King's College London

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

Degree Programmes	BSc, MSci
Module Code	5CCS2FC2
Module Title	Foundations of Computing II
Examination Period	May 2016
Time Allowed	Two hours
Rubric	ANSWER All QUESTIONS. ANSWER EACH QUESTION ON A NEW PAGE OF YOUR ANSWER BOOK AND WRITE ITS NUMBER IN THE SPACE PROVIDED. A FORMULA SHEET IS PROVIDED AT THE END OF THE EXAMINATION PAPER.
Calculators	Calculators may be used. The following models are permitted: Casio fx83 / Casio fx85.
Notes	Books, notes or other written material may not be brought into this examination

PLEASE DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

1. Turing Machines – Decidability and Recognizability.

a. Define the concepts of decidability and recognizability of languages by Turing Machines.

[5 marks]

b. Write the definition of the halting problem – the language $HALT_{TM}$ (whether a Turing Machine halts on a word). Prove that $HALT_{TM}$ is undecidable (directly, not by a reduction from another problem).

[10 marks]

c. The language E_{TM} is defined as

$$\{\langle M \rangle : M \text{ is an TM that does not accept any words}\}.$$

Prove that E_{TM} is unrecognizable. You can use the undecidability result from question 1.b.

[10 marks]

2. Complexity and Graph Algorithms.

a. Describe how time complexity is measured for computations performed by non-deterministic Turing machines. Define the complexity classes P and NP.

[5 marks]

b. If somebody finds an $O(n^2)$ algorithm for deciding the language

$$SAT = \{\varphi : \varphi \text{ is a satisfiable Boolean formula}\},$$

what will this discovery imply about the complexity classes P, NP, and co-NP? Justify your answers.

[5 marks]

QUESTION 2 CONTINUES ON NEXT PAGE

c. The following pseudo-code describes Prim's algorithm for minimum spanning trees **MST-PRIM**. The input to **MST-PRIM** is an undirected graph $G = \langle V, E \rangle$, a weight function w on E and a root vertex r.

```
1  for each u ∈ V
2      u.key = ∞
3      u.π = NIL
4  r.key = 0
5  Q = V
6  while Q ≠ ∅
7      u = EXTRACT-MIN(Q)
8      for each v ∈ G.Adj[u]
9          if v ∈ Q and w(u, v) < v.key
10             v.π = u
11             v.key = w(u, v)
```

Assume that Q is just a list of vertices without any particular order. Analyse the complexity of EXTRACT-MIN(Q), and of the entire algorithm **MST-PRIM**.

[10 marks]

d. Give the definition of a spanning tree (including a definition of a tree) for directed and undirected graphs. Prove that the algorithm in question 2.c computes a spanning tree for $G = \langle V, E \rangle$. You do not need to prove the minimality.

[5 marks]

3. a. A fruit smoothie company creates its smoothies using Banana, Strawberry and Mango. The company guarantees that 100g of smoothie (which is assumed to result from 100g of fruit) contains at least 15% of the Recommended Daily Allowance (RDA) of Vitamin A, at least 80% of their RDA of vitamin C, and at least 300mg of potassium.

- 100g of Banana contains 1% of the RDA of Vitamin A, 14% of the RDA of Vitamin C, 358mg of potassium and costs 15p.
- 100g of Strawberry contains no Vitamin A, 97% of the RDA of Vitamin C, 153mg of potassium and costs 27p.
- 100g of Mango contains 21% of the RDA of Vitamin A, 60% of the RDA of Vitamin C, 168mg of potassium and costs 30p.

In order to maximise profits the company would like to meet these constraints with minimum production cost. Express this problem as a Linear Programming Problem: you should specify the variables used, the constraints that apply and the objective function and describe the physical meaning of each variable, constraint and the objective function.

[7 marks]

b. Consider the following Linear Program (LP):

$$\begin{aligned} \text{Maximise:} \quad & -3x_1 + 2x_2 + x_3 \\ \text{Subject to:} \quad & 2x_1 + 3x_2 - x_3 \geq 10 \\ & x_1 - 2x_2 + x_3 \geq 7 \\ & -x_1 + x_2 + 2x_3 \leq 15 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

i. How can you tell that single-phase Simplex cannot solve this LP?

[1 marks]

ii. Draw the initial Simplex Tableau for this problem, make sure to show how you computed the new objective function for phase one of two-phase simplex.

[6 marks]

QUESTION 3 CONTINUES ON NEXT PAGE

c. Consider the following tableau generated during the execution of the (single phase) simplex algorithm:

$$\left(\begin{array}{ccccc|c} x & y & s & t & Z & Ans \\ 2 & 5 & 1 & 0 & 0 & 7 \\ 0.6 & 0 & -0.2 & 1 & 0 & 1.6 \\ -1.4 & 0 & 0.8 & 0 & 1 & 5.6 \end{array}\right)$$

i. State the value of each variable and of the objective function at the vertex represented by this tableau.

[3 marks]

ii. If performing an iteration of the simplex algorithm from this tableaux, what would be the value of the pivot?

[2 marks]

iii. When using Simplex to solve linear programming problems, how is the pivot column selected? What is the rationale behind this choice of pivot column?

[3 marks]

iv. What problem would occur if a negative value is chosen as the pivot in Simplex? Explain why.

[3 marks]

4. a. Use Gaussian Elimination to find the inverse of the following matrix:

[5 marks]

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & 3 \\ -2 & 3 & 1 \end{pmatrix}$$

b. Consider the Matrices A and B Below:

A: $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ B: $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

i. Give a single matrix that can be used to perform the transformation defined by A followed by that defined by B

[3 marks]

ii. Describe the effect of this matrix on a square with vertices at the points, (-1,1), (1,1), (1,-1), (-1,-1).

[3 marks]

c. The running time of an algorithm t_n on an input of size n is described by the following recurrence relation:

$t_1 = 1$

$t_n = 2t_{n-1} + 1$

i. Find the running time of the algorithm on an input of size 3. Show your working.

[2 marks]

ii. Solve the recurrence relation to find an equation in terms of n for the running time of the algorithm on a problem of size n ($n > 1$).

[6 marks]

QUESTION 4 CONTINUES ON NEXT PAGE

d. This part of the question is about statistics. A new drug has been developed that increases the infection rate with disease X by 25% but decreases the infection rate with disease Y by 5%. Clinical guidelines suggest that doctors prescribe this drug to patients as it will decrease the total infection rate. Could this be true given the information above? Explain your answer and illustrate it with some example numbers.

[5 marks]

Formulæ:

Sum of an Arithetic Series: $S_n = \frac{n(2a+(n-1)d)}{2}$

Sum of a Geometric Series: $S_n = \frac{a(1-r^n)}{(1-r)}$

Cosine/Sines of Common Angles:

Angle	Cosine	Sine
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	0	1