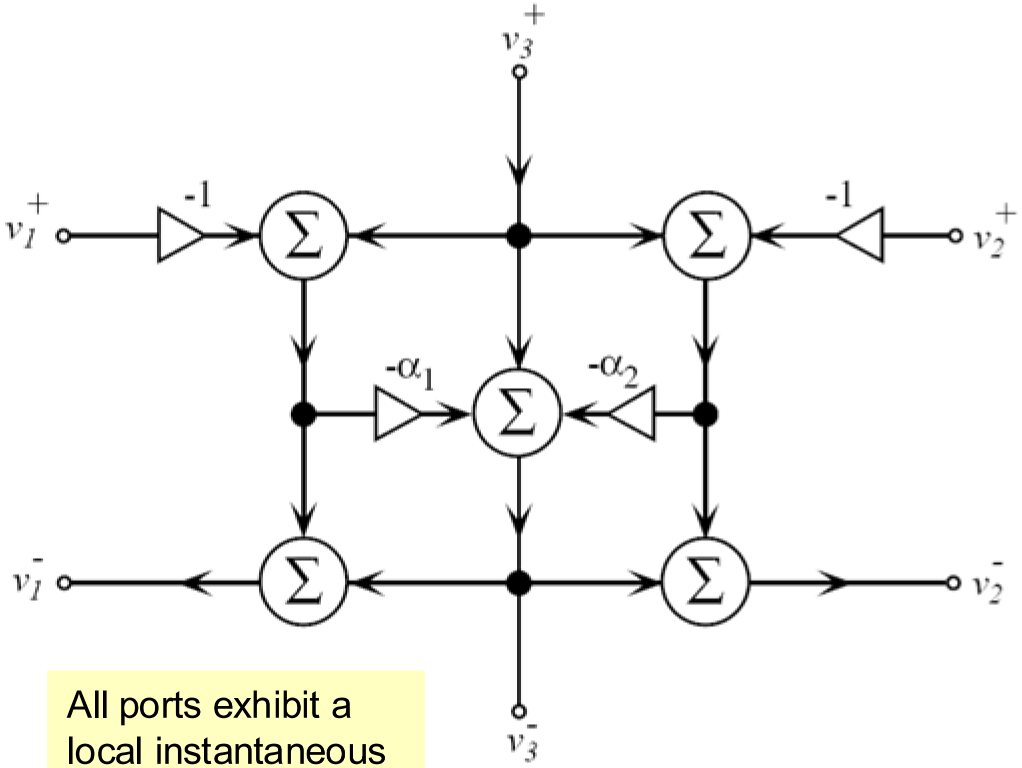
- Once again, the three ports are not independent of each other, as $\alpha_1 + \alpha_2 + \alpha_3 = 2$ therefore:

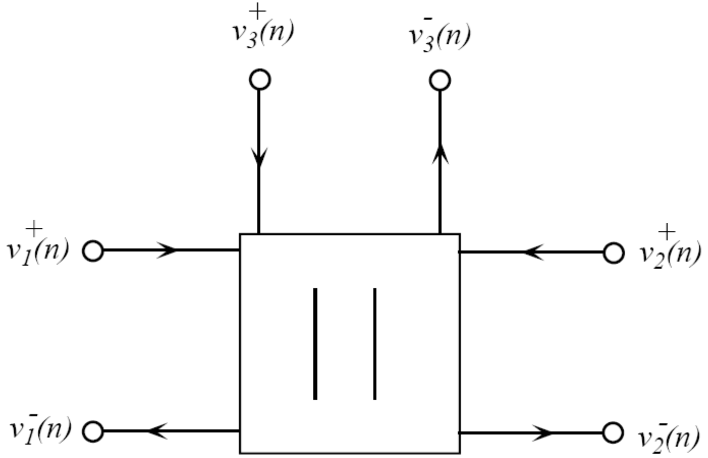
$$\begin{aligned} v_1^- &= \sum_{k=1}^{3} \alpha_k v_k^+ - v_1^+ \\ v_2^- &= \sum_{k=1}^{3} \alpha_k v_k^+ - v_2^+ \\ v_3^- &= \sum_{k=1}^{3} \alpha_k v_k^+ - v_1^+ \end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned} v_1^- &= v_3^+ + (v_3^- - v_1^+) \\ v_2^- &= v_3^+ + (v_3^- - v_2^+) \\ v_3^- &= v_3^+ - \alpha_1 (v_3^+ - v_1^-) - \alpha_2 (v_3^+ - v_2^-) \end{aligned}$$

Parallel Junction





$$
\begin{aligned}
v_1^- &= v_3^+ + (v_3^- - v_1^+) \\
v_2^- &= v_3^+ + (v_3^- - v_2^+) \\
v_3^- &= v_3^+ - \alpha_1(v_3^+ - v_1^-) - \alpha_2(v_3^+ - v_2^-)
\end{aligned}
$$

All ports exhibit a local instantaneous reflection!

Turning a Parallel Junction Into an Adaptor

- The 3-port parallel junction is fully equivalent to the multiport parallel junction described in the case of DWGs for N=3
- As mentioned before, all ports exhibit an instantaneous reflection unless we add a constraint on the three reference port resistances
- For example, in order to have a zero-reflection coefficient at port 3 we need

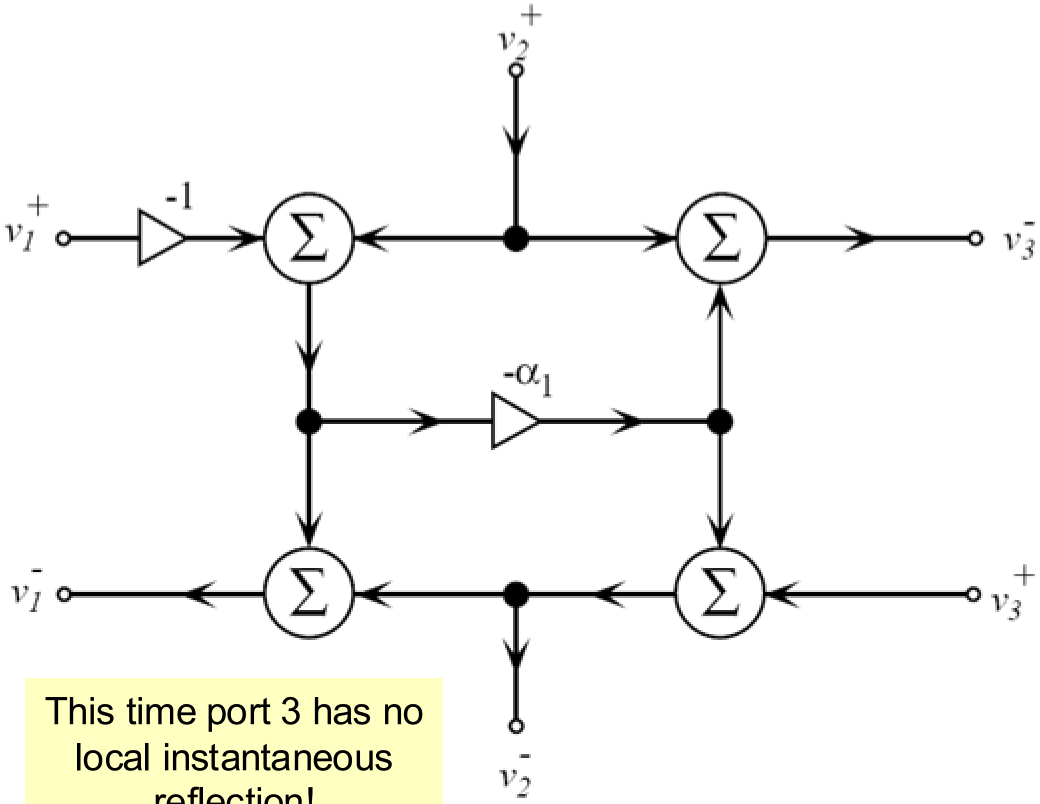
$$1 - \alpha_3 = 0\,, \qquad \alpha_1 + \alpha_2 + \alpha_3 = 2 \Longleftrightarrow \alpha_2 = 1 - \alpha_1$$

$$\Updownarrow$$

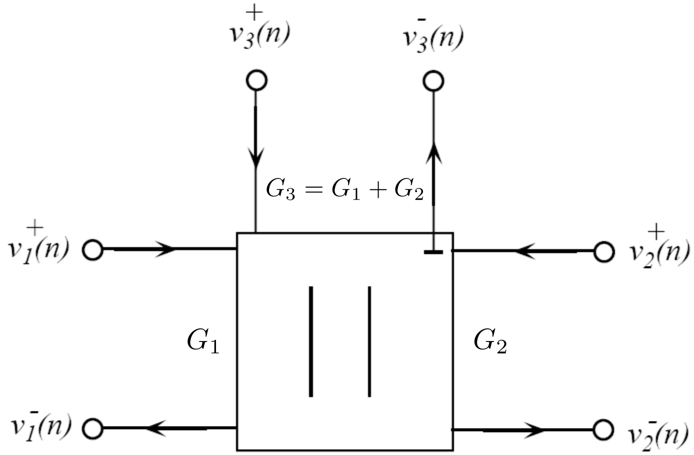
$$\boxed{G_3 = G_1 + G_2}$$

➔ In a series junction, choosing one of the port resistences to be equal to the parallel of the other two port resistances makes that port reflection free (adapted)!

Parallel adaptor



This time port 3 has no local instantaneous reflection!



$$\alpha_3 = 1 \quad \alpha_2 = 1 - \alpha_1$$

$$\Downarrow$$

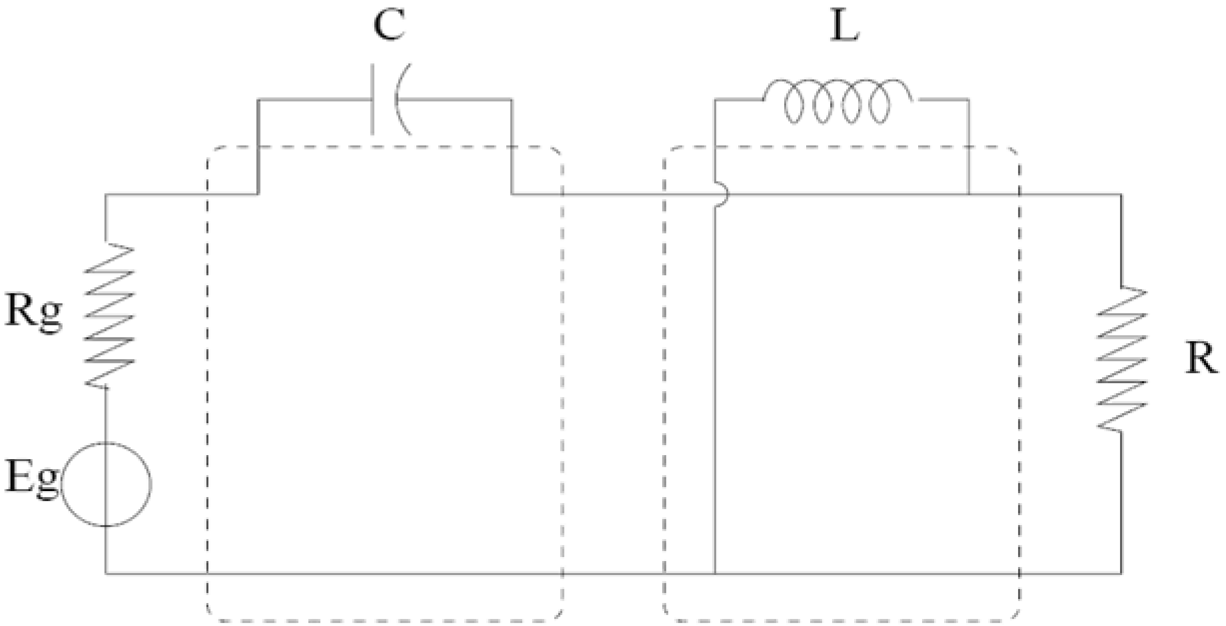
$$\begin{bmatrix} v_1^- \\ v_2^- \\ v_3^- \end{bmatrix} = \begin{bmatrix} \alpha_1 - 1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 - 1 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 - 1 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 - 1 & 1 - \alpha_1 & 1 \\ \alpha_1 & -\alpha_1 & 1 \\ \alpha_1 & 1 - \alpha_1 & 0 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \\ v_3^+ \end{bmatrix}$$

$$G_3 = G_1 + G_2\,,\ R_3 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

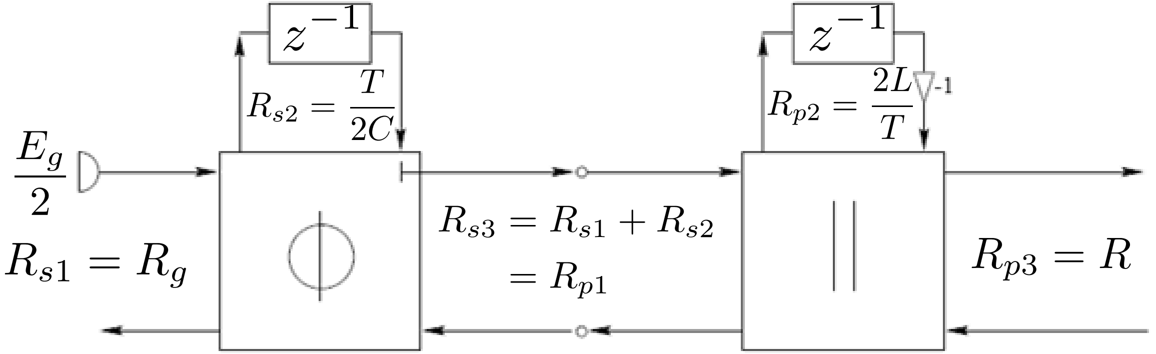
Modeling an Analog Circuit

- let us apply WDF principles to the physical modeling of a circuit, which we conveniently rearrange in order to emphasize the topological interconnection of series and parallel adaptors

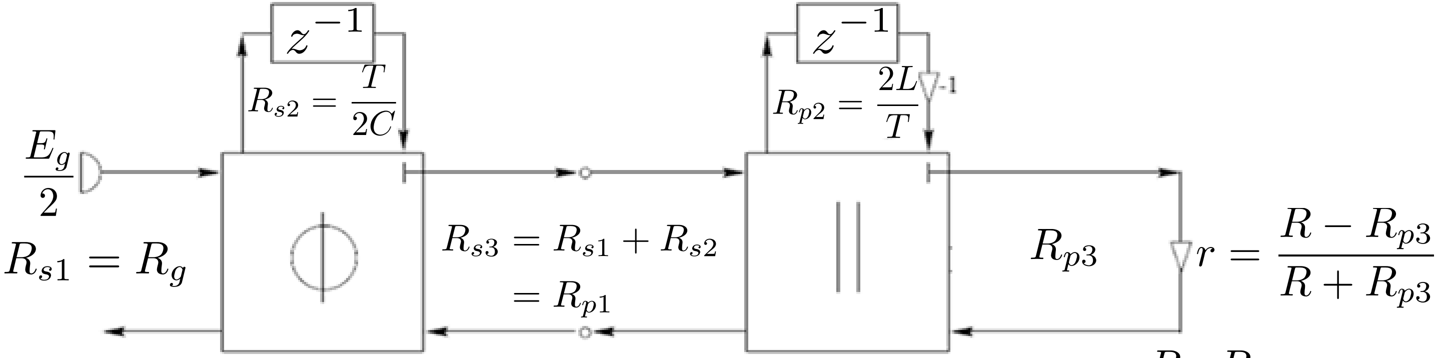


Two possible WD implementations

Adapting the resistor



Adapting the port facing the resistor



$$R_{p3} = \frac{R_{p1}R_{p2}}{R_{p1} + R_{p2}}$$

Remarks

- Underlying assumptions
  - Simple and tree-like interconnections (no funny “triconnected” elements)
  - Linear circuit: is it strictly true that we need this assumption?
- Extensions
  - Additional elements can be modeled in the WD domain (see Fettweis’ article*)
    - Transformers, Nullors, gyrators, ecc.
  - A single nonlinearity can be accommodated and modeled in the WD
    - After using up all degrees of freedom for breaking all non-computable loops, we are left with a single adapted port, which can be used for connecting a NL element
  - Further generalizations have been brought by PoliMI to the WDF theory, which makes it general enough to model arbitrary circuits
    - Digital Waves “with memory” and algebraic nonlinearities
    - Bi-parametric waves
    - Integration of graph theory and WDF theory to accommodate more general topologies
    - Vector-WDFs and nonlinear multiport elements
    - SIM method
    - ...

A. Fettweis, “Wave Digital Filters: Theory and Practice“, Proceesings of the IEEE, Vol. 74, No.2, Feb. 1986, pp. 270-329

Modeling NL elements in the WD domain

- A general class of instantaneous nonlinearities is defined by an algebraic I/O relationship btw across and through variables. In the case of an n-port NLE we have:

$$\boxed{F_k(v_1, \ldots, v_n; i_1, \ldots, i_n) = 0\,, \quad k = 1, \ldots, n}$$

- Single-port case $F(v, i) = 0$
  - Voltage-controlled resistor $i = i(v)$
  - Current-controlled resistor $v = v(i)$

Modeling a NLE in the WD

Recalling the definition of waves

$$v = v^+ + v^- \qquad i = \frac{v^+ + v^-}{R}$$

$$\Downarrow$$

We can find the WD equivalent of the NLE

$$F(v, i) = 0 \Rightarrow F\left(v^+ + v^-, \frac{v^+ - v^-}{R}\right) = f(v^+, v^-) = 0$$

We need to find the conditions that allow to write the NLE in explicit form in the WD domain

$$v^- = g(v^+)$$

⌦ Supplemental slide
Not part of the program

Explicitability of NLEs in the WD domain

- **Implicit function theorem**

Let $f(v_0^+, v_0^-)$ be a smooth function.

If (v_0^+, v_0^-) lies on the NL characteristics, i.e. $f(v_0^+, v_0^-) = 0$

$$\text{then } \left.\frac{\partial f}{\partial v^-}\right|_{(v_0^+, v_0^-)} \neq 0 \Longrightarrow \exists g : f(v_0^+, g(v_0^+)) = 0 \text{ around } v_0^+$$

If this condition holds true for all the pts of the characteristics, then we have global explicitability

Current-Controlled Resistor

⌦ Supplemental slide
Not part of the program

$$F(v,i) = v - v(i) = 0$$

$$f(v^+,v^-) = v^+ + v^- - v\left(\frac{v^+ - v^-}{R}\right) = 0$$

$$\begin{aligned}
\frac{\partial f}{\partial v^-} &= \frac{\partial}{\partial v^-}\left\{v^+ + v^- - v\left(\frac{v^+ - v^-}{R}\right)\right\} \\
&= 1 - \frac{\partial v}{\partial i}\frac{\partial i}{\partial v^-} \\
&= 1 - \frac{\partial v}{\partial i}\frac{\partial}{\partial v^-}\left\{\frac{v^+ - v^-}{R}\right\} \\
&= 1 + \frac{1}{R}v', \quad \text{where} \quad v'\left(\frac{v^+ - v^-}{R}\right) = \left.\frac{\partial v(i)}{\partial i}\right|_{i=\frac{v^+ - v^-}{R}}
\end{aligned}$$

Therefore global explicitability is guaranteed when $\boxed{v'(i) \neq -R}$

Voltage-Controlled Resistor

⌦ Supplemental slide
Not part of the program

$$F(v, i) = i - i(v)$$

$$f(v^+, v^-) = v^+ - v^- - Ri\left(v^+ + v^-\right) = 0$$

$$\frac{\partial f}{\partial v^-} = -1 - Ri' \qquad i'\left(v^+ + v^-\right) = \left.\frac{\partial i(v)}{\partial v}\right|_{v=v^+ + v^-}$$

- Therefore global explicitability is guaranteed when $\boxed{i'(v) \neq -\frac{1}{R}}$

Discontinuous NLE

⌦ Supplemental slide
Not part of the program

- Voltage-controlled resistor
  - In this case explicitability of v^{-} is guaranteed if $i = i(v)$

$$\boxed{\inf_{v_2 \neq v_1} \frac{i(v_2) - i(v_1)}{v_2 - v_1} > -\frac{1}{R}}$$

  - or

$$\boxed{\sup_{v_2 \neq v_1} \frac{i(v_2) - i(v_1)}{v_2 - v_1} < -\frac{1}{R}}$$

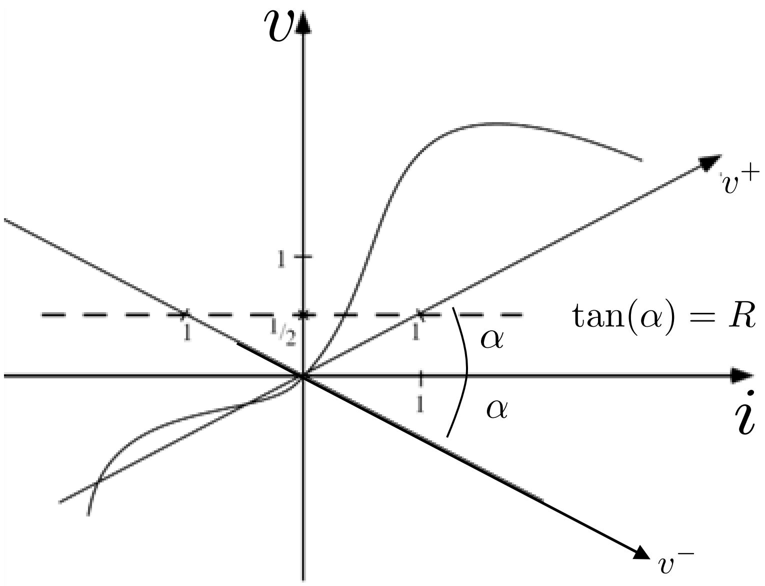
Graphical method

- We start from the NL characteristics $F(v,i)=0$
  on the K plane (*v* , *i*) and we apply the change of variables

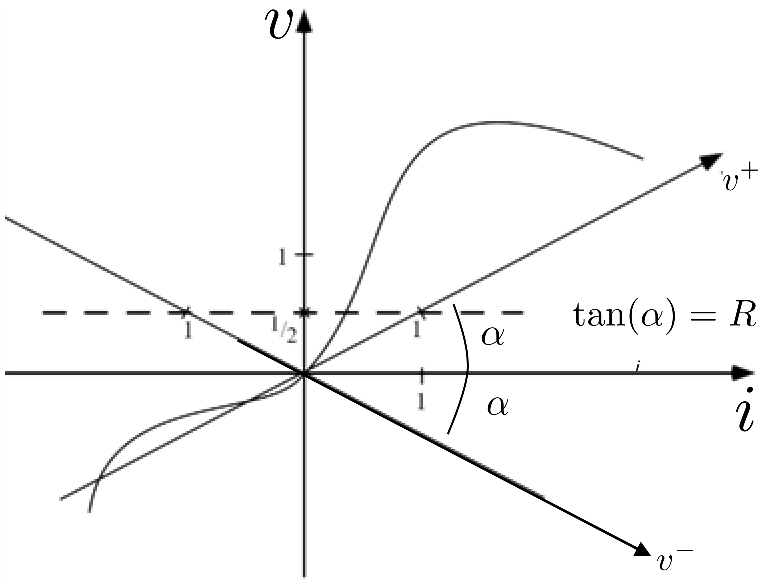
$$\begin{bmatrix} v^+ \\ v^- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & R \\ 1 & -R \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix}$$

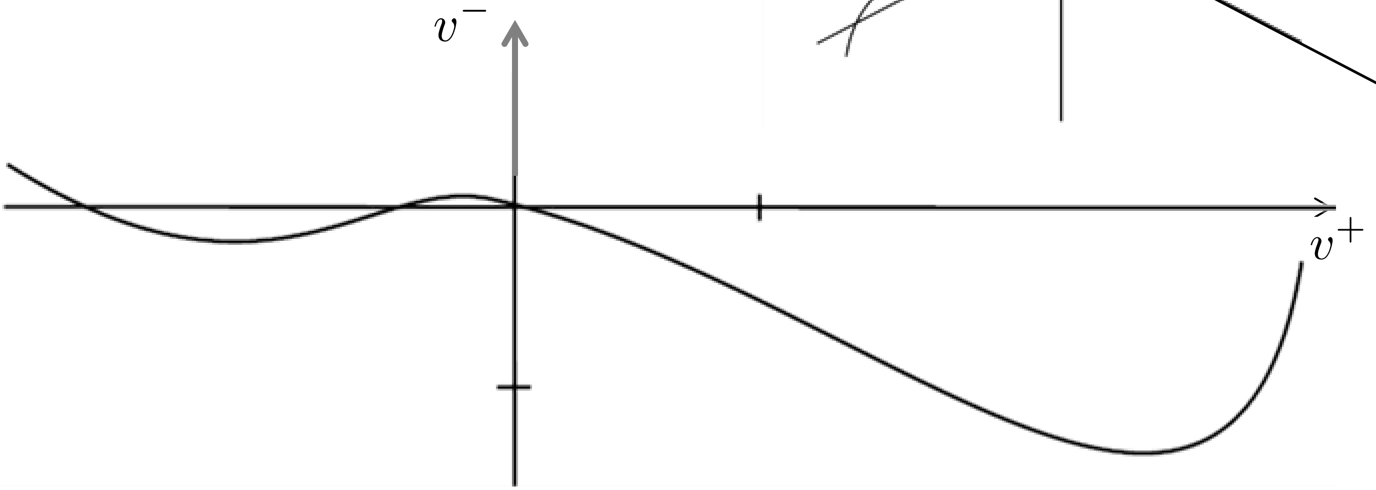
$$\begin{aligned} v^+ \text{ axis} &= \{(v,i) : v^- = 0\} \\ &= \{(v,i) : v = Ri\} \end{aligned}$$

$$\begin{aligned} v^- \text{ axis} &= \{(v,i) : v^+ = 0\} \\ &= \{(v,i) : v = -Ri\} \end{aligned}$$

