Completeness of

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Harald Ganzinger and Konstantin Korovin.
Integrating Equational Reasoning into Instantiation-Based Theorem Proving.
In *18th CSL 2004. Proceedings*, volume 3210 of *LNCS*, pages 71–84, 2004.

- a clause C is a multiset of literals
- literals are (in)equations of first order terms
- a closure $C \cdot \sigma$ is a pair of clause C and substitution σ
- orderings

$\succ_{gr}$ order on ground terms, literals, and clauses defined by a total, well-founded, and monotone extension of a total simplification ordering $\succ'_{gr}$ on ground terms

$\succ_{\ell}$ an arbitrary total well-founded extension of $\succ_{gr}$ such that

$$L\sigma \succ_{gr} L'\sigma' \Rightarrow L \cdot \sigma \succ_{\ell} L' \cdot \sigma'$$

$\succ_{cl}$ an arbitrary total well-founded extension of $\succ_{gr}$ such that

$$C\tau \succ_{gr} D\rho) \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

$$(C\tau = D\rho \text{ and } C\theta = D) \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

Unit Paramodulation

Definition

$$\frac{(\ell \approx r) \cdot \sigma \quad L[\ell'] \cdot \sigma'}{L[r]\theta \cdot \rho}\ \theta \qquad\qquad \frac{(s \not\approx t) \cdot \tau}{\Box}\ \mu$$

where

- $\ell\sigma \succ_{gr} r\sigma$, $\theta = \mathrm{mgu}(\ell, s)$, $\ell\sigma = \ell'\sigma' = \ell'\theta\rho$, $\ell' \notin \mathcal{V}$
- $s\tau = t\tau$, $\mu = \mathrm{mgu}(s, t)$

Remark

The set of literal closures $\{\,(\mathsf{f}(x) \approx \mathsf{b}) \cdot \{x \to \mathsf{a}\}, \mathsf{a} \approx \mathsf{b}, \mathsf{f}(\mathsf{b}) \not\approx \mathsf{b}\,\}$ is inconsistent, but with $\mathsf{a} \succ_{gr} \mathsf{b}$ the empty clause is not derivable.

Definition

For a set of literal closures $\mathcal{L}$ and arbitrary ground term rewrite system R we define

$$\mathit{irred}_R(\mathcal{L}) = \{L' \cdot \sigma' \mid L' \cdot \sigma' \in \mathcal{L}, \sigma' \text{ is irreducible w.r.t. } R\}$$

A literal closure $L \cdot \sigma$ is UP-redundant $\mathcal{L}$ if

$$R \cup \mathit{irred}_R(\mathcal{L}_{L\cdot\sigma\succ_\ell}) \models L\sigma$$

for any ground term rewrite system R and

$$\mathcal{L}_{L\cdot\sigma\succ_\ell} = \{L' \cdot \sigma' \mid L' \cdot \sigma' \in \mathcal{L}, L \cdot \sigma \succ_\ell L' \cdot \sigma'\}$$

oriented by $\succ_{gr}$ where σ is irreducible w.r.t. R.

$\mathcal{R}_{UP}$ denotes the set of all UP-redundant closures in $\mathcal{L}$.

Saturation I

Definition

A UP- saturation process is a sequence $\{\mathcal{L}_i\}_{i=0}^{\infty}$ of sets of literal closures where $\mathcal{L}_{i+1}$ can be obtained from $\mathcal{L}_i$ by adding a conclusion of an UP-inference with premises in $\mathcal{L}_i$ or by removing a UP-redundant w.r.t. $\mathcal{L}_i$ closure:

$$
\mathcal{L}_{i+1} = \begin{cases}
\mathcal{L}_i \cup \square & \text{if} \quad \mathcal{L}_i \ni (s \not\approx t) \cdot \tau,\ s\tau = t\tau,\ \mu = \mathrm{mgu}(s,t) \\
\mathcal{L}_i \backslash L \cdot \sigma & \text{if} \quad R \cup \mathrm{irred}_R(\mathcal{L}_{L \cdot \sigma \succ_\ell}) \vDash L\sigma \\
\mathcal{L}_i \cup L[r]\theta \cdot \rho & \text{if} \quad \begin{cases} (\ell \approx r) \cdot \sigma \in \mathcal{L}_i,\ L[\ell'] \cdot \sigma' \in \mathcal{L}_i \\ \ell\sigma \succ_{gr} r\sigma,\ \theta = \mathrm{mgu}(\ell, \ell'), \\ \ell' \notin \mathcal{V},\ \ell\sigma = \ell'\sigma' = \ell'\theta\rho, \end{cases} \\
\mathcal{L}_i & \quad\ \ \text{otherwise}
\end{cases}
$$

