Supplement to Algorithms For Shaping a Particle Swarm With a Shared Control Input Using Boundary Interaction

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***Abstract*—Includes algorithms and equations too lengthy for main paper, but potentially useful for the community. Also links to videos and demonstration code for the algorithms.**

Consider a swarm of agents that are controlled by the same global inputs and have no autonomy. This paper presents algorithms for shaping such swarms in 2D.

This model is common for current micro- and nano-robots, whose small size makes it difficult to perform onboard computation or contain a power and propulsion source. For this reason these robots are usually powered and controlled by global inputs, such as a uniform external electric or magnetic field, and every robot receives exactly the same control inputs. Due to their small size, large numbers of micro-robots are required to deliver sufficient payloads. Nevertheless, these applications require precision control of the shape and position of the robot swarm. Precision control requires breaking the symmetry caused by the global input.

A promising technique uses collisions with boundary walls to shape the swarm, however, the range of configurations created by conforming a swarm to a boundary wall is limited. This paper describes the set of stable configurations of a swarm in two canonical workspaces, a circle and a square.

To increase the diversity of configurations, we add boundary interaction to our model. We provide algorithms using friction with walls to place two robots at arbitrary locations in a rectangular workspace. Next, we extend this algorithm to place n robots at desired locations. We conclude with efficient techniques to control the covariance of a swarm not possible without wall-friction. Simulations and hardware implementations with 100 robots validate these results.

I. Introduction

This supplement gives overviews of the videos and code in §II, provides the algorithm for y position control of two robots in §III, and gives gull analytical models for fluid settling in square-shaped tanks in §IV.

II. Supplementary Videos

Five videos animate the key algorithms in this paper.

A. Robot Swarm in a Circle under Gravity

The video *Robot Swarm in a Circle under Gravity* shows the stable configuration of a swarm under a constant global input. Animated plots show mean, variance, covariance, and correlation for a swarm in a circular workspace. Full resolution video: https://youtu.be/nPFAjVIOxYc. An online demonstration and source code of the algorithm are at Zhao and Becker [4].

B. Distribution of Robot Swarm in Square under Gravity

The video *Distribution of Robot Swarm in Square under Gravity* shows the stable configuration of a swarm under a constant global input. Animated plots show mean, variance, covariance, and correlation for a swarm in a square workspace. Full resolution video: https://youtu.be/ZEksDxLpAzg. An online demonstration and source code of the algorithm are at Zhao and Becker [3].

C. Steering 2 Particles with Shared Controls Using Wall Friction

Animates Algs. 1, 2, 3 in Mathematica to show how two robots can be arbitrarily positioned in a square workspace. In this video the desired initial and ending positions of the two robots are manipulated, and the path that the robots should follow is drawn. The video ends with an extreme case where the robots must exchange positions. Full resolution video: https://youtu.be/5TWlw7vThsM. An online demonstration and source code of the algorithm are at Shahrokhi and Becker [2].

D. Arranging a robot swarm with global inputs and wall friction [discrete]

An implementation of Alg. 4 in MATLAB that illustrates how the two robots positioning algorithm is extendable to n robots. In this video all robots gets the same input, but by exploiting wall friction each robot reaches its goal, the formation "UH". Full resolution video: https://youtu.be/uhpsAyPwKeI. Full code is available at Mahadev and Becker [1]. Note that this code uses discretized version of Algorithm 3. The continuous-movement version is illustrated in Fig.**??**.

E. AutomaticCovControl.mp4

A closed-loop controller that steers a swarm of particles to a desired covariance, implemented with a box2D simulator. In this video the green ellipse is the desired covariance ellipse, the red ellipse is the current covariance ellipse of the swarm and the red dot is the mean position of the robots. Robots follow the algorithm to achieve the desired values for σ_{goalxy}, σ_x^2 and σ_y^2.

III. Algorithm for generating desired y spacing between two robots using wall friction

IV. Calculations for modeling swarm as fluid in a simple planar workspace

Two workspaces are used, a square and a circular workspace.