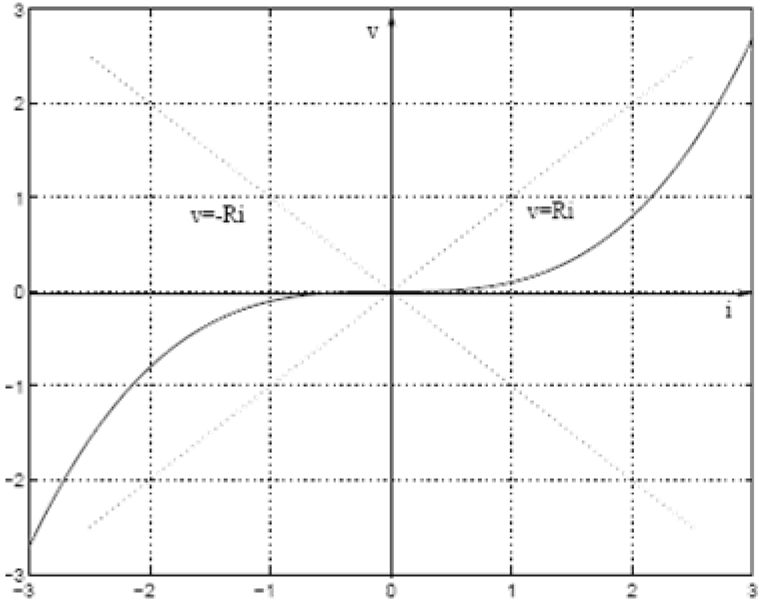


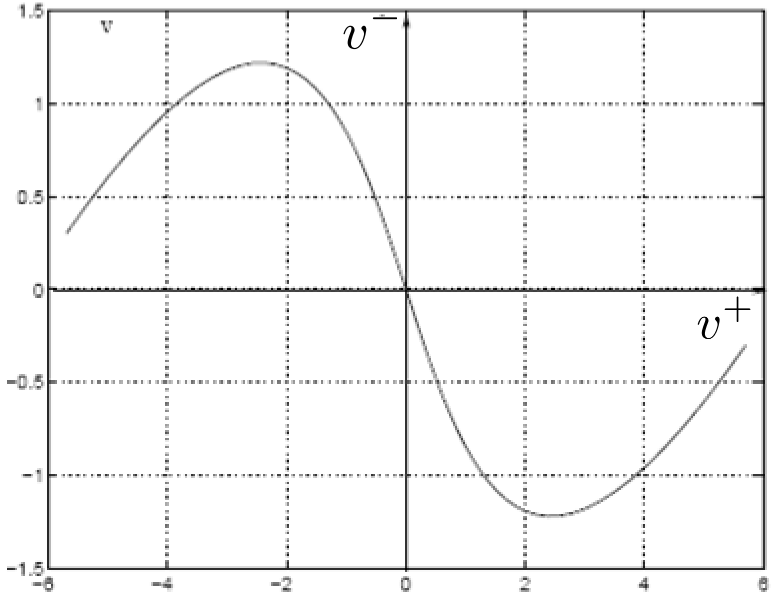
Graphical method



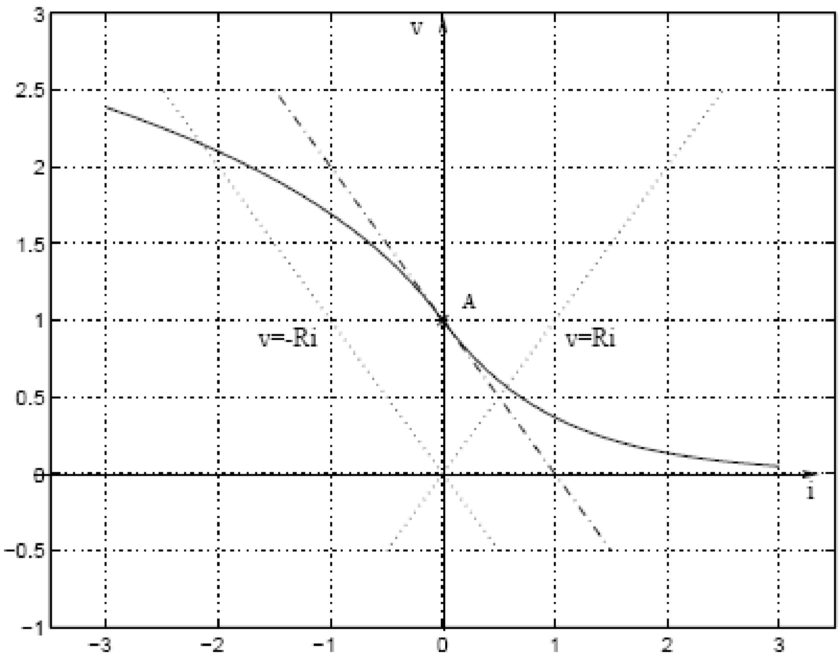


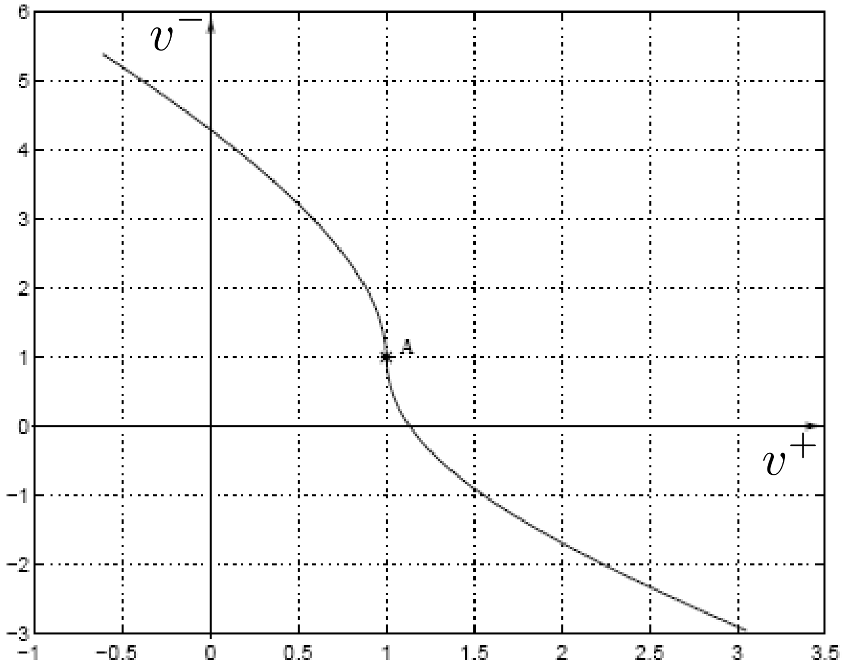
Graphical method: example



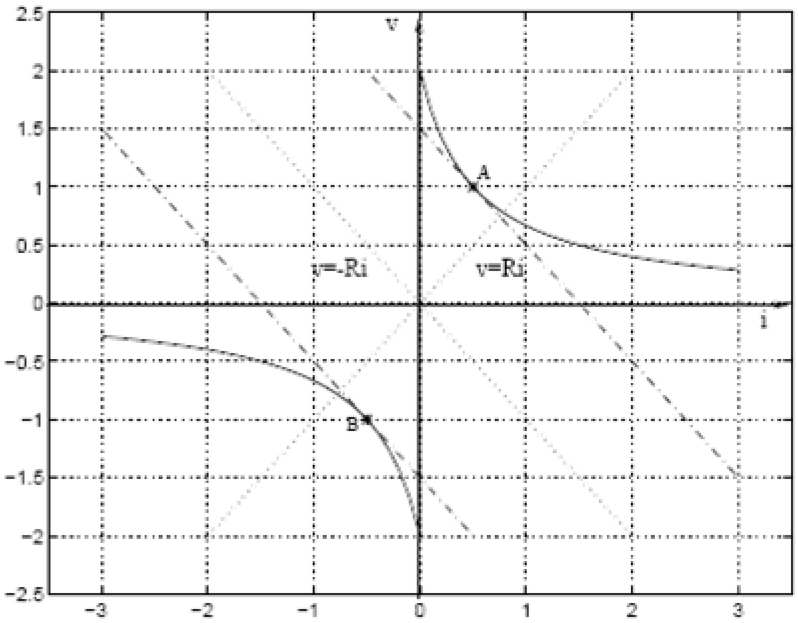


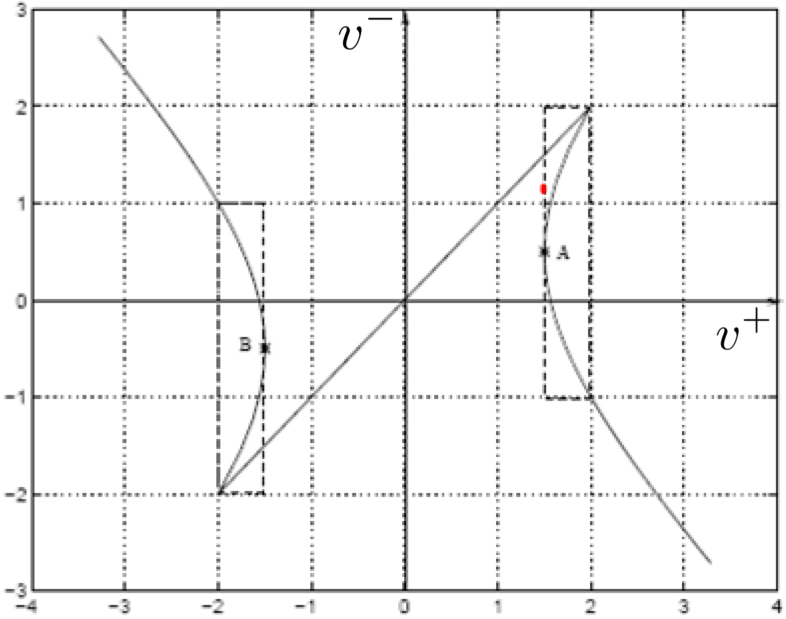
Graphical method: example



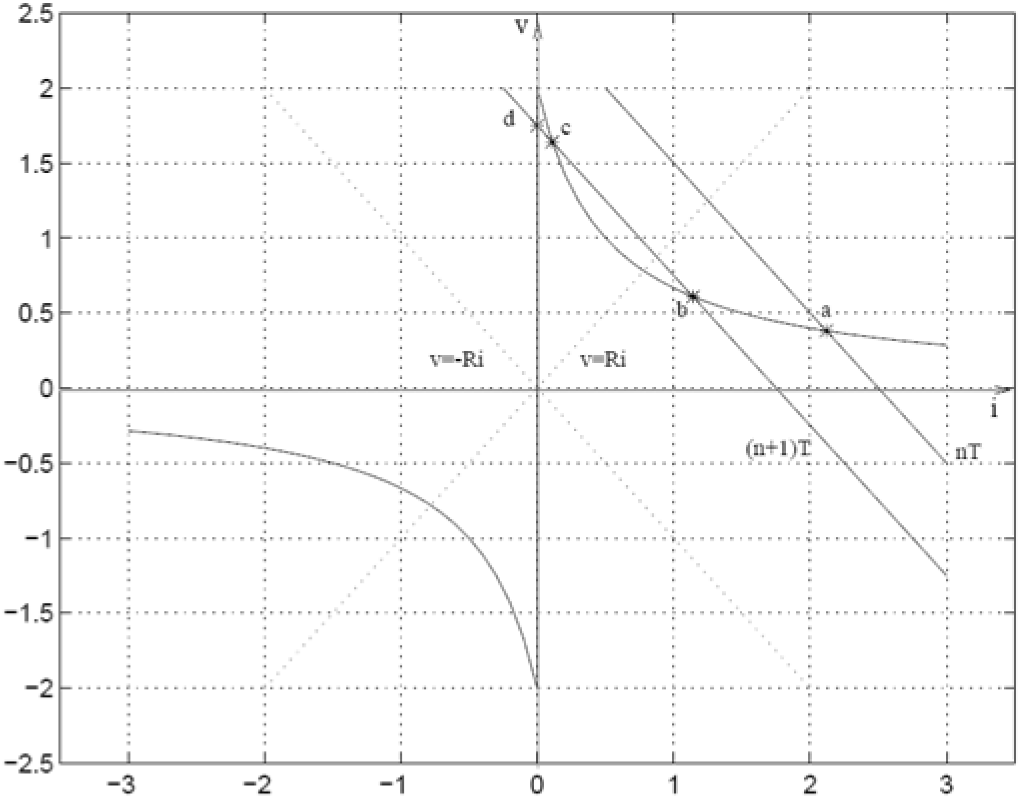


Graphical method: example





Non-explicitable NL characteristics



Reactive elements: NL capacitor

- A general definition of the NL capacitor is an algebraic relationship btw across variables (voltages, forces, pressures) and the integral of through variables (charge, displacement)

$$\boxed{F_k(v_1, \ldots, v_n; q_1, \ldots, q_n) = 0 \,, \quad k = 1, \ldots, n}$$

therefore it corresponds to an ordinary differential equation

Reactive elements: NL capacitor

- 1-port case $F(v, q) = 0$
  - Current-controlled: explicitable w.r.t. q $\quad v = v(q)$
  - Voltage-controlled: explicitable w.r.t. v $\quad q = q(v)$

$$\boxed{i = \dot{q} = \frac{\partial q}{\partial v}\frac{\partial v}{\partial t} = C(v)\dot{v}}$$

$$C(v) \stackrel{\text{def}}{=} q'(v)$$

Voltage-controlled capacity

Cannot be implemented directly in the WD domain because a NL operator and a L-tf cannot be swapped in order

Reactive elements: NL capacitor

- We overcome this difficulty by defining two special waves that allow us to treat NLE with memory as if they were memoryless (resistive)

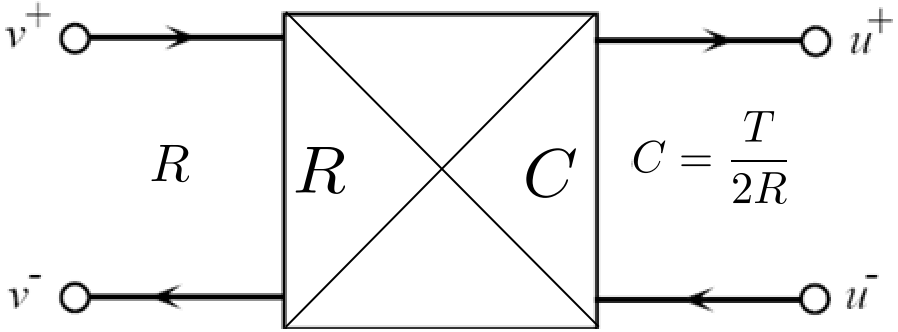
$$\boxed{u^{+} = \frac{1}{2}\left(v + \frac{q}{C}\right) \qquad u^{-} = \frac{1}{2}\left(v - \frac{q}{C}\right)}$$

$$\boxed{v = u^{+} + u^{-} \qquad q = C(u^{+} - u^{-})}$$

C is a “reference” capacity

Reactive elements: NL capacitor

- As the NL capacitor is an algebraic relationship btw *v* and *q*, we can use all results we found for NL resistors as long as we work in the WD domain of u^+ and u^-
- We need a WD tf that maps (v^+, v^-) onto (u^+, u^-)
  - We call this tf “across” integration
- Under proper conditions of computability, in the new domain the problem becomes that of finding an appropriate signal-dependent reflection coefficient



WD integration (across)

$$v = v^+ + v^- \qquad i = \frac{v^+ - v^-}{R} \qquad \dot{q} = i$$

$$\Downarrow$$

$$u^+ + u^- = v^+ + v^- \qquad C\left(\dot{u}^+ - \dot{u}^-\right) = \frac{v^+ - v^-}{R}$$

Apply Laplace tf with $\tau \stackrel{\text{def}}{=} RC$

$$\begin{aligned} -V^- + U^+ &= V^+ - U^- \\ V^- + \tau s U^+ &= V^+ + \tau s U^- \end{aligned}$$

WD integration (across)

$$\begin{aligned} V^- &= \frac{1-\tau s}{1+\tau s} V^+ + \frac{2\tau s}{1+\tau s} U^- \\ U^+ &= \frac{2}{1+\tau s} V^+ - \frac{1-\tau s}{1+\tau s} U^- \end{aligned}$$

$$H(s) \stackrel{\text{def}}{=} \frac{1-\tau s}{1+\tau s} \qquad 1 + H(s) = \frac{2}{1+\tau s} \qquad 1 - H(s) = \frac{2\tau s}{1+\tau s}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} V^- &= H(s)V^+ + (1 - H(s))U^- \\ U^+ &= (1 + H(s))V^+ - H(s)U^- \end{aligned}}$$

WD integration (across)

$$s \longrightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$\boxed{\overline{H}(z) \stackrel{\text{def}}{=} H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{p+z^{-1}}{1+pz^{-1}}}$$

$$p \stackrel{\text{def}}{=} \frac{T-2\tau}{T+2\tau}$$

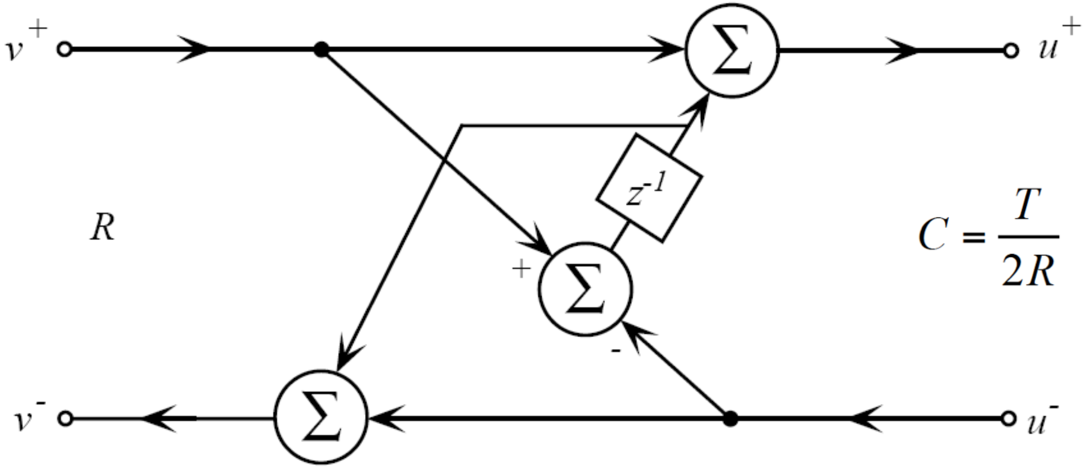
➔ We thus obtain something similar to a Kelly-Lochbaum scattering cell whose reflection coefficient is now an allpass filter

➔ Notice that the output of the filter depends instantaneously on its input unless p=0. In order to avoid computability problems we thus have to choose R in relation to C in such a way to obtain p=0

Computability

- p tells us how much T differs from 2τ
- In order to obtain $2\tau = T$ we need to have $RC = T/2$, which results in

$$\overline{H}(z) = z^{-1}$$



Implementation

- At this point we can connect the second port of the WD integrator (mutator) to a NL reflection function $u^- = u^-(u^+)$ derived from $f(v,q) = 0$
- When the capacitor is linear we find again the results proposed by Fettweis

$$C = \overline{C} \qquad\qquad R = \frac{T}{2C}$$

in this case *p*=0 and allpass + mutator combination becomes a simple delay element

Reactive elements: NL inductor

- Like before, a general definition of the NL inductor is an algebraic relationship btw the integral of across variables (voltage, force, pressure) and of through variables (current, velocity, flow)

$$\boxed{F_k(\varphi_1, \ldots, \varphi_n; i_1, \ldots, i_n) = 0 \,, \quad k = 1, \ldots, n}$$

therefore it corresponds to a differential eq.

Reactive elements: NL inductor

- 1-port case $F(\varphi, i) = 0$
  - Current-controlled: explicitable w.r.t. q $\varphi = \varphi(i)$
  - Voltage-controlled: explicitable w.r.t. v $i = i(\varphi)$

$$\boxed{v = \dot{\varphi} = \frac{\partial \varphi}{\partial i}\frac{\partial i}{\partial t} = L(i)\frac{\partial i}{\partial t}}$$

$$L(i) \stackrel{\text{def}}{=} \varphi'(i)$$

Current-controlled inductance

Cannot be implemented directly in the WD domain because a NL operator and a L-tf cannot be swapped in order

Reactive elements: NL inductor

- We overcome this difficulty by defining two special waves that allow us to treat NLE with memory as if they were memoryless (resistive)

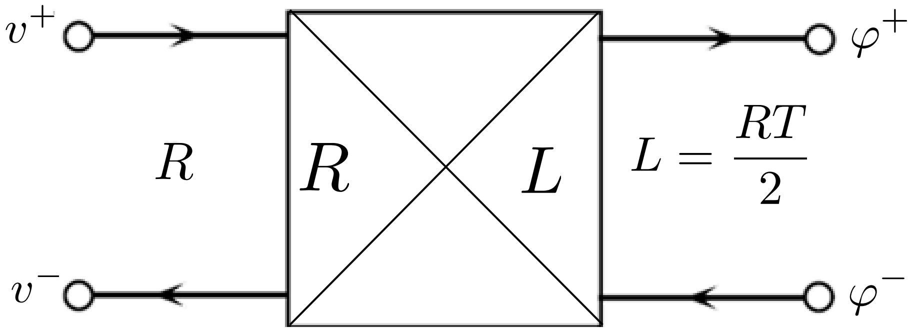
$$\boxed{\varphi^+ = \frac{1}{2}(\varphi + Li) \qquad \varphi^- = \frac{1}{2}(\varphi - Li)}$$

$$\boxed{\varphi = \varphi^+ + \varphi^- \qquad i = \frac{\varphi^+ - \varphi^-}{L}}$$

L is a “reference” inductance

Reactive elements: NL inductor

- As the NL inductor is an algebraic relationship btw φ and i, we can use all results we found for NL resistors as long as we work in the WD domain of φ^+ and φ^-
- We need a WD tf that maps (v^+, v^-) onto (φ^+, φ^-)
    - We call this tf “through” integration
- Under proper conditions of computability, in the new domain the problem becomes that of finding an appropriate signal-dependent reflection coefficient



WD integration (through)

$$v = v^+ + v^- \qquad i = \frac{v^+ - v^-}{R} \qquad \dot{\varphi} = v$$

$$\Downarrow$$

$$v^+ + v^- = \dot{\varphi}^+ + \dot{\varphi}^- \qquad \frac{\varphi^+ - \varphi^-}{L} = \frac{v^+ - v^-}{R}$$

Apply Laplace tf with $\tau \stackrel{\text{def}}{=} L/R$

$$\begin{aligned} V^- - s\Phi^+ &= -V^+ + s\Phi^- \\ \tau V^- + \Phi^+ &= \tau V^+ + \Phi^- \end{aligned}$$

WD integration (through)

$$
\begin{aligned}
V^- &= -\frac{1-\tau s}{1+\tau s}V^+ + \frac{2s}{1+\tau s}\Phi^- \\
\Phi^+ &= \frac{2\tau}{1+\tau s}V^+ + \frac{1-\tau s}{1+\tau s}\Phi^-
\end{aligned}
$$

$$H(s) \stackrel{\text{def}}{=} \frac{1-\tau s}{1+\tau s} \qquad 1+H(s) = \frac{2}{1+\tau s} \qquad 1-H(s) = \frac{2\tau s}{1+\tau s}$$

$$
\boxed{
\begin{aligned}
V^- &= -H(s)V^+ + \frac{1}{\tau}\left[1-H(s)\right]\Phi^- \\
\Phi^+ &= \tau\left[1+H(s)\right]V^+ + H(s)\Phi^-
\end{aligned}
}
$$

WD integration (through)

- Notice that, if we define

$$W^- \stackrel{\text{def}}{=} \tau V^- \qquad W^+ \stackrel{\text{def}}{=} \tau V^+$$

we obtain

$$\boxed{\begin{aligned} W^- &= -H(s)W^+ + [1 - H(s)]\,\Phi^- \\ \Phi^+ &= [1 + H(s)]\,W^+ + H(s)\Phi^- \end{aligned}}$$

therefore, w.r.t.

$$W^+, W^-, \Phi^+, \Phi^-$$

we obtain a Kelly-Lochbaum scattering cell (with a sign change w.r.t. the NL capacitor)

WD integration (through)

- Also in this case the output of the filter depends instantaneously on its input unless p=0. In order to avoid computability problems we thus have to choose R in relation to L in such a way to obtain p=0

$$s \longrightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

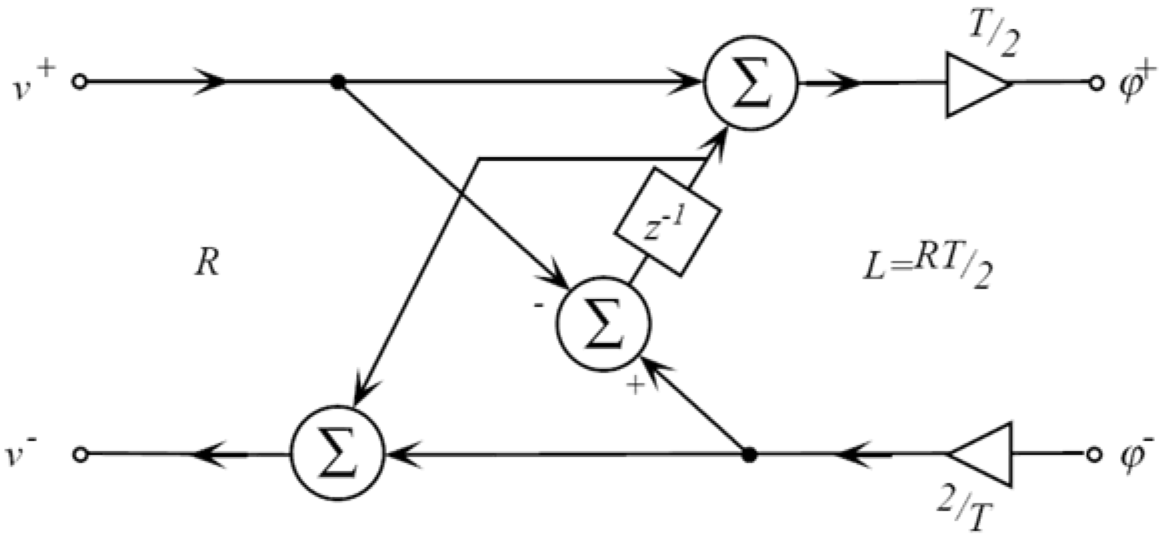
$$\boxed{\overline{H}(z) \stackrel{\text{def}}{=} H(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{p+z^{-1}}{1+pz^{-1}}}$$

$$p \stackrel{\text{def}}{=} \frac{T-2\tau}{T+2\tau}$$

Computability

- p tells us how much T differs from 2τ
- In order to obtain $2\tau = T$ we need to have $R=2L/T$, which results in

$$\overline{H}(z) = z^{-1}$$



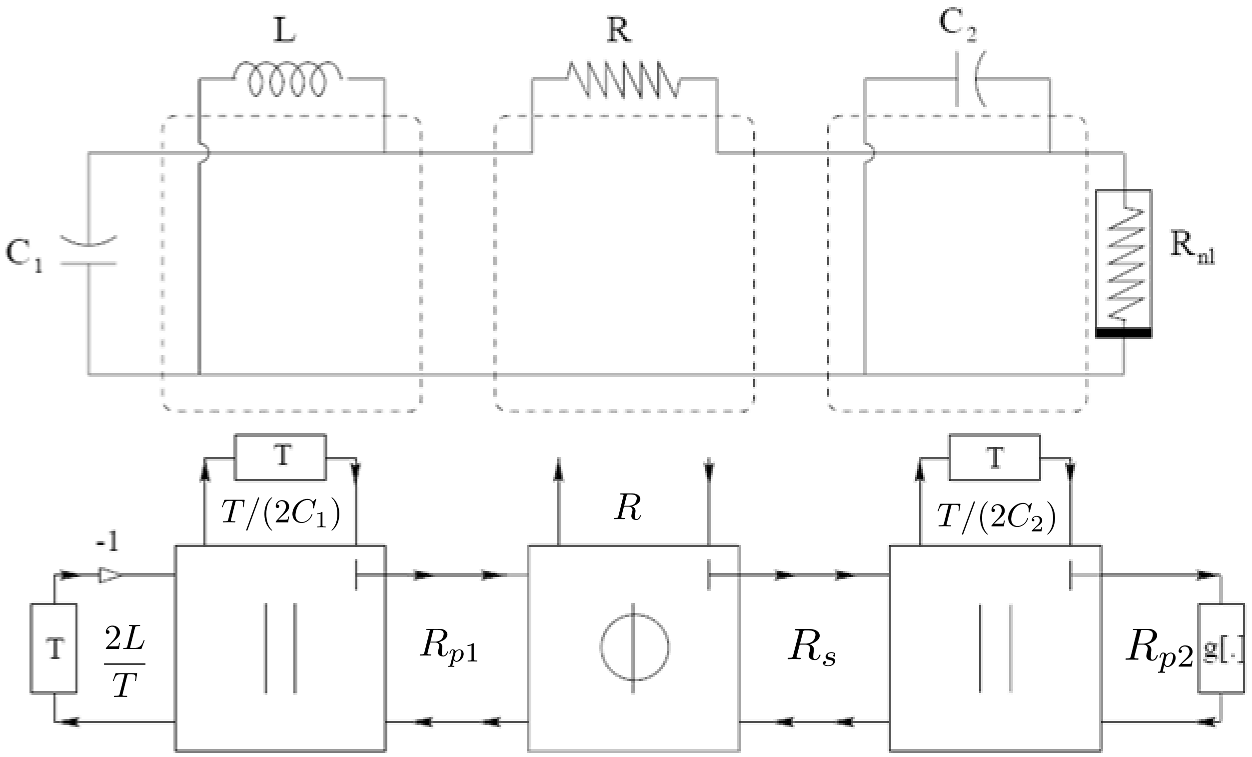
Implementation

- At this point we can connect the second port of the WD integrator (mutator) to a NL reflection function $\varphi^- = \varphi^-(\varphi^+)$ derived from $f(\varphi, i) = 0$
- When the inductor is linear we find again the results proposed by Fettweis

$$L = \overline{L} \qquad R = 2\overline{L}/T$$

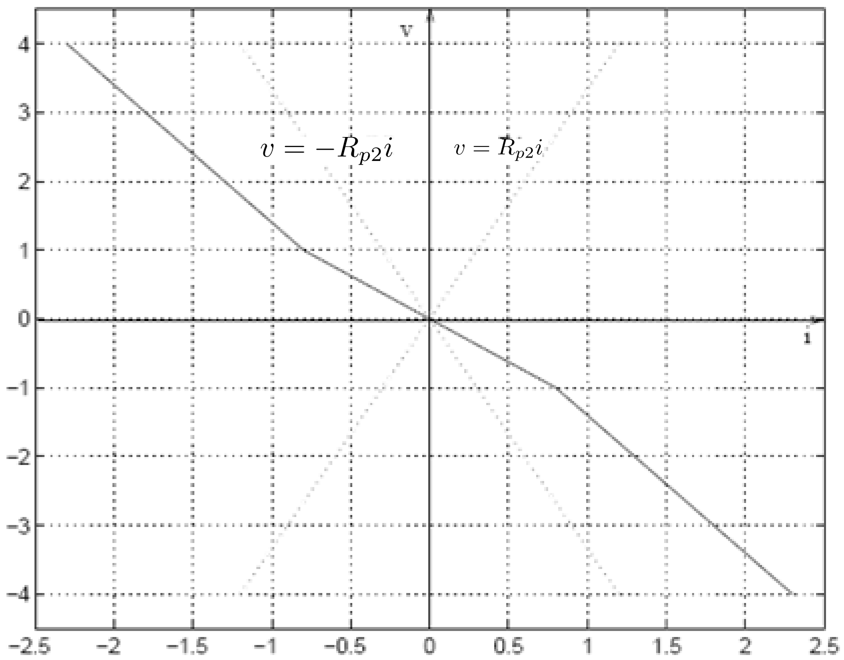
in this case p=0 and allpass + mutator combination becomes a simple delay element with a sign change

