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COORDINATING CLEARANCE MARKDOWN SALES OF SEASONAL PRODUCTS IN RETAIL CHAINS

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In this paper we propose a methodology to set prices of perishable items in the context of a retail chain with coordinated prices among its stores and compare its performance with actual practice in a real case study. We formulate a stochastic dynamic programming problem and develop heuristic solutions that approximate optimal solutions satisfactorily. To compare this methodology with current practices in the industry, we conducted two sets of experiments using the expertise of a product manager of a large retail company in Chile. In the first case, we contrast the performance of the proposed methodology with the revenues obtained during the 1995 autumn–winter season. In the second case, we compare it with the performance of the experienced product manager in a “simulation-game” setting. In both cases, our methodology provides significantly better results than those obtained by current practices.

Nowadays retail managers face rapid changes in fashion and customers' preferences. The “perishability” of fashionable clothing leads to short selling periods (on the order of three months), during which inventory management and pricing strategies are central to success. Ordering decisions at the beginning of the season are critical: short seasons together with the fact that products are often manufactured overseas make it difficult for managers to place reorders in case of demand underestimation. On the other hand, if a product does not sell well, then pricing must be used aggressively to maximize the expected return of each unit in inventory.

This problem is further complicated for retail chains that need to coordinate inventories and prices of multiple stores that are geographically dispersed. Retail chains manage inventories in different ways. Some of the mechanisms most commonly encountered in practice are (1) an initial assignment of products is made to the stores without further redistributions, (2) an initial assignment of products is made to the stores and further reallocations among stores are used to respond to sales imbalances, and (3) a central warehouse distributes products to the stores on a periodic basis according to their sales patterns. All merchandise interchanges imply management and transportation costs, and potential losses due to merchandise deterioration and pilferage. Thus, in general, managers are reluctant to move products as a regular practice (these

comments are based on conversations with a product manager of a large fashion retail chain in Chile).

Retail chains usually manage their prices centrally, i.e., all the stores have the same prices for a given product during the season. Although this adds a degree of inflexibility to management at the store level, it is desirable from a global point of view because it helps to maintain a corporate image.

Retail chains use different strategies to locate their stores. However, independently of the selected location strategy, the stores usually face customers who differ mainly in their willingness to pay for the product. The maximum price that a customer is willing to pay for a product is called the *reservation price*. Another important variable that determines the demand function is the flow of customers who visit the stores every day. Many factors influence the flow into the stores; for example, promotions of particular products or families of products, special days such as Mother's Day or Columbus Day, and liquidation sales at the end of the season. The composition between the flow of customers into the store and their willingness to pay for a product largely determines the demand function for the product. It is not unusual to find different selling patterns among the stores of a retail chain. Therefore, a dynamic pricing policy that reacts to changes in the purchasing pattern and appropriate inventory assignment

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procedures play a key role in maximizing expected revenue.

There is a vast literature in economics, marketing, and operations management on pricing strategies and their impact on customers' purchasing behavior. However, only a small number of papers develop formal tools to make pricing decisions. In the last few years manuscripts have been published in the operations management literature that deal with the operational problem of pricing a seasonal product over a fixed planning horizon. Belobaba (1987, 1989) in his work on the airline industry makes assumptions similar to those in our problem: stochastic demand, a finite planning horizon, and a fixed number of units in inventory. He develops a model with n classes of customers and a fixed price structure, i.e., each class is characterized by a price that can vary through time. The decision variables correspond to the target number of seats made available to each class of customers. They are computed, in the general case, on a rolling horizon basis.

Lazear (1986) uses a simplified model of the stochastic demand to analyze the effect of the finite planning horizon and initial inventory on pricing policies. The author considers a distribution function for the customers' reservation price to represent the seller's lack of information. He assumes that all customers have an identical reservation price (customers are completely homogeneous). Therefore, if a customer does not buy a product at a certain price, nobody will buy it. The model assumes a planning horizon of one or two periods and that the seller knows in advance the number of potential buyers. This work is intended primarily to develop economic insights into pricing strategies; however, the main assumptions of the model prevent us from using it as a practical tool to define pricing policies in real cases.

Rajan et al. (1992) develop a deterministic model for optimal pricing and inventory policies for a product whose value decreases with time. Smith and Achaval (1997) develop clearance pricing and inventory policies for a situation which the sales rate depends on time, inventory, and price. The model considers deterministic demand, and therefore the optimal price path is determined at the beginning of the planning horizon; they allow the possibility of revisiting the pricing policy on a rolling horizon basis.

The research presented in this paper is motivated by the previous work in Gallego and Van Ryzin (1994) and Bitran and Mondschein (1997). Both consider a single store, stochastic demand, a finite planning horizon, and a fixed amount of initial inventory. The former studies a pricing policy where prices can be modified continuously over the planning horizon, while the latter analyses periodic policies where prices can be changed only at specific points in time. The goal of this paper is to extend the periodic pricing policies to the case of retail chains with several stores with coordinated prices and to contrast them with the performance of current practices in the industry.

Similar to Bitran and Mondschein (1997), in this paper we study the operational problem of determining pricing

policies for clearance markdown sales (permanent markdowns), where prices are lowered for a period of time until the next markdown. The pricing discounts are operational decisions that are made dynamically during the season. On the other hand, in the case of promotions (temporary markdowns), such as Mother's Day and Columbus Day, a discount is offered for a fixed period of time and then prices return to their original level. Usually the date for promotions is set in advance, independently of how well products sell; managers announce, for example, a range for price discounts for a family of products. However, they can exclude a particular product from the sale if the product has been a success. Usually promotions are scheduled at the beginning of the season, so they do not interfere with "liquidations" toward the end of the planning horizon. Promotions are tactical decisions because they involve families of products that are promoted at specific times during the year. These decisions take into account the behavior of competitors with similar products. However, the effect of promotions on clearance markdown sales is captured in our models by a nonhomogeneous arrival rate of customers to the store: the number of people visiting the store usually increases significantly during promotions. We also assume that there is no correlation among the demand of different products; allowing for correlation is part of ongoing research.

Bitran and Wadhwa (1996) extend the periodic pricing model for a single store to incorporate a mechanism for demand learning. They assume that some parameters of the reservation price distribution are unknown to the retailer at the beginning of the season. These parameters are revised at the end of each period based on observed sales in that period using a Bayesian approach.

The remainder of this paper is organized as follows. In Section 1 we develop two mathematical formulations for the dynamic pricing of perishable items in retail chains. Due to the dimension of the mathematical formulations in Section 2, we develop heuristics to find approximations for the optimal solutions. We illustrate the performance of the heuristics through a set of computational experiments. In Section 3 we present a real case study in which we compare the results given by the methodology proposed in this paper with (1) the revenues of a large fashion retail company in Chile during the 1995 autumn–winter season and (2) the results of a "simulation–game" where a product manager makes decisions under uncertain demand scenarios. Finally, in Section 4 we present our conclusions.

1. MODEL DESCRIPTION

We consider a retail chain that sells seasonal products in several stores during a fixed planning horizon. The retail chain changes prices periodically according to inventory levels and the time left until the end of the planning horizon. Prices are the same in all stores for a given product. In general, each store faces a market characterized by a flow of customers and a willingness to pay for the product.

The purchasing process is the result of an arrival process and a reservation price distribution. We describe the arrival process by a Poisson distribution with a time-dependent arrival rate. Therefore, the flow of customers into the store varies with time, responding, for example, to special days (Mother's Day or Valentine's Day) or promotions (white-sale day). The Poisson assumption is made for simplicity in the models' description. However, the models are general and the formulations hold for any distribution of the arrival process. The purchasing process is characterized by a reservation price, which is the maximum price that a customer would pay for the product. The customer purchases the product only if the price is lower than his reservation price. To capture the heterogeneity among customers and the lack of information of the seller about the customers' purchasing behavior, we consider a probability distribution for the reservation price, i.e., every time a seller faces a customer, the seller knows only the probability distribution associated with the customer's reservation price.

We consider two cases for inventory management that are common practices in the retail industry. In the first case, total inventory is initially distributed among stores, and no merchandise exchanges are allowed during the planning horizon. According to some managers in the industry, the rationale behind this policy is the high cost associated with the transfer of merchandise. In the second case, transfers of merchandise among stores can occur at a given cost. This cost includes, for example, potential losses, pilferage, and transportation and management costs.

We also assume that reorders are not allowed during the planning horizon. This is usually observed in practice due mainly to the relatively short planning horizon (of the order of three months), which prevents managers from reordering successful products.

