5.2 两因素方差分析

一、双因素非重复试验的方差分析

二、双因素等重复试验的方差分析

一、两因素非重复试验的方差分析

1、两因素试验

设有两个因素A,B,因素A有r个不同的水平；B有s个不同水平，A,B的每一种组合水平(A_i,B_j)下做一次试验，试验结果为X_{ij},所有X_{ij}相互独立，所有试验结果为

因素A \ 因素B	B_1	B_2	$\cdots$	B_s
A_1	X_{11}	X_{12}	$\cdots$	X_{1s}
A_2	X_{21}	X_{22}	$\cdots$	X_{2s}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_r	X_{r1}	X_{r2}	$\cdots$	X_{rs}

2、两因素非重复试验

对两个因素的每一组合只做一次试验,。

3、数学模型

假设 $X_{ij} \sim N(\mu_{ij},\sigma^2), i=1,\cdots,r, j=1,\cdots,s.$各$X_{ij}$独立,

μ_{ij},σ^2均为未知参数.

$$\left.\begin{array}{l} X_{ij} = \mu_{ij} + \varepsilon_{ij}, \\ i = 1,2,\cdots,r, j = 1,2,\cdots,s, \\ \varepsilon_{ij} \sim N(0,\sigma^2), \text{各}\varepsilon_{ij}\text{独立}, \end{array}\right\}$$

为了分析此模型，需要做如下变换

设 $\boxed{\mu = \frac{1}{rs}\sum_{i=1}^{r}\sum_{j=1}^{s}\mu_{ij}}$ —总平均

$$\mu_{i\bullet} = \frac{1}{s}\sum_{j=1}^{s}\mu_{ij}, i = 1,\cdots,r \qquad \mu_{\bullet j} = \frac{1}{r}\sum_{i=1}^{r}\mu_{ij}, j = 1,\cdots,s$$

$\boxed{\alpha_i = \mu_{i\bullet} - \mu, \quad i = 1,\cdots,r}$ —水平A_i的效应，表示A_i在总体平均数上引起的偏差

$\boxed{\beta_j = \mu_{\bullet j} - \mu, \quad j = 1,\cdots,s}$ —水平B_j的效应，表示B_j在总体平均数上引起的偏差

$$\sum_{i=1}^{r}\alpha_i = 0, \qquad \sum_{j=1}^{s}\beta_j = 0.$$

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \boxed{(\mu_{ij} - \alpha_i - \beta_j - \mu)}$$

$$= \mu + \alpha_i + \beta_j + \delta_{ij},$$

因素A, B在组合水平(A_i, B_j)交互效应

其中$\delta_{ij} = \mu_{ij} - \alpha_i - \beta_j - \mu$, $\sum_{i=1}^{r} \delta_{ij} = 0, j = 1, \cdots, s$

$\sum_{j=1}^{s} \delta_{ij} = 0, i = 1, \cdots, r$.又由于只做一次试验，没有

交互效应，因而$\delta_{ij} = 0$.因而原模型可以转化为

$$
\begin{cases}
X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \\
\varepsilon_{ij} \sim N(0, \sigma^2), \varepsilon_{ij} \text{相互独立}, \\
i = 1, 2, \cdots, r, j = 1, 2, \cdots, s, \\
\sum_{i=1}^{r} \alpha_i = 0, \sum_{j=1}^{s} \beta_j = 0.
\end{cases}
$$

两因素非重复试验方差分析的数学模型

推断因素A的影响是否显著，等价于假设检验

$$H_{01}: \alpha_1 = \cdots = \alpha_r = 0, \leftrightarrow H_{11}: \alpha_1, \cdots, \alpha_r \text{不全为零}.$$

推断因素B的影响是否显著，等价于假设检验

$$H_{02}: \beta_1 = \cdots = \beta_s = 0, \leftrightarrow H_{12}: \beta_1, \cdots, \beta_s \text{不全为零}.$$

3. 离差平方和分解

$$\bar{X}_{i\bullet} = \frac{1}{s}\sum_{j=1}^{s} X_{ij}, \quad (i = 1, 2, \cdots, r) \quad \bar{X}_{\bullet j} = \frac{1}{r}\sum_{j=1}^{r} X_{ij}, \quad (j = 1, 2, \cdots, s)$$

$$\bar{X} = \frac{1}{rs}\sum_{i=1}^{r}\sum_{j=1}^{s} X_{ij} = \frac{1}{r}\sum_{i=1}^{r} \bar{X}_{i\bullet} = \frac{1}{s}\sum_{j=1}^{s} \bar{X}_{\bullet j}$$

$$\boxed{Q_T = \sum_{i=1}^{r}\sum_{j=1}^{s}(X_{ij} - \bar{X})^2}$$

总离差平方和(总变差)

$$= \sum_{i=1}^{r}\sum_{j=1}^{s}[(X_{ij} - \bar{X}_{i\bullet} - \bar{X}_{\bullet j} + \bar{X}) + (\bar{X}_{i\bullet} - \bar{X}) + (\bar{X}_{\bullet j} - \bar{X})]^2$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s}[(X_{ij}-\bar{X}_{i\bullet}-\bar{X}_{\bullet j}+\bar{X})]^2+\sum_{i=1}^{r}\sum_{j=1}^{s}[(\bar{X}_{i\bullet}-\bar{X})]^2$$

$$+\sum_{i=1}^{r}\sum_{j=1}^{s}[(\bar{X}_{\bullet j}-\bar{X})]^2$$

$$\left.\begin{aligned}
&+2\sum_{i=1}^{r}\sum_{j=1}^{s}(X_{ij}-\bar{X}_{i\bullet}-\bar{X}_{\bullet j}+\bar{X})\cdot(\bar{X}_{i\bullet}-\bar{X})\\
&+2\sum_{i=1}^{r}\sum_{j=1}^{s}(X_{ij}-\bar{X}_{i\bullet}-\bar{X}_{\bullet j}+\bar{X})\cdot(\bar{X}_{\bullet j}-\bar{X})\\
&+2\sum_{i=1}^{r}\sum_{j=1}^{s}(\bar{X}_{i\bullet}-\bar{X})\cdot(\bar{X}_{\bullet j}-\bar{X})
\end{aligned}\right\}\text{交互项}$$

$$= \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{s}[(X_{ij} - \bar{X}_{i\bullet} - \bar{X}_{\bullet j} + \bar{X})]^2}_{Q_E}$$

