Notes on Quantum Field Theory

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Lectures

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Disclaimer*: the material contained in these notes (still work in progress) is taken from the textbooks and lectures notes quoted in the bibliography. It has been written and updated during the lectures held in previous academic years, starting from 2012-2013. It is used as supporting material for the lectures on Quantum Field Theory, held at the University of Pavia, a.a. 2019-2020.*

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Introduction

This lecture notes are for the Quantum Field Theory course of the University of Pavia. It is a one-semester course and it is meant to follow and complete the course on QED. In that course you have learned how to quantize a field with the canonical formalism: the classical fields are interpreted as operators and commutation relations are imposed on the latters: commutation for fields describing bosonic fields and anticommutation for fields describing fermionic fields. Then you have seen the formalism applied to QED, i.e. fermionic and vectorial fields coupled by interaction. In presence of interaction only approximate methods can be adopted and you have developed QED at first order in perturbation theory, together with the powerful method of Feynman diagrams. In the end you were able to calculate several scattering processes cross sections in QED at tree level. Written in a sentence, the final goal of the present course is to get familiar with next term in the perturbative expansion in QED , i.e. in the up to one-loop approximation. Before entering the discussion of the loop diagrams, it is worth mentioning that the canonical method of field quantization becomes unpractical as soon as you move from QED to more involved field theories, such as non-abelian gauge theories, like QCD, the theory of strong interactions and the electroweak Standard Model, which are able to describe high energy data with high precision. These difficulties can be circumvented by adopting an alternative approach to quantization, developed by R. Feynman, following an idea of Dirac: the path-integral approach. This will be illustrated at the beginning in some detail for ordinary quantum mechanics, to move in the following to the bosonic fields, treated with functional integral methods. The Feynman rules will be rederived within this new formalism. The interaction will be introduced first for the case of the $\lambda\phi^4$

model, because the problems connected with the loop diagrams are already present here, without additional complications. We will see that, as soon as we try to calculate loop Feynman diagrams, we encounter the problem of ultraviolet divergences: the integrals are divergent for large momenta. Physical results can only be obtained after adopting a regularization procedure and carrying on the renormalization program of redefinition of parameters and fields. We will discuss in detail the dimensional regularization scheme and apply it to the $\lambda\phi^4$ model. Then we will move to QED, where we will see a powerful and elegant way of quantize the theory in the presence of gauge invariance. Concerning the fermionic part, we will go back to the beginning and develop the functional integral formalism through the use of Grassman variables. At this point we are ready to see the calculation of loop diagrams in QED. The first example will be the calculation of the $\mathcal{O}(\alpha)$ contribution to the electron anomalous moment, which is one of the most extraordinary tests of QED. Then we will see the complete one-loop renormalization program for QED and the calculation of a scattering process at one-loop order. The final part of the course will cover the renormalization group and its link to statistical mechanics.

1.1 Conventions

1.1.1 Natural units

In the c.g.s. system, the fundamental quantities are mass (M), length (L) and time (T). The system of natural units takes as fundamental quantities mass (M), action (A) and velocity (V), with units of action and velocity $\hbar$ and c, respectively. So in natural units

$$\begin{aligned} \hbar &= 1 \\ c &= 1\,. \end{aligned} \tag{1.1}$$

Since

$$A = ET = FLT = LMT^{-2}LT = MVL\,, \tag{1.2}$$

$$\begin{aligned} L &= \frac{A}{MV} \\ T &= \frac{A}{MV^2}\,, \end{aligned} \tag{1.3}$$

a quantity that in c.g.s. has dimensions

$$M^p L^q T^r = M^{p-q-r} A^{q+r} V^{-q-2r}\,, \tag{1.4}$$

in natural units has dimensions

$$M^{p-q-r}\,. \tag{1.5}$$

Many quantities have the same dimensions with natural units, e.g. energy, mass, momentum have dimension M, because

$$E^2 = m^2 + |\vec{p}|^2 = m^2 + |\vec{k}|^2\,. \tag{1.6}$$

Another relevant example is the fine structure constant

$$\alpha = \frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137}(c.g.s) \tag{1.7}$$

in natural units becomes dimensionless

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}(n.u.)\,. \tag{1.8}$$

In order to convert the dimension of a quantity in n.u. to c.g.s. it is enough to multiply by powers of $\hbar$ and c, respecting Eq. (1.4). The relevant numerical conversion factors are

$$\begin{aligned} \hbar &= 6.58 \times 10^{-22}\,\mathrm{MeV}\cdot\mathrm{s} \\ \hbar c &= 1.973 \times 10^{-11}\,\mathrm{MeV}\cdot\mathrm{cm}\,. \end{aligned} \tag{1.9}$$

1.1.2 Relativistic notation

A (contravariant) four-vector in Minkowsky space is denoted as

$$a^\mu = (a^0, a^1, a^2, a^3) = (a^0, \vec{a}) \tag{1.10}$$

The coordinate vector in c.g.s. units would be

$$x^\mu = (t/c, a_x, a_y, a_z)|_{\mathrm{c.g.s.}} \tag{1.11}$$

but in natural units it is simply

$$x^\mu = (t, a_x, a_y, a_z). \tag{1.12}$$

The metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.13}$$

Note that

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \tag{1.14}$$

Covariant vectors are

$$a_\mu = g_{\mu\nu} a^\nu = (a^0, -a^1, -a^2, -a^3) = (a^0, -\vec{a}), \tag{1.15}$$

Scalar products are

$$a \cdot b = a^\mu b_\mu = a^0 b^0 - \vec{a} \cdot \vec{b}. \tag{1.16}$$

Lorentz transformations will be denoted by $L^\mu{}_\nu$. The effect of a Lorentz transformation is to modify the vectors in this way

$$a^\mu \longrightarrow a'^\mu = L^\mu{}_\nu a^\nu. \tag{1.17}$$

Lorentz transformations leave scalar products unchanged

$$\begin{aligned} a^\mu a_\nu &= a'^\mu a'_\mu = L^\mu{}_\nu a^\nu L_\mu{}^\sigma a_\sigma \\ &\Rightarrow L^\mu{}_\nu L_\mu{}^\sigma = \delta^\sigma_\nu \\ &\Rightarrow L^{\mu\nu} L_{\mu\sigma} = \delta^\nu_\sigma \end{aligned} \tag{1.18}$$

Important: the four-dimensional gradient operator is defined as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) = (\partial_t, \vec{\nabla}). \tag{1.19}$$

Its tranformation property under a Lorentz transformation is

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \left(\Lambda^{-1} \right)^\mu_{\ \nu} \frac{\partial}{\partial x^\nu} \,. \tag{1.20}$$

Even if it is a covariant vector, the relation with the three-dimensional gradient operator is different from that of a normal vector. This guarantees that the differential of a scalar function $\phi(x)$ is a scalar

$$\delta\phi(x) = \partial_\mu \phi(x) \delta x^\mu. \tag{1.21}$$

The operator

$$\partial^\mu \partial_\mu = \partial_t^2 - \vec{\nabla}^2 \tag{1.22}$$

1.2 Brief summary: single particle wave equations

Quantum Field Theory stems from the necessity of reconciling Quantum Mechanics with the Special Relativity Theory. In fact the latter requires that the laws of physics describing a physical processes are formally the same in every inertial reference frame. In addition, all inertial reference frames are related by Lorentz transformations. The cornerstone of Quantum Mechanics, the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = \mathcal{H} \psi(\vec{x}, t) \,, \tag{1.23}$$

is not invariant under Lorentz tranformations. Space and time variables are not treated on the same ground.

Remember that the Schrödinger equation (e.g. for a free particle) can be obtained by taking the classical non-relativistic relation between energy and momentum

$$E = \frac{p^2}{2m} \tag{1.24}$$

and energy and momentum variables with the corresponding quantum operators

$$E \rightarrow i \frac{\partial}{\partial t}; \quad \vec{p} \rightarrow -i \vec{\nabla} \,. \tag{1.25}$$

1.2.1 The Klein Gordon equation

The first attempt is to apply the substitutions of Eq. (1.25) on the relativistic energy-momentum relation, which is in natural units ($\hbar = 1, c = 1$):

$$E^2 = |\vec{p}|^2 + m^2 \,. \tag{1.26}$$

This gives rise to the Klein Gordon wave equation

$$(\partial_\mu \partial^\mu + m^2)\psi(x) = 0 \,. \tag{1.27}$$

This equation, as a single particle wave equation presents some serious problems:

- both positive and negative energies are allowed. This is not acceptable when we introduce interactions because the particle, exchanging energy with the environment, could pass from positive energy states to arbitrary negative energies, emitting an infinite amount of energy;
- the temporal component of the conserved current

$$j^\mu = (\rho, \vec{j}) = \frac{i}{m}\left[\psi^* \partial^\mu \psi - (\partial^\mu \psi^*)\,\psi\right] \tag{1.28}$$

  is not positive definite, preventing a probabilistic interpretation (unlike $\rho = |\psi|^2$ for the Schrödinger equation);
- the second order derivative w.r.t. time conflicts with the principle of quantum mechanics, according to which the wave function contains all the information on the state of a physical system and therefore should be completely determined by its value at the initial time.

1.2.2 The Dirac equation

If the wave function at a certain time must contain all the information on the state, the wave equation should be of the first order w.r.t. time. On the other hand the relativistic framework requires that time and spatial coordinates appear in a simmetric way, so that only first order spatial derivatives should appear. On the other hand the solution of the hypothetic equation should be compatible with the Klein Gordon equation, which satisfies automatically the relativistic energy-momentum equation. To satisfy the above requirements, Dirac proposed the following wave equation

$$i\frac{\partial \psi}{\partial t} = \left(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m\right)\psi \,, \tag{1.29}$$

where α_i $(i = 1, 2, 3)$ and β are hermitean (since the Hamiltonian is hermitean) matrices $N \times N$ to be determined, and the wave function is a column vector with N components.

In order to comply with Eq. (1.29), the α_i and β matrices must obey the following relations

$$\begin{aligned}\{\alpha_i, \alpha_j\} &= 2\delta_{ij} \\ \{\alpha_i, \beta\} &= 0 \\ \alpha_i^2 &= \beta^2 = 1\,.\end{aligned} \tag{1.30}$$

This can be seen by requiring

$$(\vec{\alpha}\cdot\vec{p}+\beta m)\,(\vec{\alpha}\cdot\vec{p}+\beta m) = E^2|\vec{p}|^2 + m^2\,, \tag{1.31}$$

i.e.

$$\begin{aligned}&\alpha_i\alpha_j p_i p_j + m\,(\alpha_i\beta+\beta\alpha_i)\,p_i + \beta^2 m^2 \\ &= \frac{1}{2}\left(\{\alpha_i,\alpha_j\} + [\alpha_i\alpha_j]\right) p_i p_j + m\,\{\alpha_i,\beta\}\,p_i + \beta^2 m^2 \\ &= p_i p_j \delta_{ij} + m^2\,.\end{aligned} \tag{1.32}$$

The third relation of Eq. (1.30) implies that the eigenvalues of matrices α_i and β are all $= \pm 1$. From the first and third relations, instead, we have

$$\begin{aligned}\alpha_i &= \alpha_i\alpha_j\alpha_j = -\alpha_j\alpha_i\alpha_j \\ \beta &= \beta\alpha_j\alpha_j = -\alpha_j\beta\alpha_j\,,\end{aligned} \tag{1.33}$$

where the repeated indices are not summed. Taking the traces:

$$\begin{aligned}\mathrm{Tr}\,(\alpha_i) &= \mathrm{Tr}\,(\alpha_i\alpha_j\alpha_j) = \mathrm{Tr}\,(\alpha_j\alpha_i\alpha_j) = -\mathrm{Tr}\,(\alpha_j\alpha_i\alpha_j) = -\mathrm{Tr}\,(\alpha_i) \\ \mathrm{Tr}\,(\beta) &= \mathrm{Tr}\,(\beta\alpha_j\alpha_j) = \mathrm{Tr}\,(\alpha_j\beta\alpha_j) = -\mathrm{Tr}\,(\alpha_j\beta\alpha_j) = -\mathrm{Tr}\,(\beta)\end{aligned} \tag{1.34}$$

Since the matrices α_i and β have to be traceless and have eigenvalues $= \pm 1$, the dimension N can be only even. $N = 2$ is excluded because the standard 2×2 Pauli matrices σ_i

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,,\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\,,\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,, \tag{1.35}$$

satisfy the same anticommutation of Eq. (1.30), but they are a basis, together with the Identity matrix, for the 2×2 matrices. So that it is impossible to find a fourth independent matrix which anticommutes with σ_i. So the minimum dimensions for the α_i and β matrices is $N = 4$.

An explicit representation is the following one:

$$\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\,, \qquad \beta = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\,. \tag{1.36}$$

Usually the Dirac equation is written in terms of the γ matrices, defined in the following way:

$$\gamma^0 = \beta\,, \quad \gamma^i = \beta\alpha_i = \gamma^0\alpha_i\,, \tag{1.37}$$

which satisfy the anticommutation rules

$$\{\gamma^\mu\gamma^\nu\} = 2g^{\mu\nu}\,. \tag{1.38}$$

The γ matrices have the following properties

$$\left(\gamma^0\right)^2 = \mathbb{1}\,, \tag{1.39}$$
$$\left(\gamma^i\right)^2 = -\mathbb{1} \tag{1.40}$$
$$\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0\,. \tag{1.41}$$

Note that the γ matrices are not hermitean.
A possible explicit representation of the γ matrices is the Dirac-Pauli representation (useful for massive particles and for taking the non relativistic limit)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{1.42}$$

In terms of the γ matrices the Dirac equation assumes the form

$$\left(i\gamma^\mu\partial_\mu - m\mathbb{1}\right)\psi = 0\,. \tag{1.43}$$

What kind of particles are described by the Dirac equation? If we consider the Hamiltonian of Eq. (1.29)

$$H = \vec{\alpha}\cdot\vec{p} + \beta m\,, \tag{1.44}$$

it is easy to check that it does not commute with the angular momentum operator $\vec{L} = \vec{x}\times\vec{p}$. E.g.

$$[H, L_3] = [\alpha_i p_i, x_1 p_2 - p_2 x_1] = \alpha_1 p_2 [p_1, x_1] - \alpha_2 p_1 [p_2, x_2] = -i\left(\alpha_1 p_2 - \alpha_2 p_1\right)\,. \tag{1.45}$$

However, let us consider the operator

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{1.46}$$

which, heuristically, can be interpreted as the extension to four dimensions of the quantum mechanical spin operator, represented by the Pauli matrices [1]. By defining the antisymmetric tensor

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] , \tag{1.47}$$

its spatial components are

$$\sigma^{ij} = \frac{i}{2}\left[\gamma^{i}, \gamma^{j}\right] . \tag{1.48}$$

Using the Dirac-Pauli representation of the γ matrices, and considering that

$$\begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^{i}\sigma^{j} & 0 \\ 0 & -\sigma^{i}\sigma^{j} \end{pmatrix} , \tag{1.49}$$

we get

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] - \frac{i}{2}\begin{pmatrix} \left[\sigma^{i}, \sigma^{j}\right] & 0 \\ 0 & \left[\sigma^{i}, \sigma^{j}\right] \end{pmatrix} = \varepsilon^{ijk}\begin{pmatrix} \sigma_{k} & 0 \\ 0 & \sigma_{k} \end{pmatrix} = \varepsilon^{ijk}\Sigma_{k} . \tag{1.50}$$

From Eq. (1.50) we can write

$$\begin{aligned} \sigma^{12} &= \varepsilon^{123}\Sigma_{3} \\ \sigma^{21} &= \varepsilon^{213}\Sigma_{3} = -\varepsilon^{123}\Sigma_{3} . \end{aligned} \tag{1.51}$$

From the above equation and form Eq. (1.48) we can write

$$\sigma^{21} - \sigma^{21} = 2\Sigma_{3} = \frac{i}{2}\left[\gamma^{1}, \gamma^{2}\right] - \frac{i}{2}\left[\gamma^{2}, \gamma^{1}\right] = i\left[\gamma^{1}, \gamma^{2}\right] , \tag{1.52}$$

i.e., using Eq. (1.37)

$$\Sigma_{3} = \frac{i}{2}\left[\gamma_{1}, \gamma_{2}\right] = \frac{i}{2}\left[\gamma^{0}\alpha_{1}, \gamma^{0}\alpha_{2}\right] = -\frac{i}{2}\left(\alpha_{1}\alpha_{2} - \alpha_{2}\alpha_{1}\right) . \tag{1.53}$$

Remembering Eq. (1.45), let us calculate the following commutator of Σ_3 with the Hamiltonian

$$\begin{aligned} \frac{1}{2}\left[H, \Sigma_{3}\right] &= -\frac{i}{4}\left[\alpha_{i}p_{i}, \alpha_{1}\alpha_{2} - \alpha_{2}\alpha_{1}\right] = -\frac{i}{2}\left[\alpha_{i}p_{i}, \alpha_{1}\alpha_{2}\right] \\ &= -\frac{i}{2}\left[\alpha_{1}p_{1}, \alpha_{1}\alpha_{2}\right] - \frac{i}{2}\left[\alpha_{2}p_{2}, \alpha_{1}\alpha_{2}\right] \\ &= -\frac{i}{2}\left(\alpha_{1}^{2}\alpha_{2} - \alpha_{1}\alpha_{2}\alpha_{1}\right)p_{1} - \frac{i}{2}\left(\alpha_{2}\alpha_{1}\alpha_{2} - \alpha_{1}\alpha_{2}^{2}\right)p_{2} \\ &= -i\left(\alpha_{2}p_{1} - \alpha_{1}p_{2}\right) = -\left[H, L_{3}\right] . \end{aligned} \tag{1.54}$$

[1] Actually the correct relativistic operator is given by the Pauli-Lubanski operator $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^{\sigma}$, where $J_{\mu\nu}$ is the angular momentum operator, generator of rotations, and P_μ is the generator of the spatial traslations. For a particle of mass m we can perform a Lorentz transformation to go to the rest frame, where the only contribution is $\sigma = 0$

By defining the **total angular momentum** as

$$\vec{J} \equiv \vec{L} + \frac{1}{2}\vec{\Sigma}\,, \tag{1.55}$$

we have

$$[H, J_3] = \left[H, L_3 + \frac{1}{2}\Sigma_3\right] = 0\,, \tag{1.56}$$

i.e. $\frac{1}{2}\Sigma_3$ is the third component of the spin of of the particle described by the Dirac equation. Since the eigenvalues of Σ_3 are ± 1, the Dirac equation describes particles with spin $\frac{1}{2}$.

Eq. (1.54) shows that, when the momentum $\vec{p}$ is different from zero, the projection of spin on a generic axis is not a constant of the motion.

An important quantity is the **projection of the spin on the momentum direction, known as the helicity**

$$\sigma_p = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\,. \tag{1.57}$$

which is a constant of the motion, i.e. it commutes with the Hamiltonian:

$$\left[H, \sigma_p\right] = \frac{1}{|\vec{p}|}\left[\alpha_i p_i + \beta m, \Sigma_j p_j\right] = \frac{1}{|\vec{p}|}\left[\alpha_i p_i, \Sigma_j p_j\right] = \frac{1}{|\vec{p}|}\left[\alpha_i, \Sigma_j\right] p_i p_j\,. \tag{1.58}$$

From Eq. (1.53), by permutation of the indices we have

$$\begin{aligned}
\Sigma_1 &= -\frac{i}{2}\left(\alpha_2\alpha_3 - \alpha_3\alpha_2\right) = -i\alpha_2\alpha_3 \\
\Sigma_2 &= -\frac{i}{2}\left(\alpha_3\alpha_1 - \alpha_1\alpha_3\right) = -i\alpha_3\alpha_1 \\
\Sigma_3 &= -\frac{i}{2}\left(\alpha_1\alpha_2 - \alpha_2\alpha_1\right) = -i\alpha_1\alpha_2\,.
\end{aligned} \tag{1.59}$$

In Eq. (1.58), the three terms of the form $[\alpha_i, \Sigma_i]$ are 0 because they are of the form $-i(\alpha_i\alpha_j\alpha_k - \alpha_j\alpha_k\alpha_i) = 0$ by anticommutation. Then we have the following six terms

$$\begin{aligned}
[\alpha_1, \Sigma_2] &= -i\,[\alpha_1, \alpha_3\alpha_1] = +2i\alpha_3 \\
[\alpha_1, \Sigma_3] &= -i\,[\alpha_1, \alpha_1\alpha_2] = -2i\alpha_2 \\
[\alpha_2, \Sigma_1] &= -i\,[\alpha_2, \alpha_2\alpha_3] = -2i\alpha_3 \\
[\alpha_2, \Sigma_3] &= -i\,[\alpha_2, \alpha_1\alpha_2] = +2i\alpha_1 \\
[\alpha_3, \Sigma_1] &= -i\,[\alpha_3, \alpha_2\alpha_3] = +2i\alpha_2 \\
[\alpha_3, \Sigma_2] &= -i\,[\alpha_3, \alpha_3\alpha_1] = -2i\alpha_1\,.
\end{aligned} \tag{1.60}$$

So the terms $\left[\alpha_i, \Sigma_j\right]$ are antisymmetric w.r.t. exchange of i and j. Since they are multiplied (and summed over i and j) by the symmetric term $p_i p_j$, we can conclude

$$\left[H, \sigma_p\right] = 0\,. \tag{1.61}$$

So in the end, the solutions of the Dirac equation are four-component spinors, describing states with spin $\frac{1}{2}$ and positive and *negative* energy.

1.3 Canonical field quantization

Another heuristic argument conflicting with the hypothesis of a wave function for a relativistic particle is the following: if we try to measure the position of an electron with a microscope, we have to use light with wave length λ (the greater the precision, the smaller λ). On the other hand the electron receives a random momentum of the order of the momentum of the photon $k = \frac{\hbar}{\lambda}$. The indeterminacy on the position and momentum must satisfy the indetermination principle

$$\Delta x \sim \lambda = \frac{\hbar}{\Delta p} . \tag{1.62}$$

When the energy of the photon is greater than the threshold $2m_e$ we can have the creation of a pair e^+e^-, so that the concept of position of the electron becomes ambiguous since we have two electrons.
In essence, our theoretical formulation should be able to describe processes with variable number of particles.
The only way out is to abandon the idea of single particle wave equation and consider that the equation (Klein-Gordon or Dirac) describes a (classical) field. In this way we are going to pass from a system with a discrete number of degrees of freedom to a continuum system, with an infinite number of degrees of freedom. **In the field theory approach, space $\vec{x}$ and time t are parameters and the value of the field $\phi(\vec{x}, t)$ is the dynamical variable (the analogous role played by $\vec{x}(t)$ in non-relativistic quantum mechanics).** Actually the fields are distributions and not observables. The latter are bilinear in the fields.

1.3.1 The real Klein-Gordon field

The classical field theory is defined when we spcify the Lagrangian density $\mathcal{L}$. For the case of the real scalar field, we have

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) , \tag{1.63}$$

which gives the Klein-Gordon equation as Euler-Lagrange equation, from the requirement of minimum variation of the Action $S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$. Since the Lagrangian density is Lorentz-invariant and has no explicit dependence on x, the Noether theorem guarantees the conservation of four-momentum and angular momentum. The energy-momentum tensor is

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L} . \tag{1.64}$$

The (classical) energy of the field is given by the space integral of T^{00}:

$$H = \int T^{00} \, d^3x = \frac{1}{2} \int \left[(\partial_0 \phi)^2 + \vec{\nabla} \phi \cdot \vec{\nabla} \phi + m^2 \phi^2 \right] . \tag{1.65}$$

From Eq. (1.65), we can see that the scalar field, whose equation of motion is the Klein-Gordon equation, is not affected by the negative energy problem.
Up to now we don't have yet a connection between the field and particles. This is achieved through the field quantization procedure, named also **second quantization**: the field is regarded as an Hermitean linear operator (in the Heisenberg representation), which satisfies canonical equal time commutation relations, analogously to the commutation relations of ordinary quantum mechanics ($[x_i, p_j] = i\delta_{ij}$).

It is convenient to use the following Fourier representation

$$\phi(x) = \phi^+(x) + \phi^-(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2V2\omega_k}} \left[a(k)e^{-ikx} + a^\dagger(k)e^{+ikx} \right] , \tag{1.66}$$

with $\omega_k^2 = |\vec{k}|^2 + m^2$, as imposed by the equation of motion.
Observations:

- ϕ^+ contains the positive frequencies while ϕ^- contains the negative frequencies;
- $a(k)$ and $a^\dagger(k)$ are operators which inherit the commutation rules from the field and its canonically conjugate momentum.

The **canonically conjugate momentum** is given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} . \tag{1.67}$$

The equal time commutation relations are

$$\begin{aligned} \left[\phi(\vec{x}, t), \pi(\vec{x}', t)\right] &= i\delta^{(3)}(\vec{x} - \vec{x}') \\ \left[\phi(\vec{x}, t), \phi(\vec{x}', t)\right] &= \left[\pi(\vec{x}, t), \pi(\vec{x}', t)\right] = 0 . \end{aligned} \tag{1.68}$$

Eq. (1.68) implies the following commutation relations for the ladder operators a and $a^\dagger$

$$\begin{aligned} \left[a(k), a^\dagger(k')\right] &= \delta_{\vec{k}\vec{k}'} \\ \left[a(k), a(k')\right] &= \left[a^\dagger(k), a^\dagger(k')\right] = 0 . \end{aligned} \tag{1.69}$$

The above commutation relations are exactly the ones of the harmonic oscillator. **The scalar quantum field is equivalent to a collection of harmonic oscillators, one for each mode** $\vec{k}$. We can define the operator

$$N(\vec{k}) = a^\dagger(\vec{k})a(\vec{k}) . \tag{1.70}$$

with the properties

$$
\begin{aligned}
\left[N(\vec{k}), a^{\dagger}(\vec{k})\right] &= a^{\dagger}(\vec{k}) \\
\left[N(\vec{k}), a(\vec{k})\right] &= -a(\vec{k})\,.
\end{aligned}
\tag{1.71}
$$

We can take the eigenstates of the operator N, $|n(\vec{k})\rangle$ and verify that if $|n(\vec{k})\rangle$ has eigenvalue $n(\vec{k})$, the states $a^{\dagger}(\vec{k})(\vec{k})$ and $a(\vec{k})(\vec{k})$ have eigenvalues $n(\vec{k})+1$ and $n(\vec{k})-1$, respectively.

In terms of ladder operators, the Hamiltonian of the field of Eq. (1.65) becomes

$$
H = \frac{1}{2}\sum_{\vec{k}} \omega_k \left(a(\vec{k})a^{\dagger}(\vec{k}) + a^{\dagger}(\vec{k})a(\vec{k})\right) = \sum_{\vec{k}} \omega_k \left(N(\vec{k}) + \frac{1}{2}\right), \tag{1.72}
$$

and the momentum of the field

$$
P^i = \int d^3x\, T^{0i} = \int d^3x\, \partial_0\phi\partial^i\phi = \frac{1}{2}\sum_{\vec{k}} k^i \left(a(\vec{k})a^{\dagger}(\vec{k}) + a^{\dagger}(\vec{k})a(\vec{k})\right) = \sum_{\vec{k}} k^i \left(N(\vec{k}) + \frac{1}{2}\right). \tag{1.73}
$$

Eqs. (1.72,1.73) suggest that the operator $N(\vec{k})$ is the number operator of particles with momentum $\vec{k}$ and energy $\omega_{\vec{k}}$, provided the eigenvalues $n(\vec{k})$ never become negative. This can be shown by looking at the norm of the generic state $a(\vec{k})|n(\vec{k})\rangle$:

$$
\begin{aligned}
\left[a(\vec{k})|n(\vec{k})\rangle\right]^{\dagger}\left[a(\vec{k})|n(\vec{k})\rangle\right] &= \langle n(\vec{k})|a^{\dagger}(\vec{k})a(\vec{k})|n(\vec{k})\rangle \\
&= n(\vec{k})\langle n(\vec{k})|n(\vec{k})\rangle > 0\,.
\end{aligned}
\tag{1.74}
$$

The Hilbert space on which the field operator act is given by the tensorial product of the states of the different oscillators. To this space belong:

- the vacuum state $|0\rangle$, where all oscillators are in the ground state; the vacuum state is determined by the condition of being annihilated by the destruction operator $a(\vec{k})|0\rangle = 0$

- states with different excitation numbers of the various oscillators, obtained by acting on the vacuum with the *creation operators* $a^{\dagger}$

$$
|n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2!, \ldots}}\left[a^{\dagger}(\vec{k}_1)\right]^{n_1}\left[a^{\dagger}(\vec{k}_2)\right]^{n_2}\ldots|0\rangle\,. \tag{1.75}
$$

Since there is no limit to the occupation numbers, **the real scalar field describes a system of bosonic particles**. This is a direct consequence of the canonical **commutation relations**.

The step from the classical observables (e.g. energy, momentum) to the quantum versions has an intrinsic ambiguity, since we transform products of commuting variables into products of non-commuting operators. The consequence is for instance that we have an infinite energy, given by the sum of all ground state energies of the oscillators. Since we are free to fix the zero energy level, we can remove this infinity by adopting the **Normal ordering prescription**: in each normal ordered product of operators the absorption operators always stand to the right of creation operators. Example (remember that $\phi^+(x)$ contains the absorption operator):

$$\begin{aligned} N\left[\phi(x)\phi(y)\right] &= \left(\phi^+(x) + \phi^-(x)\right)\left(\phi^+(y) + \phi^-(y)\right) \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) \\ &= +\phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \end{aligned} \tag{1.76}$$

1.3.2 The complex Klein Gordon field

The classical field theory of a complex scalar field is equivalent to the one of two real scalar fields ϕ_1 and ϕ_2. In fact we may consider

$$\begin{aligned} \phi &= \frac{(\phi_1 + i\phi_2)}{\sqrt{2}} \\ \phi^* &= \frac{(\phi_1 - i\phi_2)}{\sqrt{2}} \end{aligned} \tag{1.77}$$

Treating ϕ and ϕ^* as two independent fields, the Lagrangian density (which is real, to give a real Action) is

$$\mathcal{L} = (\partial_\mu \phi)\,(\partial^\mu \phi^*) - m^2 \phi^* \phi\,. \tag{1.78}$$

In terms of the real fields ϕ_1 and ϕ_2, the Lagrangian density would be

$$\mathcal{L} = \left(\partial_\mu \vec{\phi}\right) \cdot \left(\partial^\mu \vec{\phi}\right) - m^2 \vec{\phi} \cdot \vec{\phi}\,, \tag{1.79}$$

where $\vec{\phi}$ is a two dimensional vector with components $\phi_1(x)$ and $\phi_2(x)$.

The corresponding Euler-Lagrange equations are two independent Klein-Gordon equations for ϕ and ϕ^*. The treatment is similar to the case of the real scalar field, with two degrees of freedom instead of one. However, the Lagrangian density of Eq. (1.78) shows an **additional "internal" simmetry w.r.t. to Eq. (1.63) of the real Klein-Gordon field**: the Lagrangian density is invariant under the transformation

$$\begin{aligned} \phi(x) &\to e^{-i\alpha}\phi(x) \\ \phi^*(x) &\to e^{i\alpha}\phi^*(x)\,, \end{aligned} \tag{1.80}$$

where α is a "global" phase, (global because it does not depend on x). The transformation $\in$ group $U(1)$. Adopting the degrees of freedom ϕ_1 and ϕ_2 the same tranformation would be a rotation in two dimensions ($\in$ group $SO(2)$) by an angle α, which mixes the components ϕ_1 and ϕ_2.

Through the Noether theorem, this symmetry has an associated conserved current ($\partial_\mu J^\mu = 0$):

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^*\right)}\delta\phi^* = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) . \tag{1.81}$$

The corresponding conserved quantity is the space integral of the temporal component J^0

$$Q = \int J^0\, d^3x = i\int\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) . \tag{1.82}$$

At this point Q is a purely classical charge. The connection with the electromagnetic charge can be seen in the following way: if we perform a global transformation we are propagating a faster than light signal, which is not compatible with special relativity. Instead we should admit the possibility of changing the phase locally. In this way the Lagrangian density of Eq. (1.78) is not invariant, but for the time being let us stay with the global tranformation.
Observation: Charged scalar fields can only be described by complex scalar fields. For a real field the conserved current does not exist, $J^\mu = 0$.

The quantization proceeds in the same way as for the real scalar field, but the quantum field is not Hermitean anymore:

$$\begin{aligned}
\phi(x) &= \phi^+(x) + \phi^-(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2V2\omega_k}}\left[a(k)e^{-ikx} + b^\dagger(k)e^{+ikx}\right] \\
\phi^\dagger(x) &= \phi^{\dagger+}(x) + \phi^{\dagger-}(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2V2\omega_k}}\left[b(k)e^{-ikx} + a^\dagger(k)e^{+ikx}\right] ,
\end{aligned} \tag{1.83}$$

session to be continued

1.3.3 The Dirac field

session to be written

1.4 Propagators, Green functions and causality in quantum field theory

In the canonical formalism, the propagator is defined as the vacuum expectation value of the time-ordered product of two field operators. For instance, the propagator for the scalar field is

$$i\Delta(x,y) = \langle 0|T\left(\varphi(x)\varphi^\dagger(y)\right)|0\rangle , \tag{1.84}$$

i.e.

$$
\begin{aligned}
i\Delta(x,y) &= \langle 0|\varphi(x)\varphi^\dagger(y)|0\rangle \quad \text{for } x^0 > y^0\,,\\
i\Delta(x,y) &= \langle 0|\varphi^\dagger(y)\varphi(x)|0\rangle \quad \text{for } y^0 > x^0\,.
\end{aligned}
$$

It gives the quantum amplitude for the creation of a quantum by a source localized at y and its absorption in x, if $x^0 > y^0$. Instead, if $x^0 < y^0$, it describes the creation of an antiparticle in x and its absorption in y. A general property of the propagator in a translationally invariant quantum field theory is its dependence on the difference on coordinates. In fact, if we insert in Eq. (1.84) the identity given by the product $U^\dagger U$, where U is the translation operation of $-y$ and take into account that the vacuum is translationally invariant, we get

$$
\begin{aligned}
i\Delta(x,y) &= \langle 0|U^\dagger U T\left(\varphi(x)U^\dagger U\varphi^\dagger(y)\right)U^\dagger U|0\rangle\\
&= \langle 0|T\left(\varphi(x-y)\varphi^\dagger(0)\right)|0\rangle\\
&= i\Delta(x-y,0) \equiv i\Delta(x-y)\,. \qquad (1.85)
\end{aligned}
$$

Introducing the positive and negative frequency parts of the field $\varphi(x)^{(+)}$ [2] given by [3]

$$
\varphi^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_{\mathbf{k}} e^{-i(k\cdot x)}\,, \qquad (1.86)
$$

$$
\varphi^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} b^\dagger_{\mathbf{k}} e^{+i(k\cdot x)}\,, \qquad (1.87)
$$

we have

$$
\begin{aligned}
i\Delta(x) &= \langle 0|\left[\varphi^{(+)}(x)\left(\varphi^\dagger\right)^{(-)}(0)\right]|0\rangle \quad \text{for } x^0 > 0\,,\\
i\Delta(x) &= -\langle 0|\left[\varphi^{(-)}(x)\left(\varphi^\dagger\right)^{(+)}(0)\right]|0\rangle \quad \text{for } x^0 < 0\,.
\end{aligned}
$$

It is useful to introduce the functions $\Delta^{(+)}(x)$ and $\Delta^{(-)}(x)$, defined by means of the above equations:

$$
i\Delta^{(+)}(x) = \langle 0|\left[\varphi^{(+)}(x),\left(\varphi^\dagger\right)^{(-)}(0)\right]|0\rangle\,, \qquad (1.88)
$$

$$
i\Delta^{(-)}(x) = -\langle 0|\left[\varphi^{(-)}(x),\left(\varphi^\dagger\right)^{(+)}(0)\right]|0\rangle\,. \qquad (1.89)
$$

[2]The positive frequency part is the one with temporal evolution $e^{-i\omega t}$ (and is multiplied by the annihilation operator), while the negative frequency part has the temporal evolution $e^{+i\omega t}$ (and is multiplied by the creation operator).

[3]Notice that we use here the normalization of continuum.

By using Eqs. (1.86) and (1.87), we can verify that $\Delta^{(+)}$ and $\Delta^{(-)}$ are solution of the Klein-Gordon equation of Eq. (1.27). In particular, using the Cauchy theorem, we have:

$$i\Delta^{(+)}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega}e^{-i(k\cdot x)}, \tag{1.90}$$
$$i\Delta^{(-)}(x) = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{2\omega}e^{+i(k\cdot x)}. \tag{1.91}$$

At this point it is useful to make a **digression on Green functions**. We can consider the equation of motion of the Klein-Gordon field in the presence of a given external source

$$(-\partial_\mu\partial^\mu - m^2)\varphi(x) = J(x)\,. \tag{1.92}$$

The Klein-Gordon equation of Eq. (1.27) is the omogeneus equation associated to Eq. (1.92). The solutions of Eq. (1.92) can be obtained through the Green function of the problem, which is the solution of the same equation, with the external source given by a Dirac delta localized at the origin of space-time:

$$(-\partial_\mu\partial^\mu - m^2)G(x) = \delta(x)\,. \tag{1.93}$$

Knowing $G(x)$, the solution of Eq. (1.92) is given by

$$\varphi(x) = \int dy\, G(x-y)J(y)\,. \tag{1.94}$$

The number of Green functions associated with Eq. (1.92) is infinite. A particular Green function can be singled out fixing the boundary conditions. Eq. (1.93) can be solved through Fourier transforms:

$$\tilde{f}(k) = \int d^4x\, f(x)e^{i(k\cdot x)}$$
$$f(x) = \frac{1}{(2\pi)^4}\int d^4k\tilde{f}(k)e^{-i(k\cdot x)}\,.$$

Eqs. (1.93) and (1.94) become

$$\tilde{G}(k) = \frac{1}{k^2-m^2}, \tag{1.95}$$
$$\tilde{\varphi}(k) = \tilde{G}(k)\cdot\tilde{J}(k)\,. \tag{1.96}$$

By means of Eq. (1.96) we can write a formal particular solution of Eq. (1.92):

$$\varphi(x) = \int d^4k\, e^{-i(k\cdot x)}\frac{1}{k^2-m^2}\tilde{J}(k)\,. \tag{1.97}$$

The integration of Eq. (1.97) presents singularities along the real axis in the points corresponding to the propagation of free waves: $k^0 = \pm\omega$. Every particular solution can be found by giving a particular path in the complex plane $(\mathrm{Re}(k^0), \mathrm{Im}(k^0))$ to define the integral. The Feynman prescription is the one for which $i\Delta = +i\Delta^{(+)}(x)$ when $t > 0$ and $i\Delta = -i\Delta^{(-)}(x)$ when $t < 0$. This prescription can be obtained by shifting the singularity at $k^0 = \omega$ by a negative infinitesimal imaginary part and the one at $k^0 = -\omega$ by a positive infinitesimal imaginary part. For $x^0 > 0$ we can choose a closed path enclosing the singularity at $k^0 = \omega$ (we close the path in the lower semiplane), finding

$$i\Delta_F(x) = i\Delta^{(+)}(x)\,. \tag{1.98}$$

For $x^0 < 0$ we can choose a closed path enclosing the singularity at $k^0 = -\omega$ (we close the path in the upper semiplane), finding

$$i\Delta_F(x) = -i\Delta^{(-)}(x)\,. \tag{1.99}$$

The general form of the Feynman propagator is

$$i\Delta_F(x) = \vartheta(x^0)i\Delta^{(+)}(x) - \vartheta(-x^0)i\Delta^{(-)}(x)\,. \tag{1.100}$$

In summary, the Feynman propagator is given by

$$i\Delta_F(x) \quad = \quad \langle 0|T\left(\varphi(x)\varphi^\dagger(0)\right)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\varepsilon}e^{-i(k\cdot x)}\,. \tag{1.101}$$

The Feynman prescription amounts to give an infinitesimal negative imaginary part to the mass: $m^2 \to m^2 - i\varepsilon$. This prescription fixes the way of handling the singularities on the real axis imposing that positive frequencies are propagated forward in time and negative frequencies are propagated backward in time. In fact, for positive time, the path has to be closed in the lower semiplane and the solution singles out the pole at $k^0 = +\omega$; for negative time, the path has to be closed in the upper semiplane and the solution singles out the pole at $k^0 = -\omega$. This complies with the Feynman-Stueckelberg interpretation: a particle with positive energy is propagated forward in time and a quantum with negative energy backward in time, i.e. an antiparticle with positive energy forward in time. We will see that the same prescription will ensure the convergence of the path-integral formulation of quantum field theory.

2

Feynman path-integral quantization in quantum mechanics

2.1 Introduction

During the QED course you have seen how to quantize field theories with the canonical quantization approach, with particular reference to QED. Since in this formalism the manifest Lorentz invariance is broken, it becomes very difficult to apply to more complex theories, like gauge theories. For this reason an alternative method has been developed by Feynman, based on an idea of Dirac: the path integral approach. *The main idea of the method is based on the superposition principle, which is at the roots of quantum mechanics*. For this reason we will firstly illustrate the path integral approach for quantum mechanics, with particular reference to the harmonic oscillator. The importance of this system for quantum field theory is due to the fact that a bosonic field is dynamically equivalent to a set of an infinite number of harmonic oscillators (normal modes). We remember that the classical Hamiltonian of the free oscillator is

$$\mathcal{H}_{cl} = \frac{p^2}{2m} + \frac{1}{2}kq^2 \,, \tag{2.1}$$

which, upon quantization, becomes the operator

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{1}{2}m\omega_0^2 q^2 \,. \tag{2.2}$$

For simplicity of notation, we will subtract to Eq. (2.2) the zero point energy $\frac{1}{2}\hbar\omega_0$ and use natural units and $m = 1$. Hence we have

$$\mathcal{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}\omega_0^2 q^2 - \frac{1}{2}\omega_0 \,. \tag{2.3}$$

The eigenfunctions of $\mathcal{H}$ are expressed in terms of the Hérmite polinomials as follows

$$\langle q|n\rangle = \Phi_n(q) = \omega_0^{\frac{1}{4}}\pi^{-\frac{1}{4}}2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\omega_0 q^2\right)H_n\left(\omega_0^{\frac{1}{2}}q\right) \tag{2.4}$$

and the Hérmite polinomial of order n is given by

$$H_n(z) = (-)^n \exp\left(z^2\right)\left(\frac{d}{dz}\right)^n \exp\left(-z^2\right). \tag{2.5}$$

The eigenvalues of $\mathcal{H}$ are

$$E_n = n\omega_0 \tag{2.6}$$

2.2 The propagator as the Green function of the Schrödinger equation

Before entering the details of the Feynman quantization of the harmonic oscillator, we show that **the knowledge of the solution of the Schrödinger equation for a generic non-relativistic system, the wave function $\psi(\vec{q},t)$, is equivalent to the knowledge of the Green function of the Schrödinger equation with a particular initial condition**. Limiting ourselves to a one-dimensional system, for instance a particle in a one-dimensional potential, the Schrödinger equation reads

$$\left(i\hbar\frac{\partial}{\partial t} - H\right)\psi(q,t) = 0 \,, \tag{2.7}$$

where $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)$. Next we consider the function $K(q',t';q,t)$, also called the **propagator**, defined as the solution of the equation

$$\left(i\hbar\frac{\partial}{\partial t'} - H_{q'}\right)K(q',t';q,t) = i\hbar\delta(q'-q)\delta(t'-t)\,. \tag{2.8}$$

with initial condition

$$K(q',t;q,t) = \delta(q'-q)\,. \tag{2.9}$$

By means of the Green function $K(q', t'; q, t)$, we can write the wave function, solution of the Schrödinger equation of Eq. (2.7), as

$$\psi(q', t') = \vartheta(t' - t) \int dq\, K(q', t'; q, t)\psi(q, t)\,. \tag{2.10}$$

Eq. (2.10) expresses the Huygen's principle. We can verify that the function of Eq. (2.10) is solution of Eq. (2.7) by substitution:

$$\begin{aligned}
\left(i\hbar\frac{\partial}{\partial t} - H\right)\psi(q,t) &= \vartheta(t - t_i)\left(i\hbar\frac{\partial}{\partial t} - H\right)\int dq_i\, K(q, t; q_i, t_i)\psi(q_i, t_i)\\
&= \vartheta(t - t_i)\int dq_i \left(i\hbar\frac{\partial}{\partial t} - H_q\right) K(q, t; q_i, t_i)\psi(q_i, t_i)\\
&= \vartheta(t - t_i)\int dq_i i\hbar\delta(q - q_i)\delta(t - t_i)\psi(q_i, t_i)\\
&= i\hbar\delta(t - t_i)\psi(q, t_i)\vartheta(t - t_i) = 0 \quad \text{for every } t > t_i\,.
\end{aligned} \tag{2.11}$$

Thus the ψ defined by Eq. (2.10) is a solution of the Schrödinger equation for all times $t > t_i$. The restriction on the times preserves causality.
From the second line of Eq. (2.11) we can see that for $t' < t$ must obey the following equation:

$$\left(i\hbar\frac{\partial}{\partial t'} - H_{q'}\right) K(q', t'; q, t) = 0 \quad \text{for every } t\,, \tag{2.12}$$

with the initial condition of Eq. (2.9).
We can find an **explicit form for the propagator** by means of the solutions of the stationary Schrödinger equation $\varphi_n(q)$ and the corresponding eigenvalues E_n. Since the $\varphi_n(q)$ form a complete system, $K(q', t'; q, t)$ can be expanded in this basis:

$$K(q', t'; q, t) = \vartheta(t' - t)\sum_n a_n \varphi_n(q') e^{-\frac{i}{\hbar}E_n t'}\,, \tag{2.13}$$

where we have directly introduced the constraints on forward times. The expansion coefficients depend in general on q and t: $a_n = a_n(q, t)$. Because of the initial condition of Eq. (2.9), we have

$$K(q', t; q, t) = \delta(q' - q) = \sum_n a_n(q, t)\varphi_n(q')e^{-\frac{i}{\hbar}E_n t}\,. \tag{2.14}$$

Since $\delta(q' - q)$ is time-independent, the same is true also for the r.h.s. of Eq. (2.14). This implies that

$$a_n(q, t) = a_n(q)e^{+\frac{i}{\hbar}E_n t}\,, \tag{2.15}$$

i.e.

$$\delta(q' - q) = \sum_n a_n(q)\varphi_n(q')\,, \tag{2.16}$$

which is fulfilled by

$$a_n(q) = \varphi_n^*(q)\,. \tag{2.17}$$

Putting everything together,

$$K(q',t';q,t) = \vartheta(t'-t)\sum_n \varphi_n^*(q)\varphi_n(q')e^{-\frac{i}{\hbar}E_n(t'-t)}\,, \tag{2.18}$$

Observing that $\varphi_n(q) = \langle q|n\rangle$, we can write

$$\begin{aligned}
K(q',t';q,t) &= \vartheta(t'-t)\sum_n \varphi_n^*(q)\varphi_n(q')e^{-\frac{i}{\hbar}E_n(t'-t)} \\
&= \vartheta(t'-t)\sum_n \langle n|q\rangle e^{-\frac{i}{\hbar}E_n(t'-t)}\langle q'|n\rangle \\
&= \vartheta(t'-t)\sum_n \langle n|e^{+\frac{i}{\hbar}Ht}|q\rangle\langle q'|e^{-\frac{i}{\hbar}Ht'}|n\rangle \\
&= \vartheta(t'-t)\langle q'|e^{-\frac{i}{\hbar}H(t'-t)}|q\rangle \equiv \vartheta(t'-t)\langle q'|U(t',t)|q\rangle\,.
\end{aligned} \tag{2.19}$$

Thus the propagator is the time devolpment operator for $t' > t$ in position representation.

2.3 Temporal evolution in position representation

Since the Feynman quantization is based on the study of the temporal evolution of the field in the position representation, **we consider now the temporal evolution of the harmonic oscillator**. Given at $t = 0$ the system in the position eigenstate $|q\rangle$, let's evolve the system for a time t and consider the probability amplitude of finding the system at time t in the position eigenstate $|q'\rangle$:

$$\begin{aligned}
\langle q'|\exp(-itH)|q\rangle &= \sum_{n,m=0}^{\infty}\langle q'|m\rangle\langle m|\exp(-itH)|n\rangle\langle n|q\rangle \\
&= \sum_{n=0}^{\infty}\exp(-in\omega_0 t)\langle q'|n\rangle\langle n|q\rangle \\
&= \sum_{n=0}^{\infty}\exp(-in\omega_0 t)\Phi_n(q')\Phi_n^*(q)\,,
\end{aligned} \tag{2.20}$$

which is equivalent to Eq. (2.18). In order to calculate Eq. (2.20) for an **arbitrary finite time** t, from Eq. (2.4) and Eq.(2.5) we get

$$\Phi_n(q) = (-)^n\omega_0^{\frac{1}{4}}\pi^{-\frac{1}{4}}2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}\omega_0^{-\frac{n}{2}}\exp\left(\frac{1}{2}\omega_0 q^2\right)\left(\frac{\partial}{\partial q}\right)^n\exp\left(-\omega_0 q^2\right)\,. \tag{2.21}$$

Substituting Eq. (2.21) in Eq. (2.20) we get

$$\begin{aligned}
\langle q'| \exp(-itH)|q\rangle &= \left(\frac{\omega_0}{\pi}\right)^{\frac{1}{2}} \exp\left[\frac{1}{2}\omega_0\left(q^2+q'^2\right)\right] \\
&\times \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\exp(-i\omega_0 t)}{2\omega_0}\frac{\partial}{\partial q'}\frac{\partial}{\partial q}\right)^n \exp\left[-\omega_0\left(q^2+q'^2\right)\right] \\
&= \left(\frac{\omega_0}{\pi}\right)^{\frac{1}{2}} \exp\left[\frac{1}{2}\omega_0\left(q^2+q'^2\right)\right] \\
&\times \exp\left(\frac{\exp(-i\omega_0 t)}{2\omega_0}\frac{\partial}{\partial q'}\frac{\partial}{\partial q}\right) \exp\left[-\omega_0\left(q^2+q'^2\right)\right].
\end{aligned} \tag{2.22}$$

Now it is convenient to express the last gaussian factor in terms of plane waves (eigenfunctions of the position operator) through the following integral representation (which can be verified by means of Eq. (2.202))

$$\exp\left(-\omega_0 q^2\right) = \frac{1}{2}\left(\pi\omega_0\right)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} dk\, \exp\left(ikq - \frac{k^2}{4\omega_0}\right) \tag{2.23}$$

By substitution of Eq. (2.23) in Eq. (2.22) we get

$$\begin{aligned}
\langle q'| \exp(-itH)|q\rangle &= \frac{1}{4}\pi^{-\frac{3}{2}}\omega_0^{-\frac{1}{2}} \exp\left[\frac{1}{2}\omega_0\left(q^2+q'^2\right)\right] \exp\left(\frac{\exp(-i\omega_0 t)}{2\omega_0}\frac{\partial}{\partial q'}\frac{\partial}{\partial q}\right) \\
&\int_{-\infty}^{+\infty} dk'\, \exp\left(ik'q' - \frac{k'^2}{4\omega_0}\right)\int_{-\infty}^{+\infty} dk\, \exp\left(ikq - \frac{k^2}{4\omega_0}\right) \\
&= \frac{1}{4}\pi^{-\frac{3}{2}}\omega_0^{-\frac{1}{2}} \exp\left[\frac{1}{2}\omega_0\left(q^2+q'^2\right)\right] \\
&\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk'\, \exp\left[-\frac{\exp(-i\omega_0 t)}{2\omega_0}kk' + ikq + ik'q' - \frac{k^2}{4\omega_0} - \frac{k'^2}{4\omega_0}\right].
\end{aligned} \tag{2.24}$$

The two-dimensional integral of Eq. (2.24) can be solved through Eq. (2.203). In fact the argument of the exponential can be written in the form

$$-x^T\cdot A\cdot x + b^T\cdot x\,,$$

where

$$\vec{x} = \begin{pmatrix} k \\ k' \end{pmatrix}, \qquad A = \frac{1}{4\omega_0}\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}, \qquad \vec{b} = i\begin{pmatrix} q \\ q' \end{pmatrix}, \qquad \gamma = \exp(-i\omega_0 t)\,. \tag{2.25}$$

From the above equations we get

$$\det A = \frac{1}{(4\omega_0)^2}(1-\gamma^2)\,, \qquad A^{-1} = \frac{1}{\det A}\frac{1}{4\omega_0}\begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix}. \tag{2.26}$$

The eigenvalues of A are

$$\lambda_{\pm} = \frac{1 \pm \gamma}{4\omega_0},$$

which are positive and allow to apply the n-dimensional gaussian integration formula of Eq. (2.203). The result of the bidimensional integral in Eq. (2.24) is

$$\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \, [\ldots] = \frac{4\pi\omega_0}{\sqrt{1-\gamma^2}} \exp\left[-\frac{\omega_0}{1-\gamma^2} \left(q^2 + q'^2 - 2\gamma q q' \right) \right], \tag{2.27}$$

so that Eq. (2.24) becomes

$$\langle q' | \exp(-itH) | q \rangle = \left[\frac{\omega_0}{\pi(1-\gamma^2)} \right]^{\frac{1}{2}} \exp\left\{ -\frac{1}{1-\gamma^2} \left[\frac{1}{2}\omega_0 \left(1+\gamma^2\right) \left(q^2 + q'^2\right) - 2\omega_0 \gamma q q' \right] \right\}. \tag{2.28}$$

Observing that

$$2qq' = \frac{1}{2}\left(q+q'\right)^2 - \frac{1}{2}\left(q-q'\right)^2, \tag{2.29}$$

we can manipulate the argument of the exponential in Eq. (2.28) in the following way:

$$\begin{aligned}
& -\frac{1}{1-\gamma^2} \left[\frac{1}{2}\omega_0 \left(1+\gamma^2\right) \left(q^2+q'^2\right) - 2\omega_0\gamma q q' \right] \\
&= -\frac{\omega_0}{1-\gamma^2} \left[\frac{1}{2} \left(1+\gamma^2\right) \left(q^2+q'^2\right) - 2\gamma q q' \right] \\
&= -\frac{\omega_0}{1-\gamma^2} \left[\frac{1}{2} \left(1+\gamma^2\right) \left(q^2+q'^2\right) - \frac{\gamma}{2}\left(q+q'\right)^2 + \frac{\gamma}{2}\left(q-q'\right)^2 \right] \\
&= -\frac{\omega_0}{1-\gamma^2} \left\{ \frac{1}{4} \left(1+\gamma^2\right) \left[\left(q+q'\right)^2 + \left(q-q'\right)^2 \right] - \frac{\gamma}{2}\left(q+q'\right)^2 + \frac{\gamma}{2}\left(q-q'\right)^2 \right\} \\
&= -\frac{\omega_0}{1-\gamma^2} \left\{ \frac{1}{4}(1-\gamma)^2 \left(q+q'\right)^2 + \frac{1}{4}(1+\gamma)^2 \left(q-q'\right)^2 \right\} \\
&= -\frac{\omega_0}{4} \left\{ \frac{1+\gamma}{1-\gamma}\left(q-q'\right)^2 + \frac{1-\gamma}{1+\gamma}\left(q+q'\right)^2 \right\}.
\end{aligned} \tag{2.30}$$

So Eq. (2.28) can be rewritten in the following convenient form

$$\langle q' | \exp(-itH) | q \rangle = \left[\frac{\omega_0}{\pi(1-\gamma^2)} \right]^{\frac{1}{2}} \exp\left\{ -\frac{\omega_0}{4} \left[\frac{1+\gamma}{1-\gamma}\left(q'-q\right)^2 + \frac{1-\gamma}{1+\gamma}\left(q'+q\right)^2 \right] \right\}. \tag{2.31}$$

This equation can be given a physical interpretation. Let's define

$$A_M(t) = \left[\frac{\omega_0}{\pi(1-\gamma^2)} \right]^{\frac{1}{2}} = \left[\frac{\omega_0}{\pi(1-\exp(-2i\omega_0 t))} \right]^{\frac{1}{2}}, \tag{2.32}$$

and the following variable with the dimension of time

$$\vartheta_M \equiv \frac{2}{i\omega_0}\frac{1-\gamma}{1+\gamma} = \frac{2}{i\omega_0}\frac{1-\exp(-i\omega_0 t)}{1+\exp(-i\omega_0 t)} = \frac{2}{\omega_0}\tan\left(\frac{1}{2}\omega_0 t\right). \tag{2.33}$$

Through the above definitions, Eq. (2.31) can be written as

$$\langle q'|\exp(-itH)|q\rangle = A_M(t)\exp\left\{i\vartheta_M\left[\frac{1}{2}\left(\frac{q'-q}{\vartheta_M}\right)^2 - \frac{1}{2}\omega_0^2\left(\frac{q'+q}{2}\right)^2\right]\right\}. \tag{2.34}$$

In this equation, the term $\frac{1}{2}\left(\frac{q'-q}{\vartheta_M}\right)^2$ can be interpreted as the classical kinetic energy, with mean velocity

$$v = \frac{q'-q}{\vartheta_M}, \tag{2.35}$$

where ϑ_M is the time. The term $\frac{1}{2}\omega_0^2\left(\frac{q'+q}{2}\right)^2$ represents the classical potential energy in the mean position between the initial point q and the final point q'. At this point three observations are in order:

Observation 1: the quantity between $[\ldots]$ in Eq. (2.34) would represent the classical lagrangian $L = E_{\text{cin}} - E_{\text{pot}}$.

Observation 2: ϑ_M is not the time t but a periodical function of t (with period $\frac{2\pi}{\omega_0}$), as it is clear from its previous definition.

