AE352 Homework #3:
Newton's Law and Euler's Law

(due at the beginning of class on Wednesday, September 23)

In what follows, please assume that frame 0 is an inertial frame ("fixed in space") and that frame 3 is fixed to a rigid body. Please also assume that frame 3 is chosen so that o_3 is at the center of mass of this body. Finally, please assume that n forces $f_1, \ldots, f_n$ are applied to the body at the points $p_1, \ldots, p_n$. You will now derive Newton's Equation and Euler's Equation, yourself, from scratch.

Newton's Law

1. The linear momentum of the body in the coordinates of frame 0 is

$$\ell^0 = \int \dot{p}^0 dm,$$

where the integral is taken over the entire body, and where p^0 and dm vary over this body—each "piece of mass" dm has coordinates p^0 in frame 0. Prove that

$$\ell^0 = m\dot{o}_3^0.$$

Newton's Law is

$$\dot{\ell}^0 = \sum_{i=1}^{n} f_i^0,$$

i.e., "the time rate of change of linear momentum equals the force." Suppose we denote the linear velocity of frame 3 with respect to frame 0, written in the coordinates of frame 0, by

$$v_{0,3}^0 = \dot{o}_3^0.$$

Then, as an immediate consequence of Problem 1, we have:

$$\textsc{Newton's Equation:} \qquad \boxed{m\dot{v}_{0,3}^0 = \sum_{i=1}^{n} f_i^0}$$

Euler's Law

You will derive Euler's Equation from Euler's Law in three steps. The first step is to justify the application of Euler's Law about the center of mass instead of about a fixed point.

2. The angular momentum of the body about the fixed point o_0 and written in the coordinates of frame 0 is

$$h_0^0 = \int \widehat{p^0}\dot{p}^0 dm.$$

The angular momentum of this same body about the center of mass o_3, but still written in the coordinates of frame 0, is

$$h_3^0 = \int \widehat{R_3^0 p^3}\dot{p}^0 dm.$$

Prove that

$$h_0^0 = \widehat{o_3^0}\ell^0 + h_3^0.$$

3. Suppose that n forces $f_1, \ldots, f_n$ are applied to the body at the points $p_1, \ldots, p_n$. Prove that

$$\overbrace{\sum_{i=1}^{n} \widehat{p_i^0} f_i^0}^{\text{torque about } o_0} = \widehat{o_3^0} \sum_{i=1}^{n} f_i^0 + \overbrace{R_3^0 \sum_{i=1}^{n} \widehat{p_i^3} f_i^3}^{\text{torque about } o_3}.$$

4. Euler's Law is

$$\dot{h}_0^0 = \sum_{i=1}^{n} \widehat{p_i^0} f_i^0,$$

i.e., "the time rate of change of angular momentum about a fixed point equals the torque about this fixed point." Prove that

$$\dot{h}_3^0 = R_3^0 \sum_{i=1}^{n} \widehat{p_i^3} f_i^3,$$

i.e., "the time rate of change of angular momentum about the center of mass equals the torque about the center of mass."

You have now justified the application of Euler's Law about the center of mass instead of about a fixed point, which is usually much more convenient. The second step is to derive a simple expression for the angular momentum about the center of mass in terms of the moment of inertia matrix.

5. Prove that

$$h_3^0 = R_3^0 J^3 w_{0,3}^3$$

where

$$J^3 = \int \widehat{p^3}^T \widehat{p^3} dm$$

is the moment of inertia matrix written in the coordinates of frame 3.

You now have a simple expression for the angular momentum about the center of mass. The third and final step toward Euler's Equation is to find the time rate of change of angular momentum and to equate it with the applied torque.

6. Prove:

$$\text{EULER'S EQUATION:} \qquad \boxed{J^3 \dot{w}_{0,3}^3 + \widehat{w_{0,3}^3} J^3 w_{0,3}^3 = \sum_{i=1}^{n} \widehat{p_i^3} f_i^3}$$

Equations of Motion

Last week, you saw that

$$\dot{o}_3^0 = v_{0,3}^0$$
$$\dot{R}_3^0 = R_3^0 \widehat{w_{0,3}^3}.$$

Equivalently, if you parameterize R_3^0 using an Euler Angle sequence, you saw that

$$\dot{o}_3^0 = v_{0,3}^0$$
$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = B w_{0,3}^3$$

for some matrix B that depends on θ_1, θ_2, and θ_3. Combining these equations with what you derived in Problems 1-6, we have:

EOMs for a Single Rigid Body:

$$\boxed{\begin{aligned} \dot{o}_3^0 &= v_{0,3}^0 \\ m\dot{v}_{0,3}^0 &= \sum_{i=1}^{n} f_i^0 \\ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} &= Bw_{0,3}^3 \\ J^3\dot{w}_{0,3}^3 + \widehat{w_{0,3}^3}J^3 w_{0,3}^3 &= \sum_{i=1}^{n} \widehat{p_i^3} f_i^3 \end{aligned}}$$

These equations can be integrated—just like last week—to simulate the translational and rotational motion of a single rigid body.

Implementation

7. The point of last week's homework was to visualize the motion of a rigid body when it has arbitrary linear and angular velocity. The point of this week's homework is to visualize the motion of a rigid body when it is subject to applied forces. To do so, you will be adding code to the MATLAB script `hw3.m`, available on the course website. This script makes clear exactly what you should and should not change. In particular, every line you should change is marked as follows:

   ```
   <-- CHANGE THIS LINE (PROBLEM XXXX)
   ```

   As usual, you are asked to reimplement several things from last week first.

   (a) Compute R_3^0 in terms of θ_1, θ_2, and θ_3 for a body-axis XYZ Euler Angle sequence.

   (b) Compute o_0^3 and R_0^3 in terms of o_3^0 and R_3^0.

   (c) Perform coordinate transformation to go back and forth between representations of points in frame 0 and in frame 3.

   Now, we get to this week:

   (d) You derived the equations of motion for a single rigid body that is subject to n forces $f_1, \ldots, f_n$ applied at the points $p_1, \ldots, p_n$, assuming that frame 3 is fixed to the body and that o_3 is at the body's center of mass. Implement these equations of motion—in other words, compute

   $$\dot{o}_3^0, \quad \dot{v}_{0,3}^0, \quad \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}, \quad \text{and} \quad \dot{w}_{0,3}^3$$

   in terms of

   $$o_3^0, \quad v_{0,3}^0, \quad \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad w_{0,3}^3, \quad m, \quad J^3, \quad p_1^3, \ldots, p_n^3, \quad \text{and} \quad f_1^3, \ldots, f_n^3.$$

You are provided with m and J^3 in the code. You are also provided with a list of forces—in this case, five forces (i.e., $n = 5$), which are generated by thrusters. Each thruster is located at a point on the surface of the body and applies a force in the outward normal direction at this point.

(e) Run the script. The code allows you to toggle thrusters on and off by pressing number keys ('1', '2', and so forth). Do so! What happens? Can you "fly" the spacecraft so frame 3 is aligned with frame 0? (For me, this is a big challenge. For you, perhaps not!) Turn in at least one snapshot of your figure after the simulation has ended.

Play around!!!