交互项等于零后面给出证明

$$\underbrace{+ s\sum_{i=1}^{r}(\bar{X}_{i\bullet} - \bar{X})^2}_{Q_A} \underbrace{+ r\sum_{j=1}^{s}(\bar{X}_{\bullet j} - \bar{X})^2}_{Q_B}$$

则

$$Q_T = \boxed{Q_E} \boxed{+Q_A} \boxed{+Q_B}$$

随机误差平方和

因素A引起的离差平方和

因素B引起的离差平方和

证明三个交互项都等于零

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\left(X_{ij}-\bar{X}_{i\cdot}-\bar{X}_{\cdot j}+\bar{X}\right)\left(\bar{X}_{i\cdot}-\bar{X}\right)$$

$$=\sum_{i=1}^{r}\Big[\sum_{j=1}^{s}X_{ij}-\sum_{j=1}^{s}\bar{X}_{i\cdot}-\sum_{j=1}^{s}\bar{X}_{\cdot j}+\sum_{j=1}^{s}\bar{X}\Big](\bar{X}_{i\cdot}-\bar{X})$$

$$=\sum_{i=1}^{r}[s\bar{X}_{i\cdot}-s\bar{X}_{i\cdot}-s\bar{X}+s\bar{X}]\left(\bar{X}_{i\cdot}-\bar{X}\right)=0$$

$$\sum_{j=1}^{s}\sum_{i=1}^{r}\left(X_{ij}-\bar{X}_{i.}-\bar{X}_{.j}+\bar{X}\right)\left(\bar{X}_{.j}-\bar{X}\right)$$

$$=\sum_{j=1}^{s}[r\bar{X}_{\bullet j}-r\bar{X}-r\bar{X}_{\bullet j}+r\bar{X}]\left(\bar{X}_{\bullet j}-\bar{X}\right)=0$$

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\left(\bar{X}_{i.}-\bar{X}\right)\left(\bar{X}_{.j}-\bar{X}\right)$$

$$=\left[\sum_{j=1}^{s}\left(\bar{X}_{.j}-\bar{X}\right)\right]\left[\sum_{i=1}^{r}\left(\bar{X}_{i.}-\bar{X}\right)\right]$$

$$=(s\bar{X}-s\bar{X})(r\bar{X}-r\bar{X})=0$$

3. 离差平方和的统计特性 $\begin{cases} X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \\ \varepsilon_{ij} \sim N(0, \sigma^2), \varepsilon_{ij}\text{相互独立}, \\ i = 1, 2, \cdots, r, j = 1, 2, \cdots, s, \\ \sum_{i=1}^{r} \alpha_i = 0, \sum_{j=1}^{s} \beta_j = 0. \end{cases}$

$$Q_A = s\sum_{i=1}^{r}(\bar{X}_{i\cdot} - \bar{X})^2$$

$$= s\sum_{i=1}^{r}\left(\frac{1}{s}\sum_{j=1}^{s}X_{ij} - \frac{1}{rs}\sum_{i=1}^{r}\sum_{j=1}^{s}X_{ij}\right)^2 = s\sum_{i=1}^{r}\left[\frac{1}{s}\sum_{j=1}^{s}\left(X_{ij} - \frac{1}{r}\sum_{i=1}^{r}X_{ij}\right)\right]^2$$

$$= s\sum_{i=1}^{r}\left\{\frac{1}{s}\sum_{j=1}^{s}\left[(\mu + \alpha_i + \beta_j + \varepsilon_{ij}) - \frac{1}{r}\sum_{i=1}^{r}(\mu + \alpha_i + \beta_j + \varepsilon_{ij})\right]\right\}^2$$

$$=s\sum_{i=1}^{r}\left\{\frac{1}{s}\sum_{j=1}^{s}\left[(\mu+\alpha_i+\beta_j+\varepsilon_{ij})-\frac{1}{r}\sum_{i=1}^{r}(\mu+\alpha_i+\beta_j+\varepsilon_{ij})\right]\right\}^2$$

$$=s\sum_{i=1}^{r}\left\{\frac{1}{s}\sum_{j=1}^{s}\left[(\cancel{\mu}+\alpha_i+\cancel{\beta_j}+\varepsilon_{ij})-\cancel{\mu}-\cancel{\beta_j}-\frac{1}{r}\sum_{i=1}^{r}(\alpha_i+\varepsilon_{ij})\right]\right\}^2$$

$$=s\sum_{i=1}^{r}\left\{\alpha_i+\frac{1}{s}\sum_{j=1}^{s}\left[\varepsilon_{ij}-\underbrace{\frac{1}{r}\sum_{i=1}^{r}\alpha_i}_{0}-\frac{1}{r}\sum_{i=1}^{r}\varepsilon_{ij})\right]\right\}^2$$

$$=s\sum_{i=1}^{r}\left\{\alpha_i+\frac{1}{s}\sum_{j=1}^{s}\varepsilon_{ij}-\frac{1}{rs}\sum_{j=1}^{s}\sum_{i=1}^{r}\varepsilon_{ij}\right\}^2$$

令 $\bar{\varepsilon}_{i.}=\frac{1}{s}\sum_{j=1}^{s}\varepsilon_{ij},\quad i=1,\cdots,r,\ \ \bar{\varepsilon}_{.j}=\frac{1}{r}\sum_{i=1}^{r}\varepsilon_{ij},\ \ j=1,\cdots,s,$

$$\bar{\varepsilon}=\frac{1}{rs}\sum_{i=1}^{r}\sum_{j=1}^{s}\varepsilon_{ij}=\frac{1}{r}\sum_{i=1}^{r}\bar{\varepsilon}_{i.}=\frac{1}{s}\sum_{j=1}^{s}\bar{\varepsilon}_{.j}$$