Observation 3: The transition amplitude of Eq. (2.34) must be solution of the Schrödinger equation

$$\left(-i\partial_t - \frac{1}{2}\frac{d^2}{dq'^2} + \frac{1}{2}\omega_0^2 q'^2 - \frac{1}{2}\omega_0\right)\langle q'|\exp(-itH)|q\rangle = 0, \tag{2.36}$$

with the initial condition

$$\lim_{t\to 0}\langle q'|\exp(-itH)|q\rangle = \langle q'|q\rangle = \delta(q'-q). \tag{2.37}$$

This condition is incompatible with the expression of Eq. (2.34), unless we adopt a mathematical regularization which introduces a negative imaginary part to the time t. In fact, for $t \to 0$, Eq. (2.34) we have

$$\langle q'|\exp(-itH)|q\rangle \to \frac{1}{\sqrt{2\pi i t}} e^{i\frac{(q'-q)^2}{2t}}. \tag{2.38}$$

2.3.1 Infinitesimal time evolution

Let's now specify Eq. (2.34) for an infinitesimal time interval Δt. From the definition of ϑ_M, Eq. (2.33), we have

$$\Delta\vartheta_M \simeq \Delta t. \tag{2.39}$$

The limit of $A_M(\Delta t)$ for small Δt becomes:

$$A_M(\Delta t) = \left[\frac{\omega_0}{\pi(1-\exp(-2i\omega_0 t))}\right]^{\frac{1}{2}} \simeq \frac{1}{\sqrt{2\pi i\Delta t}} = \begin{pmatrix} \frac{1}{\sqrt{2\pi\Delta t}}\exp\left(-i\frac{\pi}{4}\right) & (\Delta t > 0) \\ \frac{1}{\sqrt{2\pi\Delta t}}\exp\left(i\frac{\pi}{4}\right) & (\Delta t < 0) \end{pmatrix} \tag{2.40}$$

Observe that $A_M(\Delta t)$ diverges in the limit $\Delta t \to 0$. The velocity of Eq. (2.35), in the limit $\Delta t \to 0$ becomes

$$v = \frac{q' - q}{\Delta t} = \dot{q} \,. \tag{2.41}$$

As a consequence, the terms in square brackets in Eq. (2.34) represent the Lagrangian:

$$L(q,\dot{q}) = \frac{1}{2}\left(\frac{q'-q}{\Delta t}\right)^2 - \frac{1}{2}\omega_0^2\left(\frac{q'+q}{2}\right)^2 \tag{2.42}$$

and we have

$$\langle q'|\exp\left(-i\Delta t H\right)|q\rangle \equiv A_M(\Delta t)e^{i\Delta t L(q,\dot{q})} \,. \tag{2.43}$$

Observation: the exponent of Eq. (2.43) *does not* coincide with the power series expansion of Δt of the exponent of Eq. (2.34).

2.3.2 Path-integral

Let's consider the evolution of the system from the initial state $|q_0\rangle$ to the final state $|q_N\rangle$ in a finite time T. We can divide T in N steps Δt, all equal in size,

$$T = N\Delta t \,, \tag{2.44}$$

with N large so that we can apply the equations of the previous section. We have

$$\begin{aligned}
\langle q_N|\exp\left(-iTH\right)|q_0\rangle & \\
&= \langle q_N|e^{-i\sum\Delta t H}|q_0\rangle \\
&= \langle q_N|e^{-i\Delta t H}Ie^{-i\Delta t H}I\ldots Ie^{-i\Delta t H}|q_0\rangle \,,
\end{aligned} \tag{2.45}$$

where we have introduced $N-1$ Identity operators. Then, by using the completeness relation of the position eigenstates

$$I = \int_{-\infty}^{+\infty} dq\, |q\rangle\langle q| \,, \tag{2.46}$$

and Eq. (2.43), we get

$$\begin{aligned}
\langle q_N|e^{(-iTH)}|q_0\rangle &= [A_M(\Delta t)]^N \prod_{i=1}^{N-1}\int_{-\infty}^{+\infty} dq_i\, e^{i\Delta t L(q_{N-1},\dot{q}_{N-1})}\ldots e^{i\Delta t L(q_0,\dot{q}_0)} \\
&= [A_M(\Delta t)]^N \prod_{i=1}^{N-1}\int_{-\infty}^{+\infty} dq_i\, e^{i\Delta t \sum_{j=0}^{N-1} L(q_j,\dot{q}_j)} \\
&\simeq [A_M(\Delta t)]^N \prod_{i=1}^{N-1}\int_{-\infty}^{+\infty} dq_i\, e^{\{iS(q_0,\ldots,q_i,\ldots,q_N)\}} \,,
\end{aligned} \tag{2.47}$$

where we have introduced the Action S, approximated by a discrete sum:

$$S(q_0, \ldots, q_i, \ldots, q_N) \simeq \Delta t \sum_{j=0}^{N-1} L(q_j, \dot{q}_j)\,. \tag{2.48}$$

The Action of the above equation is an ordinary function of the $N+1$ variables $q_0, \ldots, q_N$, which define, for discrete time intervals Δt, the classical path between q_0 and q_N during the time T. In Eq. (2.47) the endpoints q_0 and q_N are fixed, while on all intermediate $N-1$ q_i positions we integrate. **The probability amplitude for the evolution of the system from the initial state $|q_0\rangle$ to the final state $|q_N\rangle$ is proportional to a weighted sum over all classical paths kinematically allowed, where each path is weighted with iS (with S being the classical action corresponding to the path).** The larger is N, the number of intermediate steps, the better is the approximation of the classical Action with the discrete sum of Eq. (2.48). However it has to be noted that the limit $N \to \infty$ ($\Delta t \to 0$) in Eq. (2.47) is not possible because $A_M(\Delta t)$ diverges for $\Delta t \to 0$ (cfr. Eq. 2.40)). Bypassing for the time being this problem, Eq. (2.47) is formally written as

$$\langle q_N | e^{(-iTH)} | q_0 \rangle \simeq [A_M(\Delta t)]^N \int [dq] e^{iS(q)}\,, \tag{2.49}$$

where the integration is understood as **functional integration** over the classical paths

We stress again that the knowledge of $\langle q' | e^{(-iTH)} | q_0 \rangle$ is equivalent to the knowledge of the Schrödinger equation. In fact the wave function at q' is $\psi(q', t) \equiv \langle q' | \psi(t) \rangle$. But $|\psi(t)\rangle = \exp(-itH) |\psi(0)\rangle$. By projecting on the bra $\langle q'|$ both members of the above equation and inserting the completeness relation of Eq. (2.46), we have

$$\psi(q', t) = \int dq\, \langle q' | e^{(-iTH)} | q \rangle | \psi(q, 0) \rangle\,. \tag{2.50}$$

The quantity $\langle q' | e^{(-iTH)} | q \rangle$ is called the **propagator**.

2.4 Mathematical difficulties

If we consider Eq. (2.43) and try to obtain the initial condition of Eq. (2.37), $\langle q' | q \rangle = \delta(q' - q)$, we get

$$\lim_{\Delta t \to 0} \langle q' | e^{-i\Delta t H} | q \rangle \simeq \lim_{\Delta t \to 0} \frac{1}{\sqrt{2\pi i \Delta t}} e^{\frac{i}{2} \frac{(q'-q)^2}{\Delta t}} \neq \delta(q' - q)\,, \tag{2.51}$$

due to the presence of the imaginary unit i, unless we introduce a mathematical regularization. If Δt has a small imaginary part the initial condition is recovered.

Another problem is related to the group properties of the temporal translation (implicitely used in Eq. (2.45)). To illustrate this point, we can rewrite Eq. (2.43) in the following way:

$$\langle q' | \exp(-i\Delta t H) | q \rangle \equiv A_M(\Delta t) e^{\left[\frac{i}{2}\left(\frac{1}{\Delta t} - \frac{1}{4}\omega_0^2 \Delta t\right)(q^2 + q'^2) - i\left(\frac{1}{\Delta t} + \frac{1}{4}\omega_0^2 \Delta t\right) q' q\right]}\,. \tag{2.52}$$

The problem consists in reproducing the following identity

$$\langle q'|e^{(-2i\Delta t H)}|q\rangle = \int_{-\infty}^{+\infty} dq'' \,\langle q'|e^{(-i\Delta t H)}|q''\rangle\langle q''|e^{(-i\Delta t H)}|q\rangle \tag{2.53}$$

trhough Eq. (2.52). In fact, using Eq. (2.52), we get for the right hand side of Eq. (2.53)

$$\begin{aligned}&\int_{-\infty}^{+\infty} dq'' \,\langle q'|e^{(-i\Delta t H)}|q''\rangle\langle q''|e^{(-i\Delta t H)}|q\rangle \\ &= [A_M(\Delta t)]^2 \exp\left[\frac{i}{2}\left(\frac{1}{\Delta t} - \frac{1}{4}\omega_0^2\Delta t\right)\left(q^2 + q'^2\right)\right] \\ &\int_{-\infty}^{+\infty} dq'' \,\exp\left[i\left(\frac{1}{\Delta t} - \frac{1}{4}\omega_0^2\Delta t\right)q''^2 - i\left(\frac{1}{\Delta t} + \frac{1}{4}\omega_0^2\Delta t\right)(q'+q)\,q''\right].\end{aligned} \tag{2.54}$$

The integral of the above equation is not defined because the coefficient of q''^2 is purely imaginary. Also in this case a methematical regularization is required (if Δt has a negative imaginary part the integration converges).

2.5 Euclidean Time

We have seen in the previous section that the mathematical difficulties can be avoided by giving the time a negative imaginary part. Usually a trick is to apply the following transformation to the time

$$t \to -i\tau\,, \tag{2.55}$$

with τ real, which is nothing else than the Wick rotation you have already encountered during the QED course. The time τ is the *Euclidean time*. At the end of the calculations the physical results are obtained through analytic continuation (inverse Wick rotation), which allows to recover the real time t.

With euclidean time, Eq. (2.34) becomes

$$\langle q'|\exp\left(-\tau H\right)|q\rangle = A_E(\tau)\exp\left\{-\vartheta_E\left[\frac{1}{2}\left(\frac{q'-q}{\vartheta_E}\right)^2 + \frac{1}{2}\omega_0^2\left(\frac{q'+q}{2}\right)^2\right]\right\}, \tag{2.56}$$

where

$$A_E(\tau) = \left[\frac{\omega_0}{\pi(1-\gamma_E^2)}\right]^{\frac{1}{2}} = \left[\frac{\omega_0}{\pi(1-\exp(-2\omega_0\tau))}\right]^{\frac{1}{2}},$$

and

$$\gamma_E = e^{-\omega_0\tau}\,. \tag{2.57}$$

The variable ϑ_E is defined in analogy with Eq. (2.33)

$$\vartheta_E = \frac{2}{\omega_0}\frac{1-\gamma}{1+\gamma} = \frac{2}{\omega_0}\frac{1-\exp\left(-\omega_0\tau\right)}{1+\exp\left(-\omega_0\tau\right)} = \frac{2}{\omega_0}\tanh\left(\frac{1}{2}\omega_0\tau\right) \tag{2.58}$$

and is not a periodical function of τ.
Observation: In analogy with Eq. (2.36), the transition amplitude of Eq. (2.56) satisfies the following equation

$$\left(\partial_\tau - \frac{1}{2}\frac{d^2}{dq'^2} + \frac{1}{2}\omega_0^2 q'^2 - \frac{1}{2}\omega_0\right)\langle q'|\exp\left(-\tau H\right)|q\rangle = 0\,, \tag{2.59}$$

with the initial condition

$$\lim_{\tau\to 0}\langle q'|\exp\left(-\tau H\right)|q\rangle = \langle q'|q\rangle = \delta(q'-q)\,. \tag{2.60}$$

If we consider infinitesimal time steps, we have

$$\Delta\vartheta_E \simeq \Delta\tau \tag{2.61}$$

and

$$A_E(\Delta\tau) \simeq \frac{1}{\sqrt{2\pi\Delta\tau}}\,, \tag{2.62}$$

which is still divergent for $\Delta\tau \to 0$.
The analogous of Eq. (2.43) for euclidean time is

$$\begin{aligned}\langle q'|\exp\left(-\Delta\tau H\right)|q\rangle &= A_E(\Delta\tau)\exp\left\{-\Delta\tau\left[\frac{1}{2}\left(\frac{q'-q}{\Delta\tau}\right)^2 + \frac{1}{2}\omega_0^2\left(\frac{q'+q}{2}\right)^2\right]\right\}\\ &\equiv A_E(\Delta\tau)e^{-\Delta\tau L_E(q,\dot{q})}\,,\end{aligned} \tag{2.63}$$

where the Euclidean Lagrangian is

$$L_E(q,\dot{q}) \equiv \frac{1}{2}\left(\frac{q'-q}{\Delta\tau}\right)^2 + \frac{1}{2}\omega_0^2\left(\frac{q'+q}{2}\right)^2\,. \tag{2.64}$$

Notice that L_E is positive definite.
With the euclidean time, the initial condition of Eq. (2.37) is recovered. In fact

$$\lim_{\Delta\tau\to 0}\langle q'|\exp\left(-\Delta\tau H\right)|q\rangle = \lim_{\Delta\tau\to 0}\frac{1}{\sqrt{2\pi\Delta\tau}}e^{-\frac{1}{2\Delta\tau}(q'-q)^2} = \delta(q'-q)\,. \tag{2.65}$$

Also the second mathematical problem raised in the previous section is solved with the adoption of the Euclidean Time. In fact, instead of Eq. (2.52), we get

$$\langle q'|\exp\left(-\Delta\tau H\right)|q\rangle \equiv A_E(\Delta\tau)e^{\left[-\frac{1}{2}\left(\frac{1}{\Delta\tau}+\frac{1}{4}\omega_0^2\Delta\tau\right)\left(q^2+q'^2\right)^2+\left(\frac{1}{\Delta\tau}-\frac{1}{4}\omega_0^2\Delta\tau\right)q'q\right]}\,. \tag{2.66}$$

and Eq. (2.53) becomes

$$\langle q'|e^{(-2\Delta\tau H)}|q\rangle = \int_{-\infty}^{+\infty} dq''\,\langle q'|e^{(-\Delta\tau H)}|q''\rangle\langle q''|e^{(-\Delta\tau H)}|q\rangle\,. \tag{2.67}$$

From the above equations we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dq'' \, \langle q'|e^{(-\Delta\tau H)}|q''\rangle\langle q''|e^{(-\Delta\tau H)}|q\rangle \\
&= [A_E(\Delta\tau)]^2 \exp\left[-\frac{1}{2}\left(\frac{1}{\Delta\tau}+\frac{1}{4}\omega_0^2\Delta\tau\right)\left(q^2+q'^2\right)^2\right] \\
&\int_{-\infty}^{+\infty} dq'' \, \exp\left[-\left(\frac{1}{\Delta\tau}+\frac{1}{4}\omega_0^2\Delta\tau\right)q''^2+\left(\frac{1}{\Delta\tau}-\frac{1}{4}\omega_0^2\Delta\tau\right)(q'+q)\,q''\right].
\end{aligned} \tag{2.68}$$

In this case the integral is well defined. The path-integral of Eq. (2.47), with Euclidean Time, becomes

$$\begin{aligned}
\langle q_N|e^{(-TH)}|q_0\rangle &= [A_E(\Delta\tau)]^N \prod_{i=1}^{N-1}\int_{-\infty}^{+\infty} dq_i \, e^{-\Delta\tau\sum_{j=0}^{N-1} L_E(q_j,\dot{q}_j)} \\
&\simeq [A_E(\Delta\tau)]^N \prod_{i=1}^{N-1}\int_{-\infty}^{+\infty} dq_i \, e^{\{-S_E(q_0,\ldots,q_i,\ldots,q_N)\}} \,,
\end{aligned} \tag{2.69}$$

where the Euclidean Action S_E is related to the Euclidean Lagrangian as follows:

$$S_E(q_0,\ldots,q_N) \simeq \Delta\tau \sum_{j=0}^{N-1} L_E(q_j,\dot{q}_j) \,. \tag{2.70}$$

Eq. (2.69) gives a positive measure to the classical paths, contrary to Eq. (2.47).

However, the divergence of $A_E(\Delta\tau)$ for $\Delta\tau \to 0$ is not solved by going from the Minkowskian to the Euclidean Time. The usual way of writing Eq. (2.69) is

$$\langle q_N|e^{(-TH)}|q_0\rangle \simeq [A_E(\Delta\tau)]^N \int [dq]\, e^{-S_E[q]} \,. \tag{2.71}$$

2.5.1 The classical limit and the semiclassical approximation

According to Eq. (2.71) we can recover the classical dynamics. In fact the path for which the Action is minimized is the one that maximizes the integral.
To see this more directly, even if in a heuristical way, let us reinsert explicitely the Planck constant and use Minkowskian time in Eq. (2.49)

$$\langle q_N|e^{(-\frac{i}{\hbar}TH)}|q_0\rangle \simeq [A_M(\Delta t)]^N \int [dq] e^{\frac{i}{\hbar}S(q)} \,. \tag{2.72}$$

The classical limit is defined as the limit of the propagator for $\hbar \to 0$. By inspection of Eq. (2.72), we see that the limit is fullfilled when $S(q) \gg \hbar$. In fact suppose that a trajectory $q_c(t)$ exists, such that $q_c(t_0) = q_0$ and $q_c(t_0+T) = q_N$, which extremizes the Action. The condition $\delta S = 0$ implies that trajectories close to $q_c(t)$ contribute to the integral of Eq. (2.72) with equal or very similar phases, thus interfering in constructive way. On the contrary, close to any trajectory which does not extremize the Action we will find other trajectories with very different phases and therefore interfering in destructive way. So the main contribution to the integral comes from a set of trajectories close to $q_c(t)$, where "close" means that the related Action differs at most from $S(q_c)$ by about $\hbar$. This means that in the limit $\hbar \to 0$ the motion of the system is well described by the classical trajectory. Moreover, the principle of least Action of classical mechanics can be thought of as a particular limit of quantum mechanics.

According to the above argument, quantum mechanics describes the fluctuations of the Action in a narrow range around the classical path. Thus we can expand the Action functional in terms of fluctuations around the classical path $q_c(t)$:

$$S[q,\dot{q}] = S_{\rm cl} + \frac{1}{2}\int\left(\frac{\delta^2 L}{\delta q^2}(\delta q)^2 + 2\frac{\delta^2 L}{\delta q \delta \dot{q}}\delta q \delta \dot{q} + \frac{\delta^2 L}{\delta \dot{q}^2}(\delta \dot{q})^2\right) + \ldots \equiv S_{\rm cl} + \frac{1}{2}\delta^2 S + \ldots, \tag{2.73}$$

where δq is the fluctuation around the classical path. The derivatives have to be taken at the classical path. Since the Action is stationary at the classical path, Eq. (2.73) does not include first derivative terms. The propagator of Eq. (2.72) becomes

$$\langle q_N | e^{(-\frac{i}{\hbar}TH)} | q_0 \rangle \simeq [A_M(\Delta t)]^N \, e^{\frac{i}{\hbar}S_{\rm cl}} \int [dq] \, e^{\frac{i}{2\hbar}\delta^2 S} + \ldots . \tag{2.74}$$

The approximation of calculating the propagator according to Eq. (2.74) is called *semiclassical approximation*. If the Lagrangian depends at most quadratically on q and $\dot{q}$, Eq. (2.74) is exact, without higher-order terms.

2.5.2 The free particle propagator

With the results of the previous sections, we can easily obtain the propagator for a free particle of mass m moving in one dimension. In fact we can refer to the Hamiltonian of Eq. (2.3), introducing the mass m and taking the limit $\omega_0 \to 0$. From Eqs.(2.33) and (2.32), we get

$$\begin{aligned} &\lim_{\omega_0 \to 0} \vartheta = t \\ &A_M(t) = \sqrt{\frac{m}{2i\pi\hbar t}} \, . \end{aligned} \tag{2.75}$$

The analogous of Eq. (2.34) is

$$K(t;q',q) \equiv \langle q' | \exp\left(-\frac{i}{\hbar}tH_0\right) | q \rangle = \sqrt{\frac{m}{2i\pi\hbar t}} \exp\left\{ i\frac{m\,(q'-q)^2}{2\hbar t} \right\}. \tag{2.76}$$

The phase factor in the square root should be taken as in Eq. (2.40), according to the inverse Wick rotation, starting from the euclidean propagator

$$K_E(\tau;q',q) = \sqrt{\frac{m}{2\pi\hbar\tau}} \exp\left\{ -\frac{m\,(q'-q)^2}{2\hbar\tau} \right\}. \tag{2.77}$$

The generalization to three dimensions is easily obtained as

$$K(t;\vec{r}',\vec{r}) =\equiv \langle\vec{r'}|\exp\left(-\frac{i}{\hbar}tH_0\right)|\vec{r}\rangle = \left(\frac{m}{2i\pi\hbar t}\right)^{\frac{3}{2}}\exp\left\{+i\frac{m\,|\vec{r}'-\vec{r}|^2}{2\hbar t}\right\}. \tag{2.78}$$

The free particle propagator in momentum representation

Since for a free particle the momentum is conserved, it is instructive to take the Fourier transform of Eq. (2.76)

$$\begin{aligned}\tilde{K}(p,t) &= \int_{-\infty}^{+\infty} d(\Delta q)\,\exp\left\{-i\frac{p}{\hbar}\Delta q\right\}K(\Delta q,t)\\ &= \sqrt{\frac{m}{2i\pi\hbar t}}\int_{-\infty}^{+\infty} d(\Delta q)\,\exp\left\{-i\frac{p}{\hbar}\Delta q\right\}\exp\left\{i\frac{m(\Delta q)^2}{2\hbar t}\right\}.\end{aligned} \tag{2.79}$$

The equivalent expression with euclidean time is

$$\tilde{K}_E(p,\tau) = \sqrt{\frac{m}{2\pi\hbar\tau}}\int_{-\infty}^{+\infty} d(\Delta q)\,\exp\left\{-i\frac{p}{\hbar}\Delta q\right\}\exp\left\{-\frac{m(\Delta q)^2}{2\hbar\tau}\right\}. \tag{2.80}$$

The integration of Eq. (2.80) can be performed by Eq. (2.202), with $a = \frac{m}{2\hbar\tau}$ and $b = -i\frac{p}{\hbar}$, obtaining:

$$\tilde{K}_E(p,t) = \exp\left\{-\frac{1}{\hbar}\frac{p^2}{2m}\tau\right\} \tag{2.81}$$

and [1]

$$\tilde{K}(p,t) = \exp\left\{-\frac{i}{\hbar}\frac{p^2}{2m}t\right\}. \tag{2.83}$$

We can now make a Fourier antitranform

$$\begin{aligned}K(\Delta q,t) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp\,\exp\left\{+i\frac{p}{\hbar}\Delta q\right\}K(\tilde{p},t)\\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp\,\exp\left\{+i\frac{p}{\hbar}\Delta q\right\}\exp\left\{-\frac{i}{\hbar}\frac{p^2}{2m}t\right\}\\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp\,\exp\left\{+\frac{i}{\hbar}\left(p\Delta q-\frac{p^2}{2m}t\right)\right\}.\end{aligned} \tag{2.84}$$

As discussed in Section 2.2, the boundary condition $K = 0$ for $t < 0$ is understood. We can implement it multiplying the integrand of Eq. (2.84) by the step function $\vartheta(t)$ and

[1] The same result of Eq. (2.83) can be obtained starting from the definition

$$K(p,p',t) = \langle p'|\exp\left\{-\frac{i}{\hbar}tH_0\right\}|p\rangle = \langle p'|\exp\left\{-\frac{i}{\hbar}t\frac{(\hat{p})^2}{2m}\right\}|p\rangle = \exp\left\{-\frac{i}{\hbar}\frac{p^2}{2m}t\right\}\delta(p-p'). \tag{2.82}$$

recalling the integral representation (which can be verified through application of the residue theorem)

$$\vartheta(t) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\omega\,\frac{e^{i\omega t}}{\omega - i\varepsilon} \quad \text{with } \varepsilon > 0\ : \tag{2.85}$$

$$K(\Delta q, t) \;=\; \frac{\hbar}{i}\int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}\frac{d\omega}{2\pi\hbar}\frac{\exp\left\{+\frac{i}{\hbar}\left[p\Delta q - \left(\frac{p^2}{2m} - \hbar\omega\right)t\right]\right\}}{\omega - i\varepsilon}\,. \tag{2.86}$$

The coefficient of t in the plane wave is the energy, so we can make the substitution

$$E = \frac{p^2}{2m} - \hbar\omega\,, \tag{2.87}$$

obtaining for Eq. (2.86)

$$K(\Delta q, t) = \frac{\hbar}{i}\int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}\frac{dE}{2\pi\hbar}\frac{\exp\left\{+\frac{i}{\hbar}\left[p\Delta q - \left(\frac{p^2}{2m} - E\right)t\right]\right\}}{E - \frac{p^2}{2m} + i\varepsilon}\,. \tag{2.88}$$

Eq. (2.88) says that the propagation takes place also for energies $E \neq \frac{p^2}{2m}$. The classical dispersion relation $E = \frac{p^2}{2m}$ is the pole of the propagator.

2.5.3 Particle in one dimension with generic potential

In this section we give the treatment for a generic one dimensional quantum system, of mass m, described by the coordinate q, the conjugate momentum p and the Hamiltonian

$$\mathcal{H}(\hat{q},\hat{p}) = K(\hat{p}) + V(\hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q})\ . \tag{2.89}$$

Let us recall the transition amplitude from the initial state $|q_0\rangle$ to the final state $|q_N\rangle$ during the finite time T, as in Eq. (2.45), which we report here

$$\begin{aligned}\langle q_N|\exp\left(-\frac{i}{\hbar}TH\right)|q_0\rangle&\\ &= \langle q_N|e^{-\frac{i}{\hbar}\Sigma\Delta t H}|q_0\rangle\\ &= \langle q_N|e^{-\frac{i}{\hbar}\Delta t H}Ie^{-\frac{i}{\hbar}\Delta t H}I\ldots Ie^{-\frac{i}{\hbar}\Delta t H}|q_0\rangle\,,\end{aligned}$$

and we replace the identity operators I with the completeness relation of the position eigenstates of Eq. (2.46)

$$I = \int_{-\infty}^{+\infty} dq\,|q\rangle\langle q|\,.$$

Since the kinetic part K and the potential V of the Hamiltonian don't commute, we have,

$$\begin{aligned} e^{-\frac{i}{\hbar}\Delta t(V+K)} &= 1 - \frac{i}{\hbar}\Delta t(K+V) - \frac{(\Delta t)^2}{2\hbar^2}\Big(K^2+V^2+VK+KV\Big) + \mathcal{O}[(\Delta t)^3] \\ e^{-\frac{i}{\hbar}\Delta tV}e^{-\frac{i}{\hbar}\Delta tK} &= 1 - \frac{i}{\hbar}\Delta t(K+V) - \frac{(\Delta t)^2}{2\hbar^2}\Big(K^2+V^2+2VK\Big) + \mathcal{O}[(\Delta t)^3]\,, \end{aligned} \tag{2.90}$$

so that we can write,

$$e^{-\frac{i}{\hbar}\Delta tH} = e^{-\frac{i}{\hbar}\Delta tV}e^{-\frac{i}{\hbar}\Delta tK} + \mathcal{O}[(\Delta t)^2]\,. \tag{2.91}$$

An error of order $(\Delta t)^2$, iterated N times, gives a globar error of order Δt, which can be neglected when we take the limit of small Δt. Eq. (2.45) can be written as

$$\begin{aligned} \langle q_N|\exp{(-\frac{i}{\hbar}TH)}|q_0\rangle &= \left[\prod_{i=1}^{N-1}\int_{-\infty}^{+\infty}dq_i\right]\langle q_N|e^{-\frac{i}{\hbar}\Delta tH}|q_{N-1}\rangle\ldots\langle q_1|e^{-\frac{i}{\hbar}\Delta tH}|q_0\rangle \\ &\simeq \left[\prod_{i=1}^{N-1}\int_{-\infty}^{+\infty}dq_i\right]\langle q_N|e^{-\frac{i}{\hbar}\Delta tV}e^{-\frac{i}{\hbar}\Delta tK}|q_{N-1}\rangle\ldots\langle q_1|e^{-\frac{i}{\hbar}\Delta tV}e^{-\frac{i}{\hbar}\Delta tK}|q_0\rangle\,. \end{aligned} \tag{2.92}$$

Now let us consider a single transition amplitude

$$\langle q_{j+1}|e^{-\frac{i}{\hbar}\Delta tV(\hat{q})}e^{-\frac{i}{\hbar}\Delta tK(\hat{p})}|q_j\rangle = e^{-\frac{i}{\hbar}\Delta tV(q_{j+1})}\langle q_{j+1}|e^{-\frac{i}{\hbar}\Delta tK(\hat{p})}|q_j\rangle\,. \tag{2.93}$$

Now, remembering that

$$\begin{aligned} \langle q|p\rangle &= \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}pq} \\ \int dp\,|p\rangle\langle p| &= I \\ \langle p'|p\rangle &= \delta(p'-p)\,, \end{aligned} \tag{2.94}$$

we have

$$\begin{aligned} \langle q_{j+1}|e^{-\frac{i}{\hbar}\Delta tK(\hat{p})}|q_j\rangle &= \int\frac{dp'_j}{\sqrt{2\pi\hbar}}\int\frac{dp_j}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}\left[p'_jq_{j+1}-p_jq_j\right]}e^{-\frac{i}{\hbar}\Delta tK(p_j)}\delta(p_j-p'_j) \\ &= \int\frac{dp_j}{2\pi\hbar}\,e^{\frac{i}{\hbar}\left[\left(p_jq_{j+1}-p_jq_j\right)-\Delta tK(p_j)\right]}\,, \end{aligned} \tag{2.95}$$

where p_j is the momentum between q_j and q_{j+1}. By means of the above result, Eq. (2.93) can be written as

$$\begin{aligned} \langle q_{j+1}|e^{-\frac{i}{\hbar}\Delta tV(\hat{q})}e^{-\frac{i}{\hbar}\Delta tK(\hat{p})}|q_j\rangle &= \frac{1}{2\pi\hbar}\int dp_j\,e^{\frac{i}{\hbar}\left[p_j\left(q_{j+1}-q_j\right)-\Delta t\left(K(p_j)+V(\bar{q}_j)\right)\right]} \\ &= \frac{1}{2\pi\hbar}\int dp_j\,e^{\frac{i}{\hbar}\left[p_j\left(q_{j+1}-q_j\right)-\Delta tH\left(p_j,\bar{q}_j\right)\right]}\,, \end{aligned} \tag{2.96}$$

where we have defined $\bar{q}_j = \frac{1}{2}\left(q_j + q_{j+1}\right)$. Inserting Eq. (2.96) in Eq. (2.92), we get

$$\langle q_N| \exp\left(-\frac{i}{\hbar}TH\right)|q_0\rangle = \left[\prod_{i=1}^{N-1}\int dq_i\right]\left[\prod_{j=0}^{N-1}\int\frac{dp_j}{2\pi\hbar}\right]\exp\left\{\frac{i}{\hbar}\sum_{i=1}^{N-1}\left[p_i\left(q_{i+1}-q_i\right)-\Delta t H\left(p_i,\bar{q}_i\right)\right]\right\}. \tag{2.97}$$

Eq. (2.97) is the defining (discretized) equation for the "functional integral" (in the continuum limit)

$$\langle q_N| \exp\left(-\frac{i}{\hbar}TH\right)|q_0\rangle = \int\frac{[dq][dp]}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_N}dt\,\left[p\dot{q}-H(p,q)\right]\right\}. \tag{2.98}$$

If the kinetic term is of the form $K(\hat{p}) = \frac{\hat{p}^2}{2m}$ and $V(q)$ is a quadratic polinomial, then the integration over p_i in Eq. (2.97) can be performed analytically by completing the square in the argument of the exponential, obtaining

$$\langle q_N| \exp\left(-\frac{i}{\hbar}TH\right)|q_0\rangle = \left[\frac{m}{2\pi h\Delta t}\right]^{\frac{N}{2}}\left[\prod_{i=1}^{N-1}\int dq_i\right]\exp\left\{\frac{i\Delta t}{\hbar}\sum_{i=0}^{N-1}\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta t}\right)^2 - V\left(\bar{q}_i\right)\right]\right\}. \tag{2.99}$$

Eq. (2.99) is the defining (discretized) equation for the "functional integral" (in the continuum limit)

$$\langle q_N| \exp\left(-\frac{i}{\hbar}TH\right)|q_0\rangle = \mathcal{N}\int[dq]\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_N}dt\,L\left(q,\dot{q}\right)\right\}, \tag{2.100}$$

where $\mathcal{N}$ is the constant (already seen in previous sections) which becomes ill defined in the continuum limit. The integral $\int L(q,\dot{q})dt = \int\left(\frac{m\dot{q}^2}{2} - V(q)\right)dt$ at the exponent gives the classical action along the given path.

Eq. (2.100) was adopted by Feynman as the starting point to derive the Schrödinger equation. However, the most general expression is Eq. (2.98), while Eq. (2.100) relyes on the form of the Hamiltionian $H(\hat{p},\hat{q}) = \frac{\hat{p}^2}{2m} + V(q)$. If, for instance, we have a velocity-dependent potential, the Lagrangian at in Eq. (2.100) needs to be modified to get the correct result. Something similar happens with non-Abelian gauge theories.

Observation: the propagator is written at the beginning in terms of non-commuting operators $\hat{q}$ and $\hat{p}$, while in the end the functional integral is expressed in terms of c–numbers only, so that ambiguities can originate if we have products of $\hat{q}$ and $\hat{p}$ in the Hamiltonian. This ambiguities are not present if we start from a "Weyl ordered" Hamiltonian, where the $\hat{q}$ operators appear symmetrically at the left and right of $\hat{p}$. For instance

$$\langle q_{k+1}|\frac{1}{4}\left(\hat{q}^2\hat{p}^2 + 2\hat{q}\hat{p}^2\hat{q} + \hat{p}^2\hat{q}^2\right)|q_k\rangle = \left(\frac{q_{k+1}+q_k}{2}\right)^2\langle q_{k+1}|\hat{p}^2|q_k\rangle, \tag{2.101}$$

without ambiguity.

Observation: the time integral in Eq. (2.100) does not converge without a regularization: the integrand $e^{(i/\hbar)S}$ has modulus equal 1. A possible solution is to adopt Euclidean time $t = -i\tau$ and perform an inverse analytic continuation at the end, which is the solution adopted in the previous sections. Here we illustrate the alternative possibility of giving a small negative imaginary part to the time: $t = (1 - i\chi)\tau$

$$\begin{aligned} dt &= (1 - i\chi)d\tau \\ \dot{q} = \frac{dq}{dt} &= (1 + i\chi)\frac{dq}{d\tau} \end{aligned} \tag{2.102}$$

With this transformation, the integrand Eq. (2.100) becomes

$$e^{\frac{i}{\hbar}S_\chi} = \exp\left\{\frac{i}{\hbar}\int d\tau \left[\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 - V(q)\right]\right\} \cdot \exp\left\{-\chi\int d\tau \left[\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)\right]\right\}. \tag{2.103}$$

The integrand $\exp\left(\frac{i}{\hbar}S_\chi\right)$ has modulus equal to $\exp(-\chi I)$, where

$$I = \int d\tau \left[\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)\right] = \int d\tau\, \mathcal{H}(q, \dot{q})\,. \tag{2.104}$$

Concerning the convergence of the functional integral, we can distinguish the following cases:

- $\underline{V(q) = 0}$ this is the free particle, which we have already treated in detail; the functional integral is convergent.
- $\underline{V(q)}$ positively defined: $I > I_0$, where I_0 is the value calculated with the same trajectory for $V(q) = 0$. The degree of convergence is the same as for $V(q) = 0$.
- $\underline{V(q)}$ bounded from below: $V(q) > V_0$. In this case $I > I_0 + V_0 T$. Adding the constant $V_0 T$ does not change the convergence w.r.t. the two previous cases.
- $\underline{V(q)}$ not bounded from below: there is no general rule. For instance, if $V(q) = -q^n$, the convergence of the functional integral depends on the value of n. We only quote in passing that the integral converges if $-1 \le n \le 0$ and does not converge if $n < -1$ or $n > 0$. The Coulomb potential is a limiting case.

The cases not covered by the above classes fail also with other formulations of quantum mechanics.

2.5.4 Periodical paths in Euclidean Time

Consider the limit of the following trace:

$$\lim_{T\to\infty} \sum_{n=0}^{\infty} \langle n|e^{-2TH}|n\rangle = \lim_{T\to\infty} \sum_{n=0}^{\infty} \left(e^{-2T\omega_0}\right)^n = \lim_{T\to\infty} \frac{1}{1 - e^{-2T\omega_0}} = 1\,. \tag{2.105}$$

Since the trace is independent of the representation, we can compute it by means of the basis of position eigenstate $|q\rangle$:

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} dq \, \langle q|e^{-2TH}|q\rangle = 1 \,. \tag{2.106}$$

We can now express the above equation by means of the path-integral in the following way:

$$\begin{aligned}
1 &= \lim_{T\to\infty} \int_{-\infty}^{\infty} dq \, \langle q|e^{-2TH}|q\rangle \\
&= \lim_{T\to\infty} \int_{\infty}^{\infty} dq \, \langle q|e^{-\Delta TH} I e^{-\Delta TH} \dots I e^{-\Delta TH}|q\rangle \\
&= \lim_{T\to\infty} \int_{-\infty}^{\infty} dq_{-N+1} \dots \int_{-\infty}^{\infty} dq_N \langle q_N|e^{-\Delta TH}|q_{N-1}\rangle\langle q_{N-1}| \dots |q_{-N+1}\rangle\langle q_{-N+1}|e^{-\Delta TH}|q_{-N} = q_N\rangle \\
&= \lim_{T\to\infty} [A_E(\Delta\tau)]^{2N} \int_{-\infty}^{+\infty} dq_{-N+1} \dots \int_{-\infty}^{\infty} dq_N e^{-\Delta\tau L_E(q_{N-1},\dot{q}_{N-1})} \dots e^{-\Delta\tau L_E(q_{-N},\dot{q}_{-N})} \\
&= \lim_{T\to\infty} [A_E(\Delta\tau)]^{2N} \int_{-\infty}^{+\infty} dq_{-N+1} \dots \int_{-\infty}^{\infty} dq_N e^{-S_E(q_{-N}=q_N,\dots,q_N)} \,.
\end{aligned} \tag{2.107}$$

Observation 1: because of the trace operation, the classical paths over which we have to sum are periodic in the euclidean time $2T$.

Observation 2: because of the term $[A_E(\Delta\tau)]^{2N}$ the limit $\Delta T \to 0$ can not be taken.
We will see in the following section how to use Eq. (2.107) and circumvent the problem.

2.6 Feynman path integral for Euclidean Green Fuctions

We recall that a Green Function is defined as the vacuum expectation value of a time ordered product of field operators:

$$G(t_1, \dots, t_n) \equiv \langle 0|T[\hat{q}_H(t_1) \dots \hat{q}_H(t_n)]|0\rangle \,. \tag{2.108}$$

We will start from Euclidean Green Functions and express them in terms of the Feynman path-integral. Let's define the Heisenberg description for euclidean time (in analogy with the Minkowskian case):

$$\hat{q}_H(\tau) = e^{\tau H} \hat{q}_s e^{-\tau H} \,, \tag{2.109}$$

where $\hat{q}_s$ is the position operator in Schrödinger description. By observing that

$$\begin{aligned}
\lim_{T\to\infty} \sum_{n=0}^{\infty} \langle n|e^{-TH} \hat{q}_H(\tau_1)\hat{q}_H(\tau_2) e^{-TH}|n\rangle &= \lim_{T\to\infty} \sum_{n=0}^{\infty} e^{-2nT\omega_0} \langle n|\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)|n\rangle \\
&= \langle 0|\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)|0\rangle \,,
\end{aligned} \tag{2.110}$$

we can express the two points euclidean Green function as follows:

$$\begin{aligned}
G_E^{(2)}(\tau_1,\tau_2) &\equiv \langle 0|T\left[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)\right]|0\rangle \\
&= \lim_{T\to\infty}\sum_{n=0}^{\infty}\langle n|e^{-TH}T\left[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)\right]e^{-TH}|n\rangle \\
&= \lim_{T\to\infty}\int_{-\infty}^{+\infty}dq\,\langle q|e^{-TH}T\left[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)\right]e^{-TH}|q\rangle \\
&= \vartheta(\tau_1-\tau_2)\lim_{T\to\infty}\int_{-\infty}^{+\infty}dq\,\langle q|e^{-TH}\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)e^{-TH}|q\rangle \\
&+ \vartheta(\tau_2-\tau_1)\lim_{T\to\infty}\int_{-\infty}^{+\infty}dq\,\langle q|e^{-TH}\hat{q}_H(\tau_2)\hat{q}_H(\tau_1)e^{-TH}|q\rangle\,.
\end{aligned} \tag{2.111}$$

The trace of the above two matrix elements can be performed by means of the Feynman path integral. Taking the time interval $2T$ as in Eq. (2.107) divided in $2N$ equal subintervals ($2T = 2N\Delta\tau$), and Eq. (2.109), we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty}dq\,\langle q|e^{-TH}\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)e^{-TH}|q\rangle \\
&= \int_{-\infty}^{+\infty}dq\,\langle q|e^{-(T-\tau_1)H}\hat{q}_s e^{-(\tau_1-\tau_2)H}\hat{q}_s e^{-(T+\tau_2)H}|q\rangle\,.
\end{aligned} \tag{2.112}$$

We can assume without loss of generality that τ_1 and τ_2 coincide with points of the temporal grid spanning from $-T$ to T:

$$\begin{aligned}
\tau_1 &= i\Delta\tau \\
\tau_2 &= j\Delta\tau\,,
\end{aligned}$$

with $i, j \in [-N, N]$ and $i > j$. We can then rewrite Eq. (2.112) as follows:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}dq\,\langle q|e^{-TH}\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)e^{-TH}|q\rangle \\
&= \int_{-\infty}^{+\infty}dq\,\langle q_N|e^{-(N-i)\Delta\tau H}\hat{q}_s e^{-(i-j)\Delta\tau H}\hat{q}_s e^{-(N+j)\Delta\tau H}|q_{-N}=q_N\rangle \\
&= \int_{-\infty}^{+\infty}dq_N\int_{-\infty}^{+\infty}dq_i\int_{-\infty}^{+\infty}dq_j\,\langle q_N|e^{-(N-i)\Delta\tau H}|q_i\rangle q_i\langle q_i|e^{-(i-j)\Delta\tau H}|q_j\rangle q_j\langle q_j|e^{-(N+j)\Delta\tau H}|q_{-N}=q_N\rangle \\
&= \left[A_E(\Delta\tau)\right]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\,q_iq_je^{-\Delta\tau L_E(q_{N-1},\dot{q}_{N-1})}\ldots e^{-\Delta\tau L_E(q_{-N},\dot{q}_{-N})} \\
&= \left[A_E(\Delta\tau)\right]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\,q_iq_je^{-S_E(q_{-N}=q_N,\ldots,q_N)}\,.
\end{aligned} \tag{2.113}$$

If we had assumed from the beginning $i < j$, we would have obtained the same result. This means that we can write Eq. (2.111) as follows:

$$\begin{aligned}
G_E^{(2)}(\tau_1,\tau_2) &\equiv \langle 0|T\left[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)\right]|0\rangle \\
&= \lim_{T\to\infty}\left[A_E(\Delta\tau)\right]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\,q_iq_j\,e^{-S_E(q_{-N}=q_N,\ldots,q_N)}\,.
\end{aligned} \tag{2.114}$$

As before, the presence of the factor $A_E(\Delta\tau)$ does not allow to take the limit $\Delta\tau \to 0$. However we can consider Eq. (2.107) to introduce a regularization of Eq. (2.114), which consists in expressing a factor of 1 in the denominator of Eq. (2.114):

$$\begin{aligned} G_E^{(2)}(\tau_1,\tau_2) &\equiv \langle 0|T\,[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)]\big|0\rangle \\ &= \frac{\lim_{T\to\infty}[A_E(\Delta\tau)]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\, q_iq_j\, e^{-S_E(q_{-N}=q_N,\ldots,q_N)}}{\lim_{T\to\infty}[A_E(\Delta\tau)]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\, e^{-S_E(q_{-N}=q_N,\ldots,q_N)}}\,. \end{aligned} \tag{2.115}$$

At this point we interchange the ratio of the limits with the limit of the ratio

$$\begin{aligned} G_E^{(2)}(\tau_1,\tau_2) &\equiv \langle 0|T\,[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)]\big|0\rangle \\ &= \lim_{T\to\infty}\frac{[A_E(\Delta\tau)]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\, q_iq_j\, e^{-S_E(q_{-N}=q_N,\ldots,q_N)}}{[A_E(\Delta\tau)]^{2N}\int_{-\infty}^{+\infty}dq_{-N+1}\ldots\int_{-\infty}^{+\infty}dq_N\, e^{-S_E(q_{-N}=q_N,\ldots,q_N)}}\,. \end{aligned} \tag{2.116}$$

Since in the ratio the divergent factor $A_E(\Delta\tau)$ disappears, we can take the limit $\Delta\tau \to 0 (N\to\infty)$ before performing the limit $T\to\infty$:

$$\begin{aligned} G_E^{(2)}(\tau_1,\tau_2) &\equiv \langle 0|T\,[\hat{q}_H(\tau_1)\hat{q}_H(\tau_2)]\big|0\rangle \\ &= \frac{\int [dq]\, e^{-S_E[q]}q(\tau_1)q(\tau_2)}{\int [dq]\, e^{-S_E[q]}} \equiv \mathcal{N}\int [dq]\, e^{-S_E[q]}q(\tau_1)q(\tau_2)\,. \end{aligned} \tag{2.117}$$

Observation: remember that the classical paths over which we have to sum in the functional integrals are periodic on a euclidean time $2T\to\infty$.

Eq. (2.117) can be generalized to Green functions of arbitrary order n:

$$\begin{aligned} G_E^{(n)}(\tau_1,\ldots,\tau_n) &\equiv \langle 0|T\,[\hat{q}_H(\tau_1)\ldots\hat{q}_H(\tau_n)]\big|0\rangle \\ &= \frac{\int [dq]\, e^{-S_E[q]}q(\tau_1),\ldots,q(\tau_n)}{\int [dq]\, e^{-S_E[q]}} \\ &\equiv \mathcal{N}\int [dq]\, e^{-S_E[q]}q(\tau_1)\ldots q(\tau_n)\,. \end{aligned} \tag{2.118}$$

Observation 1: note that Eq. (2.118) allows to calculate a typically quantum object, as $G_E^{(n)}(\tau_1,\ldots,\tau_n)$ in terms of classical quantities which appear on the right-hand side of the above equation, *avoiding the operator formalism*.

Observation 2: at the right-hand side of Eq. (2.118) contribute all kinematically classical paths (weighted with the exponential of minus the Classical Euclidean Action), not only the ones allowed by the classical dynamics.

2.7 Inverse Wick rotation

Given a Euclidean Green function, we need to perform an inverse analytic continuation (w.r.t. Eq. (2.55)) to go back to the Minkowskian time. To illustrate this point, In this section we compare, for the case of the harmonic oscillator, the two-points Euclidean and Minkowskian Green functions. The last one is defined as

$$G_M(t) = \langle 0 | T\left[\hat{q}_H(t)\hat{q}_H(0)\right] | 0 \rangle = \vartheta(t)\langle 0|\hat{q}_H(t)\hat{q}_H(0)|0\rangle + \vartheta(-t)\langle 0|\hat{q}_H(0)\hat{q}_H(t)|0\rangle\,. \tag{2.119}$$

Remembering that $\hat{q}_H(t) = \exp(itH)\,\hat{q}_s\,\exp(-itH)$, we explicitely calculate the first term of the right-hand side:

$$\begin{aligned}
\langle 0|\hat{q}_H(t)\hat{q}_H(0)|0\rangle &= \sum_{n_{0,1,2,3,4}=0}^{\infty} \langle 0|n_0\rangle\langle n_0|e^{itH}|n_1\rangle\langle n_1|\hat{q}_s|n_2\rangle\langle n_2|e^{-itH}|n_3\rangle\langle n_3|\hat{q}_s|n_4\rangle\langle n_4|0\rangle \\
&= \sum_{n_{0,1,2,3,4}=0}^{\infty} e^{i(n_1-n_3)\omega_0 t}\langle 0|n_0\rangle\langle n_0|n_1\rangle\langle n_1|\hat{q}_s|n_2\rangle\langle n_2|n_3\rangle\langle n_3|\hat{q}_s|n_4\rangle\langle n_4|0\rangle \\
&= \sum_{n_{0,1,2,3,4}=0}^{\infty} e^{i(n_1-n_3)\omega_0 t}\delta_{n_0,0}\delta_{n_0,n_1}\langle n_1|\hat{q}_s|n_2\rangle\delta_{n2,n3}\langle n_3|\hat{q}_s|n_4\rangle\delta_{n_4,0} \\
&= \sum_{n=0}^{\infty} e^{-in\omega_0 t}\langle 0|\hat{q}_s|n\rangle\langle n|\hat{q}_s|0\rangle\,.
\end{aligned} \tag{2.120}$$

Remembering now that

$$\hat{q}_s = \frac{1}{\sqrt{2\omega_0}}\left(a + a^\dagger\right)\,, \tag{2.121}$$

we have

$$\begin{aligned}
\langle 0|\vartheta(t)\hat{q}_H(t)\hat{q}_H(0)|0\rangle &= \frac{1}{2\omega_0}\sum_{n=0}^{\infty} e^{-in\omega_0 t}\langle 0|\left(a+a^\dagger\right)|n\rangle\langle n|\left(a+a^\dagger\right)|0\rangle \\
&= \frac{1}{2\omega_0}\sum_{n=0}^{\infty} e^{-in\omega_0 t}\delta_{n,1}\delta_{n,1} \\
&= \frac{1}{2\omega_0}e^{-i\omega_0 t}\,.
\end{aligned} \tag{2.122}$$

By inspection of the above calculation, the second term on the right-hand side of Eq. (2.119) reads

$$\langle 0|\hat{q}_H(0)\hat{q}_H(t)|0\rangle = \frac{1}{2\omega_0}e^{+i\omega_0 t}\,. \tag{2.123}$$

Hence Eq. (2.119) becomes

$$G_M(t) = \frac{1}{2\omega_0}\left[\vartheta(t)e^{-i\omega_0 t} + \vartheta(-t)e^{i\omega_0 t}\right] \tag{2.124}$$

Consider now the Fourier transform of $G_M(t)$:

$$\begin{aligned}
\hat{G}_M(\omega) &\equiv \int_{-\infty}^{+\infty} dt\, e^{i\omega t} G_M(t) \\
&= \frac{1}{2\omega_0} \int_{-\infty}^{+\infty} dt\, \left[\vartheta(t) e^{i(\omega-\omega_0)t} + \vartheta(-t) e^{i(\omega+\omega_0)t}\right] \\
&= \frac{1}{2\omega_0} \left\{ \int_{-\infty}^{+\infty} dt\, \vartheta(t) e^{i(\omega-\omega_0)t} + \int_{-\infty}^{+\infty} dy\, \vartheta(y) e^{-i(\omega+\omega_0)y} \right\} .
\end{aligned} \tag{2.125}$$

Since the Fourier transform of the Heaviside theta function is[2]

$$\hat{\vartheta}(\omega) \equiv \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \vartheta(t) = \frac{i}{\omega + i\varepsilon}, \tag{2.128}$$

we can write

$$\begin{aligned}
\hat{G}_M(\omega) &= \frac{1}{2\omega_0} \left\{ \hat{\vartheta}(\omega-\omega_0) + \hat{\vartheta}(-\omega-\omega_0) \right\} \\
&= \frac{1}{2\omega_0} \left\{ \frac{i}{\omega-\omega_0+i\varepsilon} + \frac{i}{-\omega-\omega_0+i\varepsilon} \right\} \\
&= \frac{i(\omega_0 - i\varepsilon)}{\omega_0(\omega+\omega_0-i\varepsilon)(\omega-\omega_0+i\varepsilon)} \simeq \frac{i}{(\omega+\omega_0-i\varepsilon)(\omega-\omega_0+i\varepsilon)} .
\end{aligned} \tag{2.129}$$

By Fourier antitransforming we can obtain $G_M(t)$:

$$G_M(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega\, \frac{e^{-i\omega t}}{(\omega+\omega_0-i\varepsilon)(\omega-\omega_0+i\varepsilon)} \tag{2.130}$$

The analytical structure of $G_M(t)$ presents two poles, slightly displaced from the real axis:

$$\begin{aligned}
\omega_1 &= -\omega_0 + i\varepsilon \\
\omega_2 &= \omega_0 - i\epsilon .
\end{aligned} \tag{2.131}$$

Observation: remembering that

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk \quad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx}\, dk , \tag{2.132}$$

---

[2]Remember the integral representation of the Heaviside theta function

$$\vartheta(t) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\exp(-i\omega t)}{\omega + i\varepsilon} d\omega , \tag{2.126}$$

as can be shown by application of the residue theorem, and the definition of Fourier inverse tranform

$$\vartheta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega t) \hat{\vartheta}(\omega) d\omega . \tag{2.127}$$

we can verify that $G_M(t)$ of Eq. (2.130) satisfies the following inhomogeneus differential equation:

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)G_M(t) = -i\delta(t) \tag{2.133}$$

Proof:

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{(-\omega^2+\omega_0^2)e^{-i\omega t}}{(\omega+\omega_0-i\varepsilon)(\omega-\omega_0+i\varepsilon)}d\omega = \frac{-i}{2\pi}\int_{-\infty}^{+\infty}e^{-i\omega t}d\omega \tag{2.134}$$

The Euclidean Green function is defined as follows

$$G_E(\tau) = \langle 0|T\left[\hat{q}_H(\tau)\hat{q}_H(0)\right]|0\rangle = \vartheta(\tau)\langle 0|\hat{q}_H(\tau)\hat{q}_H(0)|0\rangle + \vartheta(-\tau)\langle 0|\hat{q}_H(0)\hat{q}_H(\tau)|0\rangle\,, \tag{2.135}$$

where $\hat{q}_H(\tau) = \exp\left(\tau H\right)\hat{q}_s\,\exp\left(-\tau H\right)$, we explicitely calculate the first term of the right-hand side:

$$\begin{aligned}
\langle 0|\hat{q}_H(\tau)\hat{q}_H(0)|0\rangle &= \sum_{n_{0,1,2,3,4}=0}^{\infty}\langle 0|n_0\rangle\langle n_0|e^{\tau H}|n_1\rangle\langle n_1|\hat{q}_s|n_2\rangle\langle n_2|e^{-\tau H}|n_3\rangle\langle n_3|\hat{q}_s|n_4\rangle\langle n_4|0\rangle \\
&= \sum_{n_{0,1,2,3,4}=0}^{\infty} e^{(n_1-n_3)\omega_0\tau}\langle 0|n_0\rangle\langle n_0|n_1\rangle\langle n_1|\hat{q}_s|n_2\rangle\langle n_2|n_3\rangle\langle n_3|\hat{q}_s|n_4\rangle\langle n_4|0\rangle \\
&= \sum_{n_{0,1,2,3,4}=0}^{\infty} e^{(n_1-n_3)\omega_0\tau}\delta_{n_0,0}\delta_{n_0,n_1}\langle n_1|\hat{q}_s|n_2\rangle\delta_{n2,n3}\langle n_3|\hat{q}_s|n_4\rangle\delta_{n_4,0} \\
&= \sum_{n=0}^{\infty} e^{-n\omega_0\tau}\langle 0|\hat{q}_s|n\rangle\langle n|\hat{q}_s|0\rangle\,.
\end{aligned} \tag{2.136}$$

Introducing Eq. (2.121), we have

$$\begin{aligned}
\langle 0|\vartheta(t)\hat{q}_H(\tau)\hat{q}_H(0)|0\rangle &= \frac{1}{2\omega_0}\sum_{n=0}^{\infty}e^{-n\omega_0\tau}\langle 0|\left(a+a^\dagger\right)|n\rangle\langle n|\left(a+a^\dagger\right)|0\rangle \\
&= \frac{1}{2\omega_0}\sum_{n=0}^{\infty}e^{-n\omega_0\tau}\delta_{n,1}\delta_{n,1} \\
&= \frac{1}{2\omega_0}e^{-\omega_0\tau}\,.
\end{aligned} \tag{2.137}$$

By inspection of the above calculation, the second term on the right-hand side of Eq. (2.135) reads

$$\langle 0|\hat{q}_H(0)\hat{q}_H(\tau)|0\rangle = \frac{1}{2\omega_0}e^{\omega_0\tau}\,. \tag{2.138}$$

Hence Eq. (2.135) becomes

$$G_E(\tau) = \frac{1}{2\omega_0}\left[\vartheta(\tau)e^{-\omega_0\tau}+\vartheta(-\tau)e^{\omega_0\tau}\right] = \frac{e^{-|\tau|\omega_0}}{2\omega_0} \tag{2.139}$$

Consider now the Fourier transform of $G_E(\tau)$:

$$\begin{aligned}
\hat{G}_E(\omega_E) &\equiv \int_{-\infty}^{+\infty} d\tau\, e^{i\omega_E\tau} G_E(\tau) \\
&= \frac{1}{2\omega_0}\int_{-\infty}^{+\infty} d\tau\, \left[\vartheta(\tau)e^{(i\omega_E-\omega_0)\tau} + \vartheta(-\tau)e^{(i\omega_E+\omega_0)\tau}\right] \\
&= \frac{1}{2\omega_0}\int_{-\infty}^{+\infty} d\tau\, \left[\vartheta(\tau)e^{(i\omega_E+i^2\omega_0)\tau} + \vartheta(-\tau)e^{(i\omega_E-i^2\omega_0)\tau}\right] \\
&= \frac{1}{2\omega_0}\left\{\int_{-\infty}^{+\infty} d\tau\, \vartheta(\tau)e^{i(\omega_E+i\omega_0)\tau} + \int_{-\infty}^{+\infty} dy\, \vartheta(y)e^{i(-\omega_E+i\omega_0)y}\right\}.
\end{aligned} \tag{2.140}$$

Recalling the Fourier transform of the Heaviside theta function, Eq. (2.128), we can write

$$\begin{aligned}
\hat{G}_E(\omega_E) &= \frac{1}{2\omega_0}\left\{\hat{\vartheta}(\omega_E+i\omega_0) + \hat{\vartheta}(-\omega_E+i\omega_0)\right\} \\
&= \frac{1}{2\omega_0}\left\{\frac{i}{\omega_E+i\omega_0} - \frac{i}{\omega_E-i\omega_0}\right\} \\
&= \frac{1}{(\omega_E+i\omega_0)(\omega_E-i\omega_0)} = \frac{1}{\omega_E^2+\omega_0^2}.
\end{aligned} \tag{2.141}$$

By Fourier antitransforming we can obtain $G_E(\tau)$:

$$G_E(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega_E\, \frac{e^{-i\omega_E\tau}}{(\omega_E+i\omega_0)(\omega_E-i\omega_0)} \tag{2.142}$$

The analytical structure of $G_E(\tau)$ presents two poles, on the imaginary axis of ω_E:

$$\begin{aligned}
\omega_{E_1} &= -i\omega_0 \\
\omega_{E_2} &= i\omega_0\,.
\end{aligned} \tag{2.143}$$

Observation: remembering that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\, dk\,, \tag{2.144}$$

we can verify that $G_E(\tau)$ of Eq. (2.142) satisfies the following inhomogeneus differential equation:

$$\left(\frac{d^2}{d\tau^2} - \omega_0^2\right) G_E(t) = -\delta(\tau) \tag{2.145}$$

Proof:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega_E\, \frac{(-\omega_E^2-\omega_0^2)e^{-i\omega_E\tau}}{(\omega_E+i\omega_0)(\omega_E-i\omega_0)} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega\tau} d\omega\,. \tag{2.146}$$

Now the problem is to find the relation which allows to recover $\hat{G}_M(\omega)$ from $\hat{G}_E(\omega_E)$. By inspection of Eqs. (2.129,2.141), if ω is real and

$$\omega_E = \omega e^{-i\left(\frac{\pi}{2}-\varepsilon\right)} = \omega\left[\cos\left(\frac{\pi}{2}-\varepsilon\right) - i\sin\left(\frac{\pi}{2}-\varepsilon\right)\right] \simeq \omega(\varepsilon - i)\,, \tag{2.147}$$

we have

$$\begin{aligned}\hat{G}_E\left(\omega e^{-i\left(\frac{\pi}{2}-\varepsilon\right)}\right) &\simeq \frac{1}{[\omega(-i+\varepsilon)+i\omega_0][\omega(-i+\varepsilon)-i\omega_0]} \\ &= \frac{1}{[-i(\omega-\omega_0)+\omega\varepsilon][-i(\omega+\omega_0)+\omega\varepsilon]}\,.\end{aligned} \tag{2.148}$$

In the above equation, the term $\omega\varepsilon$ can be put equal to $\omega_0\varepsilon$ in the first factor and equal to $-\omega_0\varepsilon$ because it matters only in proximity of the poles:

$$\begin{aligned}\hat{G}_E\left(\omega e^{-i\left(\frac{\pi}{2}-\varepsilon\right)}\right) &\simeq \frac{1}{[-i(\omega-\omega_0)+\omega_0\varepsilon][-i(\omega+\omega_0)-\omega_0\varepsilon]} \\ &= \frac{1}{[-i(\omega-\omega_0)-i^2\omega_0\varepsilon][-i(\omega+\omega_0)+i^2\omega_0\varepsilon]} \\ &= \frac{1}{(-i)[\omega-\omega_0+i\omega_0\varepsilon](-i)[(\omega+\omega_0)-i\omega_0\varepsilon]} \\ &= \frac{-1}{[\omega-\omega_0+i\omega_0\varepsilon][(\omega+\omega_0)-i\omega_0\varepsilon]} \\ &= +i\hat{G}_M(\omega)\,.\end{aligned} \tag{2.149}$$

Eq. (2.149) gives the relation which allows us to calculate the Minkowskian Green function once we know the Euclidean one:

$$\hat{G}_M(\omega) = -i\hat{G}_E\left(\omega e^{-i\left[\frac{\pi}{2}-\varepsilon\right]}\right) = -i\hat{G}_E\left(\omega[-i+\varepsilon]\right)\,. \tag{2.150}$$

Once we know $\hat{G}_M(\omega)$, we can calculate $G_M(t)$ through inverse Fourier transform. Also $G_M(t)$ and $G_E(\tau)$ are related by an equation similar to Eq. (2.150):

$$G_M(t) = G_E\left(te^{i\left[\frac{\pi}{2}-\varepsilon\right]}\right) = G_E\left(t[i+\varepsilon]\right)\,. \tag{2.151}$$

A pragmatic way to obtain Eq. (2.151) is to modify Eq. (2.139) in this way:

$$G_E(\tau) = \frac{1}{2\omega_0}\left[\vartheta(\mathrm{Re}\tau)e^{-\omega_0\tau} + \vartheta(-\mathrm{Re}\tau)e^{\omega_0\tau}\right]\,, \tag{2.152}$$

valid for complex τ. By inspection of the analytical structure of $G_E(\tau)$ as given above, we see that it has a cut along the imaginary axis. In particular, if $\tau = t\exp\left(i\left[\frac{\pi}{2}-\varepsilon\right]\right)$ the cut is not crossed, because

$$\vartheta(\mathrm{Re}\tau) = \vartheta(t)\,.$$

Inserting this result and the above expression of τ in Eq. (2.152) gives, in the limit $\varepsilon \to 0$, the right-hand side of Eq. (2.124).