The goal of the retail chain is to maximize the total discounted expected revenue over the planning horizon, which is equivalent to maximizing the profits after the inventory decisions are made (the cost of the products is a sunk cost). The model makes decisions at the individual item level, which mimics current industry practice. According to a product manager of a large retail chain in Chile, the store management is divided into sections (women's clothing, men's clothing, electronics, etc.). A product manager is in charge of a complete section. She typically has a profitability target to reach through inventory and pricing decisions. For this purpose, the set of products is divided into families and each family consists of several items related by, for example, type of product (blouses, coats, shoes), quality, and brand. During each time period, the product manager examines the profitability of the products already sold, the inventory, and the time left until the end of the planning horizon. If the profitability is lower than her expectations, she examines the profitability by family and finally by item to identify those items that have underperformed. For those items, she determines price changes. Although correlations of demands for products is not considered explicitly in current practices (one of the few ex-

PLICIT considerations consists of not pricing a higher quality product lower than a lesser quality product), we believe that a study of demand correlation among products could lead to improvements in pricing policies. This topic is part of our ongoing research.

We divide the planning horizon into K periods at which prices can be revised. The length of these time intervals can vary, allowing for more frequent price revisions toward the end of the planning horizon.

1.1. The Models

We now present the two mathematical formulations of the optimization problems described above. We introduce the following notation:

T = length of the planning horizon,

K = number of times that the price can be modified during the planning horizon,

k = index denoting the k th period during the planning horizon (we count the periods in the planning horizon backwards, i.e., 1 is the last period),

T_k = length of the k th interval of time where the price remains unchanged, $k = 1, \dots, K$, and $\sum_{k=1}^K T_k = T$,

n = total number of stores that belong to the retail chain,

λ_{ik} = arrival rate of customers to store i in period k , $i = 1, \dots, n$, and $k = 1, \dots, K$,

$F_{ik}(p)$ = cumulative distribution function for the reservation price in store i in period k , $i = 1, \dots, n$, $k = 1, \dots, K$,

$D_k(p)$ = demand vector in period k when price is equal to p . (the element $D_{ik}(p)$ corresponds to the demand in store i at price p in period k),

c_{ik} = total number of units in inventory in store i at the beginning of period k , and

$V_k(c_{1k}, \dots, c_{nk})$ = total expected revenue from period k to the end of the planning horizon if in period k the initial inventory in store i is c_{ik} , $\forall i = 1, \dots, n$, $\forall k = 1, \dots, K$, when the optimal pricing policy is implemented.

1.1.1. No Inventory Transfers Are Allowed. In this case, no merchandise exchanges are allowed during the planning horizon. At the beginning of a period, the manager makes the pricing decision considering the initial inventory in each store; this price is kept constant during that period and it is the same in all stores. Given this decision, the purchasing process is observed and finally the inventories are updated at the end of the period according to the sales. The model can be written as:

$$\begin{aligned} V_k(c_{1k}, \dots, c_{nk}) = & \max_{p \geq 0} \sum_{j_1=0}^{\infty} \dots \sum_{j_n=0}^{\infty} [p(\min(c_{1k}, j_1) \\ & + \dots + \min(c_{nk}, j_n)) + V_{k-1}(c_{1k} \\ & - \min(c_{1k}, j_1), \dots, c_{nk} - \min(c_{nk}, j_n))] \Pr(D_k(p) \\ & = j_1, \dots, j_n). \end{aligned} \quad (1)$$

Boundary conditions:

$$V_k(0, \dots, 0) = 0, \quad \forall k = 1, \dots, K,$$

$$V_0(c_{10}, \dots, c_{n0}) = 0, \quad \forall c_{10}, \dots, c_{n0}.$$

In this model, $\Pr(D_k(p) = (j_1, \dots, j_n))$ corresponds to the joint probability that j_1, \dots, j_n customers are willing to buy the product at price p in period k in stores $1, \dots, n$, respectively. Given the Poisson arrival process with parameter λ_{ik} for store i in period k and its corresponding reservation price distribution $F_{ik}(p)$, the probability mass function for the demand is equal to:

$$\Pr(D_k(p) = j_1, \dots, j_n) = \prod_{i=1}^n \frac{(\lambda_{ik} T_k (1 - F_{ik}(p)))^{j_i} e^{-\lambda_{ik} T_k (1 - F_{ik}(p))}}{j_i!}. \quad (2)$$

This joint distribution assumes that the demand functions are independent among the stores. Based on conversations with a product manager of a large retail chain in Chile, this assumption is close to reality because the stores are usually located in local malls that are geographically dispersed. Although in practice there are correlations among demand functions due to factors such as weather or macroeconomic conditions, these represent second-order effects. We observe, however, that our two models can be formulated using general joint demand distribution functions, and therefore they do not depend on this assumption.

1.1.2. Inventory Transfers Are Allowed. In this case, at the end of each period, products can be moved from one store to another to avoid potential inventory imbalances. To capture the costs of merchandise interchanges, we consider a unit cost v associated with moving merchandise between two stores. To simplify the presentation we assume that v is independent of the stores. We also define I_{ik+1} as the final inventory in store i at the end of period $k + 1$. Therefore, I_{ik+1} also corresponds to the initial inventory in store i at the beginning of period k , before inventory exchanges take place.

The sequence of events is as follows. At the beginning of a period, the manager simultaneously makes the pricing and inventory redistribution decisions considering the initial inventory in each store; the price is kept constant during that period and it is the same in all stores. Then, physical inventory reallocation takes place; for simplicity we assume that inventory transfers occur instantaneously. Given these decisions, the purchasing process is observed, and finally the inventories are updated at the end of the period according to the sales. The mathematical formulation in this case is:

$$\begin{aligned} & V_k(I_{1k+1}, \dots, I_{nk+1}) \\ &= \max_{p, c_{1k}, \dots, c_{nk}} \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} [p(\min(c_{1k}, j_1) \\ &+ \cdots + \min(c_{nk}, j_n)) \\ &+ V_{k-1}(c_{1k} - \min(c_{1k}, j_1), \dots, c_{nk} \\ &- \min(c_{nk}, j_n))] \Pr(D_k(p) = j_1, \dots, j_n) \\ &- \sum_{i=1}^n v \max(0, c_{ik} - I_{ik+1}), \end{aligned} \quad (3)$$

s.t.

$$\sum_{i=1}^n c_{ik} = \sum_{i=1}^n I_{ik+1}, \quad (4)$$

$$p \geq 0, c_{ik} \geq 0, \forall i = 1, \dots, n. \quad (5)$$

Boundary conditions:

$$V_k(0, \dots, 0) = 0, \quad \forall k = 1, \dots, K,$$

$$V_0(c_{10}, \dots, c_{n0}) = 0, \quad \forall c_{10}, \dots, c_{n0}.$$

In some retail chains, prices differ among stores. Usually, the stores have different names and are oriented to different market segments. Although prices can be different, the pricing policies are linked by a coordinated system of inventory management. For example, merchandise can be transferred among stores. In this case, the model presented above can be modified to include a pricing decision for each individual store; the rest of the model remains unchanged.

2. HEURISTICS

The dimension of the state space of the dynamic programming formulations described in the previous section prevents us from solving the problems optimally for a realistically sized case. If we consider a retail chain with n stores with initial inventory of c_1, \dots, c_n in stores $1, \dots, n$ respectively, then the dimension of the state space is equal to $[\prod_{i=1}^n c_i]$ for each period of the planning horizon. For example, if a retail chain starts with 100 units in each of 5 stores, then 10^{10} maximization subproblems must be solved in each period of the planning horizon. Because of this limitation we develop heuristics to find approximate solutions for the optimal pricing policy. These heuristics are based on the optimal pricing policies for simpler problems where either there are no price changes or at most one change is allowed during the planning horizon. We also present computational experiments that show a satisfactory performance of the heuristics.

2.1. Heuristics for the No-Inventory-Transfer Case

2.1.1. Heuristic: HEUR1. In every period when prices are revised, this heuristic assumes that the price will not be modified in the remaining time of the planning horizon, i.e., there is a time aggregation. Therefore, given the inventories in each store (c_{1k}, \dots, c_{nk}) , the price at the beginning of period k is determined by solving the following problem:

$$\begin{aligned} & VH1(c_{1k}, \dots, c_{nk}, H^k) \\ &= \max_{p \geq 0} [VH1(c_{1k}, \dots, c_{nk}, p, H^k)], \end{aligned}$$

where $H^k = \sum_{l=1}^k T_l$, and

$$\begin{aligned} & VH1(c_{1k}, \dots, c_{nk}, p, H^k) \\ &= \sum_{i=1}^n \sum_{j_i=0}^{\infty} p \min(c_{ik}, j_i) \Pr(\hat{D}_{ik}(p) = j_i). \end{aligned}$$

The probability distribution for the aggregate demand corresponds to:

$$\Pr(\hat{D}_{ik}(p) = j_i) = \frac{[\hat{\lambda}_{ik}(p)]^{j_i} e^{-\hat{\lambda}_{ik}(p)}}{j_i!},$$

with $\hat{\lambda}_{ik}(p) = \sum_{l=1}^k \lambda_{il} T_l (1 - F_{il}(p))$.

Thus, at the beginning of each period k , and on a rolling horizon basis, it is necessary to solve a nonlinear problem, where the decision variable is the price for period k . For this purpose, we use the Fibonacci algorithm.

2.1.2. Heuristic: HEUR2. This heuristic differs from HEUR1 only in the procedure to compute the total expected revenue in the remainder of the planning horizon; HEUR2 considers an approximation for the total expected sales.