则 $Q_A=s\sum_{i=1}^{r}(\alpha_i+\bar{\varepsilon}_{i.}-\bar{\varepsilon})^2$

又因为 $\varepsilon_{ij} \sim N(0,\sigma^2), \quad \bar{\varepsilon}_{i\cdot} \sim N(0,\dfrac{\sigma^2}{s}), \quad \bar{\varepsilon}_{\cdot j} \sim N(0,\dfrac{\sigma^2}{r})$

$$\bar{\varepsilon} \sim N(0,\frac{\sigma^2}{sr})$$

因而

$$E\left(Q_A\right) = sE\left[\sum_{i=1}^{r}\left(\alpha_i + \bar{\varepsilon}_{i\cdot} - \bar{\varepsilon}\right)^2\right]$$

$$= s\sum_{i=1}^{r}\alpha_i^2 + sE\left[\sum_{i=1}^{r}\left(\bar{\varepsilon}_{i\cdot} - \bar{\varepsilon}\right)^2\right] + 2s\sum_{i=1}^{r}\alpha_i E\left(\bar{\varepsilon}_{i\cdot} - \bar{\varepsilon}\right)$$

$$= s\sum_{i=1}^{r}\alpha_i^2 + sE[\sum_{i=1}^{r}\bar{\varepsilon}_{i\cdot}^2 - r\bar{\varepsilon}^2] + 2s\sum_{i=1}^{r}\alpha_i[E\bar{\varepsilon}_{i\cdot} - E\bar{\varepsilon}]$$

$$= s\sum_{i=1}^{r}\alpha_i^2 + \left(r-1\right)\sigma^2$$

同理

$$Q_B = r\sum_{j=1}^{s}(\beta_j + \overline{\varepsilon}_{\cdot j} - \overline{\varepsilon})^2$$

$$E(Q_B) = r\sum_{j=1}^{s}\beta_i^{\,2} + (s-1)\sigma^2$$

$$Q_E = \sum_{i=1}^{r}\sum_{j=1}^{s}[\varepsilon_{ij} - \overline{\varepsilon}_{i\cdot} - \overline{\varepsilon}_{\cdot j} + \overline{\varepsilon}]^2$$

$$E(Q_E) = (r-1)(s-1)\sigma^2$$

$$E\bar{Q}_A=\sigma^2+\frac{s}{r-1}\sum_{i=1}^{r}\alpha_i^2,\ \mathrm{E}\bar{Q}_B=\sigma^2+\frac{r}{s-1}\sum_{j=1}^{s}\beta_j^2,$$

$$E(\bar{Q}_E)=\sigma^2$$

由于H_{01}成立时，$E(\bar{Q}_A)=E\bar{Q}_E$,否则$E(\bar{Q}_A)>E\bar{Q}_E$;

由于H_{02}成立时，$E(\bar{Q}_B)=E\bar{Q}_E$,否则$E(\bar{Q}_B)>E\bar{Q}_E$,

因此 构造统计量 $F_A=\dfrac{\bar{Q}_A}{\bar{Q}_E},\quad F_B=\dfrac{\bar{Q}_B}{\bar{Q}_E},$

$$EQ_A = s\sum_{i=1}^{r}\alpha_i^2 + (r-1)\sigma^2$$

对于 $$E(Q_B) = r\sum_{j=1}^{s}\beta_i^2 + (s-1)\sigma^2$$

$$E(Q_E) = (r-1)(s-1)\sigma^2$$

令 $\bar{Q}_A = \dfrac{1}{r-1}Q_A,\ \bar{Q}_B = \dfrac{1}{s-1}Q_B,\ \bar{Q}_E = \dfrac{1}{(r-1)(s-1)}Q_E,$

则 $E\bar{Q}_A = \sigma^2 + \dfrac{s}{r-1}\sum_{i=1}^{r}\alpha_i^2,\ \mathrm{E}\bar{Q}_B = \sigma^2 + \dfrac{r}{s-1}\sum_{j=1}^{s}\beta_j^2,$

$$E(\bar{Q}_E) = \sigma^2$$

$$Q_A = s\sum_{i=1}^{r}(\alpha_i + \bar{\varepsilon}_{i\cdot} - \bar{\varepsilon})^2 \qquad Q_B = r\sum_{j=1}^{s}(\beta_j + \bar{\varepsilon}_{\cdot j} - \bar{\varepsilon})^2$$

$$Q_E = \sum_{i=1}^{r}\sum_{j=1}^{s}[\varepsilon_{ij} - \bar{\varepsilon}_{i\cdot} - \bar{\varepsilon}_{\cdot j} + \bar{\varepsilon}]^2 \quad Q_T = \sum_{i=1}^{r}\sum_{j=1}^{s}(X_{ij} - \bar{X})^2$$

4. 统计量的分布

由于H_{01}, H_{02}成立时， $\alpha_i = \beta_j = 0(i = 1,\cdots,r,\ j = 1,\cdots,s)$

因而$X_{ij} = \mu + \varepsilon_{ij}$,则离差平方和可以改写为

$$Q_A = s\sum_{i=1}^{r}(\bar{\varepsilon}_{i\cdot} - \bar{\varepsilon})^2 \qquad Q_B = r\sum_{j=1}^{s}(\bar{\varepsilon}_{\cdot j} - \bar{\varepsilon})^2$$

$$Q_E = \sum_{i=1}^{r}\sum_{j=1}^{s}[\varepsilon_{ij} - \bar{\varepsilon}_{i\cdot} - \bar{\varepsilon}_{\cdot j} + \bar{\varepsilon}]^2$$

$$Q_T = \sum_{i=1}^{r}\sum_{j=1}^{s}(\varepsilon_{ij} - \bar{\varepsilon})^2 = Q_A + Q_B + Q_E$$

又由于$\frac{\varepsilon_{ij}}{\sigma} \sim N(0,1)$,由定理1.12可知，

$$\frac{Q_T}{\sigma^2} = \sum_{i=1}^{r}\sum_{j=1}^{s}\left(\frac{\varepsilon_{ij} - \overline{\varepsilon}}{\sigma}\right)^2 \sim \chi^2(rs-1).$$

$$\frac{Q_A}{\sigma^2} = \sum_{i=1}^{r}\left(\frac{\overline{\varepsilon}_{i\cdot} - \overline{\varepsilon}}{\sigma/\sqrt{s}}\right)^2 \sim \chi^2(r-1).$$

$$\frac{Q_B}{\sigma^2} = \sum_{j=1}^{S}\left(\frac{\overline{\varepsilon}_{\cdot j} - \overline{\varepsilon}}{\sigma/\sqrt{r}}\right)^2 \sim \chi^2(s-1).$$

$$Q_E=\sum_{i=1}^{r}\sum_{j=1}^{s}[\varepsilon_{ij}-\bar{\varepsilon}_{i\cdot}-\bar{\varepsilon}_{\cdot j}+\bar{\varepsilon}]^2$$

而$\frac{1}{\sigma^2}Q_E$具有约束$\sum_{i=1}^{r}(\varepsilon_{ij}-\bar{\varepsilon}_{i\cdot}-\bar{\varepsilon}_{\cdot j}+\bar{\varepsilon})=0(j=1,\cdots,s)$

以及约束$\sum_{j=1}^{s}(\varepsilon_{ij}-\bar{\varepsilon}_{i\cdot}-\bar{\varepsilon}_{\cdot j}+\bar{\varepsilon})=0(i=1,\cdots,r)$,而最后一个约束可以由前$s+r-1$得到，因而其独立约束条件共$rs-s-r+1=(r-1)(s-1)$.