2.8 Green functions for the forced harmonic oscillator

Let us consider the harmonic forced oscillator, i.e. under the action of a given external force $F(t)$. We calculate the transition amplitude from $q = Q$ at time T and $q = Q'$ at time t:

$$\langle Q'_t | Q_T \rangle_F = \int d[q] \, e^{i \int_T^t dt' \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega_0^2 q^2 + F(t') q(t') \right]} , \tag{2.153}$$

where the states $|Q_T\rangle$ and $|Q'_t\rangle$ are in the Heisenberg representation. In order to regularize the integral we add a damping term of the form $\frac{i}{2}\varepsilon \int_T^t dt' q^2(t')$, with $\varepsilon > 0$. Eq. (2.153) becomes

$$\langle Q'_t | Q_T \rangle_F = \int d[q] \, e^{i \int_T^t dt' \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\left(\omega_0^2 - i\varepsilon\right) q^2 + F(t') q(t') \right]} . \tag{2.154}$$

Let us analyze Eq. (2.154) for $T \to -\infty$ and $t \to +\infty$, *i.e.* we calculate the transition amplitude for the system to be in the groundstate in the distant past and for it to be in the ground state in the distant future, after an arbitrary external source term $F(t)q(t)$ has been switched on between T and t. To this aim it is useful to consider the Fourier transforms:

$$\begin{aligned} \tilde{q}(\omega) &= \int_{-\infty}^{+\infty} dt \, e^{i\omega t} q(t) , \\ q(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \tilde{q}(\omega) . \end{aligned} \tag{2.155}$$

$$\begin{aligned} q^2(t) &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \, e^{-i(\omega+\omega')t} \tilde{q}(\omega)\tilde{q}(\omega') , \\ \dot{q}(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \, (-i\omega) \, e^{-i\omega t} \tilde{q}(\omega) , \\ \left(\dot{q}(t)\right)^2 &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \, \left(-\omega\omega'\right) e^{-i(\omega+\omega')t} \tilde{q}(\omega)\tilde{q}(\omega') , \\ F(t)q(t) &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \, e^{-i(\omega+\omega')t} \, \frac{\left[\tilde{q}(\omega)\tilde{F}(\omega') + \tilde{q}(\omega')\tilde{F}(\omega)\right]}{2} , \end{aligned} \tag{2.156}$$

where in the last equation we have symmetrized the product $\tilde{q}(\omega)\tilde{F}(\omega')$ w.r.t. ω and ω'. Substituting the above Fourier transforms in Eq. (2.154) we have

$$\begin{aligned} & \int_T^t dt' \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\left(\omega_0^2 - i\varepsilon\right) q^2 + F(t')q(t') \right] \\ = \; & \frac{1}{4\pi^2}\frac{1}{2} \int_T^t dt' \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' e^{-i(\omega+\omega')t'} \\ \times \; & \left\{ \left[-\omega\omega' - \omega_0^2 + i\varepsilon \right] \tilde{q}(\omega)\tilde{q}(\omega') + \tilde{q}(\omega)\tilde{F}(\omega') + \tilde{q}(\omega')\tilde{F}(\omega) \right\} . \end{aligned} \tag{2.157}$$

Interchanging the order of integration, taking the limits $T \to -\infty$ and $t \to \infty$, and considering that

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dt' \, e^{-i(\omega+\omega')t'} = \delta\left(-(\omega+\omega')\right) = \delta\left((\omega+\omega')\right) , \tag{2.158}$$

$$\int_T^t dt' \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\left(\omega_0^2 - i\varepsilon\right) q^2 + F(t')q(t')\right]$$
$$= \frac{2\pi}{4\pi^2}\frac{1}{2}\int_{\infty}^{+\infty} d\omega \left\{(\omega^2-\omega_0^2+i\varepsilon)\tilde{q}(\omega)\tilde{q}(-\omega) + \tilde{q}(\omega)\tilde{F}(-\omega) + \tilde{q}(-\omega)\tilde{F}(\omega)\right\} \quad (2.159)$$

If we name $g(\omega)$ the integrand function in Eq. (2.159), Eq. (2.154) becomes

$$\langle Q'_\infty | Q_{-\infty}\rangle_F = \int d[q]\, e^{\frac{i}{2\pi}\frac{1}{2}\int_{-\infty}^{+\infty} d\omega\, g(\tilde{q}(\omega))}\,. \quad (2.160)$$

The function $g\left(\tilde{q}(\omega)\right)$ contains terms quadratic as well as linear in $\tilde{q}(\omega)$. In order to cancel the linear terms, it is useful to perform the following "shift":

$$\tilde{q}(\omega) = \tilde{x}(\omega) - \frac{\tilde{F}(\omega)}{\left(\omega^2-\omega_0^2+i\varepsilon\right)}\,. \quad (2.161)$$

Expressing $g\left(\tilde{q}(\omega)\right)$ in terms of $\tilde{x}(\omega)$, Eq. (2.160) becomes

$$\begin{aligned}\langle Q'_\infty | Q_{-\infty}\rangle_F &= \int d[q]\, e^{\frac{i}{2\pi}\frac{1}{2}\int_{-\infty}^{+\infty} d\omega \left\{(\omega^2-\omega_0^2+i\varepsilon)\tilde{x}(\omega)\tilde{x}(-\omega) - \frac{\tilde{F}(\omega)\tilde{F}(-\omega)}{(\omega^2-\omega_0^2+i\varepsilon)}\right\}} \\ &= e^{\frac{i}{2\pi}\left(-\frac{1}{2}\right)\int_{-\infty}^{+\infty} d\omega\, \frac{\tilde{F}(\omega)\tilde{F}(-\omega)}{(\omega^2-\omega_0^2+i\varepsilon)}} \int d[q]\, e^{\frac{i}{2\pi}\frac{1}{2}\int_{-\infty}^{+\infty} d\omega\, \tilde{x}(\omega)(\omega^2-\omega_0^2+i\varepsilon)\tilde{x}(-\omega)} \end{aligned} \quad (2.162)$$

Considering the Fourier antitransforms of both members of Eq. (2.161), according to Eq.(2.155), we can write

$$q(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\frac{\tilde{F}(\omega)}{\left(\omega^2-\omega_0^2+i\varepsilon\right)}\,. \quad (2.163)$$

Eq. (2.163) can be seen as a functional change of variable in the functional integration of Eq. (2.162), with

$$d\left[q(t)\right] = d\left[x(t)\right]\,. \quad (2.164)$$

Therefore Eq. (2.162) can be written, with further renaming of x with q, as

$$\langle Q'_\infty | Q_{-\infty}\rangle_F = e^{\frac{i}{2\pi}\left(-\frac{1}{2}\right)\int_{-\infty}^{+\infty} d\omega\, \frac{\tilde{F}(\omega)\tilde{F}(-\omega)}{(\omega^2-\omega_0^2+i\varepsilon)}} \int d[q(t)]\, e^{\frac{i}{2\pi}\frac{1}{2}\int_{-\infty}^{+\infty} d\omega\, \tilde{q}(\omega)(\omega^2-\omega_0^2+i\varepsilon)\tilde{q}(-\omega)}\,. \quad (2.165)$$

We observe now that the functional integral in Eq. (2.165) is the groundstate-to-groundstate transition with the external force set to zero. Hence we can write

$$\langle Q'_\infty | Q_{-\infty}\rangle_F = \langle Q'_\infty | Q_{-\infty}\rangle_{F=0}\, e^{\left(-\frac{i}{2}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} d\omega\, \frac{\tilde{F}(\omega)\tilde{F}(-\omega)}{(\omega^2-\omega_0^2+i\varepsilon)}}\,. \quad (2.166)$$

We write Eq. (2.166) expressing the function $\tilde{F}(\omega)$ in terms of $F(t)$:

$$\begin{aligned}\langle Q'_\infty | Q_{-\infty}\rangle_F &= \langle Q'_\infty | Q_{-\infty}\rangle_{F=0}\, e^{\left(-\frac{i}{2}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} d\omega\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt'\, \frac{F(t)e^{i\omega(t-t')}F(t')}{(\omega^2-\omega_0^2+i\varepsilon)}} \\ &= \langle Q'_\infty | Q_{-\infty}\rangle_{F=0}\, e^{\left(-\frac{i}{2}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt'\int_{-\infty}^{+\infty} d\omega\, \frac{F(t)e^{i\omega(t-t')}F(t')}{(\omega^2-\omega_0^2+i\varepsilon)}} \\ &= \langle Q'_\infty | Q_{-\infty}\rangle_{F=0}\, e^{\left(-\frac{i}{2}\right)\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dt'\, F(t)D(t-t')F(t')}\,, \end{aligned} \quad (2.167)$$

where we have defined

$$D(t-t') = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} d\omega\, \frac{e^{i\omega(t-t')}}{(\omega^2-\omega_0^2+i\varepsilon)}\,. \tag{2.168}$$

We remind that $|Q_t\rangle_F$ is the state describing the oscillator at position Q at time t, in the presence of the driving force. So $|Q_\infty\rangle_F$ describes the oscillator at position Q in the distant past. Let us call $|\Omega_{\pm\infty}\rangle$ the ground states in the infinite past and future, before and after the action of the driving force. We can write

$$\langle\Omega_{+\infty}|\Omega_{-\infty}\rangle_F = \int dQ'dQ\,\langle\Omega_{+\infty}|Q'_{+\infty}\rangle\langle Q'_{+\infty}|Q_{-\infty}\rangle_F\langle Q_{-\infty}|\Omega_{-\infty}\rangle \tag{2.169}$$

$$\begin{aligned} &= \int dQ'dQ\,\langle\Omega_{+\infty}|Q'_{+\infty}\rangle\langle Q'_{+\infty}|Q_{-\infty}\rangle_{F=0}\langle Q_{-\infty}|\Omega_{-\infty}\rangle \\ &\quad\times e^{\left(-\frac{i}{2}\right)\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dt'\,F(t)D(t-t')F(t')}\,, \end{aligned} \tag{2.170}$$

where we have used Eq. (2.167). On the other hand, from Eq. (2.169) (as well as from Eq. (2.170)), we can write

$$\langle\Omega_{+\infty}|\Omega_{-\infty}\rangle_{F=0} = \int dQ'dQ\,\langle\Omega_{+\infty}|Q'_{+\infty}\rangle\langle Q'_{+\infty}|Q_{-\infty}\rangle_{F=0}\langle Q_{-\infty}|\Omega_{-\infty}\rangle\,. \tag{2.171}$$

From Eqs. (2.170) and (2.171) se can write the groundstate-to-groundstate transition amplitude from $t=-\infty$ to $t=+\infty$, after the action for a finite time of the driving force, w.r.t. the free dynamics, as

$$\frac{\langle\Omega'_\infty|\Omega_{-\infty}\rangle_F}{\langle\Omega'_\infty|\Omega_{-\infty}\rangle_{F=0}} = e^{\left(-\frac{i}{2}\right)\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dt'\,F(t)D(t-t')F(t')}\,. \tag{2.172}$$

Remark 1: $\frac{\langle\Omega'_\infty|\Omega_{-\infty}\rangle_F}{\langle\Omega'_\infty|\Omega_{-\infty}\rangle_{F=0}}$ is a *functional of the driving force F, $Z[F]$, satisfying the condition* $Z[0]=1$.

Remark 2: $\langle\Omega'_\infty|\Omega_{-\infty}\rangle_{F=0}$ is the transition amplitude to pass from the groundstate at $t=-\infty$ to the groundstate at $t=+\infty$, without any external driving force, i.e. a phase factor.

2.8.1 Meaning of the function $D(t)$

We note that we have already calculated the integral of Eq. (2.168),

$$\begin{aligned} D(t-t') &= \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} d\omega\, \frac{e^{i\omega(t-t')}}{(\omega^2-\omega_0^2+i\varepsilon)} \\ &= \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} d\omega\, \frac{e^{i\omega(t-t')}}{(\omega+\omega_0-i\frac{\varepsilon}{2})(\omega-\omega_0+i\frac{\varepsilon}{2})}\,. \end{aligned}$$

In fact we had seen that

$$G_M(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega \, \frac{e^{-i\omega t}}{(\omega+\omega_0-i\varepsilon)(\omega-\omega_0+i\varepsilon)} = \frac{1}{2\omega_0}\left[\vartheta(t)e^{-i\omega_0 t} + \vartheta(-t)e^{i\omega_0 t}\right] . \tag{2.173}$$

The application of the residue theorem shows that we obtain the same result if at the numerator of the integrand we have $\exp\{+i\omega t\}$ or $\exp\{-i\omega t\}$. Therefore we have

$$D(t) = \frac{1}{2i\omega_0}\left[\vartheta(t)e^{-i\omega_0 t} + \vartheta(-t)e^{i\omega_0 t}\right] . \tag{2.174}$$

We had also seen that

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) G_M(t) = -i\delta(t) , \tag{2.175}$$

hence

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) D(t) = -\delta(t) . \tag{2.176}$$

Eq. (2.176) says that $D(t)$ is a Green function of the operator $\frac{d^2}{dt^2} + \omega_0^2$ and the infinitesimal imaginary part (with its sign) fixes the boundary conditions. $D(t)$ **is the Feynman propagator: according to Eq. (2.174) it propagates forward in time the positive frequencies and backward in time the negative frequencies, because of the Heavyside** $\vartheta(t)$ **functions.**

2.8.2 Functional derivatives of the Groundstate-to-Groundstate transition amplitudes

For the harmonic oscillator, let us consider the ratio of the propagator in the presence of an external driving force between time t_i and t_f over the same propagator without external driving force:

$$\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle} = \frac{\int [dq] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt' \, \mathcal{L}(q,\dot{q}) + \hbar F(t') q(t')}}{\int [dq] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt' \, \mathcal{L}(q,\dot{q})}} . \tag{2.177}$$

The ratio of transition amplitudes in Eq. (2.177) is defined starting from its discretized form as follows:

$$\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle} = \frac{[A_M(\Delta t)]^N \left[\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dq_j\right] \exp\left[\frac{i}{\hbar}\Delta t \sum_{k=0}^{N-1}\{L(q_k,\dot{q}_k) + \hbar F_k q_k\}\right]}{[A_M(\Delta t)]^N \left[\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dq_j\right] \exp\left[\frac{i}{\hbar}\Delta t \sum_{k=0}^{N-1}\{L(q_k,\dot{q}_k)\}\right]} . \tag{2.178}$$

Simplifying the overall factors $A_M(\Delta t)$ between numerator and denominator and taking the continuum limit, Eq. (2.177) follows from Eq. (2.178). The ratio of transition amplitudes is a functional of the driving force $Z[F]$. We can consider its functional derivative [3]

[3]See Appendix B for the definition of functional derivatives.

w.r.t. the driving force:

$$\frac{\delta}{\delta F(t_1)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right) = \frac{\int [dq]\ (iq(t_1))\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})+\hbar F(t')q(t')}}{\int [dq]\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})}}\,. \tag{2.179}$$

If we set $F = 0$ after the functional derivation, Eq. (2.179) becomes

$$\left[\frac{\delta}{\delta F(t_1)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right)\right]_{F=0} = \frac{\int [dq]\ (iq(t_1))\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})}}{\int [dq]\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})}}\,. \tag{2.180}$$

Let us now introduce n times the identity in the transition amplitude

$$\langle q(t_f)|q(t_i)\rangle = \int dq(t_1)\dots dq(t_n)\, \langle q(t_f)|q(t_n)\rangle \dots \langle q(t_1)|q(t_i)\rangle\,. \tag{2.181}$$

If we multiply both sides of Eq. (2.181) by $q(t_1)$, we can write:

$$\begin{aligned}
&\int dq(t_1)\dots dq(t_n)\, \langle q(t_f)|q(t_n)\rangle \dots \langle q(t_1)|q(t_i)\rangle q(t_1) \\
=\ &\int dq(t_1)\dots dq(t_n)\, \langle q(t_f)|q(t_n)\rangle \dots \langle q(t_1)|\hat{q}(t_1)|q(t_i)\rangle \\
=\ &\langle q(t_f)|\hat{q}(t_1)|q(t_i)\rangle\,.
\end{aligned} \tag{2.182}$$

Thus the numerator of Eq. (2.180) is equal to $i\langle q(t_f)|\hat{q}(t_1)|q(t_i)\rangle$ and we can write

$$\left[\frac{\delta}{\delta F(t_1)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right)\right]_{F=0} = \frac{\int [dq]\ (iq(t_1))\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})}}{\int [dq]\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})}} = i\frac{\langle q(t_f)|\hat{q}(t_1)|q(t_i)\rangle}{\langle q(t_f)|q(t_i)\rangle}\,. \tag{2.183}$$

The functional derivative of the propagator w.r.t. the external force gives the transition matrix element of the coordinate $\hat{q}$, with both sides normalized to the propagator without external force.

By applying higher-order functional derivatives to the ratio of amplitudes of Eq. (2.177), we obtain

$$\frac{\delta^n}{\delta F(t_1)\dots\delta F(t_n)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right) = \frac{(i)^n\int [dq]\ (q(t_1)\dots q(t_n))\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})+\hbar F(t')q(t')}}{\int [dq]\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\, \mathcal{L}(q,\dot{q})}}\,, \tag{2.184}$$

i.e. each functional derivative "brings down" a factor $iq(t)$. We could guess that

$$\left[\frac{\delta^n}{\delta F(t_1)\dots\delta F(t_n)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right)\right]_{F=0} = (i)^n\frac{\langle q(t_f)|\hat{q}(t_1)\dots\hat{q}(t_n)|q(t_i)\rangle}{\langle q(t_f)|q(t_i)\rangle}\,, \tag{2.185}$$

but there is a subtlety which we illustrate for the case $n = 2$:

$$\left[\frac{\delta^2}{\delta F(t_\alpha)\delta F(t_\beta)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right)\right]_{F=0} = \frac{(i)^2\int [dq]\ (q(t_\alpha)q(t_\beta))\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\,\mathcal{L}(q,\dot q)}}{\int [dq]\ e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt'\,\mathcal{L}(q,\dot q)}}. \tag{2.186}$$

If $t_\alpha > t_\beta$, following the same reasoning of Eq. (2.182), we can write

$$\begin{aligned}
&\int dq(t_1)\dots dq(t_n)\,\langle q(t_f)|q(t_n)\rangle\dots\langle q(t_1)|q(t_i)\rangle q(t_\alpha)q(t_\beta)\\
=\ &\int dq(t_1)\dots dq(t_n)\,\langle q(t_f)|q(t_n)\rangle\dots\langle q(t_{\alpha+1})|\hat q(t_\alpha)|q(t_\alpha)\rangle\\
&\dots\langle q(t_{\beta+1})|\hat q(t_\beta)|q(t_\beta)\rangle\langle q(t_1)|\hat q(t_1)|q(t_i)\rangle\\
=\ &\langle q(t_f)|\hat q(t_\alpha)\hat q(t_\beta)|q(t_i)\rangle\,.
\end{aligned} \tag{2.187}$$

For this case the guess of Eq. (2.185) is correct.
However, if $t_\alpha < t_\beta$ the operators $\hat q(t_\alpha)$ and $\hat q(t_\beta)$ would appear in reversed order in Eq. (2.187). So, in order to obtain the correct result, we need to insert the time-ordering operator:

$$\left[\frac{\delta^2}{\delta F(t_\alpha)\delta F(t_\beta)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right)\right]_{F=0} = (i)^2\frac{\langle q(t_f)|T\left[\hat q(t_\alpha)\hat q(t_\beta)\right]|q(t_i)\rangle}{\langle q(t_f)|q(t_i)\rangle}. \tag{2.188}$$

For generic n we have

$$\left[\frac{\delta^n}{\delta F(t_1)\dots\delta F(t_n)}\left(\frac{\langle q(t_f)|q(t_i)\rangle_F}{\langle q(t_f)|q(t_i)\rangle}\right)\right]_{F=0} = (i)^n\frac{\langle q(t_f)|T\left[\hat q(t_1)\dots\hat q(t_n)\right]|q(t_i)\rangle}{\langle q(t_f)|q(t_i)\rangle}. \tag{2.189}$$

We consider now the limit $t_i \to -\infty$ and $t_f \to +\infty$ in both numerator and denominator:

$$\begin{aligned}
\langle q(t_f)|T\left[\hat q(t_1)\dots\hat q(t_n)\right]|q(t_i)\rangle &= \sum_{n,m=0}^{\infty}\langle q(t_f)|n\rangle\langle n|T\left[\hat q(t_1)\dots\hat q(t_n)\right]|m\rangle\langle m|q(t_i)\rangle\\
&= \sum_{n,m=0}^{\infty}\langle q(t_f)|e^{-\frac{i}{\hbar}t_f\hat H}|n\rangle\langle n|T\left[\hat q(t_1)\dots\hat q(t_n)\right]|m\rangle\langle m|e^{+\frac{i}{\hbar}t_i\hat H}|q(t_i)\rangle\\
&= \sum_{n,m=0}^{\infty} e^{-\frac{i}{\hbar}\omega_n t_f}\varphi_n(q_f)\langle n|T\left[\hat q(t_1)\dots\hat q(t_n)\right]|m\rangle e^{+\frac{i}{\hbar}\omega_m t_i}\varphi_m^*(q_i)\,.
\end{aligned}$$

According to Eq. (2.154) the Hamiltonian is $\frac{p^2}{2} + \frac{1}{2}\left(\omega_0^2 - i\varepsilon\right)q^2$ and the eigenvalues are $\omega_n = n\omega_0 - i\varepsilon$. As a consequence

$$e^{-\frac{i}{\hbar}\omega_n t_f} = e^{-\frac{i}{\hbar}n\omega_0 t_f}e^{-n\varepsilon t_f} \tag{2.190}$$

$$e^{+\frac{i}{\hbar}\omega_m t_i} = e^{+\frac{i}{\hbar}m\omega_0 t_f}e^{+m\varepsilon t_i} \tag{2.191}$$

and we have

$$\begin{aligned}\langle q(t_f)|T[\hat{q}(t_1)\ldots\hat{q}(t_n)]|q(t_i)\rangle &= \varphi_0(q_f)\langle 0|T[\hat{q}(t_1)\ldots\hat{q}(t_n)]|0\rangle\varphi_0^*(q_i)\,, &(2.192)\\ \langle q(t_f)|q(t_i)\rangle &= \varphi_0(q_f)\varphi_0^*(q_i)\langle 0|0\rangle\,. &(2.193)\end{aligned}$$

In the end, for $t_f \to +\infty$ and$t_i \to -\infty$, from Eqs. (2.177) and (2.189), with

$$S[q(t)] = \int_{-\infty}^{+\infty} L(q(t),\dot{q}(t))dt\,, \tag{2.194}$$

we have

$$\begin{aligned}\langle 0|T[\hat{q}(t_1)\ldots\hat{q}(t_n)]|0\rangle &= \frac{\int d[q]q(t_1)\ldots q(t_n)e^{\frac{i}{\hbar}S[q(t)]}}{\int d[q]e^{\frac{i}{\hbar}S[q(t)]}} &(2.195)\\ &= \left(\frac{1}{i}\right)^n \frac{\delta^n Z[F]}{\delta F(t_1)\ldots\delta F(t_n)}|_{F=0}\,. &(2.196)\end{aligned}$$

Eqs. (2.195) and (2.196) say that the ground-state expectation value of a time-ordered product of position operators (also called a correlation function, or Green function), evaluated as a path-integral according to Eq. (2.195), can be obtained as a functional derivative of the functional $Z[F]$, with $F = 0$ after functional derivation. The ground state $|0\rangle$ of $\hat{H}$ appearing in Eq. (2.195) is the same as the one contained in the functional $Z[F]$: if $Z[F]$ contains only a free Hamiltonian $|0\rangle$ is the groundstate of the free theory, otherwise it is the groundstate of the interacting theory.

2.8.3 Appendix A: Gaussian Integrals for ordinary functions

The one-dimensional simplest gaussian integral is given by

$$\int_{-\infty}^{+\infty} dx\, \exp\left(-ax^2\right) = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \qquad \text{with Re}a > 0\,. \tag{2.197}$$

It is then useful to remember the integration of a quadratic form

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx+c}dx \equiv \int_{-\infty}^{+\infty}\,. \tag{2.198}$$

For $a > 0$ the value of x which minimizes $q(x)$ is

$$\bar{x} = \frac{b}{2a}\,, \qquad q(\bar{x}) = \frac{b^2}{4a} + c\,. \tag{2.199}$$

If we write $q(x)$ in terms of x and $\bar{x}$ as

$$q(x) = q(\bar{x}) - a(x-\bar{x})^2\,, \tag{2.200}$$

$$\int_{-\infty}^{+\infty} e^{q(x)}dx = e^{q(\bar{x})}\int_{-\infty}^{+\infty} e^{-a(x-\bar{x})^2} = e^{q(\bar{x})}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\,. \tag{2.201}$$

As an application of the above result,

$$\int_{-\infty}^{+\infty} dx \, \exp\left(-ax^2 + bx\right) = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(\frac{b^2}{4a}\right) \qquad \text{with } \mathrm{Re}a > 0\,. \tag{2.202}$$

The extension of the previous formula to N dimension is

$$\int d^N x \, \exp\left(-x^T \cdot A \cdot x + b^T \cdot x\right) = \pi^{\frac{N}{2}} \left(det A\right)^{-\frac{1}{2}} \exp\left(\frac{1}{4} b^T \cdot A^{-1} \cdot b\right), \tag{2.203}$$

with the condition $\mathrm{Re}a_n > 0$, where a_n are the eigenvalues of A. More details at pag. 186-187 of the textbook of Ryder.

2.8.4 Appendix B: the functional derivative

The functional derivatives generalizes the concept of the derivative of ordinary functions to functionals, defined on a space of functions. If $G[f]$ is a functional of the function $f(x)$, we can define the variation of the functional as

$$\delta G = G\left[f(x) + \varepsilon\delta(x-y)\right] - G[f(x)]\,. \tag{2.204}$$

The functional derivative is defined as

$$\frac{\delta G}{\delta f(y)} = \lim_{\varepsilon\to 0} \frac{G\left[f(x) + \varepsilon\delta(x-y)\right] - G[f(x)]}{\varepsilon}\,. \tag{2.205}$$

The functional derivatives generalizes the rules for discrete variables:

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij}\,, \tag{2.206}$$

or, equivalently,

$$\frac{\partial}{\partial x_i} \sum_j x_j k_j = k_i\,. \tag{2.207}$$

If we have a functional $F[f]$ defined by

$$F[f] = \int dy\, f(y)\phi(y)\,, \tag{2.208}$$

then

$$\frac{\delta F[f]}{\delta f(x)} = \frac{\delta}{\delta f(x)} \int dy\, f(y)\phi(y) = \phi(x)\,. \tag{2.209}$$

If we have a functional $F[f(x)]$ and consider a variation $f(x) \to f(x) + \varepsilon(x)$, we can expand $F[f+\varepsilon]$ as a power series of ε:

$$F[f+\varepsilon] = F[f] + \int dx \, \frac{\delta F}{\delta \varepsilon(x)} \varepsilon(x) + \frac{1}{2!} \int dx \int dy \, \frac{\delta^2 F}{\delta \varepsilon(x) \delta \varepsilon(y)} \varepsilon(x) \varepsilon(y) + \ldots \tag{2.210}$$

Also a function can be considered a functional. In fact, we can write

$$f(x) = \int dy \, \delta(y-x) f(y) \,. \tag{2.211}$$

Then Eq. (2.209) gives

$$\frac{\delta f(y)}{\delta f(z)} = \delta(z-y) \,. \tag{2.212}$$

The functional derivatives of more complicated functionals are obtained using the ordinary rules for derivatives of composite functions.
As an example, we analyse the Action for a classical 1-dimensional system, which is a functional over the classical paths $q(t)$ which are kinematically allowed:

$$S[q] = \int_0^T dt L\left(q(t), \dot{q}(t); t\right) \,. \tag{2.213}$$

According to the Hamilton principle, the classical dynamics selects, among all possible paths with fixed initial and final points ($q(0)$ and $q(T)$), the one for which the Action S is stationary for small variations of the path $q(t)$. To implement this idea, we can introduce an approximation of the Action for discrete times, i.e. the Action becomes an action of a finite discrete set of variables. If we slice the time interval $[0, T]$ in N equal subintervals (such that $T = N\tau$), the generic classical path $q(t)$ is approximated by a linear, not differentiable, path and the Action is a function of $N-1$ variables (q_0 and q_N are fixed):

$$S[q] \simeq S(q_1, \ldots, q_i, \ldots, q_{N-1}) = \tau \sum_{i=1}^{N-1} L_i(q_i, \dot{q}_i) \,. \tag{2.214}$$

The time derivative $\dot{q}_i$ is understood as a finite difference approximation

$$\dot{q}_i \simeq \frac{1}{\tau} \left(q_{i+1} - q_i\right) \,. \tag{2.215}$$

The index i of L_i parametrizes the possible explicit dependence of the original Lagrangian on the time. If the Lagrangian is not explicitely dependent on the time, then $L_i = L$. The Hamilton principle states that the generic $\frac{\partial S}{\partial q_i} = 0$:

$$\frac{1}{\tau} \frac{\partial S}{\partial q_i} = \frac{\partial L_i}{\partial q_i} - \frac{1}{\tau} \left(\frac{\partial L_i}{\partial \dot{q}_i} - \frac{\partial L_{i-1}}{\partial \dot{q}_{i-1}} \right) \,. \tag{2.216}$$

The **functional derivative** is defined when we take the continuum limit $q_i \to q(t)$:

$$\frac{\delta S}{\delta q(t)} = \lim_{\tau \to 0} \frac{1}{\tau} \frac{\partial S}{\partial q_i} = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \,. \tag{2.217}$$

We can see that in passing from the discrete approximation to the continuum limit, we have the following relation

$$dS = \sum_i \frac{\partial S}{\partial q_i} dq_i \quad \to \quad \delta S = \int dt \, \frac{\delta S}{\delta q(t)} \delta q(t) \,. \tag{2.218}$$

3

Functional quantization for free fields

In this chapter we discuss the functional approach to the quantization of free fields, i.e. fields describing particles which don't interact. This means that the Lagrangian contains at most two power of the fields. As a consequence the equations of motion are linear in the fields. We have seen these equations with different kinds of fields: Klein-Gordon equation (scalar fields), Dirac equation (fermionic field) and Maxwell/Proca equation (massless/massive vectorial fields). We will start with functional quantization of the free scalar field, which can be viewed directly as a generalization of the path integral approach to the quantization of the harmonic oscillator treated in the previous chapter. Then we will see what are the difficulties in the functional quantization of the free Dirac field and the solution given by the functional integral over Grassman variables. In the last section we will treat the difficulties given by the gauge symmetry in the functional quantization of the gauge fields.

3.1 The scalar field

3.1.1 The generating functional $\mathbf{Z}[J]$

The starting point is the Feynman formula for the Euclidean Green functions of Eq. (2.135), where we replace the time coordinate with the space-time coordinate and the coordinate operators $\hat{q}_H$ with the field operators $\hat{\varphi}_H$:

$$
\begin{aligned}
G_E^{(n)}(x_1,\ldots,x_n) &\equiv \langle 0|T\left[\hat{\varphi}_H(x_1)\ldots\hat{\varphi}_H(x_n)\right]|0\rangle \\
&= \frac{\int [d\varphi]\, e^{-S_E[\varphi]}\varphi(x_1),\ldots,\varphi(x_n)}{\int [d\varphi]\, e^{-S_E[\varphi]}} \\
&\equiv \mathcal{N}\int [d\varphi]\, e^{-S_E[\varphi]}\varphi(x_1)\ldots\varphi(x_n)\,.
\end{aligned}
\tag{3.1}
$$

All Green functions of any order (n) can be summarized by means of a functional, named Z functional, defined as follows:

$$
Z_E[J] \equiv \mathcal{N}\int [d\varphi]\, e^{-S_E[\varphi]} e^{\int dx\, J(x)\varphi(x)}\,, \tag{3.2}
$$

where J is an arbitrary classical fixed, called *external current*, while the normalization factor $\mathcal{N}$ is the same of Eq. (3.1). This choice allows automatically that

$$
Z[0] = 1\,. \tag{3.3}
$$

Formally every Green function of order n can be obtained from the Z functional by functional derivation w.r.t. the external current and setting to zero the current after derivation:

$$
G_E^{(n)}(x_1,\ldots,x_n) = \frac{\delta^n Z_E[J]}{\delta J(x_1)\ldots\delta J(x_n)}|_{J=0}\,. \tag{3.4}
$$

For the above property, the functional $Z[J]$ is called the **Green functions generating functional**.
It has an analogous in statistical mechanics, the partition function **(see later for the analogy between QFT and stat mech)**.
The term containing the external current J represents an interaction of the field φ with the classical source.

We note a useful relation: Eq. (3.4) can be obtained from Eq. (3.1) with the following substitution:

$$
\begin{aligned}
\varphi(x) &\to \frac{\delta}{\delta J(x)} \\
-S_E[\varphi(x)] &\to -S_E[\varphi(x)] + J(x)\varphi(x)\,.
\end{aligned}
\tag{3.5}
$$

In general, for an arbitrary functional $F[\varphi]$ we have:

$$
\int [d\varphi]\, e^{-S_E[\varphi]} F[\varphi(x)] = \left[F\left[\frac{\delta}{\delta J}\right] Z_E[J]\right]_{|J=0}\,. \tag{3.6}
$$

3.1.2 The Generating Functional Z_{0E} for the free scalar field

In order to calculate the unperturbed functional Z_{0E}, we start from the Euclidean Action S_{0E} defined as

$$S_{0E}[\varphi] = \frac{1}{2}\int dx\,\left(\partial_\mu\varphi\partial_\mu\varphi + m^2\varphi^2\right)\,. \tag{3.7}$$

The functional is defined as

$$Z_{0E}[J] = \mathcal{N}_0\int[d\varphi]\,e^{-S_{0E}[\varphi]}e^{\int dx\,J\varphi}\,. \tag{3.8}$$

We perform the following change of variables (functional translation)

$$\varphi(x) = \varphi'(x) + \bar{\varphi}(x)\,, \tag{3.9}$$

where $\bar{\varphi}(x)$ is a fixed function. Since the jacobian related to the above transformation is 1, we have

$$[d\varphi] = [d\varphi']\,. \tag{3.10}$$

Using the identity

$$\partial_\mu\varphi_1(x)\partial_\mu\varphi_2(x) = -\varphi_1(x)\Box_E\varphi_2(x) + \partial_\mu\left[\varphi_1(x)\partial\varphi_2(x)\right]\,, \tag{3.11}$$

and neglecting the second term because it gives rise to a surface integral (the fields are assumed to vanishing at the boundary), we can rewrite Eq. (3.8) as follows:

$$Z_{0E}[J] = \mathcal{N}_0 e^{\int dx\,\bar{\varphi}\left[\frac{1}{2}(\Box_E - m^2)\bar{\varphi} + J(x)\right]}\int[d\varphi']\,e^{-S_{0E}[\varphi']}e^{\int dx\,\varphi'\left[(\Box_E - m^2)\bar{\varphi} + J(x)\right]}\,. \tag{3.12}$$

We can choose $\bar{\varphi}(x)$ to be solution of the following equation

$$\left(\Box_E - m^2\right)\bar{\varphi}(x) = -J(x)\,, \tag{3.13}$$

so that

$$\begin{aligned} Z_{0E}[J] &= \mathcal{N}_0 e^{\frac{1}{2}\left[\int dx\,J(x)\bar{\varphi}(x)\right]}\int[d\varphi']\,e^{-S_{0E}[\varphi']} \\ &= \mathcal{N}e^{\frac{1}{2}\left[\int dx\,J(x)\bar{\varphi}(x)\right]} \end{aligned} \tag{3.14}$$

The solution $\bar{\varphi}$ of the non-homogeneus differential equation (3.13) is given in term of the Euclidean Green function $\triangle_E(x)$

$$\bar{\varphi}(x) = \int dx'\,\triangle_E(x-x')\,J(x')\,, \tag{3.15}$$

where $\triangle_E(x)$ satisfies the equation

$$\left(\Box_E - m^2\right)\triangle_E(x) = -\delta(x)\,. \tag{3.16}$$

By using the Fourier representation of the Dirac δ (in 4 dimensions)

$$\delta(x) = \frac{1}{(2\pi)^4} \int d^4p \, e^{ip\cdot x} \tag{3.17}$$

and of $\Delta_E(x)$

$$\Delta_E(x) = \frac{1}{(2\pi)^4} \int d^4p_E \, e^{ip\cdot x} \hat{\Delta}(p_E^2) \, , \tag{3.18}$$

Eq. (3.16) becomes

$$(-p_E^2 - m^2) \frac{1}{(2\pi)^4} \int d^4p \, e^{ip_E\cdot x} \hat{\Delta}\left(p_E^2\right) = -\frac{1}{(2\pi)^4} \int d^4p_E \, e^{ip_E\cdot x} \, . \tag{3.19}$$

This means that

$$\hat{\Delta}\left(p^2\right) = \frac{1}{p^2 + m^2} \, . \tag{3.20}$$

By Fourier antitransforming we get

$$\Delta_E(x) = \frac{1}{(2\pi)^4} \int d^4p \frac{e^{i(p\cdot x)_E}}{p_E^2 + m^2} \, . \tag{3.21}$$

Inserting Eq (3.15) in Eq. (3.14), and assuming the Eq. (3.3) for the normalization, we have the following expression for the Z functional of the free scalar field:

$$Z_{0E}\left[J\right] = e^{\left[\frac{1}{2}\int dx \int dx' \, J(x)\Delta_E(x-x')J(x')\right]} \, . \tag{3.22}$$

Observation 1: from the above expression we see that the Z_E functional is redundant (at least for the free field). In fact we can write the exponential of Eq. (3.22) as a series as follows

$$\begin{aligned} Z_{0E}\left[J\right] = 1 &+ \frac{1}{2} \int dx \int dx' \, J(x)\Delta_E(x-x')J(x') \\ &+ \frac{1}{2!}\left[\frac{1}{2} \int dx \int dx' \, J(x)\Delta_E(x-x')J(x')\right]^2 \\ &+ \frac{1}{3!}\left[\frac{1}{2} \int dx \int dx' \, J(x)\Delta_E(x-x')J(x')\right]^3 + \dots \, . \end{aligned} \tag{3.23}$$

Now we compute $G_E^{(2)}$ and $G_E^{(4)}$ by means of Eq. (3.4):

$$G_E^{(2)}(x_1, x_2) = \frac{\delta^2 Z_E[J]}{\delta J(x_1)\delta J(x_2)}_{|J=0} = \Delta_E(x_1 - x_2) \, , \tag{3.24}$$

where only the first line of Eq. (3.23) contributes;

$$
\begin{aligned}
G_E^{(4)}(x_1,x_2,x_3,x_4) &= \frac{\delta^4 Z_E[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}_{|J=0} \\
&= \frac{1}{8}\frac{\delta^4}{\delta J_1 \delta J_2 \delta J_3 \delta J_4}\left\{\left[\int dx \int dx'\, J(x)\Delta_E(x-x')J(x')\right]\left[\int dy \int dy'\, J(y)\Delta_E(y-y')J(y')\right]\right\}_{|J=0} \\
&= \Delta_E(x_1-x_2)\Delta_E(x_3-x_4) + \Delta_E(x_1-x_3)\Delta_E(x_2-x_4) + \Delta_E(x_1-x_4)\Delta_E(x_2-x_3)\,,
\end{aligned}
\tag{3.25}
$$

where J_i stands for $J(x_i)$. $G_E^{(4)}$ is determined once we know $G_E^{(2)}$. In the following section we will see that an important simplification can be obtained taking the logarithm of Z_E (this works also in the interacting case).

Observation 2: both $G_E^{(2)}$ and $G_E^{(4)}$ depend only on difference of space-time points, as required by translational invariance (see the following section).

Observation 3: all Green functions of odd degree vanish because an odd number of functional derivatives with respect to J leaves always a term proportional to the current, which vanishes once we set $J = 0$.

3.1.3 The Generating Functional $\mathbf{Z}_0$ for the free scalar field

In this section we calculate the unperturbed functional Z_0 with Minkowskian time, defined from the Action S_0 as

$$S_0[\varphi] = \frac{1}{2}\int dx\, \left(\partial_\mu\varphi\partial^\mu\varphi - m^2 - i\varepsilon\varphi^2\right)\,. \tag{3.26}$$

The functional is defined as

$$Z_0[J] = \mathcal{N}_0 \int [d\varphi]\, e^{iS_0[\varphi]} e^{i\int dx\, J\varphi}\,. \tag{3.27}$$

We perform the following change of variables (functional translation)

$$\varphi(x) = \varphi'(x) + \bar{\varphi}(x)\,, \tag{3.28}$$

where $\bar{\varphi}(x)$ is a fixed function. Since the jacobian related to the above transformation is 1, we have

$$[d\varphi] = [d\varphi']\,. \tag{3.29}$$

Using the identity

$$\partial_\mu\varphi_1(x)\partial_\mu\varphi_2(x) = -\varphi_1(x)\Box_E\varphi_2(x) + \partial_\mu\left[\varphi_1(x)\partial\bar{\varphi}_2(x)\right]\,, \tag{3.30}$$

and neglecting the second term because it gives rise to a surface integral (the fields are assumed to vanishing at the boundary), we can rewrite Eq. (3.27) as follows:

$$Z_0[J] = \mathcal{N}_0 e^{-i\int dx\,\bar{\varphi}\left[\frac{1}{2}(\Box+m^2-i\varepsilon)\bar{\varphi}+J(x)\right]}\int [d\varphi']\, e^{iS_0[\varphi']} e^{-i\int dx\,\varphi'\left[(\Box+m^2-i\varepsilon)\bar{\varphi}+J(x)\right]}\,. \tag{3.31}$$

We can choose $\bar{\varphi}(x)$ to be solution of the following equation

$$\left(\Box + m^2 - i\varepsilon\right)\bar{\varphi}(x) = J(x)\,, \tag{3.32}$$

so that

$$\begin{aligned} Z_0[J] &= \mathcal{N}_0 e^{\frac{i}{2}\left[\int dx\,J(x)\bar{\varphi}(x)\right]}\int [d\varphi']\, e^{iS_0[\varphi']} \\ &= \mathcal{N} e^{\frac{i}{2}\left[\int dx\,J(x)\bar{\varphi}(x)\right]} \end{aligned} \tag{3.33}$$

The solution $\bar{\varphi}$ of the non-homogeneus differential equation (3.32) is given in term of the Euclidean Green function $\triangle_E(x)$

$$\bar{\varphi}(x) = -\int dx'\,\triangle_F(x-x')\,J(x')\,, \tag{3.34}$$

where $\triangle_F(x)$ satisfies the equation

$$\left(\Box + m^2 - i\varepsilon\right)\triangle_F(x) = -\delta(x)\,. \tag{3.35}$$

By using the Fourier representation of the Dirac δ (in 4 dimensions)

$$\delta(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x} \tag{3.36}$$

and of $\Delta_F(x)$

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x}\hat{\Delta}(p^2)\,, \tag{3.37}$$

Eq. (3.35) becomes

$$\frac{1}{(2\pi)^4}\int d^4p\,(-p^2+m^2-i\varepsilon)e^{-ip\cdot x}\hat{\Delta}\left(p^2\right) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x}\,. \tag{3.38}$$

This means that

$$\hat{\Delta}\left(p^2\right) = \frac{1}{p^2-m^2+i\varepsilon}\,. \tag{3.39}$$

By Fourier antitransforming we get

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-i(p\cdot x)}}{p^2-m^2+i\varepsilon}\,. \tag{3.40}$$

Inserting Eq (3.34) in Eq. (3.33), and assuming the Eq. (3.3) for the normalization, we have the following expression for the Z functional of the free scalar field:

$$Z_0\left[J\right] = e^{-i\left[\frac{1}{2}\int dx \int dx'\, J(x)\Delta_F(x-x')J(x')\right]} . \tag{3.41}$$

Eqs. (3.34) and (3.35) identifies the Feynman propagator as the Green function of the operator $\partial_\mu\partial^\mu + m^2$ with boundary conditions determined by the $-i\varepsilon$ prescription. $\Delta_F(x_1 - x_2)$ propagates a signal from x to y. The signals it propagates are single particle and antiparticle states, since these are solutions of the Klein-Gordon equation

$$\left(\partial_\mu\partial^\mu + m^2 - i\varepsilon\right)\varphi = 0 . \tag{3.42}$$

The $-i\varepsilon$ prescription tells us that positive energy solutions of the Klein-Gordon equation are propagated forward in time and negative energy solutions are propagated backward in time.

3.1.4 Translation invariance and four-momentum conservation

Eq. (3.1) allows to show in a simple way the translation invariance of Green functions. In fact, performing a shift $x_i^\mu \to x_i^\mu + a^\mu$, we have

$$\begin{aligned}
G_E^{(n)}(x_1+a,\ldots,x_n+a) &= \frac{\int[d\varphi(x)]\, e^{-S_E[\varphi(x)]}\varphi(x_1+a),\ldots,\varphi(x_n+a)}{\int[d\varphi]\, e^{-S_E[\varphi(x)]}} \\
&= \frac{\int[d\varphi(x)]\, e^{-S_E[\varphi(x+a)]}\varphi(x_1+a),\ldots,\varphi(x_n+a)}{\int[d\varphi]\, e^{-S_E[\varphi(x+a)]}} \\
&= \frac{\int[d\varphi(x+a)]\, e^{-S_E[\varphi(x+a)]}\varphi(x_1+a),\ldots,\varphi(x_n+a)}{\int[d\varphi]\, e^{-S_E[\varphi(x+a)]}} \\
&= \frac{\int[d\varphi(x)]\, e^{-S_E[\varphi(x)]}\varphi(x_1),\ldots,\varphi(x_n)}{\int[d\varphi]\, e^{-S_E[\varphi(x)]}} \\
&= G_E^{(n)}(x_1,\ldots,x_n) .
\end{aligned} \tag{3.43}$$

In the second line we have used the translation invariance of the (Euclidean) Action and in the third line we have used the invariance of the functional integration measure for the change $\varphi(x) \to \varphi(x+a)$.
If we consider the Fourier transform of $G^n(x_1,\ldots,x_n)$, using the translational invariance of Eq. (3.43), we have

$$\begin{aligned}
&\int dx_1 \ldots \int dx_n\, e^{i(p_1x_1+\ldots+p_nx_n)} G_E^{(n)}(x_1,\ldots,x_n) \\
&= \int dx_1 \ldots \int dx_n\, e^{i[p_1(x_1+a)+\ldots+p_n(x_n+a)]} G_E^{(n)}(x_1,\ldots,x_n) \\
&= e^{ia\cdot(p_1+\ldots+p_n)} \int dx_n\, e^{i(p_1x_1+\ldots+p_nx_n)} G_E^{(n)}(x_1,\ldots,x_n) \Longrightarrow p_1+\ldots+p_n = 0 .
\end{aligned} \tag{3.44}$$

Thus the Fourier transform of the Green function contains a Dirac delta function of the total four momentum, ensuring four momentum conservation. It is customarily written as

$$\int dx_1 \ldots \int dx_n \, e^{i(p_1x_1+\ldots+p_nx_n)} G_E^{(n)}(x_1,\ldots,x_n) = \tilde{G}_E^{(n)}(p_1,\ldots p_n)(2\pi)^4\delta^{(4)}(p_1+\ldots+p_n)\,. \tag{3.45}$$

3.2 The Dirac field

3.2.1 Introduction: the Fermi-Dirac oscillator

In analogy with what we have seen for bosonic fields, a decomposition in normal modes is also possible for a field describing fermions. However, since the Pauli exclusion principle must hold for fermions, the normal modes will be of Fermi-Dirac oscillator. The Hamiltonian of the Fermi-Dirac oscillator, in "natural units" ($\hbar = c = 1$), is given by

$$H_0 = \omega_0 a^\dagger a\,, \tag{3.46}$$

c and $c^\dagger$ satisfy the **anti-commuting** relations

$$\begin{aligned} \{a,a\} = \left\{a^\dagger, a^\dagger\right\} = 0\,, \\ \left\{a, a^\dagger\right\} = 1\,. \end{aligned} \tag{3.47}$$

The anticommutation rules allow to automatically satisfy the Pauli exclusion principle. Infact, given an eigenstate $|E\rangle$ of the Hamiltonian H_0 with eigenvalue E

$$H_0|E\rangle = E|E\rangle\,, \tag{3.48}$$

we have

$$\begin{aligned} H_0 a|E\rangle = \omega_0 a^\dagger a a|E\rangle = \omega_0\left(-aa^\dagger + 1\right)a|E\rangle = \omega_0 a|E\rangle - aH_0|E\rangle = (\omega_0 - E)\,a|E\rangle\,, \\ H_0 a^\dagger|E\rangle = \omega_0 a^\dagger a a^\dagger|E\rangle = \omega_0 a^\dagger\left(-a^\dagger a + 1\right)|E\rangle = \omega_0 a^\dagger|E\rangle + a^\dagger H_0|E\rangle = (\omega_0 + E)\,a^\dagger|E\rangle\,. \end{aligned} \tag{3.49}$$

If we assume $E = 0$ for the ground state, from the above equations we get

$$\begin{aligned} c|0\rangle = 0\,, \\ c^\dagger|0\rangle = |1\rangle\,. \end{aligned} \tag{3.50}$$

It follows also

$$H|1\rangle = \omega_0 a^\dagger a a^\dagger|0\rangle = \omega_0 a^\dagger\left(1 - a^\dagger a\right)|0\rangle = \omega_0|1\rangle\,. \tag{3.51}$$

The relations of Eq. (3.49) give account of the name creation operator and destruction operator for $a^\dagger$ and a, respectively.
Given the anticommutation relations of Eq. (3.47) the state $|1\rangle$ is annihilated by the creation operator $c^\dagger$:

$$a^\dagger|1\rangle = a^\dagger a^\dagger|0\rangle = 0\,. \tag{3.52}$$

The above results satisfies automatically the Pauli exclusion principle [1].

Remark: how to give a path-integral formulation of the Fermi-Dirac oscillator?
The Hamiltonian of Eq. (3.46 can be derived from the Lagrangian [2]

$$L = i\hbar a^\dagger \dot{a} - \hbar\omega_0 a^\dagger a\,, \tag{3.53}$$

from which we have

$$\pi = \frac{\partial L}{\partial \dot{a}} = i\hbar a^\dagger\,. \tag{3.54}$$

We note that $\frac{\partial L}{\partial \dot{a}^\dagger} = 0$, since L does not depend on $\dot{a}^\dagger$. Actually the Action is simmetric in a and $a^\dagger$, because through an integration by parts we can express the action in two equivalent ways, where the roles of a and $a^\dagger$ are interchanged

$$S = \hbar \int dt\,\left(i a^\dagger \dot{a} - \omega_0 a^\dagger a\right) = \hbar \int dt\,\left(-\dot{a}^\dagger a - \omega_0 a^\dagger a\right)\,. \tag{3.55}$$

Imposing the commutation rules

$$\left[a, a^\dagger\right] = \frac{1}{i\hbar}\left[a,\pi\right] = \frac{1}{i\hbar} i\hbar = 1\,. \tag{3.56}$$

Instead of a and $a^\dagger$, we can consider $\tilde{a} = \sqrt{\hbar} a$ and $\tilde{a}^\dagger = \sqrt{\hbar} a^\dagger$:

$$\begin{aligned} \tilde{a} &= \sqrt{\frac{\omega_0}{2}}\left(x + i\frac{p}{\omega_0}\right) \\ \tilde{a}^\dagger &= \sqrt{\frac{\omega_0}{2}}\left(x + i\frac{p}{\omega_0}\right)\,, \end{aligned} \tag{3.57}$$

which do not depend on the Planck constant and are therefore classical quantities. The Lagrangian becomes

$$L = i\tilde{a}^\dagger \dot{\tilde{a}} - \omega_0 \tilde{a}^\dagger \tilde{a}\,, \tag{3.58}$$

with the commutation rules

$$\left[\tilde{a}, \tilde{a}^\dagger\right] = \hbar\,. \tag{3.59}$$

[1]Dirac noticed that the symmetry (w.r.t the exchange of a and $a^\dagger$) of the anticommutation relations (Eq. (3.49)), together with the first relation of Eq. (3.50) and Eq. (3.52), allow the exchange of the vacuum state $|0\rangle$ with the "filled" state $|1\rangle$, describing a system in terms of "holes", contrary to the Bose-Einstein harmonic oscillator.