For a given period k , considering the remaining time to the end of the planning horizon, H^k , the expected number of customers that request the product at price p is equal to $\hat{\lambda}_{ik}(p)$. Therefore, we approximate the expected sales in store i by:

$$\hat{N}_i(c_{ik}, p, H^k) = \min[c_{ik}, \hat{\lambda}_{ik}(p)].$$

We determine the price at the beginning of period k by solving the maximization problem:

$$VH2(c_{1k}, \dots, c_{nk}, H^k) = \max_{p \geq 0} (p \sum_{i=1}^n \hat{N}_i(c_{ik}, p, H^k)).$$

Similar to HEUR1, we apply the HEUR2 on a rolling horizon basis.

2.1.3. Heuristic: HEUR3. This heuristic is based on a combination of the two preceding heuristics. We assume that the planning horizon is divided into two periods: the first period corresponds to the current period of length T_k , and the second corresponds to the aggregation of the remaining periods (H^{k-1}). We first consider an approximation for the number of expected sales during the first period at a given price p , which is used to update the inventory available for the second aggregate period. Then, the expected revenue for the second aggregate period is computed using HEUR2. Finally, the initial price for period k is computed by maximizing the expected revenue for the current period plus the expected revenue for the remaining periods. We note that the expected revenue for the second period depends on the current price through the initial inventories for that period.

We determine the price at the beginning of period k as follows.

STEP 1. Given a price p we approximate the number of expected sales in store i in period k by:

$$N_i(c_{ik}, p, T_k) = \min[c_{ik}, \lambda_{ik} T_k (1 - F_{ik}(p))].$$

STEP 2. We determine the available inventory at the end of period k as a function of the number of sales given by Step 1.

$$I_{ik}(p) = c_{ik} - N_i(c_{ik}, p, T_k).$$

STEP 3. We determine the total expected revenue from period $k - 1$ onward using HEUR2 with initial inventories equal to $I_{1k}(p), \dots, I_{nk}(p)$ for stores $1, \dots, n$ respectively; we call this total expected revenue $VH2(I_{1k}(p), \dots, I_{nk}(p), H^{k-1})$.

STEP 4. We compute the total expected revenue as the immediate expected revenue in period k given by heuristic HEUR1 plus the expected revenue from period $k - 1$ onward obtained from Step 3. We call this total expected revenue $VH3(c_{1k}, \dots, c_{nk}, p, H^k)$.

$$\begin{aligned} VH3(c_{1k}, \dots, c_{nk}, p, H^k) \\ = VH1(c_{1k}, \dots, c_{nk}, p, T_k) \\ + VH2(I_{1k}(p), \dots, I_{nk}(p), H^{k-1}). \end{aligned}$$

STEP 5. Finally, we determine the price at the beginning of period k by solving:

$$\begin{aligned} VH3(c_{1k}, \dots, c_{nk}, H^k) \\ = \max_{p \geq 0} [VH3(c_{1k}, \dots, c_{nk}, p, H^k)]. \end{aligned}$$

For the last period, this heuristic coincides with HEUR1, and only Steps 4 and 5 must be considered.

2.2. Heuristic for the Case with Inventory Transfers Allowed

This section describes a heuristic for the problem where it is possible to move products from one store to another at the end of each period. The general idea behind this heuristic is to reformulate the problem as a single-period problem and to determine the price and the inventory redistribution through iterations of a two-step algorithm. For a given inventory distribution, the heuristic determines the optimal price using Fibonacci's algorithm, and for a given price, we use a simple characterization of the optimal inventory reallocation, which is presented in Proposition 1, to compute the optimal inventory assignment in I steps, where I is the total number of units in inventory.

Below we present a proposition that characterizes the optimal inventory reallocation. Due to the structure of the expected total revenue for a single-period problem, which corresponds to the summation of independent revenue functions, the optimality condition, assuming that the inventory variables are continuous, establishes that the marginal benefit of the last unit in inventory should be equal in all the stores. However, in our problem, the products cannot be divided, and therefore we use a more general characterization of the optimal inventory assignment.

Consider a family of functions $F = \{F_i\}_{i=1}^n$ such that $F_i : \mathcal{X}^+ \rightarrow \mathcal{R}$ satisfies:

$$F_i(c + 1) - F_i(c) \geq F_i(c + 2) - F_i(c + 1) \quad \forall c \in \mathcal{X}^+, \quad (6)$$

and the optimization problem:

$$\begin{aligned}
\mathbf{P1}: \quad & \max_{c_1, \dots, c_n} \sum_{i=1}^n F_i(c_i), \\
\text{s.t.} \quad & \sum_{i=1}^n c_i = I, \\
& c_i \in \mathcal{L}^+, \quad \forall i = 1, \dots, n.
\end{aligned} \tag{7}$$

Then the following proposition characterizes the optimal solution of problem **P1**.

Proposition 1. *The vector (c_1, \dots, c_n) is the optimal solution of problem **P1** if and only if it is feasible and satisfies the following condition:*

$$\begin{aligned}
& F_i(c_i) - F_i(c_i - 1) \\
& \geq F_j(c_j + 1) - F_j(c_j) \quad \forall i, j \text{ and } c_i \geq 1.
\end{aligned}$$

Proof. The proof can be found in Appendix 1. \square

Using Proposition 1, we develop an efficient algorithm to compute the optimal solution of problem **P1**.

Algorithm to solve problem **P1**: **ALG1**

Step 1. Initialization. Set $X^0 = (0, 0, \dots, 0)$ and $k = 1$.

Step 2.

- If $k \leq I$ assign the next unit to the store j that satisfies:

$$\begin{aligned}
& F_j(X_j^{k-1} + 1) - F_j(X_j^{k-1}) \\
& \geq F_i(X_i^{k-1} + 1) - F_i(X_i^{k-1}) \quad \forall i = 1, \dots, n,
\end{aligned}$$

and update the variable to:

$$X^k = (X_1^{k-1}, \dots, X_j^{k-1} + 1, \dots, X_n^{k-1}).$$

- If $k > I$ then go to Step 4.

Step 3. Set $k = k + 1$, and go to Step 2.

Step 4. The optimal solution corresponds to $C = X^{k-1}$.

This algorithm stops in exactly I iterations with a vector $C = X^I = (c_1, \dots, c_n)$ that satisfies $\sum_i c_i = I$, and according to Step 2, satisfies the condition:

$$\begin{aligned}
& F_i(c_i) - F_i(c_i - 1) \\
& \geq F_j(c_j + 1) - F_j(c_j) \quad \forall i, j, \text{ and, } c_i \geq 1.
\end{aligned}$$

At period k , given a price, considering time aggregation, the reallocation problem that we have to solve is:

$$\begin{aligned}
& \max_{c_1, \dots, c_n} \sum_{i=1}^n B_{ik}(c_i, p, I_{ik+1}), \\
\text{s.t.} \quad & \sum_{i=1}^n c_i = \sum_{i=1}^n I_{ik+1} = I, \\
& c_i \in N^+, \quad \forall i = 1, \dots, n,
\end{aligned}$$

where

$$\begin{aligned}
& B_{ik}(c_i, p, I_{ik+1}) \\
& = p \sum_{j_i=0}^{\infty} \min(j_i, c_i) \Pr(\hat{D}_{ik}(p) = j_i) \\
& \quad - v \max(0, c_i - I_{ik+1}),
\end{aligned}$$

and $\hat{D}_{ik}(p)$ is defined in Section 2.1.1. Therefore, we can apply ALG1 to find the optimal solution because this formulation satisfies the conditions specified in (6). See Appendix 1 for a detailed proof.

We now describe the two-step heuristic to find a price-inventory solution for the case where reallocations are allowed.

Description of Heuristic: **HEUR4**.

STEP 0. Initialize the inventory at its current level, i.e., with no reallocations. With this inventory, determine the optimal price, p_0 , as in HEUR1. Set $k = 1$.

STEP 1. Determine an optimal inventory distribution, C^k , using algorithm ALG1, given p_{k-1} .

STEP 2. Determine the optimal price, p_k , using HEUR1, given C^k . Set $k = k + 1$ and go to Step 1.

The algorithm stops when the objective function does not change from one iteration to the next. We note that the heuristic always converges within a finite number of iterations. This is due to the finite number of inventory reallocations and the fact that at each step of the heuristic, the objective function monotonically improves.

Similar heuristics to those described above can be developed for the case where prices differ among stores.

2.3. Computational Experiments

In this subsection we present a set of computational experiments in which we compare the performance of the heuristics described above with respect to the optimal expected revenue.