显然，离差平方和公式的左右两边自由度满足：

$$rs-1=(r-1)+(s-1)+(rs-r-s+1)$$

由柯赫伦因子分解定理(p16定理1.7)可知:

$$\frac{1}{\sigma^2}Q_E \sim \chi^2(rs-r-s+1)$$

因而

$$F_A = \frac{\dfrac{Q_A}{\sigma^2(r-1)}}{\dfrac{Q_E}{\sigma^2(r-1)(s-1)}} = \frac{\bar{Q}_A}{\bar{Q}_E} \sim F\big(r-1,(r-1)(s-1)\big)$$

$$F_B = \frac{\dfrac{Q_B}{\sigma^2(s-1)}}{\dfrac{Q_E}{\sigma^2(r-1)(s-1)}} = \frac{\bar{Q}_B}{\bar{Q}_E} \sim F\big(s-1,(r-1)(s-1)\big)$$

5. 方差分析对应的拒绝域

在给定显著性水平α下，因素A对试验结果有显著影响的拒绝域为

$$W_A = \{F_A \mid F_A \geq F_\alpha(r-1,(r-1)(s-1))\}$$

在给定显著性水平α下，因素B对试验结果有显著影响的拒绝域为

$$W_{\mathrm{B}} = \{F_B \mid F_B \geq F_\alpha(s-1,(r-1)(s-1))\}$$

表5.9　双因素非重复试验的方差分析表

方差来源	平方和	自由度	均　方	F　比
因素A	Q_A	$r-1$	$\bar{Q}_A = \frac{Q_A}{r-1}$	$F_A = \frac{\bar{Q}_A}{\bar{Q}_E}$
因素B	Q_B	$s-1$	$\bar{Q}_B = \frac{Q_B}{s-1}$	$F_B = \frac{\bar{Q}_B}{\bar{Q}_E}$
误　差	Q_E	$(r-1)(s-1)$	$\bar{Q}_E = \frac{Q_E}{(r-1)(s-1)}$	
总　和	Q_T	$rs-1$		

为了计算方便，通常可以采用如下公式：令

$$T=\sum_{i=1}^{r}\sum_{j=1}^{s}X_{ij},\quad P=\frac{1}{rs}T^2,\quad R=\sum_{i=1}^{r}\sum_{j=1}^{s}X_{ij}^2,$$

$$Q_I=\frac{1}{s}\sum_{i=1}^{r}\left(\sum_{j=1}^{s}X_{ij}\right)^2,\quad Q_{II}=\frac{1}{r}\sum_{j=1}^{s}\left(\sum_{i=1}^{r}X_{ij}\right)^2,$$

则 $$\begin{cases} Q_A=Q_I-P, \\ Q_B=Q_{II}-P, \\ Q_E=R-Q_I-Q_{II}+P, \\ Q_T=R-P \end{cases}$$

例1 (p165例5.5)为了提高某种合金钢的强度，需要同时考察炭C以及Ti的含量对强度的影响，以便选取合理的成份组合使得强度达到最大，在试验中分别取因素A(C的含量)3个水平，因素B(Ti的含量)4个水平，在组合$(A_i, B_j)(i=1,2,3, j=1,2,3,4)$条件下各炼一炉测得的强度为:

A水平 \ B水平	B_1	B_2	B_3	B_4
A_1	63.1	63.9	65.6	66.8
A_2	65.1	66.4	67.8	69.0
A_3	67.2	71.0	71.9	73.5

试问：炭与钛的含量对合金钢的强度是否有显著影响

(α=0.01).

解 $r = 3, s = 4, rs = 12,$ 经计算

$$Q_T = 113.29,\quad Q_A = 74.91,\quad Q_B = 35.7,\quad Q_E = 3.21$$

$$F_A = \frac{\overline{Q}_A}{\overline{Q}_E} = 70.02 > F_{0.01}(2,6) = 10.9$$

$$F_B = \frac{\overline{Q}_B}{\overline{Q}_E} = 21.91 > F_{0.01}(3,6) = 9.78$$

因而炭与钛的含量对合金钢的强度是有显著影响.

二、双因素等重复试验的方差分析

因素$A: A_1, A_2, \cdots, A_r$, 因素$B: B_1, B_2, \cdots, B_s$,每一个组合水平(A_i, B_j)下重复试验t次，测得的数据为X_{ijk},如表表.5.12

因素A \ 因素B	B_1	B_2	$\cdots$	B_s
A_1	$X_{111}, X_{112}, \cdots, X_{11t}$	$X_{121}, X_{122}, \cdots, X_{12t}$	$\cdots$	$X_{1s1}, X_{1s2}, \cdots, X_{1st}$
A_2	$X_{211}, X_{212}, \cdots, X_{21t}$	$X_{221}, X_{222}, \cdots, X_{22t}$	$\cdots$	$X_{2s1}, X_{2s2}, \cdots, X_{2st}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_r	$X_{r11}, X_{r12}, \cdots, X_{r1t}$	$X_{r21}, X_{r22}, \cdots, X_{r2t}$	$\cdots$	$X_{rs1}, X_{rs2}, \cdots, X_{rst}$

1. 数学模型

假设 $X_{ijk} \sim N(\mu_{ij}, \sigma^2), i = 1, \cdots, r, j = 1, \cdots, s, k = 1, \cdots, t.$

各 X_{ijk} 独立, μ_{ij}, σ^2 均为未知参数.

$$\begin{cases} X_{ijk} = \mu_{ij} + \varepsilon_{ijk}, \\ \varepsilon_{ijk} \sim N(0, \sigma^2), \text{各} \varepsilon_{ijk} \text{独立}, \\ i = 1, 2, \cdots, r, j = 1, 2, \cdots, s, \\ k = 1, 2, \cdots, t. \end{cases}$$