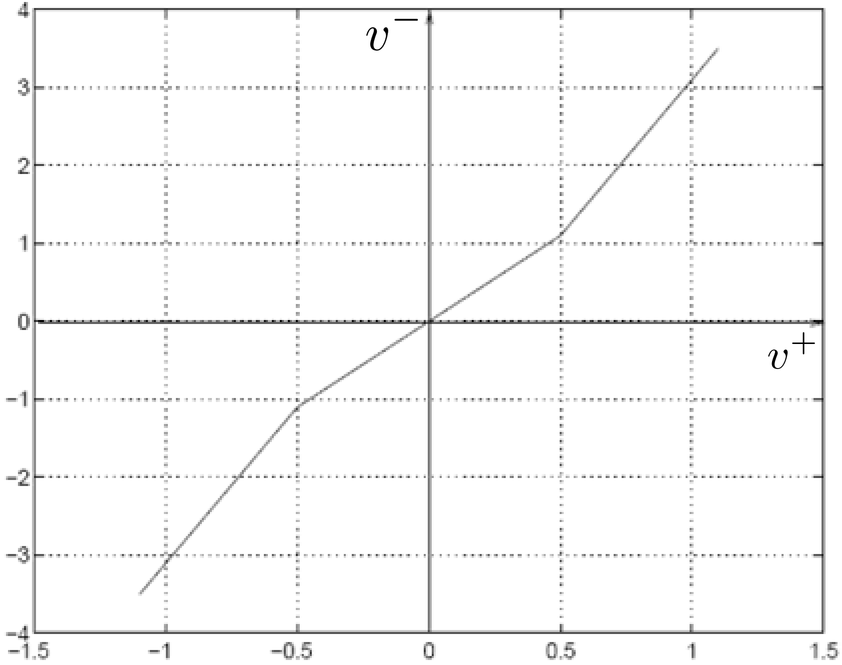
Example: Chua's circuit



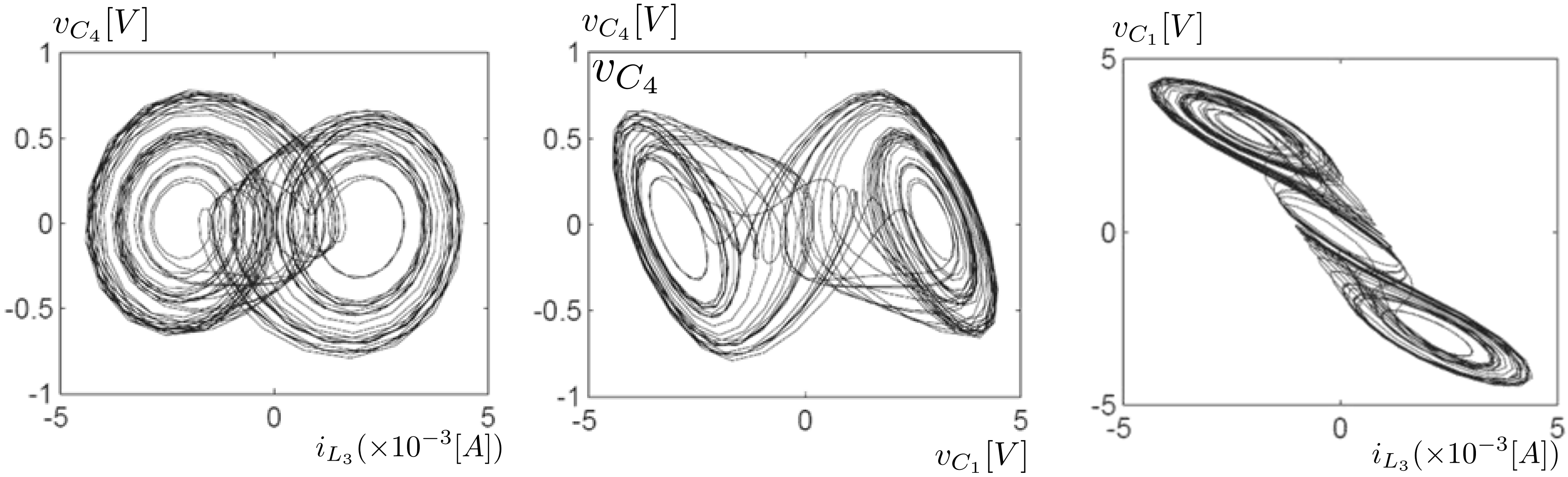
$$R_{p1} = \frac{2L}{T} // \frac{T}{2C_1} = \frac{\frac{L}{C_1}}{\frac{2L}{T} + \frac{T}{2C_1}} \qquad R_s = R_{p1} + R \qquad R_{p2} = R_s // \frac{T}{2C_2} = \frac{\frac{TR_s}{2C_2}}{R_s + \frac{T}{2C_2}}$$

Chua’s circuit: NLE

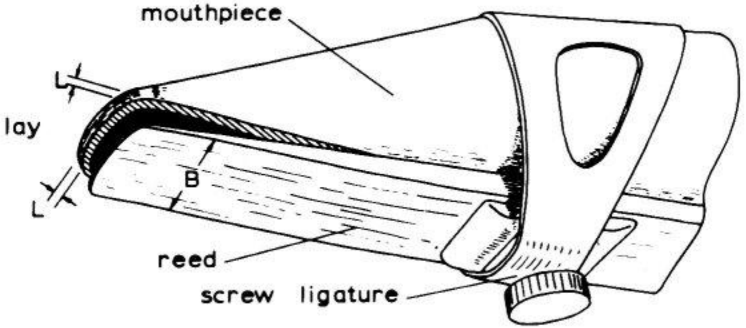


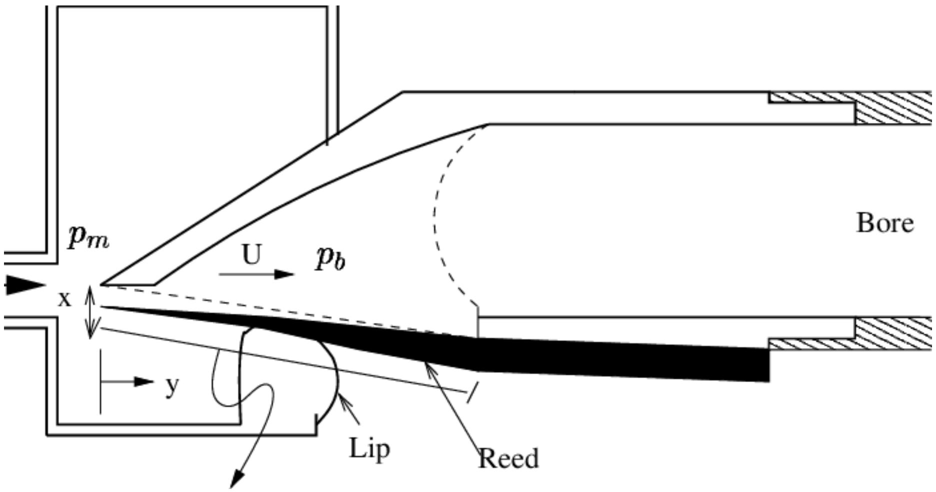


Chua's circuit: orbitations

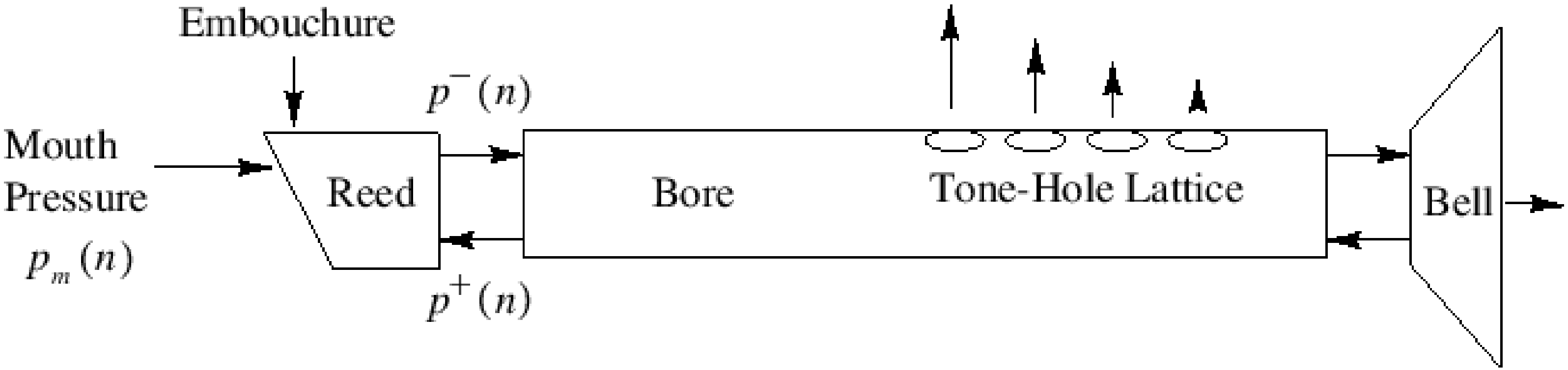


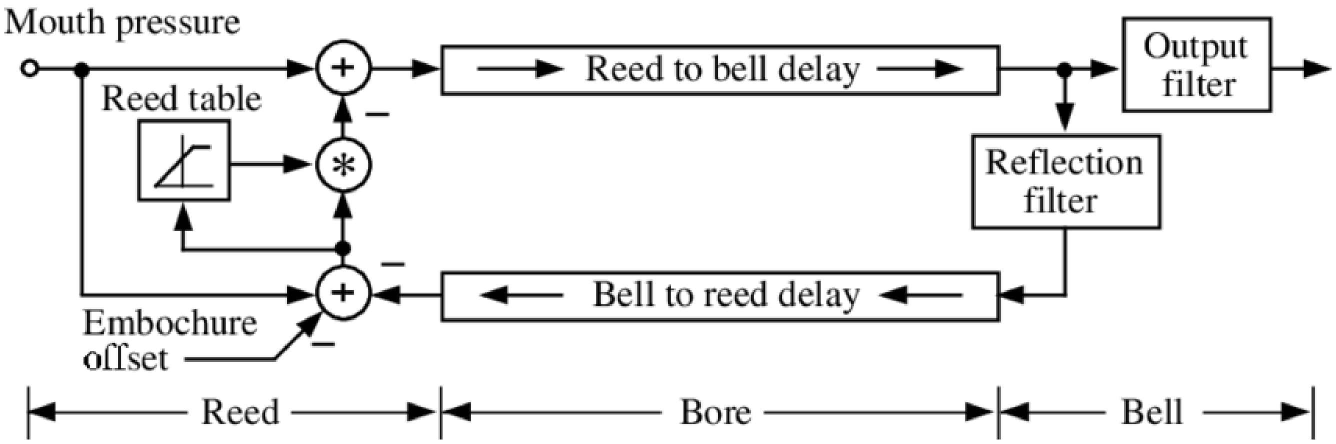
Reed-bore model



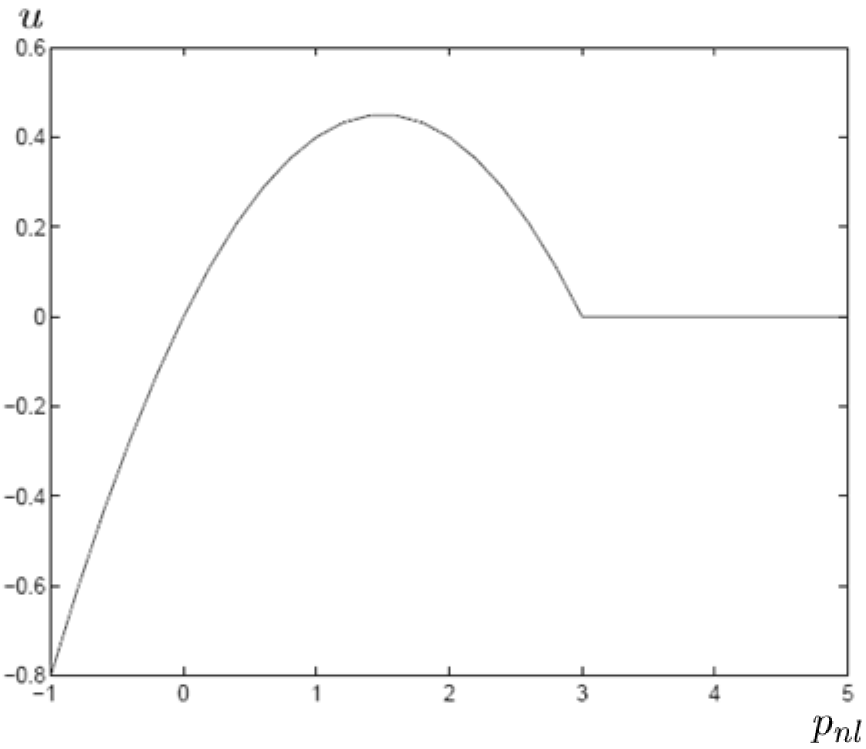


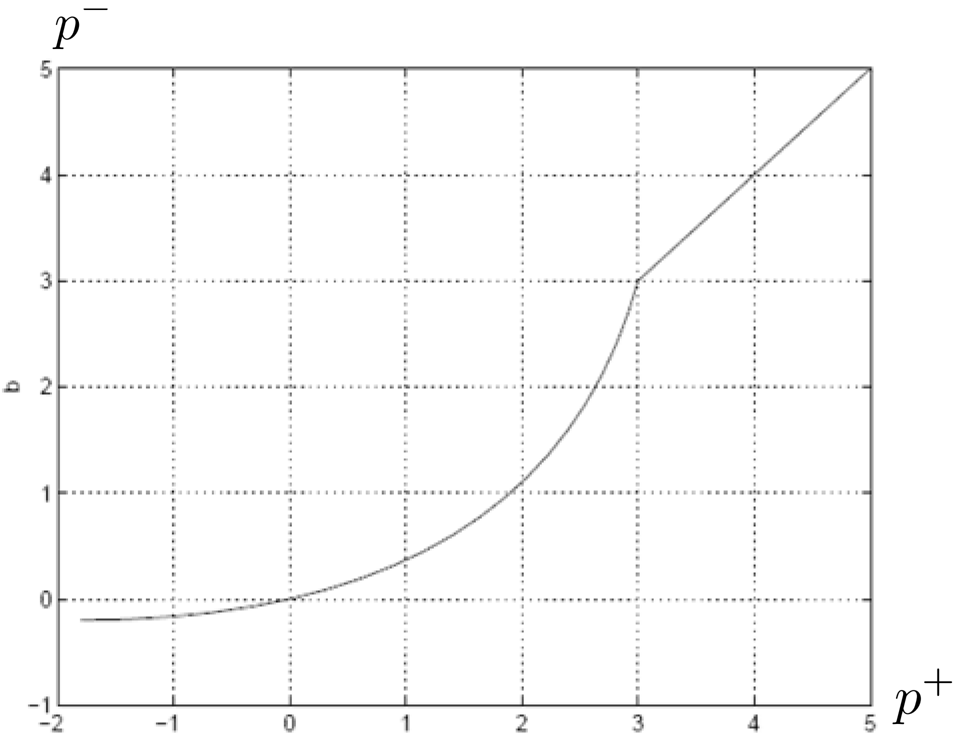
Reed-bore model



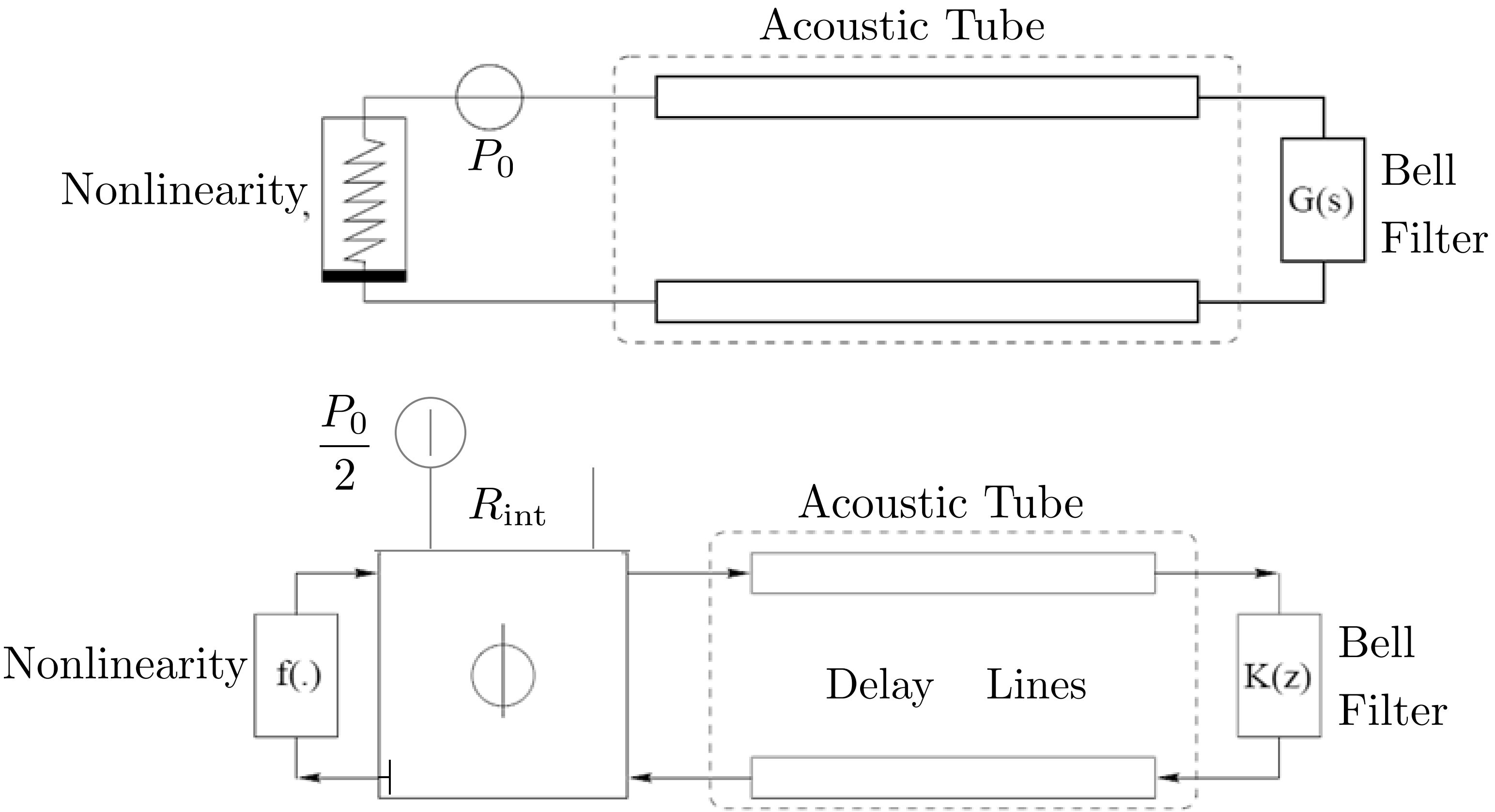


Reed-bore model

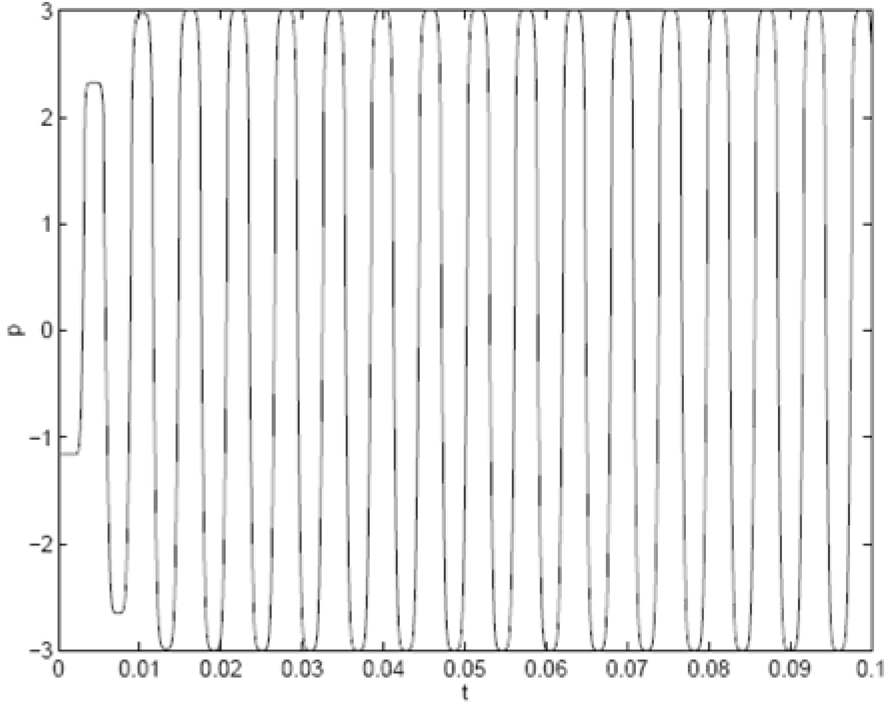


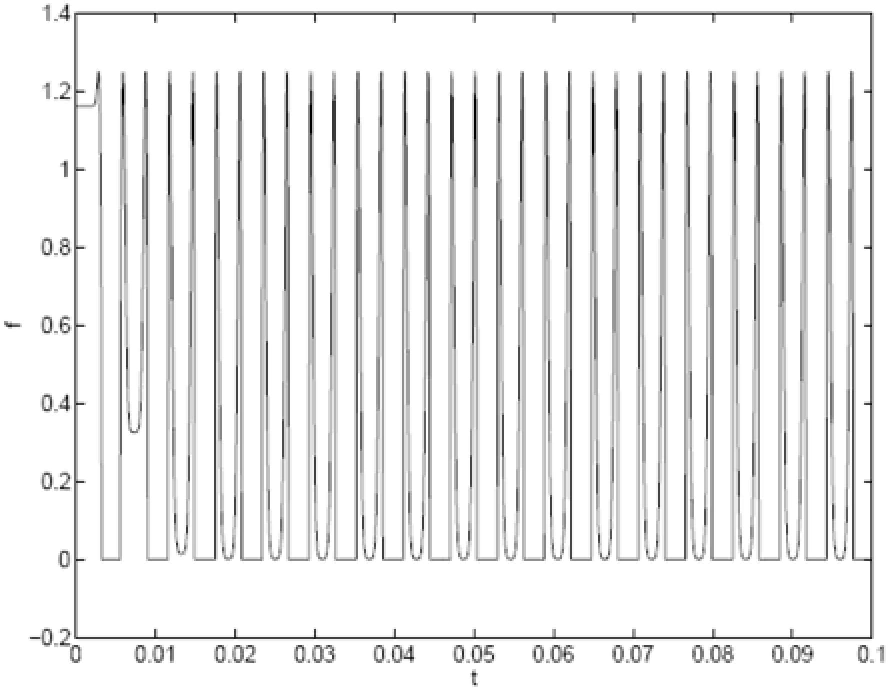


Reed-bore model

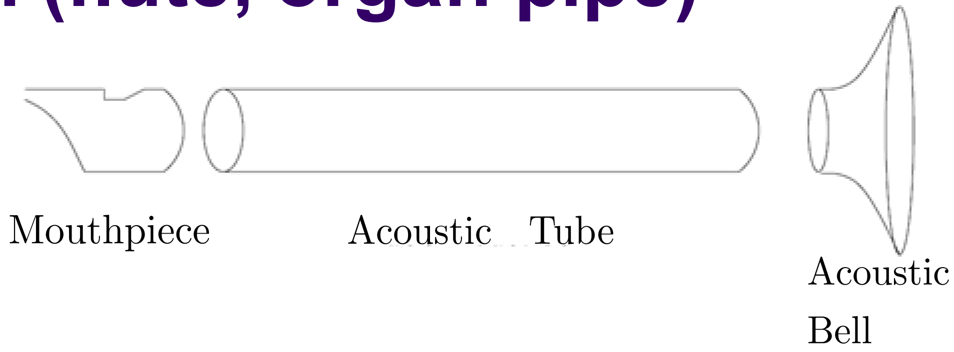


Reed-bore model

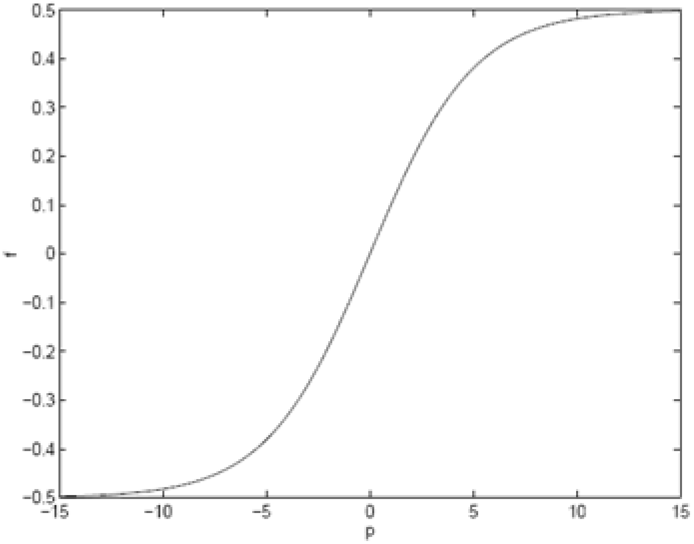


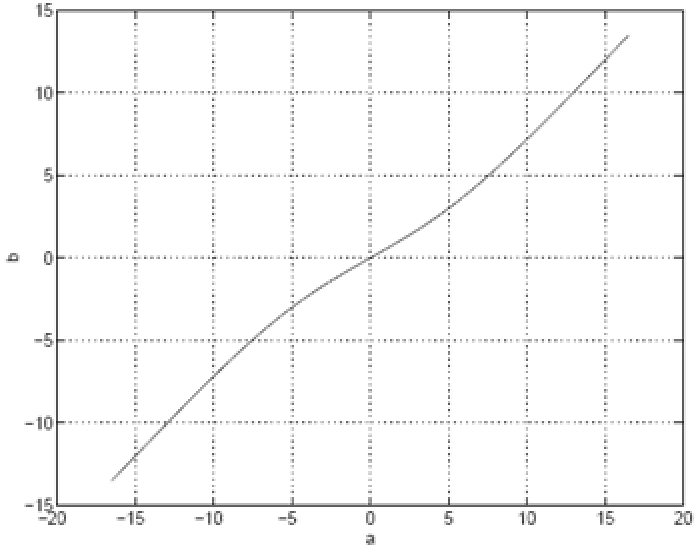


Jet switch (flute, organ pipe)

