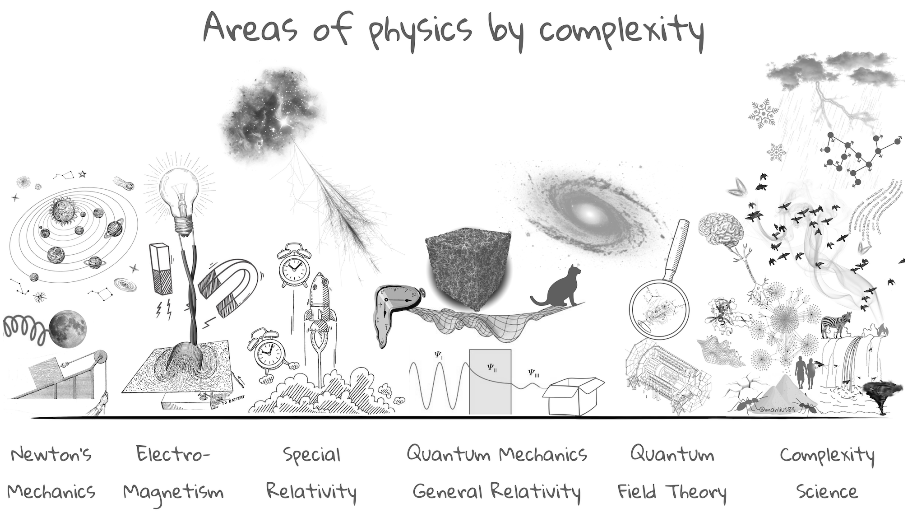
Final Report

Physics of Complex Networks: Structure and Dynamics

UNIVERSITÀ DEGLI STUDI DI PADOVA



PoCN: project report

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Contents

1 | Voter model - Task 28

The Voter model is a simple model for describing consensus in opinion dynamics as a Markov process. For our purposes, the stage is a network represented by a symmetric adjacency matrix $A \in Mat(N \times N, \{0,1\})$. On top of the network, a random variable $\sigma_i \in \{-1, 1\}$ is then defined (just two possible opinions), with i going from 1 to N; this is the state of the system, which, in the noisy version of the model [3], evolves stochastically with a discrete-time update probability given by

$$P\Big[\sigma_i(t+1) = -\sigma_i(t)\Big] = a + \left(\frac{1}{2} - a\right)\left(1 - \frac{\sigma_i(t)}{k_i}\sum_{j=1}^{N} A_{ij}\ \sigma_j(t)\right), \tag{1.1}$$

i.e., the probability to change opinion in the next time-step is exactly the current fraction of neighbors with different opinion[1] plus a spontaneous transition probability given by the noise parameter $a \in (0, \frac{1}{2})$. For $a = 0$ the classic Voter model is recovered, while for $a = \frac{1}{2}$ the update is purely random; interestingly, going beyond $\frac{1}{2}$ would result in a noisy contrarian Voter model.
The evolution of the dynamics can be described by the average interface density, defined as the density of links connecting nodes with different opinions:

$$\rho(t) = \left(\sum_{i,j=1}^{N} A_{ij}\frac{1 - \sigma_i(t)\sigma_j(t)}{2}\right) \Big/ \sum_{i=1}^{N} k_i \,. \tag{1.2}$$

This quantity is, in fact, the order parameter of the system: it is zero for the fully ordered state, or full consensus, and different from zero in a disordered state.

The main objective is to study the behavior of $\rho(t)$ as the network topology changes, and to find the structural characteristics that affect the ordering of the system (or not) in the infinite time limit. For regular lattices, it is known that, in the thermodynamic limit, the Voter model does not reach consensus for $d > 2$ [1], and for networks, it has been shown that the degree distribution alone does not alter the ordering process in a meaningful way [6]. The structural investigation pursued is therefore a combination of difference in degree heterogeneity and disorder; the latter has the role of effective dimension, with strong hints that what is important is in fact a change in the spectral dimension.

[1] Note that the update trial is carried out sequentially: each time-step, a single node is randomly chosen and (maybe) updated; nevertheless, both in the following and in the literature, a time unit consists of N such operations.