设 $\boxed{\mu = \frac{1}{rs}\sum_{i=1}^{r}\sum_{j=1}^{s}\mu_{ij}}$ $\boxed{\text{—总平均}}$

$$\mu_{i\bullet} = \frac{1}{s}\sum_{j=1}^{s}\mu_{ij}, i = 1,\cdots,r \qquad \mu_{\bullet j} = \frac{1}{r}\sum_{i=1}^{r}\mu_{ij}, j = 1,\cdots,s$$

$\boxed{\alpha_i = \mu_{i\bullet} - \mu, \quad i = 1,\cdots,r}$ —水平A_i的效应，表示A_i在总体平均数上引起的偏差

$\boxed{\beta_j = \mu_{\bullet j} - \mu, \quad j = 1,\cdots,s}$ —水平B_j的效应，表示B_j在总体平均数上引起的偏差

$$\delta_{ij} = \boxed{\mu_{ij} - \mu - \alpha_i - \beta_j,}$$ —组合水平(A_i, B_j)的交互作用效应

则 $\sum_{i=1}^{r} \alpha_i = 0,\qquad \sum_{j=1}^{s} \beta_j = 0.$

$$\sum_{i=1}^{r} \delta_{ij} = 0, j = 1, \cdots, s, \qquad \sum_{j=1}^{s} \delta_{ij} = 0, i = 1, \cdots, r.$$

证明

$$\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \frac{1}{s} \sum_{j=1}^{s} (\mu_{ij} - \mu)$$

$$= \frac{1}{s} \sum_{i=1}^{r} \sum_{j=1}^{s} \mu_{ij} - \frac{1}{s} \sum_{i=1}^{r} \sum_{j=1}^{s} \mu = \frac{1}{s} rs\mu - \frac{1}{s} rs\mu = 0$$

$$\sum_{j=1}^{s}\beta_j = \sum_{j=1}^{s}\frac{1}{r}\sum_{i=1}^{r}(\mu_{ij}-\mu)$$

$$=\frac{1}{r}\sum_{j=1}^{s}\sum_{i=1}^{r}\mu_{ij}-\frac{1}{r}\sum_{j=1}^{s}\sum_{i=1}^{r}\mu=\frac{1}{r}sr\mu-\frac{1}{r}sr\mu= 0$$

$$\sum_{i=1}^{r}\delta_{ij}=\sum_{i=1}^{r}(\mu_{ij}-\mu-\alpha_i-\beta_j)$$

$$=\sum_{i=1}^{r}\mu_{ij}-\sum_{i=1}^{r}\mu-\sum_{i=1}^{r}\alpha_i-\sum_{i=1}^{r}\beta_j$$

$$=r\mu_{.j}-r\mu-r\beta_j$$

$$=r(\mu_{.j}-\mu-\beta_j)=0$$

$$\boxed{\beta_j=\mu_{\bullet j}-\mu,\ \ j=1,\cdots,s}$$

$$\sum_{j=1}^{s}\delta_{ij} = \sum_{j=1}^{s}(\mu_{ij} - \mu - \alpha_i - \beta_j)$$

$$= \sum_{j=1}^{s}\mu_{ij} - \sum_{j=1}^{s}\mu - \sum_{j=1}^{s}\alpha_i - \sum_{j=1}^{s}\beta_j$$

$$= \sum_{j=1}^{s}(\mu_{ij}) - s\mu - s\alpha_i = 0$$

$$= s\mu_{i\bullet} - s\mu - s\alpha_i = 0$$

于是双因素等重复试验方差分析的数学模型等价为

$$
\begin{cases}
X_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \varepsilon_{ijk}, \\
\varepsilon_{ijk} \sim N(0, \sigma^2), \text{各} \varepsilon_{ijk} \text{相互独立}, \\
i = 1, 2, \cdots, r, j = 1, 2, \cdots, s, k = 1, 2, \cdots, t, \\
\sum_{i=1}^{r} \alpha_i = 0, \sum_{j=1}^{s} \beta_j = 0, \sum_{i=1}^{r} \delta_{ij} = 0, \sum_{j=1}^{s} \delta_{ij} = 0.
\end{cases}
$$

判断因素以及因素的交互作用对试验结果是否有显著影响等价于检验假设:

$$\begin{cases} H_{01}: \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0, \\ H_{11}: \alpha_1, \alpha_2, \cdots, \alpha_r \text{不全为零}. \end{cases}$$

$$\begin{cases} H_{02}: \beta_1 = \beta_2 = \cdots = \beta_s = 0, \\ H_{12}: \beta_1, \beta_2, \cdots, \beta_s \text{不全为零}. \end{cases}$$

$$\begin{cases} H_{03}: \delta_{11} = \delta_{12} = \cdots = \delta_{rs} = 0, \\ H_{13}: \delta_{11}, \delta_{12}, \cdots, \delta_{rs} \text{不全为零}. \end{cases}$$

2.分解离差平方和

$$\bar{X}_{ij\bullet} = \frac{1}{t}\sum_{k=1}^{t} X_{ijk} \qquad \bar{X}_{i\bullet\bullet} = \frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t} X_{ijk}$$

$$\bar{X}_{\bullet j\bullet} = \frac{1}{rt}\sum_{i=1}^{r}\sum_{k=1}^{t} X_{ijk}$$

$$\bar{X} = \frac{1}{rst}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t} X_{ijk}$$

$$= \frac{1}{r}\sum_{i=1}^{r}\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t} X_{ijk} = \frac{1}{r}\sum_{i=1}^{r}\bar{X}_{i\bullet\bullet} = \frac{1}{s}\sum_{j=1}^{s}\bar{X}_{\bullet j\bullet}$$

$$\boxed{Q_T = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(X_{ijk} - \bar{X})^2}$$ 总离差平方和(总变差)

$$= \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}[(X_{ijk} - \bar{X}_{ij\bullet}) + (\bar{X}_{i\bullet\bullet} - \bar{X}) + (\bar{X}_{\bullet j\bullet} - \bar{X})$$

$$+ (\bar{X}_{ij\bullet} - \bar{X}_{i\bullet\bullet} - \bar{X}_{\bullet j\bullet} + \bar{X})]^2$$

$$= \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(X_{ijk} - \bar{X}_{ij\bullet})^2}_{Q_E} + \underbrace{st\sum_{i=1}^{r}(\bar{X}_{i\bullet\bullet} - \bar{X})^2}_{Q_A}$$

