

Completeness of Inst-saturated Sets of Clauses with Equality

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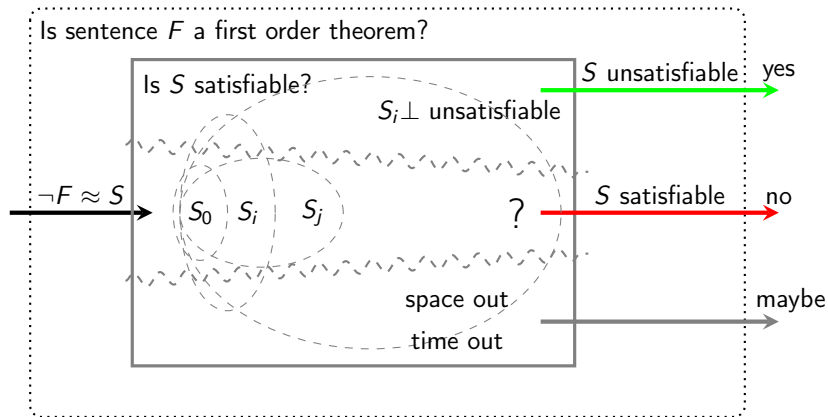
Harald Ganzinger and Konstantin Korovin.

Integrating Equational Reasoning into Instantiation-Based Theorem Proving.

In *18th CSL 2004. Proceedings*, volume 3210 of *LNCS*, pages 71–84, 2004.

Instantiation-based first order theorem proving

The big picture



$S_0 = S$, S_{i+1} is inferred from S_i by a sound calculus.

Preliminaries I

Equational First Order Logic

- ▶ first order signature with function (and predicate) symbols
- ▶ terms s, t, ℓ, r (and predicates P, Q, \bullet)
- ▶ atoms are equations of terms $s \approx t$ (or predicates $P \approx \bullet$)
- ▶ literals are atoms or negated atoms
- ▶ clauses are a multisets of literals
- ▶ closures are pairs of clauses and ground substitutions

$$(f(x) \approx b \vee x \not\approx a) \cdot \{x \mapsto f(a)\}$$

Preliminaries II

Equational First Order Logic

► orderings

\succ_{gr} order on ground terms, literals, and clauses defined by
 a total, well-founded, and monotone extension of
 a total simplification ordering \succ'_{gr} on ground terms

$$s \not\approx t \succ_{gr} s \approx t, \quad L \vee L \succ_{gr} L \quad (P \succ_{gr} \bullet)$$

\succ_{ℓ} an arbitrary total well-founded extension of \succ_{gr} such that

$$L\sigma \succ_{gr} L'\sigma' \Rightarrow L \cdot \sigma \succ_{\ell} L' \cdot \sigma'$$

\succ_{cl} an arbitrary total well-founded extension of \succ_{gr} such that

$$C\tau \succ_{gr} D\rho \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

$$(C\tau = D\rho \text{ and } C\theta = D) \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

Unit Paramodulation

$$\frac{(\ell \approx r) \cdot \sigma \quad L[\ell'] \cdot \sigma'}{L[r]\theta \cdot \rho} \theta \qquad \frac{(s \not\approx t) \cdot \tau}{\square} \mu$$

where

- ▶ $\ell\sigma \succ_{gr} r\sigma$, $\theta = \text{mgu}(\ell, \ell')$, $\ell\sigma = \ell'\sigma' = \ell'\theta\rho$, $\ell' \notin \mathcal{V}$
- ▶ $s\tau = t\tau$, $\mu = \text{mgu}(s, t)$

Example 1

The set of literal closures $\{ (f(x) \approx b) \cdot \{x \mapsto a\}, a \approx b, f(b) \not\approx b \}$ is inconsistent, but the empty clause is not derivable if $a \succ_{gr} b$.

Lemma 2

If σ, σ' are irreducible by a ground rewrite system R then ρ is irreducible by R .

UP-Redundancy

- ▶ We define the set

$$\text{irred}_R(\mathcal{L}) = \{ L \cdot \sigma \in \mathcal{L} \mid \sigma \text{ is irreducible w.r.t. } R \}$$

for a set of literal closures \mathcal{L} and a ground rewrite system R .

- ▶ Let $\mathcal{L}_{L \cdot \sigma \succ_\ell} = \{ L' \cdot \sigma' \in \mathcal{L} \mid L \cdot \sigma \succ_\ell L' \cdot \sigma' \}$.
- ▶ A literal closure $L \cdot \sigma$ is UP-redundant in \mathcal{L} if

$$R \cup \text{irred}_R(\mathcal{L}_{L \cdot \sigma \succ_\ell}) \models L\sigma$$

for every ground rewrite system R
oriented by \succ_{gr} where σ is irreducible w.r.t. R .