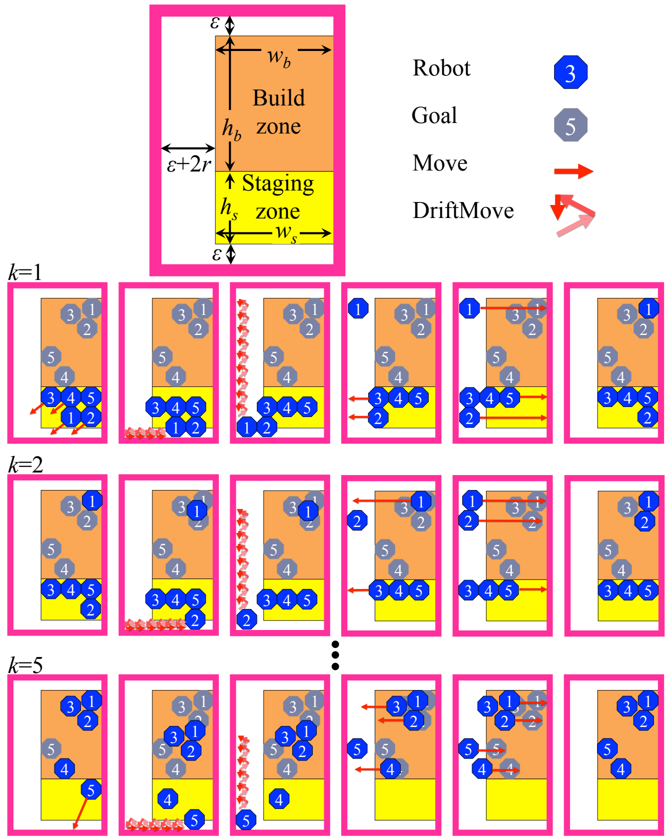

Fig. 1. Illustration of Alg. 1, n robot position control using wall friction.

Algorithm 1 GenerateDesired$y$-spacing(s_1, s_2, e_1, e_2, L)

Require: Knowledge of starting (s_1, s_2) and ending (e_1, e_2) positions of two robots. $(0, 0)$ is bottom corner, s_1 is rightmost robot, L is length of the walls. Current position of the robots are (r_1, r_2).

Ensure: $r_{1x} - r_{2x} \equiv s_{1x} - s_{2x}$

$\Delta s_y \leftarrow s_{1y} - s_{2y}$
$\Delta e_y \leftarrow e_{1y} - e_{2y}$
$r_1 \leftarrow s_1,\ r_2 \leftarrow s_2$
if $\Delta e_y < 0$ **then**
    $m \leftarrow (L - \max(r_{1y}, r_{2y}), 0)$ ▷ Move to top wall
else
    $m \leftarrow (-\min(r_{1y}, r_{2y}), 0)$ ▷ Move to bottom wall
end if
$m \leftarrow m + (0, -\min(r_{1x}, r_{2x}))$ ▷ Move to left
$r_1 \leftarrow r_1 + m,\ r_2 \leftarrow r_2 + m$ ▷ Apply move
if $\Delta e_y - (r_{1y} - r_{2y}) > 0$ **then**
    $m \leftarrow (\min(|\Delta e_y - \Delta s_y|, L - r_{1y}), 0)$ ▷ Move top
else
    $m \leftarrow (-\min(|\Delta e_y - \Delta s_y|, r_{1y}), 0)$ ▷ Move bottom
end if
$m \leftarrow m + (0, \epsilon)$ ▷ Move right
$r_1 \leftarrow r_1 + m,\ r_2 \leftarrow r_2 + m$ ▷ Apply move
$\Delta r_y = r_{1y} - r_{2y}$
if $\Delta r_y \equiv \Delta e_y$ **then**
    $m \leftarrow (e_{1x} - r_{1x}, e_{1y} - r_{1y})$
    $r_1 \leftarrow r_1 + m,\ r_2 \leftarrow r_2 + m$ ▷ Apply move
    return (r_1, r_2)
else
    return GenerateDesired$y$-spacing(r_1, r_2, e_1, e_2, L)
end if

A. Square Workspace

This section provides formulas for the mean, variance, covariance and correlation of a very large swarm of robots as they move inside a square workplace under the influence of gravity pointing in the direction β. The swarm is large, but the robots are small in comparison, and together cover an area of constant volume A. Under a global input such as gravity, they flow like water, moving to a side of the workplace and forming a polygonal shape. The workspace is

The range of possible angles for the global input angle β is [0,2π). In this range of angles, the swarm assumes eight different polygonal shapes. The shapes alternate between triangles and trapezoids when the area A<1/2, and alternate between squares with one corner removed and trapezoids when A>1/2.

Two representative formulas are attached, the outline of the swarm shapes in (II) and $\bar{x}(\beta, A)$ in (I).

