AE352 Homework #2:
how to describe the linear and angular velocity of a rigid body

(due at the beginning of class on Friday, September 11)

1. Consider again the XYZ body-axis Euler Angle sequence. Suppose that θ_1, θ_2, and θ_3 are functions of time.

   (a) Write $w_{0,1}^1$ in terms of $\dot{\theta}_1$.

   (b) Write $w_{1,2}^2$ in terms of $\dot{\theta}_2$.

   (c) Write $w_{2,3}^3$ in terms of $\dot{\theta}_3$.

   (d) Express $w_{0,1}$ and $w_{1,2}$ in the coordinates of frame 3.

   (e) Find the angular velocity $w_{0,3}^3$ in terms of the angular rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$. Write your answer as

   $$w_{0,3}^3 = A \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

   for some 3×3 matrix $A \in \mathbb{R}^{3\times 3}$ that depends only on the Euler Angles, not on their angular rates or on components of the angular velocity.

   (f) Find the angular rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of the angular velocity $w_{0,3}^3$. Write your answer as

   $$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = B w_{0,3}^3$$

   for some 3×3 matrix $B \in \mathbb{R}^{3\times 3}$ that depends only on the Euler Angles, not on their angular rates or on components of the angular velocity.

   (g) Give an example of an orientation R_3^0 and an angular velocity $w_{0,3}^3$ with unit magnitude for which at least one of the angular rates $\dot{\theta}_1$, $\dot{\theta}_2$, or $\dot{\theta}_3$ is very large (i.e., has magnitude much bigger than 1). What is special about this orientation? Are the elements of $\dot{R}_3^0$ also very large? Briefly explain.

   To solve this problem, you need to remember that $w_{j,k}^k$ is the angular velocity of frame k with respect to frame j, written in the coordinates of frame k. By "angular velocity," we mean the unique vector $w_{j,k}$ for which

   $$\dot{R}_k^j = R_k^j \widehat{w_{j,k}^k}$$

   where $\dot{R}_k^j$ is obtained by taking the time derivative of each element of R_k^j.

   You also need to remember that angular velocities add:

   $$w_{i,k} = w_{i,j} + w_{j,k}$$

   for any three frames i, j, and k.

   Finally, you need to remember how to invert a 3×3 matrix (look it up or use software that does symbolic math).

2. Suppose a point p has coordinates p^3 in frame 3. From last week, you know that this same point has coordinates

$$p^0 = o_3^0 + R_3^0 p^3$$

in frame 0. Compute $\dot{p}^0$ in terms of $\dot{p}^3$. Your expression may involve one or more of the terms o_3^0, R_3^0, $\dot{o}_3^0$, $w_{0,3}^3$, and p^3, but nothing else.

To solve this problem, you need to remember the product rule for taking derivatives:

$$\frac{d}{dt}(fg) = \frac{df}{dt}g + f\frac{dg}{dt}.$$

Yes, this rule applies equally well to matrices.

3. The point of last week's homework was to visualize the placement of a rigid body at an arbitrary position and orientation. The point of this week's homework is to visualize the *motion* of this rigid body (and of a point on this rigid body) when it has an arbitrary linear and angular velocity. To do so, you will be adding code to the MATLAB script `hw2.m`, available on the course website. This script makes very clear exactly what you should and should not change. In particular, every line you should change is marked as follows:

```
<-- CHANGE THIS LINE (PROBLEM XXXX)
```

As you'll soon see, you'll need to change a lot of lines. However, none of them is hard. We will take them one at a time.

(a) Last week, you saw how to compute R_3^0 in terms of θ_1, θ_2, and θ_3 for a body-axis XYZ Euler Angle sequence. Implement this.

(b) Last week, you saw how to compute o_0^3 and R_0^3 in terms of o_3^0 and R_3^0. Implement this.

(c) Last week, you saw how to perform coordinate transformation to go back and forth between representations of points in frame 0 and in frame 3. Implement this.

(d) In Problem 1(f), you saw how to compute $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of $w_{0,3}^3$ for a body-axis XYZ Euler Angle sequence. Implement this.

(e) In Problem 2, you saw how to compute $\dot{p}^0$ in terms of $\dot{p}^3$. Implement this.

Your MATLAB script doesn't do anything yet. However, everything you've implemented so far will stay the same for all of the examples you'll be asked to consider—everything you'll implement next will change from one example to another. So, let's take a break. Submit your answer to this problem by printing out each line of code that you changed in the script.

4. In this problem, you will complete your changes to `hw2.m` and visualize three different examples of rigid body motion.

(a) Consider the following example:

i. Suppose frames 0 and 3 are initially aligned, so when the simulation starts, we have

$$o_3^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \theta_1 = \theta_2 = \theta_3 = 0.$$

Implement this.

ii. Suppose you want to simulate motion for 2π seconds. Implement this. (Hint: try typing "`pi`" in MATLAB.)

iii. Suppose you want a point p on the spacecraft to have constant coordinates

$$p^3 = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.125 \end{bmatrix}$$

in frame 3 for all time. Compute $\dot{p}^3$ as well, and implement.

iv. Suppose you want the spacecraft to have constant linear and angular velocity

$$\dot{o}_3^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_{0,3}^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Implement.

v. Run the script. What happens? Will the spacecraft ever return to its original position and orientation? What is the shape of the path followed by the point on the spacecraft? Along with your answers, turn in a snapshot of your figure after the simulation has ended.

It is very important that you complete all of Problem 3 and all of Problem 4(a).i-iv before running the simulation! Otherwise nothing will happen.

(b) Consider the following example:

i. Suppose frames 0 and 3 are initially aligned, exactly as before. Implement.

ii. Suppose you want to simulate motion for 2π seconds, exactly as before. Implement.

iii. Suppose you want p^3 and $\dot{p}^3$ to be the same as before, in part (a)-iii. Implement.

iv. Suppose you want the spacecraft to have constant linear and angular velocity

$$\dot{o}_3^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad w_{0,3}^3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Implement.

v. Run the script. What happens? Will the spacecraft ever return to its original position and orientation? What is the shape of the path followed by the point on the spacecraft? Along with your answers, turn in a snapshot of your figure after the simulation has ended. Again, it is important that you complete all of Problems 3 and 4(b).i-iv before running the simulation.

(c) Consider the following example:

i. Suppose frames 0 and 3 are initially aligned, exactly as before. Implement.

ii. Suppose you want to simulate motion for 2π seconds, exactly as before. Implement.

iii. Suppose you want

$$p^3 = \begin{bmatrix} 0.3\cos(2t) \\ 0.3\sin(2t) \\ 0.125 \end{bmatrix}.$$

Compute $\dot{p}^3$ as well, and implement.

iv. Suppose you want the spacecraft to have linear and angular velocity

$$\dot{o}_3^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad\qquad w_{0,3}^3 = 10e^{-t} \begin{bmatrix} \sin(t) \\ \sin(2t) \\ \sin(3t) \end{bmatrix}$$

Implement.

v. Run the script. What happens? What is the shape of the path followed by the point on the spacecraft in frame 3? What is it in frame 0? Along with your answers, turn in a snapshot of your figure after the simulation has ended. Again, it is important that you complete all of Problems 3 and 4(c).i-iv before running the simulation.

Play around! You have a really nice tool for visualization of rigid body motion.