Reshaping Particles by Pushing Against Rigid Objects

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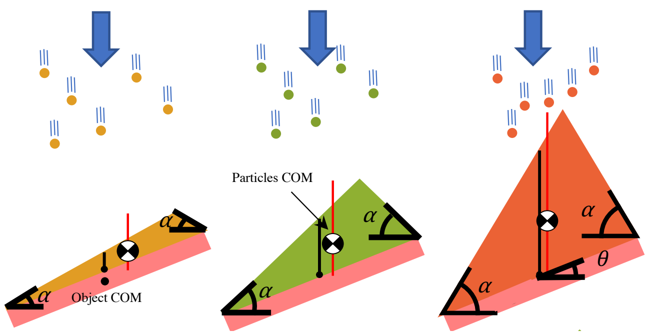

Fig. 1. Different values of angle of repose is shown when granular particles move faster than the rod.

I. ANGLE OF REPOSE

Consider a swarm of granular particles applying force to a rod. If the rod moves slower than the particles, these particles will build up behind the rod in a characteristic triangular shape defined by an apex angle. This piling up is common to all granular media, and the angle formed is a function of the *angle of repose*. Three different values of angle of repose is shown in Fig. 1. The center of mass of the rod is in the middle of the rod, but center of mass of the granular particles changes for different values of angle of repose. By measuring the angle of repose for the particles shown in the top plot of Fig. 2, we can estimate the force and torque that the swarm is applying to the rod as a function of the rod's length, the angle of repose, and orientation of the rod. We define the angle of repose as α, the rod's orientation relative to $90°$ from the particle movement vector as θ, and the rod's length as ℓ. By integrating over the triangular shape, the force applied to the rod (when a unit area of particles produces 1 N of force) is:

$$F(\theta,\alpha,\ell) = \begin{cases} \dfrac{-\ell^2\Big(\cos(2\theta)-\cos(2\alpha)\Big)}{8\cos\alpha\sin\theta} & -\alpha<\theta<\alpha \\ 0 & \text{otherwise}\ . \end{cases} \tag{1}$$

(2)

The force for different angle of repose values are shown in the middle plot of Fig. 2. Torque will also be similarly

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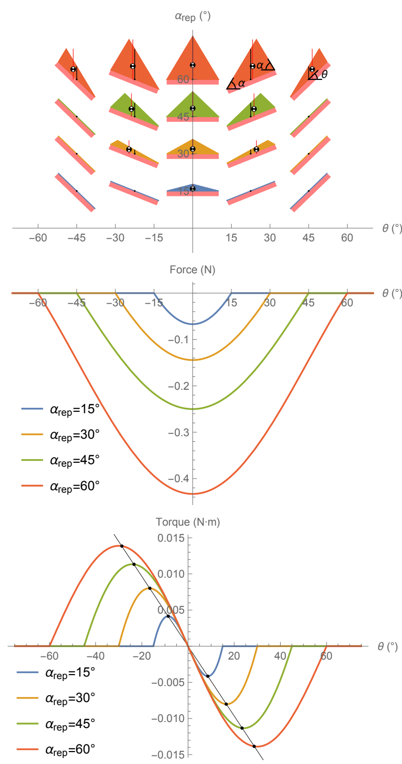

Fig. 2. Top plot shows colored particulate heaped up on pink-colored long rods. Middle plot shows the force applied to the rod and bottom the torque as a function of θ for four angle of repose values. The maximum torque values are shown with black dots.

defined as

$$\tau(\theta,\alpha,\ell) = \begin{cases} \dfrac{\ell^3\Big(\cos(2\alpha)-\cos(2\theta)\Big)\sin\theta}{48\sin^2\alpha} & -\alpha<\theta<\alpha \\ 0 & \text{otherwise}\ . \end{cases} \tag{3}$$

Torque is shown in the bottom plot of Fig. 2. Given sufficient particles to pile up to the angle of repose, this torque tends to stabilize the object to be perpendicular to the pushing direction. Force is maximized with $\theta = 0$, but the θ value that maximizes torque is a function of α:

$$\theta_{t_{\max}} = \frac{\sin(\alpha)}{\sqrt{3}}. \tag{4}$$

To maximize the torque a particulate swarm applies on a thin rod, the swarm should move in the direction $-\theta_{t_{\max}} - 90°$ with respect to the long axis of the rod.