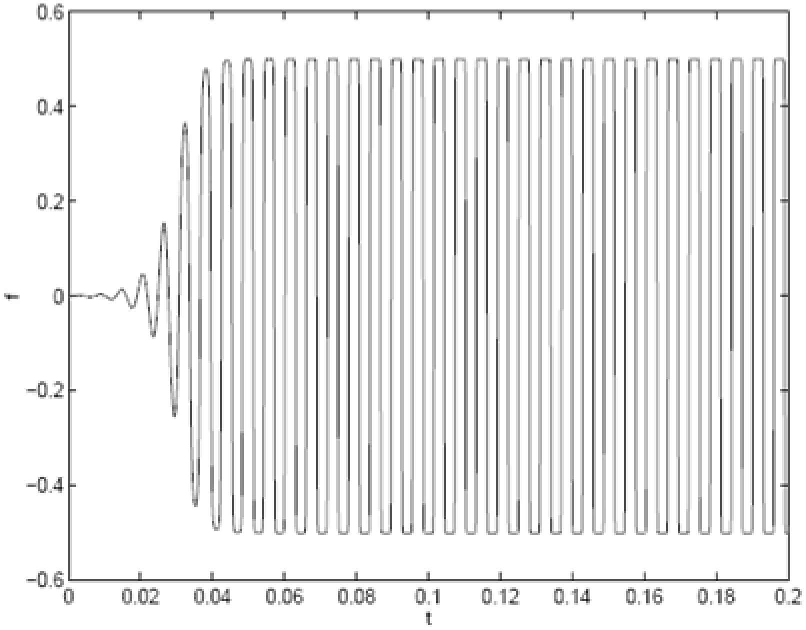


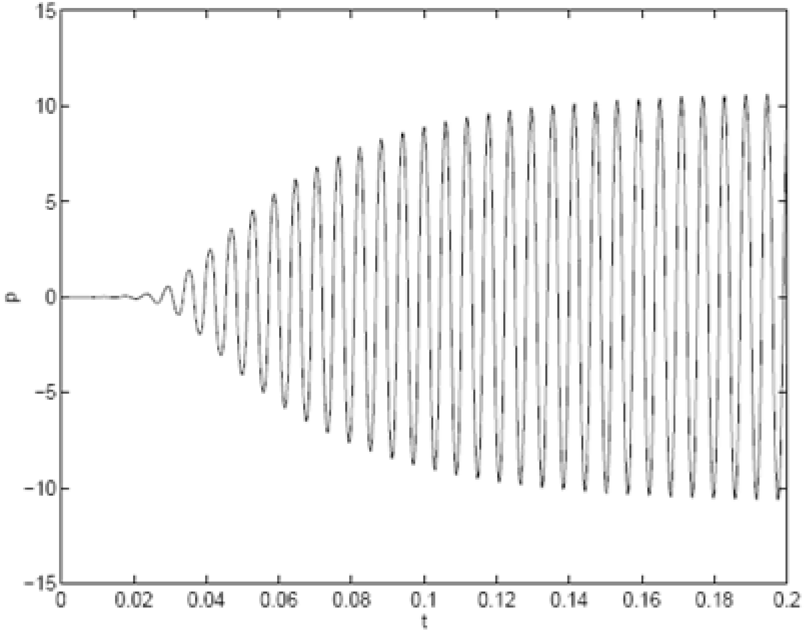
$$u = f(P_0 - p) = f(p_{nl})$$





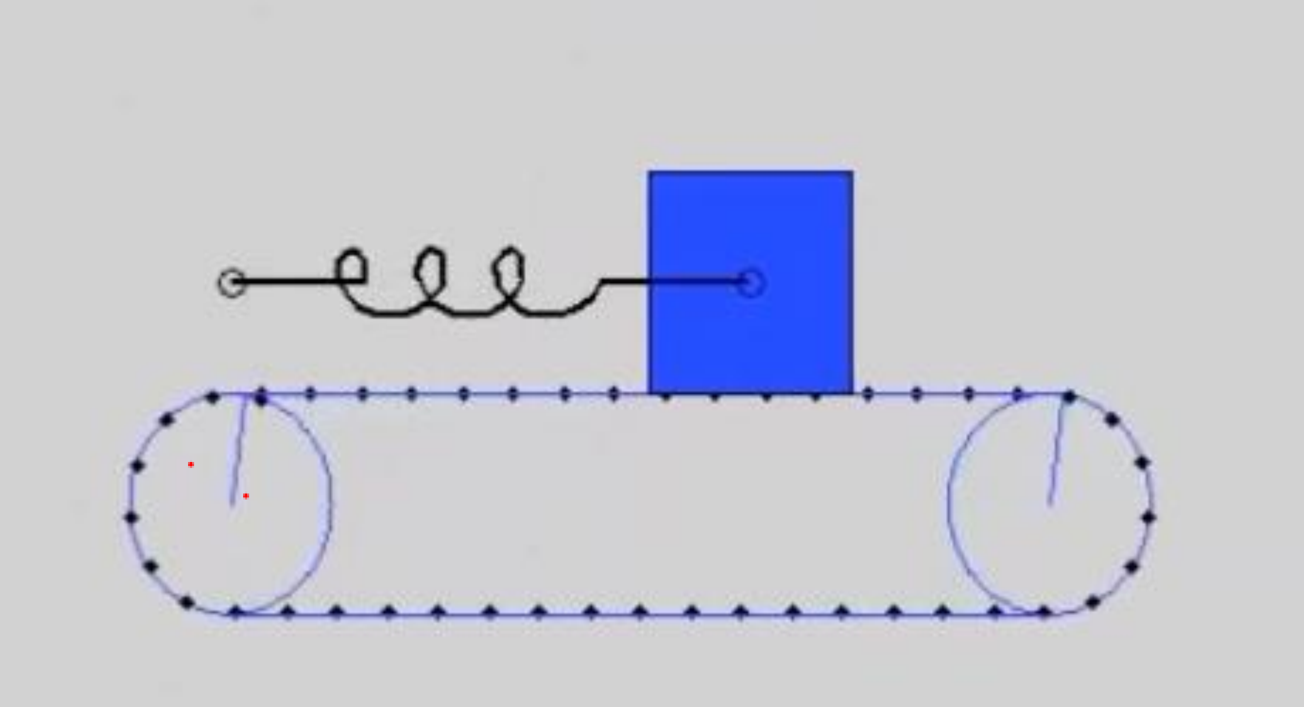
Jet switch


0.6
0.4
0.2
0
−0.2
−0.4
−0.6
f
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
t

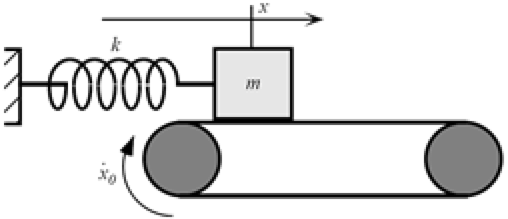

15
10
5
0
−5
−10
−15
p
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
t

Reileigh model

- Simplified bow-string interaction

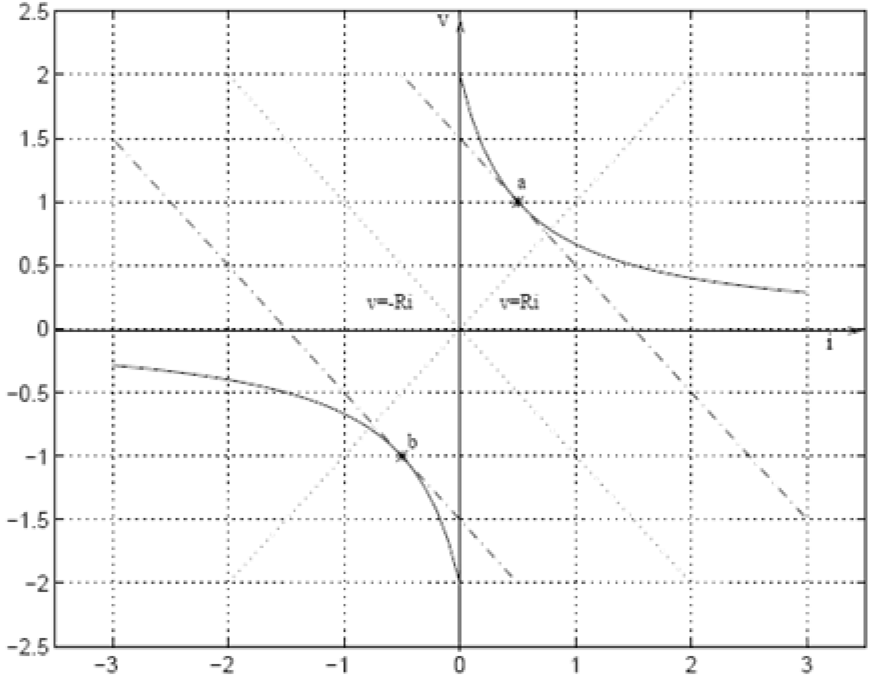


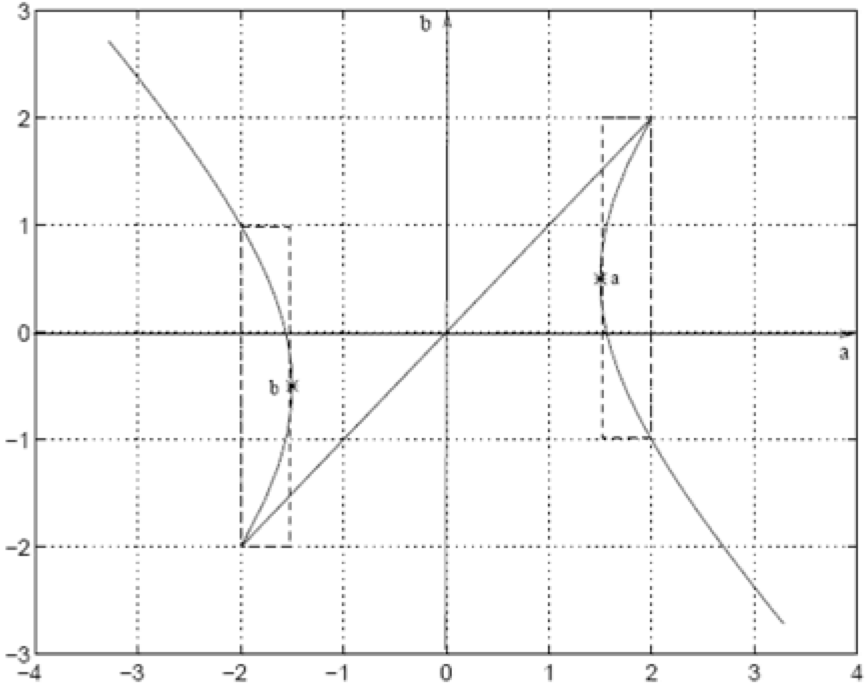
https://youtu.be/TcebgBomjRs



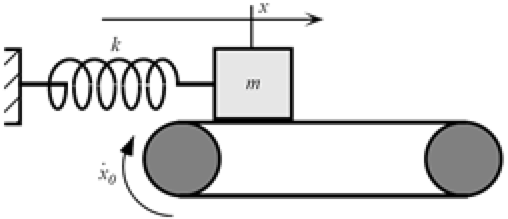
Reileigh model

- Reileigh model: NL characteristics

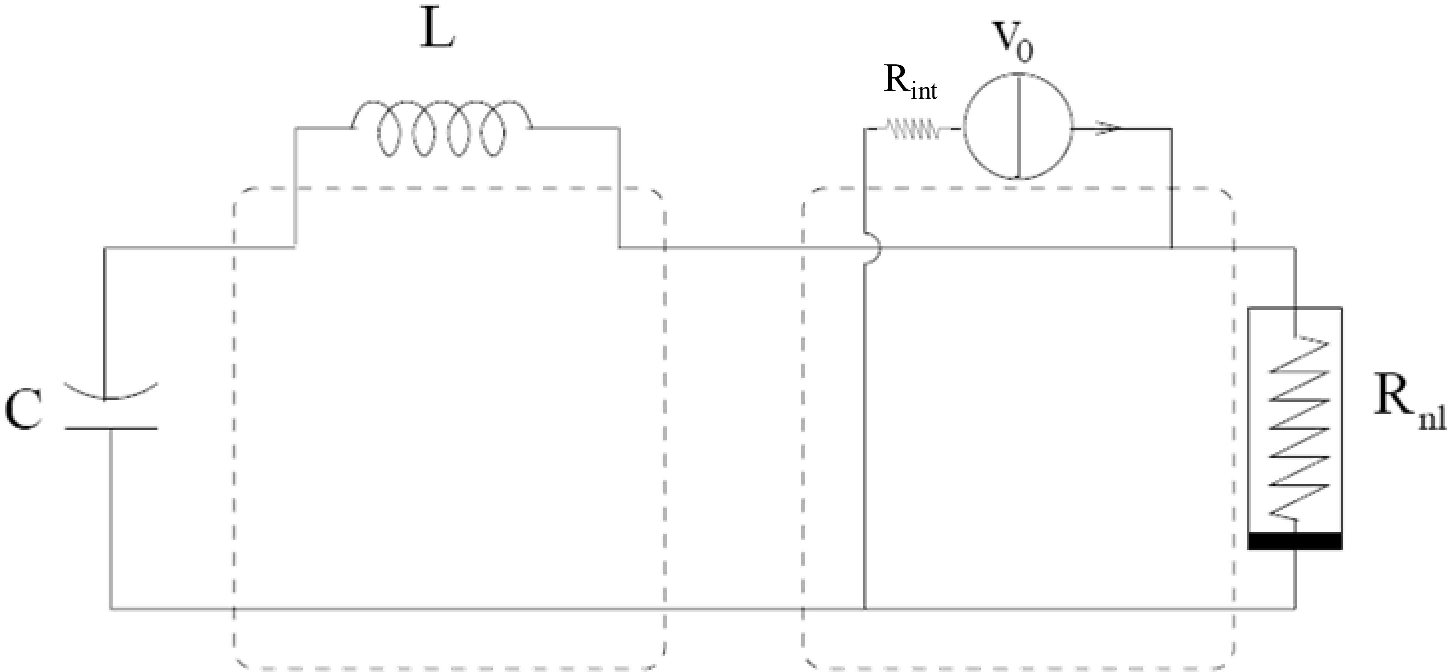




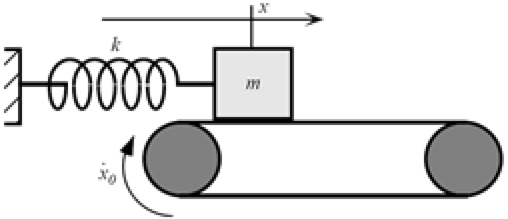
Reileigh model



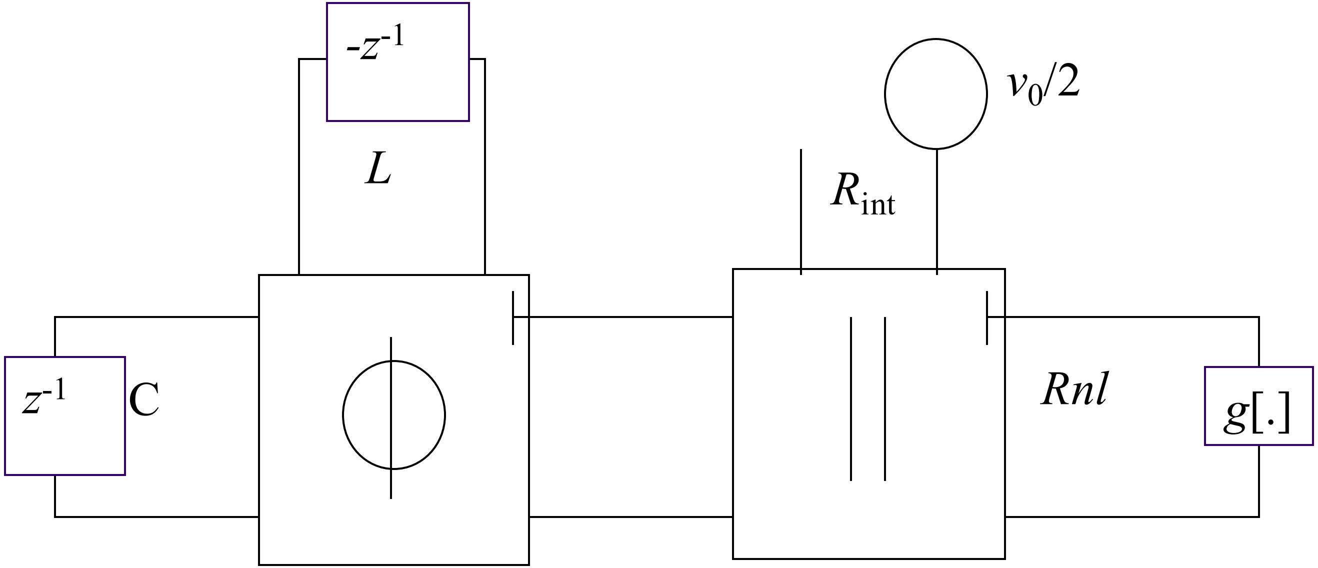
- K-domain model (equivalent electrical circuit)



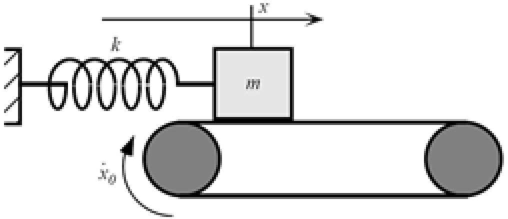
Reileigh model

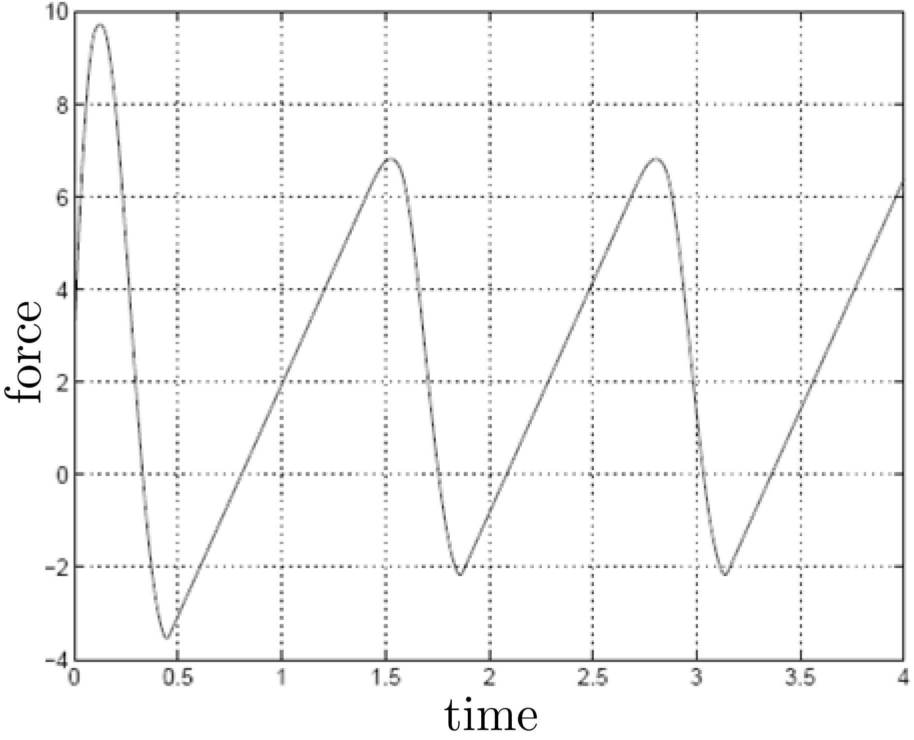


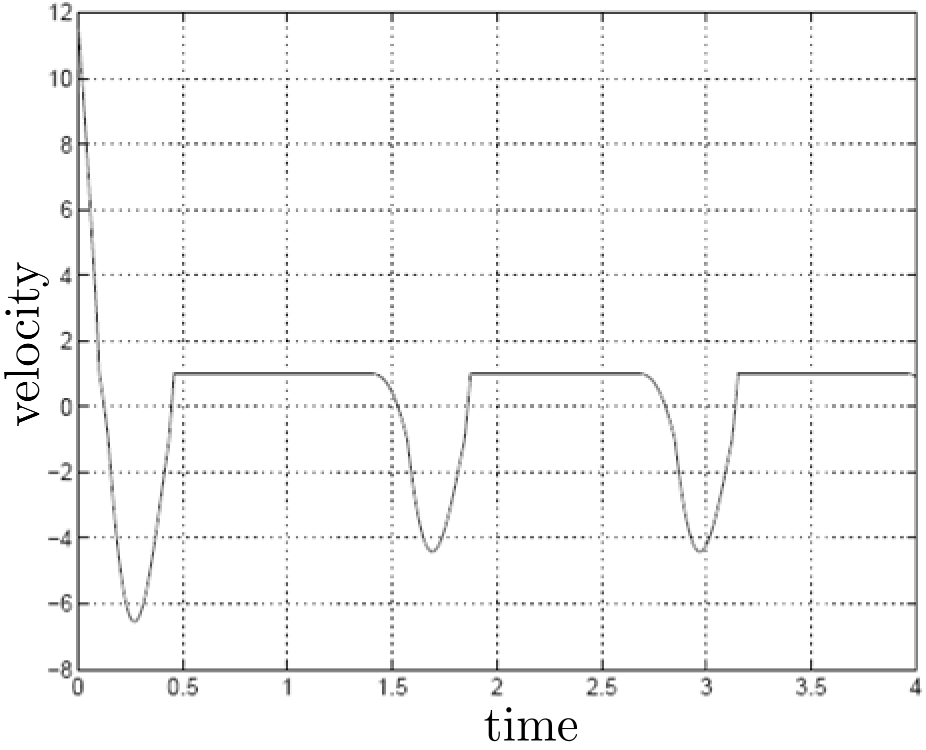
- Reileigh model in the WD domain



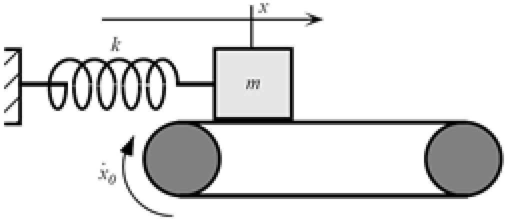
Reileigh model



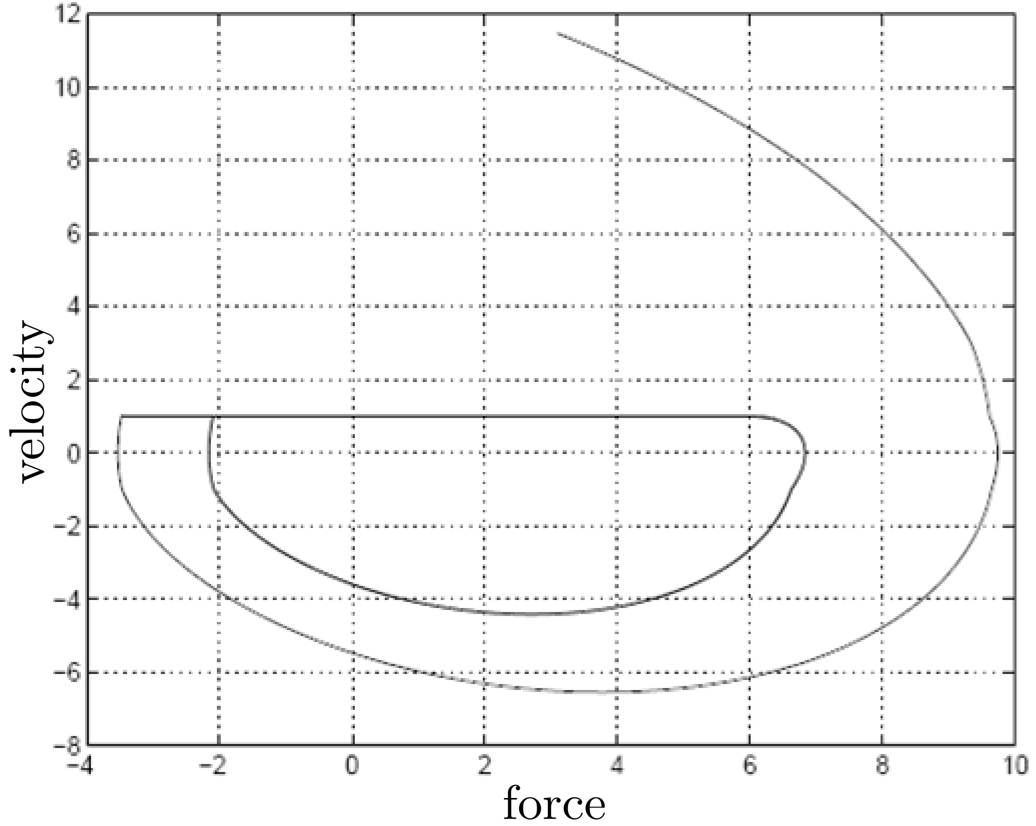




Reileigh model

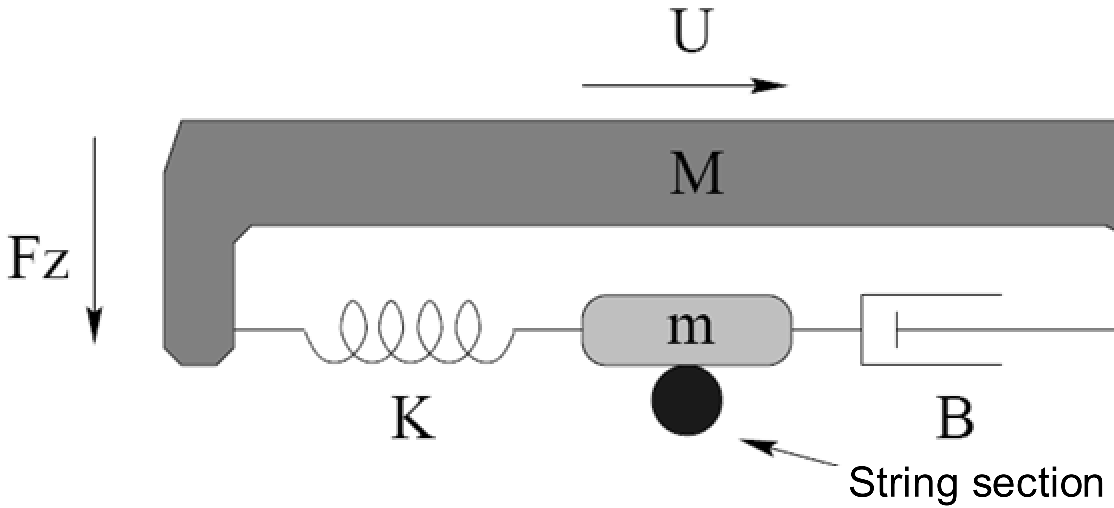


- Phase diagram



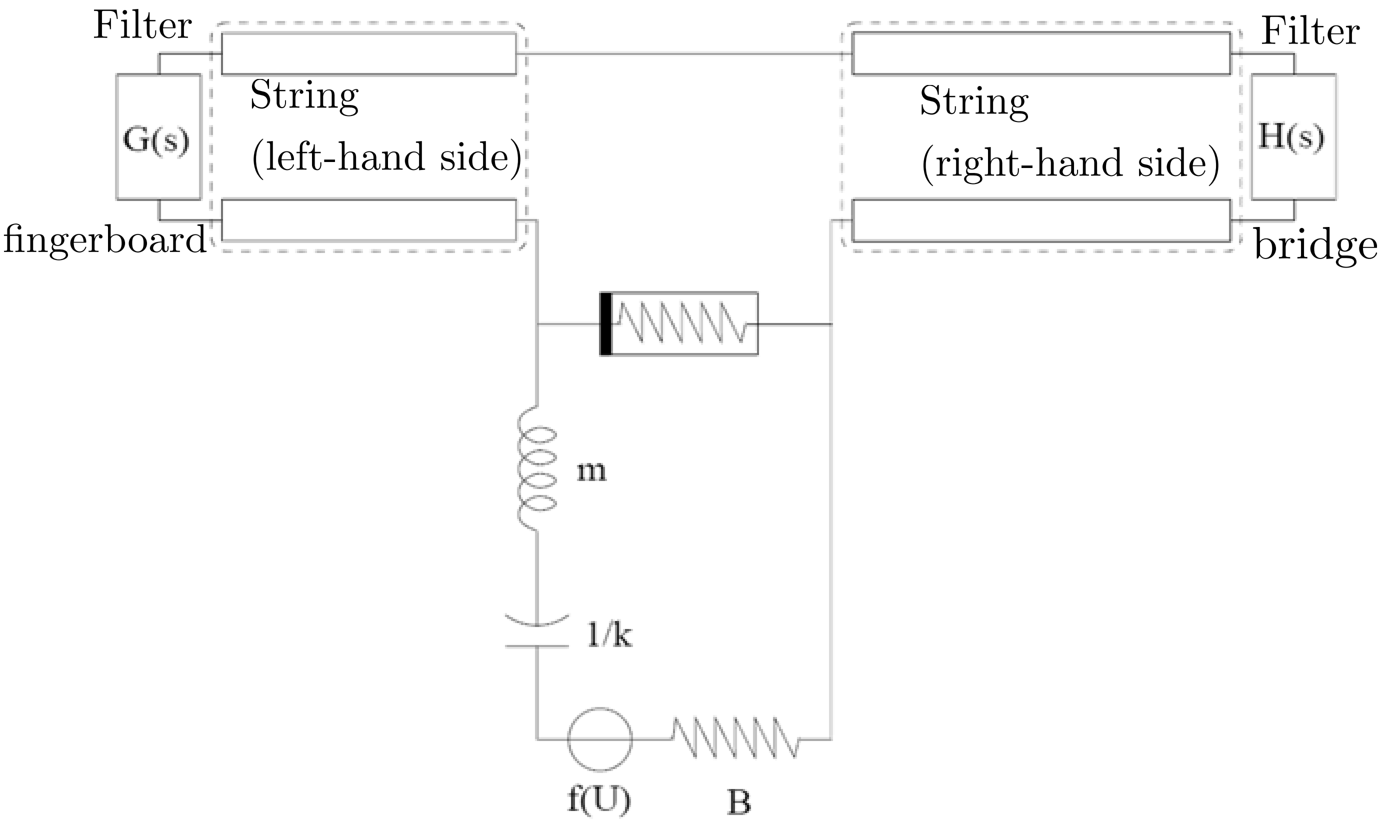
Bow-string interaction

- Dynamical bow model



Example: bow-string interaction

- Equivalent electric circuit (involving both lumped and distributed parameters)



