Valuation of Fixed Income Total Return Swaps

REN-RAW CHEN, PhD
Professor of Finance, Graduate School of Business, Fordham University

FRANK J. FABOZZI, PhD, CFA, CPA
Professor of Finance, EDHEC Business School

Abstract: A total return swap is a swap in which one party makes periodic floating rate payments to a counterparty in exchange for the total return realized on a reference asset (or underlying asset). The reference asset could be a credit-risky bond, a loan, a reference portfolio consisting of bonds or loans, an index representing a sector of the bond market, or an equity index. A total return swap can be used by asset managers for leveraging purposes and/or a transactionally efficient means for implementing a portfolio strategy. Bank managers use a total return swap as an efficient vehicle for transferring credit risk and as a means for reducing credit risk exposures. The Duffie-Singleton model can be used to value total return swaps.

In this entry we explain the valuation of total return swaps.[1] We begin with an intuitive approach.

AN INTUITIVE APPROACH

A typical *total return swap* is to swap the return on a reference asset for a risk-free return, usually the London Interbank Offered Rate (LIBOR). The cash flows for the swap buyer (that is, the total return receiver) are shown in Figure 1. In the figure, L_t is LIBOR at time t, s is the spread to LIBOR, and R_t is the total return at time t. The cash outlay at time t per \$1 of notional amount that must be made by the swap buyer is $L_t + s$; the cash inflow at time t per \$1 of notional amount is R_t.

As a result, the pricing of a total return swap is to decide the right spread, s, to pay on the funding (that is, LIBOR) leg. Formally,

$$\hat{E}_0\left\{\sum_{j=1}^{n} \exp\left(-\int_0^{T_j} r(t)dt\right)[R_j - (L_j + s)]\right\} = 0$$

where r is the risk-free discount rate.

In words, the spread should be set so that the expected payoff of the total return swap is equal to zero. (We employ the standard risk-neutral pricing and discounting at the risk-free rate.) To make the matter simple (we shall discuss more rigorous cases later), we view r, R, and

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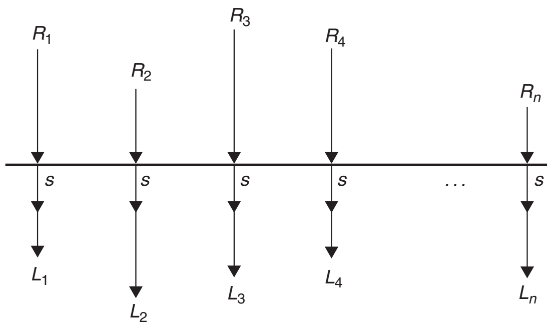

Figure 1 Cash Flows for the Total Return Receiver

L as three separate random variables. We then rearrange the above equation as

$$\hat{E}_0\left\{\sum_{j=1}^{n}\exp\left(-\int_0^{T_j} r(t)dt\right)(R_j - L_j)\right\} = \hat{E}_0\left\{\sum_{j=1}^{n}\exp\left(-\int_0^{T_j} r(t)dt\right)\right\} s$$

Exchanging expectation and summation of the right-hand side gives

$$\hat{E}_0\left\{\sum_{j=1}^{n}\exp\left(-\int_0^{T_j} r(t)dt\right)\right\} = \sum_{j=1}^{n}\hat{E}_0\left[\exp\left(-\int_0^{T_j} r(t)dt\right)\right] = \sum_{j=1}^{n} P(0,\ T_j)$$

as the sum of risk-free pure discount bond prices. This implies

$$\sum_{j=1}^{n}\hat{E}_0\left[\exp\left(-\int_0^{T_j} r(t)dt\right) R_j - L_j\right] = \sum_{j=1}^{n} P(0,\ T_j)s$$

The next step is to use the forward measure to simplify the left-hand side of the above equation:

$$\sum_{j=1}^{n} P(0,\ T_j)E_0^{F(j)}[R_j - L_j] = \sum_{j=1}^{n} P(0,\ T_j)s$$

Later, we show that the forward measure expectation of an asset gives the forward price of the asset. Hence, the left-hand side of the above equation gives two forward curves, one on the asset return, R, and the other on LIBOR, L:

$$\sum_{j=1}^{n} P(0,\ T_j)\left[f_j^R - f_j^L\right] = \sum_{j=1}^{n} P(0,\ T_j)s$$

where f_j^i is the forward rate of i ($i = R$ or L) for j periods ahead. Therefore, the spread can be solved easily as

$$s = \frac{\sum_{j=1}^{n} P(0,\ T_j)\left[f_j^R - f_j^L\right]}{\sum_{j=1}^{n} P(0,\ T_j)}$$

The result is intuitive: the spread is a weighted average of the expected difference between two floating-rate indexes. The weight is

$$\frac{P(0,\ T_j)}{\sum_{j=1}^{n} P(0,\ T_j)}$$

Note that all the weights should sum to one.