II. COVARIANCE CONTROL

Covariance control as well as mean position and variance control is needed when the swarm is moving through narrow

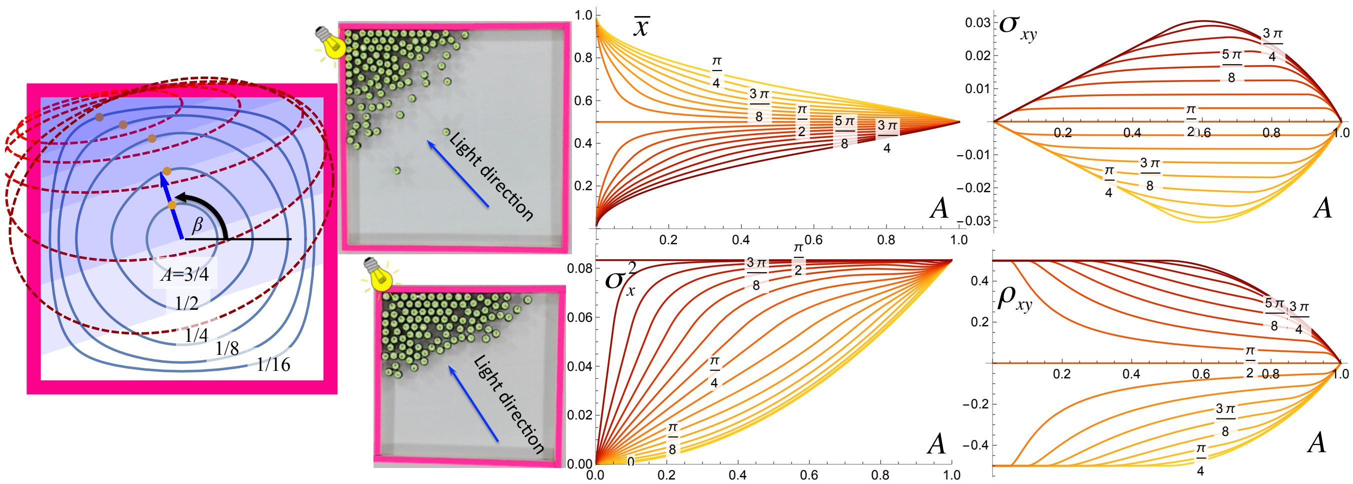

Fig. 3. Pushing the swarm against a square boundary wall allows limited control of the shape of the swarm, as a function of swarm area A and the commanded movement direction β. Left plot shows locus of possible mean positions for five values of A. Center shows two corresponding arrangements of kilobots.

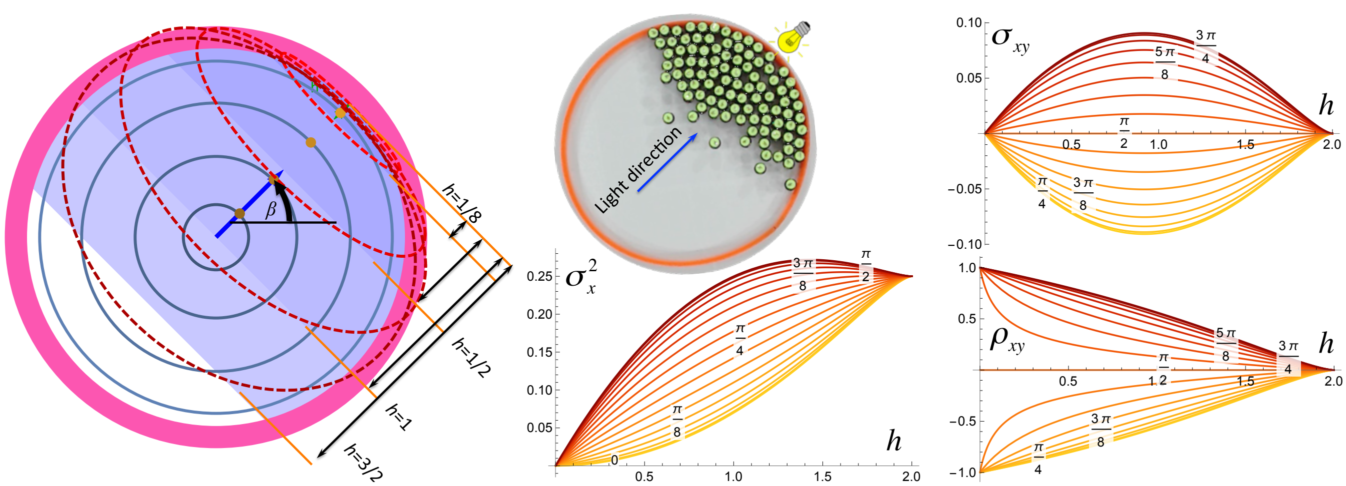

Fig. 4. Pushing the swarm against a circular boundary wall allows limited control of the shape of the swarm, as a function of the fill level h and the commanded movement direction β. Left plot shows locus of possible mean positions for four values of h.

passageways. This section discusses ideas to control covariance of the swarm using boundaries.