$$\underbrace{+rt\sum_{j=1}^{s}(\bar{X}_{\bullet j\bullet} - \bar{X})^2}_{Q_B} \quad \underbrace{+t\sum_{i=1}^{r}\sum_{j=1}^{s}(\bar{X}_{ij\bullet} - \bar{X}_{i\bullet\bullet} - \bar{X}_{\bullet j\bullet} + \bar{X})^2}_{Q_{A\times B}}$$

$$Q_T = \boxed{Q_E} \boxed{+Q_A} \boxed{+Q_B} \boxed{+Q_{A\times B}}$$

误　差平方和

因素 A 的效应平方和

因素 B 的效应平方和

因素A,B的交互效应平方和

这里仅证明两个交叉项相乘等于零，其余类似可证）

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(\bar{X}_{i\bullet\bullet}-\bar{X})(\bar{X}_{\bullet j\bullet}-\bar{X})$$

$$=\sum_{j=1}^{s}\sum_{k=1}^{t}(\bar{X}_{\bullet j\bullet}-\bar{X})\left[\sum_{i=1}^{r}(\bar{X}_{i\bullet\bullet}-\bar{X})\right] \qquad \bar{X}_{i\bullet\bullet}=\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}$$

$$=\sum_{j=1}^{s}\sum_{k=1}^{t}(\bar{X}_{\bullet j\bullet}-\bar{X})\left[\sum_{i=1}^{r}\left(\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}-\bar{X}\right)\right]$$

$$=\sum_{j=1}^{s}\sum_{k=1}^{t}(\bar{X}_{\bullet j\bullet}-\bar{X})\left[\frac{1}{st}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}-\sum_{i=1}^{r}\bar{X}\right]$$

$$=\sum_{j=1}^{s}\sum_{k=1}^{t}(\bar{X}_{\bullet j\bullet}-\bar{X})\left[r\frac{1}{rst}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}-r\bar{X}\right]=0$$

$$
\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(\bar{X}_{i\cdot\cdot}-\bar{X})(\overline{X}_{ij\cdot}-\bar{X}_{i\cdot\cdot}-\bar{X}_{\cdot j\cdot}+\bar{X}) \\
&=\sum_{i=1}^{r}(\bar{X}_{i\cdot\cdot}-\bar{X})\left[\sum_{j=1}^{s}\sum_{k=1}^{t}(\overline{X}_{ij\cdot}-\bar{X}_{i\cdot\cdot}-\bar{X}_{\cdot j\cdot}+\bar{X})\right] \\
&=\sum_{i=1}^{r}(\bar{X}_{i\cdot\cdot}-\bar{X})\left[\sum_{j=1}^{s}\sum_{k=1}^{t}\overline{X}_{ij\cdot}-\sum_{j=1}^{s}\sum_{k=1}^{t}\bar{X}_{i\cdot\cdot}-\sum_{j=1}^{s}\sum_{k=1}^{t}\bar{X}_{\cdot j\cdot}+\sum_{j=1}^{s}\sum_{k=1}^{t}\bar{X}\right] \\
&=\sum_{i=1}^{r}(\bar{X}_{i\cdot\cdot}-\bar{X})\left[\sum_{j=1}^{s}t\overline{X}_{ij\cdot}-st\bar{X}_{i\cdot\cdot}-\sum_{j=1}^{s}t\bar{X}_{\cdot j\cdot}+st\bar{X}\right] \\
&=\sum_{i=1}^{r}(\bar{X}_{i\cdot\cdot}-\bar{X})\left[\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}-st\bar{X}_{i\cdot\cdot}-\sum_{j=1}^{s}\frac{1}{r}\sum_{i=1}^{r}\sum_{k=1}^{t}X_{ijk}+st\bar{X}\right] \\
&=\sum_{i=1}^{r}(\bar{X}_{i\cdot\cdot}-\bar{X})\left[st\bar{X}_{i\cdot\cdot}-st\bar{X}_{i\cdot\cdot}-st\bar{X}+st\bar{X}\right]=0
\end{aligned}
$$

3. 离差平方和的统计特性

令 $\overline{\varepsilon}=\dfrac{1}{rst}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}\varepsilon_{ijk}$，$\overline{\varepsilon}_{ij\cdot}=\dfrac{1}{t}\sum_{k=1}^{t}\varepsilon_{ijk}$，

$\overline{\varepsilon}_{i\cdot\cdot}=\dfrac{1}{s}\sum_{j=1}^{s}\overline{\varepsilon}_{ij\cdot}$，$i=1,\cdots,r$，$\overline{\varepsilon}_{\cdot j\cdot}=\dfrac{1}{r}\sum_{i=1}^{r}\overline{\varepsilon}_{ij\cdot}$，$j=1,\cdots,s$，

易得

$$
\left.
\begin{aligned}
Q_A &= st\sum_{i=1}^{r}(\alpha_i+\overline{\varepsilon}_{i\cdot\cdot}-\overline{\varepsilon})^2 \\
Q_B &= rt\sum_{j=1}^{s}(\beta_j+\overline{\varepsilon}_{\cdot j\cdot}-\overline{\varepsilon})^2 \\
Q_{A\times \mathrm{B}} &= t\sum_{i=1}^{r}\sum_{j=1}^{s}(\delta_{ij}+\overline{\varepsilon}_{ij\cdot}-\overline{\varepsilon}_{i\cdot\cdot}-\overline{\varepsilon}_{\cdot j\cdot}+\overline{\varepsilon})^2 \\
Q_E &= \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(\varepsilon_{ijk}-\overline{\varepsilon}_{ij\cdot})^2
\end{aligned}
\right\}
$$

这里仅给出 Q_A 的推导

$$Q_A = st\sum_{i=1}^{r}\Big(\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk} - \frac{1}{rst}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}\Big)^2$$

$$= st\sum_{i=1}^{r}\left[\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}\Big(X_{ijk} - \frac{1}{r}\sum_{i=1}^{r}X_{ijk}\Big)\right]^2$$

$$= st\sum_{i=1}^{r}\Bigg\{\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}\Big[(\cancel{\mu} + \alpha_i + \cancel{\beta_j} + \delta_{ij} + \varepsilon_{ijk})$$

$$- \frac{1}{r}\sum_{i=1}^{r}(\mu + \alpha_i + \beta_j + \delta_{ij} + \varepsilon_{ijk})\Big]\Bigg\}^2$$

$$\cancel{-\mu} \cancel{- \beta_j} - \frac{1}{r}\sum_{i=1}^{r}\alpha_i - \frac{1}{r}\sum_{i=1}^{r}\delta_{ij} - \frac{1}{r}\sum_{i=1}^{r}\varepsilon_{ijk}\Big]\Bigg\}^2$$

