Completeness of Inst-saturated Sets of Clauses with Equality

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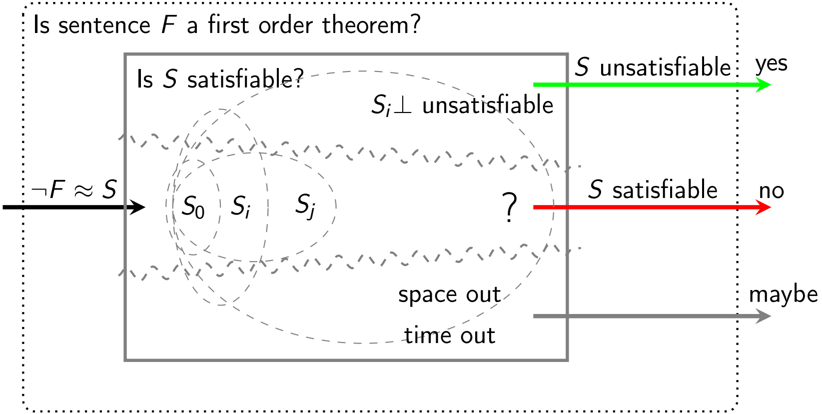
Harald Ganzinger and Konstantin Korovin.
Integrating Equational Reasoning into Instantiation-Based Theorem Proving.
In *18th CSL 2004. Proceedings*, volume 3210 of *LNCS*, pages 71–84, 2004.

Instantiation-based first order theorem proving

The big picture



$S_0 = S$, S_{i+1} is inferred from S_i by a sound calculus.

Preliminaries I

Equational First Order Logic

- first order signature with function (and predicate) symbols
- terms s, t, ℓ, r (and predicates $P, Q, \bullet$)
- atoms are equations of terms $s \approx t$ (or predicates $P \approx \bullet$)
- literals are atoms or negated atoms
- clauses are a multisets of literals
- closures $C \cdot \sigma$ are pairs of clauses and substitutions

Preliminaries II

Equational First Order Logic

- orderings

$\succ_{gr}$ order on ground terms, literals, and clauses defined by a total, well-founded, and monotone extension of a total simplification ordering $\succ'_{gr}$ on ground terms

$$s \not\approx t \succ_{gr} s \approx t, \; L \vee L \succ_{gr} L \qquad (P \succ_{gr} \bullet)$$

$\succ_{\ell}$ an arbitrary total well-founded extension of $\succ_{gr}$ such that

$$L\sigma \succ_{gr} L'\sigma' \Rightarrow L \cdot \sigma \succ_{\ell} L' \cdot \sigma'$$

$\succ_{cl}$ an arbitrary total well-founded extension of $\succ_{gr}$ such that

$$C\tau \succ_{gr} D\rho \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

$$(C\tau = D\rho \text{ and } C\theta = D) \Rightarrow C \cdot \tau \succ_{cl} D \cdot \rho$$

Unit Paramodulation

$$\frac{(\ell \approx r) \cdot \sigma \quad L[\ell'] \cdot \sigma'}{L[r]\theta \cdot \rho}\ \theta \qquad\qquad \frac{(s \not\approx t) \cdot \tau}{\square}\ \mu$$

where

- $\ell\sigma \succ_{gr} r\sigma$, $\theta = \mathrm{mgu}(\ell, s)$, $\ell\sigma = \ell'\sigma' = \ell'\theta\rho$, $\ell' \notin \mathcal{V}$
- $s\tau = t\tau$, $\mu = \mathrm{mgu}(s, t)$

Example

The set of literal closures $\{\, (\mathsf{f}(x) \approx \mathsf{b}) \cdot \{x \to \mathsf{a}\}, \mathsf{a} \approx \mathsf{b}, \mathsf{f}(\mathsf{b}) \not\approx \mathsf{b} \,\}$ is inconsistent, but the empty clause is not derivable if $\mathsf{a} \succ_{gr} \mathsf{b}$.

Lemma

If σ, σ' are irreducible by an TRS R then ρ is irreducible by R.

UP-Redundancy

- We define the set

$$\mathit{irred}_R(\mathcal{L}) = \{\, L \cdot \sigma \in \mathcal{L} \mid \sigma \text{ is irreducible w.r.t. } R \,\}$$

for a set of literal closures $\mathcal{L}$ and a ground rewrite system R.

