Completeness of

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Harald Ganzinger and Konstantin Korovin.
Integrating Equational Reasoning into Instantiation-Based Theorem Proving.
In *18th CSL 2004. Proceedings*, volume 3210 of *LNCS*, pages 71–84, 2004.

Clauses and Closures

Orderings

$\succ_{gr}$ order on ground terms, literals, and clauses defined by a total, well-founded, and monotone extension of a total simplification ordering $\succ'_{gr}$ on ground terms

$\succ_{\ell}$ an arbitrary total well-founded extension of $\succ_{gr}$ such that

$$L\sigma \succ_{gr} L'\sigma' \Rightarrow L \cdot \sigma \succ_{\ell} L' \cdot \sigma'$$

$\succ_{cl}$ an arbitrary total well-founded extension of $\succ_{gr}$ such that

$$C\tau \succ_{gr} D\rho) \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

$$(C\tau = D\rho \text{ and } C\theta = D) \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

Unit Paramodulation

Definition

$$\frac{(\ell \approx r) \cdot \sigma \quad L[\ell'] \cdot \sigma'}{L[r]\theta \cdot \rho}\ \theta \qquad\qquad \frac{(s \not\approx t) \cdot \tau}{\square}\ \mu$$

where

- $\ell\sigma \succ_{gr} r\sigma$, $\theta = \mathrm{mgu}(\ell, s)$, $\ell\sigma = \ell'\sigma' = \ell'\theta\rho$, $\ell' \notin \mathcal{V}$
- $s\tau = t\tau$, $\mu = \mathrm{mgu}(s, t)$

Definition

Let $L \cdot \sigma$ be a literal closure, $\mathcal{L}$ be a set of literal closures and R a ground rewrite system.

$$\begin{aligned} \mathit{irred}_R(\mathcal{L}) &= \{L' \cdot \sigma' \mid L' \cdot \sigma' \in \mathcal{L}, \sigma' \text{ is irreducible w.r.t. } R\} \\ \mathcal{L}_{L \cdot \sigma \succ_\ell} &= \{L' \cdot \sigma' \mid L' \cdot \sigma' \in \mathcal{L}, L \cdot \sigma \succ_\ell L' \cdot \sigma'\} \end{aligned}$$

Definition

A literal closure $L \cdot \sigma$ is UP-redundant in a set of literal closures $\mathcal{L}$ if

$$R \cup \mathit{irred}_R(\mathcal{L}_{L \cdot \sigma \succ_\ell}) \models L\sigma$$

for any ground rewrite System R
oriented by $\succ_{gr}$ where σ is irreducible w.r.t. R.
$\mathcal{R}_{UP}$ denotes the set of all UP-redundant closures in $\mathcal{L}$.

Saturation I

Definition

A UP- saturation process is a sequence $\{\mathcal{L}_i\}_{i=0}^{\infty}$ of sets of literal closures where $\mathcal{L}_{i+1}$ can be obtained from $\mathcal{L}_i$ by adding a conclusion of an UP-inference with premises in $\mathcal{L}_i$ or by removing a UP-redundant w.r.t. $\mathcal{L}_i$ closure:

$$\mathcal{L}_{i+1} = \begin{cases} \mathcal{L}_i \cup \square & \text{if} \quad \mathcal{L}_i \ni (s \not\approx t) \cdot \tau,\ s\tau = t\tau,\ \mu = \mathrm{mgu}(s,t) \\ \mathcal{L}_i \backslash L \cdot \sigma & \text{if} \quad R \cup \mathrm{irred}_R(\mathcal{L}_{L\cdot\sigma \succ_\ell}) \vDash L\sigma \\ \mathcal{L}_i \cup L[r]\theta \cdot \rho & \text{if} \quad \begin{cases} (\ell \approx r) \cdot \sigma \in \mathcal{L}_i,\ L[\ell'] \cdot \sigma' \in \mathcal{L}_i \\ \ell\sigma \succ_{gr} r\sigma,\ \theta = \mathrm{mgu}(\ell, \ell'), \\ \ell' \notin \mathcal{V},\ \ell\sigma = \ell'\sigma' = \ell'\theta\rho, \end{cases} \\ \mathcal{L}_i & \qquad \text{otherwise} \end{cases}$$

Saturation II

Definition

Let $\mathcal{L}^\infty$ be the set of persistent closures, i.e. the lower limit of the sequence. The process is fair if for every UP-inference with premesis in $\mathcal{L}^\infty$ the conclusion is UP-redundant w.r.t. $\mathcal{L}_j$ for some j. For a set of literals $\mathcal{L}$ we define the saturated set of literal closures $\mathcal{L}^{sat} = \mathcal{L}^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}^\infty)$ for some UP-saturation process $\{\mathcal{L}_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}$.

Lemma

The set $\mathcal{L}^{sat}$ is unique because for any two UP-fair saturation processes $\{\mathcal{L}_i\}_{i=0}^\infty$ and $\{\mathcal{L}'_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}'_0$ we have

$$\mathcal{L}^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}^\infty) = \mathcal{L}'^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}'^\infty)$$

Assume $\mathcal{L}\bot$ is satisfiable and $\Box \notin \mathcal{L}^{sat}$.
We define by induction on $\succ_\ell$. Assume $L = L' \cdot \sigma \in \mathcal{L}^{sat}$

$$I_L = \bigcup_{L \succ_\ell M} \epsilon_M$$

$$R_L = \{s \to t \mid s \approx t \in I_L, s \succ_{gr} t\}$$

$$\epsilon_L = \begin{cases} \emptyset & \text{if } L'\sigma \text{ reducible by } R_L \\ \emptyset & \text{if } I_L \vDash L'\sigma \text{ or } I_L \vDash \overline{L'}\sigma \\ \{L'\sigma\} & L'\sigma \text{ irreducible by } R_L \text{ and undefined in } I_L \end{cases}$$

$$I_S = \bigcup_{L \in \mathcal{L}^{sat}} \epsilon_L \qquad R_S = \bigcup_{L \in \mathcal{L}^{sat}} R_L$$

Lemma

Let $M \cdot \tau = \min_{\succ_\ell} \{M' \cdot \tau' \mid M' \cdot \tau' \in \mathit{irred}_{R_S}(\mathcal{L}_S^{sat}), \mathcal{I} \not\models M'\tau'\}$
Then, $M \cdot \tau$ *is irreducible by* R_S*.*

Proof.

Assume $M \cdot \tau$ is reducible by $(\ell \to r) \in R_S$ and $(\ell \to r)$ is produced by $(\ell' \approx r') \cdot \rho \in \mathcal{L}_S^{sat}$. Bei construction τ is irreducible by R_S. Hence UP-inference is applicable:

$$\frac{(\ell' \approx r) \cdot \rho \quad M[\ell''] \cdot \tau}{M[r']\theta \cdot \mu}\ UP$$

$$\ell'\rho = \ell''\tau = \ell''\theta\mu,\ \theta = \mathrm{mgu}(\ell', \ell''),\ \mathcal{I} \not\models M[r']\theta\mu$$

- Assume $M[r']\theta \cdot \mu$ is UP-redundant in $\mathcal{L}_S^{sat}$.
  α is irreducible (lemma ..) by R_S. From definiton:
- Assume $M[r']\theta \cdot \mu$ is not UP-redundant in $\mathcal{L}_S^{sat}$.

$\square$