1.1 | Classic VM - dimension and degree distribution

The analysis is carried out as follows (see section A.1 for a pictorial representation):

- Construct a structured scale-free network as in [4] by Klemm and Eguíluz, and a regular 1D lattice (open boundary conditions), with the same average degree.
- Rewire both graphs with probability p, preserving the degree distribution.
- Simulate the voter dynamics and compare with Barabási–Albert networks, which are representative of random scale-free topologies.

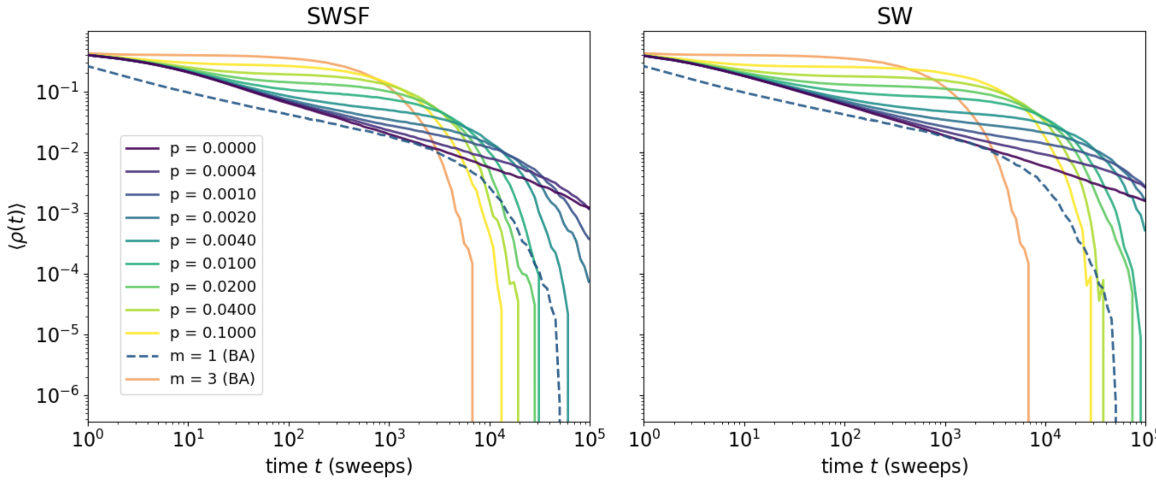

Figure 1.1: Time evolution of the interface density for $N = 4000$, averaged over 200 realizations of the dynamics and 12 graph realizations, $< k >= 6$ except for BA $m = 1$.

The BA network ($m = 3$) has an effective dimension $d = \infty$, consistently, the ordering process stops to reach a metastable state, however, given the finite size of the system, a finite fluctuation can bring it to one of the absorbing states, this is reflected in the late stage exponential decay, giving a finite lifetime to the plateau.
The structured scale free ($p = 0$ on the left) has an effective dimension $d \simeq 1$ [6], and tends to order indefinitely, decaying as a power law with exponent 1/2 as in one-dimensional regular lattices($p = 0$ on the right); both are still affected by finite size induced exponential decay, but it kicks in much later. The rewiring procedure smoothly interpolates between $d = 1$ and $d = \infty$: as p increases, the pre-plateau transient shortens and the plateau value itself grows. Degree homogeneity only extends the lifetime of the disordered phase.
Up to this point, *effective dimension* has not been rigorously defined, but there are two options: the topological dimension (Hausdorff) or the spectral dimension. The behavior of the $m = 1$ (dashed) BA suggests that the relevant concept of dimensionality is indeed the spectral dimension, in fact, ρ tends to zero even if the topological dimension is $d = \infty$, but $d_s = 4/3 < 2$ [2], this is consistent with the idea that the interface edges are performing a random walk in the line graph L(G) and annihilating upon contact [2].

[2]This is just a naive idea, a more accurate statement would be that the interface edges perform a parity-preserving branching-annihilating random walk (BARW) on L(G).