- Let $\mathcal{L}_{L \cdot \sigma \succ_\ell} = \{\, L' \cdot \sigma' \in \mathcal{L} \mid L \cdot \sigma \succ_\ell L' \cdot \sigma' \,\}$.

- A literal closure $L \cdot \sigma$ is UP-redundant in $\mathcal{L}$ if

$$R \cup \mathit{irred}_R(\mathcal{L}_{L \cdot \sigma \succ_\ell}) \vDash L\sigma$$

for every ground rewrite system R
oriented by $\succ_{gr}$ where σ is irreducible w.r.t. R.

- $\mathcal{R}_{UP}(\mathcal{L})$ denotes the set of all UP-redundant closures in $\mathcal{L}$.

UP-Saturation

The UP-saturation process for $\mathcal{L}$ is a sequence $\{\mathcal{L}_i\}_{i=0}^{\infty}$ where

- $\mathcal{L}_0 = \mathcal{L}$
- $$\mathcal{L}_{i+1} = \begin{cases} \mathcal{L}_i \backslash L \cdot \sigma & \text{if} \quad R \cup \mathrm{irred}_R(\mathcal{L}_{i,L\cdot\sigma\succ_\ell}) \vDash L\sigma \\[2ex] \mathcal{L}_i \cup \square & \text{if} \quad \begin{cases} (s \not\approx t)\cdot\tau \in \mathcal{L}_i \\ s\tau = t\tau,\ \mu = \mathrm{mgu}(s,t) \end{cases} \\[4ex] \mathcal{L}_i \cup L[r]\theta\cdot\rho & \text{if} \quad \begin{cases} (\ell \approx r)\cdot\sigma,\ L[\ell']\cdot\sigma' \in \mathcal{L}_i \\ \ell\sigma \succ_{gr} r\sigma,\ \theta = \mathrm{mgu}(\ell,\ell'), \\ \ell' \notin \mathcal{V},\ \ell\sigma = \ell'\sigma' = \ell'\theta\rho, \end{cases} \\[4ex] \mathcal{L}_i & \quad \text{otherwise} \end{cases}$$

Let $\mathcal{L}^\infty$ be the set of persistent closures.

UP-Fairness

The UP-saturation process is UP-fair if for every UP-inference with premises in $\mathcal{L}^\infty$ the conclusion is UP-redundant w.r.t. $\mathcal{L}_j$ for some j.
For a set of literals $\mathcal{L}$ we define the saturated set of literal closures $\mathcal{L}^{sat} = \mathcal{L}^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}^\infty)$ for some UP-saturation process $\{\mathcal{L}_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}$.

Lemma

The set $\mathcal{L}^{sat}$ is unique because for any two UP-fair saturation processes $\{\mathcal{L}_i\}_{i=0}^\infty$ and $\{\mathcal{L}'_i\}_{i=0}^\infty$ with $\mathcal{L}_0 = \mathcal{L}'_0$ we have

$$\mathcal{L}^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}^\infty) = \mathcal{L}'^\infty \backslash \mathcal{R}_{UP}(\mathcal{L}'^\infty)$$

Inst-Redundancy

Let S be a set of clauses.

- A ground closure C is Inst-redundant in S if for some k
  - $C_i' \in S$, $C_i = C_i' \cdot \sigma_i'$, $C \succ_{cl} C_i$ for $i \in 1 \ldots k$
  - such that $C_1, \ldots, C_k \models C$
- A (possible non-ground) clause C is called Inst-redundant in S if each ground closure $C \cdot \sigma$ is Inst-redundant in S.
- $R_{Inst}(S)$ denotes the set of all Inst-redundant clauses in S.

Example

$S = \{\, \mathsf{f}(x) \approx x,\ \mathsf{f}(\mathsf{a}) \approx \mathsf{a},\ \mathsf{f}(\mathsf{f}(x)) \approx \mathsf{f}(x) \,\}$
$R_{Inst}(S) = \{\, \mathsf{f}(\mathsf{f}(x)) \approx \mathsf{f}(x) \,\}$

Selection

Let S be a set of clauses S, let $I_\perp$ be a model of $S\perp$.