Equations

- Bow dynamics

$$f_{arc}(t) = -m\dot{u}(t) - k\int u(t)dt - Bu(t) + k\int U(t)dt + BU(t)$$

- Dynamic friction model

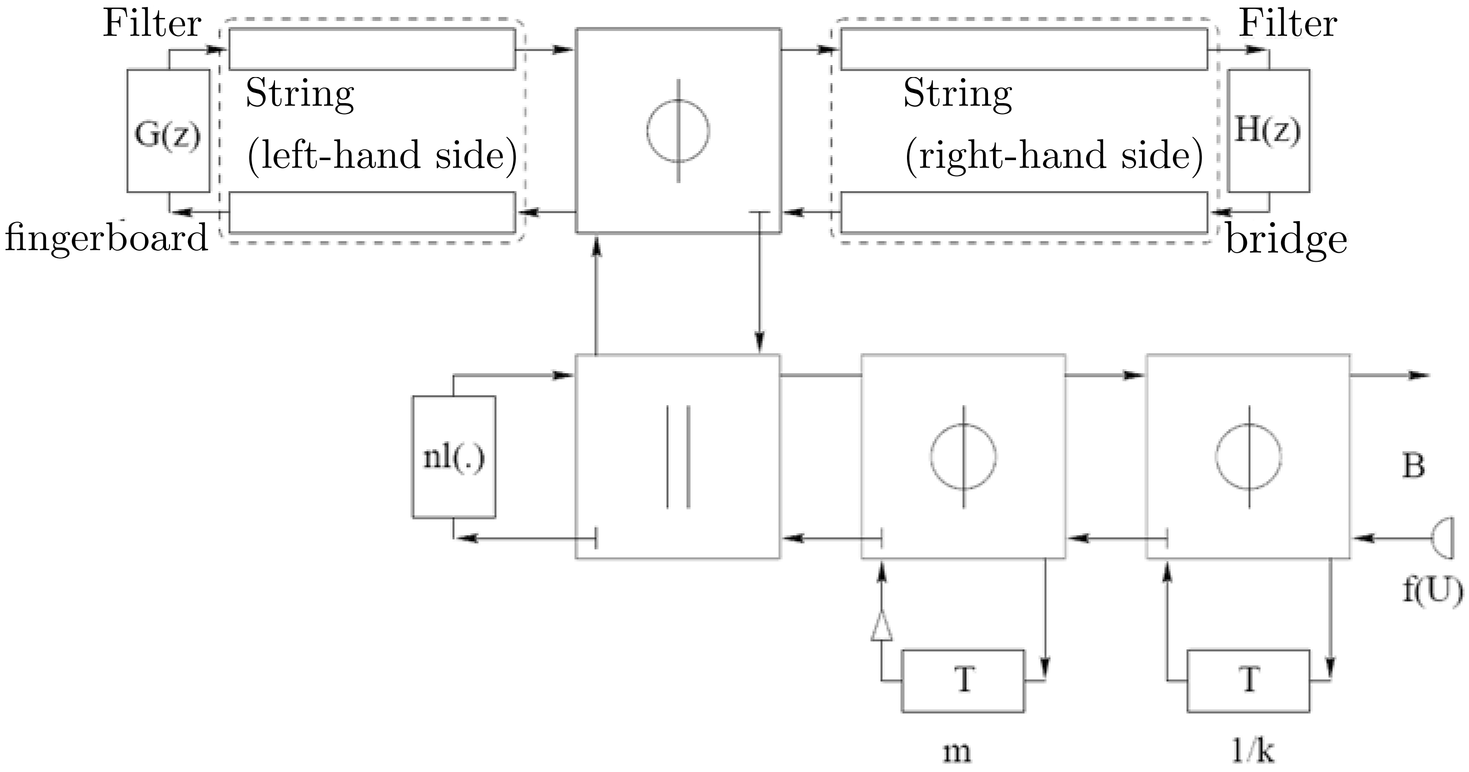
$$f_{nl} = \frac{\Delta\mu F_z}{-(u-v)/\alpha + 1} + \mu_d F_z$$

- Forces at contact pt $f_{arc} = f_{nl} = f$

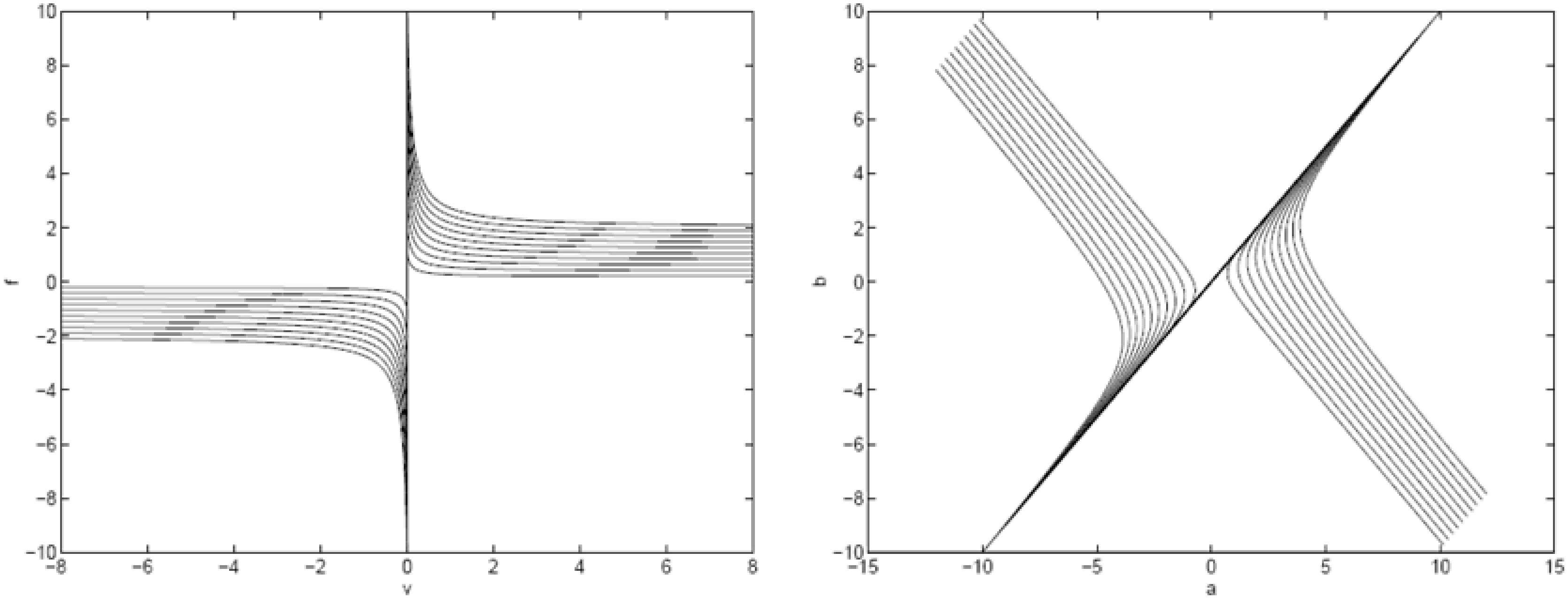
- Velocities at contact pt $v_{nl} = -v - u'$ $\Rightarrow$ $v_{nl} + v + u' = 0$

Example: bow-string interaction

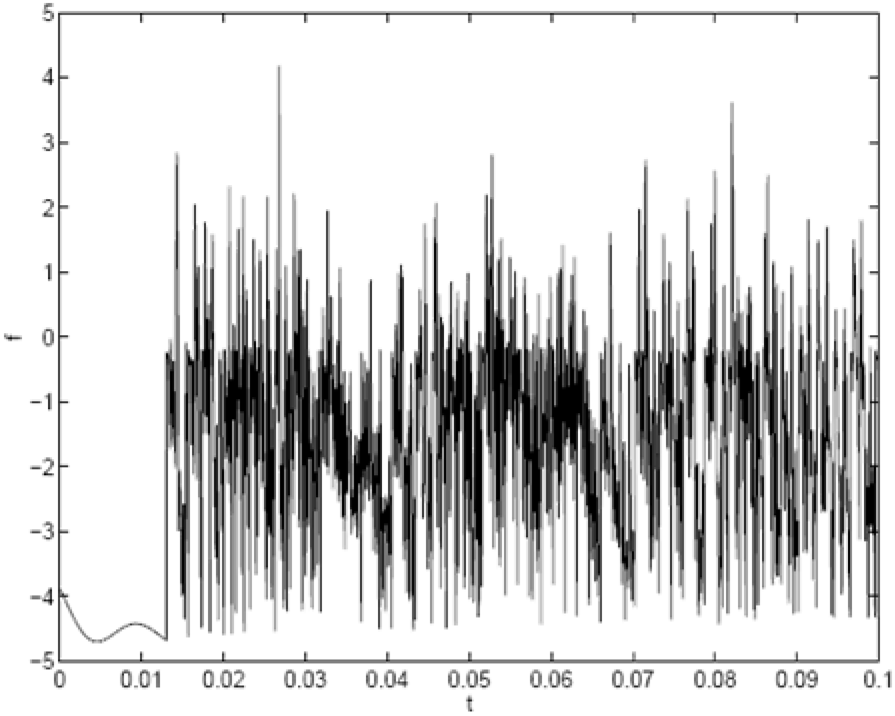
- WD implementation – hybrid WDF-DWG

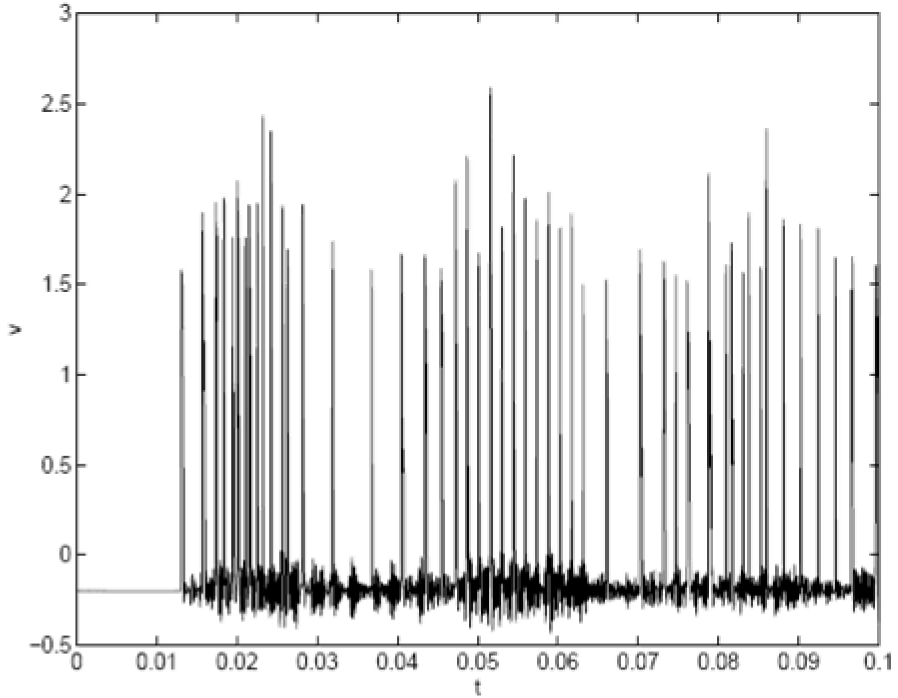


Example: bow-string interaction



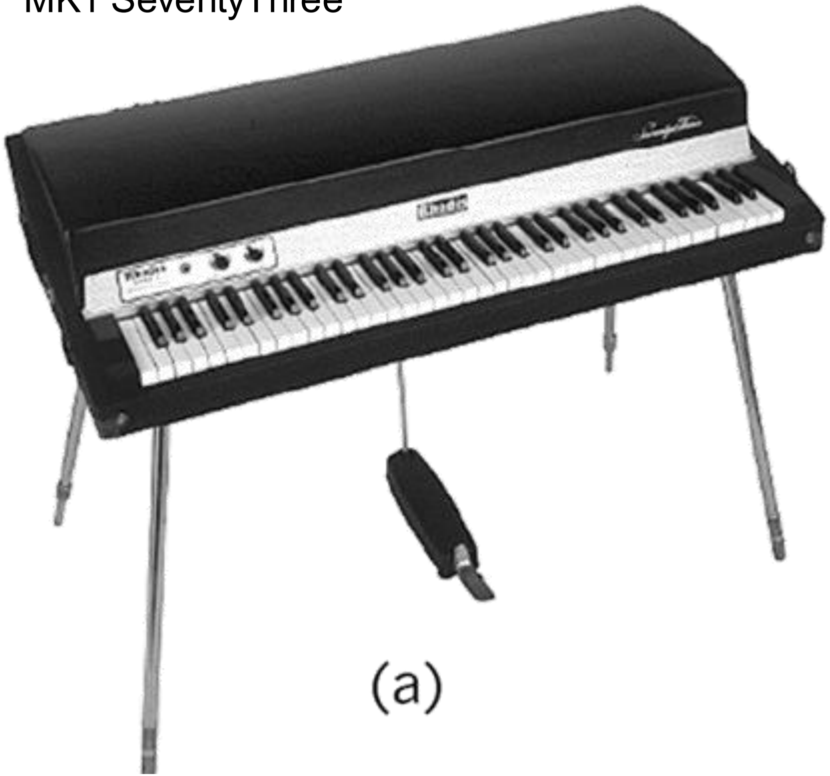
Example: bow-string interaction



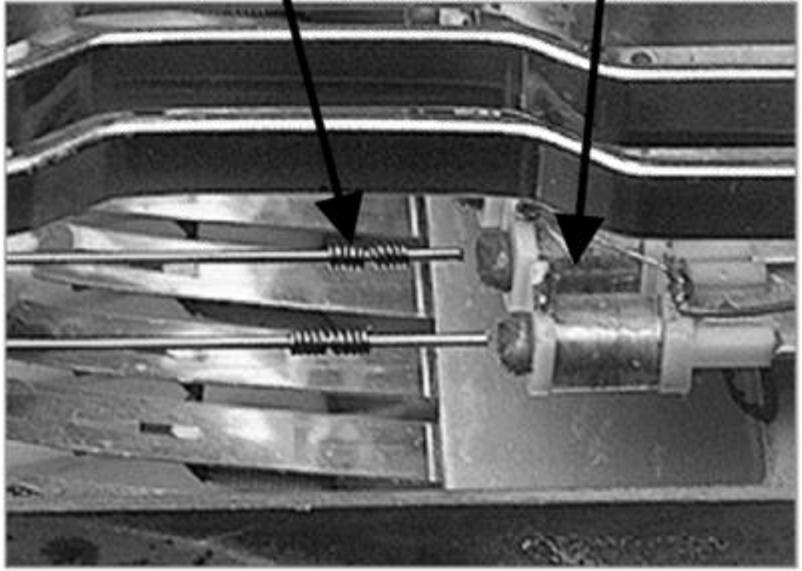


Modeling the Electric piano

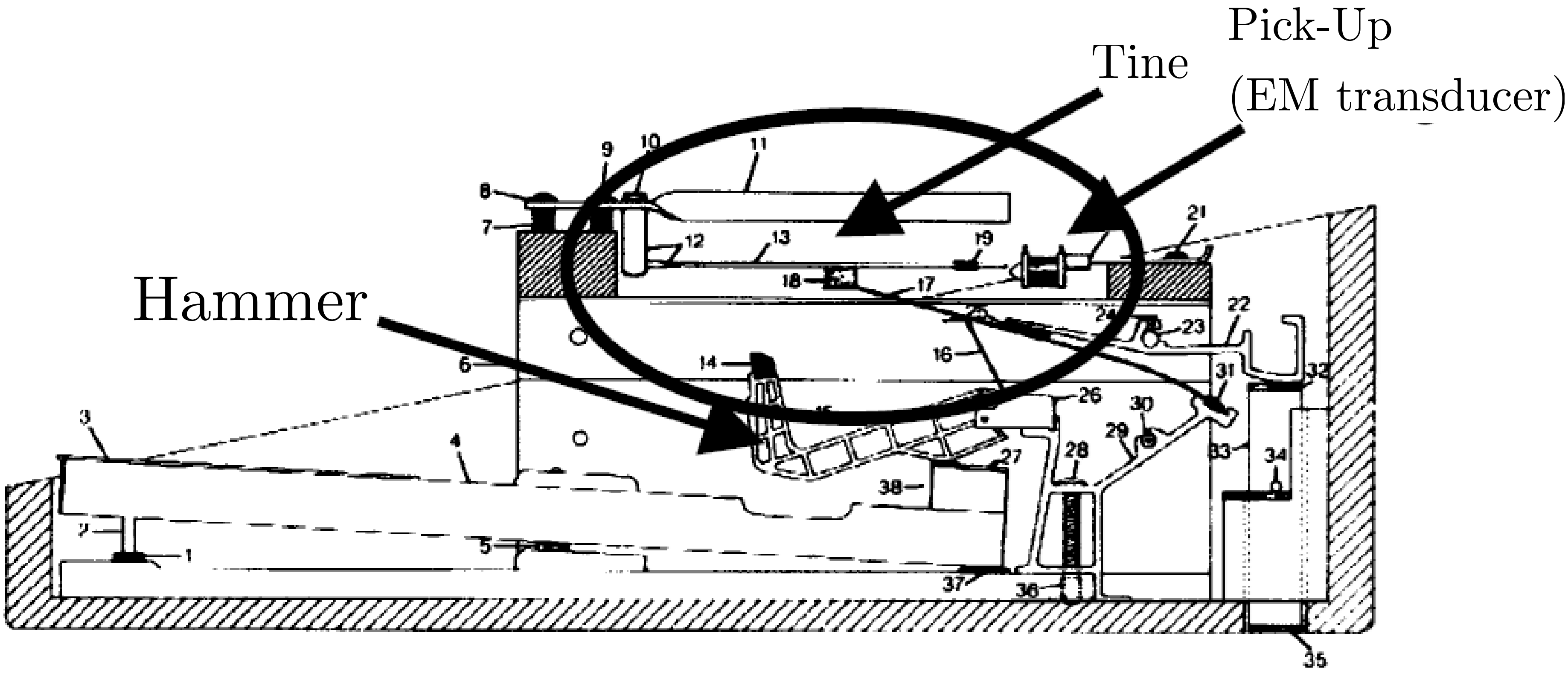
Fender Rhodes
MK1 SeventyThree



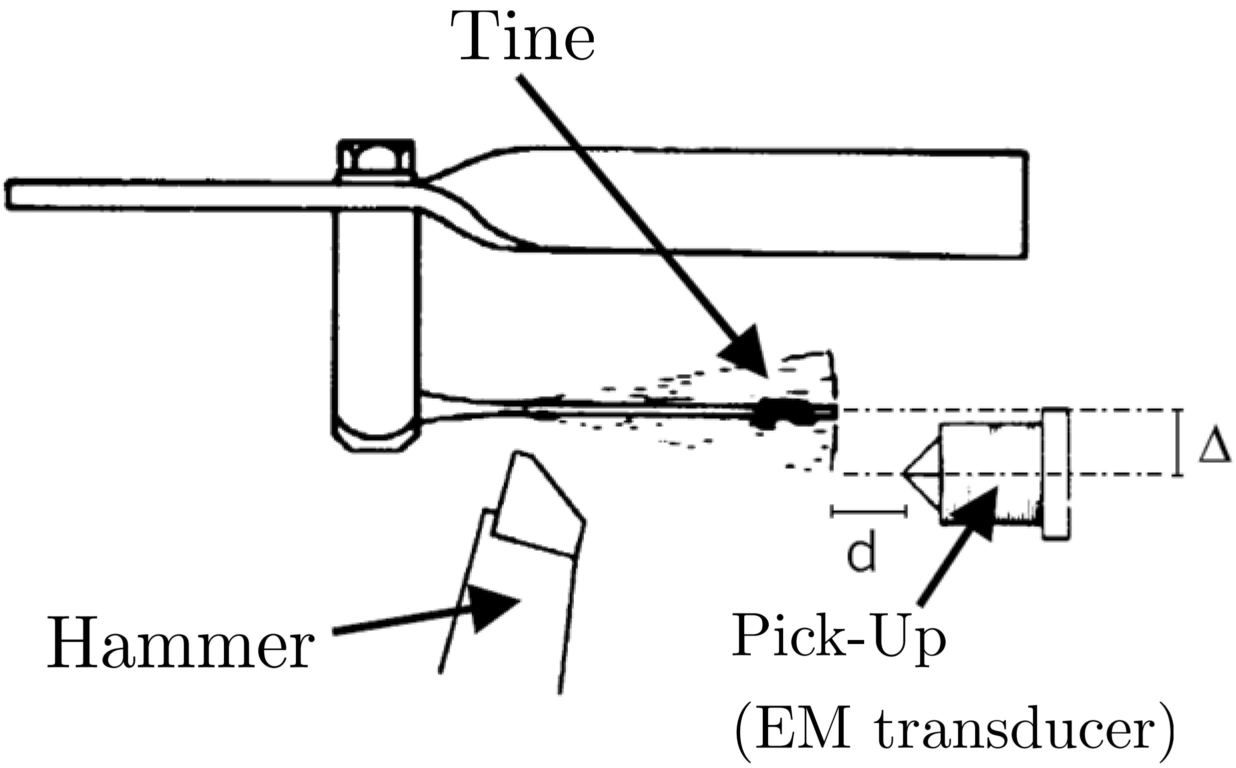
(a)