- ▶ $\mathcal{R}_{UP}(\mathcal{L})$ denotes the set of all UP-redundant closures in \mathcal{L} .

UP-Saturation

The UP-saturation process for \mathcal{L} is a sequence $\{\mathcal{L}_i\}_{i=0}^{\infty}$ where

$$\begin{aligned}
 & \blacktriangleright \mathcal{L}_0 = \mathcal{L} \\
 & \blacktriangleright \mathcal{L}_{i+1} = \begin{cases} \mathcal{L}_i \setminus L \cdot \sigma & \text{if } R \cup \text{irred}_R(\mathcal{L}_i, L \cdot \sigma \succ_{\ell}) \models L\sigma \\ \mathcal{L}_i \cup \square & \text{if } \begin{cases} (s \not\approx t) \cdot \tau \in \mathcal{L}_i \\ s\tau = t\tau, \mu = \text{mgu}(s, t) \end{cases} \\ \mathcal{L}_i \cup L[r]\theta \cdot \rho & \text{if } \begin{cases} (\ell \approx r) \cdot \sigma, L[\ell'] \cdot \sigma' \in \mathcal{L}_i \\ \ell\sigma \succ_{gr} r\sigma, \theta = \text{mgu}(\ell, \ell'), \\ \ell' \notin \mathcal{V}, \ell\sigma = \ell'\sigma' = \ell'\theta\rho, \end{cases} \\ \mathcal{L}_i & \text{otherwise} \end{cases}
 \end{aligned}$$

Let \mathcal{L}^{∞} be the set of persistent closures, i.e. the lower limit of \mathcal{L}_i .

UP-Fairness

The UP-saturation process is UP-fair if for every UP-inference with premises in \mathcal{L}^∞ the conclusion is UP-redundant w.r.t. \mathcal{L}_j for some j . For a set of literals \mathcal{L} we define the saturated set of literal closures $\mathcal{L}^{sat} = \mathcal{L}^\infty \setminus \mathcal{R}_{UP}(\mathcal{L}^\infty)$ for some UP-saturation process $\{\mathcal{L}_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}$.

Lemma 3

The set \mathcal{L}^{sat} is unique because for any two UP-fair saturation processes $\{\mathcal{L}_i\}_{i=0}^\infty$ and $\{\mathcal{L}'_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}'_0$ we have

$$\mathcal{L}^\infty \setminus \mathcal{R}_{UP}(\mathcal{L}^\infty) = \mathcal{L}'^\infty \setminus \mathcal{R}_{UP}(\mathcal{L}'^\infty)$$

Inst-Redundancy

Let S be a set of clauses.

- ▶ A ground closure C is Inst-redundant in S if for some k
 - ▶ $C'_i \in S$, $C_i = C'_i \cdot \sigma'_i$, $C \succ_{cl} C_i$ for $i \in 1 \dots k$
 - ▶ such that $C_1, \dots, C_k \models C$
- ▶ A (possible non-ground) clause C is called Inst-redundant in S if each ground closure $C \cdot \sigma$ is Inst-redundant in S .
- ▶ $R_{Inst}(S)$ denotes the set of all Inst-redundant clauses in S .

Example 4

$$S = \{ f(x) \approx x, f(a) \approx a, f(f(x)) \approx f(x) \}$$

$$R_{Inst}(S) = \{ f(f(x)) \approx f(x) \}$$

Selection

Let S be a set of clauses S , let I_{\perp} be a model of S_{\perp} .

- ▶ A selection function sel maps clauses to literals such that

$$\text{sel}(C) \in C \qquad I_{\perp} \models \text{sel}(C)_{\perp}$$

- ▶ The set of S -relevant literal closures

$$\mathcal{L}_S = \left\{ L \cdot \sigma \mid \begin{array}{l} L \vee C \in S, L = \text{sel}(L \vee C) \\ (L \vee C) \cdot \sigma \text{ is not Inst-redundant in } S, \end{array} \right\}$$

- ▶ $\mathcal{L}_S^{\text{sat}}$ denotes the saturation process of \mathcal{L}_S .
- ▶ A set of clauses S is Inst-saturated w.r.t. a selection function, if $\mathcal{L}_S^{\text{sat}}$ does not contain the empty clause.

Completeness

Theorem 5

If a set of clauses S is Inst-saturated, and $S \perp$ is satisfiable, then S is also satisfiable.

Proof.

1. Construction of a candidate model
2. Proof that candidate is a model by contradiction



Model Construction I

Let S be an Inst-saturated set of clauses, i.e.