- A selection function sel maps clauses to literals such that

$$\mathrm{sel}(C) \in C \qquad\qquad I_\perp \models \mathrm{sel}(C)\perp$$

- The set of S-relevant literal closures

$$\mathcal{L}_S = \left\{ L \cdot \sigma \;\middle|\; \begin{array}{l} L \vee C \in S,\, L = \mathrm{sel}(L \vee C) \\ (L \vee C) \cdot \sigma \text{ is not Inst-redundant in S,} \end{array} \right\}$$

- $\mathcal{L}_S^{sat}$ denotes the satuarion process of $\mathcal{L}_S$.
- A set of clauses S is Inst-saturated w.r.t. a selection function, if $\mathcal{L}_S^{sat}$ does not contain the empty clause.

Completeness

Theorem

If a set of clauses S is Inst-saturated, and $S\bot$ is satisfiable, then S is also satisfiable.

Proof.

1. Construct candidate model
2. Assumed counterexample fails

Conclude candidate is model $\square$

Model Construction I

Let S be an Inst-saturated set of clauses.

- $S\bot$ is satisfiable
- $\square \notin \mathcal{L}_S^{sat}$

Let $L = L' \cdot \sigma \in \mathcal{L}_S^{sat}$. We define by induction on $\succ_\ell$

- $I_L = \bigcup_{L \succ_\ell M} \epsilon_M$ $\qquad$ ϵ_M allready defined for all M with $L \succ_\ell M$
- $R_L = \{s \to t \mid s \approx t \in I_L, s \succ_{gr} t\}$
- $\epsilon_L = \begin{cases} \emptyset & \text{if } L'\sigma \text{ reducible by } R_L \\ \emptyset & \text{if } I_L \vDash L'\sigma \text{ or } I_L \vDash \overline{L'}\sigma \text{ (defined)} \\ \{L'\sigma\} & \text{if } L'\sigma \text{ is productive (irreducible, undefined)} \end{cases}$

Model Construction II

- $R_S = \bigcup_{L \in \mathcal{L}_S^{sat}} R_L$ — R_S is convergent and interreduced
- $I_S = \bigcup_{L \in \mathcal{L}_S^{sat}} \epsilon_L$ — I_S is consistent, $L\sigma \in L_S$ is irreducible by R_S
- Let $\mathcal{I}$ be an arbitrary total consistent extension of I_S.

Lemma

$\mathcal{I}$ is a model for all ground instances of clauses in S.

Assumed Counterexample I

Assume $\mathcal{I}$ is not a model of S.

$$\text{Let } D = \min_{\succ_{cl}} \{ C' \cdot \sigma \mid C' \in S, \mathcal{I} \not\models C'\sigma \}$$

Then

- $D = D' \cdot \sigma$ is not Inst-redundant. Otherwise $D_1, \ldots, D_n \models D$, $D \succ_{cl} D_i$ for all i, and $\mathcal{I} \not\models D_j$ for one j contradicts minimality.
- $x\sigma$ irreducible by R_S for every variable x in D'. Otherwise let $(\ell \to r)\tau \in R_L$ and $x\sigma = x\sigma[l\tau]_p$ for some variable x in D'. We define substitution σ' with $x\sigma' = x\sigma[r\tau]_p$ and $y\sigma' = y\sigma$ for $y \neq x$. $\mathcal{I} \not\models D'\sigma'$ and $D \succ_{cl} D' \cdot \sigma'$ contradicts minimality.

Assumed Counterexample II

Since D is not Inst-redundant in S, we have for some literal L, that $D' = L \vee D''$, $\mathrm{sel}(D') = L$, $L \cdot \sigma \in \mathcal{L}_S$, $L\sigma$ is false in $\mathcal{I}$

Assume $L \cdot \sigma$ is UP-redundant in $\mathcal{L}_S^{sat}$.

By construction σ is irreducible by R_S. Then we have

$$R_S \cup irred_{R_S}(\{L' \cdot \sigma' \in \mathcal{L}_S^{sat} \mid L \cdot \sigma \succ_\ell L' \cdot \sigma'\}) \models L\sigma$$

Therefore there is $L' \cdot \sigma' \in irred_{R_S}(\mathcal{L}_S^{sat})$ that is false in I.

Assumed Counterexample III

We define

$$M \cdot \tau = \min_{\succ_\ell} \{ \, L' \cdot \tau' \mid L' \cdot \sigma' \in \mathit{irred}_{R_S}(\mathcal{L}_S^{sat}), \, \mathcal{I} \not\models M'\tau' \, \}$$

Assume $M \cdot \tau$ is reducible by $(\ell \to r) \in R_S$
and $(\ell \to r)$ is produced by $(\ell' \approx r') \cdot \rho \in \mathcal{L}_S^{sat}$

UP-inference is applicable because τ is irreducible by R_S

$$\frac{(\ell' \approx r') \cdot \rho \quad M[\ell''] \cdot \tau}{M[r']\theta \cdot \mu} \; UP$$

$M[r']\theta\mu$ is false in $\mathcal{I}$.