A. *Using boundaries: stable configurations of a swarm*

One method to control a swarm's shape in a bounded workspace is to simply push in a given direction until the swarm conforms to the boundary. Like fluid settling in a tank, the stable final configuration minimizes potential energy.

a) Square workspace: We first examine the mean $(\bar{x}, \bar{y})$, covariance $(\sigma_x^2, \sigma_y^2, \sigma_{xy})$, and correlation ρ_{xy} of a large swarm of robots as they move inside a square workspace under the influence of a force pulling in the direction β. The swarm is large, but the robots are small in comparison, and together occupy a constant area A, $A \in [0, 1]$. Under a global input, the swarm moves to a side of the workspace and forms a polygonal shape that minimizes potential energy, as shown in Fig. 3, see also [**?**].

The range for the global input angle β is $[0,2\pi)$. In this range, the swarm assumes eight different polygonal shapes. These shapes alternate between triangles and trapezoids when the area A<1/2, and between squares with one corner removed and trapezoids when A>1/2.

Computing means, variances, covariance, and correlation requires integrating over R, the region containing the swarm:

$$\bar{x} = \frac{\iint_R x \, dx \, dy}{A}, \qquad \bar{y} = \frac{\iint_R y \, dx \, dy}{A}, \tag{5}$$

$$\sigma_x^2 = \frac{\iint_R (x-\bar{x})^2 \, dx \, dy}{A}, \; \sigma_y^2 = \frac{\iint_R (y-\bar{y})^2 \, dx \, dy}{A}, \text{ and} \tag{6}$$

$$\sigma_{xy} = \frac{\iint_R (x-\bar{x})(y-\bar{y}) \, dx \, dy}{A}, \; \rho_{xy} = \frac{\sigma_x^2}{\sigma_x \sigma_y}. \tag{7}$$

The region of integration R is the polygon containing the swarm. For example, if $A < 1/2$ and the force angle is β, the mean when R is a triangular region in the lower-left corner is:

$$\bar{x}(A,\beta) = \frac{\int_0^{\sqrt{2A\tan(\beta)}} \left(\int_0^{\cot(\beta)\left(\sqrt{2A\tan(\beta)}-x\right)} dy \right) x\,dx}{A}$$
$$= \frac{\sqrt{2}}{3}\sqrt{A\tan(\beta)} \tag{8}$$
$$\bar{y}(A,\beta) = \frac{\int_0^{\sqrt{2A\tan(\beta)}} \left(\int_0^{\cot(\beta)\left(\sqrt{2A\tan(\beta)}-x\right)} y\,dy \right) dx}{A}$$
$$= \frac{\sqrt{2}}{3}\sqrt{A\cot(\beta)}. \tag{9}$$

The full equations are included in the appendix [**?**], and are summarized in Fig. 3. A few highlights are that the correlation is maximized ±1/2 when the swarm is in a triangular shape. The covariance of a triangle is always $\pm(A/18)$. Variance is minimized in the direction of β and maximized orthogonal to β when the swarm is in a rectangular shape. The range of mean positions are maximized when A is small.

b) Circular workspace: Though rectangular boundaries are common in artificial workspaces, biological workspaces are usually rounded. Similar calculations can be made for a circular workspace. The workspace is a circle centered at (0,0) with radius 1 and thus area π. For notational simplicity, the swarm is parameterized by the global control input signal β and the fill-level h. Under a global input, the robot swarm fills the region under a chord with area

$$A(h) = \arccos(1-h) - (1-h)\sqrt{(2-h)h}. \tag{10}$$

For a circular workspace, the locus of mean positions are aligned with β and the mean position is at radius $r(h)$ from the center:

$$r(h) = \frac{2(-(h-2)h)^{3/2}}{3\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)}. \tag{11}$$

Variance $\sigma_x^2(\beta, h)$ is maximized at $\beta = \pi/2 + n\pi$ and $h \approx 1.43$, while covariance is maximized at $\beta = \pi 3/4 + n\pi$ and $h \approx 0.92$. For small h values, correlation approaches ±1. Results are displayed in Fig. 4, see also [**?**].

B. Using boundaries: friction and boundary layers

Global inputs move a swarm uniformly. Shape control requires breaking this uniform symmetry. A swarm inside an axis-aligned rectangular workspace can reduce variance normal to a wall by simply pushing the swarm into the boundary. If the swarm can flow around each other, pushing the swarm into a boundary produces the limited set of configurations presented in Sec. II-A. Instead of pushing our robots directly into a wall, the following sections examine an oblique approach using boundaries that generate friction with the robots. These frictional forces are sufficient to break the symmetry caused by uniform inputs. Robots touching a wall have a friction force that opposes movement along the boundary. This causes robots along the boundary to move more slowly than robots in free-space.