[2]For later convenience we introduce here explicitely $\hbar$.

In the path-integral formulation [3] of the bosonic oscillator, $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ are considered as ordinary functions, *i.e.* as commuting quantities. This means that everything goes as if $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ become commuting quantities $\left[\tilde{a}, \tilde{a}^\dagger\right] = 0$ in the classical limit $\hbar \to 0$.

The path-integral formulation of the fermionic oscillator would imply to apply anticommutation rules in the limit $\hbar \to 0$:

$$\left\{\tilde{a}, \tilde{a}^\dagger\right\} = 0\,. \tag{3.60}$$

3.2.2 Grassman algebra

For the bosonic fields we have seen that in the functional approach the generating functional for the Green's functions is written as a functional integral over the fields, considered as classical functions, *i.e.* c-numbers, contrarily to the canonical approach where the fields are operators.
Following the heuristic argument of the last remark, we can expect that if we aim at extending the functional method to fermionic fields we need anticommuting **c**-numbers [4]. Actually these mathematical entities have been introduced at the end of XIX century by the mathematician H. Grassmann. In this section we give a brief account of an n-dimensional **Grassmann algebra**[5]. It is an abstract algebra, whose generators satisfy the anticommutation relations

$$\left\{\theta_i \theta_j\right\} = 0 \quad \text{for}\, \theta_i, \theta_j = 1, \ldots, n\,. \tag{3.61}$$

This implies that

$$\theta_i^2 = \theta_i\theta_i = 0 \quad i = 1, \ldots, n\,. \tag{3.62}$$

A product of generators $\theta_{i_1}\theta_{i_2}\ldots\theta_{i_n}$ is equal to zero if any two of the indices are the same. The Grassman algebra generators can be multiplied by ordinary c-numbers with which they are commuting: $c\theta_i = \theta_i c$. We can form linear combinations of generators with standard rules as for ordinary numbers. The most general element of a Grassmann algebra with n generators is of the form

$$f(p) = p_0 + \sum p_i\theta_i + \sum p_{ij}\theta_i\theta_j + \sum p_{i,2,\ldots,n}\theta_1\theta_2\ldots\theta_n\,, \tag{3.63}$$

where $i, j = 1, 2, \ldots, n$ with $i \le j \le \ldots \le n$. The coefficients $p_0, p_i, p_{ij}, \ldots$ are ordinary numbers. p on the left-hand side stands for $p_0, p_i, p_{ij}, \ldots$. We can *define* the product of two elements of the Grassmann algebra in a simple way. Consider for instance two elements of the algebra with $n = 2$:

$$\begin{aligned} f(p) &= p_0 + p1\theta_1 + p_2\theta_2 + p_{12}\theta_1\theta_2 = p_0 + p1\theta_1 + p_2\theta_2 - p_{12}\theta_2\theta_1 \\ f(q) &= q_0 + q1\theta_1 + q_2\theta_2 + q_{12}\theta_1\theta_2 = q_0 + q1\theta_1 + q_2\theta_2 - q_{12}\theta_2\theta_1\,. \end{aligned} \tag{3.64}$$

[3]W.r.t. Eq. (2.97) we can think of changing the integration variables $(x, p) \to \left(\tilde{a}, \tilde{a}^\dagger\right)$.

[4]It seems a contradiction since we don't want operators, but we want numbers and ordinary numbers are c-numbers

[5]In this section we present an introduction to the use of Grassman variables. The reader interested in the mathematical foundations of a Grassman algebra should have a look at Ref. [16].

The product $f(p)f(q)$ is defined by the rule

$$f(p)f(q) = p_0q_0 + (p_0q_1 + p_1q_0)\,\theta_1 + (p_0q_2 + p_2q_0)\,\theta_2 + (p_0q_{12} + p_1q_2 - p_2q_1 + p_{12}q_0)\,\theta_1\theta_2\,, \tag{3.65}$$

which is associative $[f(p)\,(f(q)+f(r)) = f(p)f(q) + f(p)f(r)]$ and is another element of the algebra. **Two elements $\mathbf{f(p)}$ and $\mathbf{f(q)}$ of the algebra do not either commute or anticommute**.

A function of Grassmann variables is defined through its (finite) power series. **With n Grassmann variables, the most general function is a polinomial of order n.**

If we assume the coefficients $p_{ij\ldots n}$ to be totally antisymmetric in their indices, we can also write

$$f(p) = p_0 + \sum_i p_i\theta_i + \frac{1}{2!}\sum_{i,j} p_{ij}\theta_i\theta_j + \frac{1}{n!}\sum_{i,j,\ldots,n} p_{i,2,\ldots,n}\theta_1\theta_2\ldots\theta_n\,. \tag{3.66}$$

Differentiation

We can define a linear operator $\frac{\partial}{\partial\theta_i}$ with the properties

$$\begin{aligned} &\frac{\partial\theta_i}{\partial\theta_j} = \delta_{ij}\,, \quad i,j = 1,\ldots,n \\ &\frac{\partial f}{\partial\theta_j} = 0\,, \quad \text{with } f \text{ an } \textit{ordinary function}\,. \end{aligned} \tag{3.67}$$

Consider the case $n = 2$

$$\frac{\partial f(p)}{\partial\theta_1} = p_1 + p_{12}\frac{\partial}{\partial\theta_1}(\theta_1\theta_2)\,. \tag{3.68}$$

But we can also write

$$\frac{\partial f(p)}{\partial\theta_1} = p_1 - p_{12}\frac{\partial}{\partial\theta_1}(\theta_2\theta_1)\,. \tag{3.69}$$

Hence

$$\frac{\partial}{\theta_1}(\theta_1\theta_2) = \frac{\partial}{\theta_1}(-\theta_2\theta_1) = -\frac{\partial}{\theta_1}(\theta_2\theta_1) \tag{3.70}$$

The general rule for the derivation of a product of generators is to anticommute them until the differentiated one is moved to the left-hand end of the product and then remove it on differentiation. Example:

$$\begin{aligned} &\frac{\partial}{\partial\theta_1}(\theta_1\theta_2) = \frac{\partial\theta_1}{\partial\theta_1}\theta_2 = \theta_2\,, \\ &\frac{\partial}{\partial\theta_1}(\theta_2\theta_1) = -\frac{\partial\theta_1}{\partial\theta_1}\theta_2 = -\theta_2\,. \end{aligned} \tag{3.71}$$

Let us consider the function of two variables

$$f(\{p\}) = p_0 + p_1\vartheta_1 + p_2\vartheta_2 + p_{12}\vartheta_1\vartheta_2\,. \tag{3.72}$$

If we multiply on the left by ϑ_1 we get

$$\vartheta_1 f(\{p\}) = p_0\vartheta_1 + p_2\vartheta_1\vartheta_2\,, \tag{3.73}$$

$$\frac{\partial}{\partial\vartheta_1}\left(\vartheta_1 f(\{p\})\right) = p_0 + p_2\vartheta_2\,, \tag{3.74}$$

$$\vartheta_1\frac{\partial}{\partial\vartheta_1}\left(f(\{p\})\right) = \vartheta_1\left[p_1 + p_{12}\vartheta_2\right] = p_1\vartheta_1 + p_{12}\vartheta_1\vartheta_2\,. \tag{3.75}$$

By summation of both members of Eqs. (3.74) and (3.75), we get

$$\vartheta_1\frac{\partial}{\partial\vartheta_1}\left(f(\{p\})\right) + \frac{\partial}{\partial\vartheta_1}\left(\vartheta_1 f(\{p\})\right) = p_0 + p_1\vartheta_1 + p_2\vartheta_2 + p_{12}\vartheta_1\vartheta_2 = f(\{p\})\,. \tag{3.76}$$

Hence we can write the general property

$$\left\{\vartheta_i, \frac{\partial}{\partial\vartheta_i}\right\} = 1\,. \tag{3.77}$$

For two different Grassman variables we can write

$$\left\{\vartheta_i, \frac{\partial}{\partial\vartheta_j}\right\} = \delta_{ij}\,. \tag{3.78}$$

Another property can be derived by further derivation of Eq. (3.74):

$$\frac{\partial}{\partial\vartheta_2}\frac{\partial}{\partial\vartheta_1}\left(\vartheta_1 f(\{p\})\right) = \frac{\partial}{\partial\vartheta_2}\left[p_0 + p_{12}\vartheta_2\right] = p_{12}\,, \tag{3.79}$$

$$\frac{\partial}{\partial\vartheta_1}\frac{\partial}{\partial\vartheta_2}\left(\vartheta_1 f(\{p\})\right) = \frac{\partial}{\partial\vartheta_2}\left[p_0 - p_{12}\vartheta_2\right] = -p_{12}\,. \tag{3.80}$$

By summation of both members of Eqs. (3.79) and (3.80), we have

$$\left\{\frac{\partial}{\partial\vartheta_i}, \frac{\partial}{\partial\vartheta_j}\right\} = 0\,. \tag{3.81}$$

Eq. (3.81) for $i = j$ says that the inverse of the derivative does not exist. In fact we can multiply the defining equation of the inverse

$$\frac{\partial}{\partial\vartheta_i}\left(\frac{\partial}{\partial\vartheta_i}\right)^{-1} f(\vartheta_i) = f(\vartheta_i) \tag{3.82}$$

from the left by $\frac{\partial}{\partial\vartheta_i}$. This gives

$$\frac{\partial}{\partial\vartheta_i}\frac{\partial}{\partial\vartheta_i}\left(\frac{\partial}{\partial\vartheta_i}\right)^{-1} f(\vartheta_i) = 0 = \frac{\partial}{\partial\vartheta_i} f(\vartheta_i)\,. \tag{3.83}$$

Integration

The integration on Grassman variables is defined in an abstract way. As basic property for the definition of the integration, we require invariance under shift of the integration variable

$$\int d\eta\, f(\eta+\xi) = \int d\eta\, f(\eta)\,, \tag{3.84}$$

where ξ is an arbitrary Grassman variable. Since the most general form of a function of a Grassman variable is $f(\eta) = a + b\eta$, we must have

$$\int d\eta\, f(\eta+\xi) = \int d\eta\, [a + b(\eta+\xi)] = \int d\eta\, [a + b\eta]\ , \tag{3.85}$$

i.e.

$$\int d\eta\, b\xi = 0\,. \tag{3.86}$$

Since ξ is arbitrary, we must impose

$$\int d\eta\, b = 0\,. \tag{3.87}$$

In particular, for $b = 1$, we have

$$\int d\eta\ = 0\,. \tag{3.88}$$

Since the product of two Grassman numbers is a commuting number, it will be an ordinary number. Hence $\int d\eta\eta$ is an ordinary number, which we fix in the following way

$$\int d\eta\, \eta = 1\,. \tag{3.89}$$

The choice of Eq. (3.89) fixes only the normalization. In fact, we could choose

$$\int d\eta\, \eta = X\,, \tag{3.90}$$

with X an ordinary number and redefine η according to $\eta = \eta'\sqrt{X}$. With this susbtitution we would have

$$\int d\eta'\, \eta' = 1\,. \tag{3.91}$$

With the rules given by Eqs. (3.87) and (3.89), we have

$$\int d\eta\, f(\eta) = \int d\eta\ (a + b\eta) = b\,, \tag{3.92}$$

if b is an ordinary number and

$$\int d\eta\, f(\eta) = \int d\eta\ (a + b\eta) = \int d\eta\ (a - b\eta) = -b\,, \tag{3.93}$$

if b is a Grassman number.

Multiple integration

The multiple integration is defined iteratively:

$$\int d\vartheta_1 \int d\vartheta_2 \vartheta_1 \vartheta_2 = -\int d\vartheta_1 \left(\int d\vartheta_2 \vartheta_2 \right) \vartheta_1 = -1\,. \tag{3.94}$$

The "nested" multiple integration is compatible with Eq. (3.88):

$$\left(\int d\vartheta_1 \right)^2 = \int d\vartheta_1 \int d\vartheta_2 = -\int d\vartheta_2 \int d\vartheta_1 = -\left(\int d\vartheta_1 \right)^2 \Longrightarrow \int d\vartheta_1 = \int d\vartheta_2 = 0\,. \tag{3.95}$$

Let us integrate the function of Eq. (3.72):

$$\int d\vartheta_1 \, f\left(\{p\}\right) = \int d\vartheta_1 \left(p_0 + p_1\vartheta_1 + p_2\vartheta_2 + p_{12}\vartheta_1\vartheta_2\right) = p_1 + p_{12}\vartheta_2\,, \tag{3.96}$$

$$\int d\vartheta_2 \, f\left(\{p\}\right) = \int d\vartheta_2 \left(p_0 + p_1\vartheta_1 + p_2\vartheta_2 + p_{12}\vartheta_1\vartheta_2\right) = p_2 - p_{12}\vartheta_1\,. \tag{3.97}$$

By inspection, Eqs. (3.96) and (3.97) give the same results as the derivatives w.r.t. ϑ_1 and ϑ_2, respectively.
Hence, differentiation and integration on Grassman variables give the same results.
As an exercise, let us consider the function $\exp(-\vartheta_1\vartheta_2)$:

$$\begin{aligned} \int d\vartheta_1 \int d\vartheta_2 e^{-\vartheta_1\vartheta_2} &= \int d\vartheta_1 \int d\vartheta_2 \left(1 - \vartheta_1\vartheta_2\right) \\ &= -\int d\vartheta_1 \int d\vartheta_2 \vartheta_1\vartheta_2 \\ &= +\int d\vartheta_1 \int d\vartheta_2 \vartheta_2\vartheta_1 = 1\,. \end{aligned} \tag{3.98}$$

We report in Appendix A some useful integrals with Grassman variables, in particular the analogous of ordinary gaussian integrals, which will be used in the following.

3.2.3 Grassman functionals

A Grassman field is a function of each spacetime point, whose value is a Grassman variable. We have seen how to write a generic function in terms of the Grassman generators. We can say that in a Grassman field we have an infinity of Grassman generators $\vartheta(x)$, associated with each spacetime point, with the property $\{\vartheta(x), \vartheta(y)\} = 0$. In order to develop the functional integral formalism for the Dirac field, we need to introduce the concept of functional of a Grassman field, which can be defined as

$$F[\vartheta] = f_0 + \int d^4x \, f_1(x)\vartheta(x) + \int d^4x \int d^4y f_2(x,y)\vartheta(x)\vartheta(y) + \ldots\,, \tag{3.99}$$

where f_i are ordinary functions.

Functional derivative

The functional derivative is defined as a linear operator $\frac{\delta}{\delta\vartheta(x)}$, with the following property:

$$\begin{aligned}\frac{\delta}{\delta\vartheta(x)}\{\vartheta(x_1)\vartheta(x_2)\cdots\vartheta(x_n)\} = {} & \delta^4(x-x_1)\{\vartheta(x_2)\cdots\vartheta(x_n)\} \\ & -\delta^4(x-x_2)\{\vartheta(x_1)\cdots\vartheta(x_n)\} \\ & +(-1)^n\delta^4(x-x_n)\{\vartheta(x_1)\vartheta(x_2)\cdots\vartheta(x_{n-1})\}\,.\end{aligned} \tag{3.100}$$

From the above definition, the following properties follow:

$$\frac{\delta\vartheta(x)}{\delta\vartheta(y)} = \delta^4(x-y)\,, \tag{3.101}$$

$$\left\{\frac{\delta}{\delta\vartheta(x)}, \frac{\delta}{\delta\vartheta(y)}\right\} = 0 \tag{3.102}$$

$$\left\{\vartheta(x), \frac{\delta}{\delta\vartheta(y)}\right\} = \delta^4(x-y)\,, \tag{3.103}$$

which are the analogous of the results obtained for the differentiation of Grassman functions.

Functional integration

We will need functional integrals over two independent Grassman fields. **Two Grassman fields $\vartheta_1(x)$ and $\vartheta_2(x)$ are independent if**

$$\frac{\delta\vartheta_1(x)}{\delta\vartheta_2(y)} = 0 \qquad \frac{\delta\vartheta_2(x)}{\delta\vartheta_1(y)} = 0\,. \tag{3.104}$$

The anticommutators involving Grassman fields and their functional derivatives are vanishing:

$$\{\vartheta_1(x), \vartheta_2(y)\} = 0 \qquad \left\{\frac{\delta}{\delta\vartheta_1(x)}, \frac{\delta}{\delta\vartheta_2(y)}\right\} = 0\,. \tag{3.105}$$

$$\left\{\vartheta_1(x), \frac{\delta}{\delta\vartheta_2(y)}\right\} = 0 \qquad \left\{\vartheta_2(x), \frac{\delta}{\delta\vartheta_1(y)}\right\} = 0\,. \tag{3.106}$$

We will use "gaussian" functional integral over independent Grassman fields of the form

$$\int [d\vartheta(x)]\,[d\tilde{\vartheta}(x)]\,e^{-\vartheta A\tilde{\vartheta}} \equiv \lim_{n\to\infty}\int\left(\prod_{i=1}^{n} d\vartheta_i d\tilde{\vartheta}_i\right) e^{-\sum_{i,j=1}^{n}\vartheta_i A_{ij}\tilde{\vartheta}_j}\,, \tag{3.107}$$

where

$$A_{ij} = \int d^4x \int d^4y\, u_i(x)A(x,y)u_j(y) \tag{3.108}$$

$$\vartheta(x) = \sum_{i=1}^{\infty}\vartheta_i u_i(x) \tag{3.109}$$

$$\tilde{\vartheta}(x) = \sum_{i=1}^{\infty}\tilde{\vartheta}_i u_i(x) \tag{3.110}$$

and $u_i(x)$ are a set of ortonormal functions, *i.e.* $\int d^4x\, u_i(x)u_j(x) = \delta_{ij}$.

3.2.4 The Generating Functional for the free Dirac field

In analogy with the Fermi-Dirac oscillator, we define the generating functional through the introduction of two external currents $\eta_\rho(x)$ and $\bar\eta_\rho(x)$:

$$Z_0\left[\eta,\bar\eta\right] = \mathcal{N}_0 \int \left[d\bar\psi(x)\right]\left[d\psi(x)\right]\, e^{i\int d^4x(\bar\psi(x)D\psi(x)+\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x))}\,, \tag{3.111}$$

where D is the Dirac operator

$$D = i\gamma^\mu\partial_\mu - m \tag{3.112}$$

and $\psi(x)$ and $\bar\psi(x)$ are Grassman fields, i.e. elements of an infinite-dimensional Grassman algebra. The currents $\eta(x)$ and $\bar\eta(x)$ are four-component source functions corresponding to the classical fields $\bar\psi(x)$ and $\psi(x)$, respectively. $\eta(x)$ and $\bar\eta(x)$ are also taken to be Grassman fields that anticommute among themselves as well as with the fields $\psi(x)$ and $\bar\psi(x)$:

$$\begin{aligned}
\{\psi(x),\psi(x')\} &= \{\psi(x),\bar\psi(x')\} = \{\bar\psi(x),\bar\psi(x')\} = 0 &(3.113)\\
\{\eta(x),\eta(x')\} &= \{\eta(x),\bar\eta(x')\} = \{\bar\eta(x),\bar\eta(x')\} = 0 &(3.114)\\
\{\eta(x),\psi(x')\} &= \{\eta(x),\bar\psi(x')\} = \{\eta(x),\bar\psi(x')\} = 0 &(3.115)
\end{aligned}$$

In order to carry out the integration, we follow the method used in Secion (3.1.2), by shifting the integration fields according to

$$\begin{aligned}
\psi(x) &= \psi'(x) + \psi_0(x)\\
\bar\psi(x) &= \bar\psi'(x) + \bar\psi_0(x)\,.
\end{aligned} \tag{3.116}$$

The jacobian associated with the above integration field transformation is 1. Expressing the fields $\psi(x)$ and $\bar\psi(x)$ in Eq. (3.111) in terms of $\psi'(x)$, $\bar\psi'(x)$ and $\psi_0(x)$, in the argument of the exponential we have a sum of terms (integrals) where some of them do not depend on the integration fields $\psi'(x)$, $\bar\psi'(x)$. Since each addendum is a product of two Grassman fields, we can write the exponential of the sum as a product of exponentials and bring out of the functional integration the factors that do not depend on the integration fields:

$$\begin{aligned}
Z_0\left[\eta,\bar\eta\right] &= \mathcal{N}_0 e^{i\int d^4x\,\bar\psi_0(x)D\psi_0(x)+\bar\eta(x)\psi_0(x)+\bar\psi_0(x)\eta(x)}\\
&\times \int \left[d\bar\psi'(x)\right]\left[d\psi'(x)\right]\, e^{i\int d^4x(\bar\psi'(x)D\psi'(x)+\bar\psi'(x)D\psi_0(x)+\bar\psi_0(x)D\psi'(x)+\bar\eta(x)\psi'(x)+\bar\psi'(x)\eta(x))}\\
&= \mathcal{N}_0 e^{i\int d^4x\,\bar\psi_0(x)D\psi_0(x)+\bar\eta(x)\psi_0(x)+\bar\psi_0(x)\eta(x)} \int \left[d\bar\psi(x)\right]\left[d\psi(x)\right] \Big[e^{iS_0[\psi(x),\bar\psi(x)]}\\
&\times\, e^{i\int d^4x(\bar\psi(x)D\psi_0(x)+\bar\psi_0(x)D\psi(x)+\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x))}\Big]\,,
\end{aligned} \tag{3.117}$$

where we have introduced the "classical action" of the Dirac field

$$\begin{aligned}
S_0\left[\psi(x),\bar\psi(x)\right] &= \int d^4x\,\mathcal{L}_D(x)\,, &(3.118)\\
\mathcal{L}_D(x) &= \bar\psi(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x)\,. &(3.119)
\end{aligned}$$

Up to now $\psi_0(x)$ is an arbitrary function. We can choose $\psi_0(x)$ to be solution of the following non-homogeneous differential equation

$$D\psi_0(x) = -\eta(x)\,. \tag{3.120}$$

whose solution can be written in terms of the Green function $S_F(x)$

$$\psi_0(x) = -\int dx'\, S_F(x-x')\,\eta(x')\,, \tag{3.121}$$

where $S_F(x)$ satisfies the equation

$$DS_F(x) = \delta^4(x)\,. \tag{3.122}$$

A solution $S_F(x)$ can be obtained, by resorting to Eq. (3.40) for the scalar field:

$$S_F(x) = (i\gamma^\mu\partial_\mu + m)\,\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{-ipx}\frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + i\varepsilon}\,. \tag{3.123}$$

As a check, if we substitute Eq. (3.123) into Eq. (3.122), we get

$$(i\gamma^\mu\partial_\mu - m)\,S_F(x) = (i\gamma^\mu\partial_\mu - m)\,(i\gamma^\nu\partial_\nu + m)\,S_F(x) = -\left(\Box + m^2\right)\Delta_F(x) = \delta^4(x)\,. \tag{3.124}$$

With the expression of $\psi_0(x)$ of Eq. (3.121), the argument of the last exponential (inside the functional integral) in Eq. (3.117) becomes vanishing and the argument of the exponential outside integration is

$$\int d^4x\, \bar\psi_0(x)D\psi_0(x) + \bar\eta(x)\psi_0(x) + \bar\psi_0(x)\eta(x) = -\int d^4x \int d^4x'\bar\eta(x')S_F(x'-x)\eta(x)\,, \tag{3.125}$$

so that the expression of the generating functional $Z_0\,[\eta,\bar\eta]$ becomes

$$Z_0\,[\eta,\bar\eta] = \mathcal{N}_0 e^{-i\int d^4x\int d^4x'\,\bar\eta(x')S_F(x'-x)\eta(x)}\int [d\bar\psi(x)]\,[d\psi(x)]\; e^{iS_0[\psi(x),\bar\psi(x)]}\,. \tag{3.126}$$

Defining

$$\mathcal{N}_0^{-1} \equiv \int [d\bar\psi(x)]\,[d\psi(x)]\; e^{iS_0[\psi(x),\bar\psi(x)]}\,, \tag{3.127}$$

we obtain the final expression for the generating functional, normalized to one for vanishing external currents:

$$Z_0\,[\eta,\bar\eta] = e^{-i\int d^4x\int d^4x'\,\bar\eta(x')S_F(x'-x)\eta(x)} \tag{3.128}$$

satisfying the normalization condition $Z[0,0] = 1$.

Observation: while $\Delta_F(x-y)$ is symmetric under exchange $x \leftrightarrow y$, $S_F(x-y)$ is not. However, $S_F(x-y)$ is symmetric under simultaneous exchange $x \leftrightarrow y$ and hermitean conjugation:

$$\gamma_0 S_F^\dagger(y-x)\gamma_0 + S_F(x-y)\,, \tag{3.129}$$

since $\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu$.

3.2.5 Green functions for the Dirac field

The n-point Green functions for fermions can be obtained as functional derivatives of $Z[\eta]$, paying attention to the fact that we have two fields: $\psi(x)$ and $\bar{\psi}(x)$:

$$G_{\alpha,\beta,\ldots,2\nu}(x_1, x_2, \ldots, x_{2n}) = \left(\frac{1}{i}\right)^{2n} \frac{\delta^{2n} Z[\eta, \bar{\eta}]}{\delta\eta_{2\nu}(x_{2n}) \cdots \delta\eta_{\nu+1}(x_{n+1})\delta\eta_\nu(x_n) \cdots \delta\bar{\eta}_\alpha(x_1)}_{|\eta=\bar{\eta}=0}, \tag{3.130}$$

where $Z[\eta, \bar{\eta}]$ is given by Eq. (3.111).
Observation 1: the order of the functional derivatives with respect to η and $\bar{\eta}$ is fixed by convention.
Observation 2: the Green's functions for fermions depend also on the Dirac spinor-indices.
Observation 3: the expression of Eq. (3.130) for the Green functions corresponds to the following ones, as path integrals and vacuum expectation value of time-ordered product of field operators:

$$\begin{aligned} G_{\alpha,\beta,\ldots,2\nu}(x_1, x_2, \ldots, x_{2n}) &= \frac{\int [d\bar{\psi}]\,[d\psi]\; \psi_\nu(x_n) \cdots \psi_\alpha(x_1)\bar{\psi}_{2\nu}(x_{2n}) \cdots \bar{\psi}_{\nu+1}(x_{n+1}) e^{iS[\psi,\bar{\psi}]}}{\int [d\bar{\psi}]\,[d\psi]\; e^{iS[\psi,\bar{\psi}]}} \\ &= \langle 0| \left[\hat{\psi}_\nu(x_n) \cdots \hat{\psi}_\alpha(x_1)\hat{\bar{\psi}}_{2\nu}(x_{2n}) \cdots \hat{\bar{\psi}}_{\nu+1}(x_{n+1})\right] |0\rangle\,. \end{aligned} \tag{3.131}$$

Let us verify Eq. (3.131) for the two-point Green function, starting from Eq. (3.130) and Eq. (3.111):

$$\begin{aligned} G_{\alpha,\beta}(x_1, x_2) &= -\frac{\delta^2 Z[\eta, \bar{\eta}]}{\delta\eta_\beta(x_2)\bar{\eta}_\alpha(x_1)}_{|\eta=\bar{\eta}=0} \\ &= -\left\{\frac{\delta}{\delta\eta_\beta(x_2)}\mathcal{N}_0 \int [d\bar{\psi}(x)]\,[d\psi(x)]\; i\psi_\alpha(x_1) e^{i\int d^4x(\bar{\psi}(x)D\psi(x)+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x))}\right\}_{|\eta=\bar{\eta}=0} \\ &= -\left\{\mathcal{N}_0 \int [d\bar{\psi}(x)]\,[d\psi(x)]\; i\psi_\alpha(x_1)(-)(-)i\bar{\psi}_\beta(x_2) e^{i\int d^4x(\bar{\psi}(x)D\psi(x)+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x))}\right\}_{|\eta=\bar{\eta}=0} \\ &= \mathcal{N}_0 \int [d\bar{\psi}(x)]\,[d\psi(x)]\; \psi_\alpha(x_1)\bar{\psi}_\beta(x_2) e^{i\int d^4x(\bar{\psi}(x)D\psi(x))}, \end{aligned} \tag{3.132}$$

where

$$\mathcal{N}_0 = \left\{\int [d\bar{\psi}(x)]\,[d\psi(x)]\; e^{i\int d^4x\bar{\psi}(x)D\psi(x)}\right\}^{-1}. \tag{3.133}$$

In Eq. (3.132) we have shown explicitely the minus signs coming from the anticommutation of the functional derivative w.r.t $\eta_\beta(x_2)$ with $\psi(x_1)$ and from the anticommutation of $\bar{\psi}(x)$ and $\eta(x)$ in the argument of the exponential.
By means of the expression of Eq. (3.128) for $Z_0[\eta, \bar{\eta}]$ we can derive the relation between

the fermionic Feynman propagator $S_F(x_1 - x_2)$ and the two-point Green function:

$$\begin{aligned}
G_{\alpha,\beta}(x_1,x_2) &= -\frac{\delta^2 Z\,[\eta,\bar\eta]}{\delta\eta_\beta(x_2)\bar\eta_\alpha(x_1)}_{|\eta=\bar\eta=0} \\
= & -\frac{\delta}{\delta\eta_\beta(x_2)}\left\{\left[-i\int d^4x\,(S_F(x_1-x))_{\alpha\gamma}\,\eta_\gamma(x)\right]e^{-i\int d^4y\int d^4x'\,\bar\eta(x')S_F(x'-y)\eta(y)}\right\}_{|\eta=\bar\eta=0} \\
= & +i\,(S_F(x_1-x_2))_{\alpha\beta}\ . \qquad (3.134)
\end{aligned}$$

Exercise: Verify the relations between the various forms given above for the fermionic four-point Green function.
Solution:

$$\begin{aligned}
G_{\alpha,\beta,\gamma,\delta}(x_1,x_2,x_3,x_4) &= \frac{\delta^4\int[d\bar\psi(x)]\,[d\psi(x)]\;e^{i\int d^4x\mathcal{L}+\bar\eta\psi+\bar\psi\eta}}{\delta\eta_\delta(x_4)\delta\eta_\nu(x_3)\delta\bar\eta_\beta(x_2)\delta\bar\eta_\alpha(x_1)} \\
&= \frac{\delta^3\int[d\bar\psi(x)]\,[d\psi(x)]\;\psi_\alpha(x_1)e^{i\int d^4x\mathcal{L}+\bar\eta\psi+\bar\psi\eta}}{\delta\eta_\delta(x_4)\delta\eta_\nu(x_3)\delta\bar\eta_\beta(x_2)} \\
&= \frac{\delta^2\int[d\bar\psi(x)]\,[d\psi(x)]\;(-)\psi_\alpha(x_1)\psi_\beta(x_2)e^{i\int d^4x\mathcal{L}+\bar\eta\psi+\bar\psi\eta}}{\delta\eta_\delta(x_4)\delta\eta_\nu(x_3)} \\
&= \frac{\delta\int[d\bar\psi(x)]\,[d\psi(x)]\;(-)\psi_\alpha(x_1)\psi_\beta(x_2)(-)\bar\psi_\nu(x_3)e^{i\int d^4x\mathcal{L}+\bar\eta\psi+\bar\psi\eta}}{\delta\eta_\delta(x_4)} \\
&= \int[d\bar\psi(x)]\,[d\psi(x)]\;(-)\psi_\alpha(x_1)\psi_\beta(x_2)(-)\bar\psi_\nu(x_3)\bar\psi_\delta(x_4)e^{i\int d^4x\mathcal{L}+\bar\eta\psi+\bar\psi\eta} \\
&= \int[d\bar\psi(x)]\,[d\psi(x)]\;\psi_\beta(x_2)\psi_\alpha(x_1)\bar\psi_\delta(x_4)\bar\psi_\nu(x_3)e^{i\int d^4x\mathcal{L}+\bar\eta\psi+\bar\psi\eta}\,. \qquad (3.135)
\end{aligned}$$

3.3 The Electromagnetic field

The aim of the present Section is to introduce the quantization of the free electromagnetic field through functional integration. In the previous Sections, we have seen that the propagator of the (bosonic and fermionic) field was obtained through the inversion of the operator quadratic in the fields in the generator functional. For instance, in the case of the scalar field, we have obtained the following result:

$$\int d\varphi\, e^{-\frac{1}{2}\varphi K\varphi + J\varphi} = e^{\frac{1}{2}JK^{-1}J} \qquad (3.136)$$

and similarly for the fermionic field. The free electromagnetic field presents an additional problem: because of the gauge invariance property, the photon propagator in not defined. Because of gauge invariance, also the functional integration of the generating functional is not defined.

3.3.1 Propagator and gauge fixing

The free Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3.137}$$

Therefore the classical Action is

$$\begin{aligned} S_0 &= -\int dx \, \frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= -\int dx \, \frac{1}{4} \left[\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) - \left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right) - \left(\partial_\nu A_\mu\right)\left(\partial^\mu A^\nu\right) + \left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right)\right] \\ &= -\int dx \, \frac{1}{4} \left[2\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) - \left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right) - \left(\partial_\nu A_\mu\right)\left(\partial^\mu A^\nu\right)\right] \,. \end{aligned} \tag{3.138}$$

Let us analyse the first term of Eq. (3.138):

$$\begin{aligned} \left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) &= -A_\nu \left(\partial_\mu \partial^\mu\right) A^\nu + \partial_\mu \left(A_\nu \partial^\mu A^\nu\right) \\ &= -A_\nu \Box A^\nu + \text{fourdivergence} \, ; \end{aligned} \tag{3.139}$$

$$\begin{aligned} \left(\partial_\mu A_\nu\right)\left(\partial^\nu A_\mu\right) &= -A_\nu \left(\partial_\mu \partial^\nu\right) A^\mu + \partial_\mu \left(A_\nu \partial^\nu A^\mu\right) \\ &= -A_\nu \left(\partial^\nu \partial_\mu\right) A^\mu + \partial_\mu \left(A_\nu \partial^\nu A^\mu\right) \\ &= -A_\nu \left(\partial^\nu \partial_\mu\right) A^\mu + \text{fourdivergence} \, ; \end{aligned} \tag{3.140}$$

$$\left(\partial_\nu A_\mu\right)\left(\partial^\mu A_\nu\right) = -A_\nu \left(\partial^\nu \partial_\mu\right) A^\mu + \text{fourdivergence} \,. \tag{3.141}$$

Through the above equations, we see that, up to a fourdivergence, we can write Eq. (3.138) as follows

$$\begin{aligned} \mathcal{S}_0 &= -\frac{1}{2} \int dx \, \left[-A_\nu \Box A^\nu + A_\nu \partial^\nu \partial_\mu A^\mu\right] \\ &= +\frac{1}{2} \int dx \, \left[+A_\nu \Box g^{\mu\nu} A_\mu - A_\nu \partial^\nu \partial^\mu A_\mu\right] \\ &= +\frac{1}{2} \int dx \, A_\nu \left[g^{\mu\nu} \Box - \partial^\nu \partial^\mu\right] A_\mu = +\frac{1}{2} \int dx \, A^\mu \left[g_{\mu\nu} \Box - \partial_\mu \partial_\nu\right] A^\nu \,. \end{aligned} \tag{3.142}$$

According to Eq. (3.142) we have rewritten the Lagrangian density in the form $A^\mu K_{\mu\nu} A^\nu$, where $K_{\mu\nu}$ is the operator $g_{\mu\nu}\Box - \partial_\mu \partial_\nu$. The photon propagator is the inverse of the operator $K_{\mu\nu}$, *i.e.*

$$K_{\mu\nu} D^{\nu\lambda}(x-y) = \delta_\mu^\lambda \delta^4(x-y) \,. \tag{3.143}$$

$D^{\nu\lambda}$ in Eq. (3.143) is the two-point Green function. In fact, we can consider the Maxwell equations

$$\partial_\mu F^{\mu\nu} = J^\nu \,. \tag{3.144}$$

On the other hand, from the definition of $F^{\mu\nu}$ we have

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \Box A^\nu - \partial^\nu \partial_\mu A^\mu \\ &= \left(g^{\mu\nu} \Box - \partial^\nu \partial^\mu\right) A_\mu = K^{\mu\nu} A_\mu \,, \end{aligned} \tag{3.145}$$

i.e. the Maxwell equations would be

$$K_{\mu\nu} A^{\mu} = J^{\mu} \, . \tag{3.146}$$

which would be solved by finding the Green functions of the operator $K_{\mu\nu}$.
We note that if we apply the operator $K_{\mu\nu}$ to an arbitrary four-gradient $\partial^{\nu} G$ we obtain:

$$K_{\mu\nu} \partial^{\nu} G = (\partial_{\nu} \Box - \partial_{\nu} \Box) \, G = 0 \, . \tag{3.147}$$

Eq. (3.147) means that $K_{\mu\nu}$ **has zero eigenvalue and its inverse does not exist.**
Observation 1: the above feature is due to the fact that the photon is massless. In fact, if it had a mass, in the above equations we would have $\Box \to \Box + m^2$ and gauge invariance woul not hold [6].
Observation 2: the gauge invariance has non-trivial consequences also on the functional integration of the generating functional

$$Z = \int [dA_{\mu}] \, e^{i \int dx \, \mathcal{L}(x)} \tag{3.148}$$

where $\mathcal{L}(x)$ is invariant under gauge transformations $A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu} \Lambda(x)$. **The functional integration is done over all $A_{\mu}(x)$, including those connected by a gauge tranformation**. This means that we are doing multiple counting of the same (from a physical point of view) path.

Gauge Fixing

A solution to the above problem is to restrict the functional integral over the functions $A_{\mu}(x)$ connected by a gauge transformation. For instance we can impose the Lorentz condition $\partial_{\mu} A^{\mu} = 0$ and the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} A^{\mu} g_{\mu\nu} \Box A^{\nu} \tag{3.149}$$

and the inverse of the operator $g_{\mu\nu}\Box$ is the Feynman propagator

$$D_F(x, y)_{\mu\nu} = -g_{\mu\nu} \Delta_F(x, y)|_{m=0} \tag{3.150}$$

where $\Delta_F(x, y)$ is the scalar propagator. The Lagrangian density reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{GF} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(\partial_{\mu} A^{\mu} \right)^2 \, . \tag{3.151}$$

It is useful to study the problem in momentum space, where

$$g_{\mu\nu} \Box - \partial_{\mu} \partial_{\nu} \to -g_{\mu\nu} k^2 + k_{\mu} k_{\nu} \, . \tag{3.152}$$

[6] While for a massive vector we can always find a restframe where we can easily identify three polarization directions, for a massless vector we have only two polarization directions, which are in the plane perpendicular to the motion direction. So a massless vector is intrinsecally different from the limit $m \to 0$ of a massive vector because of the change in the number of polarization degrees of freedom.

The inverse of the operator of Eq. (3.152) would have the form $Ag^{\nu\lambda} + Bk^\nu k^\lambda$, with A and B such that

$$\left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right)\left(Ag^{\nu\lambda} + Bk^\nu k^\lambda\right) = \delta^\lambda_\mu\,. \tag{3.153}$$

Working out the product in the above equation, we have

$$-A\left(k^2\delta^\lambda_\mu + k_\mu k^\lambda\right) = \delta^\lambda_\mu\,, \tag{3.154}$$

which has no solution. On the other hand, with the Lorentz condition $\partial_\mu A^\mu = 0$, the substitution of Eq. (3.152) becomes

$$g_{\mu\nu}\Box \to -g_{\mu\nu}k^2\,, \tag{3.155}$$

where the operator on the right-hand side admits the inverse

$$-\frac{g^{\mu\nu}}{k^2}\,. \tag{3.156}$$

Hence the Feynman propagator is

$$\left(D_F(k)\right)_{\mu\nu} = -\frac{g^{\mu\nu}}{k^2}\,. \tag{3.157}$$

More in general w.r.t. Eq. (3.151), conventionally the gauge fixing Lagrangian is multiplied by a finite arbitrary constant ξ:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{GF} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 = \frac{1}{2}A^\mu\left[g_{\mu\nu}\Box + \left(\frac{1}{\xi} - 1\right)\partial_\mu\partial_\nu\right]A^\nu\,. \tag{3.158}$$

The operator quadratic in the field $A^\mu(x)$, in momentum space, is

$$-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)\,, \tag{3.159}$$

whose inverse gives the propagator

$$\left(D(k)\right)_{\mu\nu} = -\frac{1}{k^2}\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]\,. \tag{3.160}$$

Different choices of the constant ξ correspond to different gauges:

$$\xi \to 1 \implies \text{Feynman gauge} \tag{3.161}$$
$$\xi \to 0 \implies \text{Landau gauge}\,. \tag{3.162}$$

3.3.2 The Generating Functional for the free electromagnetic field

Equipped with the expression of Eq. (3.160) for the photon propagator, we can write the Z generating functional as

$$Z_A[J] = \int [dA_\mu]\, e^{i\int d^4x \left(\frac{1}{2}A^\mu D^{-1}_{\mu\nu}A^\nu + J_\mu A^\mu\right)} = \int [dA_\mu]\, e^{i\int d^4x\,(\mathcal{L}_0+\mathcal{L}_{GF}+J_\mu A^\mu)}, \tag{3.163}$$

where the notation $[dA_\mu]$ stands for $\prod_{\mu=1}^{4}[dA_\mu]$.
The equivalent of Eq. (3.163) with Euclidean metrics is

$$Z_A^E[J] = \int [dA_\mu]\, e^{-\int d^4x_E\left[\frac{1}{2}A^\mu\left(\delta_{\mu\nu}\Box^E+\partial^E_\mu\partial^E_\nu\right)A^\nu + J_\mu A^\mu\right]} e^{-\frac{1}{2\xi}\int d^4x_E\left(\partial^E_\mu A^\mu\right)^2}. \tag{3.164}$$

Eq. (3.164) shows that the gauge fixing term gives a suppression factor for the field configurations which do not fulfill the Lorentz condition.
The normalized generating functional for the free electromagnetic field is

$$Z_0[J] = e^{-\frac{i}{2}\int d^4x d^4y\, J^\mu(x)D_{\mu\nu}(x-y)J^\nu(x)}. \tag{3.165}$$

Eq. (3.165) shows that the integral of the gauge fixing term and the integral of the source term are gauge-dependent. Therefore the generating functional for the free electromagnetic field is gauge dependent (it depends on the arbitrary parameter ξ), together with all Green functions obtained by functional derivation. However, when computing physical transition amplitudes, the field $A_\mu(x)$ is always coupled to a conserved current, for which $\partial_\mu j^\mu = 0$ ($k_\mu j^\mu(k) = 0$, in momentum representation). Since, by inspection, the gauge dependent terms are proportional to $k_\mu k_\nu$, they give no contribution to physical quantities, which are, correctly, gauge independent.

3.3.3 The Faddeev and Popov method

The basic idea of the method [7] is to find a coordinate system in the path space, such that, along one of the coordinates, only non equivalent gauge field configurations are integrated (*i.e.not related by gauge transformations of the kind* $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$) and the integral along the remaining coordinates becomes completely factorized:

$$\int [dA_\mu(x)] \rightarrow \int [d\Lambda] \int [d\bar{A}_\mu]. \tag{3.166}$$

If we succeed in finding this coordinate system, the overall factor cancels out in the Green function calculations.
We can illustrate the basic idea through a well known example:

$$I = \int dx \int dy\, e^{-\left(x^2+y^2\right)}. \tag{3.167}$$

[7] We illustrate the method for QED, even if it displays its power in the quantization of the more complex Yang-Mills theories.

The integrand is function of only $r = \sqrt{x^2+y^2}$. By a change of coordinate system, from cartesian to polar coordinates, we have

$$I = \int d\vartheta \int dr\, r e^{-r^2}\,, \tag{3.168}$$

where $2\pi = \int d\vartheta$ is the volume of the rotation group in two dimensions.
We can see the above transformation in a more general way as follows:

$$I = \int d\vartheta' \int dr\, r e^{-r^2} = \int d\vartheta' \int dr \int d\vartheta\, r e^{-r^2} \delta\left(f(\vartheta)\right)\,. \tag{3.169}$$

Remembering that

$$\delta\left(f(\vartheta)\right) = \sum_i \frac{1}{\left|\frac{df(\vartheta)}{d\vartheta}\right|_{\vartheta=\vartheta_i}} \delta\left(\vartheta-\vartheta_i\right)\,, \tag{3.170}$$

if we take

$$f(\vartheta) = y\cos\vartheta - x\sin\vartheta\,, \tag{3.171}$$

we have

$$\begin{aligned} &y\cos\vartheta_i - x\sin\vartheta_i = 0\,,\\ &\vartheta_1 = \arctan\left(\frac{y}{x}\right)\,,\\ &\vartheta_2 = \pi + \arctan\left(\frac{y}{x}\right)\,,\\ &\left.\frac{df}{d\vartheta}\right|_{\vartheta_1,\vartheta_2} = -\frac{x}{\cos\vartheta}\,. \end{aligned} \tag{3.172}$$

Therefore

$$\delta\left(f(\vartheta)\right) = \frac{1}{r}\left[\delta(\vartheta-\vartheta_1)+\delta(\vartheta-\vartheta_2)\right]\,, \tag{3.173}$$

$$\int d\vartheta\,\delta\left(f(\vartheta)\right) = \frac{2}{r} = \frac{2}{\sqrt{x^2+y^2}}\,. \tag{3.174}$$

We define the quantity $\Delta(r)$ according to

$$\Delta(r)\int d\vartheta\,\delta\left(f(\vartheta)\right) \equiv 1 \Longrightarrow \Delta(r) = \frac{\sqrt{x^2+y^2}}{2}\,. \tag{3.175}$$

We observe also that Eq. (3.171) defines $f(\vartheta)$ through a rotation in the $x-y$ plane (which leaves $\sqrt{x^2+y^2} = \sqrt{x'^2+y'^2}$):

$$\begin{aligned} y' &= y\cos\vartheta - x\sin\vartheta\,,\\ x' &= x\cos\vartheta + y\sin\vartheta\,. \end{aligned} \tag{3.176}$$

Hence Eq. (3.171) can be written as

$$\Delta\left(\sqrt{x'^2+y'^2}\right)\int \delta\left(y'\right)\,d\vartheta = 1\,. \tag{3.177}$$

By inserting the above identity in Eq. (3.169), we have

$$I = \int d\vartheta \int dx' \int dy' \, e^{-\left(x'^2+y'^2\right)} \Delta\left(\sqrt{x'^2+y'^2}\right) \delta\left(y'\right) \, . \tag{3.178}$$

The separation of variables is possible thanks to the rotational invariance of the integrand. Since the integration on $dx'\,dy'$ is independent of ϑ, $\int d\vartheta$ is an overall multiplicative factor.

Moreover,

$$(\Delta(r))^{-1} = \int d\vartheta \, \delta\left(f(\vartheta)\right) = \int df \, \delta\left(f(\vartheta)\right) \det\left[\frac{d\vartheta}{df}\right] = \det\left[\frac{d\vartheta}{df}\right]_{f=0} \, . \tag{3.179}$$

Hence [8]

$$\Delta(r) = \left(\det\left[\frac{d\vartheta}{df}\right]_{f=0}\right)^{-1} \, . \tag{3.180}$$

We will now apply the idea illustrated above on the functional integral of the QED generator functional of Eq. (3.148). Let us now introduce the function $f\left(A_\mu(x)\right)$ used to fix the gauge. For instance, we have $f\left(A_\mu(x)\right) = \partial_\mu A^\mu(x)$ in the Lorentz gauge. We call $B\left[f\right]$ the functional of f which fixes the gauge. The Lorentz gauge is given with $B\left[f\right] = \delta\left[f(A)\right]$. More in general, we can use a gaussian functional:

$$B\left[f\right] = e^{-\frac{i}{2\xi}\int d^4x \left(f(A_\mu(x))\right)^2} \, . \tag{3.181}$$

The Lorentz gauge is given by

$$B\left[f\right] = e^{-\frac{i}{2\xi}\int d^4x \left(\partial_\mu A^\mu\right)^2} \, , \tag{3.182}$$

in the limit $\xi \to 0$. Let us denote with $\left(A^\Lambda\right)_\mu$ the gauge potential A_μ after a gauge transformation fixed by the function $\Lambda(x)$:

$$\left(A^\Lambda\right)_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \, . \tag{3.183}$$

We introduce also the determinant (called the Faddeev-Popov determinant) [9]

$$\Delta\left[A\right] = \det\left[\frac{\delta f\left(A^\lambda(x)\right)}{\delta\lambda(y)}\Big|_{\lambda=0}\right] \, . \tag{3.184}$$

[8] Actually the determinant comes out naturally in more than one dimension.

[9] The Faddeev-Popov determinant is a *functional determinant*. With n-discretized space-time coordinates, we would have

$$\Delta A_n = \det\left[\frac{\delta f\left(A^\lambda(x_i)\right)}{\delta\lambda(x_j)}\Big|_{\lambda=0}\right] \, .$$

We first note that the functional integral

$$\int [d\Lambda]\, \Delta\left[A^{\Lambda}\right] B\left[f\left(A^{\Lambda}\right)\right] \tag{3.185}$$

is a constant, $\mathcal{C}$, **independent of** $A(x)$ **but only on** f **and** $B[f]$. In fact, through Eq. (3.183), we can write

$$\left[\left(A^{\Lambda}\right)^{\lambda}\right]_{\mu} = \left(A^{\Lambda}\right)_{\mu} + \partial_\mu \lambda = A_\mu + \partial_\mu \Lambda + \partial_\mu \lambda = A_\mu + \partial_\mu (\Lambda + \lambda) = \left(A^{\Lambda+\lambda}\right)_{\mu} . \tag{3.186}$$

As a consequence, for the Faddeev-Popov determinant we can write

$$\Delta\left[A^{\Lambda}\right] = \det\left[\frac{\delta f\left(\left(A^{\Lambda}\right)^{\lambda}(x)\right)}{\delta\lambda(y)}\Big|_{\lambda=0}\right] = \det\left[\frac{\delta f\left(A^{\Lambda+\lambda}(x)\right)}{\delta\lambda(y)}\Big|_{\lambda=0}\right] = \det\left[\frac{\delta f\left(A^{\Lambda}(x)\right)}{\delta\Lambda(y)}\right] . \tag{3.187}$$

We can insert Eq. (3.187) into Eq. (3.185) to get

$$\int [d\Lambda] \det\left[\frac{\delta f\left(A^{\Lambda}(x)\right)}{\delta\Lambda(y)}\right] B\left[f\left(A^{\Lambda}\right)\right] = \int [df]\, B[f] , \tag{3.188}$$

which is manifestly independent of A. We have transformed the functional integral on Λ in a functional integral on f and the result is independent of A [10].
Using the constant of Eq. (3.185), we can show that the following equation holds:

$$\mathcal{C} \int [dA_\mu]\; \mathcal{O}[A_\mu]\, e^{iS_0[A_\mu]} = \mathcal{N} \int [dA_\mu]\; \mathcal{O}[A_\mu]\, \Delta[A_\mu]\, e^{iS_0[A_\mu]} , \tag{3.190}$$

where $S_0[A_\mu]$ is the gauge invariant Maxwell Action of Eq. (3.138), $\mathcal{O}[A_\mu]$ is a generic gauge invariant functional, $\mathcal{C}$ is the gauge invariant functional integral of Eq. (3.185) and $\mathcal{N}$ is an undetermined constant. In fact

$$\begin{aligned}
\mathcal{C} \int [dA]\; \mathcal{O}[A]\, e^{iS_0[A]} &= \int [d\Lambda] \int [dA]\; \mathcal{O}[A]\, e^{iS_0[A]} \Delta\left[A^{\Lambda}\right] B\left[f\left(A^{\Lambda}\right)\right] \\
&= \int [d\Lambda] \int [dA]\; \mathcal{O}\left[A^{\Lambda}\right] e^{iS_0\left[A^{\Lambda}\right]} \Delta\left[A^{\Lambda}\right] B\left[f\left(A^{\Lambda}\right)\right] \\
&= \int [d\Lambda] \int \left[dA^{\Lambda}\right]\; \mathcal{O}\left[A^{\Lambda}\right] e^{iS_0\left[A^{\Lambda}\right]} \Delta\left[A^{\Lambda}\right] B\left[f\left(A^{\Lambda}\right)\right] \\
&= \int [d\Lambda] \int [dA]\; \mathcal{O}[A]\, e^{iS_0[A]} \Delta[A]\, B[f(A)] .
\end{aligned} \tag{3.191}$$

[10] In order to ensure that Eq. (3.188) is a valid change of variable in the functional integration, we need a univoque relation between f and Λ. This means that there should not be two values of Λ corresponding to the same value of A. In other words we should not have multiple solutions of the equation

$$f(A_0) = \partial_\mu A_0^\mu = 0 . \tag{3.189}$$

If we have multiple solutions, these are named *Gribov copies*. In perturbation theory we move in the neighborhood of A_0 and we can neglect the problem.

In the second equality we have used the gauge invariance of $\mathcal{O}[A]$ and of the Maxwell Action $S_0[A]$; in the third equality we have used the gauge invariance property of the funcional measure because the transformation of Eq. (3.183) is a translation in the functional space; in the last equality we have just renamed the integration variable. Comparing Eqs. (3.190) and (3.191), we conclude that the constant factor is $\mathcal{C} = \int [d\Lambda]$. Actually it is an infinite constant! But what is important is that the original functional integral is splitted into two separate factors.

If we calculate a gauge invariant Green function [11], we have:

$$\begin{aligned}\langle 0|\mathcal{O}[A]|0\rangle &= \frac{\int [dA]\ \mathcal{O}[A]\, e^{iS_0[A]}}{\int [dA]\ e^{iS_0[A]}} = \frac{\int [dA]\ \mathcal{O}[A]\,\Delta[A]\,B[f(A)]\,e^{iS_0[A]}}{\int [dA]\ \Delta[A]\,B[f(A)]\,e^{iS_0[A]}} \\ &= \frac{\int [dA]\ \mathcal{O}[A]\,\Delta[A]\,e^{iS_{new}[A]}}{\int [dA]\ \Delta[A]\,e^{iS_{new}[A]}}\,. \end{aligned} \tag{3.192}$$

The first equality is formally the usual expression for the calculation of a Green function, but, due to the gauge invariance, numerator and denominator are divergent. Instead, on the right hand of the second equality, the functional $B[f]$ restricts the integration paths and numerator and denominator are convergent, because a common (divergent) factor has been cancelled between numerator and denominator. In the third equality we have introduced a modified Action, reabsorbing the functional $B[f]$:

$$e^{iS_{new}[A]} = e^{iS_0[A]}B[f(A)]\ . \tag{3.193}$$

Even if S_{new} is not gauge invariant, Eq. (3.192) gives, by construction, a gauge invariant result, provided $\mathcal{O}[A]$ is gauge invariant.
We can specify S_{new} introducing Eq. (3.182) in Eq. (3.193), with the result

$$S_{new}[A] = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu)\right] = \int d^4x\,[\mathcal{L}_0(x) + \mathcal{L}_{GF}(x)]\ , \tag{3.194}$$

where $\mathcal{L}_0$ and $\mathcal{L}_{GF}$ have been specified in Eq. (3.158).
We have found (as in Eq. (3.151)) the modified Lagrangian density, which defines the quantized free theory through the usual functional integral formulation.

Remark 1: the method outlined in this section is a general, very useful method for the quantization of Yang-Mills theories, which involve self-interactions of the gauge fields already at tree-level. With Eq. (3.158) we can write the generating functional as follows:

$$Z_A[J] = \int [dA]\ \Delta[A]\, e^{i\int d^4x\,(\mathcal{L}_0 + \mathcal{L}_{GF} + J_\mu A^\mu)}\,. \tag{3.195}$$

Remark 2: By inspection, we can see an important difference between Eq. (3.195) and

[11] The presented derivation is valid for gauge invariant quantities (as is the case for S-matrix elements).

Eq. (3.163), which was obtained by simply analyzing the photon propagator: the former contains the Faddeev-Popov determinant, which is absent in the latter. Actually we can prove, by explicit calculation, that the two expressions for the Z generating functional are proportional, because *the Faddeev-Popov determinant in QED is independent of the field*. In order to calculate the Faddeev-Popov determinant, we obtain from Eq. (3.183), choosing the Lorentz gauge $(f(A_\mu(x)) = \partial^\mu A_\mu(x) = 0)$:

$$\partial_\mu \left(A^\lambda\right)^\mu (x) = \partial_\mu A^\mu(x) + \Box\lambda(x)\,. \tag{3.196}$$

According to the definition of Eq. (3.184) we have:

$$\begin{aligned}\Delta\left[A\right](x,y) &= \det\left[\frac{\delta f\left(A^\lambda(x)\right)}{\delta\lambda(y)}\Big|_{\lambda=0}\right] \\ &= \frac{\delta f\left(A_\mu(x) + \partial_\mu\lambda(x)\right)}{\delta\lambda(y)}\Big|_{\lambda=0} \\ &= \frac{\Box\delta\lambda(x)}{\delta\lambda(y)} = \Box\delta^4(x-y)\,.\end{aligned} \tag{3.197}$$

Since the Faddeev-Popov determinant is independent of the field, it is factored out of the functional integration on the physically different field configurations. Therefore the normalized generating functional is the same of Eq. (3.165).