Assumed Counterexample IV

- If $M[r']\theta \cdot \mu$ is not UP-redundant in $\mathcal{L}_S^{sat}$ then $M[r']\theta \cdot \mu \in \mathcal{L}_S^{sat}$.

  $M \cdot \tau \succ_\ell M[r']\theta \cdot \mu \in irred_{R_S}(\mathcal{L}_S^{sat})$ (μ is irreducible by R_S) contradicts minimality of $M \cdot \tau$.

- If $M[r']\theta \cdot \mu$ is UP-redundant in $\mathcal{L}_S^{sat}$ then

  $$R_S \cup irred_{R_S}(\{M' \cdot \tau' \in \mathcal{L}_S^{sat} \mid M[r']\theta \cdot \mu \succ_\ell M'\tau'\} \models M[r']\theta\mu$$

  Hence there is $M' \cdot \tau' \in \mathcal{L}_S^{sat}$ false in $\mathcal{I}$ such that
  $M \cdot \tau \succ_\ell M[r']\theta \cdot \mu \succ_\ell M' \cdot \tau'$,
  $M' \cdot \tau'$ contradicts minimality of $M \cdot \tau$.

Hence $M \cdot \tau$ is irreducible by R_S.

Assumed Counterexample V

Under the assumption that $\mathcal{I}$ is not a model
a minimal ground literal $M \cdot \tau$ exists that is

- false in $\mathcal{I}$
- in $\mathcal{L}_S^{sat}$
- irreducible by R_S
- not productive.

Hence $I_{M\cdot\tau} \models \overline{M}\tau$ with two possible cases:

1. $M \cdot \tau$ is equation $(s \approx t) \cdot \tau$ $\qquad I_{M\cdot\tau} \models (s \not\approx t)\tau$
2. $M \cdot \tau$ is inequation $(s \not\approx t) \cdot \tau$ $\qquad I_{M\cdot\tau} \models (s \approx t)\tau$

Both cases lead to a contradiction (next slide).
We reject the assumption and conclude
that $\mathcal{I}$ is a model for all ground instances of S.

Assumed Counterexample VI

1. Assume $M \cdot \tau$ is equation $(s \approx t) \cdot \tau$:
   - $I_{M \cdot \tau} \models (s \not\approx t)\tau$
   - All literals in $I_{M \cdot \tau}$ are irreducible by $R_{M \cdot \tau}$
   - $s\tau$ and $t\tau$ are irreducible by $R_{M \cdot \tau}$
   - $R_{M \cdot \tau}$ is a convergent term rewrite system

   Hence $(s \not\approx t)\tau \in I_{M \cdot \tau}$ and produced to $I_{M \cdot \tau}$ by a $(s' \not\approx t') \cdot \tau'$.
   Then $(s' \not\approx t')\tau' \succ_{gr} (s \approx t)\tau$ and $(s' \not\approx t') \cdot \tau' \succ_{\ell} M \cdot \tau$
   Contradiction to minimality of $M \cdot \tau$ w.r.t. $\succ_{\ell}$.

2. Assume $M \cdot \tau$ is inequation $(s \not\approx t) \cdot \tau$. We have
   - $I_{M \cdot \tau} \models (s \approx t)\tau$
   - $s\tau$ and $t\tau$ are irreducible by $R_{M \cdot \tau}$

   Hence $s\tau = t\tau$ and equality resolution is applicable.
   Contradiction to $\square \notin \mathcal{L}_S^{sat}$.

Summary

Abstract

Instantiation-based theorem proving eventually finds a finite set of unsatisfiable ground instances for any unsatisfiable set of first order clauses (at least in theory). Satisfiability of ground instances can be effectively decided by a SAT-solver.

But satisfiability of a finite set of ground instances does not confirm satisfiability of a set of non-ground clauses.

Unit paramodulation is part of a sound instantiation-based calculus for first order logic with equality. They proving procedure may saturate without generating an unsatisfiable set of ground instances. We present the completeness proof from the literature. An Inst-saturated set of non-ground clauses with satisfiable grounded instances is satisfiable.