Let the control input be a vector force $\vec{F}$ with magnitude F and orientation θ with respect to a line perpendicular to and into the nearest boundary. N is the normal or perpendicular force between the robot and the boundary. The force of friction F_f is nonzero if the robot is in contact with the boundary and $\sin(\theta) < 0$. The resulting net force on the robot, $F_{forward}$, is aligned with the wall and given by

$$F_{forward} = F\sin(\theta) - F_f$$
$$\text{where } F_f = \begin{cases} \mu_f N, & \mu_f N < F\sin(\theta) \\ F\sin(\theta), & \text{else} \end{cases} \tag{12}$$
$$\text{and } N = F\cos(\theta).$$

Fig. 5 shows the resultant forces on two robots when one is touching a wall. Though each receives the same inputs, they experience different net forces. For ease of analysis, the following algorithms assume μ_f is infinite and robots touching the wall are prevented from sliding along the wall. This means that if one robot is touching the wall and another robot is free, the touching robot will not move when the control input is parallel or into the wall. There are many alternate models of friction that also break control symmetry. Fig. 5c shows fluid flow along a boundary. Fluid in the free-flow region moves uniformly, but flow decreases to zero in the boundary layer [**?**]. Force in such a system can be calculated as

$$F_{forward}(y) = F - F_f \begin{cases} \frac{h-y}{h}, & y < h \\ 0, & \text{else} \end{cases}. \tag{13}$$

The next section shows how a system in a rectangularly bounded workspace with friction model (12) can arbitrarily position two robots.

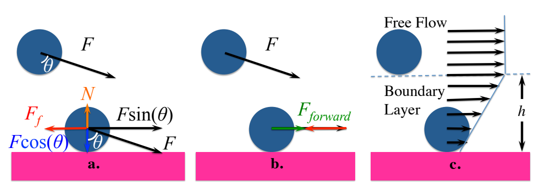

Fig. 5. (a,b) Wall friction reduces the force for going forward $F_{forward}$ on a robot near a wall, but not for a free robot. (c) velocity of a fluid reduces to zero at the boundary.

C. Maximizing correlation using wall friction

Using the environment to individually move n particles to goal positions in Alg. **??** requires $O(n^2)$ time, while Alg. **??** can only control two particles. The controllers in Section **??** are efficient because they require only a single move but the range of possible configurations are limited. This section presents an efficient technique to generate desired correlations.

Assume an obstacle-free, bounded, unit-size, square workspace. As shown in Fig. 3, the maximum correlation occurs when the swarm is pushed in the direction $\beta = 3\pi/4$. This correlation as a function of swarm area A is never larger than 1/2, and the maximum correlation decays to 0 as A

grows to 1. By (7), this maximum correlation is:

$$\rho_{xy} = \begin{cases} \frac{1}{2}, & 0 \leq A \leq \frac{1}{2} \\ \frac{3A(2(A-2)A+1)}{4A^3-24A+\sqrt{2}(12A-12)\sqrt{1-A}+17} - 1, & \frac{1}{2} \leq A \leq 1 \end{cases}. \tag{14}$$

If friction obeys the linear boundary layer model of (13) with boundary layer thickness h and maximum friction F_f equal to the maximum applied force F, we can generate larger correlations. If the swarm size is smaller than ≈ 0.43 and the boundary layer is sufficiently thick we can generate correlations larger than 1/2 using boundary friction.

Assume that the swarm is initialized in the lower-left corner, in a rectangle of width w and height A/w. Such a rectangular configuration can be accomplished using the variance controllers from [**?**]. If the swarm is then commanded to move a distance L to the right, components of the swarm outside the boundary friction layer of height h move further than components near the boundary. The swarm is contained in a region R composed of no more than three stacked components: at bottom a parallelogram inclined to the right top, at middle a rectangle, and at top a parallelogram inclined to the left top. These regions can be defined by the rectangle's left side, bottom, and top:

$$\begin{aligned} r_{\text{left}} &= \min(L, 1-w), \\ r_{\text{bottom}} &= \min\left(\frac{A}{w}, h\frac{r_{left}}{L}\right) \text{ and} \\ r_{\text{top}} &= \min\left(\frac{A}{w}, 1 - h\frac{r_{left}}{L}\right). \end{aligned} \tag{15}$$

