TD VII

7.2 $f(x,y)=\begin{cases} x^2\sin\frac{y}{x} & \text{si } x\neq 0 \\ 0 & \text{si } x=0\end{cases}$ calculer $f_{xy}(0,0)$ $f_{yx}(0,0)$

$\frac{\partial f}{\partial x} = 2x\sin\frac{y}{x} - y\cos\frac{y}{x}$ pour $x\neq 0$ d'où $\frac{\partial f}{\partial x}$ existe sur $\mathbb{R}^2$

$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \frac{h^2\sin 0}{h} = 0$

$\frac{\partial f}{\partial y} = x\cos\frac{y}{x}$ $\lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \frac{0-0}{h} = 0$

$= x^2\left[\frac{y}{x}\right]'\sin'\left(\frac{y}{x}\right) = \frac{x^2}{x}\cos\left(\frac{y}{x}\right) = x\cos\frac{y}{x}$ d'où $\frac{\partial f}{\partial y}$ existe sur $\mathbb{R}^2$

$f_{xy}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \frac{0-0}{h} = 0$

$f_{yx}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \frac{h-0}{h} = 1$

7.3 Vérifier que $f_{xx}+f_{yy}=0$

1) $f(x,y)=e^x\sin y$

$\frac{\partial f}{\partial x} = (\sin y)e^x$ $\frac{\partial f}{\partial y} = e^x\cos y$

$\frac{\partial^2 f}{\partial x^2} = (\sin y)e^x$ $\frac{\partial^2 f}{\partial y^2} = -e^x\sin y$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x\sin y - e^x\sin y = 0$

2) $f(x,y) = x^3 - 3xy^2$

$\frac{\partial f}{\partial x} = 3x^2 - 3y^2$ $\frac{\partial f}{\partial y} = -6xy$

$\frac{\partial^2 f}{\partial x^2} = 6x$ $\frac{\partial^2 f}{\partial y^2} = -6x$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x - 6x = 0$

3) $f(x,y) = x^4 - 6x^2y^2 + y^4$

$\frac{\partial f}{\partial x} = 4x^3 - 12xy^2$ $\frac{\partial f}{\partial y} = -12x^2y + 4y^3$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 12y^2$ $\frac{\partial^2 f}{\partial y^2} = -12x^2 + 12y^2$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ □

7.4 Écrire la différentielle seconde en $(0,0)$ de $f(x,y) = e^x\sin y$

Rappel : $d^2f_a(h) = h^T H_f(a)\, h$ $h=(x,y)$

$\frac{\partial f}{\partial x} = e^x\sin y$ $\frac{\partial f}{\partial y} = e^x\cos y$ $\frac{\partial^2 f}{\partial x^2} = e^x\sin y$ $\frac{\partial^2 f}{\partial y\partial x} = e^x\cos y$ $\frac{\partial^2 f}{\partial y^2} = -e^x\sin y$

$H_f(h_1,h_2) = \begin{pmatrix} e^{h_1}\sin h_2 & e^{h_1}\cos h_2 \\ e^{h_1}\cos h_1 & -e^{h_1}\sin h_2\end{pmatrix} = e^{h_1}\begin{pmatrix}\sin h_2 & \cos h_2 \\ \cos h_1 & -\sin h_2\end{pmatrix}$

$d^2_{(h_1,h_2)}f(x,y) = \overset{1\times 2}{(x\ \ y)}\, e^{h_1}\overset{2\times 2}{\begin{pmatrix}\sin h_2 & \cos h_2 \\ \cos h_1 & -\sin h_2\end{pmatrix}}\overset{2\times 1}{(x\ \ y)^T}$

$= e^{h_1}(x\sin h_2 + y\cos h_2 \quad x\cos h_2 - y\sin h_2)(x\ \ y)^T$

$= e^{h_1}(x^2\sin h_2 + xy\cos h_2 + xy\cos h_2 - y^2\sin h_2)$

$= e^{h_1}(x^2\sin h_2 + 2xy\cos h_2 - y^2\sin h_2)$

$\boxed{d^2_{(0,0)}f(x,y) = 2xy}$

7.5 2) $q_2(x,y) = 2x^2 + 4xy + 3y^2 \to \begin{pmatrix} 2 & 2 \\ 2 & 3\end{pmatrix} \to$ déf. pos $|2|>0$ $\begin{vmatrix} 2 & 2 \\ 2 & 3\end{vmatrix} = 6-4 = 2 > 0$

3) $q_3(x,y) = x^2 + 4xy + 4y^2 \to \begin{pmatrix} 1 & 2 \\ 2 & 4\end{pmatrix} \to |1|>0$, $\begin{vmatrix} 1 & 2 \\ 2 & 4\end{vmatrix} = 4-2\cdot 2 = 0$ } semi-défini positif

4) $q_4(x,y) = 3xy \to \begin{pmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0\end{pmatrix} \to |0| = 0$, $\begin{vmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0\end{vmatrix} < 0$ } indéfinie

5) $q_5(x,y) = -2x^2 + 2xy - 5y^2 \to \begin{pmatrix} -2 & 1 \\ 1 & -5\end{pmatrix} \to |-2| < 0$, $\begin{vmatrix} -2 & 1 \\ 1 & -5\end{vmatrix} = 10-1 = 9 > 0$ Déf négative

$\Uparrow$

$-\begin{pmatrix} -2 & 1 \\ 1 & -5\end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 5\end{pmatrix}$ $|2|>0$, $\begin{vmatrix} 2 & -1 \\ -1 & 5\end{vmatrix} > 0$ } Déf positif