3.4 Appendix A: useful integrals with Grassman variables

Gaussian integration
We consider now the generalization of Eq. (3.98) to N dimensions:

$$I_N = \int d\vartheta_1 \dots d\vartheta_N \, e^{-\vartheta^T M\vartheta}\,, \tag{3.198}$$

where m_{ij} are the elements of an $N \times N$ antisymmetric matrix.
Observation: for a 2×2 matrix we have

$$\vartheta^T M\vartheta = \begin{pmatrix}\vartheta_1 & \vartheta_2\end{pmatrix}\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix} = m_{12}\vartheta_1\vartheta_2 + m_{21}\vartheta_2\vartheta_1 \tag{3.199}$$

We note that the diagonal elements do not appear. If the matrix M is symmetric we have $\vartheta^T M\vartheta = 0$. If, instead, the matrix M is antisimmetric $\vartheta^T M\vartheta = 2m_{12}\vartheta_1\vartheta_2$. So, if M is antisimmetrix, we can write

$$I_2(M) = \int d\vartheta_1 \int d\vartheta_2 \left(1 - 2m_{12}\vartheta_1\vartheta_2\right) = 2m_{12} = 2\sqrt{\det(M)} \tag{3.200}$$

Consider now the case of a 3×3 matrix:

$$\begin{aligned}
\vartheta^T M \vartheta &= \begin{pmatrix} \vartheta_1 & \vartheta_2 & \vartheta_3 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{pmatrix} \\
&= m_{12}\vartheta_1\vartheta_2 + m_{13}\vartheta_1\vartheta_3 + m_{21}\vartheta_2\vartheta_1 \\
&+ m_{23}\vartheta_2\vartheta_3 + m_{31}\vartheta_3\vartheta_1 + m_{32}\vartheta_3\vartheta_2 \\
&= (m_{12} - m_{21})\vartheta_1\vartheta_2 + (m_{13} - m_{31})\vartheta_1\vartheta_3 + (m_{23} - m_{32})\vartheta_2\vartheta_3 \,. \qquad (3.201)
\end{aligned}$$

The function $\exp -\vartheta^T M \vartheta$ is defined by its formal series expansion

$$e^{-\vartheta^T M \vartheta} = 1 - \vartheta^T M \vartheta + \frac{\left(\vartheta^T M \vartheta\right)^2}{2} + \dots \,. \qquad (3.202)$$

We observe that

$$\left(\vartheta^T M \vartheta\right)^2 = \left(A\vartheta_1\vartheta_2 + B\vartheta_1\vartheta_3 + C\vartheta_1\vartheta_3\right)\left(A\vartheta_1\vartheta_2 + B\vartheta_1\vartheta_3 + C\vartheta_1\vartheta_3\right) = 0\,, \qquad (3.203)$$

because each element contains always four Grassman elements, so that one always appears two times in every term. As a consequence

$$e^{-\vartheta^T M \vartheta} = 1 - \vartheta^T M \vartheta \,. \qquad (3.204)$$

This means that

$$I_3(M) = \int d\vartheta_1 \int d\vartheta_2 \int d\vartheta_3 \, e^{-\vartheta^T M \vartheta} = 0 \qquad (3.205)$$

because for each term we have always one integration yielding 0: $\int d\vartheta_i = 0$.
We can now go to the case $N = 4$:

$$I_4(M) = \int d\vartheta_1 \int d\vartheta_2 \int d\vartheta_3 \int d\vartheta_4 \, e^{-\vartheta^T M \vartheta} \,. \qquad (3.206)$$

Eq. (3.199) becomes

$$\vartheta^T M \vartheta = \sum_{i=1,3;j=2,4} \left(m_{ij} - m_{ji}\right) \vartheta_i \vartheta_j = 2 \sum_{i=1,3;j=2,4} m_{i,j}\vartheta_i\vartheta_j = 2 \sum_{i=1,3} m_{i,i+1}\vartheta_i\vartheta_{i+1} \,, \qquad (3.207)$$

where the antisimmetry of M has been assumed. Eq. (3.203) becomes

$$\begin{aligned}
\left(\vartheta^T M \vartheta\right)^2 &= 4 \left(\sum_{i=1,3} m_{i,i+1}\vartheta_i\vartheta_{i+1} \right) \left(\sum_{j=1,3} m_{j,j+1}\vartheta_j\vartheta_{j+1} \right) \\
&= 4 \Big[m_{12}m_{34}\vartheta_1\vartheta_2\vartheta_3\vartheta_4 + m_{34}m_{12}\vartheta_3\vartheta_4\vartheta_1\vartheta_2 \\
&+ m_{13}m_{24}\vartheta_1\vartheta_3\vartheta_2\vartheta_4 + m_{24}m_{13}\vartheta_2\vartheta_4\vartheta_1\vartheta_3 \\
&+ m_{14}m_{23}\vartheta_1\vartheta_4\vartheta_2\vartheta_3 + m_{23}m_{14}\vartheta_2\vartheta_3\vartheta_1\vartheta_4 \Big] \\
&= 8 \left(m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}\right) \vartheta_1\vartheta_2\vartheta_3\vartheta_4 \\
&= 8 \sqrt{\det(M)} \vartheta_1\vartheta_2\vartheta_3\vartheta_4 \,. \qquad (3.208)
\end{aligned}$$

Using Eqs. (3.207) and (3.208), Eq. (3.206) becomes

$$
\begin{aligned}
I_4(M) &= \int d\vartheta_1 \int d\vartheta_2 \int d\vartheta_3 \int d\vartheta_4\, e^{-\vartheta^T M \vartheta} \\
&= \int d\vartheta_1 \int d\vartheta_2 \int d\vartheta_3 \int d\vartheta_4\, 4\sqrt{\det(M)}\vartheta_1\vartheta_2\vartheta_3\vartheta_4 \\
&= -4\sqrt{\det(M)} \int d\vartheta_1 \int d\vartheta_2 \int d\vartheta_3 \left(\int d\vartheta_4\, \vartheta_4\right) \vartheta_1\vartheta_2\vartheta_3 \\
&= +4\sqrt{\det(M)}\,. \qquad (3.209)
\end{aligned}
$$

The general formula for $I_N(M)$ is

$$
\begin{aligned}
I_N(M) &= 2^{\frac{N}{2}}\sqrt{\det(M)} \qquad \text{for even N}\,, \\
I_N(M) &= 0 \qquad \text{for odd N}\,.
\end{aligned} \qquad (3.210)
$$

Observation: we note that Eq. (3.210) is similar to Eq. (2.203), which we report here (with $b = 0$):

$$
\int d^N x \,\exp\left(-x^T \cdot A \cdot x\right) = \pi^{\frac{N}{2}} \left(\det A\right)^{-\frac{1}{2}}\,, \qquad (3.211)
$$

By comparison of Eq. (3.211) and Eq. (3.210) we can conclude that: **in the "gaussian" integral over Grassman variables the matrix determinant appears with a different sign in the exponent with respect to the standard gaussian integral (used for bosonic fields)** .

It is useful to consider also the following integral

$$
I_N(M;\chi) \equiv \int d\vartheta_1 \dots d\vartheta_N\, e^{-\vartheta^T M \vartheta + \chi^T \vartheta}\,, \qquad (3.212)
$$

which is formally analogous to the bosonic case, when we considered the forced harmonic oscillator. χ is a vector of Grassman variables χ_i with the properties

$$
\begin{aligned}
\{\chi_i, \chi_j\} &= 0 \\
\{\chi_i, \vartheta_j\} &= 0\,.
\end{aligned} \qquad (3.213)
$$

Let us work out the case of $N = 2$:

$$
\begin{aligned}
I_2(M;\chi) &\equiv \int d\vartheta_1 d\vartheta_2\, e^{-\vartheta^T M \vartheta + \chi^T \vartheta} \\
&= \int d\vartheta_1 d\vartheta_2 \,\left(1 - 2m_{12}\vartheta_1\vartheta_2 - \chi_1\chi_2\vartheta_1\vartheta_2\right) \\
&= 2\left(m_{12} + \frac{1}{2}\chi_1\chi_2\right)\,. \qquad (3.214)
\end{aligned}
$$

The same result of Eq. (3.214) can be obtained with a change of variables:

$$
\begin{aligned}
\vartheta' &= \vartheta + \frac{1}{2} M^{-1}\chi\,, \\
d\vartheta' &= d\vartheta\,.
\end{aligned} \qquad (3.215)
$$

Considering that, for an antisimmetric 2x2 matrix M, $\left(M^{-1}\right)^T = -M^{-1}$, [12] we have

$$\vartheta = \vartheta' - \frac{1}{2}M^{-1}\chi \Longrightarrow \vartheta^T = \vartheta'^T - \frac{1}{2}\chi^T\left(M^{-1}\right)^T = \vartheta'^T + \frac{1}{2}\chi^T M^{-1}\,. \tag{3.216}$$

By means of Eq. (3.216) wecan write:

$$\begin{aligned}
I_2(M;\chi) &\equiv \int d\vartheta_1 d\vartheta_2\, e^{-\vartheta^T M\vartheta + \chi^T\vartheta} \\
&= \int d\vartheta'_1 d\vartheta'_2 \,\exp\left\{-\left(\vartheta'^T + \frac{1}{2}\chi^T M^{-1}\right) M\left(\vartheta' - \frac{1}{2}M^{-1}\chi\right) + \chi^T\left(\vartheta' - \frac{1}{2}M^{-1}\chi\right)\right\} \\
&= \int d\vartheta'_1 d\vartheta'_2 \,\exp\left\{-\left(\vartheta'^T + \frac{1}{2}\chi^T M^{-1}\right)\left(M\vartheta' - \frac{1}{2}\chi\right) + \chi^T\left(\vartheta' - \frac{1}{2}M^{-1}\chi\right)\right\} \\
&= \int d\vartheta'_1 d\vartheta'_2 \,\exp\left\{-\vartheta'^T M\vartheta' + \frac{1}{2}\vartheta'^T\chi - \frac{1}{2}\chi^T\vartheta' + \frac{1}{4}\chi^T M^{-1}\chi + \chi^T\vartheta' - \frac{1}{2}\chi^T M^{-1}\chi\right\} \\
&= \int d\vartheta'_1 d\vartheta'_2 \,\exp\left\{-\vartheta'^T M\vartheta' + \frac{1}{2}\vartheta'^T\chi + \frac{1}{2}\chi^T\vartheta' - \frac{1}{4}\chi^T M^{-1}\chi\right\} \\
&= \int d\vartheta'_1 d\vartheta'_2 \,\exp\left\{-\vartheta'^T M\vartheta' - \frac{1}{4}\chi^T M^{-1}\chi\right\} \\
&= e^{-\frac{1}{4}\chi^T M^{-1}\chi} I_2(M)\,,
\end{aligned} \tag{3.217}$$

where we have extracted the exponential from the integral because it involves an even number of exchanges of Grassman variables.

We now calculate the exponential factor of Eq. (3.217):

$$\chi^T M^{-1}\chi = \begin{pmatrix}\chi_1 & \chi_2\end{pmatrix}\begin{pmatrix}0 & -\frac{1}{m_{12}} \\ \frac{1}{m_{12}} & 0\end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = -\frac{2}{m_{12}}\chi_1\chi_2 \tag{3.218}$$

$$e^{-\frac{1}{4}\chi^T M^{-1}\chi} = 1 + \frac{1}{2m_{12}}\chi_1\chi_2 \tag{3.219}$$

$$I_2(M,\chi) = \left(1 + \frac{1}{2m_{12}}\chi_1\chi_2\right) 2m_{12} = 2\left(m_{12} + \frac{1}{2}\chi_1\chi_2\right)\,. \tag{3.220}$$

Eq. (3.220) coincides with Eq. (3.214). The general result for arbitrary (even) N is

$$I_N(M;\chi) = e^{-\frac{1}{4}\chi^T M^{-1}\chi} I_N(M)\,. \tag{3.221}$$

[12] For a generic 2x2 antisimmetric matrix we have

$$M = \begin{pmatrix}0 & m_{12} \\ -m_{12} & 0\end{pmatrix} \qquad M^{-1} = \begin{pmatrix}0 & -\frac{1}{m_{12}} \\ \frac{1}{m_{12}} & 0\end{pmatrix}\,.$$

Complex integration
It is also useful to consider an integral of the form

$$I_n = \int \prod_{i=1}^{n} d\vartheta_i d\tilde{\vartheta}_i \, e^{-\sum_{i,j=1}^{n} \vartheta_i A_{ij} \tilde{\vartheta}_j} , \tag{3.222}$$

where A is an arbitrary matrix and ϑ_i and $\tilde{\vartheta}_i$ are two sets of independent Grassman generators (for instance θ and its complex conjugate θ^*), for which the following anticommutation rules hold

$$\{\vartheta_i, \vartheta_j\} = \{\tilde{\vartheta}_i, \tilde{\vartheta}_j\} = \{\vartheta_i, \tilde{\vartheta}_j\} = 0 . \tag{3.223}$$

We can introduce the new variables

$$\tilde{\eta}_i = \sum_{j=1}^{n} A_{ij} \tilde{\eta}_j \quad i = 1, \dots, n . \tag{3.224}$$

The new variables satisfy

$$= \{\tilde{\eta}_i, \tilde{\eta}_j\} = \{\eta_i, \tilde{\vartheta}_j\} = 0 . \tag{3.225}$$

In terms of the variables $\tilde{\eta}_i$

$$e^{-\vartheta^T A \tilde{\vartheta}} = e^{-\sum_{i=1}^{n} \vartheta_i \tilde{\eta}_i} . \tag{3.226}$$

Since the terms $(\vartheta_i \tilde{\eta}_i)$ and $(\vartheta_j \tilde{\eta}_j)$ commute, we can write the exponential of the sum as the product of exponentials:

$$e^{-\sum_{i=1}^{n} \vartheta_i \tilde{\eta}_i} = \prod_{i=1}^{n} e^{\vartheta_i \tilde{\eta}_i} = \prod_{i=1}^{n} (1 - \vartheta_i \tilde{\eta}_i) . \tag{3.227}$$

Since the only term contributing to the integral is the one where <u>all</u> variables are present (remember $\int d\vartheta_i = 0$), we have

$$I_n = \int d\vartheta_1 \int d\tilde{\vartheta}_1 \dots \int d\vartheta_n \int d\tilde{\vartheta}_n (-1)^n (\vartheta_1 \tilde{\eta}_1) \dots (\vartheta_n \tilde{\eta}_n) . \tag{3.228}$$

From the definition of $\tilde{\eta}_i$ of Eq. (3.224), we have

$$\begin{aligned} \tilde{\eta}_1 \cdots \tilde{\eta}_n &= \sum_{\alpha,\beta,\dots,\nu} A_{1\alpha} A_{2\beta} \cdots A_{n\nu} \tilde{\vartheta}_\alpha \tilde{\vartheta}_\beta \cdots \tilde{\vartheta}_\nu \\ &= \tilde{\vartheta}_1 \tilde{\vartheta}_2 \cdots \tilde{\vartheta}_n \sum_{\alpha,\beta,\dots,\nu} \varepsilon_{\alpha\beta\dots\nu} A_{1\alpha} A_{2\beta} \cdots A_{n\nu} \\ &= \tilde{\vartheta}_1 \tilde{\vartheta}_2 \cdots \tilde{\vartheta}_n \det A . \end{aligned} \tag{3.229}$$

Considering Eq. (3.229) and the relation

$$(-1)^n \left(\vartheta_1 \tilde{\vartheta}_1\right) \cdots \left(\vartheta_n \tilde{\vartheta}_n\right) = \left(\vartheta_n \tilde{\vartheta}_n\right) \cdots \left(\vartheta_1 \tilde{\vartheta}_1\right) , \tag{3.230}$$

we can write Eq. (3.228) in the following way

$$I_n = \det A \int d\vartheta_1 \int d\tilde{\vartheta}_1 \dots \int d\vartheta_n \int d\tilde{\vartheta}_n (\vartheta_n \tilde{\eta}_n) \dots (\vartheta_1 \tilde{\eta}_1) = \det A . \tag{3.231}$$

So the general result is

$$I_n = \int \prod_{i=1}^{n} d\vartheta_i d\tilde{\vartheta}_i \, e^{-\sum_{i,j=1}^{n} \vartheta_i A_{ij} \tilde{\vartheta}_j} = \det A \,, \tag{3.232}$$

Another useful integral, which we leave as an exercise is

$$\int \prod_{i=1}^{n} d\vartheta_i d\tilde{\vartheta}_i \tilde{\vartheta}_k \vartheta_l e^{-\vartheta^T A \tilde{\vartheta}} = (\det A) \left(A^{-1} \right)_{kl} \,. \tag{3.233}$$

A slightly more general form of Eq. (3.232) can be obtain adding linear terms in the exponent, *i.e.* when we generalize Eq. (3.221) to complex Grassman variables:

$$\int d\eta^\dagger d\eta \, e^{i\eta^\dagger A \eta + i\zeta^\dagger \eta + i\zeta^T \eta^*} = [\det(-iA)] \, e^{-i\zeta^\dagger A^{-1} \zeta} \,, \tag{3.234}$$

where ζ is a vector of complex Grassman variables and A is a general complex matrix.

4

Interacting fields

In this chapter we will consider interacting fields, i.e. we will add local interactions to the Lagrangian. The equations of motion become non-linear and exact solutions are not possible. We will introduce the perturbative method, with reference to the simple model of a real scalar field with an interaction of the form $\lambda\varphi^4(x)$.

4.1 Perturbative evaluation of Green functions

Eq. (3.6) is at the root of the perturbative expansion of Green functions. Indeed, when the classical Action has the following form

$$S_E[\varphi] = S_{0E}[\varphi] + gS_{\text{int}}[\varphi] \,, \tag{4.1}$$

where S_0 is the Action for the free field and S_{int} is related to the interaction (we factored out the coupling constant g for the sake of simplicity),
from Eq. (3.2) we can write

$$\begin{aligned} Z_E[J] &= \mathcal{N} \int [d\varphi]\, e^{-S_{0E}[\varphi]} e^{-gS_{int}[\varphi]} e^{\int dx\, J(x)\varphi(x)} \\ &= \mathcal{N}' e^{-gS_{int}[\frac{\delta}{\delta J}]} \mathcal{N}_0 \int [d\varphi]\, e^{-S_{0E}[\varphi]} e^{\int dx\, J(x)\varphi(x)} \\ &= \mathcal{N}' e^{-gS_{int}[\frac{\delta}{\delta J}]} Z_{0E}[J] \,, \end{aligned} \tag{4.2}$$

where Z_{0E} is the functional for the free field (unperturbed functional), while the constant $\mathcal{N}'$ is defined by

$$\mathcal{N}'^{(-1)} = \left[e^{-gS_{int}\left[\frac{\delta}{\delta J}\right]} Z_{0E}[J] \right]_{J=0} , \tag{4.3}$$

in agreement with Eq. (3.3).
Once we know the unperturbed functional Z_{0E}, we can calculate the generating functional in the presence of interaction through a power series expansion of Eq. (4.2) over the coupling constant g. For record, we report here also the expression of the Z functional with Minkwoskian time:

$$Z_[J] = \mathcal{N} \int [d\varphi]\, e^{iS_0[\varphi] - gS_{int}[\varphi] + \int dx\, J(x)\varphi(x)} . \tag{4.4}$$

4.1.1 The Normalization of the Z functional (for scalar fields)

The functional $Z\,[J]$ is normalized according to Eq. (3.3), which we report here:

$$Z[0] = 1 .$$

Through such a normalization we can get rid of the factorized vacuum contributions from the Green functions, i.e., in diagrammatic language, the non-connected vacuum diagrams. The proof of this statement is done in perturbation theory, at the first perturbative order within the $\lambda\varphi^4$ model [1].
The Z functional is given by Eq. (4.2):

$$Z[J] = \frac{\exp\left\{-gS_{int}\left[\frac{\delta}{\delta J}\right]\right\} Z_0[J]}{\left[\exp\left\{-gS_{int}\left[\frac{\delta}{\delta J}\right]\right\} Z_0[J]\right]_{J=0}} , \tag{4.5}$$

where Z_0 is the unperturbed functional and for the $\lambda\varphi^4$ model

$$gS_{int}\left[\frac{\delta}{\delta J}\right] = \frac{\lambda}{4!} \int d^4x \left[\frac{\delta}{\delta J(x)}\right]^4 . \tag{4.6}$$

We now calculate the generating functional in the presence of interaction, as a power series at first order in λ. Eq. (4.5) for the $\lambda\varphi^4$ model (with euclidean time) is

$$Z_E\,[J] = \frac{e^{-\frac{\lambda}{4!}\left[\left(\int d^4x\, \frac{\delta}{\delta J(x)}\right)^4\right]} Z_{0E}}{\left\{ e^{-\frac{\lambda}{4!}\left[\int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4\right]} Z_{0E} \right\} \Big|_{J=0}} , \tag{4.7}$$

[1] We note that the generating functional of the interacting theory can be thought of as a power series expansion in currents as well as in the coupling constant. The first one corresponds to all possible ways of exciting the vacuum. Picking up a term with a fixed number of external currents means considering a Green function with a fixed number of external points.

with Z_{0E} given by Eq. (3.22), which we report here

$$Z_{0E}[J] = e^{\left[\frac{1}{2}\int d^4x \int d^4x' \, J(x)\Delta_E(x-x')J(x')\right]} . \tag{4.8}$$

Let us see explicitely the numerator $N(Z_E)$ of Eq. (4.7) with a first order (in λ) expansion:

$$N(Z_E) = \left(1 - \frac{\lambda}{4!}\left[\int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4\right]\right) Z_{0E}[J] . \tag{4.9}$$

We now evaluate the functional derivatives (without imposing $J = 0$), introducing the notation $e^{JJ} = e^{\left[\frac{1}{2}\int d^4x' \int d^4y' \, J(x')\Delta(x'-y')J(y')\right]}$:

$$\begin{aligned} \frac{\delta}{\delta J(x)} Z_{0E}[J] &= \frac{\delta}{\delta J(x)} e^{JJ} \\ &= \left(\int dy' \, \Delta(x-y')J(y')\right) e^{JJ} ; \end{aligned} \tag{4.10}$$

$$\begin{aligned} \left(\frac{\delta}{\delta J(x)}\right)^2 Z_{0E}[J] &= \Delta(x-x) e^{JJ} \\ &+ \left(\int dy' \, \Delta(x-y')J(y')\right)\left(\int dx' \, \Delta(x-x')J(x')\right) e^{JJ} ; \end{aligned} \tag{4.11}$$

$$\begin{aligned} \left(\frac{\delta}{\delta J(x)}\right)^3 Z_{0E}[J] &= \Delta(x-x)\left(\int dy' \, \Delta(x-y')J(y')\right) e^{JJ} \\ &+ 2\Delta(x-x)\left(\int dx' \, \Delta(x-x')J(x')\right) e^{JJ} \\ &+ \left(\int dy' \, \Delta(x-y')J(y')\right)\left(\int dx' \, \Delta(x-x')J(x')\right)\left(\int dx' \, \Delta(x-x')J(x')\right) e^{JJ} ; \end{aligned} \tag{4.12}$$

$$\begin{aligned} \left(\frac{\delta}{\delta J(x)}\right)^4 Z_{0E}[J] &= [\Delta(x-x)]^2 e^{JJ} \\ &+ \Delta(x-x)\left(\int dy' \, \Delta(x-y')J(y')\right)\left(\int dx' \, \Delta(x-x')J(x')\right) e^{JJ} \\ &+ 2\,[\Delta(x-x)]^2 e^{JJ} \\ &+ 2\Delta(x-x)\left(\int dx' \, \Delta(x-x')J(x')\right)\left(\int dy' \, \Delta(x-y')J(y')\right) e^{JJ} \\ &+ 3\Delta(x-x)\left(\int dy' \, \Delta(x-y')J(y')\right)\left(\int dx' \, \Delta(x-x')J(x')\right) e^{JJ} \\ &+ \left(\int dy' \, \Delta(x-y')J(y')\right)^4 e^{JJ} \end{aligned}$$

$$\begin{aligned} &= \left\{3\left[\Delta(x-x)\right]^2 + 6\Delta(x-x)\left[\int dy'\,\Delta(x-y')J(y')\right]^2\right. \\ &+ \left.\left[\int dy'\,\Delta(x-y')J(y')\right]^4\right\}e^{JJ}\,; \end{aligned} \tag{4.13}$$

By means of Eq. (4.13), Eq. (4.9) becomes

$$\begin{aligned} N\left(Z_E\right) &= \left\{1-\frac{\lambda}{4!}\int d^4x\left[3\left[\Delta(x-x)\right]^2 + 6\Delta(x-x)\left[\int dy'\,\Delta(x-y')J(y')\right]^2\right.\right. \\ &+ \left.\left.\left[\int dy'\,\Delta(x-y')J(y')\right]^4\right]\right\}e^{JJ}\,; \end{aligned} \tag{4.14}$$

The denominator of Eq. (4.7) is equal to Eq. (4.14) with $J = 0$:

$$D\left(Z_E\right) = 1 - \frac{\lambda}{4!}3\int d^4x\left[\Delta(x-x)\right]^2 . \tag{4.15}$$

Thus, the generating functional, at first order in λ, is

$$Z_E\left[J\right] = \frac{\left\{1-\frac{\lambda}{4!}\int d^4x\left[3\left[\Delta(x-x)\right]^2 + 6\Delta(x-x)\left[\int dy'\,\Delta(x-y')J(y')\right]^2 + \left[\int dy'\,\Delta(x-y')J(y')\right]^4\right]\right\}}{\left\{1-\frac{\lambda}{4!}3\int d^4x\left[\Delta(x-x)\right]^2\right\}} \tag{4.16}$$

Remark: the above expression satisfies the normalization $Z_E[0] = 1$.

Starting from Eq. (4.16), we calculate perturbatively, at first order in λ the two- and four-point Green functions:

$$\begin{aligned} G^{(2)}(y_1,y_2) &= \left\{\frac{\delta^2}{\delta J(y_1)\delta J(y_2)}Z_E\left[J\right]\right\}_{J=0} \\ &= \frac{\left\{-\frac{\lambda}{4!}6\times 2\int dx dy_1 dy_2\,\Delta(x-x)\Delta(x-y_1)\Delta(x-y_2)\right\}e^{JJ}}{\left\{1-\frac{\lambda}{4!}3\int d^4x\left[\Delta(x-x)\right]^2\right\}} \\ &+ \frac{\left\{1-\frac{\lambda}{4!}3\int d^4x\left[\Delta(x-x)\right]^2\right\}}{\left\{1-\frac{\lambda}{4!}3\int d^4x\left[\Delta(x-x)\right]^2\right\}}\Delta(y_1-y_2) \\ &= \Delta(y_1-y_2) - \frac{\lambda}{2}\int dx\,\Delta(x-x)\Delta(x-y_1)\Delta(x-y_2) + \mathcal{O}(\lambda^2)\,. \end{aligned} \tag{4.17}$$

Remark 1: the term $3\left[\Delta(x-x)\right]^2$ has been cancelled between numerator and denominator in Eq. (4.17). This is a consequence of the choice of the normalization of the functional Z.

Remark 2: we can represent the term $\Delta(y_1 - y_2)$ as a line connecting y_1 and y_2 (a particle is created at one of the two points, it propagates to the other one and get annihilated). Hence the term $\Delta(x - x)$ represents the creation of a quantum a point x and its absorption at x again, after an arbitrary closed path. Thi can be represented by a circle.
Remark 3: the perturbative expression of $G^{(2)}(y_1, y_2)$ of Eq. (4.17) can be represented as

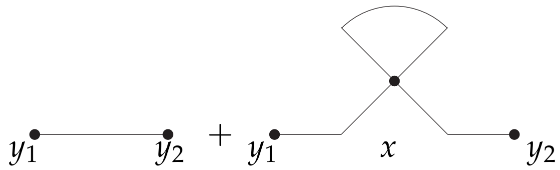

Figure 4.1: Pictorial representation with Feynman rules of the two point Green function at $\mathcal{O}(\lambda)$.

Remark 4: at the interaction vertex we associate the factor $-\frac{\lambda}{4!}\int dx$. The term of $\mathcal{O}(\lambda)$ is obtained by convoluting the factors. *The integration on x stands for the sum over all possibilities, i.e. it is the path integral;*
Remark 5: The coefficient of $\frac{\lambda}{4!}$ in Eq. (4.17), 12 is the *symmetry factor of the diagram*. It has to be calculated diagram by diagram. In the present case we have one vertex with four external points and two propagator, with two fixed external points y_1 and y_2. We can attach one end of one propagator to the vertex in four ways, while we can attach the remaining propagator in three different ways to the three free points of the vertex, amounting to a total of twelve possibilities.
Remark 6: The factor term $[\Delta(x - x)]^2$ is a vacuum diagram (there are no propagators connecting x to one of the external points). It can be represented as in Fig. (4.2).

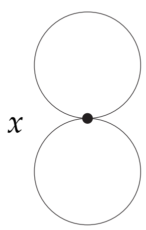

Figure 4.2: Pictorial representation of the vacuum diagram of $\mathcal{O}(\lambda)$.

For the four-point Green function $G^{(4)}(y_1, y_2, y_3, y_4)$ we have

$$
\begin{aligned}
G^{(4)}(y_1, y_2, y_3, y_4) &= \left\{ \frac{\delta^4}{\delta J(y_1)\delta J(y_2)\delta J(y_3)\delta J(y_4)} Z_E[J] \right\}_{J=0} \\
= \frac{1}{\left\{1 - \frac{\lambda}{4!} 3 \int d^4x \left[\Delta(x-x)\right]^2\right\}} &\left\{ -\frac{\lambda}{4!} 6 \times 2 \left[\int d^4x \Delta(x-y_1)\Delta(x-y_2)\right] \Delta(y_3 - y_4) \right. \\
&- \frac{\lambda}{4!} 6 \times 2 \left[\int d^4x \Delta(x-y_1)\Delta(x-y_3)\right] \Delta(y_2 - y_4) \\
&- \frac{\lambda}{4!} 6 \times 2 \left[\int d^4x \Delta(x-y_1)\Delta(x-y_4)\right] \Delta(y_2 - y_3) \\
&- \frac{\lambda}{4!} 6 \times 2 \left[\int d^4x \Delta(x-y_2)\Delta(x-y_3)\right] \Delta(y_1 - y_4) \\
&- \frac{\lambda}{4!} 6 \times 2 \left[\int d^4x \Delta(x-y_2)\Delta(x-y_4)\right] \Delta(y_1 - y_3) \\
&\left. - \frac{\lambda}{4!} 6 \times 2 \left[\int d^4x \Delta(x-y_3)\Delta(x-y_4)\right] \Delta(y_1 - y_2) \right\} \\
\frac{1}{\left\{1 - \frac{\lambda}{4!} 3 \int d^4x \left[\Delta(x-x)\right]^2\right\}} &\left\{ -\frac{\lambda}{4!} 4! \int d^4x \Delta(x-y_1)\Delta(x-y_2)\Delta(x-y_3)\Delta(x-y_4) \right\} \\
\frac{\left\{1 - \frac{\lambda}{4!} 3 \int d^4x \left[\Delta(x-x)\right]^2\right\}}{\left\{1 - \frac{\lambda}{4!} 3 \int d^4x \left[\Delta(x-x)\right]^2\right\}} &\left\{ G_0^{(4)}(y_1, y_2, y_3, y_4) \right\}.
\end{aligned}
\tag{4.18}
$$

Acording to the previously derived Feynman rules, we can give following pictorial representation of Eq. (4.18):

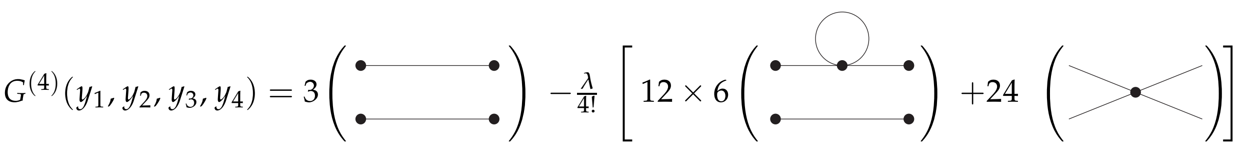

Figure 4.3: Graphical representation of the four-point Green function of Eq. (4.18).

4.1.2 The functional W $[J]$

We have seen that the Z generating functional contains the whole quantum description of a physical system; nevertheless it contains redundant information, namely, all disconnected graphs describing the independent propagation of particles, together with their interactions. In order to eliminate this redundancy, it is useful to introduce the generating functional W, which is defined as the logarithm of the Z functional:

$$W_E[J] \equiv \ln Z_E[J] \tag{4.19}$$

The corresponding equation with minkowskian metrics is

$$W_M[J] \equiv -i \ln Z_M[J] \tag{4.20}$$

In the following the suffix $_E$ is understood. We introduce some further abbreviations/definitions:

$$\begin{aligned} J_i &\equiv J(y_i) \\ \frac{\delta}{\delta J_i} &\equiv \frac{\delta}{\delta J(y_i)} \\ G^{(n)}_{1\ldots n} &\equiv G^{(n)}(y_1 \ldots y_n) = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J_1 \ldots \delta J_n} \\ W^{(n)}_{1\ldots n} &\equiv G^{(n)}_c(y_1 \ldots y_n) \equiv \frac{\delta^n W[J]}{\delta J_1 \ldots \delta J_n}\,. \end{aligned} \tag{4.21}$$

We analyze now the relations that we can obtain by iterated functional derivations of the functional W. The first derivative is

$$W^{(1)}_1 \equiv \frac{\delta W}{\delta J_1} = \frac{1}{Z}\frac{\delta Z}{\delta J_1} \equiv \varphi_c(y_1)\,, \tag{4.22}$$

which shows that $W^{(1)}_1$ is the "classical" field (in the presence of current J). In fact, from the definition of Eq. (4.22), we have:

$$\varphi_c(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\int [d\varphi]\, e^{\{-S_E[\varphi]-gS_{int}[\varphi]+\int dx\, J(x)\varphi(x)\}}\,\varphi(x)}{\int [d\varphi]\, e^{\{-S_E[\varphi]-gS_{int}[\varphi]+\int dx\, J(x)\varphi(x)\}}} = \langle 0|\hat{\varphi}(x)|0\rangle_J\,, \tag{4.23}$$

i.e. the classical field is the average of the field in presence of external current. We assume that for $J = 0$ we have $\varphi_c = 0$. In the contrary case we could have a symmetry braking (in the interaction model $\lambda\varphi^4$ the symmetry would be $\varphi \to -\varphi$).
If take two derivatives we have (*i.e.* we derive Eq. (4.22))

$$W^{(2)}_{12} \equiv \frac{\delta^2 W}{\delta J_1 \delta J_2} = \frac{1}{Z}\frac{\delta^2 Z}{\delta J_1 \delta J_2} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_1}\frac{\delta Z}{\delta J_2} = \frac{1}{Z}\frac{\delta^2 Z}{\delta J_1 \delta J_2} - \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\,. \tag{4.24}$$

Eq. (4.24) shows that the two-points function generated by the functional W is the Green function generated by the Z functional where we subtract the "factorized" contribution of two $W^{(1)}$. This feature is completely general: the functional W, w.r.t. G, does not generate the contributions which factorize in terms of lower degree W (derivative) functions. We will see that the degree is the numbers of external legs. In the perturbative expansion these factorized contributions correspond to disconnected Feynman diagrams. For this

reason the Green functions generated by W are denoted as **connected** Green functions. This is the reason of the subscript $_c$ in Eq. (4.21). Notice that in the above derivation we have not yet set the external current $J = 0$ after functional derivation.
Let us now take three derivatives (*i.e.* we derive Eq. (4.24)):

$$W^{(3)}_{123} \equiv \frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3} = \frac{1}{Z}\frac{\delta^3 Z}{\delta J_1 \delta J_2 \delta J_3} - \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J_1 \delta J_2}\frac{\delta Z}{\delta J_3} - \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta W}{\delta J_2} - \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta W}{\delta J_1}\,. \tag{4.25}$$

From Eq. (4.22) and Eq. (4.24) we have, respectively,

$$\begin{aligned} \frac{1}{Z}\frac{\delta Z}{\delta J_3} &= \frac{\delta W}{\delta J_3} \\ \frac{1}{Z}\frac{\delta^2 Z}{\delta J_1 \delta J_2} &= \frac{\delta^2 W}{\delta J_1 \delta J_2} + \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\,. \end{aligned} \tag{4.26}$$

Inserting the above expressions in Eq. (4.25) we have

$$\begin{aligned} W^{(3)}_{123} &= \frac{1}{Z}\frac{\delta^3 Z}{\delta J_1 \delta J_2 \delta J_3} - \frac{\delta W}{\delta J_3}\left(\frac{\delta^2 W}{\delta J_1 \delta J_2} + \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\right) - \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta W}{\delta J_2} - \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta W}{\delta J_1} \\ &= \frac{1}{Z}\frac{\delta^3 Z}{\delta J_1 \delta J_2 \delta J_3} - \left(\frac{\delta^2 W}{\delta J_1 \delta J_2}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta W}{\delta J_2} + \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta W}{\delta J_1}\right) - \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_3} \end{aligned} \tag{4.27}$$

Also for the $W^{(3)}_{123}$ we recognize the general feature of the subtraction from $Z^{(3)}_{123}$ of all possible lower order W factorized terms. Notice again that we have not yet set $J = 0$. (For the particular case of the $\lambda\varphi^4$ interaction we would have $W^{(3)}_{123} = 0$ after setting $J = 0$ because all Green functions of odd order are 0.
Now we make an additional functional derivation of Eq. (4.27), by deriving each term of the above equation:

$$\begin{aligned} W^{(4)}_{1234} &\equiv \frac{\delta^4 W}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \\ &= \left(\frac{1}{Z}\frac{\delta^4 Z}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} - \frac{1}{Z^2}\frac{\delta^3 Z}{\delta J_1 \delta J_2 \delta J_3}\frac{\delta Z}{\delta J_4}\right) \\ &- \left(\frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_1 \delta J_2}\frac{\delta^2 W}{\delta J_3 \delta J_4}\right) \\ &- \left(\frac{\delta^3 W}{\delta J_1 \delta J_3 \delta J_4}\frac{\delta W}{\delta J_2} + \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta^2 W}{\delta J_2 \delta J_4}\right) \\ &- \left(\frac{\delta^3 W}{\delta J_2 \delta J_3 \delta J_4}\frac{\delta W}{\delta J_1} + \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta^2 W}{\delta J_1 \delta J_4}\right) \\ &- \left(\frac{\delta^2 W}{\delta J_1 \delta J_4}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_2 \delta J_4}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_3 \delta J_4}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\right)\,. \end{aligned} \tag{4.28}$$

In Eq. (4.28) we can use Eq. (4.27) and Eq. (4.22) to eliminate the term containing $\delta^3 Z$ and at the same time we group together the additional terms containing equal order derivatives of W:

$$
\begin{aligned}
W^{(4)}_{1234} = & \frac{1}{Z}\frac{\delta^4 Z}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \\
& - \frac{\delta W}{\delta J_4}\left(\frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3} + \frac{\delta^2 W}{\delta J_1 \delta J_2}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta W}{\delta J_2} + \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta W}{\delta J_1} + \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_3}\right) \\
& - \left(\frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3}\frac{\delta W}{\delta J_3} + \frac{\delta^3 W}{\delta J_1 \delta J_3 \delta J_4}\frac{\delta W}{\delta J_2} + \frac{\delta^3 W}{\delta J_2 \delta J_3 \delta J_4}\frac{\delta W}{\delta J_1}\right) \\
& - \left(+\frac{\delta^2 W}{\delta J_1 \delta J_2}\frac{\delta^2 W}{\delta J_3 \delta J_4} + \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta^2 W}{\delta J_2 \delta J_4} + \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta^2 W}{\delta J_1 \delta J_4}\right) \\
& - \left(\frac{\delta^2 W}{\delta J_1 \delta J_4}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_2 \delta J_4}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_3 \delta J_4}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\right) .
\end{aligned}
\tag{4.29}
$$

Riorganizing terms, we have:

$$
\begin{aligned}
W^{(4)}_{1234} \equiv & \frac{\delta^4 W}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} = \frac{1}{Z}\frac{\delta^4 Z}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \\
& - \left\{\left(\frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3}\frac{\delta W}{\delta J_4} + \frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3}\frac{\delta W}{\delta J_3} + \frac{\delta^3 W}{\delta J_1 \delta J_3 \delta J_4}\frac{\delta W}{\delta J_2} + \frac{\delta^3 W}{\delta J_2 \delta J_3 \delta J_4}\frac{\delta W}{\delta J_1}\right)\right. \\
& + \left(\frac{\delta^2 W}{\delta J_1 \delta J_2}\frac{\delta^2 W}{\delta J_3 \delta J_4} + \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta^2 W}{\delta J_2 \delta J_4} + \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta^2 W}{\delta J_1 \delta J_4}\right) \\
& + \left(\frac{\delta^2 W}{\delta J_1 \delta J_2}\frac{\delta W}{\delta J_3}\frac{\delta W}{\delta J_4} + \frac{\delta^2 W}{\delta J_1 \delta J_3}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_4} + \frac{\delta^2 W}{\delta J_2 \delta J_3}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_4}\right. \\
& \quad \left. + \frac{\delta^2 W}{\delta J_1 \delta J_4}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_2 \delta J_4}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_3} + \frac{\delta^2 W}{\delta J_3 \delta J_4}\frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\right) \\
& \left. + \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\frac{\delta W}{\delta J_3}\frac{\delta W}{\delta J_4}\right\} .
\end{aligned}
\tag{4.30}
$$

Applying the above general results to the $\lambda\varphi^4$ model we have, **with external current**:

$$
\begin{aligned}
W^{(1)} &= G^{(1)} \\
W^{(2)}_{12} &= G^{(2)}_{12} - W^{(1)}_1 W^{(2)}_2 \\
W^{(3)}_{123} &= G^{(3)}_{123} - \left[W^{(2)}_{12}W^{(1)}_3 + W^{(2)}_{13}W^{(1)}_2 + W^{(2)}_{23}W^{(1)}_1 + W^{(1)}_1 + W^{(1)}_2 + W^{(1)}_3\right] \\
W^{(4)}_{1234} &= G^{(4)}_{1234} - \left[\left(W^{(3)}_{123}W^{(1)}_4 + W^{(3)}_{124}W^{(1)}_3 + W^{(3)}_{134}W^{(1)}_2 + W^{(3)}_{234}W^{(1)}_1\right)\right. \\
&+ \left(W^{(2)}_{12}W^{(2)}_{34} + W^{(2)}_{13}W^{(2)}_{24} + W^{(2)}_{14}W^{(2)}_{23}\right) \\
&+ \left(W^{(2)}_{12}W^{(1)}_3W^{(1)}_4 + W^{(2)}_{13}W^{(1)}_2W^{(1)}_4 + W^{(2)}_{14}W^{(1)}_2W^{(1)}_3\right. \\
&+ \left. W^{(2)}_{23}W^{(1)}_1W^{(1)}_4 + W^{(2)}_{24}W^{(1)}_1W^{(1)}_3 + W^{(2)}_{34}W^{(1)}_1W^{(1)}_2\right) \\
&+ \left. W^{(1)}_1W^{(1)}_2W^{(1)}_3W^{(1)}_4\right] .
\end{aligned}
\tag{4.31}
$$

With external current, in the $\lambda\varphi^4$ model we would have

$$
\begin{aligned}
W^{(1)} &= G^{(1)} = 0 \\
W^{(2)}_{12} &= G^{(2)}_{12} \\
W^{(3)}_{123} &= G^{(3)}_{123} = 0 \\
W^{(4)}_{1234} &= G^{(4)}_{1234} - \left[W^{(2)}_{12} W^{(2)}_{34} + W^{(2)}_{13} W^{(2)}_{24} + W^{(2)}_{14} W^{(2)}_{23} \right] .
\end{aligned}
\tag{4.32}
$$

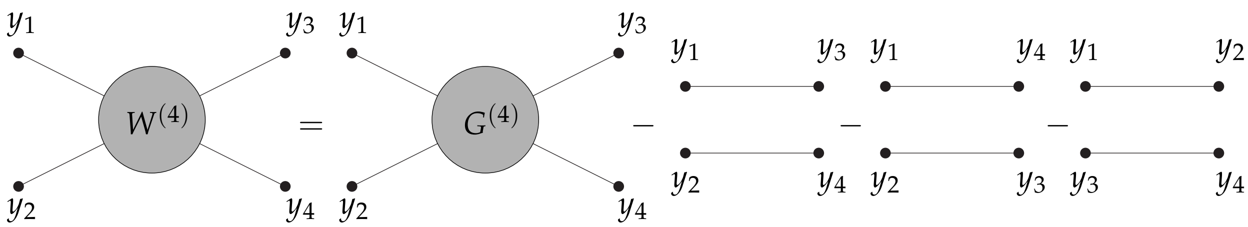

Figure 4.4: Graphical representation of the four-point Green function of Eq. (4.18) in terms of connected and disconnected terms.

We note that all the obtained results do not rely upon perturbation theory.
In statistical mechanics the analogous of the Z functional is the partition function, while the analogous of the W functional is the Helmoltz free energy.

4.1.3 The Effective Action Γ

The analogous in statistical mechanics is the Gibbs free energy. The Effective Action Γ is a functional which generates only Green functions which, **in momentum representation, factorize in terms of Green function of lower degree**. The functional Γ is related to the functional W through a Legendre transformation, in an analogous way as the Lagrangian is related to the Hamiltonian in mechanics. If we start with $L(q, \dot{q})$, we define the momentum

$$p = \frac{\partial L}{\partial \dot{q}} . \tag{4.33}$$

The Hamiltonian is defined as

$$H(q, p) = p\dot{q} - L(q, \dot{q}) , \tag{4.34}$$

which does not depend on $\dot{q}$. In fact, by differentiation, we get

$$dH = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} = \dot{q}dp - \frac{\partial L}{\partial q}dq . \tag{4.35}$$

From the above equation follows also

$$\frac{\partial H}{\partial p} = \dot{q} \,. \tag{4.36}$$

In an analogous way, the Effective Action Γ is defined as a functional Legendre transformation of the functional W:

$$\Gamma\left[\varphi_c\right] = W\left[J\right] - \int dx\, J(x)\varphi_c(x) \tag{4.37}$$

where, from Eq. (4.22)

$$\frac{\delta W}{\delta J(x)} =_J \left\langle 0|\varphi(x)|0\right\rangle_J = \varphi_c(x) \,. \tag{4.38}$$

The x integration in Eq. (4.37) is a four-dimensional integration. If we take the functional differential of Eq. (4.37), we get

$$\delta\Gamma = \int dx\, \frac{\delta W}{\delta J(x)} - \int dx\, \varphi_c(x)\delta J(x) - \int dx\, J(x)\delta\varphi_c(x) = -\int dx\, J(x)\delta\varphi_c(x) \,. \tag{4.39}$$

Eq. (4.39) tells us that Γ is a functional only of φ_c and that

$$\frac{\delta\Gamma}{\delta\varphi_c(x)} = -J(x) \,. \tag{4.40}$$

Eq. (4.40), for $J = 0$, is the analogous of the classical equation for the classical field φ_c

$$\frac{\delta S}{\delta\varphi_c(x)} = 0 \,. \tag{4.41}$$

Actually, also at the classical level, the field is described by the underlying quantum theory through Eq. (4.40).

By taking the functional derivative of both sides of Eq. (4.37) we get

$$\frac{\delta\varphi_c(x_1)}{\delta J(x_2)} = \frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} \,. \tag{4.42}$$

On the other hand, taking the functional derivative of both sides of Eq. (4.40) we get

$$\frac{\delta J(x_1)}{\delta\varphi_c(x_2)} = -\frac{\delta^2\Gamma}{\delta\varphi_c(x_1)\delta\varphi_c(x_2)} \,. \tag{4.43}$$

From the rule of derivation of function of function we have also

$$\int dy\, \frac{\delta\varphi_c(x_1)}{\delta J(y)}\frac{\delta J(y)}{\delta\varphi_c(x_2)} = \frac{\delta\varphi_c(x_1)}{\varphi_c(x_2)} = \delta(x_1 - x_2) \,. \tag{4.44}$$

Through Eq. (4.42) and Eq. (4.43), Eq. (4.44) becomes

$$\int dy \frac{\delta^2 W}{\delta J(x_1)\delta J(y)} \frac{\delta^2 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(x_2)} . = -\delta(x_1 - x_2) \tag{4.45}$$

Eq. (4.45) allows to understand clearly the meaning of

$$\Gamma^{(2)}(x_1, x_2) \equiv \frac{\delta^2 \Gamma}{\delta\varphi_c(x_1)\delta\varphi_c(x_2)} . \tag{4.46}$$

$\Gamma^{(2)}$ is minus the inverse of the propagator $\Gamma^{(2)}$.

From Eq. (4.42) follows another important relation:

$$\frac{\delta}{\delta J(x)} = \int dy \frac{\delta\varphi_c(y)}{\delta J(x)} \frac{\delta}{\delta\varphi_c(y)} = \int dy \frac{\delta^2 W}{\delta J(x)\delta J(y)} \frac{\delta}{\delta\varphi_c(y)} . \tag{4.47}$$

Let us now take the derivative w.r.t. $J(z)$ of Eq. (4.45) (using the results of Eq. (4.47) for the functional derivation of Γ):

$$\begin{aligned} &\int dy \frac{\delta^3 W}{\delta J(z)\delta J(x_1)\delta J(y)} \frac{\delta^2 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(x_2)} \\ &+ \int dy \int dt \frac{\delta^2 W}{\delta J(x_1)\delta J(y)} \frac{\delta^2 W}{\delta J(z)\delta J(t)} \frac{\delta^3 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(t)\delta\varphi_c(x_2)} = 0 . \end{aligned} \tag{4.48}$$

Multiplying now both sides of Eq. (4.48) by $\frac{\delta^2 W}{\delta J(x_2)\delta J(u)}$ and integrating over x_2, we get

$$\begin{aligned} &\int dx_2 \int dy \frac{\delta^3 W}{\delta J(z)\delta J(x_1)\delta J(y)} \frac{\delta^2 W}{\delta J(x_2)\delta J(u)} \frac{\delta^2 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(x_2)} \\ &= -\int dx_2 \int dy \int dt \frac{\delta^2 W}{\delta J(x_2)\delta J(u)} \frac{\delta^2 W}{\delta J(x_1)\delta J(y)} \frac{\delta^2 W}{\delta J(z)\delta J(t)} \frac{\delta^3 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(t)\delta\varphi_c(x_2)} . \end{aligned} \tag{4.49}$$

Using the fact that

$$\int dx_2 \frac{\delta^2 W}{\delta J(x_2)\delta J(u)} \frac{\delta^2 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(x_2)} = -\delta(u - y) ,$$

we obtain

$$\begin{aligned} &\frac{\delta^3 W}{\delta J(x_1)\delta J(z)\delta J(u)} \\ &= \int dy_2 \int dy \int dt \frac{\delta^2 W}{\delta J(y_2)\delta J(u)} \frac{\delta^2 W}{\delta J(z)\delta J(t)} \frac{\delta^2 W}{\delta J(x_1)\delta J(y)} \frac{\delta^3 \Gamma}{\delta\varphi_c(y)\delta\varphi_c(t)\delta\varphi_c(y_2)} . \end{aligned} \tag{4.50}$$

where we have made the substitution $x_2 \to y_2$. By means of the further substitutions $y \to y_1$, $t \to y_3$, $z \to x_3$ and $u \to x_2$ we obtain

$$\frac{\delta^3 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = \int dy_1 \int dy_2 \int dy_3 \frac{\delta^2 W}{\delta J(x_1)\delta J(y_1)} \frac{\delta^2 W}{\delta J(x_2)\delta J(y_2)} \frac{\delta^2 W}{\delta J(x_3)\delta J(y_3)} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} . \tag{4.51}$$

Eq. (4.51) can be given a pictorial representation, which shows that $\Gamma^{(3)}$ is equal to $W^{(3)}$ with amputated propagators on the external legs. For this reason $\Gamma^{(n)}$ with $n > 2$ is also named **proper vertex**.

Now let us take a further functional derivative, w.r.t. $J(x_4)$, of Eq. (4.51). By means of Eq. (4.47) we find:

$$\begin{aligned}
&\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \\
&= \int dy_1 \int dy_2 \int dy_3 \int dy_4 \frac{\delta^2 W}{\delta J(x_1)\delta J_1} \frac{\delta^2 W}{\delta J(x_2)\delta J_2} \frac{\delta^2 W}{\delta J(x_3)\delta J_3} \frac{\delta^2 W}{\delta J(x_4)\delta J_4} \frac{\delta^4 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)\delta\varphi_c(y_4)} \\
&+ \int dy_1 \int dy_2 \int dy_3 \frac{\delta^3 W}{\delta J(x_4)\delta J(x_1)\delta J_1} \frac{\delta^2 W}{\delta J(x_2)\delta J_2} \frac{\delta^2 W}{\delta J(x_3)\delta J_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} \\
&+ \int dy_1 \int dy_2 \int dy_3 \frac{\delta^2 W}{\delta J(x_1)\delta J_1} \frac{\delta^3 W}{\delta J(x_4)\delta J(x_2)\delta J_2} \frac{\delta^2 W}{\delta J(x_3)\delta J_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} \\
&+ \int dy_1 \int dy_2 \int dy_3 \frac{\delta^2 W}{\delta J(x_1)\delta J(y_1)} \frac{\delta^2 W}{\delta J(x_2)\delta J(y_2)} \frac{\delta^3 W}{\delta J(x_4)\delta J(x_3)\delta J(y_3)} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} .
\end{aligned} \tag{4.52}$$

Using Eq. (4.51) with $y_{123} \to z_{123}$ and the notation $W^2_{x_\alpha y_\beta} = \frac{\delta^2 W}{\delta J(x_\alpha)\delta J(x_\beta)}$, $\int dy_1 \int y_2 \int dy_3 \to \int dy_{1,2,3}$ we get

$$\begin{aligned}
&\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \\
&= \int dy_{1,2,3,4} \frac{\delta^2 W}{\delta J(x_1)\delta J_1} \frac{\delta^2 W}{\delta J(x_2)\delta J_2} \frac{\delta^2 W}{\delta J(x_3)\delta J_3} \frac{\delta^2 W}{\delta J(x_4)\delta J_4} \frac{\delta^4 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)\delta\varphi_c(y_4)} \\
&+ \int dy_{1,2,3}\, dz_{1,2,3}\, W^2_{x_4 z_1} W^2_{x_1 z_2} W^2_{y_1 z_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(z_1)\delta\varphi_c(z_2)\delta\varphi_c(z_3)} W^2_{x_2 y_2} W^2_{x_3 y_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} \\
&+ \int dy_{1,2,3}\, dz_{1,2,3}\, W^2_{x_1 y_1} W^2_{x_4 z_1} W^2_{x_2 z_2} W^2_{y_2 z_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(z_1)\delta\varphi_c(z_2)\delta\varphi_c(z_3)} W^2_{x_3 y_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} \\
&+ \int dy_{1,2,3}\, dz_{1,2,3} W^2_{x_1 y_1} W^2_{x_2 y_2} W^2_{x_4 z_1} W^2_{x_3 z_2} W^2_{y_3 z_3} \frac{\delta^3 \Gamma}{\delta\varphi_c(z_1)\delta\varphi_c(z_2)\delta\varphi_c(z_3)} \frac{\delta^3 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)} .
\end{aligned} \tag{4.53}$$

By renaming the integration variables in the following way

$$
\begin{aligned}
&y_1 \to y\,, z_2 \to y_1\,, z_1 \to y_4\,, z_3 \to z \quad \text{in the second line,}\\
&z_1 \to y_4\,, z_2 \to y_2\,, y_2 \to y\,, z_3 \to z \quad \text{in the third line,}\\
&y_3 \to y\,, z_1 \to y_4\,, z_2 \to y_3\,, z_3 \to z \quad \text{in the last line,}
\end{aligned}
$$

we get

$$
\begin{aligned}
&\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\\
&= \int dy_{1,2,3,4} \frac{\delta^2 W}{\delta J(x_1)\delta J_1}\frac{\delta^2 W}{\delta J(x_2)\delta J_2}\frac{\delta^2 W}{\delta J(x_3)\delta J_3}\frac{\delta^2 W}{\delta J(x_4)\delta J_4}\frac{\delta^4 \Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y_3)\delta\varphi_c(y_4)}\\
&+ \int dy_{1,2,3,4}\, dy\, dz\, W^2_{x_4y_4} W^2_{x_1y_1} W^2_{yz} \frac{\delta^3\Gamma}{\delta\varphi_c(y_4)\delta\varphi_c(y_1)\delta\varphi_c(z)} W^2_{x_2y_2} W^2_{x_3y_3} \frac{\delta^3\Gamma}{\delta\varphi_c(y)\delta\varphi_c(y_2)\delta\varphi_c(y_3)}\\
&+ \int dy_{1,2,3,4}\, dy\, dz\, W^2_{x_1y_1} W^2_{x_4y_4} W^2_{x_2y_2} W^2_{yz} \frac{\delta^3\Gamma}{\delta\varphi_c(y_4)\delta\varphi_c(y_2)\delta\varphi_c(z)} W^2_{x_3y_3} \frac{\delta^3\Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y)\delta\varphi_c(y_3)}\\
&+ \int dy_{1,2,3,4}\, dy\, dz\, W^2_{x_1y_1} W^2_{x_2y_2} W^2_{x_4y_4} W^2_{x_3y_3} W^2_{yz} \frac{\delta^3\Gamma}{\delta\varphi_c(y_4)\delta\varphi_c(y_3)\delta\varphi_c(z)} \frac{\delta^3\Gamma}{\delta\varphi_c(y_1)\delta\varphi_c(y_2)\delta\varphi_c(y)}\,.
\end{aligned}
\tag{4.54}
$$

The first term of Eq. (4.54) is a proper vertex with four external legs with amputated external propagators. The other three contributions contain, each, two proper vertices with three external legs, four of which are amputated of the external propagators, while the other two are connected trhough a propagator. The external points occur in three possible distinguishable ways. In momentum representation, the momentum of the internal propagator is fixed through four momentum conservation by the external momenta and each contribution can be written in factorized form. Each one of the three contributions can be cut in two separate parts by cutting only one propagator. For this reason these contributions are named **one-particle-reducible**. On the contrary, the **Effective Action generates the (excluding Γ_2) one-particle-irreducible Green functions**, which are amputated of the external leg propagators.
There is a strong connection between the scattering amplitudes and connected Green functions (generated by the functional W amputated of the propagators on the external legs, calculated in momentum representation. (It is understood the inverse Wick rotation to go back to the Minkowski metrics). We show this connection in the following sections.