If $\frac{A}{w} \leq r_{\text{top}}$ the top parallelogram has no area. Similarly, if $r_{\text{top}} \leq r_{\text{bottom}}$ the rectangle has no area. The mean, variance, and correlation are calculated using (5), (6), and (7) over the region R:

$$\begin{aligned} \iint_R f(x,y)\,dx\,dy = &\int_0^{r_{\text{bottom}}} \int_{\frac{L}{h}y}^{\frac{L}{h}y+w} f(x,y)dx\,dy \qquad (16) \\ &+ \int_{r_{\text{bottom}}}^{r_{\text{top}}} \int_{r_{\text{left}}}^{r_{\text{left}}+w} f(x,y)dx\,dy \\ &+ \int_{r_{\text{top}}}^{\frac{A}{w}} \int_{-\frac{L(y-r_{\text{top}})}{h}+r_{\text{left}}}^{r_{\text{left}}+w-\frac{L(y-r_{\text{top}})}{h}} f(x,y)dx\,dy. \end{aligned}$$

Given an environment parameterized by A and h, efficient correlation control consists of choosing the w, L pair that generates the desired positive correlation. Negative correlations can be generated by initializing the swarm in the upper left, or lower right.

D. Efficient control of correlation

This section examines maximum correlation values as a function of w, L using (16) from Section II-C. The maximum correlation using boundary layer friction $\max_{w,L}(\rho(A,h,w,L))$ can be found by gradient descent, as shown in Fig. 6. For swarms with small area, this method enables generating the full range of correlations ± 1, while the stable configuration method from Section **??** could only generate correlations $\pm 1/2$. As the swarm area A increases above ≈ 0.43, the stable configuration method is more effective. Larger boundary layers h enable more control of correlation. The multimedia attachment shows efficient control of correlation with simulations using the 2D physics engine Box2D [**?**] and 144 disc-shaped robots with boundary layer model (13).

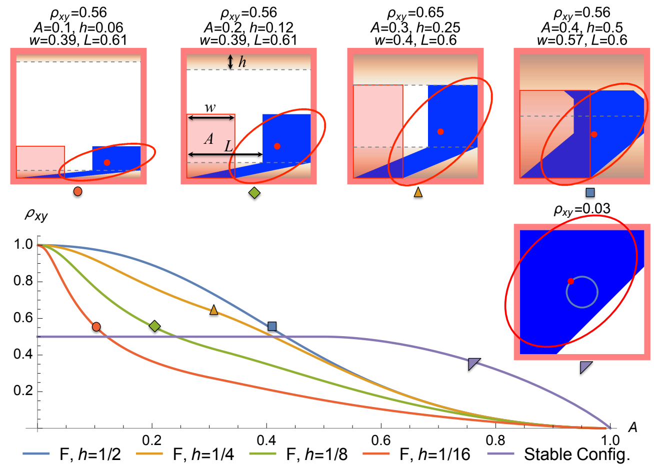

Fig. 6. Analytical results comparing maximum correlation ρ_{xy} under the boundary layer friction model of (16) with four boundary layer thicknesses h and the stable triangular configuration (14).

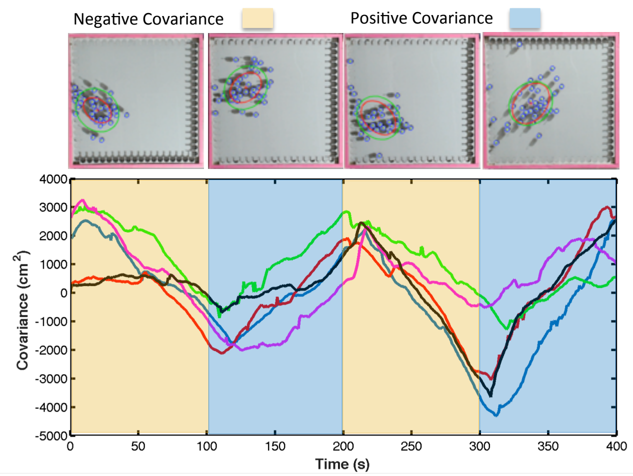

Fig. 7. Hardware demonstration steering ≈ 50 kilobot robots to desired covariance. The goal covariance is negative in first 100 seconds and is positive in the next 100 seconds. The actual covariance is shown in different trials. Frames above the plot show output from machine vision system and an overlaid covariance ellipse.

E. Hardware experiment: control of covariance

To demonstrate covariance control, up to 100 kilobots were placed on the workspace and manually steered with lights, using friction with the boundary walls to vary the covariance from -4000 to 3000 cm^2. The resulting covariance is plotted in Fig. 7, along with snapshots of the swarm.