0

$$= st\sum_{i=1}^{r}\left\{\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}\left[\alpha_i+\delta_{ij}+\varepsilon_{ijk}-\frac{1}{r}\sum_{i=1}^{r}\varepsilon_{ijk}\right]\right\}^2$$

$$= st\sum_{i=1}^{r}\left\{\alpha_i+\underbrace{\frac{1}{s}\sum_{j=1}^{s}\delta_{ij}}_{0}+\frac{1}{st}\sum_{j=1}^{s}\sum_{k=1}^{t}\varepsilon_{ijk}-\frac{1}{str}\sum_{j=1}^{s}\sum_{k=1}^{t}\sum_{i=1}^{r}\varepsilon_{ijk}\right\}$$

$$= st\sum_{i=1}^{r}(\alpha_i+\bar{\varepsilon}_{i\cdot\cdot}-\bar{\varepsilon})^2$$

又因为

$$\varepsilon_{ijk}\sim N(0,\sigma^2),\quad \bar{\varepsilon}_{i\cdot\cdot}\sim N(0,\frac{\sigma^2}{st}),\quad \bar{\varepsilon}_{\cdot j\cdot}\sim N(0,\frac{\sigma^2}{rt})$$

$$\bar{\varepsilon}_{ij\cdot}\sim N(0,\frac{\sigma^2}{t}),\quad \bar{\varepsilon}\sim N(0,\frac{\sigma^2}{srt}),$$

因此

$$EQ_A = E\left[st\sum_{i=1}^{r}(\alpha_i + \bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon})^2\right]$$

$$= stE\left[\sum_{i=1}^{r}\alpha_i^2 + 2\sum_{i=1}^{r}\alpha_i(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}) + \sum_{i=1}^{r}(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon})^2\right]$$

$$= st\sum_{i=1}^{r}\alpha_i^2 + 2st\alpha_i\sum_{i=1}^{r}E(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}) + stE\sum_{i=1}^{r}(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon})^2$$

$$= st\sum_{i=1}^{r}\alpha_i^2 + st\left[\sum_{i=1}^{r}E(\bar{\varepsilon}_{i\cdot\cdot}^2) - rE(\bar{\varepsilon}^2)\right] \qquad \bar{\varepsilon}_{i\cdot\cdot} \sim N(0, \frac{\sigma^2}{st})$$

$$\bar{\varepsilon} \sim N(0, \frac{\sigma^2}{srt}),$$

$$= (r-1)\sigma^2 + st\sum_{i=1}^{r}\alpha_i^2$$

同理 $E(Q_B)=rt\sum_{j=1}^{s}\beta_j^{\ 2}+(s-1)\sigma^2$，$E(Q_E)=rs(t-1)\sigma^2$，

$$E(Q_{A\times B})=(r-1)(s-1)\sigma^2+t\sum_{i=1}^{r}\sum_{j=1}^{s}\delta_{ij}^2,$$

令 $\bar{Q}_A=\dfrac{Q_A}{r-1}$，$\bar{Q}_B=\dfrac{Q_B}{s-1}$，$\bar{Q}_{A\times B}=\dfrac{Q_{A\times B}}{(r-1)(s-1)}$，

$$\bar{Q}_E=\frac{Q_E}{rs(t-1)}$$

则 $E\bar{Q}_A=\sigma^2+\dfrac{st}{r-1}\sum_{i=1}^{r}\alpha_i^{\ 2}$，$E\bar{Q}_B=\sigma^2+\dfrac{rt}{s-1}\sum_{j=1}^{s}\beta_j^{\ 2}$，

$$E(\bar{Q}_{A\times B})=\sigma^2+\frac{t}{(r-1)(s-1)}\sum_{i=1}^{r}\sum_{j=1}^{s}\delta_{ij}^2,\quad E(\bar{Q}_E)=\sigma^2$$

由于H_{01}成立时，$E(\bar{Q}_A) = E\bar{Q}_E$，否则$E(\bar{Q}_A) > E\bar{Q}_E$；

由于H_{02}成立时，$E(\bar{Q}_B) = E\bar{Q}_E$，否则$E(\bar{Q}_B) > E\bar{Q}_E$，

由于H_{03}成立时，$E(\bar{Q}_{A\times B}) = E\bar{Q}_E$，否则$E(\bar{Q}_{A\times B}) > E\bar{Q}_E$，

因此 构造统计量

$$F_A = \frac{\bar{Q}_A}{\bar{Q}_E}, \quad F_B = \frac{\bar{Q}_B}{\bar{Q}_E}, \quad F_{A\times B} = \frac{\bar{Q}_{A\times B}}{\bar{Q}_E}$$

4. 统计量的分布

由于H_{01}, H_{02}, H_{03}成立时，$\alpha_i = \beta_j = \delta_{ij} = 0 (i = 1, \cdots, r, j = 1, \cdots, s)$，因而$X_{ijk} = \mu + \varepsilon_{ijk}$，则离差平方和可以改写为

$$Q_A = st\sum_{i=1}^{r}(\bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon})^2,\quad Q_B = rt\sum_{j=1}^{s}(\bar{\varepsilon}_{\cdot j\cdot} - \bar{\varepsilon})^2$$

$$Q_{A\times B} = t\sum_{i=1}^{r}\sum_{j=1}^{s}[\bar{\varepsilon}_{ij\cdot} - \bar{\varepsilon}_{i\cdot\cdot} - \bar{\varepsilon}_{\cdot j\cdot} + \bar{\varepsilon}]^2$$

$$Q_E = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(\varepsilon_{ijk} - \bar{\varepsilon}_{ij\cdot})^2$$

又由于$\dfrac{\varepsilon_{ijk}}{\sigma} \sim N(0,1)$,由定理1.12可知,

$$\frac{Q_T}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}(\varepsilon_{ijk} - \bar{\varepsilon})^2 \sim \chi^2(rst-1).$$

$\dfrac{Q_A}{\sigma^2}$的自由度为$r-1$, $\dfrac{Q_B}{\sigma^2}$的自由度为$s-1$,

$\dfrac{Q_{A\times B}}{\sigma^2}$的自由度为$(s-1)(r-1)$, $\dfrac{Q_E}{\sigma^2}$的自由度为$rs(t-1)$.