- ▶ $S \perp$ is satisfiable
- ▶ $\Box \notin \mathcal{L}_S^{sat}$

Let $L = L' \cdot \sigma \in \mathcal{L}_S^{sat}$. We define by induction on \succ_ℓ

- ▶ $I_L = \bigcup_{L \succ_\ell M} \epsilon_M$ ϵ_M already defined for all M with $L \succ_\ell M$
- ▶ $R_L = \{s \rightarrow t \mid s \approx t \in I_L, s \succ_{gr} t\}$
- ▶ $\epsilon_L = \begin{cases} \emptyset & \text{if } L'\sigma \text{ reducible by } R_L \\ \emptyset & \text{if } I_L \models L'\sigma \text{ or } I_L \models \overline{L'}\sigma \text{ (defined)} \\ \{L'\sigma\} & \text{if } L'\sigma \text{ is productive (irreducible, undefined)} \end{cases}$

Model Construction II

- ▶ $R_S = \bigcup_{L \in \mathcal{L}_S^{\text{sat}}} R_L$ R_S is convergent and interreduced
- ▶ $I_S = \bigcup_{L \in \mathcal{L}_S^{\text{sat}}} \epsilon_L$ I_S is consistent, $L\sigma \in I_S$ is irreducible by R_S

Conjecture 1

An arbitrary total consistent extension \mathcal{I} of I_S is a model for all ground instances of clauses in S .

For the following slides let $\mathcal{L}_S, \mathcal{L}_S^{\text{sat}}, I_S, R_S$ be defined as above and \mathcal{I} be an arbitrary consistent extension of I_S .

Lemma 6

If any $L \cdot \sigma \in \mathcal{L}_S$ exists with $\mathcal{I} \not\models L\sigma$ then
there is a $L' \cdot \sigma' \in \text{irred}_{R_S}(\mathcal{L}_S^{\text{sat}})$ with $\mathcal{I} \not\models L'\sigma'$.

Proof.

We have two cases

- ▶ If $L \cdot \sigma$ is not UP-redundant in $\mathcal{L}_S^{\text{sat}}$, then $L' \cdot \sigma' = L \cdot \sigma$.
- ▶ If $L \cdot \sigma$ is UP-redundant in $\mathcal{L}_S^{\text{sat}}$. By construction σ is irreducible by R_S . Then we have

$$R_S \cup \text{irred}_{R_S}(\{L' \cdot \sigma' \in \mathcal{L}_S^{\text{sat}} \mid L \cdot \sigma \succ_\ell L' \cdot \sigma'\}) \models L\sigma$$

Hence there is $L' \cdot \sigma' \in \text{irred}_{R_S}$ with $\mathcal{I} \not\models L'\sigma'$.



Lemma 7

Whenever

$$M \cdot \tau = \min_{\succ_{\ell}} \{ L' \cdot \tau' \mid L' \cdot \sigma' \in \text{irred}_{R_S}(\mathcal{L}_S^{\text{sat}}), L' \sigma' \text{ false in } \mathcal{I} \}$$

is defined, then $M \cdot \tau$ is irreducible by R_S .

Proof

Assume $M \cdot \tau$ is reducible by $(\ell \rightarrow r) \in R_S$
and $(\ell \rightarrow r)$ is produced by $(\ell' \approx r') \cdot \rho \in \mathcal{L}_S^{\text{sat}}$.

Now UP-inference is applicable because τ is irreducible by R_S ,

$$\frac{(\ell' \approx r') \cdot \rho \quad M[\ell''] \cdot \tau}{M[r']\theta \cdot \mu} \text{ UP}$$

μ is irreducible by R_S , and $M[r']\theta\mu$ is false in \mathcal{I} .

- If $M[r']\theta \cdot \mu$ is not UP-redundant in \mathcal{L}_S^{sat} then $M[r']\theta \cdot \mu \in \mathcal{L}_S^{sat}$.

Now $M \cdot \tau \succ_\ell M[r']\theta \cdot \mu \in \text{irred}_{R_S}(\mathcal{L}_S^{sat})$
 contradicts minimality of $M \cdot \tau$.

- If $M[r']\theta \cdot \mu$ is UP-redundant in \mathcal{L}_S^{sat} then

$$R_S \cup \text{irred}_{R_S}(\{M' \cdot \tau' \in \mathcal{L}_S^{sat} \mid M[r']\theta \cdot \mu \succ_\ell M' \cdot \tau'\}) \models M[r']\theta \mu$$

Hence there is $M' \cdot \tau' \in \mathcal{L}_S^{sat}$ false in \mathcal{I} such that
 $M \cdot \tau \succ_\ell M[r']\theta \cdot \mu \succ_\ell M' \cdot \tau'$,
 $M' \cdot \tau'$ contradicts minimality of $M \cdot \tau$.