1.2 Introducing noise

Setting $a > 0$ in 1.1 removes the presence of the two absorbing states, since even in full consensus, there is always the possibility of a spontaneous transition, this fact alone changes the universality class of the system [5].

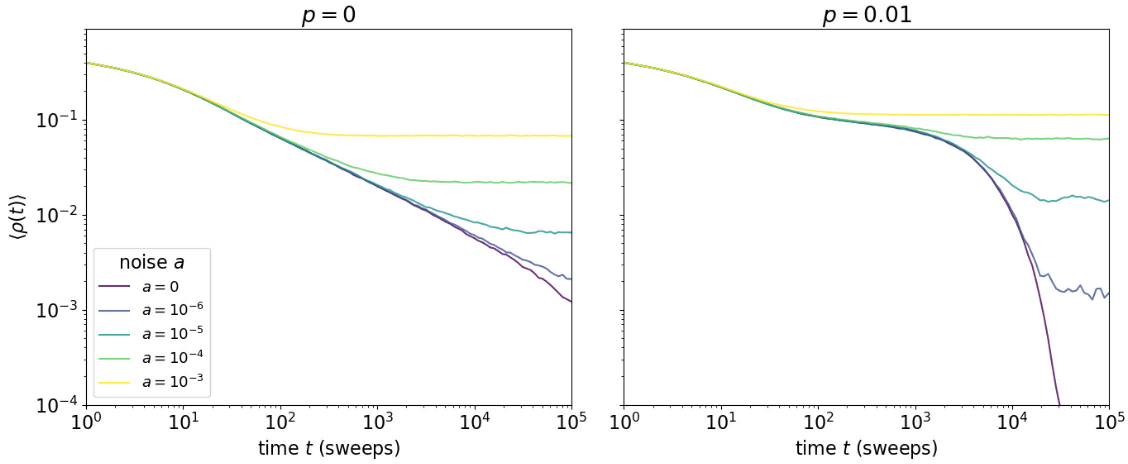

Figure 1.2: Time evolution of the interface density for SWSF,$N = 4000$, averaged over 200 realizations of the dynamics and 12 graph realizations, $< k >= 6$.

Noise generates mixed states which correspond to a plateau in the mean interface density, if the plateau value is reached before the fast exponential drop, the system becomes insensitive to finite size effects[3]. The plateau value is still an increasing function of p, indicating a collaborative effect between noise and spectral dimension. At low a, the temporal variance grows, reflecting a competition between noise and finite-size effects. Indeed, a finite-size induced phase transition is found between a unimodal and a bimodal phase, i.e. the system either is in a constant mixed state or the vast majority or nodes are either in one of the two states. This does not fully answer the high variance observed in $\langle \rho(t) \rangle$, because that trajectory was an average over both the dynamics and the different graphs, and because the high oscillations occur at noise values lower than the observed critical value. Although $a_{crit} \to 0$ for $N \to \infty$, it would be interesting to relate a_{crit} to the spectral dimension of the underlying network.

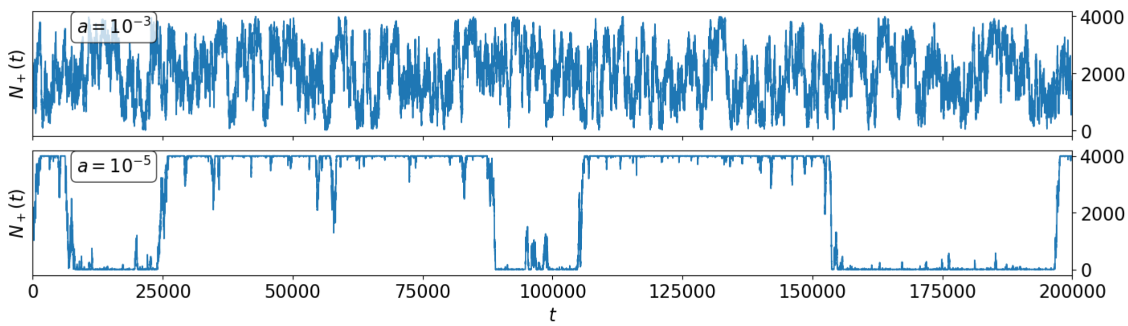

Figure 1.3: Single run plot of the number of up-spins for a BA network with $m = 2$.