显然，离差平方和公式的左右两边自由度满足：

$$rst-1=(r-1)+(s-1)+(rs-r-s-1)+rs(t-1)$$

由柯赫伦因子分解定理(p26定理1.7)可知:

$$\frac{1}{\sigma^2}Q_A \sim \chi^2(r-1)$$

$$\frac{1}{\sigma^2}Q_B \sim \chi^2(s-1)$$

$$\frac{1}{\sigma^2}Q_{A\times B} \sim \chi^2((r-1)(s-1))$$

$$\frac{1}{\sigma^2}Q_E \sim \chi^2(rs(t-1))$$

因而

$$F_A = \frac{\dfrac{Q_A}{\sigma^2(r-1)}}{\dfrac{Q_E}{\sigma^2 rs(t-1)}} = \frac{\bar{Q}_A}{\bar{Q}_E} \sim F(r-1, rs(t-1))$$

$$F_B = \frac{\dfrac{Q_B}{\sigma^2(s-1)}}{\dfrac{Q_E}{\sigma^2 rs(t-1)}} = \frac{\bar{Q}_B}{\bar{Q}_E} \sim F(s-1, rs(t-1))$$

$$F_{A\times B} = \frac{\dfrac{Q_{A\times B}}{\sigma^2(s-1)(r-1)}}{\dfrac{Q_E}{\sigma^2 rs(t-1)}} = \frac{\bar{Q}_{A\times B}}{\bar{Q}_E} \sim F((r-1)(s-1), rs(t-1))$$

5. 方差分析对应的拒绝域

在给定显著性水平α下，因素A对试验结果有显著影响的拒绝域为

$$W_A = \{F_A \mid F_A \geq F_\alpha(r-1, rs(t-1))\}$$

在给定显著性水平α下，因素B对试验结果有显著影响的拒绝域为

$$W_{\mathrm{B}} = \{F_B \mid F_B \geq F_\alpha(s-1, rs(t-1))\}$$

在给定显著性水平α下，因素A,B的交互作用对试验结果有显著影响的拒绝域为

$$W_{\mathrm{A\times B}} = \{F_{A\times B} \mid F_{A\times B} \geq F_{\alpha}((r-1)(s-1), rs(t-1))\}$$

将上述结果总结，可以得到如下表内容:

表5.13双因素等重复试验的方差分析表

方差来源	平方和	自由度	均　方	F　比
因素　A	Q_A	$r-1$	$\overline{Q}_A=\dfrac{Q_A}{r-1}$	$F_A=\dfrac{\overline{Q}_A}{\overline{Q}_E}$
因素　B	Q_B	$s-1$	$\overline{Q}_B=\dfrac{Q_B}{s-1}$	$F_B=\dfrac{\overline{Q}_B}{\overline{Q}_E}$
交互作用	$Q_{A\times B}$	$(r-1)(s-1)$	$\overline{Q}_{A\times B}=\dfrac{Q_{A\times B}}{(r-1)(s-1)}$	$F_{A\times B}=\dfrac{\overline{Q}_{A\times B}}{\overline{Q}_E}$
误　　差	Q_E	$rs(t-1)$	$\overline{Q}_E=\dfrac{Q_E}{rs(t-1)}$	
总　　和	Q_T	$rst-1$		

为了计算方便，通常可以采用如下公式：令

$$T=\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk},\quad P=\frac{1}{rst}T^2,\quad W=\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}^2,$$

$$U=\frac{1}{st}\sum_{i=1}^{r}\Big(\sum_{j=1}^{s}\sum_{k=1}^{t}X_{ijk}\Big)^2,\quad V=\frac{1}{rt}\sum_{j=1}^{s}\Big(\sum_{i=1}^{r}\sum_{k=1}^{t}X_{ijk}\Big)^2,$$

$$R=\frac{1}{t}\sum_{i=1}^{r}\sum_{j=1}^{s}\Big(\sum_{k=1}^{t}X_{ijk}\Big)^2$$

则

$$\begin{cases} Q_A=U-P,\quad Q_B=V-P,\quad Q_{A\times B}=R-U-V-P, \\ Q_E=W-R,\quad Q_T=W-P \end{cases}$$

例2(p171例5.6) 考察合成纤维中对纤维弹性有影响的两个因素，收缩率A和总拉伸倍数B，A和B各取各取4种水平，每组组合水平重复试验两次，得到下表数据问收缩率和总拉伸倍数以及这两者的交互作用对纤维弹性是否有显著的影响, 取显著性水平为0.05.

因素A / 因素B	B_1	B_2	B_3	B_4
A_1	71,73	72,73	75,73	77,75
A_2	73,75	76,74	78,77	74,74
A_3	76,73	79,77	74,75	74,73
A_4	75,73	73,72	70,71	69,69

解 $r=4, s=4, t=2, rst=32$,经计算

$Q_T=180.219,\ \ Q_A=70.594,\ \ Q_B=8.594,$

$Q_{A\times B}=79.531 Q_E=21.500$

$$F_A=\frac{\bar{Q}_A}{\bar{Q}_E}=17.5>F_{0.05}(3,16)=3.24$$

$$F_B=\frac{\bar{Q}_B}{\bar{Q}_B}=2.1<F_{0.05}(3,16)=3.24$$

$$F_{A\times B}=\frac{\bar{Q}_{A\times B}}{\bar{Q}_E}=6.6>F_{0.05}(9,16)=2.54$$

因而纤维收缩率对弹性是有显著影响，总拉伸倍数对弹性无显著影响，而它们的相互作用对弹性有显著影响.

T检验与方差分析的区别与联系

检验一个总体（方差未知）的均值是否为某一个常值，或检验两个总体（总体的方差未知且相等）的均值是否相等，用T统计量检验。T^2是F统计量。

方差分析用于检验多个总体（3个或3个以上）的均值是否相等。总体的方差是未知且相等的，用F统计量进行检验。

另外，方差分析只是知道了这几组是否有差异，但具体是哪组有差异，还需进一步分析。

三、小结

1.双因素非重复试验的方差分析步骤

(1)建立数学模型;

(2)分解平方和;

(3)研究统计特性;

(4)确定拒绝域.

2.双因素等重复试验的方差分析步骤

(1)建立数学模型;

(2)分解平方和;

(3)研究统计特性;

(4)确定拒绝域.

Thank You!