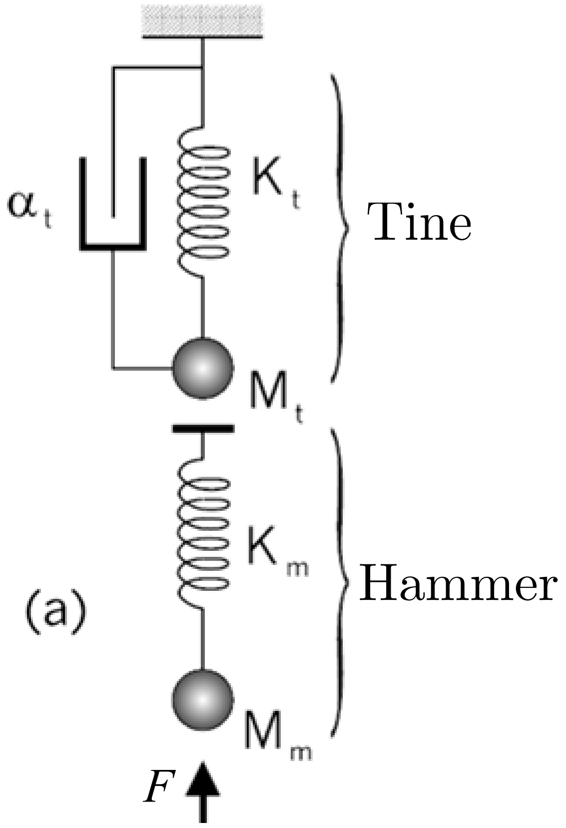
Modeling the Electric piano

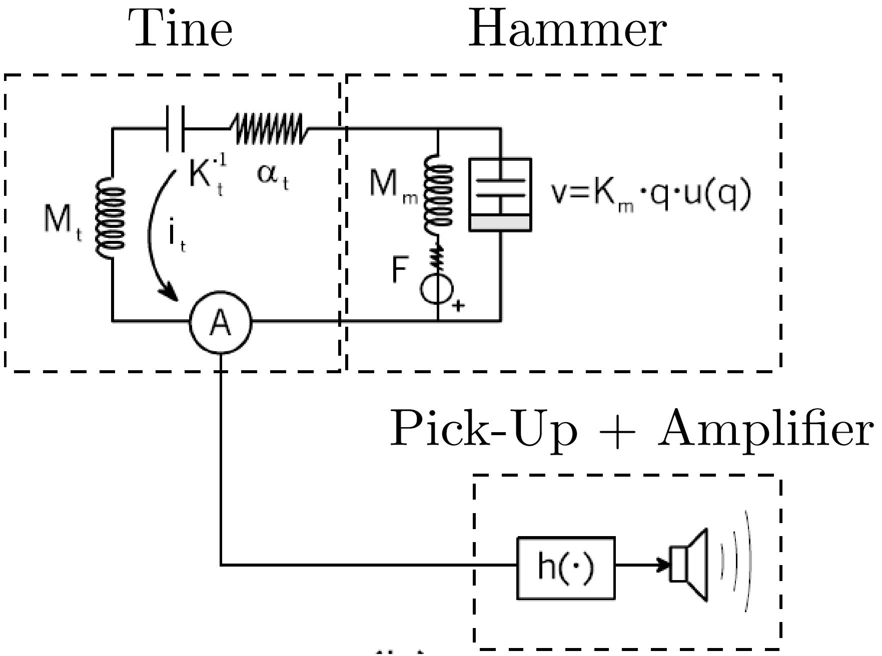


Modeling the Electric piano

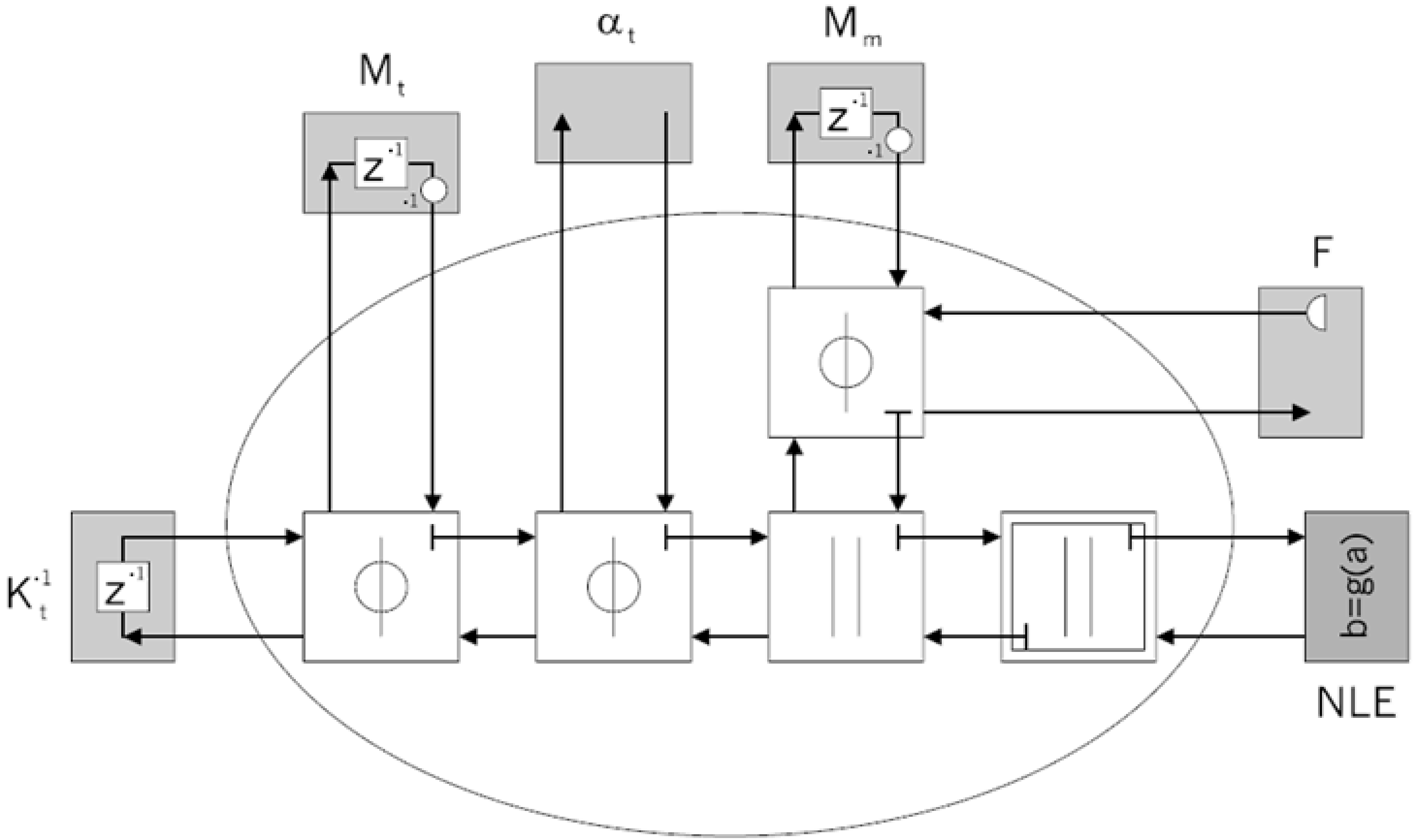


Modeling the Electric piano

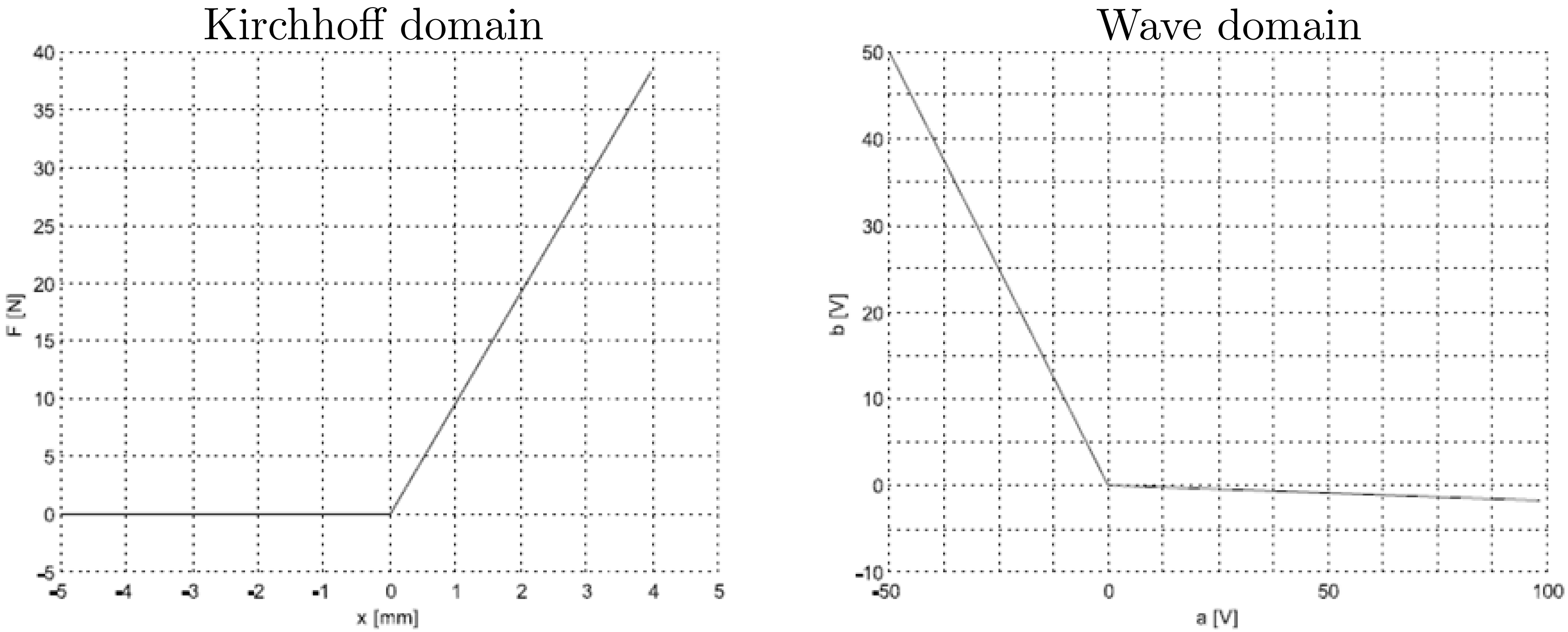




Modeling the Electric piano



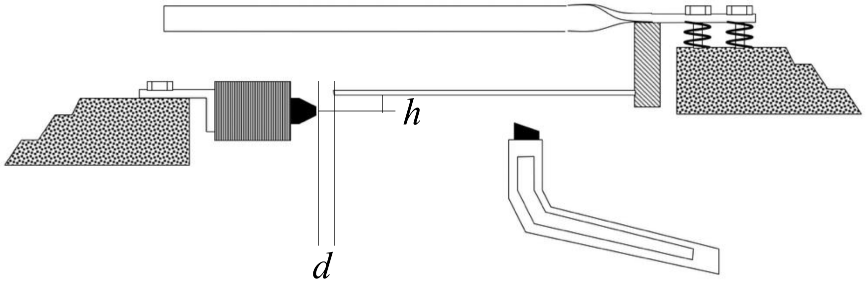
Modeling the Electric piano



The nonlinearity is all in the contact condition

Transduction NLE

- This is the most important part: responsible for timbral dynamics
- The transducer (plus amplifier) exhibits a NL characteristics due to the NL dependency of the tine-pickup distance from the vertical displacement, combined with the amplifier's saturation
- Pick-up
  - The eq. that describes the output is



$$s(t) \approx -\frac{d}{dt}\Phi(y_R(t), d, h)$$

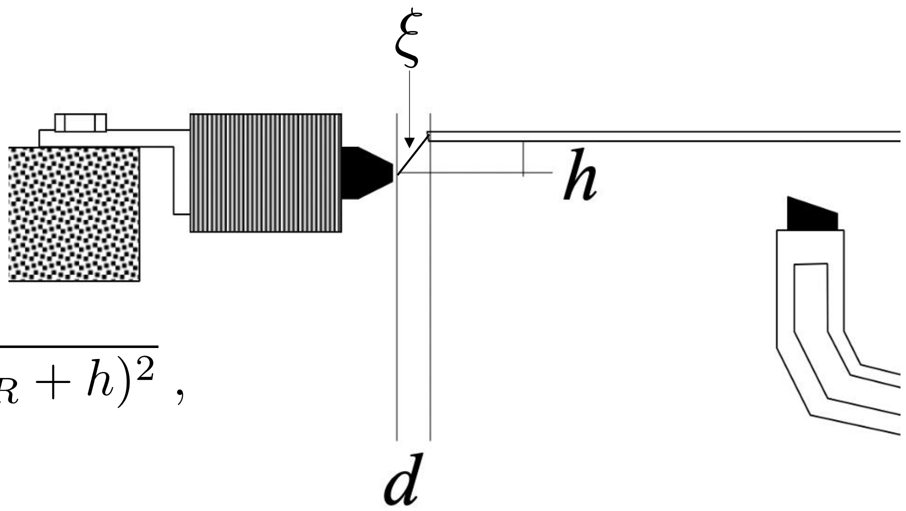
where $s(t)$ is the output voltage, $\Phi \left[\mathrm{T} \cdot \mathrm{m}^2\right]$ is the magnetic induction flux, and y_R $[\mathrm{m}]$ is displacement of the tine

  - From that we derive $$\frac{d}{dt}\Phi(y_R(t)) = \Phi'(y_R(t))y'_R(t)$$

but this modeling approach does not yield numerically stable results

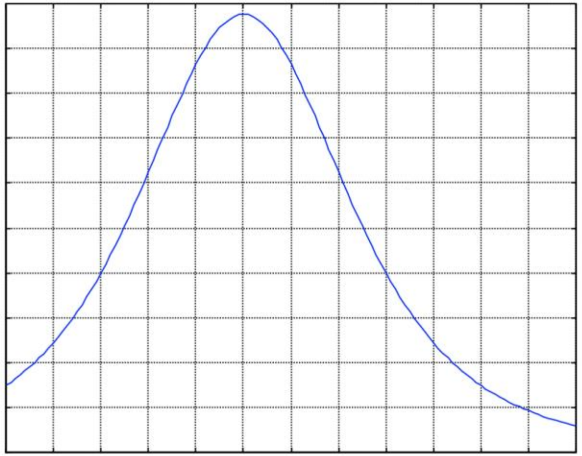
Transduction NLE

We therefore approached the modeling from a different (geometric) perspective:



$$s(t) = \dot{f}\left(\frac{1}{\xi(t)}\right)\ , \quad \xi(t) \triangleq \sqrt{d^2 + (y_R + h)^2}\ ,$$

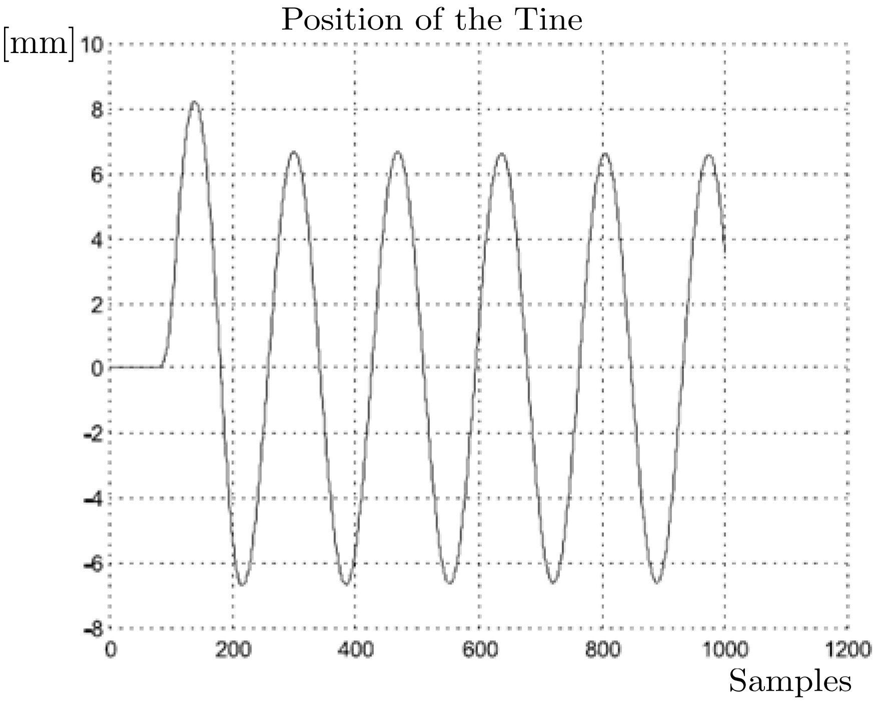
$$f(y_R(t)) = \frac{1}{\xi^4(t)} = \frac{1}{\left[d^2 + (y_R + h)^2\right]^2}$$



$$\Downarrow$$

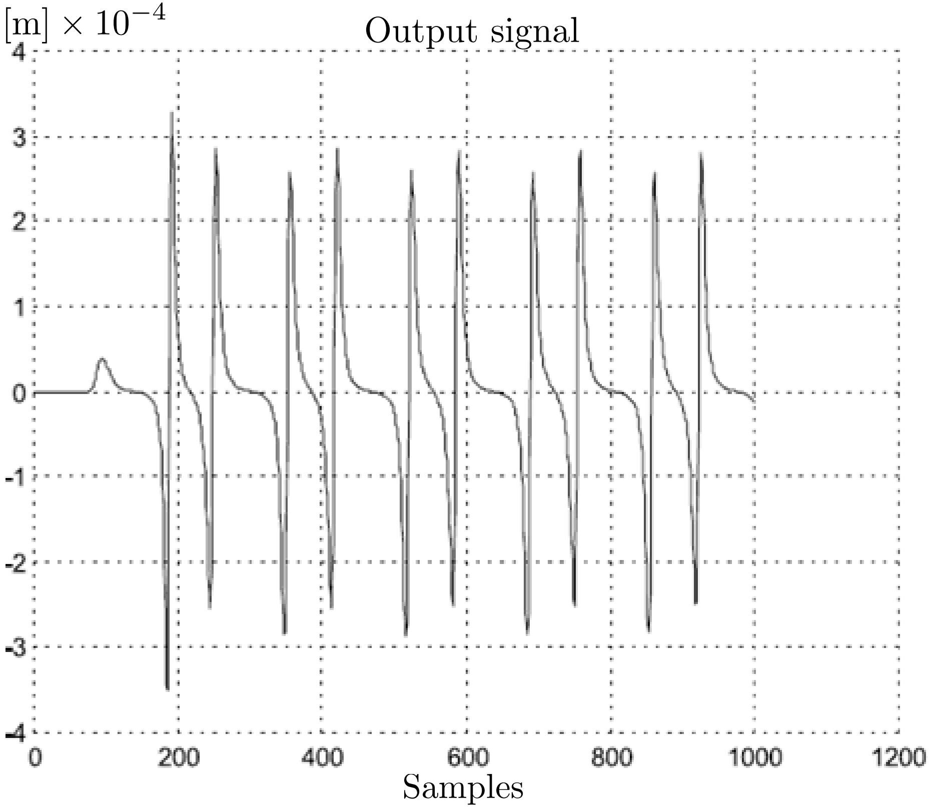
$$s(t) = -V_{\text{sat}} \tanh\left[\mu \frac{d}{dt}\left(\frac{1}{d^2 + (q_t(t) + h)^2}\right)\right]$$

Results


Position of the Tine
[mm]
10
8
6
4
2
0
-2
-4
-6
-8
0
200
400
600
800
1000
1200
Samples


Position of the Hammer
[cm]
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
200
400
600
800
1000
1200
Samples

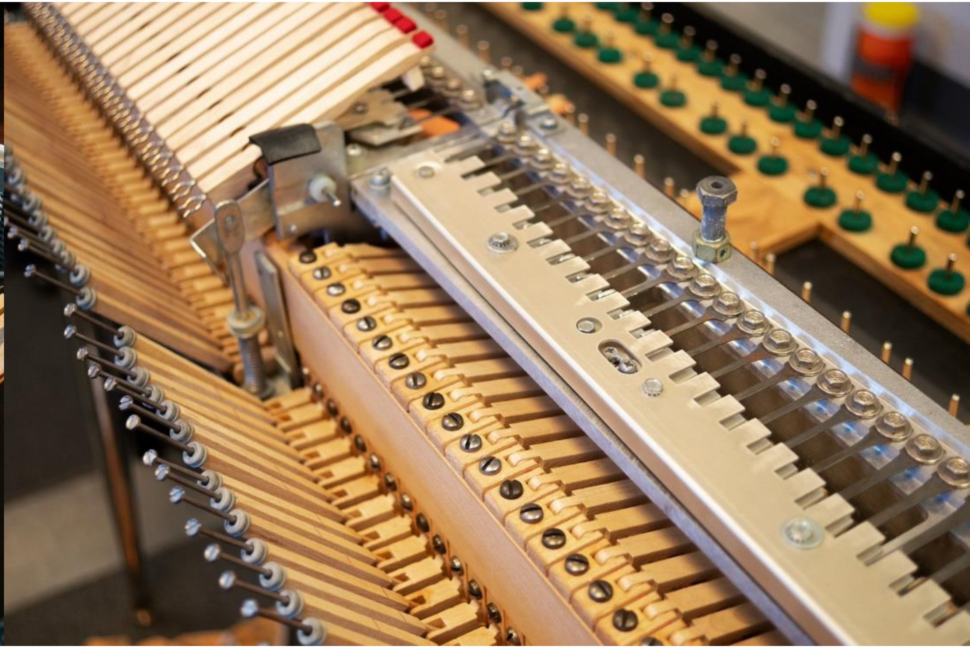
Results



[m] × 10^{-4}
Output signal
4
3
2
1
0
-1
-2
-3
-4
0
200
400
600
800
1000
1200
Samples

Wurlitzer Electric Piano

With the lid of this Wurlitzer 140b off, you can clearly see the keyboard's mechanical action. The reeds and the harp are also partially visible in this photo, particularly in the treble section where the dampers are shorter. In this model, the amplifier is located at the back



With the dampers removed from this 200-series Wurlitzer, you can clearly see the reeds and the pickup

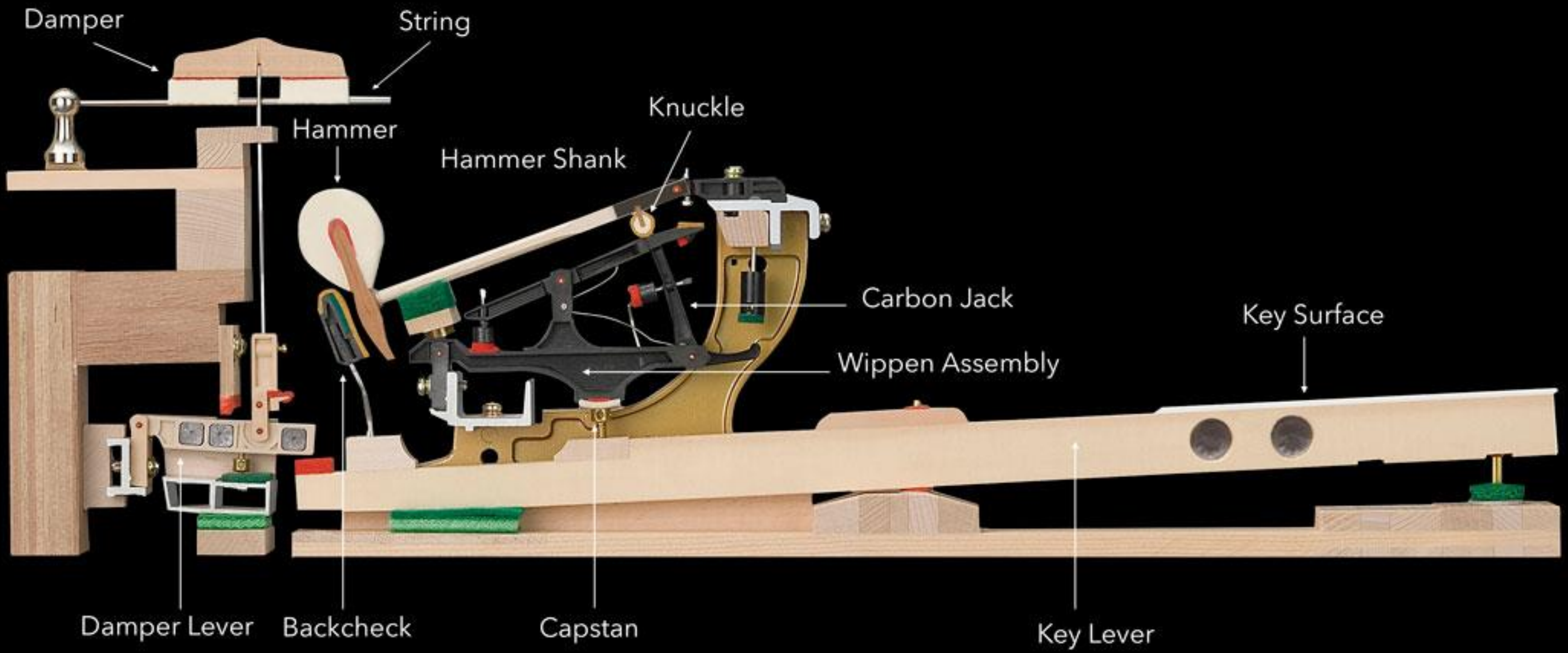
Wurlitzer Electric Piano

This Wurlitzer 200 has a different physical layout than the 140b
The action is more compact, so that the amplifier can fit in the front

Wurlitzer Electric Piano

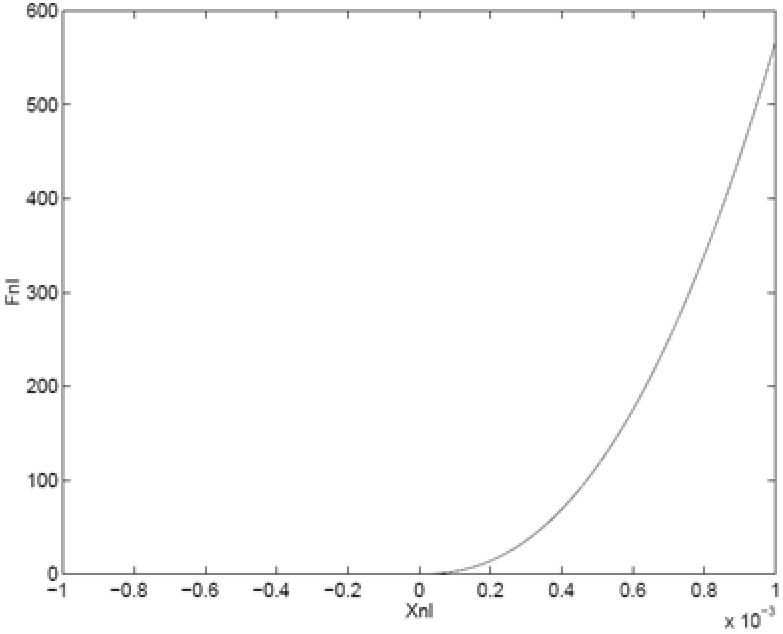
- The sound is generated by striking a metal reed with a hammer, which induces an electric current in a pickup; although conceptually similar to the Rhodes piano, the sound is different
- Invented by Benjamin Miessner, who had worked on various types of electric pianos since the early 1930s
- The first Wurlitzer was manufactured in 1954, and production continued until 1984
- The sound is generated electromechanically by **striking a metal reed with a felt hammer**, using conventional piano action. This induces an electrical current in an electrostatic pickup using a DC voltage of 170V

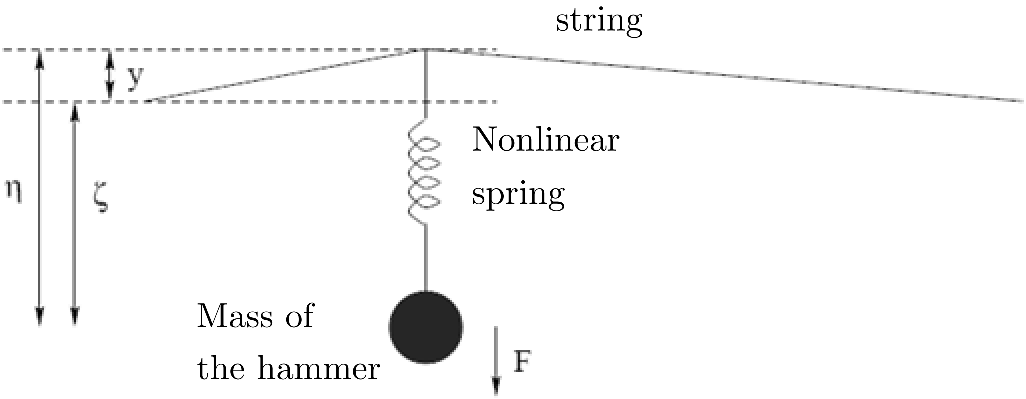
Piano model



https://youtu.be/vFXBIFyG4tU

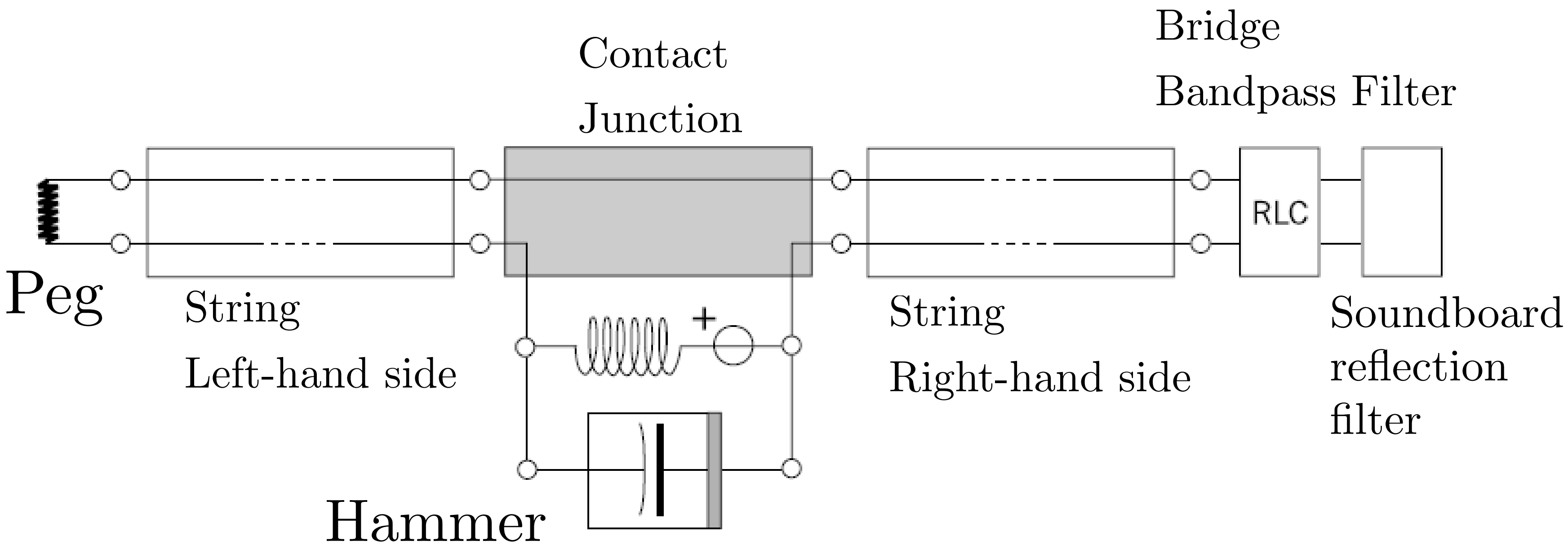
Piano Model





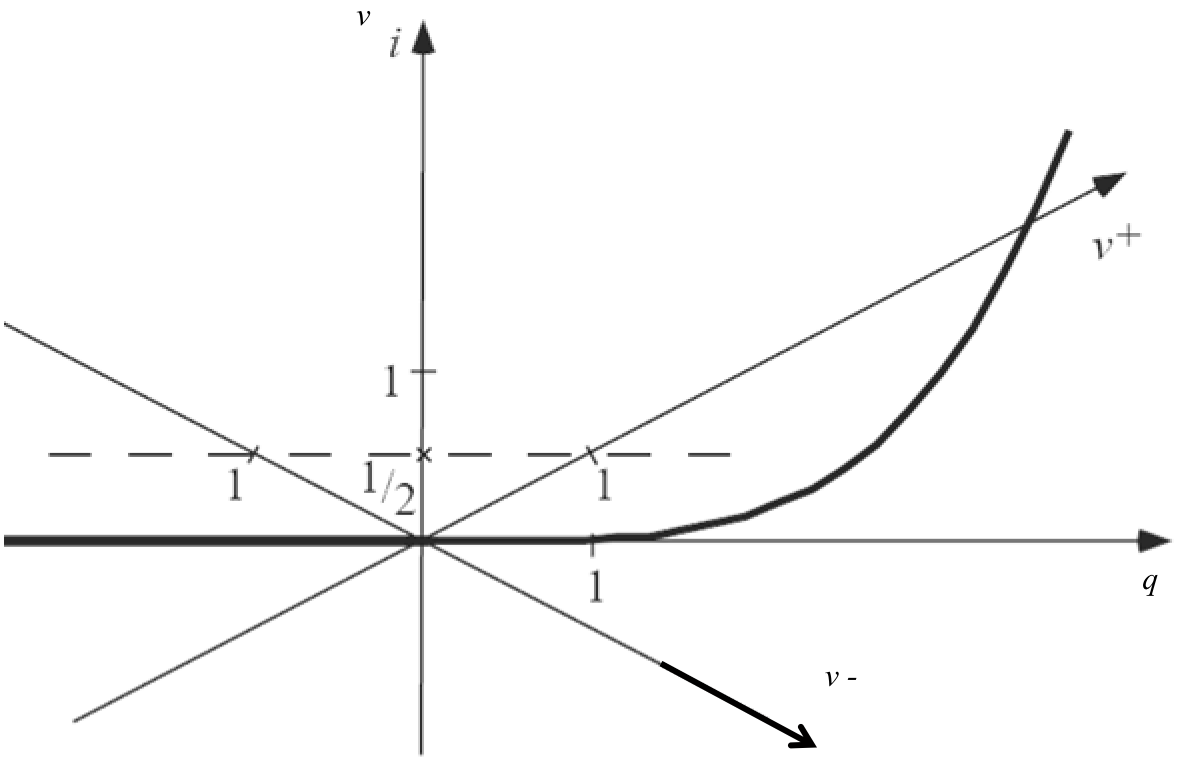
$$
\begin{cases}
f = K(x_m - x_s)^p & \text{per} \quad x_m - x_s > 0 \\
0 & \text{per} \quad x_m - x_s \leq 0
\end{cases}
$$