Remark: it is interesting to calculate the Effective Action for the free scalar theory. In this case we have

$$
\varphi_c(x) = \frac{\delta W_E[J]}{\delta J(x)} = \int d^4y \triangle_E(x-y) J(y)\,, \tag{4.55}
$$

where we have used the expression of Eq. (3.22) for the Z functional of the free scalar theory inside Eq. (4.19). Note that the expression of Eq. (4.55) is the same as $\bar{\varphi}(x)$ of

Eq. (3.15), used to solve the functional integral for the free theory. We can now apply $(\Box_E^x - m^2)$ to both members of Eq. (4.55) to obtain

$$\begin{aligned}\left(\Box_E^x - m^2\right)\varphi_c(x) &= \int d^4y \left(\Box_E^x - m^2\right)\triangle_E(x-y)J(y) \\ &= \int d^4y \left(\Box_E^{(x-y)} - m^2\right)\triangle_E(x-y)J(y) \\ &= -\int d^4y\, \delta^4(x-y)J(y) = -J(x)\,. \end{aligned} \tag{4.56}$$

From the definition of $\Gamma[\varphi_c(x)]$ of Eq. (4.37), we obtain

$$\begin{aligned}\Gamma_E[\varphi_c] &= \frac{1}{2}\int d^4x d^4y\, J(x)\triangle_E(x-y)J(y) - \int d^4x d^4y\, J(x)\triangle_E(x-y)J(y) \\ &= -\frac{1}{2}\int d^4x d^4y\, J(x)\triangle_E(x-y)J(y)\,. \end{aligned} \tag{4.57}$$

Inserting the expression of the current as a function of the classical field of Eq. (4.56) in Eq. (4.57), we get

$$\begin{aligned}\Gamma_E[\varphi_c] &= -\frac{1}{2}\int d^4x d^4y \left(\Box_E^x - m^2\right)\varphi_c(x)\triangle_E(x-y)\left(\Box_E^y - m^2\right)\varphi_c(y) \\ &= \frac{1}{2}\int d^4x\, \varphi_c(x)\left(\Box_E - m^2\right)\varphi_c(x) \\ &= -S_{0E}[\varphi_c]\,. \end{aligned} \tag{4.58}$$

Eq. (4.58) states that the generator functional of the 1PI Green functions of the free theory is the classical Action. This is the origin of the name "Effective Action" for the functional $\Gamma[\varphi_c]$.

4.2 The S matrix and its relation with Green functions

The main goal of our QFT development is to calculate cross sections and decay rates for processes involving elementary particles, by means of S matrix elements. We can think of a scattering process of two particles in the following way: at the initial time, say $-T/2$ the particles are prepared in a state corresponding to two wave-packets which are localized in space regions very far away from each other. We can consider them as non interacting particles. Then, for a finite time interval, say T, the particles interact (without external influence) producing the final state particles, which are detected at a time $+T/2$ by the experimental apparatus. At this time the final state particles are again very far away from each other and can be considered as non interacting. The time interval T is much longer than the typical time scale of the interaction. In fact the order of magnitude of T is dictated by the linear dimensions of the experimental detector, say of the order of one meter. Indeed T is given by the time necessary to light to pass through the detector: $T \sim 10^{-9}$ - 10^{-8} sec. The typical length over which the interaction between particles takes place is, instead, of the order of one Fermi (1 fm = 10^{-15} m), corresponding to times of the order of 10^{-23} sec. Hence, physically, we can consider the time interval $[-T/2, T/2]$ as extending from $-T/2 = -\infty$ to $+T/2 = +\infty$. The scattering amplitudes, which we can compute perturbatively through Quantum Field Theory methods, are described by the matrix elements of the S matrix operator (where S stands for Scattering). Since the asymptotic states appearing in ths S matrix are those of free, on-shell particles, we describe them as non-interacting quantum excitations of the vacuum of the theory with free dispersion relations. Up to now we have seen how to calculate Green functions through functional integral methods. In the following sections we illustrate the link between Green functions and S matrix elements. The calculation is illustrated in some detail for the real scalar field, for the sake of simplicity.

4.2.1 "in" and "out" states

We assume that at time $-T/2$ the initial state is given by separated non-interacting particles. We can describe the initial state by a superposition of plane waves, with given momentum of each particle

$$|p_i, \alpha_i\rangle\,, \quad \left(\text{with}\, t = -\frac{T}{2}\right)\,, \tag{4.59}$$

where α_i charcterize other possible fixed quantum numbers of the initial state. In the Schrödinger picture the state of Eq. (4.59) evolves with time (i.e. it describes a trajectory in the space of states) according to the dynamics given by the complete Hamiltonian of the system. The same trajectory is described, within the Heisenberg picture, by a state which does not evolve with time (remember that in the Heisenberg picture all the time evolution resides in the operators), characterized by the momenta p_i at time $-T/2$. Even if the state does not evolve with time, the system which it describes does evolve with time. We define an **in-state** this state in the limit of $-T \to -\infty$

$$|p_i, \alpha_i; \text{in}\rangle = \lim_{-T/2\to-\infty} |p_i, \alpha_i\rangle_H\,. \tag{4.60}$$

The set of all possible states of this kind, *i.e.* the set of all Heisenberg states, with an arbitrary number of particles is the basis "in", which is an ortonormal set. Among these states there is also the state with no particles, the vacuum state $|0\rangle$. The "in" state with n particles is obtained by acting n times the creation operator contained in the free field on the vacuum state.

The *asymptotic hypothesis* assumes that the set of "in" states is also complete, *i.e.* every state of the system can be reached starting from particles very far away in the far past [2]. Thus we write the completeness relation of the "in" states as

$$\sum_i |i, \text{in}\rangle\langle i, \text{in}| = 1\,. \tag{4.61}$$

[2]This hypothesis is at the root of the study of microscopic systems through scattering experiments.

Together with the "in" state basis, we can introduce also the "out" state basis: the set of states which, describing the motion of the system in the Heisenberg description, become, at $T/2 \to +\infty$, states with free particles, far away from each other and with definite momentum. The asymptotic hypothesis for "out" states works in the same way as for "in" states. While for a free theory, the "in" and "out" state coincide up to a phase factor, they are different in the presence of interaction, with two exceptions: 1) the vacuum state, $|0, \text{“in”}\rangle = |0, \text{“out”}\rangle = |0\rangle$ (which is, however, different from the vacuum state of the interacting theory); 2) the states with only one particle, for which the momentum and the spin component along the direction of motion are conserved: $|p, \alpha, , \text{“in”}\rangle = |p, \alpha, \text{“out”}\rangle = |p, \alpha\rangle$.

Remarks on the completeness of the "in" and "out" states

We present here some remarks on the asymptotic hypothesis. Let us consider the case of a non-relativistic particle moving within a given potential, with discrete spectrum for $E < 0$ and continuum spectrum for $E > 0$ (for instance an electron within the potential of a proton, considered as a static source). We can build normalized wave packets by means of a superposition of eigenstates with $E > 0$. For these states the motion happens mostly at infinity, because the time average of the probability of finding the particle in an arbitrary confined region is zero:

$$|\psi(\vec{x}, t)|^2 = \int \int dEdE' \, c(E)^* c(E') \psi_E(\vec{x})^* \psi_E(x) e^{i(E-E')t} \tag{4.62}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \, e^{i(E-E')t} = \frac{2\pi}{T} \delta(E - E') \, . \tag{4.63}$$

From the above equations we have

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_V dx \, |\psi(\vec{x}, t)|^2 = \frac{2\pi}{T} \int dE \, |c(E)|^2 \int_V dx \, |\psi_E(\vec{x})|^2 \to 0 \, . \tag{4.64}$$

Hence an arbitrary state with $E > 0$, for times sufficiently far away in the past, can be represented as a superposition of free states, *i.e.* it can be reached by the "in" states. On the contrary, if we consider a wave packet build up as a superposition of eigenstates of the discrete spectrum, we have

$$|\psi(\vec{x}, t)|^2 = \sum_{n,n'} c_n(E)^* c_{n'}(E') \psi_n(\vec{x})^* \psi_{n'}(x) e^{i(E_n - E_{n'})t} \tag{4.65}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \, e^{i(E_n - E_{n'})t} = \frac{2\pi}{T} \delta_{n,n'} \, , \tag{4.66}$$

and Eq. (4.64) gets replaced by

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_V dx\, |\psi(\vec{x},t)|^2 = \frac{2\pi}{T}\sum_n |c_n(E)|^2 \int_V dx\, |\psi_n(\vec{x})|^2 \neq 0\,. \tag{4.67}$$

The particle stays always in the region where it is bound by the potential. Therefore these bound states can not be reached by the "in" states. In a theory which is invariant under translations, where the proton can move, the spectrum of the total energy is always continuus and all the localized states can go to infinity. The "in" states with one electron and one proton very far from each other in the far past are not a complete basis because electron and proton can skip to infinity remaining nonetheless bound together. We can recover the completeness of the "in" states if we include also the states that contain, at $-T/2 \to -\infty$ the bound states (for instance the hydrogen atom in the ground state). This means that, in order to determine the physical properties of the electron-proton system, we should also study scattering experiments involving also the hydrogen atom in the "in" states, such as, for instance $eH \to eeP$.

4.2.2 The S-matrix

The S-matrix is defined as the probability amplitude for a process that leads from an "in" state $|\alpha, \text{in}\rangle$ to an "out" state $|\alpha, \text{out}\rangle$, *i.e.*

$$S_{\beta\alpha} = \langle \beta, \text{out}|\alpha, \text{in}\rangle\,. \tag{4.68}$$

We can introduce a unitary operator that transforms "in" states (bras) into "out" states (bras)

$$\langle \beta, \text{out}| = \langle \beta, \text{in}|\hat{S}\,, \tag{4.69}$$

so that we have [3]

$$S_{\beta\alpha} = \langle \beta, \text{in}|\hat{S}|\alpha, \text{in}\rangle\,. \tag{4.70}$$

The unitarity can be shown by considering that Eq. (4.69) and its hermitean conjugate imply

$$\langle \beta, \text{in}|\hat{S}\hat{S}^\dagger|\alpha, \text{in}\rangle = \langle \beta, \text{out}|\alpha, \text{out}\rangle = \delta_{\alpha,\beta} \Longrightarrow \hat{S}\hat{S}^\dagger = \mathbb{1}\,. \tag{4.71}$$

The unitarity of $\hat{S}$ expresses the conservation of probability. In fact, from Eq. (4.71) we can write

$$1 = \langle \alpha, \text{in}|\hat{S}\hat{S}^\dagger|\alpha, \text{in}\rangle = \sum_\beta |\langle \beta, \text{in}|\hat{S}^\dagger|\alpha, \text{in}\rangle|^2 = \sum_\beta P_{\alpha\to\beta}\,. \tag{4.72}$$

Thus the set of "out" states β coincides with all possible outcomes of scattering experiments, as expected if the set of states is a complete one. Eq. (4.70) is independent of the representation and it makes contact with the definition of the $\hat{S}$ matrix operator in the

[3]In Eq. (4.70) the "bra" and "ket" states refer to the same time, so they do not refer to a particular representation.

interaction picture (already illustrate during the QED course).

If we have a conserved physical quantity $\hat{Q}$ (*i.e.* $[\hat{Q}, \hat{H}] = 0$), we can choose the "in" states, as well as the "out" states, such that they are eigenstates of both $\hat{Q}$ and $\hat{H}$. In particular, if an "in" state corresponds to an eigenvalue q of Q, it has to be mapped by $\hat{S}$ into an "out" state corresponding to the same eigenvalue. Hence $\langle \beta, q', \mathrm{out} | \alpha, q, \mathrm{in}\rangle = 0$ if $q \neq q'$. In particular, for systems with translational invariance, the $\hat{S}$ matrix must be diagonal in the basis of the states with given four-momentum. The $\hat{S}$ matrix elements can be written as

$$S_{\beta,\alpha} = (2\pi)^4 \delta^4 \left(\sum P_\beta - \sum P_\alpha\right) \mathcal{M}_{\beta,\alpha}\,, \tag{4.73}$$

where P_β and P_α are the total final and initial state four-momentum, respectively and the factor $(2\pi)^4$ is factored out by convention.

4.2.3 The optical theorem

From the unitarity f the $\hat{S}$ matrix, through the definition

$$S = \mathbb{1} + iT\,, \tag{4.74}$$

we can derive the following relation for the T matrix:

$$-TT^\dagger = i\left(T - T^\dagger\right)\,. \tag{4.75}$$

We can easily prove Eq. (4.75):

$$\mathbb{1} = SS^\dagger = (\mathbb{1} + iT)\left(\mathbb{1} - iT^\dagger\right) = \mathbb{1} + iT - iT^\dagger + TT^\dagger\,.$$

Let us consider now the matrix elements of the operators of Eq. (4.75) between the following two-particle states $|\bar{p}_1\bar{p}_2\rangle$ and $|\bar{k}_1\bar{k}_2\rangle$ and insert the completeness relation as follows:

$$\langle \bar{p}_1\bar{p}_2|TT^\dagger|\bar{k}_1\bar{k}_2\rangle = \sum_n \left(\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3 2E_i}\right) \langle \bar{p}_1\bar{p}_2|TT^\dagger|\{\bar{q}_i\}\rangle\langle\{\bar{q}_i\}|TT^\dagger|\bar{k}_1\bar{k}_2\rangle\,. \tag{4.76}$$

Using Eq. (4.75) in Eq. (4.76), we can write

$$i\left[\langle \bar{p}_1\bar{p}_2|T|\bar{k}_1\bar{k}_2\rangle - \langle \bar{k}_1\bar{k}_2|T^\dagger|\bar{p}_1\bar{p}_2\rangle\right] = \sum_n \left(\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3 2E_i}\right) \langle \bar{p}_1\bar{p}_2|TT^\dagger|\{\bar{q}_i\}\rangle\langle\{\bar{q}_i\}|TT^\dagger|\bar{k}_1\bar{k}_2\rangle\,. \tag{4.77}$$

All the matrix elements of the above equations are scattering matrix elements between physical states. If we condiser the case $|\bar{p}_1\bar{p}_2\rangle = |\bar{k}_1\bar{k}_2\rangle$, *i.e.* the case of forward scattering (the initial and final states are not modified), we can write, denoting with $\mathcal{M}$ the scattering matrix elements:

$$-Im\mathcal{M}(k_1, k_2 \to k_1, k_2) \sim \sum_n \left(\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3 2E_i}\right) |\mathcal{M}(k_1, k_2 \to f)|^2\,. \tag{4.78}$$

The term on the right-hand side is a sum of squared scattering amplitudes of two particles with momenta k_1 and k_2 into a final with n particles. From the relations which will be derived in the following subsections, we can interpret the term on the right-hand side as proportional to the total cross section for the scattering of k_1 and k_2. The above relation is called optical theorem.

4.2.4 The asymptotic fields

We have previously defined the "in" and "out" states in the Heisenberg picture, where the dynamical variables, *i.e.* the fields, are time-dependent. While in the free theory the application of the field to the vacuum state create a single particle state for any time, this is not true anymore in the presence of interaction. However we have assumed that for $T \to -\infty$ and $T \to +\infty$ the physical system under study consists of non-interacting on-shell particles. Hence, in this limits, the interacting field should converge (in a sense to be precised below) to a free field. This requirement is called *asymptotic condition*, according to which the *matrix elements of the field*, in the limit $T \to -\infty$, are proportional to the respective matrix elements of a free field, called $\varphi_{\text{in}}(x)$. The proportionality constant is fixed by the requirement that $\varphi_{\text{in}}(x)$ is normalized as a canonical field:

$$\begin{aligned}
\lim_{T\to-\infty} \langle \beta, \text{in} | \varphi(x) | \alpha, \text{in} \rangle &= Z^{\frac{1}{2}} \langle \beta, \text{in} | \varphi_{\text{in}}(x) | \alpha, \text{in} \rangle \,, && (4.79)\\
\left(\Box + m^2 \right) \varphi_{\text{in}}(x) &= 0 \,. && (4.80)
\end{aligned}$$

Applying the field $\varphi_{\text{in}}(x)$ to the vacuum, we can generate all the states with many on-shell, non-interacting particles, *i.e.* we generate the basis of the "in" states. The convergence of the field to the asymptotic "in" field can only be weak (*i.e.* valid for each matrix element separately) [4]. Along the same line as above, we can introduce the "out" field as the limit of φ when $T \to +\infty$:

$$\begin{aligned}
\lim_{T\to-\infty} \langle \beta, \text{out} | \varphi(x) | \alpha, \text{out} \rangle &= Z^{\frac{1}{2}} \langle \beta, \text{out} | \varphi_{\text{out}}(x) | \alpha, \text{out} \rangle \,, && (4.81)\\
\left(\Box + m^2 \right) \varphi_{\text{out}}(x) &= 0 \,, && (4.82)
\end{aligned}$$

which generate the basis of the "out" states.

The fields φ_{in} and φ_{out} tranform in the standard way under the unitary tranformation given by the operator $\hat{S}$:

$$\varphi_{\text{out}} = \hat{S}^\dagger \varphi_{\text{in}} \hat{S} \,. \tag{4.83}$$

[4]If this were not the case, the commutator of two fields φ would be equal, up to Z, to the corresponding c-number commutator of free fields. In this case the canonical quantization would require $Z = 1$ and the field φ would be a free field at all times.

The field operators are obtained by quantizing the asymptotic fields $\varphi_{\rm in}$ and $\varphi_{\rm out}$ in the usual way by imposing canonical commutator relations for the fields and their momenta:

$$\begin{aligned}\left[\hat{\varphi}_{\rm in/out}(\vec{x},t),\hat{\Pi}(\vec{x'},t)_{\rm in/out}\right] &= i\delta^3(\vec{x}-\vec{x'})\\ \left[\hat{\Pi}_{\rm in/out}(\vec{x},t),\hat{\Pi}_{\rm in/out}(\vec{x'},t)\right] &= 0\\ \left[\hat{\varphi}_{\rm in/out}(\vec{x},t),\hat{\varphi}_{\rm in/out}(\vec{x'},t)\right] &= 0\,. \end{aligned} \tag{4.84}$$

The asymptotic fields can be expanded in plane waves in the usual way:

$$\begin{aligned}\hat{\varphi}_{\rm in/out}(\vec{x},t) &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_{\rm in/out}(\vec{k},0)e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}+a^\dagger_{\rm in/out}(\vec{k},0)e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}\right)\\ \hat{\Pi}_{\rm in/out}(\vec{x},t) &= -i\int\frac{d^3k}{(2\pi)^3}\frac{\omega_k}{\sqrt{2\omega_k}}\left(a_{\rm in/out}(\vec{k},0)e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}-a^\dagger_{\rm in/out}(\vec{k},0)e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}\right)\,,\end{aligned} \tag{4.85}$$

where $\omega_k=\sqrt{|\vec{k}|^2+m^2}$. Introducing the notation

$$f_k(x)=\frac{1}{\sqrt{2\omega_k}}e^{-ik\cdot x}\,, \tag{4.86}$$

Eq. (4.85) becomes

$$\begin{aligned}\hat{\varphi}_{\rm in/out}(\vec{x},t) &= \int\frac{d^3k}{(2\pi)^3}\left(a_{\rm in/out}(\vec{k},0)f_k(x)+a^\dagger_{\rm in/out}(\vec{k},0)f^*_k(x)\right)\\ \hat{\Pi}_{\rm in/out}(\vec{x},t) &= -i\int\frac{d^3k}{(2\pi)^3}\omega_k\left(a_{\rm in/out}(\vec{k},0)f_k(x)-a^\dagger_{\rm in/out}(\vec{k},0)f^*_k(x)\right)\,.\end{aligned} \tag{4.87}$$

We remind the inverse relations expressing the annihilation and creation asymptotic operators in terms of the asymptotic field (remembering that $\hat{\Pi}=\dot{\hat{\varphi}}$)

$$\begin{aligned}a^\dagger_{\rm in/out}(\vec{k},0) &= -i\int d^3x f_k(x)\left(\partial_t\hat{\varphi}_{\rm in/out}(\vec{x},t)+i\omega_k\hat{\varphi}_{\rm in/out}(\vec{x},t)\right)\\ a_{\rm in/out}(\vec{k},0) &= i\int d^3x f^*_k(x)\left(\partial_t\hat{\varphi}_{\rm in/out}(\vec{x},t)-i\omega_k\hat{\varphi}_{\rm in/out}(\vec{x},t)\right)\,.\end{aligned} \tag{4.88}$$

Since the "in/out" field is normalized as a canonical field, $a_{\rm in/out}$ and $a^\dagger_{\rm in/out}$ are the annihilation and creation operators of the "in/out" states

$$\left|p_1,\ldots,p_n;{\rm out}\right\rangle=a^\dagger_{\rm out}(p_1)\ldots a^\dagger_{\rm out}(p_n)\left|0\right\rangle\,.$$

4.2.5 The Källen-Lehmann spectral representation

We investigate in this section the general form of the propagator for an interacting theory. Unlike the case of free field theories, we can not calculate exactly the Green functions.

However, the requirements of Lorentz invariance and unitarity allow to give a *spectral representation of the two-point Green function*, which is independent of the nature of the interaction and of the perturbative order. The results will be used to obtain the relation between scattering matrix elements and Green functions. Let us start from the definition of the exact propagator (involving interacting fields) as

$$\langle 0|T[\varphi(x)\varphi(y)]|0\rangle \,. \tag{4.89}$$

Assuming $x^0 > y^0$ and using the usual form for the translation of the fields we can write:

$$\begin{aligned}
\langle 0|\varphi(x)\varphi(y)|0\rangle &= \langle 0|e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}e^{iP\cdot y}\varphi(0)e^{-iP\cdot y}|0\rangle \\
&= \langle 0|\varphi(0)e^{-iP\cdot x}e^{iP\cdot y}\varphi(0)|0\rangle \\
&= \sum_\alpha \langle 0|\varphi(0)e^{-iP\cdot x}|\alpha\rangle\langle\alpha|e^{iP\cdot y}\varphi(0)|0\rangle \\
&= \sum_\alpha \langle 0|\varphi(0)|\alpha\rangle e^{-ip_\alpha\cdot(x-y)}\langle\alpha|\varphi(0)|0\rangle \\
&= \sum_\alpha |\langle 0|\varphi(0)|\alpha\rangle|^2 e^{-ip_\alpha\cdot(x-y)}\,,
\end{aligned}$$

where we have made the assumption that the spectrum of eigenstates of the Hamiltonian is complete and p_α is the sum of the momenta (with positive energy) of the particles present in the state $|\alpha\rangle$. Integrating on an additional four-momentum q with the constraint $\delta^4(q-p_\alpha)$, Eq. (4.90) becomes

$$\begin{aligned}
\langle 0|\varphi(x)\varphi(y)|0\rangle &= \int d^4q\,\delta^4(q-p_\alpha)\sum_\alpha |\langle 0|\varphi(0)|\alpha\rangle|^2 e^{-iq\cdot(x-y)} \\
&= \int d^4q\sum_\alpha \delta^4(q-p_\alpha)|\langle 0|\varphi(0)|\alpha\rangle|^2 e^{-iq\cdot(x-y)} \\
&= \int \frac{d^4q}{(2\pi)^3}\rho(q)e^{-iq\cdot(x-y)}\,,
\end{aligned} \tag{4.90}$$

where we have introduced the function $\rho(q)$ defined as follows:

$$\rho(q) = (2\pi)^3\sum_\alpha \delta^4(q-p_\alpha)|\langle 0|\varphi(0)|\alpha\rangle|^2\,, \tag{4.91}$$

which is positive, vanishes for $q^0 < 0$ and is Lorentz invariant. Assuming $x^0 < y^0$ we obtain Eq. (4.90) with x and y interchanged, which is equivalent to Eq. (4.90) with q replaced with $-q$. Being Lorentz invariant, we can write $\rho(q) = \vartheta(q^0)\sigma(q^2)$, with $\sigma(q^2) = 0$ if $(q^2 < 0)$, *i.e.*

$$\rho(q^2) = \int_0^{+\infty} \delta(q^2-\mu^2)\sigma(\mu^2)d\mu^2\,. \tag{4.92}$$

Through Eq. (4.92) we can write Eq. (4.90), taking into account of the T-ordered product, as follows:

$$\begin{aligned}\langle 0|T[\varphi(x)\varphi(y)]|0\rangle &= \int_0^\infty \sigma(\mu^2)\int \frac{d^4q}{(2\pi)^3}\delta(q^2-\mu^2)\vartheta(q^0)\left[e^{-iq\cdot(x-y)}+e^{+iq\cdot(x-y)}\right]d\mu^2 \\ &= \int_0^\infty \sigma(\mu^2)\int \frac{d^3q}{(2\pi)^3 2E(q,\mu)}e^{-iq\cdot(x-y)}+e^{+iq\cdot(x-y)}d\mu^2 \\ &= \int_0^\infty \sigma(\mu^2)\int \frac{d^3q}{(2\pi)^3 2E(q,\mu)}e^{-iq\cdot(x-y)}+e^{+iq\cdot(x-y)}d\mu^2 \\ &= \int_0^\infty \sigma(\mu^2)\Delta_F(x-y;\mu^2)d\mu^2\,, \end{aligned} \tag{4.93}$$

where $\Delta_F(x-y;\mu^2)$ is the propagator of the free scalar field with mass μ. According to Eq. (4.93), the exact propagator of an arbitrary scalar interacting Heisenberg field can be written as a positively weighted average of the corresponding free field propagator for fields of varying mass.

We can separate the contribution of the one-particle states in Eq. (4.91) by using the asymptotic condition, to write [5]

$$\langle 0|T[\varphi(x)\varphi(y)]|0\rangle = iZ\Delta_F(x-y;m^2)+i\int_{m_{\text{thr}}}^\infty \sigma(\mu^2)\Delta_F(x-y;\mu^2)d\mu^2\,, \tag{4.96}$$

where m_{thr} is the threshold for multiparticle states.

4.2.6 The Lehmann-Symanzyk-Zimmermann reduction formulae

Every physical scattering or decay process involves interactions among particles. We assume that the interactions are switched on and off adiabatically (without energy transfer) for a limited time duration. The asymptotic "in" and "out" states are thus free states, which are described by the free field operators $a_{\text{in,out}}$ and $a^\dagger_{\text{in,out}}$. These operators have harmonic time dependence only for $t\to\pm\infty$. The free field operators create and annihilate field quanta when acting on the vacuum of the interacting theory only at times $t\to\pm\infty$. An S matrix element can be written as

$$S_{\beta\alpha} = \langle \beta,\text{out}|\alpha,\text{in}\rangle = \langle \beta,\text{out}|a^\dagger_{\text{in}}(k)|\alpha-k,\text{in}\rangle\,, \tag{4.97}$$

[5] The one-particle states are parameterized as

$$\langle 0|\varphi(x)|p\rangle = e^{-ip\cdot x}\langle 0|\varphi(0)|p\rangle = \frac{\sqrt{Z}}{\sqrt{(2\pi^3)2\omega(p)}}\,. \tag{4.94}$$

Therefore

$$\langle 0|T[\varphi(x)\varphi(0)]|0\rangle_1 = \int d^3p\frac{Z}{(2\pi)^3 2\omega(p)}e^{-ip\cdot x}\,. \tag{4.95}$$

where we have assumed that the *in* state contained a free particle with three-momentum $\vec{k}$. Then $|\alpha - k, \text{in}\rangle$ is that *in* state in which just this particle is missing. We can also write (adding and subtracting $a^\dagger_{\text{out}}(k)$ to the l.h.s. of Eq. (4.97))

$$\begin{aligned} S_{\beta\alpha} &= \langle \beta, \text{out}|\alpha, \text{in}\rangle \\ &= \langle \beta, \text{out}|a^\dagger_{\text{out}}(k)|\alpha - k, \text{in}\rangle + \langle \beta, \text{out}|a^\dagger_{\text{in}}(k) - a^\dagger_{\text{out}}(k)|\alpha - k, \text{in}\rangle \,. \end{aligned} \tag{4.98}$$

In the first term of Eq. (4.98) the operator $a^\dagger_{\text{out}}(k)$ annihilates a quantum with momentum k on the *out* state: $\langle \beta, \text{out}|a^\dagger_{\text{out}}(k) = \langle \beta - k, \text{out}|$, which is zero if the state $\langle \beta, \text{out}|$ does not contain a particle with momentum k. It is useful to write the *in* and *out* creator operator as follows:

$$a^\dagger_{\text{in,out}}(k) = -i \int d^3x\, f_k(x) \hat{\partial}_t \hat{\varphi}_{\text{in,out}}(x,t)\,, \tag{4.99}$$

where

$$f(t)\hat{\partial}_t \hat{\varphi}(x,t) = f\frac{\partial \hat{\varphi}}{\partial t} - \frac{\partial f}{\partial t}\hat{\varphi} \tag{4.100}$$

With the expression of Eq. (4.99) we can write Eq. (4.97) as follows:

$$S_{\beta\alpha} = \langle \beta - k, \text{out}|\alpha - k, \text{in}\rangle - i\langle \beta, \text{out}| \int d^3x\, f_k(x)\hat{\partial}_t \left[\hat{\varphi}_{\text{in}}(x,t) - \hat{\varphi}_{\text{out}}(x,t)\right] |\alpha - k, \text{in}\rangle\,. \tag{4.101}$$

Remark 1: for only two particles in the *in* and *out* states the first term on the r.h.s. of Eq. (4.101) represents a single particle transition matrix element: it can contribute only if both particles do not change their four-momenta, *i.e.* if *in* and *out* states are identical. This is the contribution to the forward scattering amplitude.
Remark 2: . *the r.h.s. of Eq. (4.101) is time-independent.* In fact we can calculate the time derivative of each term of the integrand:

$$\begin{aligned} \frac{\partial}{\partial t}\left(f_k(x)\hat{\partial}_t \hat{\varphi}_{\text{in,out}}(x,t)\right) &= \frac{\partial f_k(x)}{\partial t}\frac{\partial \hat{\varphi}_{\text{in,out}}(x,t)}{\partial t} + f_k(x)\frac{\partial^2 \hat{\varphi}_{\text{in,out}}(x,t)}{\partial t^2} \\ &- \frac{\partial^2 f_k(x)}{\partial t^2}\hat{\varphi}_{\text{in,out}}(x,t) - \frac{\partial f_k(x)}{\partial t}\frac{\partial \hat{\varphi}_{\text{in,out}}(x,t)}{\partial t} \\ &= f_k(x)\frac{\partial^2 \hat{\varphi}_{\text{in,out}}(x,t)}{\partial t^2} - \frac{\partial^2 f_k(x)}{\partial t^2}\hat{\varphi}_{\text{in,out}}(x,t) \qquad (4.102) \\ &= f_k(x)\boldsymbol{\nabla}^2 \hat{\varphi}_{\text{in,out}}(x,t) - \left(\boldsymbol{\nabla}^2 f_k(x)\right)\hat{\varphi}_{\text{in,out}}(x,t)\,, \qquad (4.103) \end{aligned}$$

where, in Eq. (4.102), we have used the fact that $f_k(x) = \frac{1}{\sqrt{2\omega_k}}e^{-ik\cdot x}$ and $\hat{\varphi}_{\text{in,out}}(x,t)$ both solve the free Klein-Gordon equation with the same mass and hence the terms proportional to the mass cancel out. The $\int d^3x$ integral of Eq. (4.103) vanishes by twofold by part integration of one of the two terms. In fact we can consider for instance the second term:

$$\begin{aligned}\int d^3x\,(\nabla\cdot\nabla f_k(\vec{x}))\,\hat{\varphi}_{\rm in,out}(\vec{x}) &= \int dydz\left(\frac{\partial f_k}{\partial x}\varphi_{\rm in,out}(\vec{x})\right)_{x=-\infty}^{x=+\infty}\\ &+ \int dxdz\left(\frac{\partial f_k}{\partial y}\varphi_{\rm in,out}(\vec{x})\right)_{y=-\infty}^{y=+\infty}\\ &+ \int dxdy\left(\frac{\partial f_k}{\partial z}\varphi_{\rm in,out}(\vec{x})\right)_{z=-\infty}^{z=+\infty}\\ &- \int d^3x\,\vec{\nabla} f_k\cdot\vec{\nabla}\hat{\varphi}_{\rm in,out}(\vec{x})\,. \end{aligned} \tag{4.104}$$

The three terms involving the fields evaluated at the boundaries are vanishing and we can write, with a further integration by parts:

$$\begin{aligned}\int d^3x\,(\nabla\cdot\nabla f_k(\vec{x}))\,\hat{\varphi}_{\rm in,out}(\vec{x}) = &- \int dydz\left(f_k\frac{\partial}{\partial x}\hat{\varphi}_{\rm in,out}(\vec{x})\right)_{x=-\infty}^{x=+\infty}\\ &- \int dxdz\left(f_k\frac{\partial}{\partial y}\hat{\varphi}_{\rm in,out}(\vec{x})\right)_{y=-\infty}^{y=+\infty}\\ &- \int dxdy\left(f_k\frac{\partial}{\partial z}\hat{\varphi}_{\rm in,out}(\vec{x})\right)_{z=-\infty}^{z=+\infty}\\ &+ \int d^3x\,f_k\nabla^2\hat{\varphi}_{\rm in,out}(\vec{x})\,. \end{aligned} \tag{4.105}$$

Since the three terms involving the derivatives of the fields at the boundaries vanish, we can write

$$\int d^3x\,(\nabla\cdot\nabla f_k(\vec{x}))\,\hat{\varphi}_{\rm in,out}(\vec{x}) = \int d^3x\,f_k\nabla^2\hat{\varphi}_{\rm in,out}(\vec{x})\,, \tag{4.106}$$

thus proving that

$$\int d^3x\frac{\partial}{\partial t}\left(f_k(x)\hat{\partial}_t\hat{\varphi}_{\rm in,out}(x,t)\right) = 0\,. \tag{4.107}$$

Given the time-independence of the r.h.s. of Eq. (4.101), we can take it at any time, in particular also for $t\to\pm\infty$. At these times we can replace the *in* and *out* fields by the respective limits of the field $\hat{\varphi}(x)$, obtaining for Eq. (4.101)

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta-k,\text{out}|\alpha-k,\text{in}\rangle\\ &+ \lim_{t\to+\infty} i\langle\beta,\text{out}|\int d^3x\,f_k(x)\hat{\partial}_t\hat{\varphi}(x,t)|\alpha-k,\text{in}\rangle\frac{1}{\sqrt{Z}}\\ &- \lim_{t\to-\infty} i\langle\beta,\text{out}|\int d^3x\,f_k(x)\hat{\partial}_t\hat{\varphi}(x,t)|\alpha-k,\text{in}\rangle\frac{1}{\sqrt{Z}}\\ &= \langle\beta-k,\text{out}|\alpha-k,\text{in}\rangle\\ &+ i\int_{-\infty}^{+\infty}dt\,\frac{\partial}{\partial t}\langle\beta,\text{out}|\int d^3x\,f_k(x)\hat{\partial}_t\hat{\varphi}(x,t)|\alpha-k,\text{in}\rangle\frac{1}{\sqrt{Z}} \end{aligned}$$

$$
\begin{aligned}
&= \langle \beta - k, \mathrm{out} | \alpha - k, \mathrm{in} \rangle \\
&+ i\langle \beta, \mathrm{out}| \int_{-\infty}^{+\infty} dt \int d^3x \, \frac{\partial}{\partial t} \left(f_k(x) \hat{\partial}_t \hat{\varphi}(x,t) \right) |\alpha - k, \mathrm{in}\rangle \frac{1}{\sqrt{Z}} \\
&= \langle \beta - k, \mathrm{out} | \alpha - k, \mathrm{in} \rangle \\
&+ i\langle \beta, \mathrm{out}| \int d^4x \left(f_k(x) \frac{\partial^2}{\partial t^2} \hat{\varphi}(x,t) - \frac{\partial^2 f_k(x)}{\partial t^2} \hat{\varphi}(x,t) \right) |\alpha - k, \mathrm{in}\rangle \frac{1}{\sqrt{Z}} \\
&= \langle \beta - k, \mathrm{out} | \alpha - k, \mathrm{in} \rangle \\
&+ \frac{i}{\sqrt{Z}} \langle \beta, \mathrm{out}| \int d^4x \left[f_k(x) \frac{\partial^2}{\partial t^2} - \left((\boldsymbol{\nabla}^2 - m^2) f_k(x) \right) \right] \hat{\varphi}(x,t) |\alpha - k, \mathrm{in}\rangle
\end{aligned} \tag{4.108}
$$

Remark 3: contrary to Eq. (4.102) and Eq. (4.103), . *Eq. (4.108) does not vanish because we are integrating over all times and, therefore, the field $\hat{\varphi}(x,t)$ is an interacting field and it does not solve the free Klein-Gordon equation.*

We consider now the term proportional to $\boldsymbol{\nabla}^2$ in Eq. (4.108):

$$
\begin{aligned}
\int d^4x \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} f_k(x) \right) \hat{\varphi}(x) &= \int dt \int d^3x \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} f_k(x) \right) \hat{\varphi}(x) = \int dt \int d^3x \, f_k(x) \boldsymbol{\nabla}^2 \hat{\varphi}(x) \\
&= \int d^4x \, f_k(x) \boldsymbol{\nabla}^2 \hat{\varphi}(x) \,,
\end{aligned} \tag{4.109}
$$

where we have used the result of Eq. (4.107), which is valid for generic fields, not only for $\hat{\varphi}_{\mathrm{in,out}}$. By means of Eq. (4.109), we can rewrite Eq. (4.108) as follows:

$$
\begin{aligned}
S_{\beta\alpha} &= \langle \beta - k, \mathrm{out} | \alpha - k, \mathrm{in} \rangle \\
&+ i\langle \beta, \mathrm{out}| \int d^4x \, f_k(x) \left(\Box + m^2 \right) \hat{\varphi}(x,t) |\alpha - k, \mathrm{in}\rangle \frac{1}{\sqrt{Z}} \\
&= \langle \beta - k, \mathrm{out} | \alpha - k, \mathrm{in} \rangle \\
&+ i \int d^4x \, f_k(x) \left(\Box + m^2 \right) \langle \beta, \mathrm{out} | \hat{\varphi}(x,t) |\alpha - k, \mathrm{in}\rangle \frac{1}{\sqrt{Z}} \,.
\end{aligned} \tag{4.110}
$$

In this way we have removed one particle with momentum k from the *in* state. We proceed by removing one particle with momentum k' from the *out* state following the same steps as before. Neglecting the term contributing only to forward scattering, we can write

$$
\langle \beta, \mathrm{out} | \hat{\varphi}(x,t) | \alpha - k, \mathrm{in} \rangle = \langle \beta - k', \mathrm{out} | \left[a_{\mathrm{out}}(k') \hat{\varphi}(x,t) - \hat{\varphi}(x,t) a_{\mathrm{in}}(k') \right] | \alpha - k, \mathrm{in} \rangle \,. \tag{4.111}
$$

We replace the annihilation operators by the corresponding field operators as in Eq. (4.99) [6]:

$$
\langle \beta, \mathrm{out} | \hat{\varphi}(x,t) | \alpha - k, \mathrm{in} \rangle = \langle \beta - k', \mathrm{out} | \hat{\varphi}(x,t) | \alpha - k - k', \mathrm{in} \rangle
$$

[6] We need here the hermitean conjugate of Eq. (4.99):

$$
a_{\mathrm{in,out}}(k) = +i \int d^3x \, f_k^*(x) \hat{\partial}_t \hat{\varphi}_{\mathrm{in,out}}(x,t) \,, \tag{4.112}
$$

where we have used the assumption of real scalar fields $\varphi^\dagger(x) = \varphi(x)$.

$$+ \quad i\langle\beta - k', \text{out}| \int d^3x' \, f^*_{k'}(x') \left[\hat{\partial}_{t'}\hat{\varphi}_{\text{out}}(x')\hat{\varphi}(x) - \hat{\varphi}(x)\hat{\partial}_{t'}\hat{\varphi}_{\text{in}}(x')\right] |\alpha - k, \text{in}\rangle \,. \tag{4.113}$$

Along the same reasoning as for Eq. (4.101), we can prove that the r.h.s. of Eq. (4.113) is independent of t', *i.e.* we can substitute φ_{in} with $\lim_{t\to-\infty}$ and φ_{out} with $\lim_{t\to+\infty}$, obtaining

$$\begin{aligned}
&\langle\beta, \text{out}|\hat{\varphi}(x,t)|\alpha - k, \text{in}\rangle = \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i \lim_{t'\to+\infty} \langle\beta - k', \text{out}| \int d^3x' \, f^*_{k'}(x')\hat{\partial}_{t'}\hat{\varphi}(x')\hat{\varphi}(x)|\alpha - k, \text{in}\rangle \\
- \quad & i \lim_{t'\to-\infty} \langle\beta - k', \text{out}| \int d^3x' \, f^*_{k'}(x')\hat{\varphi}(x)\hat{\partial}_{t'}\hat{\varphi}(x')|\alpha - k, \text{in}\rangle \\
= \quad & \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i \lim_{t'\to+\infty} \langle\beta - k', \text{out}| \int d^3x' \, f^*_{k'}(x')\hat{\partial}_{t'} T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]|\alpha - k, \text{in}\rangle \\
- \quad & i \lim_{t'\to-\infty} \langle\beta - k', \text{out}| \int d^3x' \, f^*_{k'}(x')\hat{\partial}_{t'} T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]|\alpha - k, \text{in}\rangle \\
= \quad & \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i \int_{-\infty}^{+\infty} dt' \, \frac{\partial}{\partial t'} \langle\beta - k', \text{out}| \int d^3x' \, f^*_{k'}(x')\hat{\partial}_{t'} T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]|\alpha - k, \text{in}\rangle \\
= \quad & \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i \int_{-\infty}^{+\infty} dt' \, \langle\beta - k', \text{out}| \int d^3x' \, \frac{\partial}{\partial t'} \left(f^*_{k'}(x')\hat{\partial}_{t'} T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]\right)|\alpha - k, \text{in}\rangle \\
= \quad & \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i\langle\beta - k', \text{out}| \int d^4x' \left(f^*_{k'}(x')\frac{\partial^2}{\partial t'^2} T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right] - \frac{\partial^2 f^*_{k'}(x')}{\partial t'^2} T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]\right)|\alpha - k, \text{in}\rangle \\
= \quad & \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i\langle\beta - k', \text{out}| \int d^4x' \left(f^*_{k'}(x')\frac{\partial^2}{\partial t'^2} - (\nabla^2 - m^2) f^*_{k'}(x')\right) T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]|\alpha - k, \text{in}\rangle
\end{aligned} \tag{4.114}$$

Applying the result of Eq. (4.109) to Eq. (4.114) we obtain

$$\begin{aligned}
&\langle\beta, \text{out}|\hat{\varphi}(x,t)|\alpha - k, \text{in}\rangle = \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & i \int d^4x' \, f^*_{k'}(x') \left(\Box_{x'} + m^2\right) \langle\beta - k', \text{out}|T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]|\alpha - k, \text{in}\rangle \,.
\end{aligned} \tag{4.115}$$

By means of Eq. (4.115) we can write Eq. (4.110) as follows

$$\begin{aligned}
S_{\beta\alpha} \quad = \quad & \langle\beta - k, \text{out}|\alpha - k, \text{in}\rangle \\
+ \quad & \frac{i}{\sqrt{Z}} \int d^4x \, f_k(x) \left(\Box_x + m^2\right) \langle\beta - k', \text{out}|\hat{\varphi}(x,t)|\alpha - k - k', \text{in}\rangle \\
+ \quad & \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x d^4x' \, f_k(x) f^*_{k'}(x') \left(\Box + m^2\right) \left(\Box_{x'} + m^2\right) \langle\beta - k', \text{out}|T\left[\hat{\varphi}(x')\hat{\varphi}(x)\right]|\alpha - k, \text{in}\rangle \,.
\end{aligned}$$

(4.116)

This *reduction procedure* can be applied on both sides until all particles have been removed from the *in* and *out* states and we have, with n particles with momenta k'_i in the *out* state and m particles with momenta k_j in the *in* state:

$$S_{\beta\alpha} = \langle \beta n k', \text{out} | \alpha m k, \text{in} \rangle =$$
$$i^{m+n} \int \prod_{i=1}^{m} d^4 x_i \int \prod_{j=1}^{n} d^4 x'_j f^*_{k'_j}(x'_j) f_{k_i}(x_i)$$
$$\left(\frac{1}{\sqrt{Z}}\right)^{m+n} \left(\Box'_j + m^2\right) \left(\Box_i + m^2\right) \langle 0 | T \left[\hat{\varphi}(x'_1) \ldots \hat{\varphi}(x'_n) \hat{\varphi}(x_1) \ldots \hat{\varphi}(x_m)\right] | 0 \rangle \quad (4.117)$$

Eq. (4.117) is the LSZ reduction theorem which allows to calculate S matrix elements in terms of Green functions. The mass m appearing in Eq. (4.117) is the physical mass, which coincides with the inertial mass of the field quanta when they are far away from each other.

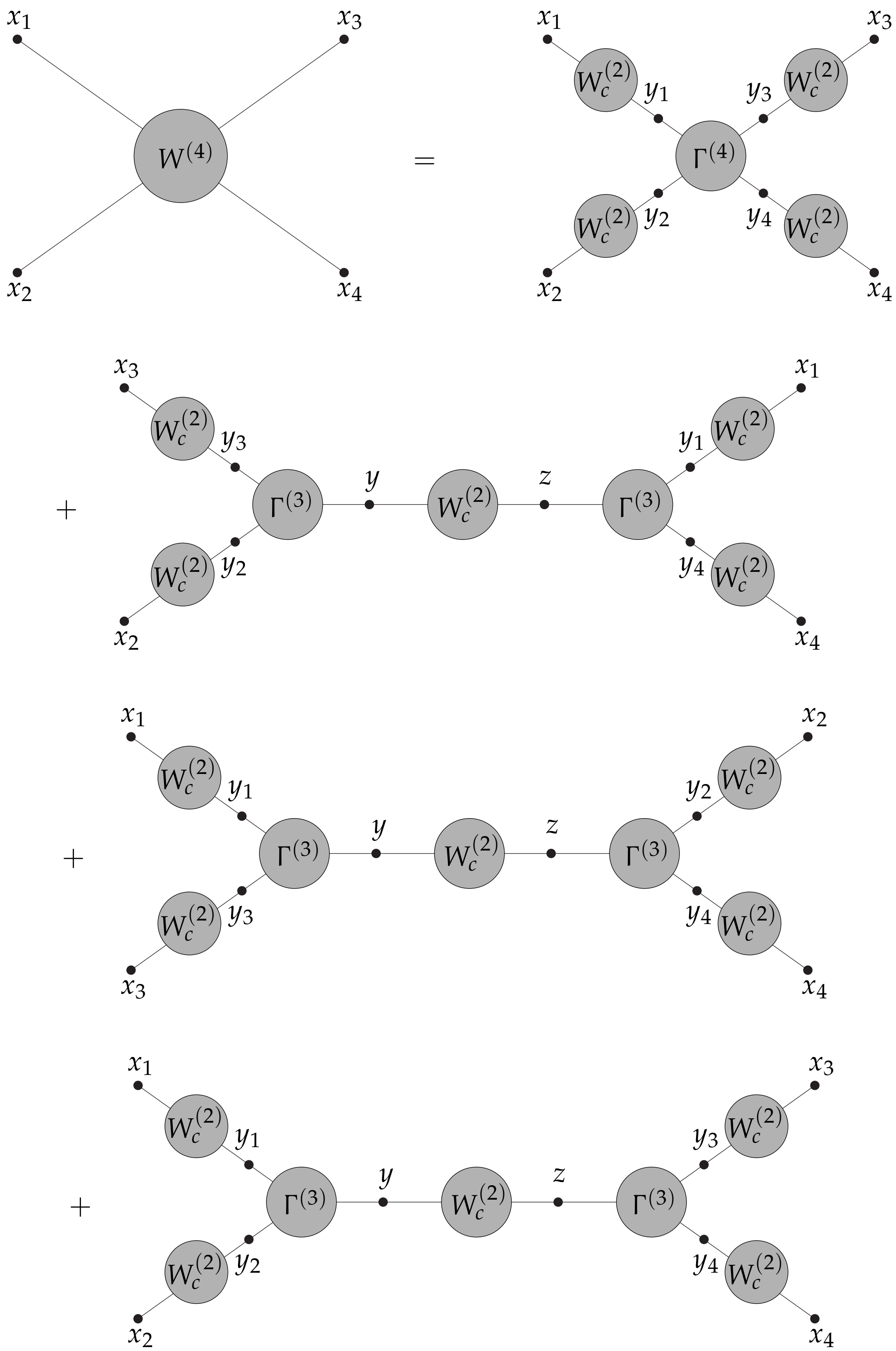

Figure 4.5: Graphical representation of the connected four-point Green function of Eq. (4.54) in terms of 1PI-contributions.

5

Renormalization

In this chapter we introduce the perturbative method at one-loop to study the interacting fields. With reference to the case of a self-interacting real scalar field, we show that divergences appear due to the continuum structure of four-dimensional space-time and some regularization scheme has to be introduced. We discuss in detail the dimensional regularization scheme and apply it to the calculation of divergent Green functions for the $\lambda\varphi^4$ model. In the final section we discuss the renormalization of the $\lambda\varphi^4$ model at one-loop, within the two equivalent approaches of "bare perturbation theory" and "renormalized perturbation theory".

5.1 The $\lambda\varphi^4$ model

It consists of a real scalar self-interacting field, with a quartic local (i.e. the fields are taken at the same point) interaction. It allows to introduce the renormalization theory in a relatively simplified way. The model has interesting applications in statistical mechanics and in QFT it is an essential component of the Higgs mechanism. The starting Lagrangian is the following:

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\varphi_b(x)\partial^\mu\varphi_b(x) - m_b^2\left(\varphi_b(x)\right)^2\right] - \frac{\lambda_b}{4!}\left(\varphi_b(x)\right)^4 . \tag{5.1}$$

Its corresponding Euclidean version is

$$\mathcal{L}_E = \frac{1}{2}\left(\partial_\mu\varphi_b(x)\partial^\mu\varphi_b(x) + m_b^2\left(\varphi_b(x)\right)^2\right] + \frac{\lambda_b}{4!}\left(\varphi_b(x)\right)^4 . \tag{5.2}$$

The subscript b in Eq. (5.1) will be clear in the following.
The Z functional is written in terms of a perturbative expansion of the unperturbed Z_0 functional as follows:

$$Z_E^b[J_b] = \mathcal{N}_b \exp\left\{-\frac{\lambda_b}{4!}\int d^4x\, \left[\frac{\delta}{\delta J_b(x)}\right]^4\right\} Z_{0E}[J_b]\,, \tag{5.3}$$

where the unperturbed functional is given by

$$Z_{0E}[J_b] = \exp\left\{\frac{1}{2}\int d^4x \int d^4x'\, J(x)\Delta_E(x-x')J(x')\right\}. \tag{5.4}$$

In Eq. (5.3) the normalization constant $\mathcal{N}_b$ is fixed by the condition

$$Z_E^b[0] = 1\,, \tag{5.5}$$

while $\Delta_E(x-x')$ in Eq. (5.4) is the Euclidean propagator

$$\Delta_E(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{i(p\cdot x)_E}}{p^2+m_b^2}\,, \tag{5.6}$$

where both p^2 and $(p\cdot x)_E$ are expressed with the euclidean metrics.

The Feynman rules for the Green functions, generated by the Z functional, can be read off from Eq. (5.3). It is convenient to use the momentum representation, where

$$\hat{G}_E^{b(n)}(p_1,\ldots,p_n)(2\pi)^4\delta^4(p_1+\ldots p_n) = \int d^4x_1\ldots\int d^4x_n e^{i[(p_1\cdot x_1)_E+\ldots+(p_n\cdot x_n)_E]}G_E^{b(n)}(x_1,\ldots,x_n)\,. \tag{5.7}$$

On the left-hand side we have factored out the Dirac δ function ensuring the total momentum conservation.

In momentum representation, and in Euclidean space-time, the Feynman rules have the following form. The perturbative order corresponds to the number of vertices. Given the interaction term, we have only vertices involving four lines. We have the following correspondence between analitic expressions and Feynman diagrams

- $$\frac{1}{p^2+m_b^2} \tag{5.8}$$

- $$-\frac{\lambda_b}{4!} \qquad (p_1+p_2+p_3+p_4=0) \tag{5.9}$$

- Integration on internal loops:
$$\int\frac{d^4q}{(2\pi)^4} \tag{5.10}$$

- Appropriate statistical factors for each diagram (which will illustrated with examples).

The Minkowskian Feynman rules corresponding to Eqs. (5.8) and (5.9) are

- $$\frac{i}{p^2 - m_b^2} \tag{5.11}$$

- $$-i\frac{\lambda_b}{4!} \qquad (p_1 + p_2 + p_3 + p_4 = 0) \tag{5.12}$$

5.2 Ultraviolet divergences

As soon as we go beyond the tree-level approximation, we have to deal with momentum loop integrations which can be infinite (or better, not defined). As first example, let us consider the first order correction to the propagator, which is given by the following Feynman diagram (also called self-energy diagram), where the momentum conservation at the vertex allows an arbitrary momentum q circulating in the loop:

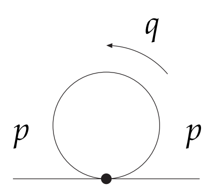

Figure 5.1: Self-energy diagram in the $\lambda\varphi^4$ theory.

The corresponding expression is

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{p^2 + m_b^2} \sim Q^2 \,, \tag{5.13}$$

where Q is an artificial upper limit ("cut off") on the integration four momentum. In quantum field theory we do not have an upper limit on the momenta. Since the momentum is proportional to the inverse wavelength, the existence of an ultraviolet cutt-off on the momenta would imply a (periodic) discretized structure of the space-time. Such an hypothesis would not be compatible with the requiremente of special relativity.

Another example is given by the correction to the vertex diagram of Eq. (5.12). It is a second order term, since the tree level is of the first order in the number of vertices.

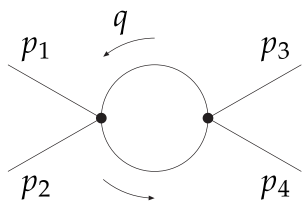

Figure 5.2: Scalar four-point vertex diagram.

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2+m_b^2)[(q+p)^2+m_b^2]} \sim \ln Q\,. \tag{5.14}$$

The procedure which allows to remove consistently these infinities is the *renormalization*. It is a two-step procedure: first we have to adopt a *regularization scheme* in terms of some parameter, in order to be able to carry out the calculations, and then the renormalization allows to obtain finite physical predictions.

5.3 Power counting for the $\lambda\varphi^4$ model

We have seen that the two- and four-point Green functios in the $\lambda\varphi^4(x)$ model are UV divergent. It would be useful to have a criterion to understand which Green functions of the theory are UV divergent. A diagram is said to have *superficial degree of divergence d* if it diverges as Λ^d. A logarithmic divergence of the form $\log\Lambda$ counts as $d=0$. The power counting theorem states that the superficial degree of divergence d, in four dimensions, of a diagram with n external legs is

$$d = 4 - n\,. \tag{5.15}$$

To prove the theorem, let us define the number of vertices with V, the number of internal lines with I and the number of loops with L. The number of loops is the number of integrations $\int \frac{d^4k}{(2\pi)^4}$ we have to perform. Each internal line carries with it a momentum to be integrated over, so we should have I integrals. But this number is reduced by the number of momentum conservation delta functions associated with the vertices, one to each vertex. In total we have V delta functions, but one of them is associated with the overall momentum conservation of the entire diagram. Thus the number of loops L is

$$L = I - (V - 1)\,. \tag{5.16}$$

Verify the above relation for the two and four point functions of $\lambda\varphi^4$.
There is also a relation among the V, n and I. In fact, each external line is attached to a vertex and internal line connects two vertices (in other words an internal line counts as two external lines):

$$4V = n + 2I\,. \tag{5.17}$$

On the other hand, for each loop there is a $\int d^4k$ while for each internal line there is a factor $\sim \frac{1}{k^2}$, bringing down the powers of momentum by 2. Hence we can write

$$d = 4L - 2I\,. \tag{5.18}$$

Combining Eqs. (5.16,5.17,5.18), we obtain Eq. (5.15). Notice that Eq. (5.15) states that, for a given number of external lines, no matter to what order of perturbation theory we go, the superficial degree of divergence remains always the same.
Since the Lagrangian of Eq. (5.2) is an even function of φ, the Green functions of order n are 0 if n is odd. As a consequence, only the two point and four point Green functions contain ultraviolet divergences.
Observation 1: it can happen that a Green function with $n > 4$ is divergent. If this is the case, it contains some subgraph with two or four legs.

Observation 2: the power counting can overestimate the real degree of divergence (never underestimate). This is not the case for the $\lambda\varphi^4$ model but is the case for QED, as we will see.

Observation 3: a general result is given by the Weinberg theorem, which we will state without proof: a Feynman diagram is convergent if its superficial degree of divergence is negative, together with the one of all its possible subdiagrams.

Observation 4: In the model $\lambda\varphi^4(x)$ the connected Green functions $G_c^{(2)}$ and $G_c^{(4)}$ are the **primitive divergences**.

Observation 5: in D dimensions Eq. (5.18) becomes

$$d = DL - 2I \tag{5.19}$$

and Eq. (5.15) becomes

$$d = 4 - n + (D - 4)L\,. \tag{5.20}$$

As an exercise we can generalize the results for a generic $\lambda\varphi^m$ model[1]. In this case we find a useful relation

$$d = D + \left[m \left(\frac{D-2}{2} \right) - D \right] V - \left(\frac{D-2}{2} \right) n \,. \tag{5.21}$$

We can check that the quantity that multiplies V in the above equation is minus the dimension of the coupling constant λ. For example, if $\dim[\lambda] > 0$, for a given number n of external legs, there exists a maximum number of vertices V_0 such that, for $V > V_0$ the superficial degree of divergence $d < 0$. The above result is valid also for other field theories. This allows to give a criterion for three degrees of renormalizability:

- Super-renormalizable theory: coupling constant with positive mass dimension. Only a finite number of Feynman diagrams superficially diverge.