[3]Fine-tuning a could indefinitely sustain an otherwise short-lived plateau.

2 | Europe rail networks - Task 45

A comprehensive dataset of European railroads (2019) from EuroGlobalMap is analyzed. The dataset consists of two objects: linestrings[1] and points, the former represent rail roads, while the latter are train stations. The resulting graph consists of two kind of nodes, station (s) and non-station (ns) nodes.
Although the fine details can be very important, the first instinct was to use the ns nodes to build a network consisting entirely of s nodes, connected if a path that does not cross any other s node exists between them. However, this task is extremely challenging while keeping an undirected graph with no additional node-metadata, the fundamental reason is because of rail junctions.

In the provided example, it is clear that in order to go from a to b, one must pass through c, and, while in this case the connection is straightforward ($a - c$; $b - c$) a lot of junctions are intertwined with each other and there are many edge cases.
Until the definition of *path* is addressed properly, centrality measures of betweenness and closeness could be misleading.

The objective is therefore to prune the network as much as possible, using only non-spatial topological features, such as degree, while retaining all the useful information on how different stations are connected with each other[2].

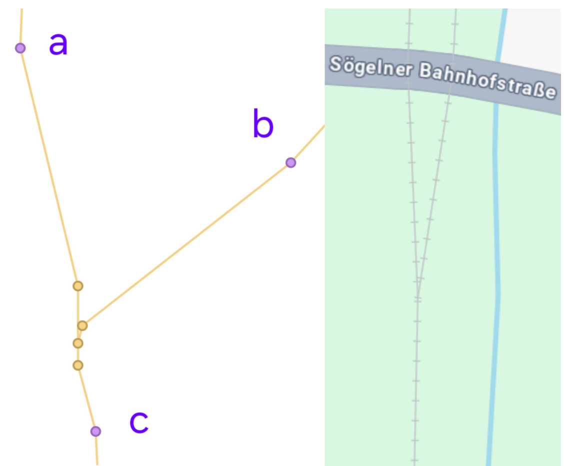

Figure 2.1: German rail junction. a, b and c are s nodes; on the right a Maps view.

[1]Linestring objects are a series of points (each with its own coordinate) connected successively with each other.

[2]Note that in the pruning process, one could keep records of the distance between ns nodes before removing them, and encode it in the edges of the pruned graph.

2.1 Pruning algorithm, overview and visualization

Upon data extraction, each linestring object is immediately reduced to just the first, second, second-last, and last points; then, a list of protected nodes is defined, its elements are:

- s nodes,
- neighbors of nodes v s.t. $k_v \geq 3$ and v does not belong in any $3 - clique$,
- all nodes that belong in a $3 - clique$.

The choice of protecting neighbors of high-degree nodes is an attempt to preserve the spatial information that would be needed to decide which paths are allowed through a junction, this can be achieved simply by forbidding the path between the nodes that form the lowest angle with that junction (although it is not that easy for 4 or 5 way junctions). The focus on 3-cliques is motivated by the fact that those types of junction always connect everyone with everyone else, without needing spatial information.
The algorithm itself first removes non protected degree-1 (dangling) nodes, than all non protected degree-2 nodes, then, intermediate nodes between almost-3-cliques are removed.
Throughout the process, an interactive visualization tool was used; here is a before and after example for the French rail network.

3 | Bibliography

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[6] Krzysztof Suchecki, Víctor M. Eguíluz, and Maxi San Miguel. Voter model dynamics in complex networks: Role of dimensionality, disorder, and degree distribution. *Physical Review E*, 72(3):036132, 2005.

[7] Duncan J. Watts and Steven H. Strogatz. Collective dynamics of “small-world” networks. *Nature*, 393(6684):440–442, 1998. doi: 10.1038/30918.