In the first set of experiments we study the performance of heuristics HEUR1, HEUR2, HEUR3 developed for the No-Transfer-Inventory Policy Case. To be able to compute the optimal pricing policy, we consider a retail chain with two stores; in this relatively small case it is feasible to compute the optimal solution in reasonable time. We consider five periods in which prices are revised. We assume a Weibull distribution to describe the reservation price. This unimodal distribution implicitly assumes that the store faces a single market segment, which is close to reality if we consider that stores are usually located in local malls. This distribution also allows for flexibility to model different reservation price behaviors. We consider average arrivals at the stores of 120 and 60 customers over the planning horizon. Since only a fraction of customers actually buy the product, these numbers allow us to study the performance of the heuristics under high and low inventory levels (we consider initial inventories in a range of 30 units for store 1 and 20 units for store 2 to 5 units for both stores). The

Table I
Performance of Heuristics HEUR1, HEUR2, and HEUR3

Inventory Store 1–Store 2	HEUR1/ Opt.	HEUR2/ Opt.	HEUR3/ Opt.	Opt. (US\$)
30–20	98.0%	97.2%	99.0%	1366.7
30–15	98.7%	98.1%	99.0%	1281.7
30–10	99.4%	99.1%	98.6%	1177.9
30–5	99.5%	99.1%	97.9%	1043.2
30–0	99.6%	99.0%	98.4%	893.2
20–5	99.3%	99.1%	97.0%	767.4
10–5	98.6%	98.3%	97.4%	471.8
5–5	97.6%	97.7%	97.8%	315.4

parameters used in the experiments can be found in Appendix 2.

Using a dynamic programming approach we compute the exact expected revenue given by the heuristics. The results are summarized in Table I. The first column shows the initial inventory in stores 1 and 2, respectively; the next three columns show the performance of the heuristics with respect to the optimal value, and the last column shows the optimal value of the objective function. We observe that the performance of the three heuristics is satisfactory, with a maximum deviation from the optimal value of 3.0 percent.

The results also show no significant differences among the three heuristics, although HEUR1 shows a slightly better performance than the results obtained using HEUR2. HEUR3 shows better results for high inventory levels and deteriorates a bit for low inventory levels. Analyzing the empirical results, we observe that HEUR3 computes initial prices that are close to the optimal prices for high inventory levels, but for low inventory levels tends to overestimate the optimal prices, and therefore fewer products are sold than the optimal number. With few units in inventory, the impact of the lost sales on the expected revenue is higher than the cases in HEUR1 and HEUR2.

In these experiments we kept constant the arrival rate of customers to the stores. We also performed computational experiments where, for a fixed level of inventory, we varied the arrival rate of customers from low to high values. The results do not differ significantly from those presented in Table I.

In the second set of experiments we study the performance of heuristic HEUR4, for the case of merchandise interchange. For this case we consider three stores and three time periods when prices are revised. We assume that no cost is associated with moving products from one store to another. The main reason for this assumption is to make computation of the optimal solution tractable. The parameters can be found in Appendix 2.

In order to compute the expected revenue given by the heuristic, we generate the arrivals to the store in each period of time using the Poisson distribution, and for each arrival we generate its reservation price; if this price is larger than the product's price then the customer buys the product. We simulate observations of the revenue over the

Table II
Performance of Heuristic HEUR4

Total Amount in Inventory	HEUR4/Opt.	Opt. (US\$)
1	98.8%	35.2
5	98.5%	172.3
10	98.8%	339.2
15	99.1%	502.7
20	99.2%	663.8
25	98.8%	822.9
30	98.9%	980.2
35	99.2%	1136.0

planning horizon until the coefficient of variation of the expected revenue is less than or equal to 0.5 percent. This criterion is equivalent to choosing a confidence interval of 95 percent for the expected revenue with a resulting width less than or equal to 2 percent of the average revenue. For example, if the computed average revenue is 100, then the 95-percent confidence interval is contained in $[100 \pm 1]$.

Table II shows a satisfactory performance of the heuristic compared with the optimal solution.

For the case of merchandise interchange we compare the inventory transfers among stores given by the heuristic to those given by the optimal solution. In Figure 1 (a) we show the average merchandise interchange at the end of the first period as a function of the initial inventory. The continuous line corresponds to the optimal solution and the dotted line corresponds to the solution given by the heuristic. Figure 1 (b) is equivalent to Figure 1 (a) at the end of the second period. Finally, Figure 2 shows the average final inventory as a function of the initial inventory. The continuous curve corresponds to the optimal solution and the dotted curve to the heuristic solution. We observe that both the optimal and heuristic solution lead to very similar results. This behavior was observed in several other computational experiments.

In the computational experiments we also observed that the larger the variance in the reservation price distribution, the higher the initial prices. This result was reported in Bitran and Mondschein (1997) for the single-store case. Empirical tests show that the result still holds for the multiple-store case. A large variance in the reservation price distribution reflects the fact that the seller has little information about customers' willingness to pay for the product or/and that the customers in the market segment are heterogeneous. New, fashion, or exclusive products fit within this category of products. Store managers therefore start with a very high price and then adjust it according to the customers' reaction. A more complete analysis can be found in Bitran and Mondschein (1997).

3. REAL-CASE ANALYSIS

3.1. Demand Estimation

In this subsection we present the demand estimation for six different products in the eight stores that belong to the

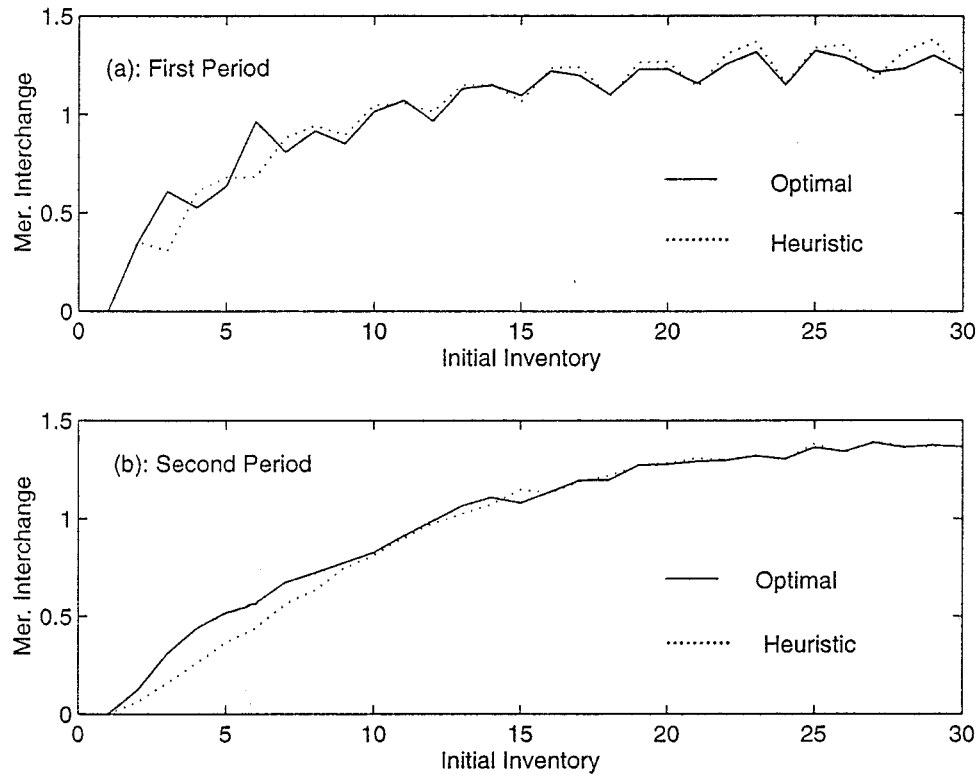


Figure 1. Average merchandise interchange as a function of initial inventory.

Chilean fashion retail chain Falabella. We collected data during 132 days, corresponding to 72.5 percent of the total 1995 autumn–winter season. (For reasons beyond our control we were not able to collect the data for the last 50 days of the season.) The demand estimation consists of the following steps: data collection, parameter estimation, and hypothesis testing for the distribution function of the demand.

3.1.1. Data Collection. The planning horizon is divided into five periods of 97, 7, 8, 14, and 6 days, respectively. Although the length of the periods is irregular, they correspond to the actual periods where the product manager said he or she revised the prices. At the end of the five periods we kept records of the price and sales for each product in each store of the retail chain. We also collected information about initial inventories. Table III shows an example of the data obtained during the season for an individual item.

3.1.2. Parameter Estimation. Before performing hypothesis testing to check that the purchasing process comes from a Poisson distribution, we estimate, using maximum likelihood procedures, the *purchasing rates* for each product j , store i , and price p as follows:

$$\hat{\lambda}_{ij}(p) = \frac{\sum_{k=1}^K x_{ijk}(p)}{\sum_{k=1}^K T_k(p)} \quad \forall i, j,$$

where $x_{ijk}(p)$ corresponds to the sales in store i in period k at price p for product j , and $T_k(p)$ is equal to the length of period k if in that period the price was equal to p and zero otherwise. Table IV shows the estimated purchasing rates for product 1.