- Renormalizable theory: dimensionless coupling constant. Even if divergences occur at all orders in perturbation theory, only a finite number of amplitudes superficially diverge.

- Non-renormalizable theory: coupling constant with negative mass dimension. All amplitudes are divergent at a sufficiently high order in perturbation theory.

5.4 Regularization schemes

Adopting a regularization scheme amounts to changing the theory with the introduction of some parameter, which allow to render mathematically well-defined the divergent integrals. In this way, the regularized theory based on the regularized integrals will violate some of the underlying physical requirements, such as Lorentz invariance, gauge symmetry, unitarity etc., which are restored only at the end of the calculations, when we take the limit to the original theory. In this sense, any regularization scheme is equivalent to others, even if some scheme can allow to preserve more physical symmetries than others. We list below some adopted regularization schemes, and then analyse in more detail the features of dimensional regularization, which is commonly adopted in QED and in gauge theories.

- **Cut-off scheme**: it consists in cutting off the high-momentum region, which is the source of divergence in the divergent integrals. It was used in the early literature in QED. However it breaks translation invariance and hence the shift of momenta in the integral, in general, changes the result. Also gauge invariance is broken in this regularization scheme

[1]See observation 3 of Sec. (5.6) for the dimensions of the field in arbitrary spacetime dimensions.

- **Pauli-Villars scheme**: it consists in replacing the propagator in the integrand in the following way:

$$\frac{1}{m^2+k_E^2} \to \frac{1}{m^2+k_E^2} - \frac{1}{M^2+k_E^2} = \frac{M^2-m^2}{(m^2+k_E^2)(M^2+k_E^2)} \,. \tag{5.22}$$

  The modified propagator reduces to the original one as $M \to \infty$. For finite M the high-momentum behaviour of the modified propagator is $\mathcal{O}(1/k^4)$, two powers less than the original one $\mathcal{O}(1/k^4)$. This scheme respects translation, Lorentz invariance and also gauge invariance (but not for massive non-abelian gauge theories).

- **Analytic regularization scheme**: it consists in changing the power of the propagator denominator:

$$\frac{1}{m^2+k_E^2} \to \frac{1}{\left(m^2+k_E^2\right)^{\alpha}} \,, \tag{5.23}$$

  with α a complex parameter with Re$\alpha > 1$. The original propagator is recovered as $\alpha \to 1$. This method violates gauge invariance.

- **Lattice regularization scheme**: it consists in discretizing the space-time structure with a minimum length a, which corresponds in momentum space to the cut-off method. Hence this scheme breaks translation and Lorentz invariance. However it is suitable for a numerical approach to quantum field theory, where the generating functional $Z[J]$ is calculated in configuration space.

- **Dimensional regularization**: the key observation is that the divergent integrals would be finite if the dimensions of space-time were less than four. More details are given in the following section.

5.5 Dimensional regularization scheme

Let us consider Eq. (5.13) and Eq. (5.14) with D dimensions. For example,

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{1}{p^2+m_b^2} \,, \tag{5.24}$$

where by convention we have set $D = 2\omega$.

The basic idea is the one of an analitic continuation in the number of space-time dimensions. Let us consider the Euler and Weierstrass representations of the $\Gamma(z)$ function. With $Re(z) > 0$

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1} \,, \tag{5.25}$$

which is an analytical function of z (Eq. (5.25)) is the Euler representation. For $Re(z) < 0$ the integrand diverges at $t = 0$. We split the integration interval

$$\Gamma(z) = \int_0^\alpha dt\, e^{-t} t^{z-1} + \int_\alpha^\infty dt\, e^{-t} t^{z-1} \,. \tag{5.26}$$

If $\alpha > 0$ the second integral of Eq. (5.26) is well defined, also for $Re(z) < 0$. We define the first integral by substituting the exponential with its power series expansion:

$$\begin{aligned}\Gamma(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\alpha} dt\, t^{n+z-1} + \int_{\alpha}^{\infty} dt\, e^{-t} t^{z-1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\alpha^{n+z}}{(z+n)} + \int_{\alpha}^{\infty} dt\, e^{-t} t^{z-1} \,. \end{aligned} \tag{5.27}$$

Eq. (5.27) has simple poles for negative integer $z \leq 0$. For complex values of z Eq. (5.27) is valid and $\Gamma(z)$ does not depend on α: $\frac{d\Gamma}{d\alpha} = 0$. Eq. (5.27) with $\alpha = 1$ is the Weierstrass representation of the Gamma function.

In doing loop calculations, we have to evaluate an integral of the kind

$$I(n, \alpha) = \int d^n q \frac{1}{(q^2 + m^2)^{\alpha}} \,, \tag{5.28}$$

where q is the euclidean momentum $q_\mu = (q_1, \ldots, q_n)$. Since the integrand depends only on the squared modulus q^2, the angular integral can be performed using a generalized polar coordinate system in n dimensions, where the vector q_μ is characterized by its modulus and $n-1$ angles. The volume element is given by

$$\int d^n q = \int_0^{\infty} dq\, q^{n-1} \int_0^{2\pi} d\vartheta_1 \int_0^{\pi} \sin \vartheta_2 d\vartheta_2 \ldots \int_0^{\pi} \sin^{n-2} \vartheta_{n-1} d\vartheta_{n-1} \,. \tag{5.29}$$

The validity of Eq. (5.29) can be shown by induction: it is true for $n = 1, 2, 3$. Assuming it is true for n dimensions, it can be shown to hold also for $n+1$ dimensions:

$$\int d^{n+1} r = \int dx_{n+1} \int d^n r \,. \tag{5.30}$$

This is left as an exercise.

The angular integration can be performed by means of the following formula

$$\int_0^{\pi} \sin^k \vartheta d\vartheta = \frac{\Gamma\left(\frac{1}{2}\right) + \Gamma\left(\frac{1}{2} + \frac{1}{2}k\right)}{\Gamma\left(1 + \frac{1}{2}k\right)} \,. \tag{5.31}$$

We remind that

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\ \Gamma(z) &= (z-1)! \\ \Gamma(z+1) &= z\Gamma(z) \\ \Gamma(2) &= \Gamma(1) = 1 \,. \end{aligned}$$

By using $n-1$ times Eq. (5.31) we obtain

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \,. \tag{5.32}$$

An alternative way to see the above result of Eq. (5.32) is by remembering that

$$\begin{aligned} \pi &= \int dxdy\, e^{-\left(x^2+y^2\right)} = \int d^2x\, e^{-|x|^2} \\ \sqrt{\pi} &= \int dx\, e^{-x^2} . \end{aligned}$$

In n dimensions we have

$$\begin{aligned} \pi^{\frac{n}{2}} &= \int d^n x\, e^{-x^2} = \int d\Omega_n \int_0^\infty dx\, x^{n-1} e^{-x^2} \\ &= \frac{1}{2} \int d\Omega_n \int_0^\infty dt\, t^{\frac{n}{2}-1} e^{-t} = \frac{1}{2} \int d\Omega_n\, \Gamma\left(\frac{n}{2}\right) , \end{aligned} \tag{5.33}$$

which is equivalent to Eq. (5.32). In the above equation we have used the integration variable substitution $x^2 \to t$.

The remaining integral to be performed, after angular integration, to find $I(n, \alpha)$ of Eq. (5.28) is

$$\begin{aligned} \int_0^\infty dq\, \frac{q^{n-1}}{\left(q^2+m^2\right)^\alpha} &= \int_0^\infty dq\, \frac{q q^{n-2}}{\left(q^2+m^2\right)^\alpha} = \frac{1}{2} \int_0^\infty dq^2\, \frac{\left(q^2\right)^{\frac{n}{2}-1}}{\left(q^2+m^2\right)^\alpha} \\ &= \frac{1}{2} \frac{1}{m^{2\alpha}} \left(m^2\right)^{\frac{n}{2}} \int_0^\infty d\left(\frac{q^2}{m^2}\right) \frac{\left(\frac{q^2}{m^2}\right)^{\frac{n}{2}-1}}{\left(1+\frac{q^2}{m^2}\right)^\alpha} \\ &= \frac{1}{2} \left(m^2\right)^{\left(\frac{n}{2}-\alpha\right)} \int_0^\infty ds\, \frac{s^{\frac{n}{2}-1}}{(s+1)^\alpha} . \end{aligned} \tag{5.34}$$

It is useful to introduce the following change of variables in the last integral

$$\begin{aligned} s &= \frac{t}{1-t} \qquad ds = \frac{dt}{(1-t)^2} \\ t &= \frac{s}{1+s} , \end{aligned}$$

which means that $s = 0 \Longrightarrow t = 0$ and $s \to \infty \Longrightarrow t \to 1$ monotonically. By means of the above susbstitution, Eq. (5.34) becomes

$$\begin{aligned} \int_0^\infty dq\, \frac{q^{n-1}}{\left(q^2+m^2\right)^\alpha} &= \frac{1}{2} \left(m^2\right)^{\left(\frac{n}{2}-\alpha\right)} \int_0^1 dt\, \frac{1}{(1-t)^2} \frac{1}{(1-t)^{-\alpha}} \frac{t^{\frac{n}{2}-1}}{(1-t)^{\frac{n}{2}-1}} \\ &= \frac{1}{2} \left(m^2\right)^{\left(\frac{n}{2}-\alpha\right)} \int_0^1 dt\, t^{\frac{n}{2}-1} (1-t)^{\alpha-\frac{n}{2}-1} . \end{aligned} \tag{5.35}$$

We remind that the special function βeta is defined as

$$B(p,q) = \int_0^1 dx\, x^{p-1}(1-x)^{q-1} = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} . \tag{5.36}$$

Hence Eq. (5.35) becomes

$$\int_0^\infty dq\, \frac{q^{n-1}}{(q^2+m^2)^\alpha} = \frac{1}{2}\left(m^2\right)^{\left(\frac{n}{2}-\alpha\right)} B\left(\frac{n}{2}, \alpha-\frac{n}{2}\right). \tag{5.37}$$

With Eqs. (5.32) and (5.37), Eq. (5.28) becomes

$$\begin{aligned} I(n,\alpha) &= \int d^n q \frac{1}{(q^2+m^2)^\alpha} \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{2}\left(m^2\right)^{\left(\frac{n}{2}-\alpha\right)} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)} \\ &= \pi^{\frac{n}{2}} \frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}\left(m^2\right)^{\left(\frac{n}{2}-\alpha\right)}, \end{aligned} \tag{5.38}$$

which is valid for $\alpha > \frac{n}{2}$. *For $\alpha > 2$ the integral $I(n,\alpha)$ is finite in 4 dimensions. For $\alpha \leq 2$ the integral $I(n,\alpha)$ is defined through the expression of Eq. (5.38).* We note that, out of the range $\alpha > \frac{n}{2}$, $I(n,\alpha)$ displays simple poles when $\alpha - \frac{n}{2} < 0$, in particular for negative integer values. As a function of the number of dimensions n, $I(n,\alpha)$ is an analytical function in the complex plane. **We can develop the theory in $n < 4$ dimensions and "parametrize the infinities" of the loop integration as simple poles proportional to $\frac{1}{n-4}$.**

Let us see as the analytic continuation works in practice fo a generic function $f(q)$. With by part integration we can write

$$\int d^n q \frac{\partial}{\partial q_\mu}\left(q_\mu f(q)\right) = \int d^n q\, q_\mu \left(\frac{\partial}{\partial q_\mu} f(q)\right) + n\int d^n q\, f(q), \tag{5.39}$$

where we have used $\frac{\partial q_\mu}{\partial q_\mu} = n$. Since the left-hand side of Eq. (5.39) is the integral of a quadridivergence, due to Gauss theorem it is equal to zero, provided that $f(q)$ goes to zero rapidly enough at infinity. *In dimensional regularization the surface term is defined equal to zero, independently of the asymptotic behaviour of the integrand* [2]*:*

$$n\int d^n q\, f(q) = -\int d^n q\, q_\mu \left(\frac{\partial}{\partial q_\mu} f(q)\right). \tag{5.40}$$

Eq. (5.40) can be used to define divergent integrals in terms of finite integrals for complex values of n. Let us consider, for instance,

$$f(q) = \frac{1}{(q^2+m^2)^\alpha}. \tag{5.41}$$

[2] If $f(q)$ does not behave at infinity as a power of q^2, multiple poles can originate, but we will not consider this case.

Application of Eq. (5.40) gives

$$\begin{aligned} n\int d^n q\,\left(q^2+m^2\right)^{-\alpha} &= 2\alpha\int d^n q\,\frac{q^2}{(q^2+m^2)^{\alpha+1}} = 2\alpha\int d^n q\,\frac{q^2+m^2-m^2}{(q^2+m^2)^{\alpha+1}} \\ &= 2\alpha\int d^n q\,\frac{1}{(q^2+m^2)^{\alpha}} - 2m^2\alpha\int d^n q\,\frac{1}{(q^2+m^2)^{\alpha+1}}\,. \end{aligned}$$

Hence

$$\int d^n q\,\frac{1}{(q^2+m^2)^{\alpha}} = \frac{m^2\alpha}{\left(\alpha-\frac{n}{2}\right)}\int d^n q\,\frac{1}{(q^2+m^2)^{\alpha+1}}\,. \tag{5.42}$$

While the integral at the left-hand side of Eq. (5.42) is finite for $n < 2\alpha$, the one on the right-hand side is finite for $n < 2\alpha+2$. Hence Eq. (5.42) defines an analytic continuation of the integral on the left-hand side in the region $2\alpha < n < 2\alpha+2$, where it is not convergent. The continuation is well defined except for the point $n = 2\alpha$, where the right-hand side has a pole with finite residuum.

We can iterate the procedure until the integral at the right-hand side is convergent for the value of n which we are interested in. For instance

$$\int d^n q\,\frac{1}{(q^2+m^2)^{\alpha}} = \frac{m^4\alpha(\alpha+1)}{\left(\alpha+1-\frac{n}{2}\right)\left(\alpha-\frac{n}{2}\right)}\int d^n q\,\frac{1}{(q^2+m^2)^{\alpha+2}}\,. \tag{5.43}$$

The integral at the right-hand side in Eq. (5.43) is finite for $n < 2\alpha+4$, except for the poles at $n = 2\alpha$ and $n = 2\alpha+2$. The outlined procedure is consistent with the starting integral $I(n,\alpha)$, as can be proved by using the properties of the Γ function, in particular $\Gamma(z+1) = z\Gamma(z)$.

We study now how to treat the integrals in the physical limit $n \to 4$. The basic point is to consider the Laurent expansion of the Γ function. To this aim it is useful to remember the Laurent expansion of $\Gamma(z)$ around $z = 0$:

$$\Gamma(z) = \frac{1}{z} - \gamma_E + \left(\frac{\pi^2}{12}+\frac{\gamma_E^2}{2}\right)z + \mathcal{O}\left(z^2\right)\,, \tag{5.44}$$

where γ_E is the Euler-Mascheroni constant, defined as

$$\gamma_E = \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right) = 0.577\ldots\,. \tag{5.45}$$

Let us consider for instance $I(n,2)$

$$\begin{aligned} I(n,2) &= \pi^{\frac{n}{2}}\frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}\left(m^2\right)^{\left(\frac{n}{2}-2\right)} = \pi^{\frac{n}{2}}\Gamma\left(\frac{4-n}{2}\right)\left(m^2\right)^{\left(\frac{n-4}{2}\right)} \\ &= \pi^{\frac{n-4}{2}}\pi^2\Gamma\left(\frac{4-n}{2}\right)\left(m^2\right)^{\left(\frac{n-4}{2}\right)}\,. \end{aligned} \tag{5.46}$$

Putting
$$\epsilon = \frac{4-n}{2}\,, \tag{5.47}$$
we have
$$\begin{aligned}
\Gamma\left(\frac{4-n}{2}\right) &= \frac{2}{4-n} - \gamma_E + \mathcal{O}(\epsilon) \\
\left(m^2\right)^{\frac{n-4}{2}} &= \left(m^2\right)^{-\epsilon} = e^{\ln\left(m^2\right)^{-\epsilon}} = e^{-\epsilon \ln m^2} = 1 - \epsilon \ln m^2 + \mathcal{O}\left(\epsilon^2\right) \\
\pi^{\frac{n-4}{2}} &= 1 - \epsilon \ln \pi + \mathcal{O}\left(\epsilon^2\right)\,.
\end{aligned} \tag{5.48}$$
Substituting the Laurent expansions of Eq. (5.48) in Eq. (5.46), we get
$$\begin{aligned}
I(n,2) &= \pi^2\left[\frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right]\left[1 - \epsilon \ln m^2 + \mathcal{O}\left(\epsilon^2\right)\right]\left[1 - \epsilon \ln \pi + \mathcal{O}\left(\epsilon^2\right)\right] \\
&= \pi^2\left[\frac{1}{\epsilon} - \gamma_E - \ln \pi - \ln m^2 + \mathcal{O}(\epsilon)\right]\,.
\end{aligned} \tag{5.49}$$

Observations on Eq. (5.49)

- we have split $I(n,2)$ into a divergent part and a finite remainder,
- we have a logarithm of a dimensionfull quantity! On this point we will come again later on.

5.6 Calculation of divergent Green functions in $\lambda\varphi^4(x)$

Let us consider the integral of Eq. (5.13) which we had associated to the one-loop diagram of the propagator in $\lambda\varphi^4$ model, going from 4 to n dimensions:
$$\begin{aligned}
\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2} \quad \Longrightarrow \quad \int \frac{d^nq}{(2\pi)^n}\frac{1}{q^2+m^2} &= \frac{1}{(2\pi)^n}\pi^{\left(\frac{n}{2}\right)}\frac{\Gamma\left(1-\frac{n}{2}\right)}{\Gamma(1)}\left(m^2\right)^{\left(\frac{n}{2}-1\right)} \\
&= \frac{1}{(2\pi)^{n-4}}\frac{1}{(2\pi)^4}\pi^{\left(\frac{n}{2}-2\right)}\pi^2\Gamma\left(2-\frac{n}{2}-1\right)m^{2\left(\frac{n}{2}-2+1\right)}\,.
\end{aligned} \tag{5.50}$$
By means of the properties of the Γ function of Eq. (5.32) and of the expansions of Eq. (5.48), we have
$$\begin{aligned}
\Gamma\left(2-\frac{n}{2}-1\right) &= \frac{\Gamma\left(2-\frac{n}{2}\right)}{2-\frac{n}{2}-1} = -\frac{\Gamma\left(\frac{4-n}{2}\right)}{1-\frac{4-n}{2}} \\
&= -\Gamma\left(\frac{4-n}{2}\right)\left(1+\frac{4-n}{2}\right) + \mathcal{O}\left(\epsilon^2\right)) = -\frac{2}{4-n} - 1 + \gamma_E + \mathcal{O}(\epsilon) \\
m^{2\left(\frac{n}{2}-2+1\right)} &= m^2 m^{2\left(\frac{n}{2}-2\right)} = m^2\left[1 - \frac{4-n}{2}\ln m^2 + \mathcal{O}\left(\epsilon^2\right)\right] \\
\pi^{\frac{n}{2}-2} &= 1 - \frac{4-n}{2}\ln \pi + \mathcal{O}(\epsilon) \\
(2\pi)^{4-n} &= e^{(4-n)\ln 2\pi} = 1 + \frac{4-n}{2}\ln (2\pi)^2 + \mathcal{O}\left(\epsilon^2\right)\,.
\end{aligned} \tag{5.51}$$

By means of the above results, Eq. (5.50) becomes

$$\begin{aligned}\int \frac{d^n q}{(2\pi)^n}\frac{1}{q^2+m^2} &= \frac{\pi^2 m^2}{16\pi^4}\left[1+\epsilon\ln(4\pi)+\mathcal{O}\left(\epsilon^2\right)\right]\left[1-\epsilon\ln m^2+\mathcal{O}\left(\epsilon^2\right)\right]\\ &\times \left[-\frac{1}{\epsilon}-1+\gamma_E+\mathcal{O}(\epsilon)\right]\\ &\simeq \frac{m^2}{16\pi^2}\left[-\frac{1}{\epsilon}+\gamma_E-\ln 4\pi-1+\ln m^2\right]. \end{aligned} \tag{5.52}$$

5.6.1 Feynman parameters

Eq. (5.52) gives the $n-$dimensional contribution of the self-energy diagram for the $\lambda\varphi^4$ model. In This kind of integral is known also as the *scalar one-point integral* (because it is characterized by the presence of one denominator). The techniques used above to calculate this integral are not enough to calculate the *scalar two-point integral*, characterized by the presence of two propagators, of the kind of the integral of Eq. (5.14). To solve this integral in $n-$dimensions we use an identity due to Feynman:

$$\prod_{i=1}^{n}\frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{j=1,n}\Gamma\left(\alpha_j\right)}\int_0^1\left(\prod_{i=1}^{n}dx_i\, x_i^{\alpha_i-1}\right)\frac{\delta(1-x)}{\left(\sum_{k=1}^{n}x_k A_k\right)^{\alpha}}. \tag{5.53}$$

where α_i $(i=1,2,\ldots,n)$ are arbitrary complex numbers and

$$\begin{aligned}\alpha &= \sum_{i=1}^{n}\alpha_i\\ x &= \sum_{i=1}^{n}x_i\,. \end{aligned} \tag{5.54}$$

Let us work out Eq. (5.53) for the simplest case $n=2$:

$$\begin{aligned}\frac{1}{AB} &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 dx_1\int_0^1 dx_2\,\frac{\delta(1-x_1-x_2)}{(x_1A+x_2B)^2}\\ &= \int_0^1 dx_1\int_{-\infty}^{+\infty}dx_2\,\frac{\delta(1-x_1-x_2)\vartheta(x_2)\vartheta(1-x_2)}{(x_1A+x_2B)^2}. \end{aligned} \tag{5.55}$$

Performing the x_2 integration of Eq. (5.55) by means of the δ distribution we get

$$\frac{1}{AB} = \int_0^1 dx_1\,\frac{\vartheta(x_1)\vartheta(1-x_1)}{[x_1A+(1-x_1)B]^2} = \int_0^1 dx_1\,\frac{1}{[x_1A+(1-x_1)B]^2}. \tag{5.56}$$

Let us consider the n−dimensional version of Eq. (5.14) by using the identity of Eq. (5.56):

$$
\begin{aligned}
& \int \frac{d^nq}{(2\pi)^n} \frac{1}{(q^2+m_1^2)\left[(q+p)^2+m_2^2\right]} = \int \frac{d^nq}{(2\pi)^n} \int_0^1 dx \, \frac{1}{\left[x\left(q^2+m_1^2\right)+(1-x)\left((q+p)^2+m_2^2\right)\right]^2} \\
= & \int \frac{d^nq}{(2\pi)^n} \int_0^1 dx \, \frac{1}{\left[q^2+2(1-x)(q\cdot p)+xm_1^2+(1-x)\left(p^2+m_2^2\right)\right]^2} \\
= & \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \frac{1}{\left[q^2+2(1-x)(q\cdot p)+(1-x)^2p^2-(1-x)^2p^2+xm_1^2+p^2(1-x)+m_2^2(1-x)\right]^2} \\
= & \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \frac{1}{\left\{[q_\mu+(1-x)p_\mu]\cdot[q^\mu+(1-x)p^\mu]+m_2^2+x(m_1^2-m_2^2)+x(1-x)p^2\right\}^2}\,, \qquad (5.57)
\end{aligned}
$$

where we have used two different masses for the sake of generality. We safely exchanged also the order of integration because the integral is convergent in n dimensions. With the change of variables $Q_\mu = q_\mu + (1-x)p_\mu$, $d^nQ = d^nq$, we have

$$
\begin{aligned}
\int \frac{d^nq}{(2\pi)^n} \frac{1}{(q^2+m_1^2)\left[(q+p)^2+m_2^2\right]} &= \int_0^1 dx \int \frac{d^nQ}{(2\pi)^n} \frac{1}{\left\{Q^2+M^2\right\}^2} && (5.58) \\
M^2 &= m_2^2 + \left(m_1^2 - m_2^2\right)x + p^2x(1-x) && (5.59) \\
Q_\mu &= q_\mu + (1-x)p_\mu\,. && (5.60)
\end{aligned}
$$

The integral on the right-hand side of Eq. (5.58) is of the form $\frac{1}{(2\pi)^n} I(n,2)$:

$$
\begin{aligned}
\int \frac{d^nQ}{(2\pi)^n} \frac{1}{\left\{Q^2+M^2\right\}^2} &= \frac{\pi^2}{(2\pi)^4} \frac{\pi^{\frac{n}{2}-2}}{(2\pi)^{n-4}} \Gamma\left(\frac{4-n}{2}\right) \left(M^2\right)^{\left(\frac{n-4}{2}\right)} \\
&= \frac{1}{16\pi^2}\left[1+\epsilon\ln 4\pi - \epsilon \ln M^2\right]\left[\frac{1}{\epsilon} - \gamma_E\right] + \mathcal{O}(\epsilon) \\
&\simeq \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln M^2\right]\,. \qquad (5.61)
\end{aligned}
$$

Substituting Eq. (5.61) into Eq. (5.58), we get

$$
\begin{aligned}
\int \frac{d^nq}{(2\pi)^n} \frac{1}{(q^2+m_1^2)\left[(q+p)^2+m_2^2\right]} &= \frac{1}{16\pi^2}\left\{\Delta_{UV} - \int_0^1 dx \, \ln\left[M^2\right]\right\} && (5.62) \\
M^2 &= p^2x(1-x) + m_2^2 + \left(m_1^2 - m_2^2\right)x && (5.63) \\
\Delta_{UV} &= \frac{1}{\epsilon} - \gamma_E + \ln 4\pi\,. && (5.64)
\end{aligned}
$$

Observation 1: Eq. (5.62) is valid for general internal masses. It simplifies when we consider the particular cases of $m_1 = m_2 = m$ (as in the $\lambda\varphi^4$ model), as well as $m_1 = 0$ or $m_2 = 0$. See later for details.
Observation 2: in both Eq. (5.52) and Eq. (5.62), together with the pole term $\sim \frac{1}{\epsilon}$, the

terms γ_E and $\ln 4\pi$ appear in the same linear combination Δ_{UV} of Eq. (5.64).

Observation 3: both Eq. (5.52) and Eq. (5.62) contain the logarithm of a quantity with dimension mass^2. This is a general feature, due to the change of dimensionality. In fact, going from 4 to n dimensions the Lagrangian density changes dimensions, because the Action $\int d^n x \mathcal{L}(\varphi, \partial_\mu \varphi)$ must be dimensionless. Hence dim $[\mathcal{L}] = n$. In the $\lambda\varphi^4$ model

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu \varphi \partial_\mu \varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 \,. \tag{5.65}$$

If we consider the kinetic term (or the mass term) of $\mathcal{L}_E$

$$2\text{dim}[\varphi] + 2 = n \Longrightarrow \text{dim}[\varphi] = \frac{n-2}{2} \,. \tag{5.66}$$

Since the interaction term proportional to φ^4 has dim $= 2(n-2)$, the coupling constant λ has dimension $4-n$ (only in 4-dim it is dimensionless). In order to highlight this feature and keep a dimensionless coupling constant, it is useful to introduce an *arbitrary* mass parameter μ^{4-n}, which multiplies the coupling constant λ, which remains in this way dimensionless. In terms of this arbritrary mass parameter, the Euclidean Lagrangian density of Eq. (5.65) reads as

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu \varphi \partial_\mu \varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda \mu^{2\left(\frac{4-n}{2}\right)}}{4!}\varphi^4 = \frac{1}{2}\partial_\mu \varphi \partial_\mu \varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda \mu^{2\epsilon}}{4!}\varphi^4 \,. \tag{5.67}$$

We can rewrite the one- and two-point integrals for the scalar $\lambda\varphi^4$ model of Eq. (5.52) and Eq. (5.62) taking into account of the above arbitrary mass. It is useful to multiply the integral by the coupling constant as it appears in the corresponding Feynman diagrams and the correct statistical factor. For the diagram of Fig. (5.1) we have:

$$\begin{aligned} 12\left(\frac{-\lambda}{4!}\right)\mu^{2\left(\frac{4-n}{2}\right)}\int \frac{d^n q}{(2\pi)^n}\frac{1}{q^2+m^2} &= -\frac{\lambda}{2(2\pi)^n}\pi^{\frac{n}{2}}\frac{\Gamma\left(1-\frac{n}{2}\right)}{\Gamma(1)}\frac{(m^2)^{\left(\frac{n}{2}-1\right)}}{(\mu^2)^{\frac{n-4}{2}}} \\ &= -\frac{\lambda m^2}{32\pi^2}\left[-\Delta_{UV} - 1 + \ln\left(\frac{m^2}{\mu^2}\right)\right] \,. \end{aligned} \tag{5.68}$$

For the two-point diagram we have

$$\begin{aligned} &\lambda^2\left(\mu^2\right)^{(4-n)}\int \frac{d^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)\left((q+p)^2+m^2\right)} \\ =\ &\lambda^2\left(\mu^2\right)^{\left(\frac{4-n}{2}\right)}\int_0^1 dx\,\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}\left(M^2\right)^{\left(\frac{n}{2}-2\right)}\left(\mu^2\right)^{\left(\frac{4-n}{2}\right)} \\ =\ &\frac{\lambda^2}{16\pi^2}\left\{\Delta_{UV} - \int_0^1 dx\,\ln\left[\frac{p^2 x(1-x)+m^2}{\mu^2}\right]\right\} \,. \end{aligned} \tag{5.69}$$

With Eq. (5.68) and Eq. (5.69) we have regularized the divergent integrals of the $\lambda\varphi^4$ model. So we can now carry out the renormalization.

5.7 Loop expansion

In the analysis of the perturbative expansion of the Green functions for the model $\lambda\varphi^4(x)$, we have seen that only the only two of them contain the primitive divergences: $G_c^{(2)}$ and $G_c^{(4)}$. We note that the divergent contributions correspond to two different perturbative orders (thought of as an expansion in the coupling constant): $\mathcal{O}(\lambda)$ for $G_c^{(2)}$ and $\mathcal{O}(\lambda^2)$ for $G_c^{(4)}$, while both of them contain one loop diagrams. It is therefore necessary to understand which is the more reliable expansion parameter, between the number of loops and the coupling constant. To this aim, let us consider the functional of Eq. (4.2), where we insert explicitely the factors $\hbar$

$$\begin{aligned} Z_E[J] &= \mathcal{N}\int [d\varphi]\, e^{-\frac{1}{\hbar}S_{0E}[\varphi]}e^{-\frac{g}{\hbar}S_{int}[\varphi]}e^{\frac{\hbar}{\hbar}\int dx\, J(x)\varphi(x)} \\ &= \mathcal{N}'e^{-gS_{int}\left[\frac{\delta}{\delta J}\right]}\mathcal{N}_0\int [d\varphi]\, e^{-S_{0E}[\varphi]}e^{\int dx\, J(x)\varphi(x)} \\ &= \mathcal{N}'e^{-gS_{int}\left[\frac{\delta}{\delta J}\right]}Z_{0E}[J] \\ &= \mathcal{N}'e^{-\frac{g}{\hbar}\int dx \mathcal{L}_{int}\left[\frac{\delta}{\delta J}\right]}Z_{0E}[J]\,, \end{aligned} \tag{5.70}$$

with

$$\begin{aligned} Z_{0E}[J] &= \mathcal{N}_0\int [d\varphi(x)]e^{-\frac{1}{\hbar}\int dx\,(\mathcal{L}[\varphi(x)]-\hbar J(x)\varphi(x))} \\ &= e^{\frac{\hbar}{2}\int dx \int dy J(x)\Delta_E(x-y)J(y)}\,. \end{aligned} \tag{5.71}$$

From Eq. (5.70) we see that each interaction vertex gives a factor $\hbar^{-1}$ while, from Eq. (5.71), every propagator gives a factor $\hbar$. If we consider only the contribution of the internal lines (i.e. we consider "amputated Green functions" w.r.t. external propagators), every diagrams gives a

$$\hbar^{(I-V)} = \hbar^{L-1}\,. \tag{5.72}$$

where we have used the general relation of Eq. (5.16) $L = I - (V-1)$. So the perturbative expansion with the number of loops as parameter is preferable because it corresponds to an expansion in $\hbar$, i.e. an expansion around the classical theory. In a theory as in QED, where the interaction Lagrangian term involves only three fields, the loop expansion is equivalent to the perturbative expansion with the coupling constant as expansion parameter.

5.8 One loop renormalization of the $\lambda\varphi^4(x)$ model

5.8.1 Bare perturbation theory

Eqs. (5.68) and (5.69) give the regularized expressions for basic divergent diagrams. We can now proceed with the *renormalization* program. Let us start with the explicit calculation of the two- and four-point Green functions at one-loop. If we call $\Pi_b(p)$ the one-loop contribution to the self-energy diagram, we can write the propagator as

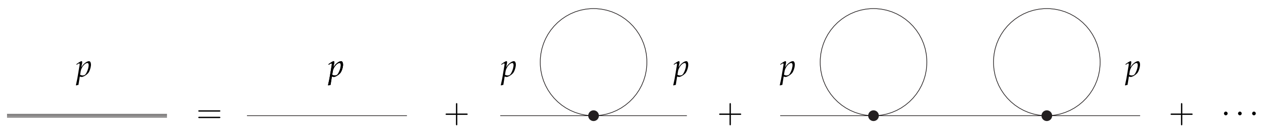

Figure 5.3: Scalar propagator in the presence of $\lambda\varphi^4$ interaction.

$$
\begin{aligned}
\hat{G}_E^{b(2)}(p,-p) = & \frac{1}{p^2+m_b^2} \\
& + \frac{1}{p^2+m_b^2}\Pi_b(p)\frac{1}{p^2+m_b^2} \\
& + \frac{1}{p^2+m_b^2}\Pi_b(p)\frac{1}{p^2+m_b^2}\Pi_b(p)\frac{1}{p^2+m_b^2} \\
& + \ldots
\end{aligned}
\tag{5.73}
$$

By considering higher-oder diagrams we would have multiple poles, which would contradict the definition of the physical mass as the (simple) pole of the propagator. Actually we can consider the iteration of the self-energy insertions as the terms of a geometric series

$$
\begin{aligned}
\hat{G}_E^{b(2)}(p,-p) &= \frac{1}{p^2+m_b^2}\sum_{n=0}^{\infty}\left(\Pi_b(p)\frac{1}{p^2+m_b^2}\right)^n \\
&= \frac{1}{p^2+m_b^2}\frac{1}{1-\frac{\Pi_b(p)}{p^2+m_b^2}} \\
&= \frac{1}{p^2+m_b^2-\Pi_b(p)},
\end{aligned}
\tag{5.74}
$$

where, considering the symmetry factor, we have

$$
\begin{aligned}
\Pi_b(p) &= \Pi_b(p^2) = 12\left(-\frac{\lambda_b}{4!}\right)\int\frac{d^nq}{(2\pi)^n}\frac{1}{q^2+m_b^2} \\
&= -\frac{\lambda_b}{32\pi^2}m_b^2\left[-\Delta_{UV}-1+\ln\left(\frac{m_b^2}{\mu^2}\right)\right],
\end{aligned}
\tag{5.75}
$$

where we have used Eq. (5.68). Since $\Pi_b(p^2)$ is a one-loop contribution, we replace the bare parameters λ_b and m_b with the corresponding physical values, λ and m:

$$\Pi_b(p^2) = -\frac{\lambda}{32\pi^2} m^2 \left[-\Delta_{UV} - 1 + \ln\left(\frac{m^2}{\mu^2}\right)\right], \tag{5.76}$$

The Minkowskian version of Eq. (5.74) is

$$\hat{G}^{b(2)}(p,-p) = \frac{i}{p^2 - m_b^2 + i\varepsilon - \Pi_b(p^2)}. \tag{5.77}$$

The above equation shows that the one-loop correction has shifted (by an "infinite" amount!) the position of the pole in the propagator from m_b^2 to $m_b^2 + \Pi_b(p^2)$. But **the position of the pole gives the mass of the particle**. At tree-level m_b is the mass of the particle, which is also the parameter multiplying the quadratic term in the field $\varphi(x)$ in the Lagrangian density. At one-loop order the pole is shifted and also the relation between the lagrangian parameter m_b and the mass of the particle is changed: the lagrangian parameter m_b is called the **bare mass**, while the position of the pole in the propagator gives the **renormalized (or physical) mass** m_R. The relation between the two quantities is (at one-loop order)

$$m_R^2 = m_b^2 + \Pi_b(m_R^2), \tag{5.78}$$

where at one-loop order $\Pi_b(m_R^2) = \Pi_b(m_b^2)$, because they differ by higher-order terms. m_b is the bare mass, which is said to be "infinite" because it receives an infinite correction from $\Pi_b(m_R^2)$ (in the limit $\varepsilon \to 0$) and the two quantities sum up to the physical mass.
Observation: the relation between the lagrangian parameter m_b and the physical mass m_R depends on the perturbative order of the calculation. Using Eq. (5.78) and a series expansion of $\Pi(p^2)$ around m_R^2, we can rewrite Eq. (5.77) as follows[3]

$$\begin{aligned} \hat{G}^{b(2)}(p,-p) &= \frac{i}{p^2 - m_R^2 + i\varepsilon + \Pi_b(m_R^2) - \Pi_b(p^2)} \\ &= \frac{1}{p^2 - m_R^2 - (p^2 - m_R^2)\frac{d\Pi_b(p^2)}{dp^2}|_{p^2=m_R^2} + \ldots + i\varepsilon}, \end{aligned} \tag{5.79}$$

where, in the last term, the ellipses stand for additional terms (powers of $(p^2 - m_R^2)$) which give no contribution to the pole of the propagator. We should remember that the physical theoretical predictions for S matrix elements are obtained as on-shell limits $(p^2 - m_R^2)$ of the amputated Green functions and for this reason we can neglect the higher order terms of the expansion. So Eq. (5.79) becomes

$$\hat{G}^{b(2)}(p,-p) = \frac{i}{(p^2 - m_R^2 + i\varepsilon)\left(1 - \frac{d\Pi(p^2)}{dp^2}|_{p^2=m_R^2}\right)}. \tag{5.80}$$

[3]Actually, in our calculation $\Pi_b(p^2)$ is a constant as a function of p^2 and the following steps are redundant. However, this feature is not true for other interactions and also in the $\lambda\varphi^4$ model at two loops. For this reason we proceed illustrating how the residuum of the propagator can be changed at higher orders.

We can see from the above equation that also the normalization of the propagator has changed by the factor $1/(1 - \frac{d\Pi(p^2)}{dp^2}|_{p^2=m_R^2}$. This does not come as a surprise, since we have seen in Section (4.2.5) that the general structure of the propagator foresees a factor $\sqrt{Z}$ as normalization, which is one at tree-level, but can be different at higher orders. In our case under study, however, the derivative is zero.

Next we come to the 1PI-four-point Green function, whose one-loop approximation is given in Fig. 5.4.

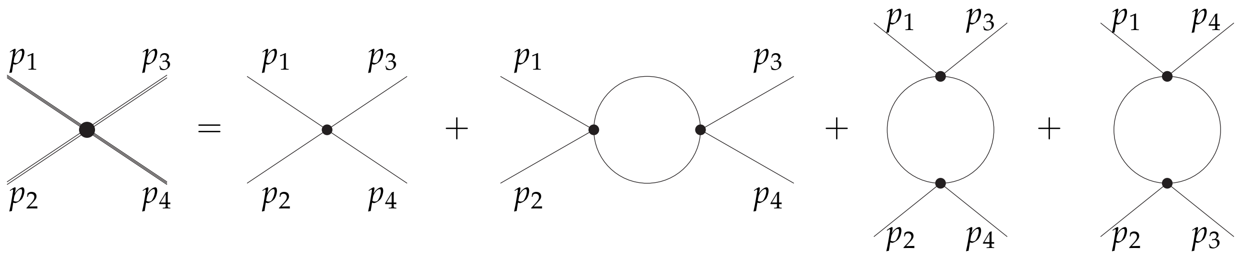

Figure 5.4: Scalar four-point vertex function in the presence of interaction, at one loop order.

The "effective" diagram on the left-hand side in Fig. 5.4 is the 1PI-four-point vertex function $\hat{\Gamma}^{b(4)}(p_1, p_2, p_3, p_4)$, where the momenta are taken incoming $(p_1 + p_2 + p_3 + p_4 = 0)$. If we calculate the second diagram of the right-hand side of Fig. 5.4, using the results of Eq. 5.69, we have:

$$
\begin{aligned}
& \frac{1}{2}(-\lambda_b)^2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{\left(q^2 + m_b^2\right)} \frac{1}{\left((q+p)^2 + m_b^2\right)} \\
= \; & \frac{\lambda_b^2}{32\pi^2} \left\{ \Delta_{UV} - \int_0^1 dx \, \ln\left[\frac{p^2 x(1-x) + m^2}{\mu^2}\right] + \mathcal{O}(\varepsilon) \right\} \\
= \; & \frac{\lambda_b^2}{32\pi^2} \left\{ \Delta_{UV} - \ln\frac{m^2}{\mu^2} + 2 - \sqrt{1 + 4\frac{m^2}{p^2}} \ln\left[\frac{1 + \sqrt{4\frac{m^2}{p^2} + 1}}{\sqrt{1 + 4\frac{m^2}{p^2}} - 1}\right] + \mathcal{O}(\varepsilon) \right\} . \qquad (5.81)
\end{aligned}
$$

The above expression becomes very simple in the limit $m \to 0$:

$$
\frac{\lambda_b^2}{32\pi^2} \left\{ \Delta_{UV} - \ln\frac{p^2}{\mu^2} + 2 \right\} . \qquad (5.82)
$$

The third and fourth diagrams of the right-hand side of Fig. 5.4 give formally the same expression, but with different values of p^2 as a function of the external momenta:

$$
\begin{aligned}
p^2 &= (p_1 + p_2)^2 \qquad I \text{ diagram} \\
p^2 &= (p_1 + p_3)^2 \qquad II \text{ diagram} \\
p^2 &= (p_1 + p_4)^2 \qquad III \text{ diagram} .
\end{aligned}
$$

Thus we have:

$$\hat{\Gamma}^{b(4)}(p_1, p_2, p_3, p_4) = -\lambda_b + \frac{3\lambda^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u) + \mathcal{O}(\varepsilon)\right\} + \mathcal{O}(\lambda^3)\,,$$
$$A(s,t,u) = \sum_{p^2=s,t,u} \ln\left[\frac{1+\sqrt{4\frac{m^2}{p^2}+1}}{\sqrt{1+4\frac{m^2}{p^2}}-1}\right]\,. \tag{5.83}$$

Before going on, we need to think about the physical meaning of the expression "measuring the coupling constant λ". We need to refer to an ideal scattering experiment between two scalar particles described by the field φ. The tree level matrix element is

$$\mathcal{M} = -i\lambda \tag{5.84}$$

The cross section is directly proportional to λ^2, which is fixed ("measured") by equating the theoretical prediction with the (ideal) experimental data. If we name λ_R the *renormalized/physical* coupling constant, we have, <u>at tree level</u>

$$\lambda_R = \lambda\,. \tag{5.85}$$

If we push the theoretical prediction at one-loop order, Eq. (5.85) becomes

$$-i\lambda_R = -i\lambda + i\frac{3\lambda^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s_0,t_0,u_0)\right\}\,, \tag{5.86}$$

where s_0, t_0 and u_0 refer to a particular kinematical point. We can use the above relation of Eq. (5.86) to express the Lagrangian parameter λ as (at one-loop order we can identify λ with λ_R):

$$-i\lambda = -i\lambda_R - i\frac{3\lambda_R^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s_0,t_0,u_0)\right\} \tag{5.87}$$

Eq. (5.87) is the defining equation of λ at on-loop order, through the chosen particular (**arbitrary**) kinematical point. Equipped with the above definition, we can make a theoretical prediction for other generic kinematical points:

$$\mathcal{M} = -i\lambda + i\frac{3\lambda_R^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right\}\,. \tag{5.88}$$

If we insert Eq. (5.87) in Eq. (5.88), we obtain the following theoretical prediction for the physical amplitude at a generic kinematical point specified by the Mandelstam variables s, t, u:

$$\begin{aligned}\mathcal{M} &= -i\lambda_R - i\frac{3\lambda_R^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s_0,t_0,u_0)\right\}\\ &\quad + i\frac{3\lambda_R^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right\}\\ &= -i\lambda_R - i\frac{3\lambda_R^2}{32\pi^2}\left\{A(s_0,t_0,u_0) - A(s,t,u)\right\}\,,\end{aligned} \tag{5.89}$$

where, in the last term, we have identified again λ and λ_R at one-loop order.
Observation 1: the physical prediction of Eq. (5.89) is independent of the UV parameter Δ_{UV}. For this to happen, it is crucial that the coefficient of the pole $\frac{1}{\varepsilon}$ is independent of any external momentum, otherwise the whole renormalization procedure would be spoiled.
Observation 2: the kinematical point s_0, t_0, u_0 is the **subtraction point** where the lagrangian parameter is defined. The choice of the subtraction points defines a particular *renormalization scheme*.
Observation 3: as for the case of the mass parameter, the relation between the lagrangian parameter λ and physical coupling constant λ_R depends on the perturbative order of the calculation.

Summary of the bare renormalization procedure
We sketch here the necessary steps to compute a physical S matrix element involving divergent diagrams (this is valid in general, not only for the $\lambda\varphi^4$ model):

- compute the diagrams within the UV regularized theory, to obtain an expression that depends on the bare parameters (mass and coupling);
- compute the physical parameters in terms of the bare ones, to the perturbative order consistent with the rest of the calculation;
- compute the residue of the propagators of the external lines, again to the needed perturbative order which is consistent with the whole calculation;
- each of the above expressions depends on the bare parameters, the UV cutoff (and the arbitrary mass μ connected with the dimensional regularization). Combining all the expressions consistently, from the perturbative point of view, and eliminating the bare parameters in favor of the renormalized ones, the resulting expression does not depend on the UV regulator (ε and μ in dimensional regularization). At this point the limit to four spacetime dimensions can safely be taken and the theoretical prediction depends only on physical quantities.
- This approach guarantees finite S matrix elements but not Green functions.

5.8.2 Renormalized perturbation theory

Another approach to implement the renormalization program is to rewrite the bare Lagrangian denisity in terms of physical (or renormalized) quantities, by means of multiplicative constants:

$$\varphi_b(x) = Z_\varphi^{\frac{1}{2}}\varphi(x) \tag{5.90}$$

$$m_b = Z_m^{\frac{1}{2}} m \tag{5.91}$$

$$\lambda_b = \mu^{2\varepsilon}\frac{Z_\lambda}{Z_\varphi^2}\lambda\,, \tag{5.92}$$

where

$$Z_i = Z_i\left(\lambda, \frac{m}{\mu}, \varepsilon\right) . \tag{5.93}$$

The dependence of Z_i on m and μ is only through their ratio because, by definition, Z_i is a dimensionless constant.
By means of Eqs.(5.90,5.91,5.92), we can write

$$\begin{aligned}\mathcal{L}_E &= \frac{1}{2}\partial_\mu\varphi_b\partial_\mu\varphi_b + \frac{m_b^2}{2}\varphi_b^2 + \frac{\lambda_b}{4!}\varphi_b^4 \\ &= \frac{1}{2}Z_\varphi\partial_\mu\varphi\partial_\mu\varphi + \frac{m^2}{2}Z_m Z_\varphi\varphi^2 + \frac{\lambda}{4!}Z_\lambda Z_\varphi^2\varphi^4 .\end{aligned} \tag{5.94}$$

The renormalized parameters Z_i are calculated perturbatively as series expansions in λ. At tree-level $Z_i = 1$, i.e. without interaction we do not need renormalization:

$$Z_i = 1 + \delta Z_i . \tag{5.95}$$

Rewriting the Z_i factors in Eq. (5.94) according to Eq. (5.95), the bare Lagrangian density $\mathcal{L}_E$ is split into a renormalized Lagrangian density and a counterterm Lagrangian density

$$\mathcal{L}_E = \mathcal{L}_E^{\text{ren}} + \mathcal{L}_E^{\text{c.t.}} , \tag{5.96}$$

with

$$\mathcal{L}_E^{\text{ren}} = \frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 , \tag{5.97}$$

$$\mathcal{L}_E^{\text{c.t.}} = \frac{1}{2}\delta Z_\varphi\partial_\mu\varphi\partial_\mu\varphi + \frac{m^2}{2}\left(Z_m Z_\varphi - 1\right)\varphi^2 + \frac{\lambda}{4!}\left(Z_\lambda Z_\varphi^2 - 1\right)\varphi^4 . \tag{5.98}$$

$\mathcal{L}_E^{\text{ren}}$ is expressed in terms of renormalized/physical fields and parameters. It gives rise formally to the same Feynman rules as the bare Lagrangian density. $\mathcal{L}_E^{\text{c.t.}}$, the counterterm Lagrangian density, gives additional Feynman graphs which cancel the divergences from the Green functions. *The fact that the theory is renormalizable means that alla Green functions can be rendered finite with a finite numbers of counterterms. Moreover, since the bare Lagrangian density is not changed, the required counterterms, in terms of fields, have to be of the form of the terms already present in the bare Lagrangian density.* From the previous subsection, we could already write the expressions for the counterterms δZ_i but we proceed looking at the conditions coming from the requirement of finiteness of the renormalized Green functions.

Before going on, however, we need to establish the relation between generating functional of the bare theory and the one of the renormalized theory. The Green functions of the bare theory are generated by the functional of Eq. (4.2), which we report here

$$Z_E^b[J_b] = \mathcal{N}_b \int [d\varphi_b]\, e^{-S_E[\varphi_b]} e^{\int d^n x\, J_b(x)\varphi_b(x)} , \tag{5.99}$$

where we have added the subscripts b and the integration on spacetime is performed in n dimensions. Since the Lagrangian density of the renormalized theory is obtained by the one of the bare theory, with the change of variables and parameters of Eqs. (5.90,5.91,5.92), we have

$$S_E[\varphi_b] = S_E[\varphi] \,. \tag{5.100}$$

The generating functional of the renormalized theory is defined, in n dimensions, by

$$Z_E[J] = \mathcal{N} \int [d\varphi] \, e^{-S_E[\varphi]} e^{\int d^n x \, J(x)\varphi(x)} \,. \tag{5.101}$$

In Eq. (5.101) we can perform the integration variable substitution given by Eq. (5.90). Considering that the functional Jacobian can be absorbed in the overall normalization constant, since it is a constant (it does not depend on φ), and considering Eq. (5.100), we have

$$Z_E[J] = \mathcal{N}' \int [d\varphi_b] \, e^{-S_E[\varphi_b]} e^{\int d^n x \, J(x) Z_\varphi^{-\frac{1}{2}} \varphi_b(x)} \,. \tag{5.102}$$

By inspection of Eq. (5.102), if we assume

$$J_b = Z_\varphi^{-\frac{1}{2}} J \,, \tag{5.103}$$

we obtain that $Z_E^b[J_b]$ and $Z_E[J]$ are proportional. Moreover $J = 0 \Longrightarrow J_b = 0$. Since both generating functional are normalized in the same way, $Z_E^b[0] = 1$ and $Z_E[0] = 1$, we get the relation between the generating functional Z of the bare theory and the one of the renormalized theory:

$$Z_E[J] = Z_E^b\left[J_b\right] \equiv Z_E^b\left[Z_\varphi^{-\frac{1}{2}} J\right] \,. \tag{5.104}$$

An analogous relation holds for the generating functionals of connected Green functions: taking the logarithms of both members of Eq. (5.104), we get

$$W_E[J] = W_E^b\left[J_b\right] \,. \tag{5.105}$$

In order to find the analogous relation for the effective Action, we observe that, thanks to Eq. (5.103) and Eq. (5.105), we can write

$$\varphi_{bc}(x) = \frac{\delta W_E^b[J_b]}{\delta J_b(x)} = Z_\varphi^{\frac{1}{2}} \frac{\delta W_E[J]}{\delta J(x)} = Z_\varphi^{\frac{1}{2}} = \varphi_c(x) \,. \tag{5.106}$$

As a consequence we have

$$\Gamma_E^b\left[\varphi_{bc}\right] \equiv W_E^b\left[J_b\right] - \int d^n x J_b(x)\varphi_{bc}(x) = W_E\left[J\right] - \int d^n x J(x)\varphi_c(x) \equiv \Gamma_E\left[\varphi_c\right] \,. \tag{5.107}$$

Since Eqs. (5.104, 5.105, 5.107) give the relations between bare and renormalized functionals, they generate important relations between the corresponding Green functions.

For instance, we can search for the relation between the n-point bare Green function and the renormalized one:

$$G_E^{b\,(n)}(x_1,\ldots,x_n) \equiv \frac{\delta^n Z_E^b\,[J_b]}{\delta J_b(x_1)\ldots\delta J_b(x_n)}|_{J_b=0} = Z_\varphi^{\frac{n}{2}} \frac{\delta^n Z_E\,[J]}{\delta J(x_1)\ldots\delta J(x_n)}|_{J=0} \equiv Z_\varphi^{\frac{n}{2}} G_E^{b\,(n)}(x_1,\ldots,x_n)\,. \tag{5.108}$$

A completely analogous relation holds also for the connected Green functions generated by the W functional:

$$W_E^{b\,(n)}(x_1,\ldots,x_n) \equiv \frac{\delta^n W_E^b\,[J_b]}{\delta J_b(x_1)\ldots\delta J_b(x_n)}|_{J_b=0} = Z_\varphi^{\frac{n}{2}} \frac{\delta^n W_E\,[J]}{\delta J(x_1)\ldots\delta J(x_n)}|_{J=0} \equiv Z_\varphi^{\frac{n}{2}} W_E^{b\,(n)}(x_1,\ldots,x_n)\,. \tag{5.109}$$

For the 1PI Green functions, generated by the effective Action, we have

$$\Gamma_E^{b\,(n)}(x_1,\ldots,x_n) \equiv \frac{\delta^n \Gamma_E^b\,[\varphi_{bc}]}{\delta\varphi_{bc}(x_1)\ldots\delta\varphi_{bc}(x_n)} = Z_\varphi^{-\frac{n}{2}} \frac{\delta^n \Gamma_E\,[\varphi_c]}{\delta\varphi_c(x_1)\ldots\delta\varphi_c(x_n)} \equiv Z_\varphi^{-\frac{n}{2}} \Gamma_E{}^{(n)}(x_1,\ldots,x_n)\,. \tag{5.110}$$

The relations of Eqs. (5.108, 5.110) hold also in momentum representation.
In order to understand the meaning of renormalization, it is useful to write more explicitely the dependencies on the arguments in Eqs. (5.108,5.110). Let us take for instance Eq. (5.108):

$$G_E{}^{(n)}(x_1,\ldots,x_n;\lambda,m,\mu,\varepsilon) = Z_\varphi^{-\frac{n}{2}}\left(\lambda,\frac{m}{\mu},\varepsilon\right) G_E^{b\,(n)}(x_1,\ldots,x_n;\lambda_b(\lambda,m,\mu,\varepsilon),m_b(\lambda,m,\mu,\varepsilon),\varepsilon)\,. \tag{5.111}$$

The aim of the renormalization procedure is to determine the Z_i functions of Eqs. (5.90, 5.91, 5.92) in such a way that the renormalized Green functions are finite in the limit $\varepsilon \to 0$, order by order in perturbation theory. The right-hand side of Eq. (5.111) shows that the power series expansion in λ is different, order by order, from the power series expansion in λ_b. In order to determine the counterterms, we can use the results of the previous section. For example, let us consider the 1PI-two point Green function, which is minus the inverse of the propagator and use the result of Eq. (5.76):

$$\begin{aligned}
(\hat{\Gamma})_E^{b\,(2)}(p^2) &= -p^2 - m_b^2 + \Pi_b(p^2) \\
&= -p^2 - m^2(1+\lambda\delta Z_m) - \frac{\lambda m^2}{32\pi^2}\left[-\Delta_{UV} - 1 + \ln\frac{m^2}{\mu^2}\right],
\end{aligned} \tag{5.112}$$

where, with respect to Eq. (5.95), we have factored out λ from δZ_m. In the following will do the same also for δZ_φ and δZ_λ. Using Eq. (5.110) we have

$$\begin{aligned}
(\hat{\Gamma})_E^{(2)}(p^2) &= Z_\varphi(\hat{\Gamma})_E^{b\,(2)}(p^2) \\
&= (1+\lambda\delta Z_\varphi)\left\{-p^2 - m^2(1+\lambda\delta Z_m) - \frac{\lambda m^2}{32\pi^2}\left(-\Delta_{UV} - 1 + \ln\frac{m^2}{\mu^2}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -p^2 - m^2 - \lambda\delta Z_\varphi p^2 - \lambda m^2 \Bigg[\delta Z_\varphi + \delta Z_m \\
&\quad + \frac{1}{32\pi^2}\left(-\Delta_{UV} - 1 + \ln\frac{m^2}{\mu^2}\right)\Bigg] , \qquad (5.113)
\end{aligned}
$$

where we have neglected terms of order λ^2, consistently with the perturbative accuracy of our calculation (otherwise we should have included also terms of $\mathcal{O}(\lambda_b^2)$, i.e. two loop terms, in the evaluation of $\Pi_b(p^2)$). Eq. (5.113) allows to fix the counterterms δZ_φ and δZ_m by requiring that the 1PI-two-point Green function be finite. **While the divergent part of the counterterms is uniquely determined, the choice of the finite part amounts to the choice of a renormalization scheme.** We illustrate in this case a renormalization scheme different from the one adopted in the case of bare perturbation theory, i.e. we use the *modified minimal subtraction scheme ($\overline{MS}$)*[4]. By inspection of Eq. (5.113), there is no term proportional to $p^2\Delta_{UV}$, so δZ_φ is finite. According to our chosen scheme $\delta Z_\varphi = 0$. Instead, δZ_m has to cancel the term proportional to Δ_{UV}. Hence, choosing

$$
\begin{aligned}
\delta Z_\varphi &= 0 , \qquad (5.114)\\
\delta Z_m &= \frac{1}{32\pi^2}\Delta_{UV} , \qquad (5.115)
\end{aligned}
$$

the 1PI-two-point Green function of Eq. (5.113) becomes

$$
(\hat{\Gamma})_E^{(2)}(p^2) = -p^2 - m^2\left[1 - \frac{\lambda}{32\pi^2}\left(-1 + \ln\frac{m^2}{\mu^2}\right)\right] , \qquad (5.116)
$$

We notice that, in this scheme, the renormalized mass m does not coincide with the physical mass, which is by definition the pole of the propagator (or equivalently the zero of the 1PI-two-point Green function):

$$
m^2_{\text{phys}} = m^2\left[1 - \frac{\lambda}{32\pi^2}\left(-1 + \ln\frac{m^2}{\mu^2}\right)\right] . \qquad (5.117)
$$

In order to fix the remaining counterterm δZ_λ we need to consider 1PI-four point Green function. We report the result of Eq. (5.83) on the bare Green function[5]:

$$
\hat{\Gamma}_E^{b(4)}(p_1, p_2, p_3, p_4) = -\lambda_b + \frac{3\lambda^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right\} . \qquad (5.118)
$$

[4]The idea of this scheme is to absorb in the counterterms only the ultraviolet quantity Δ_{UV}. It is useful when there is no precisely determined physical mass scale (as, on the contrary, will be the case in QED) in the theory to be renormalized. For this reason this scheme is commonly used in the renormalization of Quantum Chromodynamics of the strong interactions. The original proposal, introduced by G. 't-Hooft, was the *minimal subtraction scheme*, where the counterterms contain only the terms proportional to the pole $1/\varepsilon$.