A | Supplementary material

A.1 | Structural setup

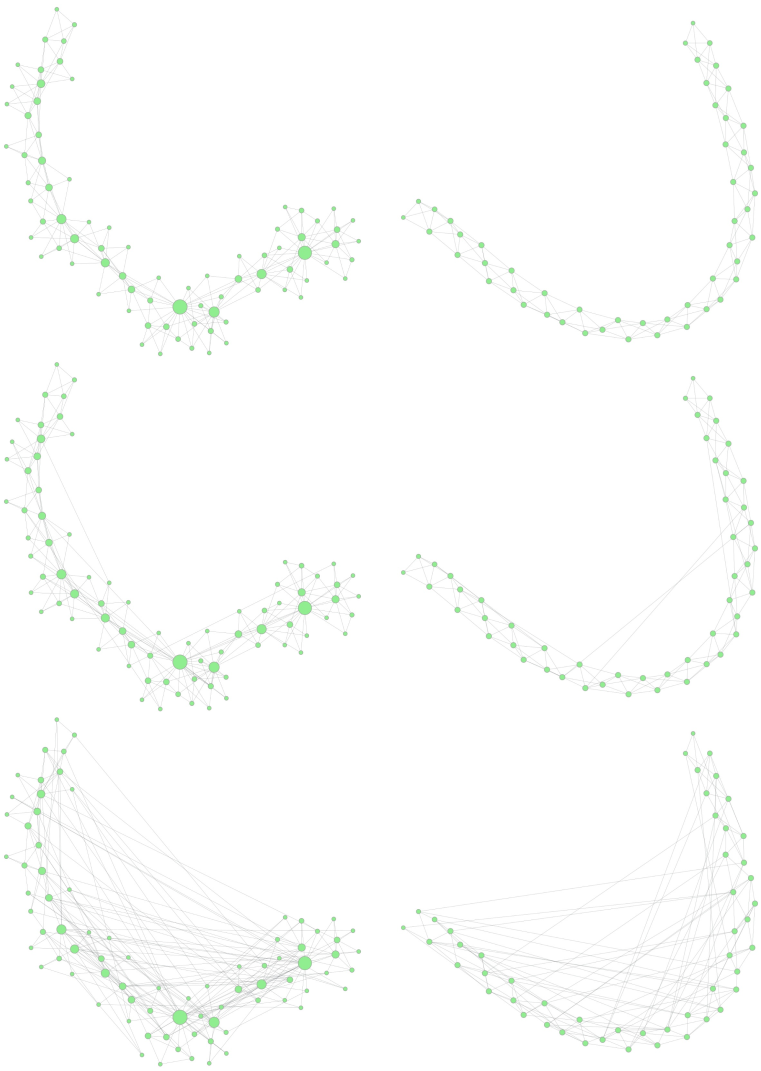

Figure A.1: At the top there is a structured scale free and a regular lattice, both 1-dimensional, then a rewiring procedure is employed, constructing a so called small world scale free (SWSF) and a small world network, for a value of the rewiring parameter p = 1 we have a scale free random graph on the left and a homogeneous random graph on the right,