Rappel : Critère de Silvester : déf pos si tous les mineurs sont pos. ; déf neg s'ils suivent le signe de $(-1)^i$

7.6 Classifier $A = \begin{pmatrix} 1+\alpha & \alpha \\ 1 & 2\alpha\end{pmatrix}$

Quand est-elle définie positive ? $1+\alpha > 0 \iff \underbrace{\alpha > -1}_{M_1 =]-1;+\infty[}$

$\begin{vmatrix} 1+\alpha & \alpha \\ 1 & 2\alpha\end{vmatrix} > 0 \iff 2\alpha + 2\alpha^2 - \alpha > 0$
$\iff \alpha(2\alpha+1) > 0$
$\iff \begin{cases}\alpha > 0 \\ 2\alpha+1 > 0\end{cases}$ ou $\begin{cases}\alpha < 0 \\ 2\alpha+1 < 0\end{cases}$
$\iff \begin{cases}\alpha > 0 \\ \alpha > -\frac{1}{2}\end{cases}$ ou $\begin{cases}\alpha < 0 \\ \alpha < -\frac{1}{2}\end{cases}$
$\iff \underbrace{]0;+\infty[\cup]-\infty;-\frac{1}{2}[}_{M_2} \ni \alpha$

$M_1 \cap M_2 =]-1;+\infty[\cap (]-\infty;-\frac{1}{2}[\cup]0;+\infty[)$
$=]-1;-\frac{1}{2}[\cup]0;+\infty[$

Par les valeurs propres : $\begin{pmatrix} 1+\alpha & \alpha \\ 1 & 2\alpha\end{pmatrix} \to \begin{vmatrix} 1+\alpha-\lambda & \alpha \\ 1 & 2\alpha-\lambda\end{vmatrix}$

$= (1+\alpha-\lambda)(2\alpha-\lambda) - \alpha$

$= \cancel{2}\alpha + 2\alpha^2 - 2\alpha\lambda - \lambda - \alpha\lambda - \lambda^2 \cancel{-\alpha}$

$= \alpha + 2\alpha^2 - 3\alpha\lambda - \lambda - \lambda^2$

$= (\alpha+2\alpha^2) - (3\alpha+1)\lambda - \lambda^2$

$\Delta = (3\alpha+1)^2 - 4(-1)(\alpha+2\alpha^2)$
$= 9\alpha^2 + 6\alpha + 1 + 4\alpha + 8\alpha^2$
$= 17\alpha^2 + 10\alpha + 1$

<u>Méthode</u> : → chercher dans quels cas $\Delta > 0$ → 2 valeurs propres → chercher les signes
$\Delta < 0$ → pas de valeur propre.
$\Delta = 0$ → 1 seule valeur propre
↳ chercher son signe

$\boxed{7.7}$ 1) $f_1(x,y) = xy\exp(-x^2-y^2)$

① On cherche des candidats extremums → Théorème de Fermat

$$\nabla f_1(x,y) = \begin{cases} y\exp(-x^2-y^2) + xy(-2x)\exp(-x^2-y^2) \\ x\exp(-x^2-y^2) + xy(-2y)\exp(-x^2-y^2) \end{cases}$$

$$= \begin{cases} y\exp(-x^2-y^2)(1-2x^2) \\ x\exp(-x^2-y^2)(1-2y^2) \end{cases}$$

$$\nabla f_1(x,y) = 0 \iff \begin{cases} y\cancel{\exp(-x^2-y^2)}(1-2x^2) = 0 \\ x\cancel{\exp(-x^2-y^2)}(1-2y^2) = 0 \end{cases} \iff \begin{cases} y(1-2x^2) = 0 \\ x(1-2y^2) = 0 \end{cases}$$

$$\iff \begin{cases} y = 0 \text{ ou } \frac{1}{2} = x^2 \\ x = 0 \text{ ou } \frac{1}{2} = y^2 \end{cases}$$

$$\iff \begin{cases} y = 0 \text{ ou } x = \sqrt{\frac{1}{2}} \text{ ou } x = -\sqrt{\frac{1}{2}} \\ x = 0 \text{ ou } y = \sqrt{\frac{1}{2}} \text{ ou } y = -\sqrt{\frac{1}{2}} \end{cases}$$

$$\frac{\partial^2 f_1}{\partial x^2} = y(-2x)\exp(-x^2-y^2)(1-2x^2) + y\exp(-x^2-y^2)(-4x)$$

$$= \underbrace{(-2x)y\exp(-x^2-y^2)\left[1-2x^2-2\right]}_{\text{en } (0,0) = 0}$$

$$\frac{\partial^2 f_1}{\partial y^2} = x(-2y)\exp(-x^2-y^2)(1-2y^2) + x\exp(-x^2-y^2)(-4y)$$

$$= \underbrace{(-2y)x\exp(-x^2-y^2)(1-2y^2-2)}_{\text{en } (0,0) = 0}$$

$$\frac{\partial^2 f_1}{\partial x \partial y} = (1-2x^2)\exp(-x^2-y^2) + y(1-2x^2)(-2y)\exp(-x^2-y^2)$$

$$= \underbrace{\exp(-x^2-y^2)(1-2x^2)(1-2y^2)}_{\text{en } (0,0) = 1}$$

Pour $(0,0)$: $H_{f_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ → indéfinie → $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \begin{cases} \lambda = 1 \\ \text{ou} \\ \lambda = -1 \end{cases}$

Rappel : la matrice Hessienne s'écrit comme $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \longleftrightarrow d_a^2 f(x,y) = x^2 f_{xx}(a) + 2 f_{xy}(a)xy + y^2 f_{yy}(a)$

pour H symétrique