3.1.3. Hypothesis Testing for the Poisson Distribution. Given the estimation of the parameters, we perform a hypothesis test to verify if it is “reasonable” to assume that the purchasing process comes from a Poisson distribution. We test the hypothesis for each individual product and store where it is sold. The most common test used for this purpose is the χ^2 . However, in this case two difficulties prevent us from using the test directly: (1) the observations of the number of sales for a given price were measured in periods of time of different length, and (2) the number of observations is relatively small. Therefore, we modified the test to deal with these limitations. We now briefly explain this procedure. To simplify notation we omit the indices i and j for the store and product, respectively.

Step 1. Let x_1, x_2, \dots, x_K denote the sales during periods T_1, T_2, \dots, T_K for a given price. We define the statistic χ_M^2 as follows:

$$\chi_M^2 = \sum_{k=1}^K \frac{(x_k - \hat{\lambda}(p)T_k(p))^2}{\hat{\lambda}(p)T_k(p)}.$$

Finally we evaluate this statistic using the raw sales data: $\hat{\chi}_M^2$.

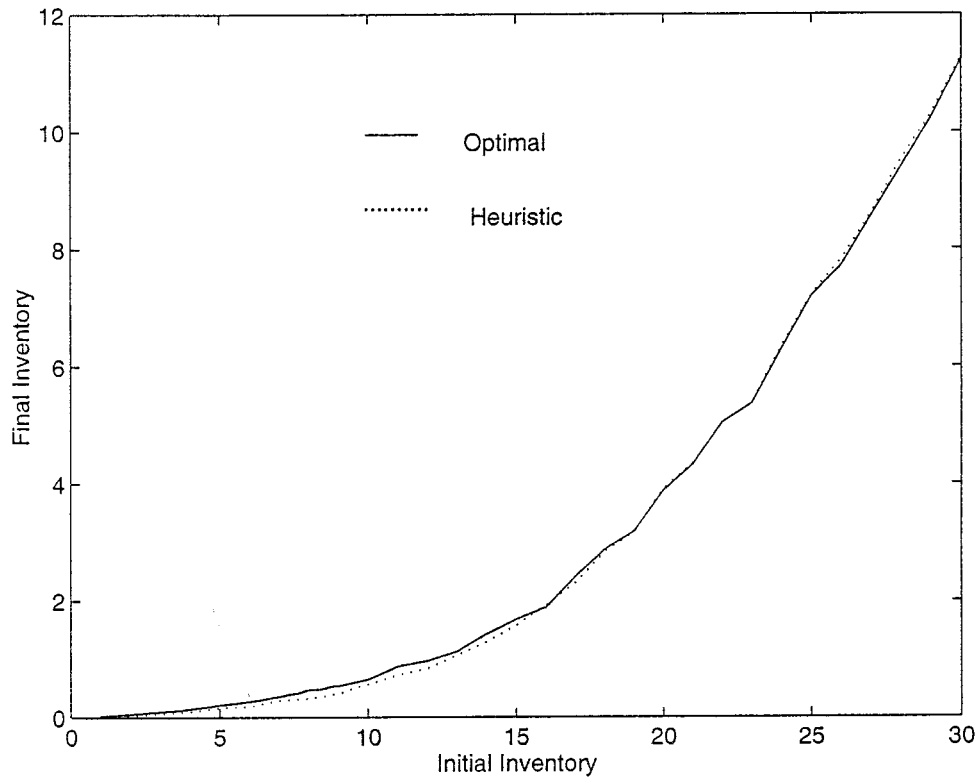


Figure 2. Average final inventory as a function of initial inventory.

Step 2. The χ_M^2 statistic does not necessarily follow a chi-squared distribution. Therefore, we determine the true distribution of this statistic using Monte Carlo simulation. For this purpose we generate sales in periods T_1, T_2, \dots, T_K according to a Poisson distribution with parameters $T_1\hat{\lambda}(p), T_2\hat{\lambda}(p), \dots, T_K\hat{\lambda}(p)$. We apply Step 1 to these generated sales to compute a value of χ_M^2 . We repeat this procedure to obtain an empirical distribution of the χ_M^2 statistic; we denote by $G(\cdot)$ this probability distribution function.

Finally, we accept the hypothesis that the sales process follows a Poisson distribution if the p -value is greater than or equal to 10 percent, where the p -value satisfies $1 - G(\hat{\chi}_M^2) = p$ -value.

Results of the Hypothesis Testing. We performed the hypothesis test for six products in the eight stores of Fala-bella following the procedure described in Steps 1 and 2. For each product and store we built the empirical distribution for this statistic and tested the hypothesis. For all stores except the eighth, we concluded that for a p -value of 10 percent we cannot reject the hypothesis that the sales come from a Poisson process. The eighth store was excluded from the computational experiments in subsection 3.2; the real contribution of this store to the total revenue associated with the set of products under consideration is less than 1 percent (the products considered are for women, and this store is located in a mining area with a small population of women).

3.1.4. *Estimation of the Purchasing Rate as a Function of Price.* Finally, we estimate the function that describes the

Table III
Prices and Sales for Product 1

Store	Period 1 97 Days	Period 2 7 Days	Period 3 8 Days	Period 4 14 Days	Period 5 6 Days
	\$29	\$20	\$20	\$20	\$20
1	177	21	28	92	17
2	87	17	14	43	21
3	96	18	17	50	17
4	52	11	10	21	16
5	71	7	6	12	11
6	68	16	27	24	23
7	18	4	9	17	7
8	8	4	1	2	0

Table IV
Estimated Purchasing Rates for Product 1

Store	Price \$29	Price \$20
1	1.82	4.51
2	0.90	2.71
3	0.99	2.91
4	0.54	1.66
5	0.73	1.03
6	0.70	2.57
7	0.19	1.06
8	0.08	0.19

Table V
Number of Customers per Week

Store	Low Flow Cust./Week	Medium Flow Cust./Week	High Flow Cust./Week
Store 1	7	16	27
Store 2	5	12	20
Store 3	4	9	15
Store 4	4	8	13
Store 5	3	7	11
Store 6	3	6	11
Store 7	2	3	6
Store 8	1	1	2

Reservation price distributions: Weibull distribution with cumulative distribution equal to $F(p) = 1 - e^{-(\rho p)^\beta}$, where:

- Stores 1, 3, 7, 8, and 9: $\beta = 3.49$, and $\rho = 0.0456$.
- Stores 2, 4, 5, and 6: $\beta = 4.39$, and $\rho = 0.0368$.

Unit cost: \$12.5

Moving costs: 0

purchasing rate as a function of price. We assume a constant elasticity in the range of prices used by the retail chain, a common assumption made in economics. This assumption implicitly considers a probability density function for the reservation price equal to $f(p) = -\delta\gamma/p^{1-\gamma}$, where γ is the constant elasticity ($\gamma < 0$). We note, however, that the models and heuristics developed previously are valid for any reservation price distribution.

For prices smaller than a lower bound we assume an elasticity equal to zero and for prices higher than an upper bound we assume an elasticity equal to infinity. Figure 3 shows a typical purchasing rate function.

Figure 3 shows the estimated purchasing rate, assuming that the range of prices that the store used during the planning horizon fluctuates between \$20 and \$29. Although we only know the customer behavior in that range, it is reasonable to assume that the constant elasticity holds for a wider range of prices, i.e., it is likely that there are customers that are willing to pay higher or lower prices than those in the range actually set by the retail chain. Thus, we selected the range of prices between \$15 and \$35. The elasticity is computed with the estimates of the purchasing rates obtained from subsection 3.1.2. We believe that these assumptions are reasonable and, in any case, lead to a conservative performance of our method compared with the real results obtained by Falabella. For example, we truncate the reservation price distribution at the upper price and assume that nobody would pay higher prices.

3.2. Comparison Between the Performance of Our Model and Falabella's Policy for the 1995 Autumn–Winter Season

In this section we contrast the performance of our method with actual results obtained by Falabella under the same selling scenarios, for the 1995 autumn–winter season. For this purpose, we compare the actual revenues obtained by Falabella for each product, aggregated by stores, with all possible revenues that the retail chain would have obtained if it had used our model to make the pricing decisions. We observed the actual sales that the retail chain obtained in each period of the season for the chosen prices; the sales that would have taken place when implementing our methodology would have been different

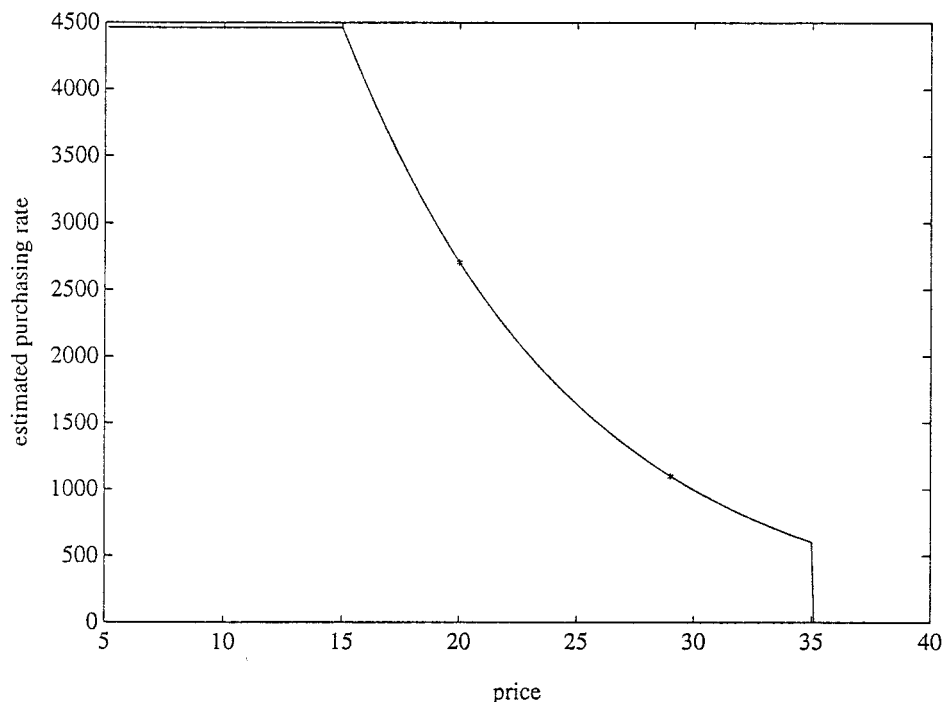


Figure 3. Estimated purchasing rate.