A.2 | Noise investigation and other

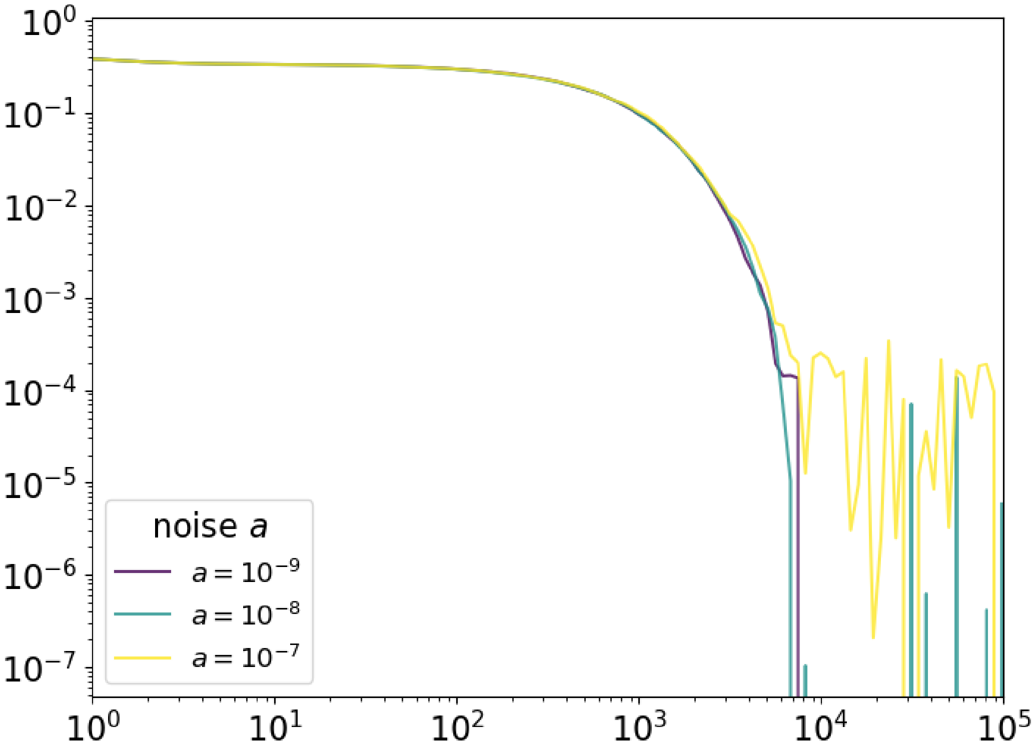

Figure A.2: Investigation of the behavior of low noise experiments on a BA network with m=2, the temporal variance is even larger, and the middle noise is characterized by spikes,they could be reflection of the transient between states in the unimodal phase, but is not a good guess

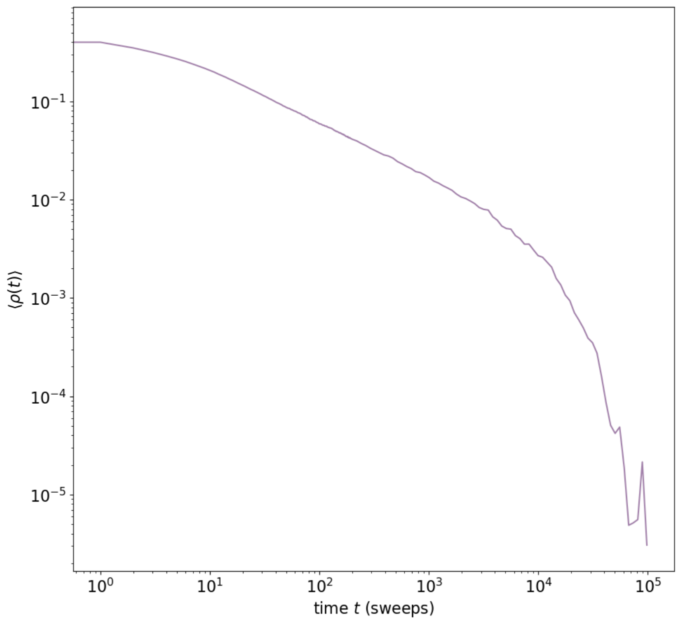

Figure A.3: run on a SSF network wirh $N = 1000$ to verify that finite size exponential decay indeed kicks in later as declared in the main section