Equivalent electrical circuit



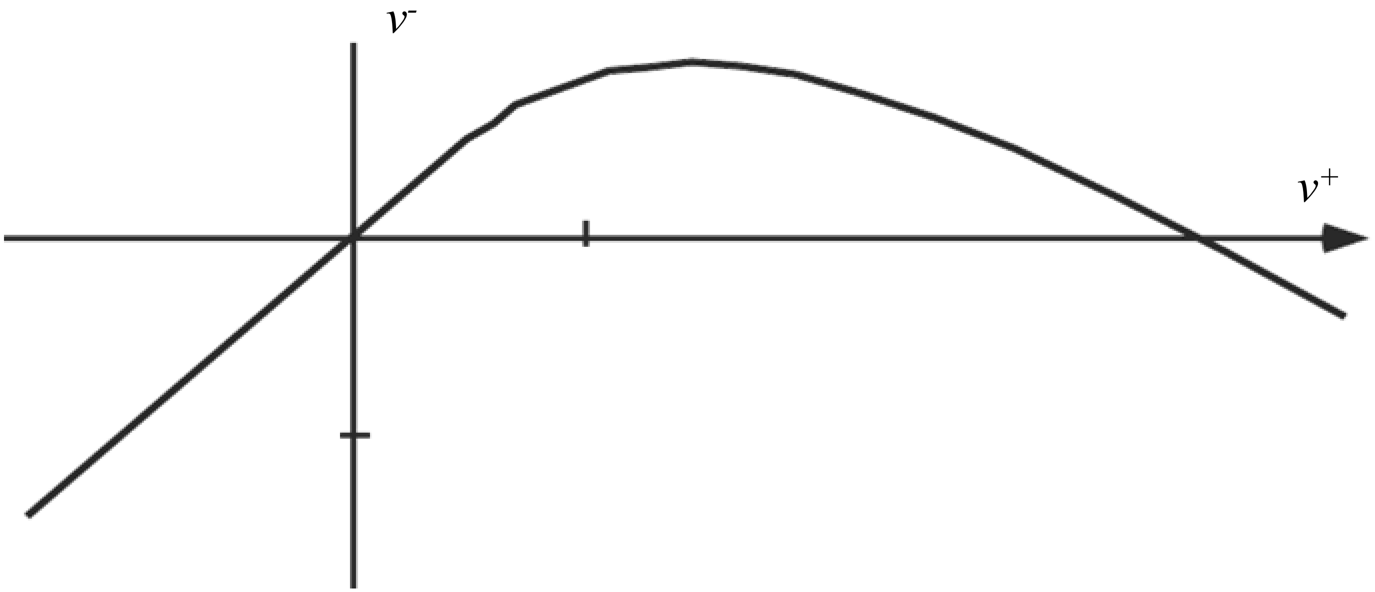
Hammer-string interaction

- Axis transformation



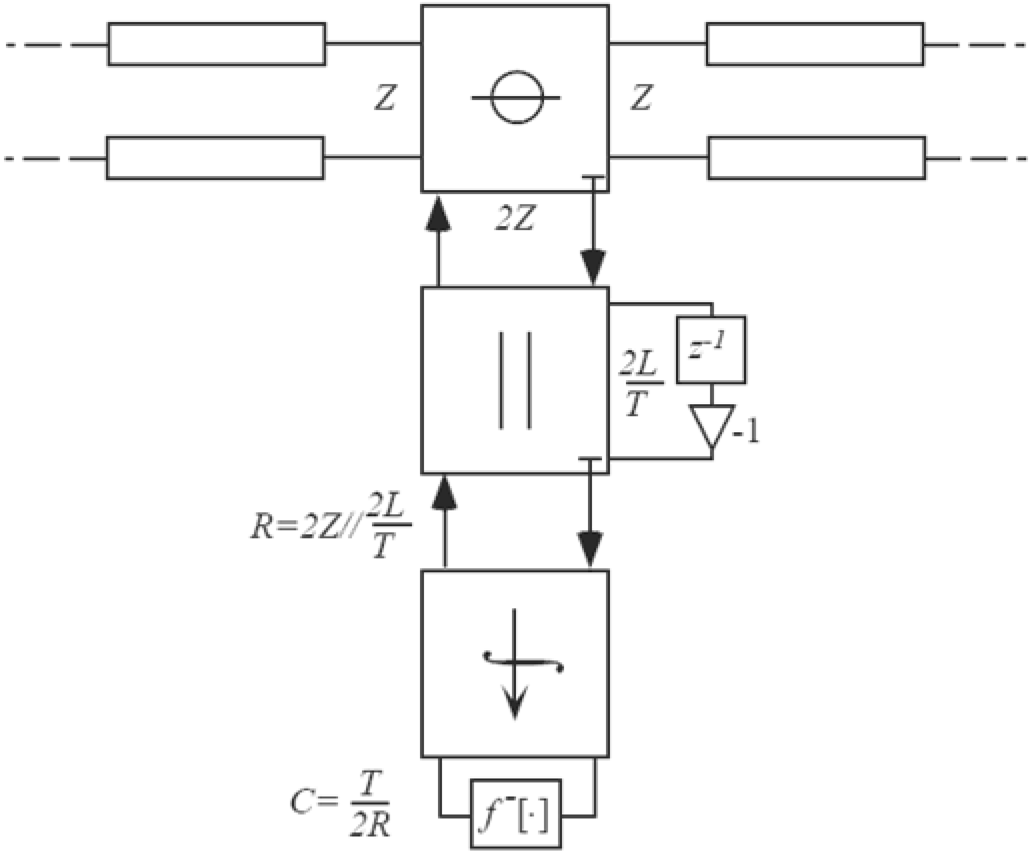
Hammer-string interaction

- NL characteristics in the WD domain

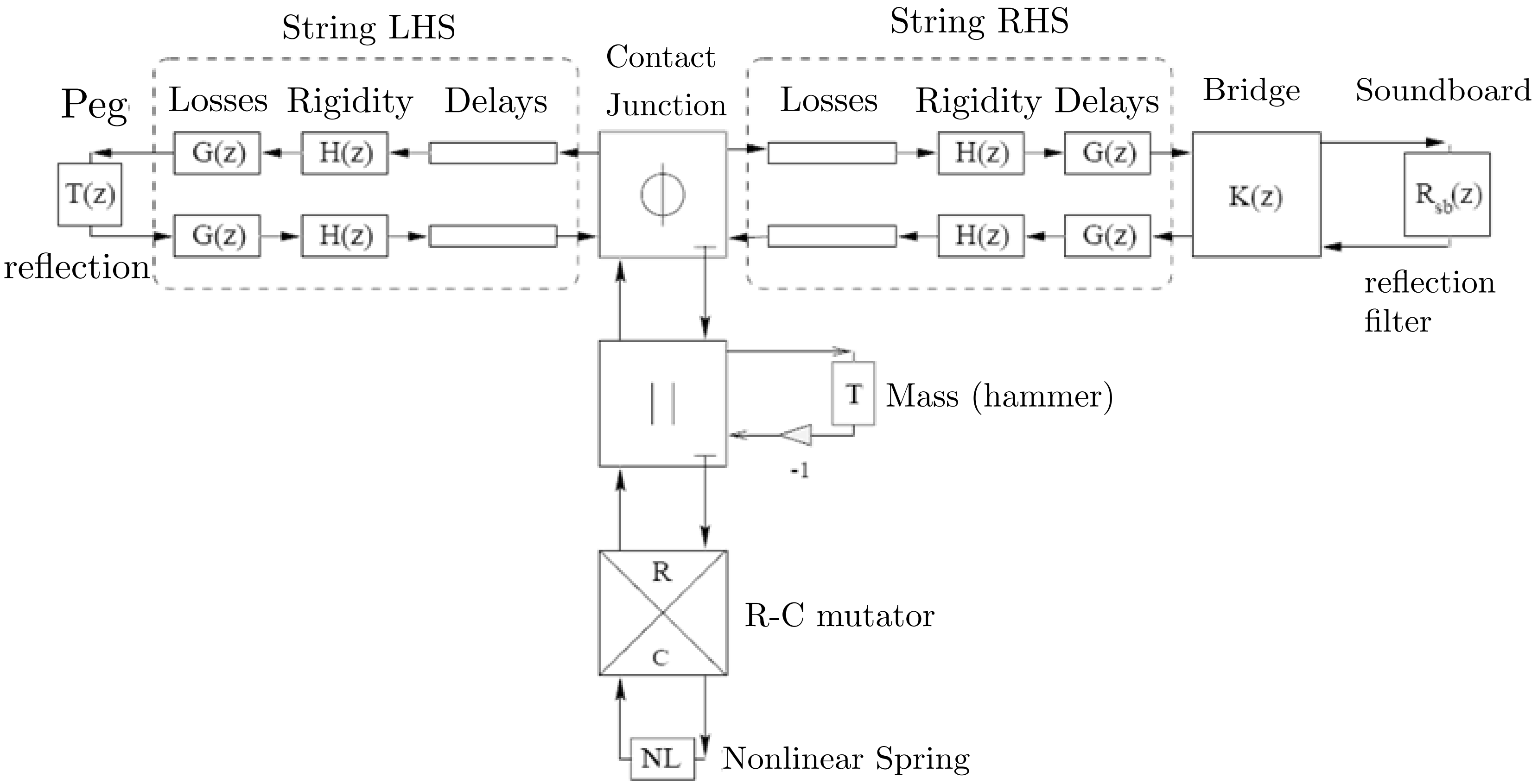


Hammer-string interaction

- WD implementation



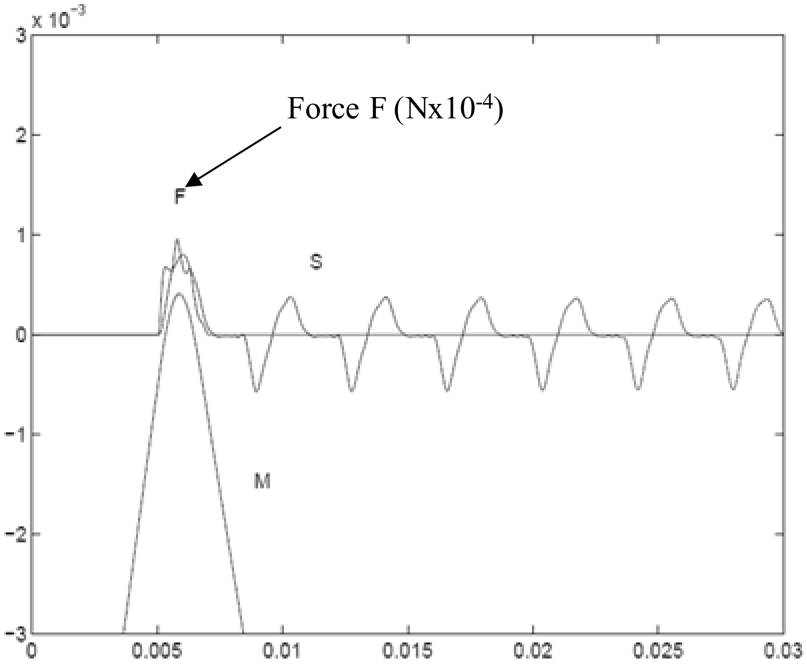
Complete model

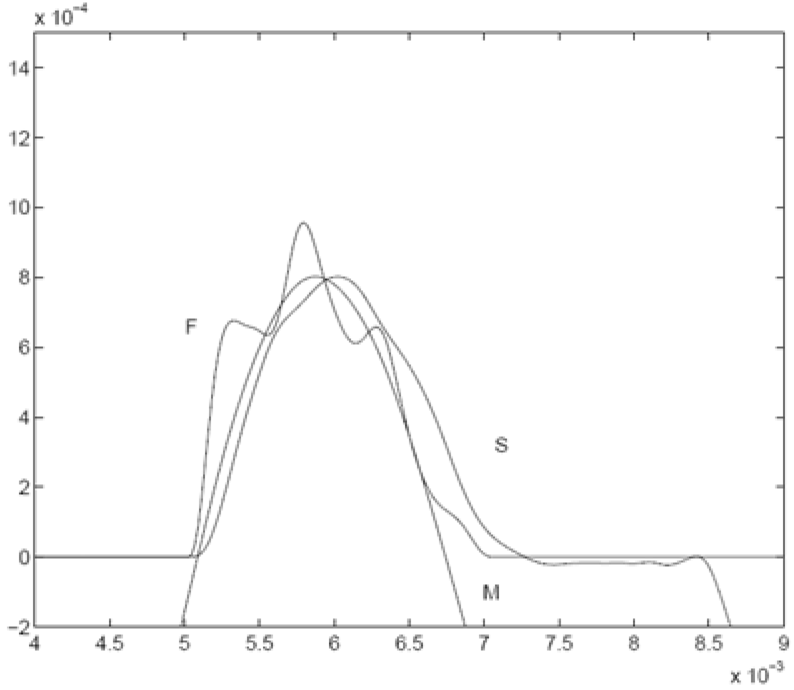


Typical Parameters for Piano Modeling

Parameters	Symbol	C2	C4	Units
Fundamental Frequency	f_0	65.4	262	Hz
String length	L	1.90	0.62	m
Total string mass	M_s	35.0	3.93	g
String tension	T	750	670	N
Coefficient of viscosity	γ	18.42×10^{-3}	2.068×10^{-3}	Kg/(m·s)
Young's modulus	Q	2.0×10^{11}	2.0×10^{11}	$\mathrm{N/m^2}$
Inharmonicity index	k	20.306×10^{-3}	9.838×10^{-3}	$\mathrm{N \cdot m^2}$
Relative striking point	α	0.12	0.12	
Hammer's initial position	y_{m0}	0.01	0.01	m
Hammer's mass	M_h	4.9	2.97	g
NL elasticity exponent	p	2.3	2.5	
Felt's elasticity coefficient	K	4.0×10^8	4.5×10^9	$\mathrm{N/m}^p$