Saturation II

Definition

Let $\mathcal{L}^\infty$ be the set of persistent closures, i.e. the lower limit of the sequence. The process is fair if for every UP-inference with premesis in $\mathcal{L}^\infty$ the conclusion is UP-redundant w.r.t. $\mathcal{L}_j$ for some j.
For a set of literals $\mathcal{L}$ we define the saturated set of literal closures $\mathcal{L}^{sat} = \mathcal{L}^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}^\infty)$ for some UP-saturation process $\{\mathcal{L}_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}$.

Lemma

The set $\mathcal{L}^{sat}$ is unique because for any two UP-fair saturation processes $\{\mathcal{L}_i\}_{i=0}^\infty$ and $\{\mathcal{L}'_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}'_0$ we have

$$\mathcal{L}^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}^\infty) = \mathcal{L}'^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}'^\infty)$$

Let $\mathcal{L}$ be a set of literal closures.
Assume $\mathcal{L}\bot$ is satisfiable and $\square \notin \mathcal{L}^{sat}$.
We define by induction on $\succ_\ell$. Assume $L = L' \cdot \sigma \in \mathcal{L}^{sat}$

$$I_L = \bigcup_{L \succ_\ell M} \epsilon_M$$

$$R_L = \{s \to t \mid s \approx t \in I_L, s \succ_{gr} t\}$$

$$\epsilon_L = \begin{cases} \emptyset & \text{if } L'\sigma \text{ reducible by } R_L \\ \emptyset & \text{if } I_L \vDash L'\sigma \text{ or } I_L \vDash \overline{L'}\sigma \\ \{L'\sigma\} & L'\sigma \text{ irreducible by } R_L \text{ and undefined in } I_L \end{cases}$$

$$I_S = \bigcup_{L \in \mathcal{L}^{sat}} \epsilon_L \qquad R_S = \bigcup_{L \in \mathcal{L}^{sat}} R_L$$

Final Step I

$$M \cdot \tau = \min_{\succ_\ell} \{ M' \cdot \tau' \mid L' \cdot \sigma' \in irred_{R_S}(\mathcal{L}_S^{sat}), \mathcal{I} \not\models M'\tau' \}$$

We have that $M \cdot \tau$

- is false in $\mathcal{I}$
- is in $\mathcal{L}_S^{sat}$
- is irreducible by R_S
- is not productive.

Hence $I_{M\cdot\tau} \models \overline{M}\tau$ with two possible cases:

1. $M \cdot \tau$ is equation $(s \approx t) \cdot \tau$ $\qquad I_{M\cdot\tau} \models (s \not\approx t)\tau$
2. $M \cdot \tau$ is inequation $(s \not\approx t) \cdot \tau$ $\qquad I_{M\cdot\tau} \models (s \approx t)\tau$

Final Step II

$M \cdot \tau = \min_{\succ_\ell} \{\, M' \cdot \tau' \mid L' \cdot \sigma' \in irred_{R_S}(\mathcal{L}_S^{sat}),\, \mathcal{I} \not\models M'\tau' \,\}$
$M \cdot \tau$ is false in $\mathcal{I}$, in $\mathcal{L}_S^{sat}$, irreducible in R_S, not productive.

1. Assume $M \cdot \tau$ is equation $(s \approx t) \cdot \tau$:
   - $I_{M\cdot\tau} \models (s \not\approx t)\tau$
   - All literals in $I_{M\cdot\tau}$ are irreducible by $R_{M\cdot\tau}$
   - $s\tau$ and $t\tau$ are irreducible by $R_{M\cdot\tau}$
   - $R_{M\cdot\tau}$ is a convergent term rewrite system

   Hence $(s \not\approx t)\tau \in I_{M\cdot\tau}$ and produced to $I_{M\cdot\tau}$ by a $(s' \not\approx t') \cdot \tau'$.
   Contradicts the minimality of $M \cdot \tau$.

2. Assume $M \cdot \tau$ is inequation $(s \not\approx t) \cdot \tau$:
   - $I_{M\cdot\tau} \models (s \approx t)\tau$
   - $s\tau$ and $t\tau$ are irreducible by $R_{M\cdot\tau}$

   Hence $s\tau = t\tau$ and equality resolution is applicable.
   Contradicts that the empty clause is not in $\mathcal{L}_S^{sat}$.