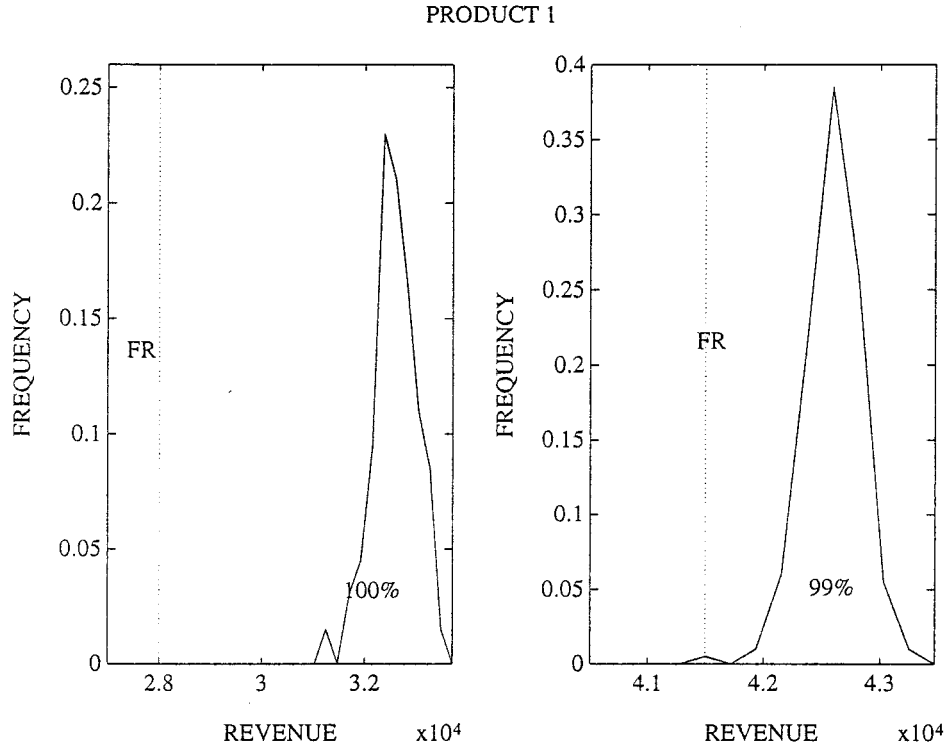


Figure 4. Histogram of revenues associated with product 1. (FR is the actual revenue obtained by Falabella.)

because of different prices, and therefore need to be estimated. We can obtain the probability distribution for the conditional sales given past observations. With these conditional sales, we generate the “probability distribution function” for the total revenues under all feasible scenarios that could have taken place. Finally, we determine the probability that we would have obtained better results than those obtained by Falabella. We use HEUR1 to solve the model because of its simplicity and satisfactory performance.

Product managers at Falabella have a target profitability to reach during the selling season for the set of products that they manage. However, they have flexibility in choosing the pricing policy to reach this goal, which depends mainly on factors such as the type of products (for example, electronics, seasonal or fashion clothing, and standard clothing) and the experience of the product manager. In particular, for the set of products considered in this case (seasonal clothing), at the beginning of each period, the product manager computed the ratio between the fraction of the unsold inventory and the fraction of time left to the end of the season. If this ratio was much greater than one, then she decreased the price using her experience (not a formal procedure). For example, let us consider an initial inventory of 100 units and 20 days to sell the product. Assume that after 5 days they have sold only 10 units. Then the above ratio is equal to

$$\frac{(100 - 10)/100}{(20 - 5)/20} = 1.2.$$

We observe that the criterion used in this case attempts to keep a constant sales rate that would get rid of inventory by the end of the season. This criterion is not necessarily optimal.

We now briefly describe the conditional distribution for the sales and the results obtained for two different products. To simplify notation we omit the indices for the product and store under consideration. We denote by x_k the real sales in period k for a given price p_F and by $D_k(p)$ the random variable that represents the demand at price p . Our purpose is to determine:

$$\Pr(D_k(p) = s/D_k(p_F) = x_k) \quad \forall p.$$

After applying basic probability concepts we obtain:

Case 1. $p > p_F$.

$$\Pr(D_k(p) = s/D_k(p_F) = x_k)$$

$$= \begin{cases} \binom{x_k}{s} \left[\frac{\hat{\lambda}(p)}{\hat{\lambda}(p_F)} \right]^s \left[1 - \frac{\hat{\lambda}(p)}{\hat{\lambda}(p_F)} \right]^{x_k - s} & \forall p \text{ if } s \leq x_k, \\ 0 & \text{if } s > x_k. \end{cases}$$

Case 2. $p \leq p_F$.

$$\Pr(D_k(p) = s/D_k(p_F) = x_k)$$

$$= \begin{cases} \frac{(\hat{\lambda}(p) - \hat{\lambda}(p_F))^{s-x_k}}{(s-x_k)!} e^{-(\hat{\lambda}(p) - \hat{\lambda}(p_F))} & \forall p, \text{ if } s \geq x_k, \\ 0 & \text{if } s < x_k. \end{cases}$$

Using this conditional probability distribution function for the demand we apply our methodology to obtain the

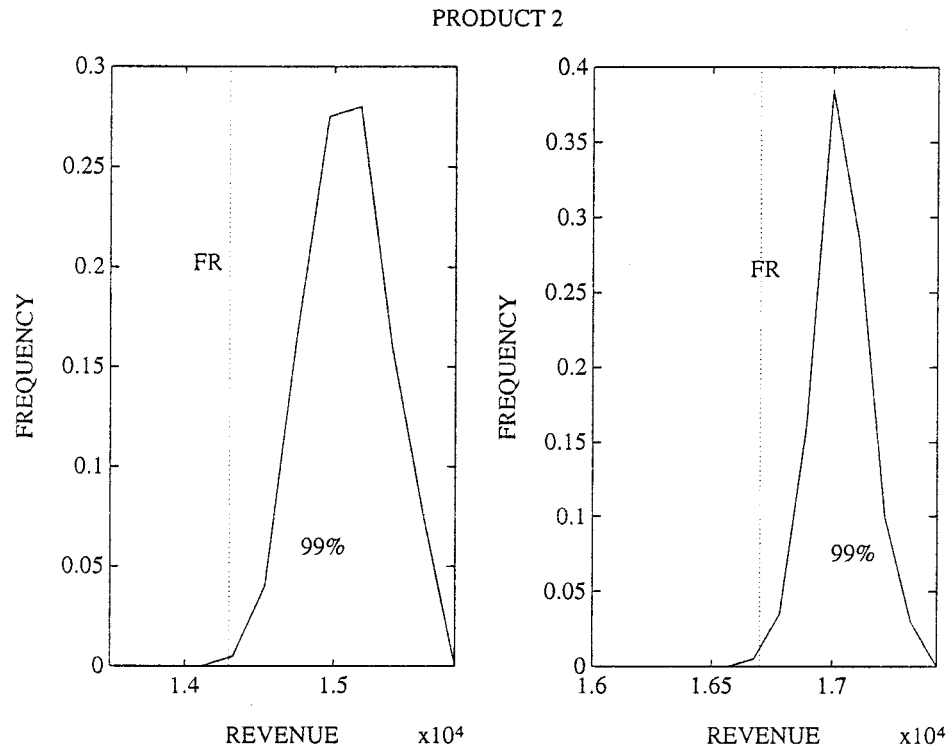


Figure 5. Histogram of revenues associated with product 2. (FR is the actual revenue obtained by Falabella.)

probability distribution function for the revenues. Figures 4 and 5 show the results obtained for the two products. The vertical line represents the actual revenues obtained by the company. As mentioned earlier, we collected information for the first 72.5 percent of the planning horizon. Therefore, we make a comparison for the first 132 days, and for the whole planning horizon, assuming that Falabella acts optimally in the last 50 days.