B. Circle Workspace

The area under a chord of a circle is the area of a sector less the area of the triangle originating at the circle center: $A = S(sector) - S(triangle) = 1/2LR - 1/2C(1-h)$, thus

$$A = (1/2)\left[LR - c(R-h)\right] \quad (3)$$

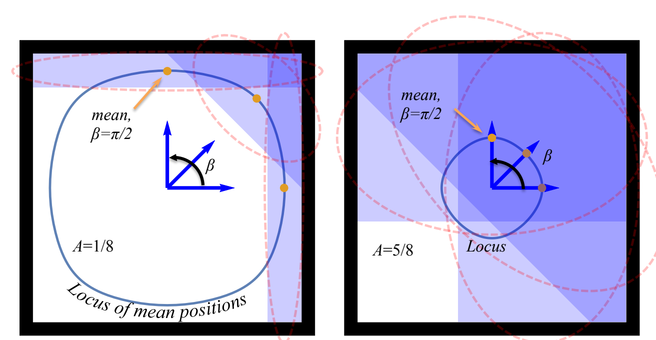

Fig. 2. A swarm in a square workspace under a constant global input assumes either a triangular or a trapezoidal shape if $A < 1/2$. If $A > 1/2$ the swarm is either a squares with one corner removed or a trapezoidal shape.

where L is arc length, c is chord length, R is radius and h is height. Solving for L and C gives

$$L = 2\cos^{-1}(1-h) \quad (4)$$

$$C = 2\sqrt{h(2-h)} \quad (5)$$

$$
\begin{aligned}
\bar{x}(\beta, A) = A \leq \frac{1}{2} : &\begin{cases}
-\frac{\tan^2(\beta)}{24A} - \frac{A}{2} + 1 & 0 \leq \beta \leq \tan^{-1}(2A) \vee 2\pi - \tan^{-1}(2A) < \beta \leq 2\pi \\
1 - \frac{1}{3}\sqrt{2}\sqrt{A\tan(\beta)} & \tan^{-1}(2A) < \beta \leq \frac{\pi}{2} - \tan^{-1}(2A) \\
\frac{\cot(\beta)}{12A} + \frac{1}{2} & \frac{\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{\pi}{2} \\
\frac{1}{3}\sqrt{2}\sqrt{-A\tan(\beta)} & \tan^{-1}(2A) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}(2A) \\
\frac{\tan^2(\beta)}{24A} + \frac{A}{2} & \pi - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \pi \\
\frac{1}{3}\sqrt{2}\sqrt{A\tan(\beta)} & \tan^{-1}(2A) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}(2A) \\
\frac{1}{2} - \frac{\cot(\beta)}{12A} & \frac{3\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{3\pi}{2} \\
1 - \frac{1}{3}\sqrt{2}\sqrt{-A\tan(\beta)} & \tan^{-1}(2A) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}(2A)
\end{cases} \\
\frac{1}{2} < A < 1 : &\begin{cases}
-\frac{\tan^2(\beta)}{24A} - \frac{A}{2} + 1 & 0 \leq \beta \leq \tan^{-1}\left(\frac{1}{2}, 1 - A\right) \vee 2\pi - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) < \beta \leq 2\pi \\
\frac{2\sqrt{2}\sqrt{(1-A)\tan(\beta)}(A-1)+3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1 - A\right) < \beta \leq \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) \\
\frac{6A+\cot(\beta)}{12A} & \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1 - A\right) + \frac{\pi}{2} \\
\frac{-2\sqrt{2}\sqrt{(A-1)\tan(\beta)}(A-1)+6A-3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1 - A\right) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) \\
\frac{\tan^2(\beta)}{24A} + \frac{A}{2} & \pi - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1 - A\right) + \pi \\
\frac{2\sqrt{2}\sqrt{(1-A)\tan(\beta)}(1-A)+6A-3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1 - A\right) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) \\
\frac{1}{2} - \frac{\cot(\beta)}{12A} & \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1 - A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1 - A\right) + \frac{3\pi}{2} \\
\frac{2\sqrt{2}\sqrt{(A-1)\tan(\beta)}(A-1)+3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1 - A\right) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}\left(\frac{1}{2}, 1 - A\right)
\end{cases} \\
A = 1 : &\frac{1}{2}
\end{aligned} \tag{1}
$$

TABLE I
$\bar{x}$ IN A UNIT-SQUARE WORKSPACE

Therefore the area under a chord is

$$\cos^{-1}(1-h) - (1-h)\sqrt{(2-h)h} \tag{6}$$

For a circular workspace, with $\beta = 0$, the variance of x and y are:

$$
\begin{aligned}
\sigma_x^2(h) = &\frac{64(h-2)^3 h^3}{144\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)^2} + \\
&\frac{9\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)\left(\sin\left(4\arcsin(1-h)\right) + 4\arccos(1-h)\right)}{144\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)^2}
\end{aligned} \tag{7}
$$

$$\sigma_y^2(h) = \frac{12\arccos(1-h) - 8\sin\left(2\arccos(1-h)\right) + \sin\left(4\arccos(1-h)\right)}{48\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)} \tag{8}$$

For $\beta = 0$, $\sigma_{xy} = 0$. These values can be rotated to calculate $\sigma_x^2(\beta, h), \sigma_y^2(\beta, h)$, and $\sigma_{xy}(\beta, h)$.