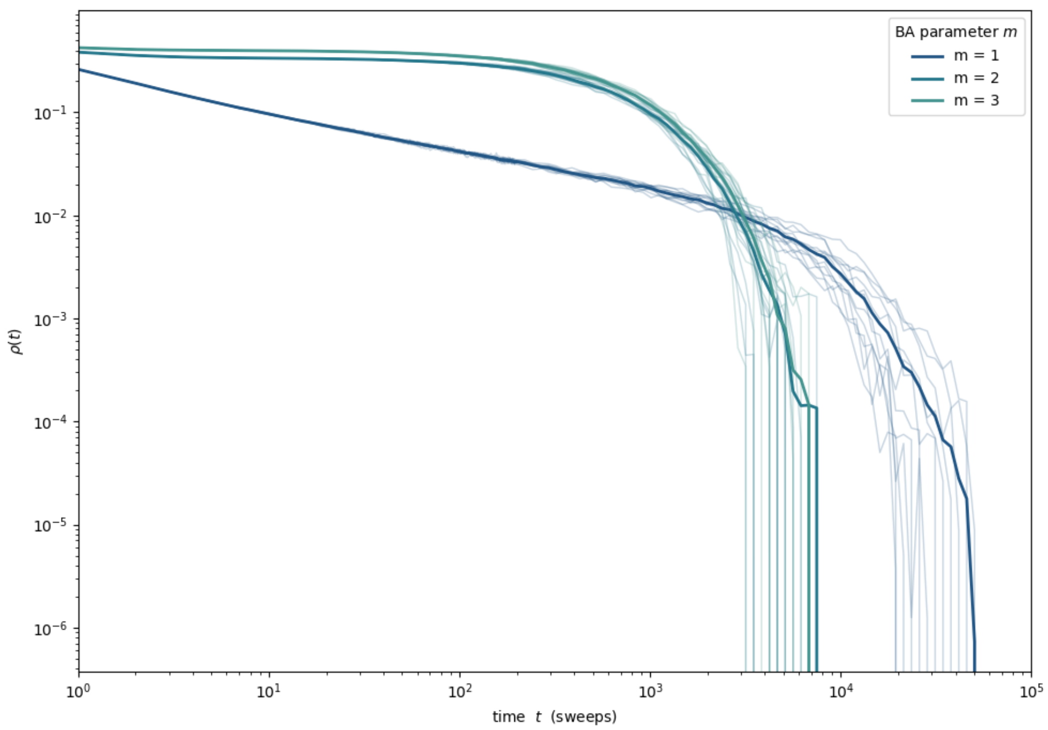

Figure A.4: Classic voter model on BA networks at increasing values of m, the light trajectories are the 12 single-graph trajectories of which the main ones are the average, when a plateau exists (m>1), its value is an increasing function of m.

B | Some remarks

B.1 | Voter model

It would be interesting to address quantitatively both the plateau value and the the lifetime of the system, for the former, single runs must be analyzed, because the plateau represents a metastable state for the system and has a precise value per graph realization; in theory it should be affected by quenched disorder, but maybe the effects vanish in the termodinamic limit.
Looking at 1.2, a perplexing phenomena is also the difference in behaviour between the SSF and the BA ($m = 1$), the only thing they share is having a spectral dimension smaller than two, but its curious how the BA seems to be a lower bound for the SSF, being just tangent to it right before the decay, this is also true for the 1d lattice, other low dimansional topologies should be studied to see how they relate with each other.

Finally, the authors of [6] never mention the spectral dimension, but they don't define clearly what dimension are they talking about, they talk about disorder (parameter p) wich can be understood to be small-worldness, and dimension separately, it would be interesting to compare the voter model dynamics while keeping in mind the three concepts of average path length, topological, and spectral dimension, finding the edge cases could confirm or refute that the spectral dimension is truly the only deciding factor for determining if the system will order or not.

B.2 | European rail network

The easiest way to address Y-junctions is to simply add metadata based on angles, some kind of node variable that tells a walker:"if you come from a you cannot go to b" (as in 2) so that the distance $d(a, b) = d(a, c) + d(b, c)$; "purer" approaches failed: I couldn't find a directed subgraph to replace the junction node with and achieve desired effect, nor is effective the idea to transform from G to $L(G)$ delete there, and then anti-transform (the anti-transorfm doesn't exist once you remove the desired nodes). For degree-4 nodes, I found it almost impossible to tell if the rails are simply crossing or there is an actual junction, angles don't always cut it.
Another thing that I saw was that stations, especially bigger ones, tend to have clumps of ns nodes around them, in reality big stations are far from being points, rather, they extend over a (possibly vast) area. A simple strategy could be just to delete all nodes close to a station and give it all their connections, although defining a universal radius is not straightforward.