2.1. Regla de Simpson de 3/8

La fórmula de Simpson de 3/8 consiste en dividir el intervalo en subintervalos de $m = 4$ puntos cada uno. Demuestre que la integral de cualquiera de estos subintervalos está dada por

$$\int_{x_i}^{x_{i+3}} f(x)dx \approx \frac{3h}{8}(f(x_i) + 3f(x_{i+1} + 3f(x_{i+2}) + f(x_{i+3}))$$

$$= -\frac{f(x_i)}{6h^2}\int_{x_i}^{x_{i+3}} (x-x_i-h)(x-x_i-2h)(x-x_i-3h)\,dx + \frac{f(x_{i+1})}{2h^3}\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_i-2h)(x-x_i-3h)\,dx$$

$$- \frac{f(x_{i+2})}{2h^3}\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_i-h)(x-x_i-3h)\,dx + \frac{f(x_{i+3})}{6h^3}\int_{x_i}^{x_{i+3}} (x-x_i)(x-x_i-h)(x-x_i-2h)\,dx$$

$u = x - x_i$ $\quad$ $x = x_i$ $\quad$ $u = 0$

$du = dx$ $\quad$ $x = x_{i+3} = x_i + 3h$ $\quad \rightarrow u = 3h$

$$= -\frac{f(x_i)}{6h^3}\int_0^{3h} (u-h)(u-2h)(u-3h)\,du + \frac{f(x_{i+1})}{2h^3}\int_0^{3h} u(u-2h)(u-3h)\,du$$

$$- \frac{f(x_{i+2})}{2h^3}\int_0^{3h} u(u-h)(u-3h)\,du + \frac{f(x_{i+3})}{6h^3}\int_0^{3h} u(u-h)(u-2h)\,du$$

$$\int_0^{3h} (u-ah)(u-bh)(u-ch)\,du$$

$$= \int_0^{3h} u^3 - hu^2(a+b+c) + (ab+ac+bc)h^2u - abc\,h^2\,du$$

$$= \frac{1}{4}u^4 - \frac{1}{3}(a+b+c)hu^3 + \frac{1}{2}(ab+ac+bc)h^2u^2 - abch^3u \Big|_0^{3h}$$

$$= \frac{1}{4}(3h)^4 - \frac{1}{3}(a+b+c)h(3h)^3 + \frac{1}{2}(ab+ac+bc)h^2(3h)^2 - abch^3(3h)$$

$$= \frac{81}{4}h^4 - 9(a+b+c)h^4 + \frac{9}{2}(ab+ac+bc)h^4 - 3abch^4$$

$$= h^4\left(\frac{81}{4} - 9(a+b+c) + \frac{9}{2}(ab+ac+bc) - 3abc\right)$$

$$\Rightarrow = -\frac{f(x_i)}{6h^3}h^4\left(\frac{81}{4} - 9(6) + \frac{9}{2}(11) - 3\cdot 6\right) + \frac{f(x_{i+1})}{2h^3}\cdot h^4\left(\frac{81}{4} - 9(5) + \frac{9}{2}(6)\right)$$

$$- \frac{f(x_{i+2})}{2h^3}\left(\frac{81}{4} - 9(4) + \frac{9}{2}(3)\right) + \frac{f(x_{i+2})}{6h^3}h^4\left(\frac{81}{4} - 9(3) + \frac{9}{2}(2)\right)$$

$$= -\frac{h}{6}f(x_i)\left(-\frac{9}{4}\right) + \frac{h}{2}f(x_{i+1})\cdot\frac{9}{4} - \frac{h}{2}f(x_{i+2})\cdot\left(-\frac{9}{4}\right) + \frac{h}{6}f(x_{i+3})\left(\frac{9}{4}\right)$$

$$= \frac{9}{4}h\left(\frac{1}{6}f(x_i) + \frac{1}{2}f(x_{i+1}) + \frac{1}{2}f(x_{i+2}) + \frac{1}{6}f(x_{i+3})\right)$$

$$= \frac{3}{8}h\left(f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_{i+3})\right)$$

$$= \int_{x_i}^{x_{i+3}} f(x)\,dx \approx \frac{3}{8}h\left(f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_{i+3})\right)$$