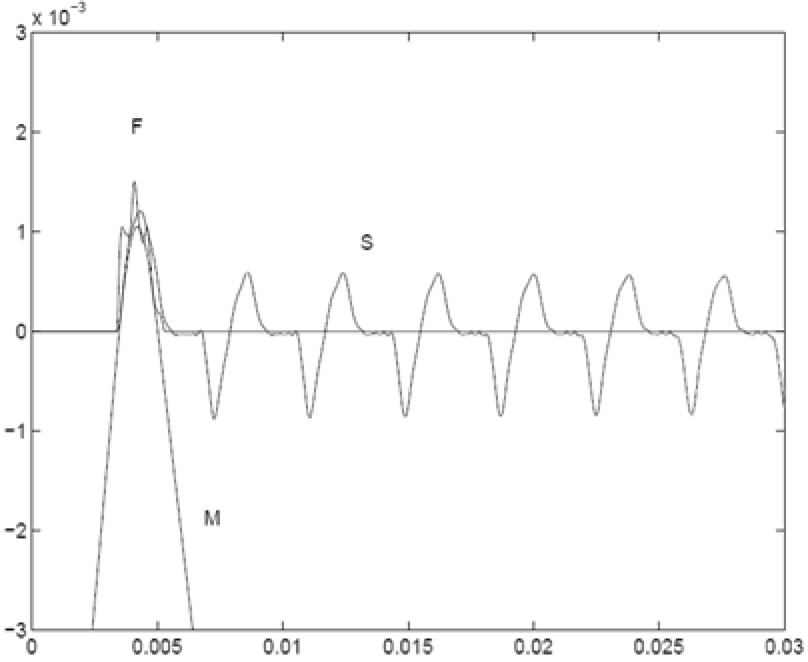
Results

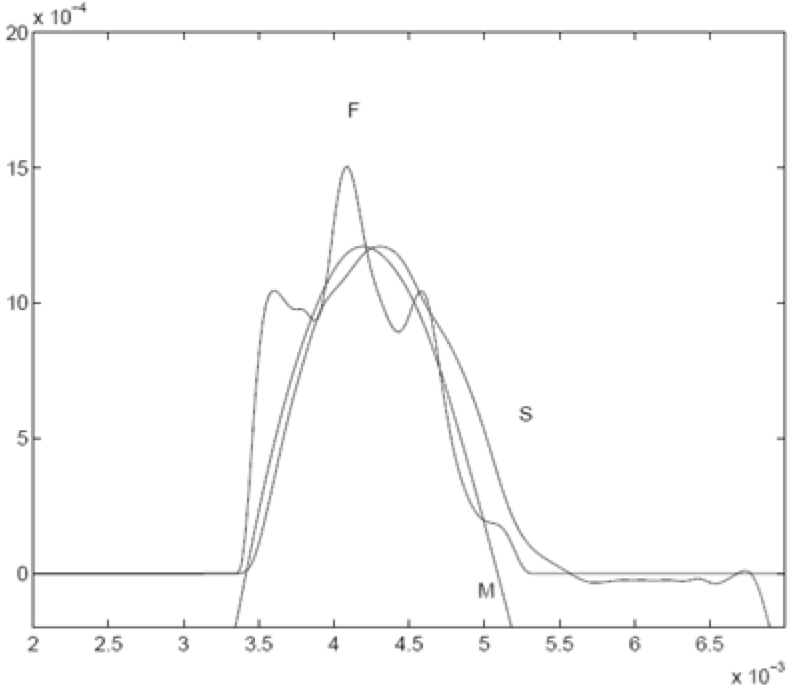




C4 0.75 m/s

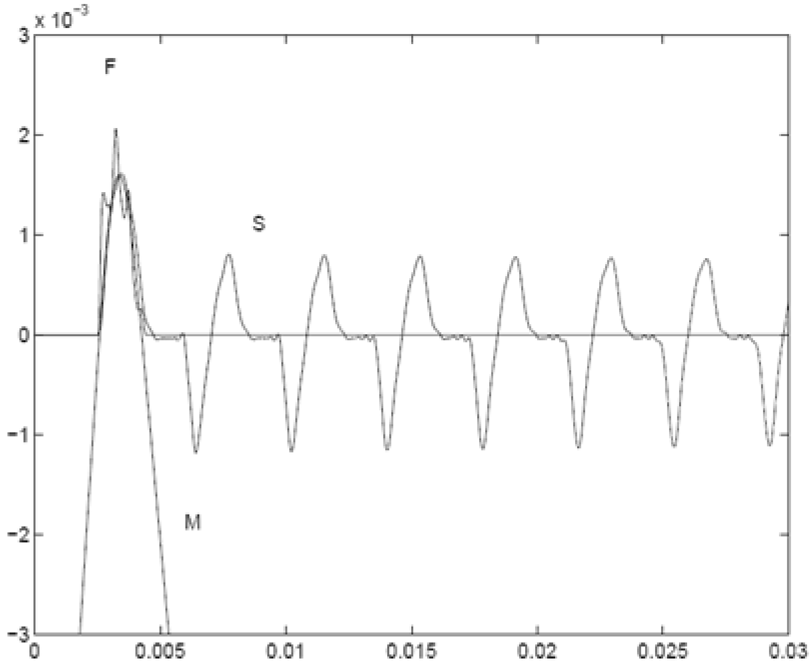
Results

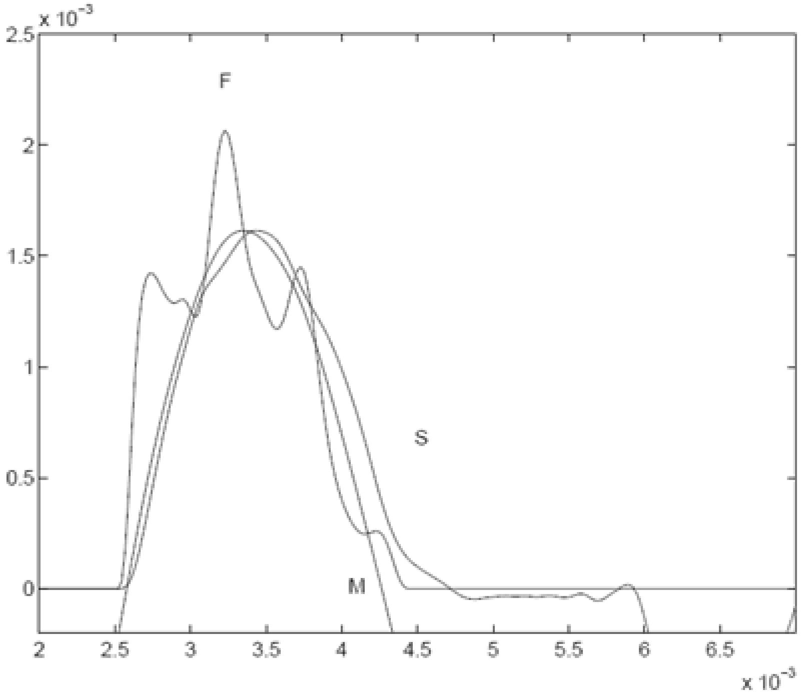




C4 1.5 m/s

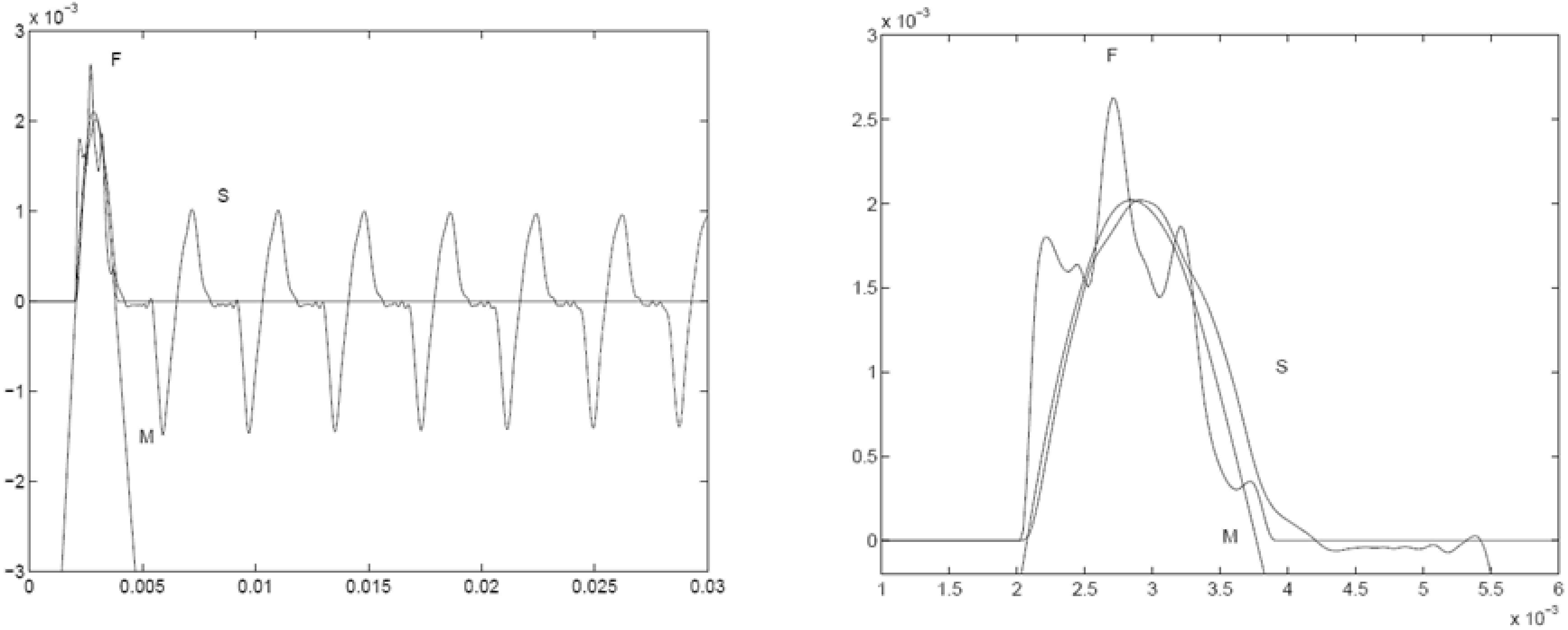
Results





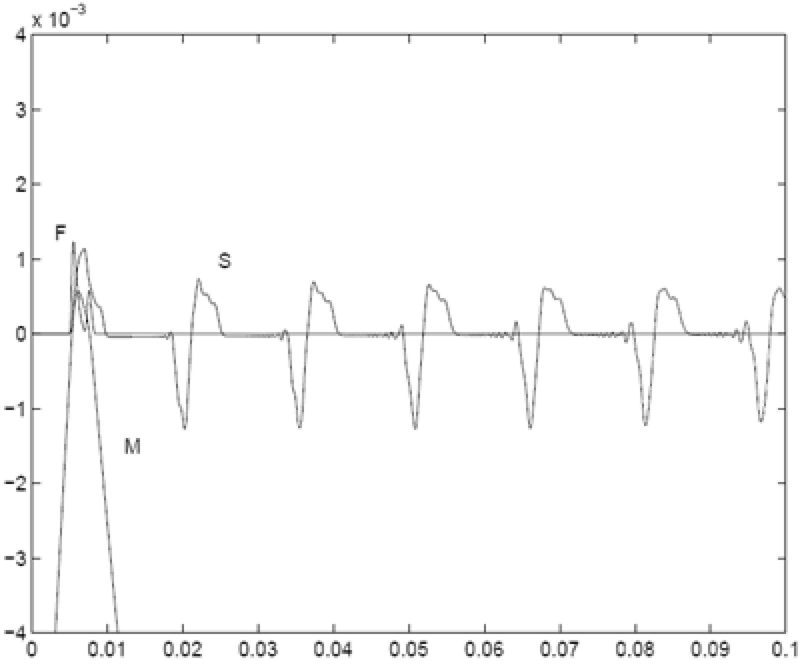
C4 2.75 m/s

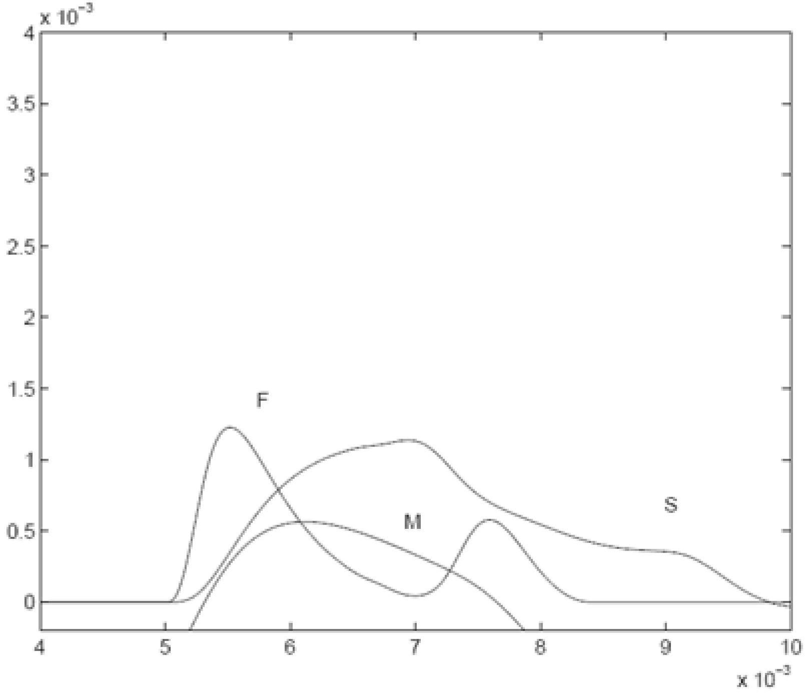
Results



C4 4.5 m/s

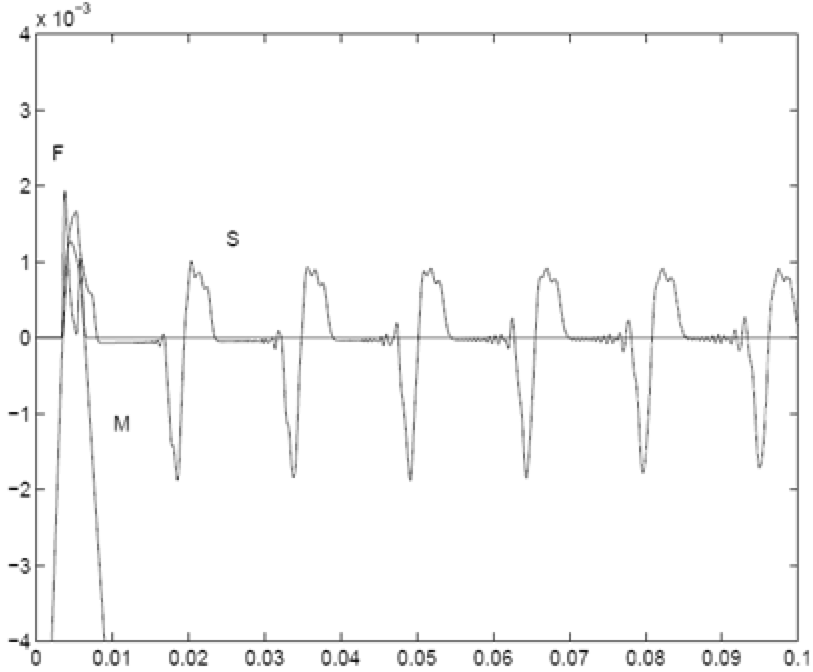
Results

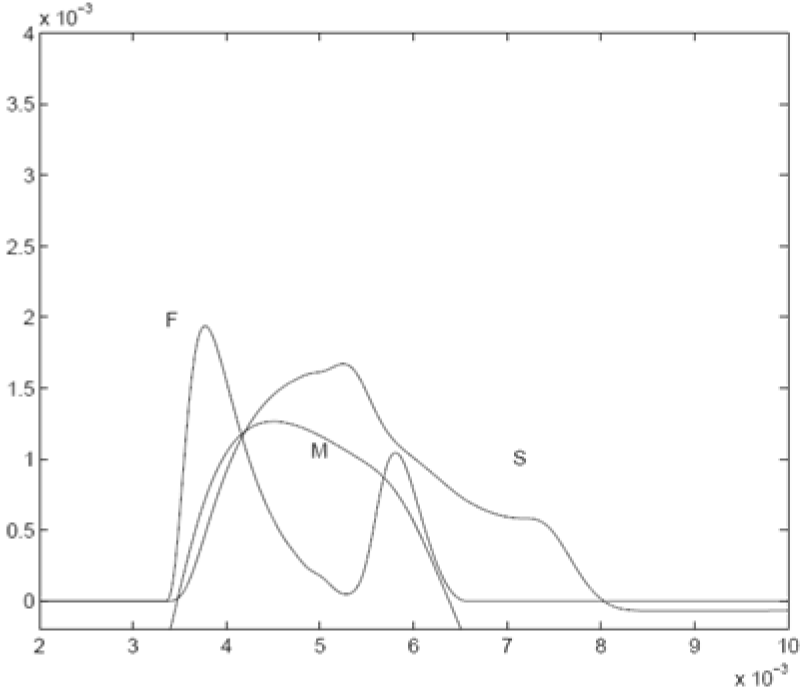




C2 0.75 m/s

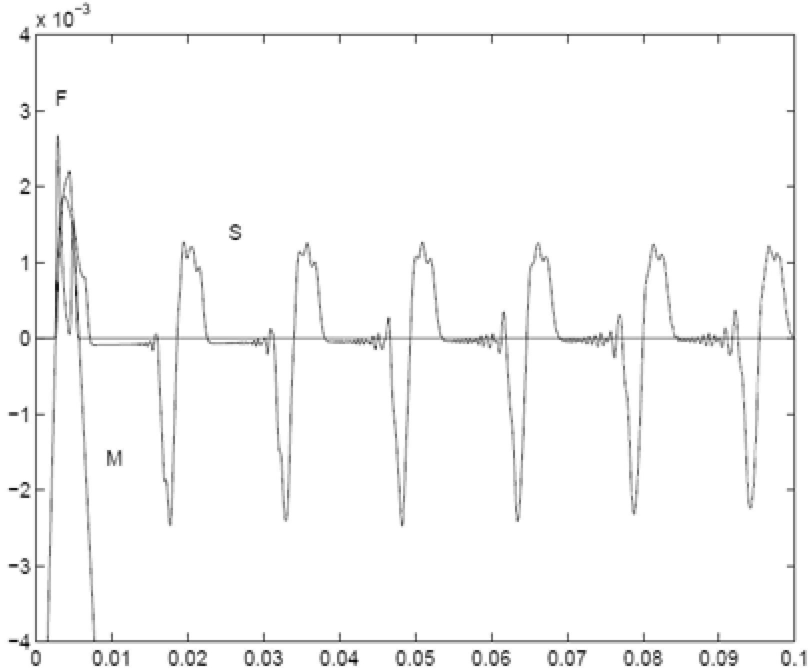
Results

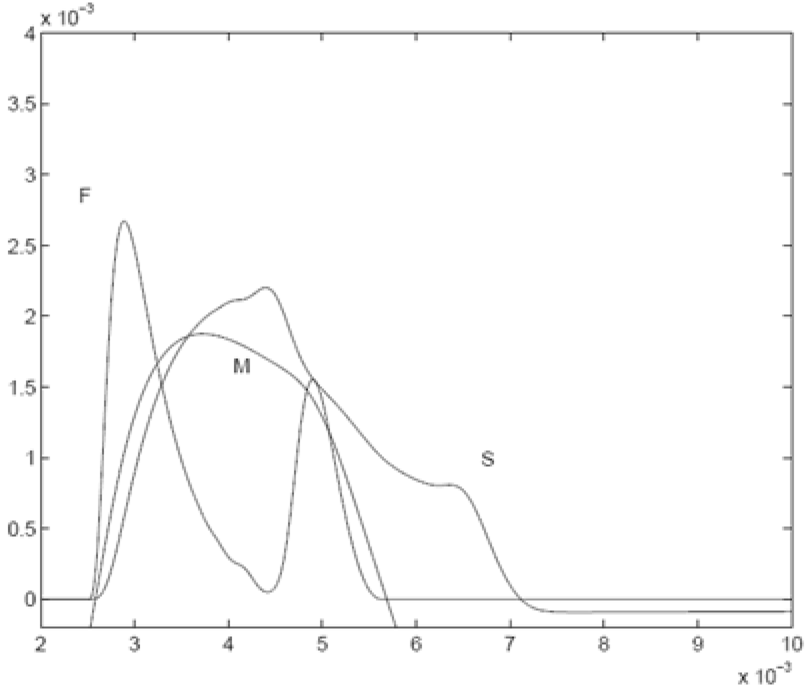




C2 1.5 m/s

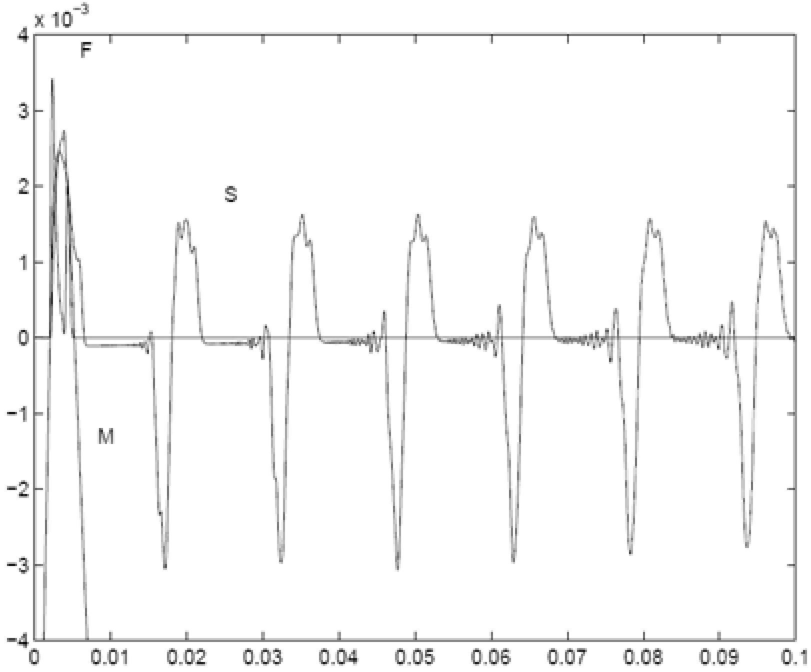
Results

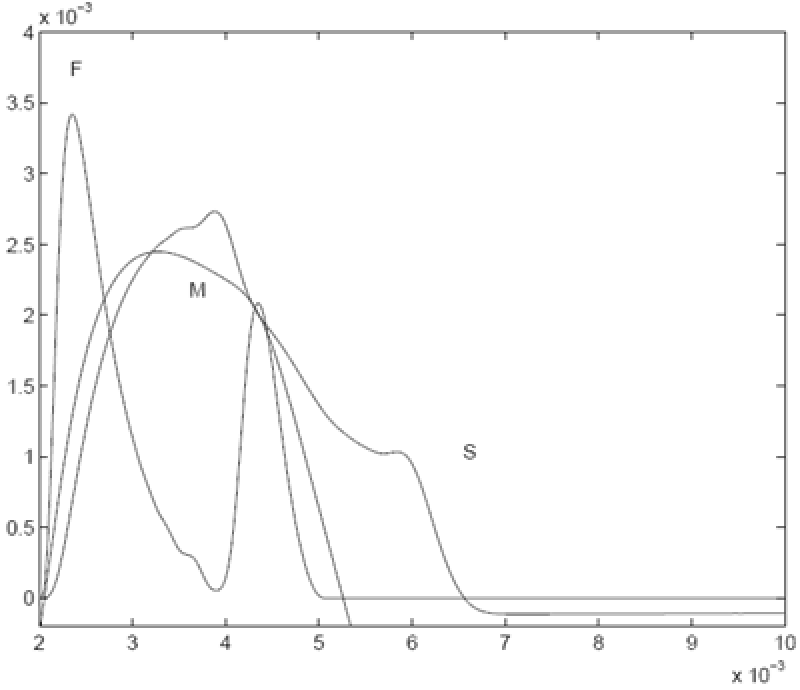




C2 2.75 m/s

Results

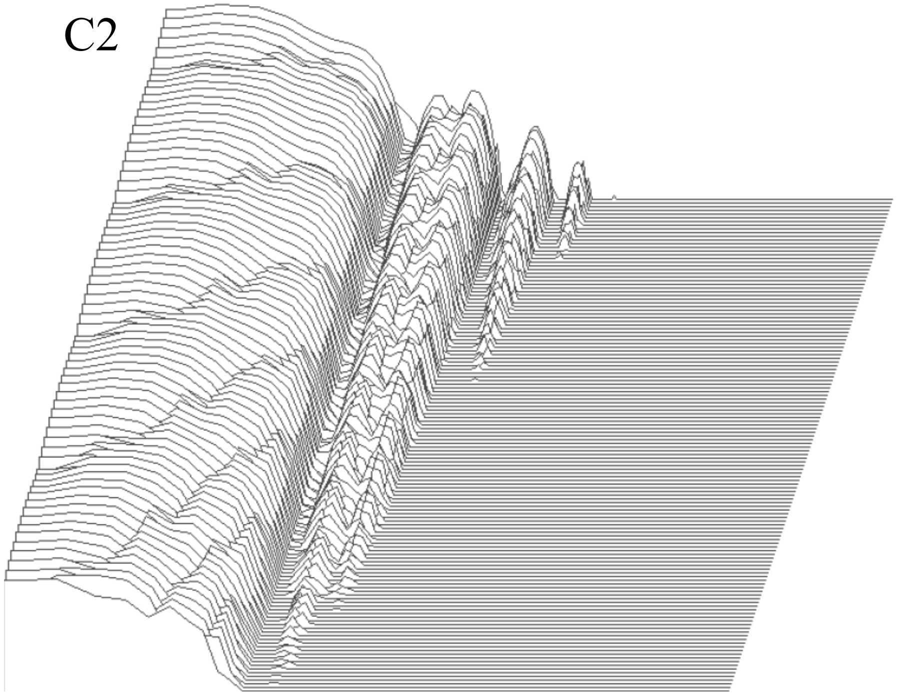




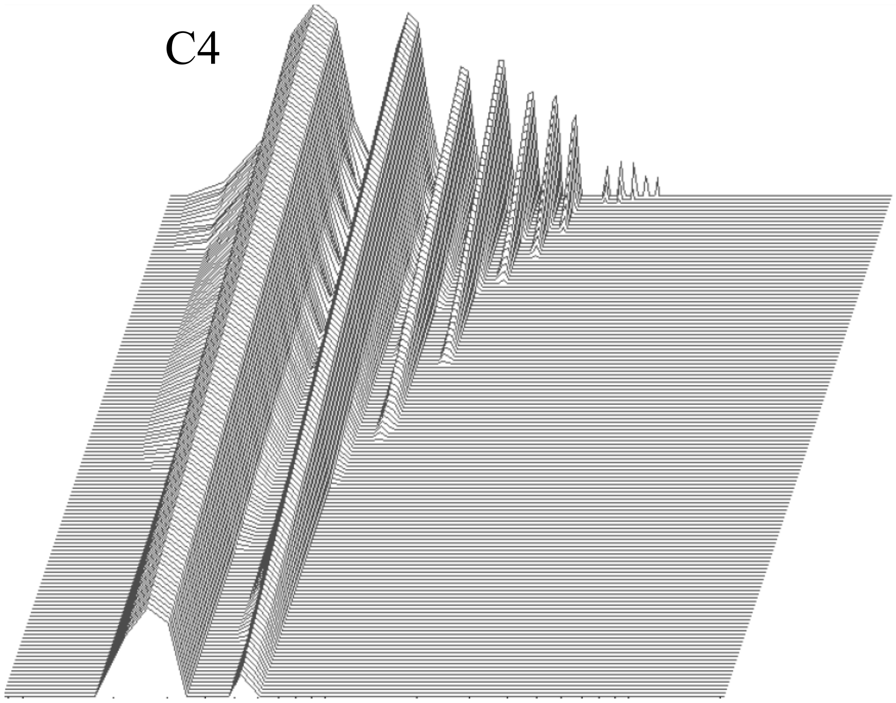
C2 4.5 m/s

Waterfall plot

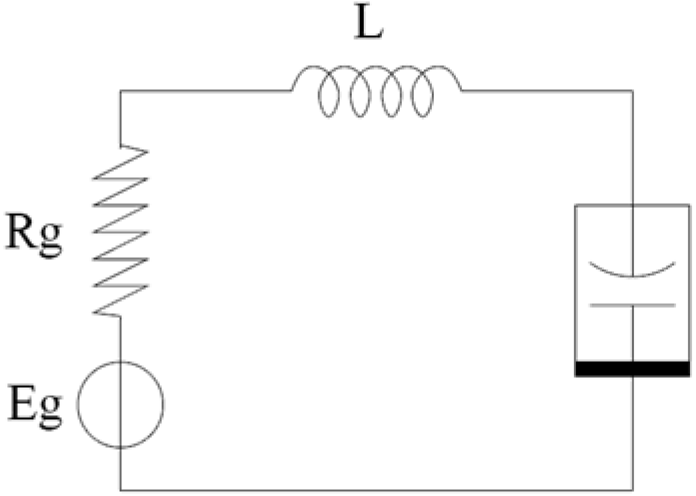
C2



C4



Subharmonic oscillator

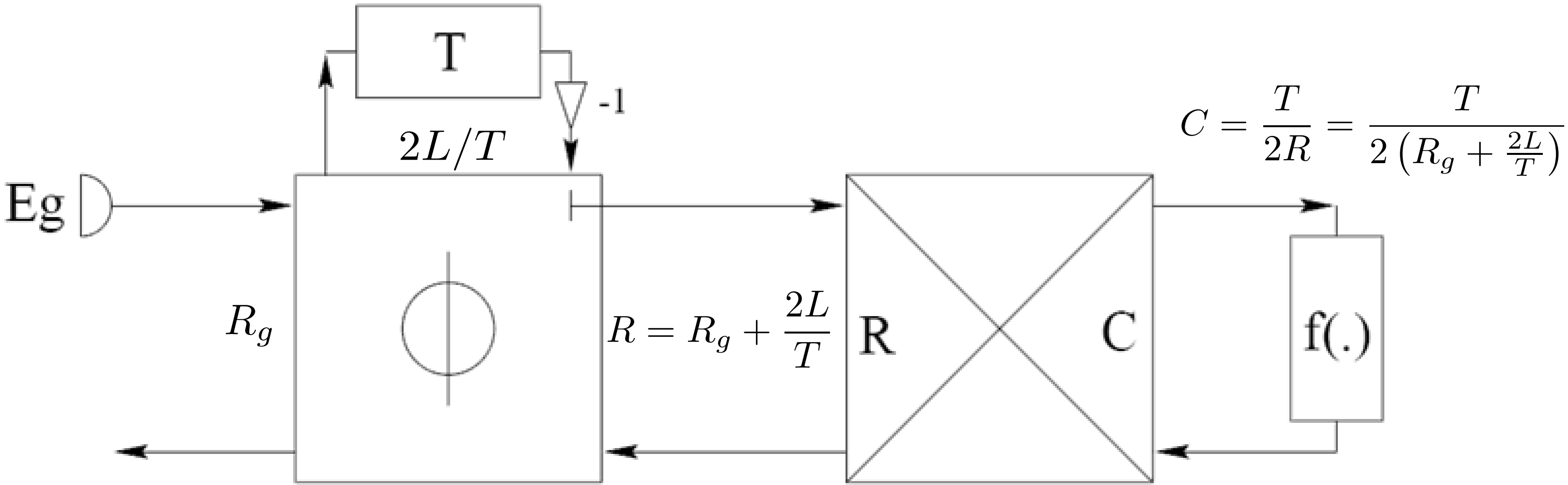


$$v_0 = 0.6V$$
$$R = 180\Omega$$
$$L = 100\mu H$$
$$C_0 = 80pF$$

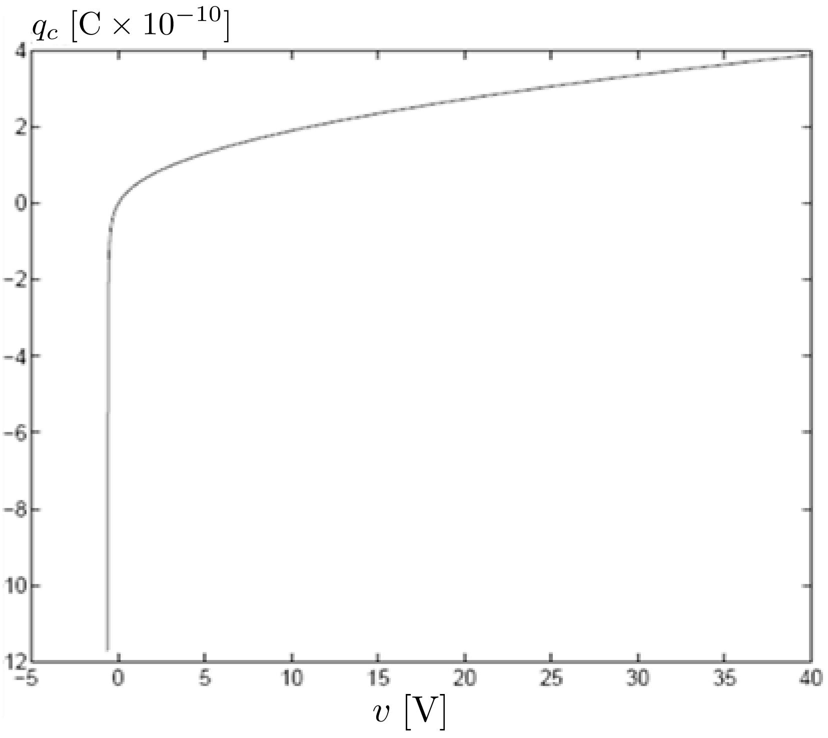
$$\begin{cases} q = C_0 \dfrac{v}{\sqrt{1 + v/v_0}} & \text{per} \quad v > -v_0 \\ 0 & \text{per} \quad v \leq -v_0 \end{cases}$$

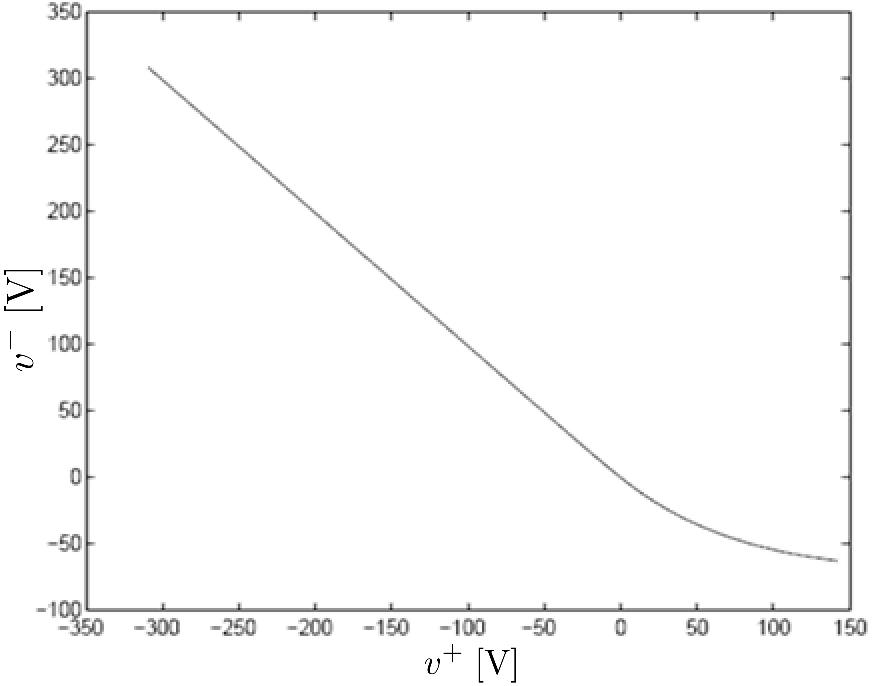
$$e(t) = e_0 \sin(2\pi f_0 t), \quad f_0 = \frac{1}{2\pi\sqrt{LC_0}}$$

Subharmonic oscillator

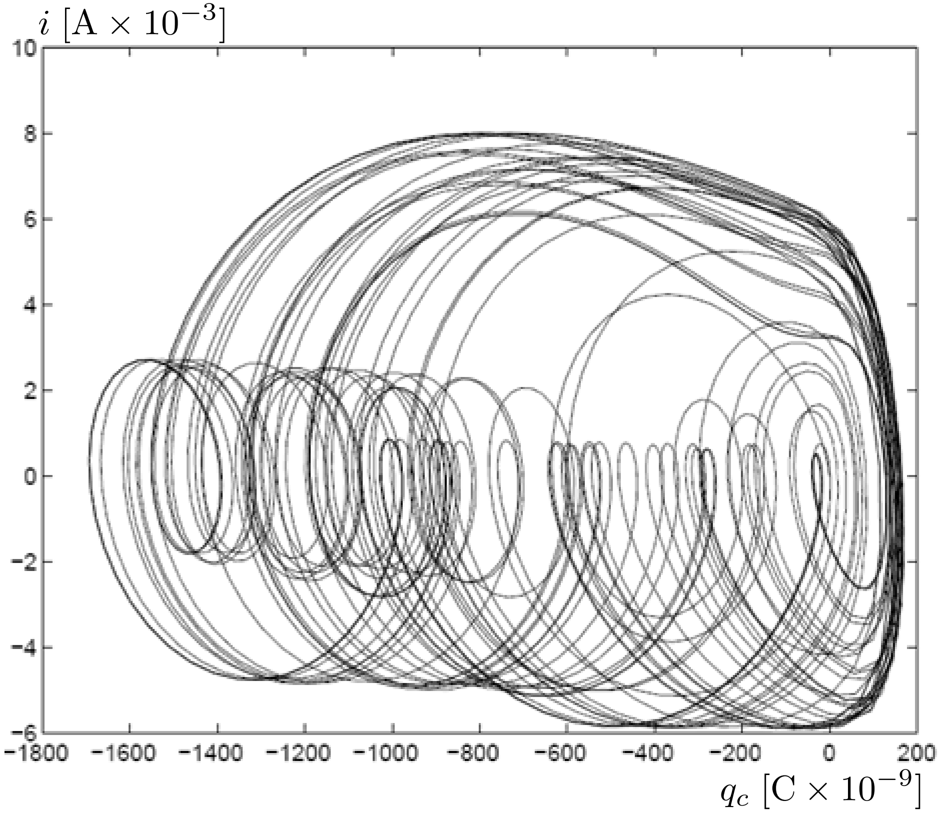


Subharmonic oscillator





Subharmonic oscillator



Phase portrait of $\epsilon_0 = 3.57\,\mathrm{V}$ using the parallel integrator and $T_s = \dfrac{1}{32 f_s}$