For product 1 we observe that in all possible scenarios our model leads to better results for the first 132 days. The average performance of the heuristic is equal to \$32,500 compared with the \$28,000 obtained by the retail chain, i.e., an average increase of 16.1 percent. When considering the total planning horizon, in 99 percent of the possible scenarios our model performs better than the company's policy; in the 1-percent probability of a poorer performance, the difference in revenues is less than 1 percent.

Product 2 has similar behavior to that of product 1. Figure 5 shows that in 99 percent of the cases our model leads to better results than that obtained by Falabella when considering the first part of the planning horizon as well as the total planning horizon. We obtained similar results for the remaining four products. (As mentioned in subsection 3.1.3, the eighth store was excluded from the computational experiments.)

3.3. Simulation Game to Compare the Performance of the Proposed Methodology Against the Current Practices at Falabella

In this section we compare the performance of our methodology against the performance of an experienced prod-

uct manager using a "simulation-game" setting. For this purpose we created a simulation software that generates the number of sales and the associated revenue as a function of prices. In the simulation game, the product manager sets a price at the beginning of each period of the planning horizon according to the inventory levels in the stores and the time left until the end of the horizon. For this price, the software generates a random number of sales and updates the inventory levels at the end of the current period. At the end of each simulation the revenue is computed. The simulation game also allows for merchandise interchange.

At the beginning of the simulation, the product manager was told about the distribution of the flow of customers into the stores and the corresponding reservation price distributions. The information was presented through graphs and tables. To simplify the information that the manager had to process to make the pricing decisions we considered three different (discrete) scenarios for the flow of customers. For the reservation price we used a Weibull distribution. The results of these simulations are compared to the performance of our methodology under the same scenarios for the demand function. The data used in the experiments can be found in Appendix 2.

The simulation-game experiments allow us to compare our methodology with the product manager's pricing policy under the same demand process, independently of the demand estimation that is involved in Section 3.2. Therefore, all factors that can affect the product manager's decision process are contained in the setting given in the simulation game.

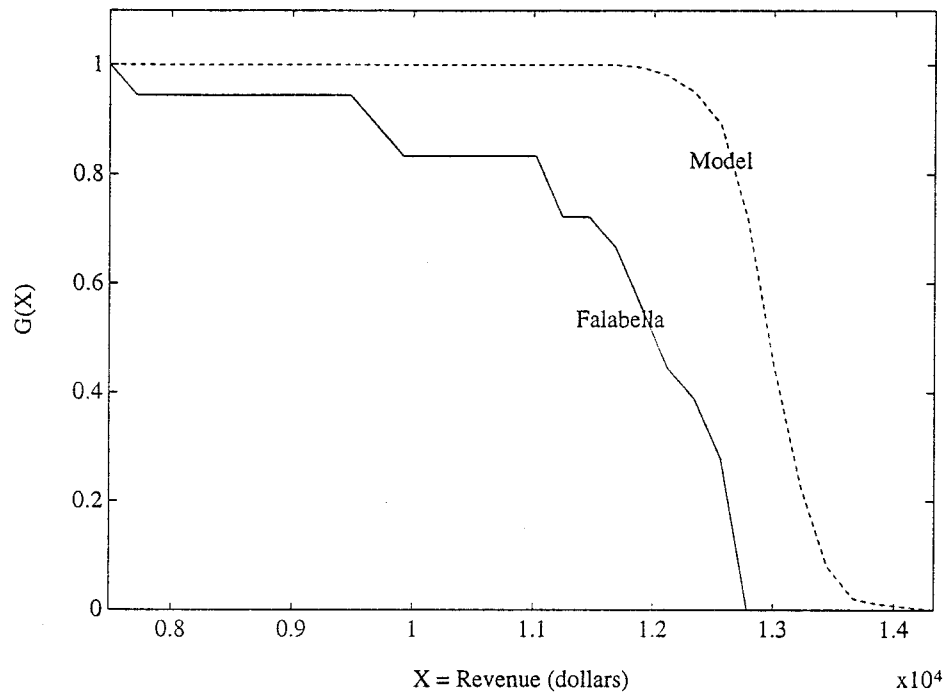


Figure 6. The curves ($G(X)$) represent the probability that the revenue will be higher than x , for a given value x for the revenue.

3.3.1. *Comparison for the No-Inventory-Transfer Case.* Similarly to section 3.2, we use HEUR1 to compute the revenue given by our methodology. The average revenue obtained in the product manager simulations is \$11,591 with a standard deviation of \$1390. The average revenue obtained using our methodology is \$12,967 with a standard deviation of \$360, i.e., approximately 12 percent higher than that obtained by the product manager. In Figure 6 we show the probability that the product manager (or the model) will obtain a higher revenue for all revenue levels; i.e., for a given value x for the revenue, the curves ($G(X)$) represent the probability that the revenue will be higher than x , given the demand functions and the initial inventory. We observe that our methodology performs better for all levels of revenue.

3.3.2. *Comparison for the Case with Inventory Transfers Allowed.* In this case the manager and the model can move merchandise among stores at a cost. We use HEUR4 to compute the revenue given by our methodology. The average revenue obtained in the simulations made by the product manager is \$13,834 with a standard deviation of \$790. Our methodology led to an average revenue of \$14,787 with a standard deviation of \$111, i.e., approximately 7 percent better than the performance of the product manager. Figure 7 shows the probability that the manager (or the model) will have obtained a higher revenue for all levels of revenue. We observe that our methodology is always better than the current practice. Furthermore, the revenue obtained in the worst selling scenario for the model is higher than the highest revenue obtained in the simulations made by the product manager. An interesting

outcome of the simulation-game experiments is the learning process experienced by the product manager that performed the simulations. As stated by the product manager, the simulations allowed her to learn the impact of different types of strategies on the profitability of the set of products under her supervision. She was able to experiment with pricing policies in the simulations that she would not have been able to try in practice due to the risk of a poor performance. According to her, during the current season she is implementing new pricing policies that start with lower initial prices and use lower discounts later on in the planning horizon.

4. CONCLUSIONS

In this paper we developed a methodology to set prices of perishable products in retail chains. We found that our methodology performs better than the current practice in a large fashion retail chain in Chile. We believe there are three main factors that explain this fact. The first factor is the lack of formal procedures for pricing decisions in the retail chain. Product managers often estimate the demand for new and old products using their experience and intuition instead of relying on available data from previous seasons. They make decisions based on average demand without taking into account explicitly the associated uncertainty. Prices are set after comparing the demand estimate, or the actual demand at the end of a period, with the rate of sales needed to sell the remaining inventory by the end of the season. A second factor is the way the product managers are evaluated by the company. They have to

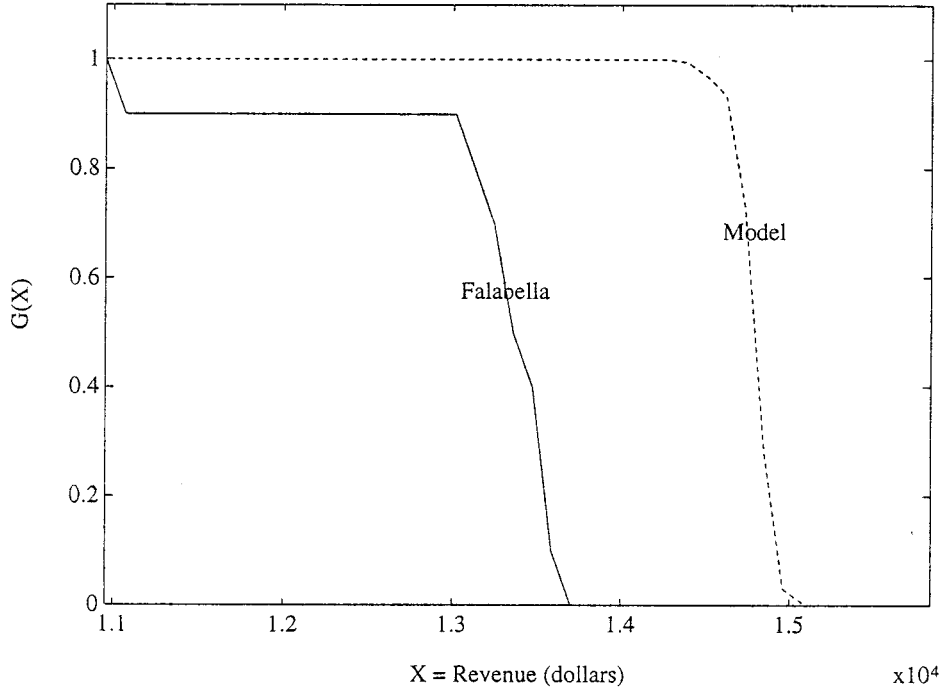


Figure 7. The curves $G(X)$ represent the probability that the revenue will be higher than x , for a given value x for the revenue.

reach a profitability target each season. This short-run incentive scheme leads managers to set “high” initial prices to guarantee the achievement of their financial goal early in the season. Finally, a third factor is that our methodology is able to handle the stochastic and dynamic structure of the problem. It attempts to maximize the profit contribution of each unit in inventory. The heuristics show a behavior close to the optimal solution in a problem that is difficult to handle without the support of a mathematical model. We expect that our methodology will have similar savings in other retail companies that have not invested in decision support systems of this nature.