USING THE DUFFIE-SINGLETON MODEL

The difference in two floating rates is mainly due to their credit risk, otherwise they should both offer identical rates and give identical forward curves. As a consequence, to be rigorous about getting the correct result, we need to incorporate the credit risk in one of the indexes.

Among various choices, the model by Duffie and Singleton (1999) suits the best for this situation. The *Duffie-Singleton model* is a popular reduced-form model that is used in credit risk modeling. In the model, the present value of any risky cash flow is defined as

$$C(t) = \left[\frac{S_{t+1} - S_t}{S_t} - L_{t+1} \right] N$$

where N is the notional, L is LIBOR, and S is the index level. As noted earlier, since both cash flows are random, it is a floating-floating swap. Also since the index is always higher than LIBOR because of credit risk, this swap requires a premium. As a result, the premium is computed as the sum of all future values, discounted and expected:

$$V = \sum_{j=1}^{n} \hat{E}_t \left[\exp\left(-\int_t^{T_j} [r(u) + q(u)]du \right) C(T_j) \right]$$

where q is the "spread" in the Duffie-Singleton model that incorporates the recovery rate and default probability.

THE FORWARD MEASURE

In this section, we show how the *forward measure* works and why a forward-adjusted expectation gives the forward value. We first state the separation principle that leads to the forward measure. Based on the no-arbitrage principle, the current value of any asset is the risk-neutral expected value of the discounted future payoff:

$$C(t) = \hat{E}_t \left[\exp\left(-\int_t^{T} r(u)du \right) C(T) \right]$$

The separation principle states that if we adopt the forward measure, then the above equation can be written as

$$C(t) = \hat{E}_t \left[\exp\left(-\int_t^{T} r(u)du \right) \right] E_t^{F(T)}[C(T)]$$

where $E_t^{F(T)}[\cdot]$ is the forward measure.[2] Note that the first term is nothing but the zero-coupon bond price:

$$P(t,\ T) = \hat{E}_t \left[\exp\left(-\int_t^{T} r(u)du \right) \right]$$

and hence

$$C(t) = P(t,\ T) E_t^{F(T)}[C(T)]$$

While we do not prove this result, we should note the intuition behind it. Let C be a zero-coupon bond expiring at time u. Then the above result can be applied directly and gives

$$P(t,\ s) = P(t,\ T) E_t^{F(T)}[P(T,\ u)]$$

or equivalently

$$E_t^{F(T)}[P(T,\ s)] = \frac{P(t,\ s)}{P(t,\ T)}$$

This is an indirect proof that the forward-adjusted expectation gives a forward value. The instantaneous forward rate can be shown to be the forward-adjusted expectation of the future instantaneous spot rate:

$$\begin{aligned} f(t,\ T) &= -\frac{d \ln P(t,\ T)}{dT} \\ &= -\frac{1}{P(t,\ T)} \hat{E}_t \left[\frac{d}{dT} \exp\left(-\int_t^{T} r(u)du \right) \right] \\ &= \frac{1}{P(t,\ T)} \hat{E}_t \left[\exp\left(-\int_t^{T} r(u)du \right) r(T) \right] \\ &= E_t^{F(T)}[r(T)] \end{aligned}$$

The discrete forward rates, $f_D(t,\ w,\ T)$ for all w and T, can also be shown to be the forward-adjusted expectations of future discrete spot rates:

$$
\begin{aligned}
f_D(t,\ w,\ T) &= \frac{1}{\Psi(t,\ w,\ T)} - 1 \\
&= \frac{P(t,\ w)}{P(t,\ T)} - 1 \\
&= \frac{1}{P(t,\ T)} \hat{E}_t \left[\exp\left(-\int_t^T r(u)du \right) \frac{1}{P(w,\ T)} \right] - 1 \\
&= E_t^{F(T)} \left[\frac{1}{P(w,\ T)} - 1 \right]
\end{aligned}
$$

where $t < w < T$.

KEY POINTS

- A total return swap is a swap in which one party makes periodic floating rate payments to a counterparty in exchange for the total return realized on a reference asset such as a credit-risky bond.
- The pricing of a total return swap is to decide the right spread to pay on the funding leg.
- Using the standard risk-neutral pricing and discounting at the risk-free rate, the spread should be set so that the expected payoff of the total return swap is equal to zero.
- A reduced form model used in valuing credit derivatives, the Duffie-Singleton model, is employed to value total return swaps.
- The forward measure expectation of an asset gives the forward price of the asset that is the underlying for a total return swap.

NOTES

1. For a discussion of total return swaps and their applications, see Anson et al. (2004).
2. The derivation of this result can be found in a number of places. See, for example, Jamshidian (1987) and Chen (1996).

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