ACKNOWLEDGMENTS

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[3] Haoran Zhao and Aaron T. Becker. "distribution of a swarm of robots in a circular workplace under gravity", wolfram demonstrations project, February 2016. URL http://demonstrations.wolfram.com/DistributionOfASwarmOfRobotsInACircularWorkplaceUnderGravity/.

[4] Haoran Zhao and Aaron T. Becker. "distribution of a robot swarm in a square under gravity", wolfram demonstrations project, January 2016. URL http://demonstrations.wolfram.com/DistributionOfARobotSwarmInASquareUnderGravity/.

$$
\mathrm{RobotRegion}(\beta, A) =
\begin{cases}
A \leq \frac{1}{2} : \begin{cases}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -A - \frac{\tan(\beta)}{2} + 1 & 1 \\ -A + \frac{\tan(\beta)}{2} + 1 & 0 \end{pmatrix} & 0 \leq \beta \leq \tan^{-1}(2A) \vee 2\pi - \tan^{-1}(2A) < \beta \leq 2\pi \\
\begin{pmatrix} 1 & 1 \\ 1 - \sqrt{2}\sqrt{A\tan(\beta)} & 1 \\ 1 & 1 - \sqrt{2}\sqrt{A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) < \beta \leq \frac{\pi}{2} - \tan^{-1}(2A) \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -A + \frac{\cot(\beta)}{2} + 1 \\ 1 & -A - \frac{\cot(\beta)}{2} + 1 \end{pmatrix} & \frac{\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{\pi}{2} \\
\begin{pmatrix} 0 & 1 \\ \sqrt{2}\sqrt{-A\tan(\beta)} & 1 \\ 0 & 1 - \sqrt{2}\sqrt{-A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}(2A) \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ A - \frac{\tan(\beta)}{2} & 1 \\ A + \frac{\tan(\beta)}{2} & 0 \end{pmatrix} & \pi - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \pi \\
\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\sqrt{A\cot(\beta)} \\ \sqrt{2}\sqrt{A\tan(\beta)} & 0 \end{pmatrix} & \tan^{-1}(2A) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}(2A) \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & A - \frac{\cot(\beta)}{2} \\ 0 & A + \frac{\cot(\beta)}{2} \end{pmatrix} & \frac{3\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{3\pi}{2} \\
\begin{pmatrix} 1 & 0 \\ 1 - \sqrt{2}\sqrt{-A\tan(\beta)} & 0 \\ 1 & \sqrt{2}\sqrt{-A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}(2A)
\end{cases} \\
\frac{1}{2} < A < 1 : \begin{cases}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ (1-A) - \frac{\tan(\beta)}{2} & 1 \\ (1-A) + \frac{\tan(\beta)}{2} & 0 \end{pmatrix} & 0 \leq \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) \vee 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq 2\pi \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & \sqrt{2}\sqrt{(1-A)\cot(\beta)} \\ \sqrt{2}\sqrt{(1-A)\tan(\beta)} & 0 \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & (1-A) - \frac{\cot(\beta)}{2} \\ 0 & (1-A) + \frac{\cot(\beta)}{2} \end{pmatrix} & \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 - \sqrt{2}\sqrt{-(1-A)\tan(\beta)} & 0 \\ 1 & \sqrt{2}\sqrt{-(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -(1-A) - \frac{\tan(\beta)}{2} + 1 & 1 \\ -(1-A) + \frac{\tan(\beta)}{2} + 1 & 0 \end{pmatrix} & \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi \\
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 - \sqrt{2}\sqrt{(1-A)\tan(\beta)} & 1 \\ 1 & 1 - \sqrt{2}\sqrt{(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -(1-A) + \frac{\cot(\beta)}{2} + 1 \\ 1 & -(1-A) - \frac{\cot(\beta)}{2} + 1 \end{pmatrix} & \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ \sqrt{2}\sqrt{-(1-A)\tan(\beta)} & 1 \\ 0 & 1 - \sqrt{2}\sqrt{-(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right)
\end{cases} \\
A = 1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}
\end{cases}
, \qquad (2)
$$

TABLE II

ROBOTREGIONS IN A UNIT-SQUARE WORKSPACE