Hence $M \cdot \tau$ is irreducible by R_S . □

Lemma 8

Let $M \cdot \tau \in \mathcal{L}_S^{sat}$, irreducible by R_S , and not productive.
 With $\mathcal{I} \not\models M\tau$ follows that M is not an equation ($s \approx t$).

Proof.

Assume $M = (s \approx t)$. Then we have

- ▶ $I_{M \cdot \tau} \models (s \not\approx t)\tau$
- ▶ All literals in $I_{M \cdot \tau}$ are irreducible by $R_{M \cdot \tau}$
- ▶ $s\tau$ and $t\tau$ are irreducible by $R_{M \cdot \tau}$
- ▶ $R_{M \cdot \tau}$ is a convergent term rewrite system

Hence there is $(s \not\approx t)\tau \in I_{M \cdot \tau}$ produced to $I_{M \cdot \tau}$ by a $(s' \not\approx t') \cdot \tau'$.
 Then $(s' \not\approx t')\tau' \succ_{gr} (s \approx t)\tau$ and $(s' \not\approx t') \cdot \tau' \succ_\ell M \cdot \tau$ □

Lemma 9

Let $M \cdot \tau \in \mathcal{L}_S^{sat}$, irreducible by R_S , and not productive.
With $\mathcal{I} \not\models M\tau$ follows that M is not an inequation ($s \not\approx t$).

Proof.

Assume $M \cdot \tau$ is inequation $(s \not\approx t) \cdot \tau$. We have

- ▶ $I_{M \cdot \tau} \models (s \approx t)\tau$
- ▶ $s\tau$ and $t\tau$ are irreducible by $R_{M \cdot \tau}$

Hence $s\tau = t\tau$ and equality resolution is applicable.

Contradiction to $\square \notin \mathcal{L}_S^{sat}$. □

Proof I

Assume \mathcal{I} is not a model of S .

$$\text{Let } D = \min_{\succ_{cl}} \{ C' \cdot \sigma \mid C' \in S, C'\sigma \text{ false in } \mathcal{I} \}$$

Then

- ▶ $D = D' \cdot \sigma$ is not Inst-redundant.

Otherwise there are $D_1, \dots, D_n \models D$, $D \succ_{cl} D_i$ for all i , and D_j false in \mathcal{I} for one j , which contradicts minimality.

- ▶ $x\sigma$ irreducible by R_S for every variable x in D' .

Otherwise let $(\ell \rightarrow r)\tau \in R_L$ and $x\sigma = x\sigma[l\tau]_p$ for some variable x in D' . We define substitution σ' with $x\sigma' = x\sigma[r\tau]_p$ and $y\sigma' = y\sigma$ for $y \neq x$. $D'\sigma'$ is false in \mathcal{I} and $D \succ_{cl} D' \cdot \sigma'$, which contradicts minimality.

Proof II

Since D is not Inst-redundant in S , we have for some literal L , that $D' = L \vee D''$, $\text{sel}(D') = L$, $L \cdot \sigma \in \mathcal{L}_S$, $L\sigma$ is false in \mathcal{I}

Hence the following literal closure

$$M \cdot \tau = \min_{\succ_\ell} \{ L' \cdot \tau' \mid L' \cdot \sigma' \in \text{irred}_{R_S}(\mathcal{L}_S^{\text{sat}}), L' \cdot \sigma' \text{ false in } \mathcal{I} \}$$

exists by Lemma 6, is irreducible by Lemma 7, and not productive.

- ▶ M is not an equation by lemma 8
- ▶ M is not an inequation by lemma 9

This is a contradiction.

Our assumption is false and \mathcal{I} is a model for all instances of S . \square

► UP for literal closures

$$\frac{(\ell \approx r) \cdot \sigma \quad L[\ell'] \cdot \sigma'}{L[r]\theta \cdot \rho} \theta \qquad \frac{(s \not\approx t) \cdot \tau}{\square} \mu$$

- $\ell\sigma \succ_{gr} r\sigma$, $\theta = \text{mgu}(\ell, \ell')$, $\ell\sigma = \ell'\sigma' = \ell'\theta\rho$, $\ell' \notin \mathcal{V}$
- $s\tau = t\tau$, $\mu = \text{mgu}(s, t)$

► UPL for literals (clauses)

$$\frac{\ell \approx r \quad L[\ell']}{L[r]\theta} \theta \qquad \frac{s \not\approx t}{\square} \mu$$

- $\ell\sigma \succ_{gr} r\sigma$ for some grounding substitution σ ,
 $\theta = \text{mgu}(\ell, \ell')$, $\ell' \notin \mathcal{V}$
- $\mu = \text{mgu}(s, t)$