An interesting learning process took place as an outcome of the simulation-game. As stated by the product manager, the simulation-game experiments allowed her to learn the impact on profitability of different types of pricing strategies, and therefore change her behavior in some types of selling scenarios. For example, she realized that under certain scenarios it is a good strategy to start with lower prices and implement lower discounts later on in the selling season; previously she was, in general, using a much more aggressive strategy setting high initial prices and large discounts during the liquidation period. We believe that the simple simulation-game is a powerful learning tool.

APPENDIX 1

Proof of Proposition 1

Necessary condition: Let $C^* = (c_1^*, \dots, c_m^*)$ be the optimal solution of problem **P1**. Therefore, C^* is feasible. Assume that there exist i and j such that $c_i^* \geq 1$ and:

$$F_i(c_i^*) - F_i(c_i^* - 1) < F_j(c_j^* + 1) - F_j(c_j^*),$$

then, we could define the vector $V = (v_1, \dots, v_m) = (c_1^*, \dots, c_i^* - 1, \dots, c_j^* + 1, \dots, c_m^*)$, which is feasible for problem **P1** and satisfies that following condition:

$$\sum_{i=1}^m F_i(v_i) > \sum_{i=1}^m F_i(c_i^*).$$

Therefore, C^* would not be the optimal solution.

Sufficient condition: Assume that $C = (c_1, \dots, c_m)$ is feasible and satisfies the condition:

$$F_i(c_i) - F_i(c_i - 1) \geq F_j(c_j + 1) - F_j(c_j). \quad (8)$$

We want to prove that C is the optimal solution of problem **P1**. Let us consider another feasible vector $V = (v_1, \dots, v_m)$ for problem **P1** and the set of indices $A = \{i : c_i > v_i\}$ and $B = \{j : c_j < v_j\}$. We have:

$$\begin{aligned} & \sum_{i=1}^m [F_i(c_i) - F_i(v_i)] \\ &= \sum_{i \in A} [F_i(c_i) - F_i(v_i)] - \sum_{j \in B} [F_j(v_j) - F_j(c_j)]. \end{aligned} \quad (9)$$

For all $i \in A$ we have:

$$\begin{aligned} F_i(c_i) - F_i(v_i) &= \sum_{k=v_i}^{c_i-1} (F_i(k+1) - F_i(k)) \\ &\geq \sum_{k=v_i}^{c_i-1} (F_i(c_i) - F_i(c_i - 1)) \\ &= (c_i - v_i)(F_i(c_i) - F_i(c_i - 1)). \end{aligned} \quad (10)$$

We define $\mathcal{F} = \operatorname{argmin}_{i \in A} (F_i(c_i) - F_i(c_i - 1))$ and $\Delta F_A = F_{\mathcal{F}}(c_{\mathcal{F}}) - F_{\mathcal{F}}(c_{\mathcal{F}} - 1)$. Hence, for all $i \in A$ we have:

$$F_i(c_i) - F_i(v_i) \geq (c_i - v_i) \Delta F_A. \quad (11)$$

Similarly, for all $j \in B$ we have:

$$\begin{aligned} F_j(v_j) - F_j(c_j) &= \sum_{k=c_j}^{v_j-1} (F_j(k+1) - F_j(k)) \\ &\leq \sum_{k=c_j}^{v_j-1} (F_j(c_j+1) - F_j(c_j)) \\ &= (v_j - c_j)(F_j(c_j+1) - F_j(c_j)). \end{aligned} \quad (12)$$

We define $\mathcal{J} = \operatorname{argmax}_{j \in B} (F_j(c_j+1) - F_j(c_j))$ and $\Delta F_B = F_{\mathcal{J}}(c_{\mathcal{J}}+1) - F_{\mathcal{J}}(c_{\mathcal{J}})$. Hence, for all $j \in B$ we have:

$$F_j(v_j) - F_j(c_j) \leq (v_j - c_j) \Delta F_B. \quad (13)$$

Replacing (11) and (13) in (9) we obtain:

$$\begin{aligned} \sum_{i=1}^m [F_i(c_i) - F_i(v_i)] \\ \geq \Delta F_A \sum_{i \in A} (c_i - v_i) - \Delta F_B \sum_{j \in B} (v_j - c_j). \end{aligned}$$

We observe that $\sum_{i \in A} (c_i - v_i) = \sum_{j \in B} (v_j - c_j)$ because both vectors are feasible for **P1**. Therefore,

$$\sum_{i=1}^m [F_i(c_i) - F_i(v_i)] \geq (\Delta F_A - \Delta F_B) \sum_{i \in A} (c_i - v_i).$$

Finally, using condition (8), we observe that $\Delta F_A \geq \Delta F_B$, and therefore,

$$\sum_{i=1}^m [F_i(c_i) - F_i(v_i)] \geq 0. \quad \square$$

Proof that the Function $B_{ik}(\cdot)$ Satisfies Condition (6)

In what follows we prove that the function

$$\begin{aligned} B_{ik}(c_i, p, I_{ik+1}) \\ = p \sum_{j_i=0}^{\infty} \min(j_i, c_i) \Pr(\hat{D}_{ik}(p) = j_i) \\ - v \max(0, c_i - I_{ik+1}) \end{aligned}$$

satisfies the property:

$$\begin{aligned} B_{ik}(c_i+1) - B_{ik}(c_i) &\geq B_{ik}(c_i+2) - B_{ik}(c_i+1). \\ B_{ik}(c_i+1, p) - B_{ik}(c_i, p) \\ &= p \sum_{h=0}^{\infty} [\min(h, c_i+1) - \min(h, c_i)] \Pr(\hat{D}_{ik}(p) = h) \\ &\quad - v \max(0, c_i+1 - I_{ik+1}) + v \max(0, c_i - I_{ik+1}) \\ &= p \sum_{h=c_i+1}^{\infty} \Pr(\hat{D}_{ik}(p) = h) \\ &\quad - v \max(0, c_i+1 - I_{ik+1}) + v \max(0, c_i - I_{ik+1}) \\ &= p \Pr(\hat{D}_{ik}(p) \geq c_i+1) - v \quad \text{if } c_i - I_{ik+1} \geq 0 \\ &= p \Pr(\hat{D}_{ik}(p) \geq c_i+1) \quad \text{if } c_i - I_{ik+1} < 0. \end{aligned} \quad (14)$$

Finally, using the previous result for the pairs (c_i+1, c_i) and (c_i+2, c_i+1) and simple algebraic manipulations, we obtain the desired inequality, considering the cases $c_i - I_{ik+1} \geq 0$, $c_i - I_{ik+1} = -1$, and $c_i - I_{ik+1} \leq -2$.

APPENDIX 2

Data Used in the Computational Experiments Described in Section 2.3.

1. Computational experiments for the no-transfer inventory policy problem.

Number of stores: 2

Arrival rates for the product under analysis:

- Store 1: 2 customers per day;
- Store 2: 1 customer per day.

The arrival rates are relatively small (120 and 60 customers during the planning horizon) in order to observe the performance of the heuristics for high and low initial inventory levels (compared to the average arrival rates). With larger arrival rates would have been infeasible to compute the optimal pricing policies in reasonable amount of computational time, for large initial inventories.

Reservation price distribution: Weibull with cumulative distribution equal to $F(p) = 1 - e^{-(\rho p)^\beta}$, where:

- Store 1: $\beta = 8$, and $\rho = 0.0344$;
- Store 2: $\beta = 5$, and $\rho = 0.0372$.

Periods for pricing updates:

- Period 1: 20 days;
- Period 2: 15 days;
- Period 3: 10 days;
- Period 4: 8 days;
- Period 5: 7 days.

2. Computational experiments for the case where merchandise interchange is allowed.

Number of stores: 3

Arrival rates for the product under analysis during the planning horizon:

- Store 1: 6;
- Store 2: 4;
- Store 3: 2.

Reservation price distribution: Weibull with cumulative distribution equal to $F(p) = 1 - e^{-(\rho p)^\beta}$, where:

- Store 1: $\beta = 8$, and $\rho = 0.0344$;
- Store 2: $\beta = 6$, and $\rho = 0.0372$;
- Store 3: $\beta = 5$, and $\rho = 0.0393$.

Periods for pricing updates: three periods of equal length.

Data Used in the Simulation-Game in Section 3.3.

Planning horizon: 180 days, divided in 7 periods of 45, 30, 45, 15, 15, 15, and 15 days.

Number of Stores: 9, with initial inventories of 90, 130, 100, 80, 60, 80, 36, 24, and 60 units.

Flow of customers: we present three flow levels (low, medium, and high) that take place with probabilities 0.3, 0.5, and 0.2, but for store 8 where the probabilities are 0.3, 0.6, 0.1.

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