[5]In Eq. (5.83) we used the minkowskian Green function, while here we are using the euclidean one. For this case, however, the result is the same.

We also have

$$\hat{\Gamma}_E^{(4)}(p_1,p_2,p_3,p_4) = Z_\varphi^2 \hat{\Gamma}_E^{b(4)}(p_1,p_2,p_3,p_4) = \hat{\Gamma}_E^{b(4)}(p_1,p_2,p_3,p_4)\,, \tag{5.119}$$

because $Z_\varphi = 1$ from Eq. (5.114). Hence we can write

$$\hat{\Gamma}_E^{(4)}(p_1,p_2,p_3,p_4) = -\lambda_b + \frac{3\lambda^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right\}\,. \tag{5.120}$$

Inserting Eq. (5.92) we have

$$\begin{aligned}\hat{\Gamma}_E^{(4)}(p_1,p_2,p_3,p_4) &= -\lambda_b + \frac{3\lambda^2}{32\pi^2}\left\{\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right\} \\ &= -\lambda\mu^{2\varepsilon}\left\{1 + \lambda\delta Z_\lambda - \frac{3\lambda}{32\pi^2}\left[\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right]\right\} \\ &= -\lambda\mu^{2\varepsilon} + \lambda^2\mu^{2\varepsilon}\left\{-\delta Z_\lambda + \frac{3}{32\pi^2}\left[\Delta_{UV} - \ln\frac{m^2}{\mu^2} - \frac{1}{3}A(s,t,u)\right]\right\}\,.\end{aligned} \tag{5.121}$$

In order to make finite $\hat{\Gamma}_E^{(4)}$, according to the $\bar{M}S$ renormalization scheme, we have to put

$$\delta Z_\lambda = \frac{3\lambda}{32\pi^2}\Delta_{UV}\,. \tag{5.122}$$

With the above choice, the 1PI-four-point Green function becomes

$$\hat{\Gamma}_E^{(4)}(p_1,p_2,p_3,p_4) = -\lambda - \frac{3}{32\pi^2}\lambda^2\left[\frac{1}{3}A(s,t,u) + \ln\frac{m^2}{\mu^2}\right]\,. \tag{5.123}$$

To summarize, in one-loop approximation, the relations between bare and renormalized parameters of Eqs.(5.90,5.91,5.92), in the $\bar{M}S$ renormalization scheme, are the following ones

$$\lambda_b = \mu^{2\varepsilon}\left(\lambda + \frac{3}{32\pi^2}\lambda^2\Delta_{UV}\right)\,, \tag{5.124}$$

$$m_b^2 = m^2\left(1 + \frac{\lambda}{32\pi^2}\Delta_{UV}\right)\,, \tag{5.125}$$

$$Z_\varphi = 1\,, \tag{5.126}$$

with $\Delta_{UV} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi$.

6

QED radiative corrections

In this chapter we will treat the renormalization of quantum electrodynamics (QED). The theory is defined by the Maxwell-Dirac Lagrangian plus the gauge fixing Lagrangian, as we saw in the treatment of the free electromagnetic field:

$$\mathcal{L}_{QED} = \mathcal{L}_{MD} + \mathcal{L}_{GF}\,, \tag{6.1}$$

where

$$\begin{aligned}
\mathcal{L}_{MD} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}_b(x)\gamma^\mu \left(\partial_\mu + ie_b A_{b\mu}(x)\right)\psi_b(x) - m_b\bar{\psi}_b(x)\psi_b(x)\,,\\
\mathcal{L}_{GF} &= -\frac{1}{2\xi_b}\left(\partial_\mu A_b^\mu(x)\right)^2 .
\end{aligned} \tag{6.2}$$

We remind that the general expression for the photon propagator (in momemtum representation) is

$$\hat{D}_{\mu\nu}(k) = -\frac{1}{k^2 + i\varepsilon}\left[g_{\mu\nu} + (\xi_b - 1)\,\frac{k_\mu k_\nu}{k^2}\right] . \tag{6.3}$$

In the following section we present the superficial degree of divergence, as given by power counting arguments. Then we define the generator functionals and we present the functional form of the Ward-Takahashi Identity (which is a direct consequence of the gauge symmetry), together with Ward identities obtained by subsequent functional derivation. In Section 6.3 we discuss the renormalization of QED through the multiplicative renormalization approach and present the general conditions which define the renormalization constants in the *on-shell renormalization scheme*. Section 6.4 is devoted to the explicit calculation of the QED renormalization constants at one-loop, which allows to check at one-loop the QED Ward Identity. In Section 6.5 we present the one-loop calculation of the anomalous magnetic moment of the electron, which is one of the quantities where the agreement between experiment and perturbative theoretical calculations has reached its highest level, showing the predictive power of QED as a quantum field theory.

In the calculations of Sec. 6.4 another regularization has to be introduced, due to the masslessness of the photon. In fact this feature is the source of the *Infrared Divergences* of QED. The last Section of this chapter presents an introductory discussion of this kind of divergences and how to treat them in a perturbative calculation. The reference scattering process is $e^+e^- \to \mu^+\mu^-$.

6.1 Power counting in QED

Given a Green function with E external fermionic lines and P external photons, the superficial degree of divergence is

$$d \equiv d(E,P) = 4 - \frac{3}{2}E - P\,. \tag{6.4}$$

By inspection we have

$$\begin{aligned} d(0,2) &= 2 \\ d(0,4) &= 0 \\ d(2,0) &= 1 \\ d(2,1) &= 0\,. \end{aligned} \tag{6.5}$$

All other Green functions are convergent. In the above Eq. (6.5) we have not included $D(0,3)$ because the corresponding Green function is zero, because of charge conjugation invariance. In fact, the effect of a charge conjugation transformation is to change sign to the field $A_\mu(x)$

$$A_\mu(x) \to A'_\mu(x) = -A_\mu(x)\,. \tag{6.6}$$

Since QED has the gauge symmetry, the superficial degree of divergence of Eq. (6.5) is an overestimate of the true degree of divergence. In fact, due to the gauge symmetry, there are cancellations among particular contributions to the Feynman diagrams of a given Green function, lowering the degree of divergence. In particular, the loop contributions to the photon propagator $(0,2)$ and to the electron propagator $(2,0)$ give only logarithmic divergencies.

By gauge invariance also the Green function with four external photons $(0,4)$ is not logarithmically divergent but finite. Were it divergent, QED would not be renormalizable because the Maxwell-Dirac Lagrangian does not contain a term corresponding to the interaction of four photons already at tree level.

As a consequence, **the only UV divergent Green functions (the primitive divergencies) in QED are the two propagators (of the electron and of the photon) and the $ee\gamma$ vertex**.

6.2 The generating functionals for QED

We will define the generating functionals directly with Minkowskian time. Let us define the "effective" Lagrangian density

$$\mathcal{L}_{eff}(x) = \mathcal{L}_{MD}(x) + \mathcal{L}_{GF}(x) + J^{\mu}_{b}(x)A_{b\mu}(x) + \bar{\eta}_b(x)\psi_b(x) + \bar{\psi}_b(x)\eta_b(x)\,, \tag{6.7}$$

where J^{μ}_{b}, $\bar{\eta}_b$ and η_b are the classical currents associated with the electromagnetic field and the Dirac fields ψ and $\bar{\psi}$, respectively, and $\mathcal{L}_{MD}$ and $\mathcal{L}_{GF}$ are defined in Eq. (6.2). ψ, $\bar{\psi}$ η, and $\bar{\eta}$ are four-components fields with values in a Grassmann algebra and are therefore anticommunting. The Z functional is defined through a functional integration of the exponential of $\mathcal{L}_{eff}$ over the three fields:

$$Z_b\left[J^{\mu}_{b}, \eta_b, \bar{\eta}_b\right] = \mathcal{N}_b \int \left[dA_{b\mu}\right]\left[d\bar{\psi}_b\right]\left[d\psi_b\right] e^{i\int dx \mathcal{L}_{eff}}\,, \tag{6.8}$$

where the dimensional regularization is understood. With Minkowskian time, the relation between $n-$point Green functions and the generating functional is

$$(i)^n\langle 0|T\left[A_{b\mu_1}(x_1)\ldots A_{b\mu_n}(x_n)\right]|0\rangle = \frac{1}{Z_b}\frac{\delta^n Z}{\delta J^{\mu_1}(x_1)\ldots\delta J^{\mu_n}(x_n)}_{|0}\,. \tag{6.9}$$

The classical field is defined as

$$A^{c}_{b\mu}(x) = -i\frac{1}{Z_b}\frac{\delta Z_b}{\delta J^{\mu}_{b}(x)} \tag{6.10}$$

and the propagator is defined as

$$D_{\mu\nu b}(x-y) \equiv -i\langle 0|T\left[A_{b\mu}(x)A_{b\nu}(y)\right]|0\rangle = i\frac{1}{Z_b}\frac{\delta^2 Z_b}{\delta J^{\mu}_{b}(x)\delta J^{\nu}_{b}(y)}\,. \tag{6.11}$$

The relation between the generating functional Z and the generating functional of the connected Green functions W is

$$Z_b = e^{iW_b} \quad \text{or equivalently} \quad W_b = -i\ln Z_b\,. \tag{6.12}$$

Expressing the classical field and the propagator in terms of the functional W, we get

$$A^{c}_{b\mu}(x) = -i\frac{1}{Z_b}\frac{\delta Z_b}{\delta J^{\mu}_{b}(x)} = \frac{\delta W_b}{\delta J^{\mu}_{b}(x)}\,, \tag{6.13}$$

and

$$D_{\mu\nu b}(x-y) \equiv -i\langle 0|T\left[A_{b\mu}(x)A_{b\nu}(y)\right]|0\rangle = i\frac{1}{Z_b}\frac{\delta^2 Z_b}{\delta J^{\mu}_{b}(x)\delta J^{\nu}_{b}(y)} = -\frac{\delta^2 W_b}{\delta J^{\mu}_{b}(x)\delta J^{\nu}_{b}(y)}\,. \tag{6.14}$$

6.2.1 Functional form of the Ward-Takahashi Identity

$\mathcal{L}_{MD}$ of Eq. (6.2) is invariant under the gauge transformation of the fields

$$\begin{aligned} A_{b\mu}(x) &= A'_{b\mu}(x) + \partial_\mu \delta\Lambda(x) \\ \psi_b(x) &= e^{-ie\delta\Lambda(x)}\psi'_b(x) \simeq \psi'_b(x) - ie\delta\Lambda(x)\psi'_b(x) \\ \bar{\psi}_b(x) &= e^{+ie\delta\Lambda(x)}\bar{\psi}'_b(x) \simeq \bar{\psi}'_b(x) + ie\delta\Lambda(x)\bar{\psi}'_b(x) \end{aligned} \tag{6.15}$$

As a consequence of gauge invariance the functional Z_b must satisfy an identity, which can be demonstrated as follows. We can consider the gauge tranformation of Eq. (6.15) as a change of integration variables in the functional integration of Eq. (6.8). Since $\left[dA_{b\mu}\right] = \left[dA'_{b\mu}\right]$, $[d\psi_b] = e^{-ie\delta\Lambda}\left[d\psi'_b\right]$ and $[d\bar{\psi}_b] = e^{-ie\delta\Lambda}\left[d\bar{\psi}'_b\right]$, the Jacobian of the transformation is equal to one. Furthermore, while S_{MD} is gauge invariant, S_{GF} changes after the transformation of Eq. (6.15) by a quantity δS_{GF} given by:

$$\delta S_{GF} = -\frac{1}{\xi_b}\int dx\,\left(\partial^\mu A_{b\mu}(x)\right)\Box\delta\Lambda(x) = -\frac{1}{\xi_b}\int dx\,\delta\Lambda(x)\Box\left(\partial^\mu A_{b\mu}(x)\right)\,, \tag{6.16}$$

where in the last step we have used two differentiations by parts. The variation of the effective action S_{eff} can therefore be written as

$$\begin{aligned} \delta S_{eff} &= \int dx\,\delta\Lambda(x) C_{eff}(x) \quad \text{with} \\ C_{eff} &= \left[-\frac{1}{\xi_b}\Box\left(\partial^\mu A_{b\mu}(x)\right) - \partial_\mu J^\mu_b(x) - ie_b\left(\bar{\eta}_b(x)\psi_b(x) - \bar{\psi}_b(x)\eta_b(x)\right)\right]. \end{aligned} \tag{6.17}$$

Eq. (6.8), after the gauge transformation of Eq. (6.15), becomes

$$\begin{aligned} Z_b\left[J^\mu_b, \eta_b, \bar{\eta}_b\right] &= \mathcal{N}_b\int\left[dA_{b\mu}\right][d\bar{\psi}_b][d\psi_b]\; e^{iS_{eff}+\delta S_{eff}} \\ &= \mathcal{N}_b\int\left[dA_{b\mu}\right][d\bar{\psi}_b][d\psi_b]\; e^{iS_{eff}}\left(1 + i\delta S_{eff}\right) \\ &= Z_b\left[J^\mu_b, \eta_b, \bar{\eta}_b\right] + i\mathcal{N}_b\int\left[dA_{b\mu}\right][d\bar{\psi}_b][d\psi_b]\; e^{iS_{eff}}\int dx\,\delta\Lambda(x) C_{eff} \end{aligned} \tag{6.18}$$

Since $\delta\Lambda(x)$ is arbitrary, Eq. (6.18) implies the identiy

$$\mathcal{N}_b\int\left[dA_{b\mu}\right][d\bar{\psi}_b][d\psi_b]\; e^{iS_{eff}}\left[-\frac{1}{\xi_b}\Box\left(\partial^\mu A_{b\mu}(x)\right) - \partial_\mu J^\mu_b(x) - ie_b\left(\bar{\eta}_b(x)\psi_b(x) - \bar{\psi}_b(x)\eta_b(x)\right)\right] = 0\,. \tag{6.19}$$

We can bring $A_{b\mu}$, ψ_b and $\bar{\psi}_b$ outside the functional integration, in agreement with Eq. (6.17), with the substitutions

$$\begin{aligned} A_{b\mu} &\to -i\frac{\delta}{\delta J^\mu_b} \\ \psi_b &\to -i\frac{\delta}{\delta\bar{\eta}_b} \\ \bar{\psi}_b &\to +i\frac{\delta}{\delta\eta_b}\,. \end{aligned} \tag{6.20}$$

The sign difference in the last two relations is due to the anticommunting Grassman variables. By means of Eq. (6.20), Eq. (6.19) becomes

$$\left[\frac{i}{\xi_b}\Box\partial^\mu\frac{\delta}{\delta J_b^\mu} - \partial_\mu J_b^\mu(x) - e_b\left(\bar{\eta}_b\frac{\delta}{\delta\bar{\eta}_b} - \eta_b\frac{\delta}{\delta\eta_b}\right)\right] Z_b\left[A_{b\mu},\eta_b,\bar{\eta}_b\right] = 0\,. \qquad (6.21)$$

Eq. (6.21), inserting the definition of the Z functional in terms of the W functional of Eq. (6.12), becomes

$$\frac{1}{\xi_b}\Box\partial^\mu\frac{\delta W_b}{\delta J_b^\mu(x)} + \partial_\mu J_b^\mu(x) + ie_b\left(\bar{\eta}_b(x)\frac{\delta W_b}{\delta\bar{\eta}_b(x)} + \frac{\delta W_b}{\delta\eta_b(x)}\eta_b(x)\right) = 0\,. \qquad (6.22)$$

We can also express the above identity in terms of the Effective Action Γ, which is defined as

$$\Gamma_b\left[A^c_{b\mu},\psi^c_b,\bar{\psi}^c_b\right] = W_b\left[J_b^\mu,\eta_b,\bar{\eta}_b\right] - \int dx\,\left(J_b^\mu A^c_{b\mu} + \bar{\eta}_b\psi^c_b + \bar{\psi}^c_b\eta_b\right)\,, \qquad (6.23)$$

where the classical fields are given by

$$\begin{aligned} A^c_{b\mu}(x) &= \frac{\delta W_b}{\delta J_b^\mu(x)} \\ \psi^c_b(x) &= \frac{\delta W_b}{\delta\bar{\eta}_b(x)} \\ \bar{\psi}^c_b(x) &= -\frac{\delta W_b}{\delta\eta_b(x)}\,, \end{aligned} \qquad (6.24)$$

in agreement with Eqs. (6.12) and (6.20). From Eqs. (6.23) and (6.24) we get

$$\begin{aligned} \frac{\delta\Gamma_b}{\delta A^c_{b\mu}(x)} &= -J_b^\mu(x) \\ \frac{\delta\Gamma_b}{\delta\psi^c_b(x)} &= \bar{\eta}_b \\ \frac{\delta\Gamma_b}{\delta\bar{\psi}^c_b(x)} &= -\eta_b \end{aligned} \qquad (6.25)$$

From the above relations and Eq. (6.22) we obtain the following identity:

$$\frac{1}{\xi_b}\Box\partial^\mu A^c_{b\mu}(x) - \partial_\mu\frac{\delta\Gamma_b}{\delta A^c_{b\mu}(x)} + ie_b\frac{\delta\Gamma_b}{\delta\psi^c_b(x)}\psi^c_b(x) + ie_b\bar{\psi}^c_b(x)\frac{\delta\Gamma_b}{\delta\bar{\psi}^c_b(x)} = 0\,. \qquad (6.26)$$

Eqs. (6.21), (6.22) and (6.26) are three equivalent functional forms of the Ward-Takahashi Identity in QED. By iterated functional derivation, they generate an infinite number of identities relating different Green functions. These identities are the true Ward-Takahashi Identities.

We remind that the Effective Action is the quantum analogous of the Classical Action for the classical theory. It is useful to understand whether the identities of Eqs. (6.21), (6.22) and (6.26) are valid also in for the Classical Action

$$S = S_{MD} + S_{GF}\,. \qquad (6.27)$$

From Eq. (6.16) we get for the variation of the Classical Action

$$\delta S = -\frac{1}{\xi_b}\int dx\, \delta\Lambda(x)\Box\left(\partial^\mu A_{b\mu}(x)\right) , \tag{6.28}$$

On the other hand, the general expression of the functional variation

$$\delta S = \int dx \left\{ \frac{\delta S}{\delta A^b_\mu(x)}\partial_\mu\delta\Lambda(x) + ie_b\delta\Lambda\frac{\delta S}{\delta\psi_b}\psi_b + ie_b\delta\Lambda\bar\psi_b\frac{\delta S}{\delta\bar\psi_b}\right\} , \tag{6.29}$$

which, after an integration by parts, becomes

$$\delta S = \int dx\, \delta\Lambda(x) \left\{ -\partial_\mu\frac{\delta S}{\delta A^b_\mu} + ie_b\delta\Lambda\frac{\delta S}{\delta\psi_b}\psi_b + ie_b\delta\Lambda\bar\psi_b\frac{\delta S}{\delta\bar\psi_b}\right\} . \tag{6.30}$$

Using Eqs. (6.28) and (6.30), and considering that $\delta\Lambda$ is arbitrary, the Classical Action S must satisfy the following identity:

$$\frac{1}{\xi_b}\Box\partial^\mu A^b{}_\mu(x) - \partial_\mu\frac{\delta S}{\delta A^b{}_\mu(x)} + ie_b\frac{\delta S}{\delta\psi_b(x)}\psi_b(x) + ie_b\bar\psi_b(x)\frac{\delta S}{\delta\bar\psi_b(x)} = 0 , \tag{6.31}$$

which is formally the same identity of Eq. (6.26) satisfied by the Effective Action Γ_b.

6.2.2 Ward-Takahashi Identities

We remind that the Effective Action Γ_b generates, through functional derivation w.r.t. the classical fields and upon setting them to zero after derivation, the 1P-I Green functions, with the exception of the two-point functions, which are the inverse of the propagators. It is useful to introduce the following notation

$$\Gamma^{(p,2q)}_{b\mu_1\ldots\mu_n}(x_1,\ldots x_p; y_1,\ldots y_q; y'_1,\ldots y'_q) = \frac{\delta^{p+2q}\Gamma_b}{\prod_{i=1}^p \delta A^c_{b\mu_i}(x_i)\prod_{j=1}^q \delta\bar\psi^c_b(y_j)\prod_{k=1}^q \delta\psi^c_b(y'_k)}_{|0} \tag{6.32}$$

By functionally deriving Eq. (6.26) w.r.t. $A^c_{b\nu}(y)$, we get

$$\frac{\partial}{\partial x^\mu}\Gamma^{(2,0)\,\mu\nu}_b(x) = \frac{1}{\xi_b}\Box_x\partial^\nu_x\delta(x) . \tag{6.33}$$

Actually, since the Green functions depend only on coordinate differences we can write explicitely

$$\frac{\partial}{\partial x^\mu}\Gamma^{(2,0)\,\mu\nu}_b(x-y) = \frac{1}{\xi_b}\Box_x\partial^\nu_x\delta(x-y) . \tag{6.34}$$

Let us consider the Fourier transform of Eq. (6.33):

$$\int dx\, e^{ik\cdot x}\frac{\partial}{\partial x^\mu}\Gamma^{(2,0)\,\mu\nu}_b(x) = \int dx\, e^{ik\cdot x}\frac{1}{\xi_b}\Box_x\partial^\nu_x\delta(x) . \tag{6.35}$$

Through integration by parts, Eq. (6.35) becomes

$$-\int dx\, \Gamma_b^{(2,0)\,\mu\nu}(x) \frac{\partial}{\partial x^\mu} e^{ik\cdot x} = -\int dx\, \delta(x) \frac{1}{\xi_b} \Box_x \partial_x^\nu e^{ik\cdot x}\,, \tag{6.36}$$

which becomes

$$-ik_\mu \int dx\, \Gamma_b^{(2,0)\,\mu\nu}(x) e^{ik\cdot x} = \frac{i}{\xi_b} k^2 k^\nu \int dx\, e^{ik\cdot x} \delta(x)\,, \tag{6.37}$$

i.e.

$$k_\mu \left(\hat{\Gamma}_b\right)^{(2,0)\,\mu\nu}(k) = -\frac{1}{\xi_b} k^2 k^\nu\,. \tag{6.38}$$

If we consider the inverse of the free photon propagator of Eq. (6.11), which is given by

$$\hat{D}^{-1}_{\mu\nu}(k) = -k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi_b}\right) k_\mu k_\nu\,, \tag{6.39}$$

we see that $\hat{D}^{-1}$ satisfies Eq. (6.38). This means that the longitudinal part of the propagator is not affected by radiative corrections, since Eq. (6.38) holds to all orders of perturbation theory. On the other hand, the longitudinal part of the free photon propagator is due to the gauge fixing term, i.e. the latter is not touched by renormalization [1].

We can derive another important Ward Identity by functionally deriving n times (with $n > 2$) Eq. (6.26):

$$\frac{\partial}{\partial x_1^{\mu_1}} \Gamma_b^{(n,0)\,\mu_1 \ldots \mu_n}(x_1, \ldots, x_n) = 0\,, \tag{6.40}$$

which, in Fourier representation, becomes

$$k_{1\,\mu_1} \left(\hat{\Gamma}_b\right)^{(n,0)\,\mu_1 \ldots \mu_n}(k_1, \ldots, k_n) = 0\,, \tag{6.41}$$

which expresses the transversity of the 1P-I Green functions with n external photons: substituting the external polarization vector of the $n-$th photon with its four-momentum gives zero.

Another Ward identity can be obtained by functionally deriving Eq. (6.26) w.r.t. $\bar{\psi}_b^c(y)$ and $\psi_b^c(y')$:

$$-\frac{\partial}{\partial x^\mu} \Gamma_b^{(1,2)\,\mu}(x, y, y') = ie_b \delta(x - y') \Gamma_b^{(0,2)}(y' - y) - ie_b \delta(x - y) \Gamma_b^{(0,2)}(y - y')\,. \tag{6.42}$$

[1] We remind that the transverse part of the propagator is the one proportional to $\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$, i.e. it gives zero when contracted with k^μ.

Let us consider the Fourier transform of Eq. (6.42). For the first member we get

$$\begin{aligned}
& -\int dx \int dy \int dy'\, e^{ik\cdot x} e^{ip\cdot y} e^{ip'\cdot y'} \frac{\partial}{\partial x^\mu} \Gamma_b^{(1,2)^\mu}(x,y,y') \\
=& -\int dy \int dy'\, e^{ip\cdot y} e^{ip'\cdot y'} \int dx\, e^{ik\cdot x} \frac{\partial}{\partial x^\mu} \Gamma_b^{(1,2)^\mu}(x,y,y') \\
=& +\int dy \int dy'\, e^{ip\cdot y} e^{ip'\cdot y'} \int dx\, \Gamma_b^{(1,2)^\mu}(x,y,y') \frac{\partial}{\partial x^\mu} e^{ik\cdot x} \\
=& +ik_\mu \int dx \int dy \int dy'\, e^{ik\cdot x} e^{ip\cdot y} e^{ip'\cdot y'} \Gamma_b^{(1,2)^\mu}(x,y,y') \\
=& +ik_\mu (2\pi)^n \delta(k+p+p') \big(\hat{\Gamma}_b\big)^{(1,2)^\mu}(k,p,p') \\
=& +ik_\mu (2\pi)^n \delta(k+p+p') \big(\hat{\Gamma}_b\big)^{(1,2)^\mu}(k,p,-p-k)\,.
\end{aligned} \tag{6.43}$$

Introducing $\big(\hat{\Gamma}_b\big)^{(1,2)^\mu}(k,p,p')$ in the above equation, we used the definition of Eq. (3.45), generalized to n dimensions.

For the first term of the second member of Eq. (6.42) we get

$$\begin{aligned}
& ie_b \int dx \int dy \int dy'\, e^{ik\cdot x} e^{ip\cdot y} e^{ip'\cdot y'} \delta(x-y') \Gamma_b^{(0,2)}(y'-y) \\
=& ie_b \int dy\, e^{ip\cdot y} \int dx \int dy'\, e^{ik\cdot x} e^{ip'\cdot y'} \delta(x-y') \Gamma_b^{(0,2)}(y'-y) \\
=& ie_b \int dy\, e^{ip\cdot y} \int dx\, e^{i(k+p')\cdot x} \Gamma_b^{(0,2)}(x-y)\,.
\end{aligned} \tag{6.44}$$

We first introduce the variable $x' = x - y$ through $\int dx'\, \delta(x' - (x-y))$ and then perform the $\int dx$, obtaining

$$\begin{aligned}
& ie_b \int dx' \int dy \int dx\, e^{ip\cdot y} e^{i(k+p')\cdot x} \delta(x'-(x-y)) \Gamma_b^{(0,2)}(x-y) \\
=& ie_b \int dy\, e^{i(p+k+p')\cdot y} \int dx'\, e^{i(k+p')\cdot x'} \Gamma_b^{(0,2)}(x') \\
=& ie_b (2\pi)^n \delta(k+p+p') \big(\hat{\Gamma}_b\big)^{(0,2)}(k+p') \\
=& ie_b (2\pi)^n \delta(k+p+p') \big(\hat{\Gamma}_b\big)^{(0,2)}(-p) = ie_b (2\pi)^n \delta(k+p+p') \big(\hat{\Gamma}_b\big)^{(0,2)}(p)\,.
\end{aligned} \tag{6.45}$$

where in the last line we have used the fact that for Lorentz invariance $\big(\hat{\Gamma}_b\big)^{(0,2)}$ depends on p^2.

For the second term of the second member of Eq. (6.42) we get (with a change of variable analogous to Eq. (6.44))

$$
\begin{aligned}
&- ie_b \int dx \int dy \int dy' \, e^{ik\cdot x} e^{ip\cdot y} e^{ip'\cdot y'} \delta(x-y) \Gamma_b^{(0,2)}(y-y') \\
=& - ie_b \int dy' \, e^{ip'\cdot y'} \int dx \int dy \, e^{ik\cdot x} e^{ip\cdot y} \delta(x-y) \Gamma_b^{(0,2)}(y-y') \\
=& - ie_b \int dy' \, e^{ip'\cdot y'} \int dx \, e^{i(k+p)\cdot x} \Gamma_b^{(0,2)}(x-y') \\
=& - ie_b \int dy' \, e^{i(p'+k+p)\cdot y'} \int dx' \, e^{i(k+p)\cdot x'} \Gamma_b^{(0,2)}(x') \\
=& - ie_b (2\pi)^n \delta(k+p+p') \big(\hat{\Gamma}_b\big)^{(0,2)}(p+k) \,.
\end{aligned}
\tag{6.46}
$$

Combining Eqs. (6.43), (6.45) and (6.46), we obtain that Eq. (6.42), through Fourier transform, corresponds to the following identity in momentum representation

$$
k_\mu \big(\hat{\Gamma}_b\big)^{(1,2)^\mu}(k,p,-p-k) = e_b \left[\big(\hat{\Gamma}_b\big)^{(0,2)}(p) - \big(\hat{\Gamma}_b\big)^{(0,2)}(p+k) \right] . \tag{6.47}
$$

In the limit $k_\mu \to 0$, Eq. (6.47) becomes

$$
\big(\hat{\Gamma}_b\big)^{(1,2)^\mu}(0,p,-p) = -e_b \frac{\partial}{\partial p^\mu} \big(\hat{\Gamma}_b\big)^{(0,2)}(p) \,. \tag{6.48}
$$

Eq. (6.48) is usually known as the QED Ward Identity, which relates the inverse electron propagator with the proper three point vertex function, calculated in the limit of zero photon momentum. It is a relation valid to all orders of perturbation theory. We will verify it explicitely at one-loop order in the following sections.

6.3 Renormalization of QED

In this section we perform the renormalization of QED in the renormalized perturbation theory approach. We recall the expression of the QED bare Lagrangian density of Eqs. (6.1) and (6.2)

$$
\begin{aligned}
\mathcal{L}_{QED} =& -\frac{1}{4} \left(\partial_\mu A_{b\nu}(x) - \partial_\nu A_{b\mu}(x)\right) \left(\partial^\mu A_b{}^\nu(x) - \partial^\nu A_b{}^\mu(x)\right) \\
& + i\bar{\psi}_b(x)\gamma^\mu \left(\partial_\mu + ie_b A_{b\mu}(x)\right) \psi_b(x) - m_b \bar{\psi}_b(x)\psi_b(x) \,, \\
& - \frac{1}{2\xi_b} \left(\partial_\mu A_b^\mu(x)\right)^2 \,.
\end{aligned}
\tag{6.49}
$$

The bare fields and parameters can be rewritten in terms of multiplicatively renormalized quantities as follows:

$$\begin{aligned} A_{b\mu}(x) &= Z_A^{\frac{1}{2}} A_\mu(x) \\ \psi_{b\mu}(x) &= Z_\psi^{\frac{1}{2}} A_\mu(x) \\ m_b &= Z_m m \\ e_b &= \mu^\varepsilon Z_e e \\ \xi_b &= Z_\xi \xi \end{aligned} \tag{6.50}$$

Note that the bare electric charge gets rescaled with μ^ε, at variance with λ of the scalar self-interacting field, which gets rescaled with $\mu^{2\varepsilon}$. The reason is the different dimensions of the fields in n dimensions: $[\psi] = M^{\frac{n-1}{2}}$, $[A_\mu] = M^{\frac{n-2}{2}}$; therefore the product of the fields in the interaction term has the dimension M^x, with $x = \frac{n-2}{2} + n - 1 = n + \frac{n}{2} - 2$ and, in order to get dimension M^n for the whole interaction term, $[e] = M^{\frac{4-n}{2}} = M^\varepsilon$.

In the literature the renormalization constants are also named differently as follows:

$$\begin{aligned} Z_A &\to Z_3 \\ Z_\psi &\to Z_2 \\ Z_e &\to \frac{Z_1}{Z_2 Z_3^{\frac{1}{2}}} \\ Z_\xi &\to Z_3 \end{aligned} \tag{6.51}$$

The last identification in Eq. (6.51) can be made because the gauge-fixing term does not get renormalized, as already shown in the previous sections. Inserting Eq. (6.50) into Eq. (6.49) we get

$$\begin{aligned} \mathcal{L}_{QED} = &-\frac{1}{4} Z_A \left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)\left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right) \\ &- \frac{1}{2\xi} Z_A Z_\xi^{-1} \left(\partial_\mu A^\mu(x)\right)^2 \\ &+ Z_\psi \bar{\psi}(x) i\partial\!\!\!/ \psi(x) - Z_\psi Z_m \bar{\psi}(x) m \psi(x) \\ &- Z_e Z_A^{\frac{1}{2}} Z_\psi e A_\mu \bar{\psi}(x) \gamma^\mu \psi(x) \,. \end{aligned} \tag{6.52}$$

Writing $Z_i = 1 + \delta Z_i$, the above Lagrangian density expression can be split into the original one of Eq. (6.49), written in terms of renormalized parameters and fields, and into the *counterterm* Lagrangian density (remember that the quantities δZ_i start at one-loop order):

$$\mathcal{L}_{QED} = \mathcal{L}_{QED}^{\text{ren}} + \mathcal{L}_{QED}^{\text{c.t.}} \,, \tag{6.53}$$

where

$$\begin{aligned}\mathcal{L}^{\text{ren}}_{QED} =& -\frac{1}{4}\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)\left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right)\\ & -\frac{1}{2\xi}\left(\partial_\mu A^\mu(x)\right)^2\\ & + i\bar{\psi}(x)\gamma^\mu\left(\partial_\mu + ieA_\mu(x)\right)\psi(x) - m\bar{\psi}(x)\psi(x)\,,\end{aligned} \tag{6.54}$$

$$\begin{aligned}\mathcal{L}^{\text{c.t.}}_{QED} =& -\frac{1}{4}\delta Z_A\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)\left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right)\\ & -\frac{1}{2\xi}\left(Z_A Z_\xi^{-1} - 1\right)\left(\partial_\mu A^\mu(x)\right)^2\\ & + \delta Z_\psi \bar{\psi}(x) i\partial\!\!\!/\psi(x) - \left(Z_\psi Z_m - 1\right)\bar{\psi}(x) m\psi(x)\\ & - \left(Z_e Z_A^{\frac{1}{2}} Z_\psi - 1\right) eA_\mu\bar{\psi}(x)\gamma^\mu\psi(x)\,.\end{aligned} \tag{6.55}$$

The Lagrangian density of Eq. (6.54) gives rise to the same Feynam rules of QED, in terms of renormalized (physical) fields and parameters, while the counterterm Lagrangian density of Eq. (6.55) gives rise to additional Feynaman rules, the ones needed to cancel the UV divergences present in the up-to-one loop theoretical predictions. **We stress again that the renormalizability of QED means that all divergencies of higher order calculations in perturbation theory can be absorbed in counterterms that contain only terms which are already present in the bare Lagrangian.** *The counterterms have to be fixed according to a choice of renormalization scheme. However, before specifying a renormalization scheme, we should consider that, due to the gauge invariance (and the consequent Ward identities discussed in previous sections), not all of the counterterms are independent.*

6.3.1 Ward Identities for renormalized Green functions

The Ward Identity for the renormalized photon propagator

$$-\frac{1}{Z_\xi}\frac{1}{\xi}q^2 q^\mu Z_A W^{(2,0)}{}_{\mu\nu}(q,-q) = iq_\nu\,. \tag{6.56}$$

implies that $\frac{Z_A}{A_\xi}$ must be finite. We can fix the finite part by choosing

$$Z_\xi = Z_A \tag{6.57}$$

The Ward Identity for the renormalized electron-photon vertex function, using Eq. (6.57), is

$$-\frac{i}{Z_A}\frac{1}{\xi}q^2 q^\mu Z_\psi Z_A^{\frac{1}{2}} W^{(1,2)}{}_{\mu\nu}(q,p,p') = Z_e e Z_\psi\left(W^{(0,2)}\left(-p',p'\right) - W^{(0,2)}\left(p,-p\right)\right)\,. \tag{6.58}$$

Therefore $Z_e Z_A^{\frac{1}{2}}$ must be finite and we can choose

$$Z_e = Z_A^{-\frac{1}{2}}\,. \tag{6.59}$$

Hence the charge renormalization can be related to the renormalization of the photon field and the total number of needed renormalization constants to renormalize QED is three: we can choose Z_A, Z_ψ and Z_m. The final expression of the counterterm Lagrangian is

$$\begin{aligned}\mathcal{L}_{QED}^{\text{c.t.}} = &-\frac{1}{4}\delta Z_A\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)\left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right) \\ &+ \delta Z_\psi \bar{\psi}(x) i\partial\!\!\!/ \psi(x) - \left(\delta Z_\psi + \delta Z_m\right)\bar{\psi}(x) m \psi(x)\,.\end{aligned} \tag{6.60}$$

6.3.2 On-shell renormalization scheme in QED

In this subsection we present the renormalization conditions imposed on the renormalization constants Z_A, Z_ψ and Z_m, in order to fix their finite parts in the *on-shell renormalization scheme*. This scheme amounts to impose that the photon and electron propagator as well as the electron mass are the same at every order of perturbation theory, *i.e.* the propagators are required to have residue one and the renormalized electron mass gives the pole of the electron propagator. In the following we show in detail how to impose the above conditions.

Photon-field renormalization constant: choice of $\mathbf{Z_A}$

The photon-field renormalization constant is fixed by requiring

$$\lim_{q^2\to 0}\left[\frac{\hat{\Gamma}^{(2,0)}_{\mu,\nu}(q,-q)\varepsilon^\nu(q)}{q^2} = -\varepsilon_\mu(q)\right]. \tag{6.61}$$

This condition enforces that the pole at $q^2 = 0$ of the renormalized photon propagator has residue one, so that the normalization of every external photon leg in the calculation of S matrix elements is kept fixed at one order by order in perturbation theory. ¡Since $\hat{\Gamma}^{(2,0)}_{\mu,\nu}(q,-q)$, written in terms of renormalized quantities, depends on Z_A, Eq. (6.61) allows to fix the wave-function renormalization constant Z_A. It is useful to introduce the decomposition of $\hat{\Gamma}^{(2,0)}_{\mu,\nu}(q,-q)$ into Lorentz tensors and scalar functions as follows

$$\hat{\Gamma}^{(2,0)}_{\mu,\nu}(q,-q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\hat{\Gamma}^{(2,0)}_T(q^2) + \frac{q_\mu q_\nu}{q^2}\hat{\Gamma}^{(2,0)}_L(q^2)\,. \tag{6.62}$$

We remind that the tensor $\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$ is transverse because $q^\mu\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) = 0$. Therefore

$$q^\mu \hat{\Gamma}^{(2,0)}_{\mu,\nu}(q,-q) = q^\mu \frac{q_\mu q_\nu}{q^2}\hat{\Gamma}^{(2,0)}_L(q^2) = q_\nu \hat{\Gamma}^{(2,0)}_L(q^2)\,. \tag{6.63}$$

The Ward Identity of Eq. (6.38) (or, equivalently, Eqs. (6.56) and (6.57)

$$q^\mu \hat{\Gamma}^{(2,0)}_{\mu,\nu}(q,-q) = q_\nu \hat{\Gamma}^{(2,0)}_L(q^2) = -\frac{1}{\xi}q^2 q_\nu\,. \tag{6.64}$$

i.e.

$$\hat{\Gamma}_L^{(2,0)}(q^2) = -\frac{1}{\xi}q^2\,, \tag{6.65}$$

which shows that the longitudinal part gets no higher-order corrections. As a consequence of Eq. (6.65) $\hat{\Gamma}_L^{(2,0)}(0) = 0$. Moreover, the absence of poles in the 1PI Green functions **(perché le 1PI Green function non devono avere poli? controllare sul denner)**, in particular $\hat{\Gamma}_{\mu\nu}^{(2,0)}(q,-q)$, enforce that the transverse part of $\hat{\Gamma}_{\mu\nu}^{(2,0)}(q,-q)$ vanishes at $q^2 = 0$:

$$\hat{\Gamma}_T^{(2,0)}(0) = \hat{\Gamma}_L^{(2,0)}(0) = 0\,, \tag{6.66}$$

which means that no photon mass term is generated by higher-order corrections. If this were the case, gauge invariance would be destroyed by higher-horder corrections. Inserting Eq. (6.62) in the condition of Eq. (6.61) we have

$$\begin{aligned}
&\lim_{q^2\to 0}\left\{\frac{\left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\hat{\Gamma}_T^{(2,0)}(q^2) + \frac{q_\mu q_\nu}{q^2}\hat{\Gamma}_L^{(2,0)}(q^2)\right]\varepsilon^\nu(q)}{q^2}\right\} = -\varepsilon_\mu(q)\\
&\lim_{q^2\to 0}\left\{\frac{\left[g_{\mu\nu}\hat{\Gamma}_T^{(2,0)}(q^2)\right]\varepsilon^\nu(q)}{q^2}\right\} = -\varepsilon_\mu(q)\,,
\end{aligned} \tag{6.67}$$

where we have exploited the transversality of the photons $q^\mu\varepsilon_\nu(q) = 0$. Eq. (6.67) yields the following **renormalization condition**

$$\lim_{q^2\to 0}\left[\frac{\hat{\Gamma}_T^{(2,0)}(q^2)}{q^2}\right] = \lim_{q^2\to 0}\left[\frac{\hat{\Gamma}_T^{(2,0)}(q^2) - \hat{\Gamma}_T^{(2,0)}(0)}{q^2}\right] = \frac{\partial\hat{\Gamma}_T^{(2,0)}(q^2)}{\partial q^2}\Big|_{q^2=0} = -1\,. \tag{6.68}$$

Electron mass and field renormalization constants: choice of $\mathbf{Z}_m$ and $\mathbf{Z}_\psi$

The factors Z_m and Z_ψ for the renormalization of the electron mass and field are fixed by

$$\lim_{p^2\to m^2}\left\{\frac{(\not{p}+m)\hat{\Gamma}^{(0,2)}(p,-p)u(p)}{p^2-m^2}\right\} = u(p)\,. \tag{6.69}$$

The above equation implies

$$\hat{\Gamma}^{(0,2)}(p,-p)u(p)|_{p^2=m^2} = 0\,, \tag{6.70}$$

which means that m **is the physical electron mass** [2]. Also in this case it is convenient to decompose $\hat{\Gamma}^{(0,2)}(p,-p)$ in Lorentz covariant form:

$$\begin{aligned}
\hat{\Gamma}^{(0,2)}(p,-p) &= \not{p}\hat{\Gamma}_V^{(0,2)}(p^2) + m\hat{\Gamma}_S^{(0,2)}(p^2)\\
&= (\not{p}-m)\hat{\Gamma}_V^{(0,2)}(p^2) + m\left(\hat{\Gamma}_V^{(0,2)}(p^2) + \hat{\Gamma}_S^{(0,2)}(p^2)\right)\,.
\end{aligned} \tag{6.71}$$

[2] We remind that at tree level $\hat{\Gamma}^{(0,2)}(p,-p) = \not{p} - m$, which satisfies Eq. (6.69) and Eq. (6.70).

Inserting Eq. (6.71) into Eq. (6.69) we have

$$\lim_{p^2\to m^2}\left\{\left[\hat\Gamma_V^{(0,2)}(p^2)+\frac{m(\not p+m)\left(\hat\Gamma_V^{(0,2)}(p^2)+\hat\Gamma_S^{(0,2)}(p^2)\right)}{p^2-m^2}\right]u(p)\right\}=u(p)\,. \tag{6.72}$$

By expanding Eq. (6.71) around the mass shell $p^2=m^2$, we have

$$\hat\Gamma_{V,S}^{(0,2)}(p^2)=\hat\Gamma_{V,S}^{(0,2)}(p^2)+\left(p^2-m^2\right)\frac{\partial\hat\Gamma_{V,S}^{(0,2)}(p^2)}{\partial p^2}|_{p^2=m^2}+\dots\,. \tag{6.73}$$

Inserting Eq. (6.73) into Eq. (6.72), neglecting terms of $\mathcal{O}(p^2-m^2)^n$ with $n\geq 2$ and suppressing the notation $^{(0,2)}$ for simplicity, the argument of the limit in Eq. (6.72) becomes

$$\hat\Gamma_V(m^2)+\frac{m(\not p+m)\left(\hat\Gamma_V(m^2)+\hat\Gamma_S(m^2)+\left(p^2-m^2\right)\left(\hat\Gamma'_V(m^2)+\hat\Gamma'_S(m^2)\right)\right)}{p^2-m^2}\,. \tag{6.74}$$

Eq. (6.69) is satisfied if the expression of Eq. (6.74), in the on-shell limit [3], becomes the identity matrix. For this to happen, the following conditions have to be imposed:

$$\hat\Gamma_V^{(0,2)}(m^2)+\hat\Gamma_S^{(0,2)}(m^2) = 0 \tag{6.75}$$

$$\hat\Gamma_V^{(0,2)}(m^2)+2m^2\left[\frac{\partial\hat\Gamma_V^{(0,2)}(p^2)}{\partial p^2}|_{p^2=m^2}+\frac{\partial\hat\Gamma_S^{(0,2)}(p^2)}{\partial p^2}|_{p^2=m^2}\right] = 1\,. \tag{6.76}$$

The mass renormalization constant Z_m is determined from Eq. (6.75) while the field renormalization constant Z_ψ is determined from Eq. (6.76), which involves the derivative of $\hat\Gamma_{V,S}^{(0,2)}(p^2)$.

Charge renormalization constant and universality of charge renormalization

We have previously fixed the charge renormalization constant with Eq. (6.59). We show now that this is equivalent to the following condition on the renormalized 1PI three-point Green function:

$$\bar u(p)\hat\Gamma_\mu^{(1,2)}(0,p,-p)u(p)=-e\bar u(p)\gamma_\mu u(p)\qquad\text{for } p^2=m^2\,. \tag{6.77}$$

This can be seen by writing Eq. (6.48) in terms of renormalized quantities:

$$\begin{aligned}\hat\Gamma_{b\mu}^{(1,2)}(0,p,-p) &= -e_b\frac{\partial}{\partial p_\mu}\hat\Gamma_b^{(0,2)}(p)\\ \Longrightarrow Z_A^{-\frac{1}{2}}Z_\psi^{-1}\hat\Gamma_\mu^{(1,2)}(0,p,-p) &= -Z_e e Z_\psi^{-1}\frac{\partial}{\partial p_\mu}\hat\Gamma^{(0,2)}(p)\,,\end{aligned} \tag{6.78}$$

[3] We remind that in the on-shell limit $p^2\to m^2$ we have also $\not p\to mI$.

where in Eq. (6.78) we have used Eq. (5.110) (valid for a general theory, provided that the factors for each different external field are taken into account) and we have dropped the dimensional parameter $\mu^{2\varepsilon}$ because the renormalized Green functions are finite and therefore the limit $\varepsilon \to 0$ can safely be taken. By means of Eq. (6.59) and Eq. (6.71), Eq. (6.78) becomes

$$\begin{aligned}
\hat{\Gamma}_\mu^{(1,2)}(0,p,-p) &= -e\frac{\partial}{\partial p_\mu}\hat{\Gamma}^{(0,2)}(p) \\
&= -e\frac{\partial}{\partial p_\mu}\left[\not{p}\hat{\Gamma}_V^{(0,2)}(p^2)+m\hat{\Gamma}_S^{(0,2)}(p^2)\right] \\
&= -e\left[\gamma_\mu\hat{\Gamma}_V^{(0,2)}(p^2)+\not{p}\frac{\partial}{\partial p_\mu}\hat{\Gamma}_V^{(0,2)}(p^2)+m\frac{\partial}{\partial p_\mu}\hat{\Gamma}_S^{(0,2)}(p^2)\right] \\
&= -e\left[\gamma_\mu\hat{\Gamma}_V^{(0,2)}(p^2)+2\not{p}p_\mu\frac{\partial}{\partial p^2}\hat{\Gamma}_V^{(0,2)}(p^2)+2mp_\mu\frac{\partial}{\partial p^2}\hat{\Gamma}_S^{(0,2)}(p^2)\right] \\
&= -e\left[\gamma_\mu\hat{\Gamma}_V^{(0,2)}(p^2)+\not{p}\left\{\not{p},\gamma_\mu\right\}\frac{\partial}{\partial p^2}\hat{\Gamma}_V^{(0,2)}(p^2)+m\left\{\not{p},\gamma_\mu\right\}\frac{\partial}{\partial p^2}\hat{\Gamma}_S^{(0,2)}(p^2)\right],
\end{aligned} \tag{6.79}$$

where we have used the identity $2p_\mu = \{\not{p},\gamma_\mu\}$. Multiplying each member of Eq. (6.79) on the right by $u(p)$ and on the left by $\bar{u}(p)$, and using the Dirac equations $(\not{p}-m)\,u(p)=0$ and $\bar{u}(p)\,(\not{p}-m)=0$, we get

$$\bar{u}(p)\hat{\Gamma}_\mu^{(1,2)}(0,p,-p)u(p) = -e\bar{u}(p)\left[\gamma_\mu\hat{\Gamma}_V^{(0,2)}(m^2)+2m^2\gamma_\mu\left(\frac{\partial\hat{\Gamma}_V^{(0,2)}(p^2)}{\partial p^2}+\frac{\partial\hat{\Gamma}_S^{(0,2)}(p^2)}{\partial p^2}\right)\bigg|_{p^2=m^2}\right]u(p)\,. \tag{6.80}$$

Inserting Eq. (6.76) in Eq. (6.80) we obtain Eq. (6.77).

All the intermediate steps to prove Eq. (6.77) are valid also when we consider the limit $q^2 \to 0$ of the scattering of a photon on a fermion with relative charge Q_f to the electron In this case Eq. (6.77) would be

$$\bar{u}(p)\hat{\Gamma}_\mu^{(1,2)}(0,p,-p)u(p) = -eQ_f\bar{u}(p)\gamma_\mu u(p) \qquad \text{for } p^2=m^2\,. \tag{6.81}$$

Eq. (6.81) expresses the *charge universality* in QED: the renormalized charge, i.e. the on-shell coupling to the photon, is independent of the fermion species.

Remark: in the on-shell renormalization scheme, the theory is fitted to physical parameters in such a way that a photon scattered with vanishing momentum transfer on a real, physical electron (i.e. $p^2=m^2$) feels a coupling strength e, with $\frac{e^2}{4\pi}=\alpha$, the fine structure constant. This guarantees also that all higher-order radiative corrections to the Compton scattering vanish in the low-energy limit (this is known as Thirring theorem [4]).

[4] W. Thirring, *Helv. Phys. Acta* 26 (1953) 33.

6.4 One-loop radiative corrections

In this section we explicitely calculate at one-loop order, the three divergent 1PI Green functions of QED and calculate the renormalization constants at one-loop order in the on-shell renormalization scheme, making contact with the general results obtained in the previous sections. In order to work out QED one-loop diagrams, we need to extend the Dirac algebra to n dimensions. We give here a brief account of the elementary rules:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \qquad \text{with } \mu = \{0, 1, \ldots, n-1\}\,. \tag{6.82}$$

The metric tensor $g^{\mu\nu}$ has $n-1$ spatial components and satisfies the relation $g^\mu_\mu = n$. Since μ runs from 0 to $n-1$, we have

$$\gamma^\mu \gamma_\mu = nI\,. \tag{6.83}$$

As a consequence of Eq. (6.83), we can deduce $\gamma^\mu\gamma^\alpha\gamma_\mu = (2-n)\gamma^\alpha$. The trace of the unit matrix can be chosen as in the four-dimensional case: $\mathrm{Tr}I = 4$.

6.4.1 Photon vacuum polarization

The photon-photon vertex function $\Gamma^{AA}_{\mu\nu}$, in momentum space, can be decomposed into its free part and into the so called photon self-energy $\Sigma^A_{\mu\nu}(p)$ as follows:

$$\Gamma^{(2,0)}_{\mu\nu} = -\left[g_{\mu\nu}q^2 - q_\mu q_\nu\left(1 - \frac{1}{\xi}\right)\right] - \Sigma^A_{\mu\nu}\,. \tag{6.84}$$

The photon self-energy is calculated from the one-loop diagram of Fig. 6.1.

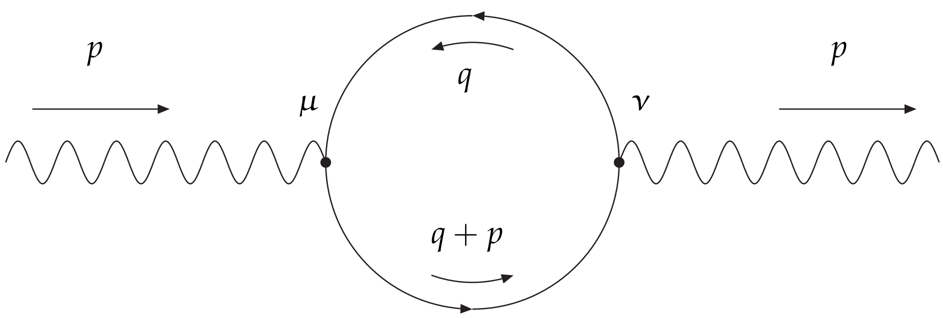

Figure 6.1: One-loop photon self-energy diagram.

With the Feynman rules of QED, we can calculate $\Sigma^A_{\mu\nu}(q)$:

$$\Sigma^A_{\mu\nu}(p) = -ie^2\mu^{4-n}\int \frac{d^nq}{(2\pi)^n}\frac{\mathrm{Tr}\left[\gamma_\mu\left(\not{q}+m\right)\gamma_\nu\left(\not{q}+\not{p}+m\right)\right]}{[q^2-m^2+i\varepsilon]\,[(q+p)^2-m^2+i\varepsilon]}\,. \tag{6.85}$$

First we work out the trace (in four dimensions) in the integrand of Eq. (6.85):

$$\begin{aligned}\mathrm{Tr}\left[\gamma_\mu\left(\not{q}+m\right)\gamma_\nu\left(\not{q}+\not{p}+m\right)\right] &= 4\left[2q_\mu q_\nu+q_\mu p_\nu+p_\mu q_\nu-g_{\mu\nu}\left(q^2+q\cdot p-m^2\right)\right]\\ &= 4\{2q_\mu q_\nu+q_\mu p_\nu+p_\mu q_\nu\\ &- \frac{1}{2}g_{\mu\nu}\left((q^2-m^2)+((q+p)^2-m^2)-p^2\right)]\}\,.\end{aligned}$$

Thanks to the above expression, Eq. (6.85) can be rewritten as

$$\begin{aligned}\Sigma^A_{\mu\nu}(p) = &- 4ie^2\mu^{4-n}2\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu q_\nu}{[q^2-m^2+i\varepsilon]\,[(q+p)^2-m^2+i\varepsilon]}\\ &- 4ie^2\mu^{4-n}p_\nu\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu}{[q^2-m^2+i\varepsilon]\,[(q+p)^2-m^2+i\varepsilon]}\\ &- 4ie^2\mu^{4-n}p_\mu\int\frac{d^nq}{(2\pi)^n}\frac{q_\nu}{[q^2-m^2+i\varepsilon]\,[(q+p)^2-m^2+i\varepsilon]}\\ &+ 4ie^2\mu^{4-n}\frac{1}{2}g_{\mu\nu}\int\frac{d^nq}{(2\pi)^n}\frac{1}{[(q+p)^2-m^2+i\varepsilon]}\\ &+ 4ie^2\mu^{4-n}\frac{1}{2}g_{\mu\nu}\int\frac{d^nq}{(2\pi)^n}\frac{1}{[q^2-m^2+i\varepsilon]}\\ &- 4ie^2p^2\mu^{4-n}\frac{1}{2}g_{\mu\nu}\int\frac{d^nq}{(2\pi)^n}\frac{1}{[q^2-m^2+i\varepsilon]\,[(q+p)^2-m^2+i\varepsilon]}\,.\end{aligned} \tag{6.86}$$

With respect to the case of the self-energy calculation of the $\lambda\varphi^4$ model, we have also *tensor* integrals, where the integration momentum appears in the numerator. Let us consider the first three integrals on the r.h.s. of Eq. (6.86): since they are Lorent-covariants, they can be obtained from a complete set of Lorentz tensors and invariant scalar coefficient functions. In particular we can write (for simplicity of notation we assum that the term m^2 contains the small imaginary part $-i\varepsilon$):

$$\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu q_\nu}{[q^2-m^2]\,[(q+p)^2-m^2]} = g_{\mu\nu}B_{00}(p^2,m^2)+p_\mu p_\nu B_{11}(p^2,m^2) \tag{6.87}$$

$$\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{q_\mu}{[q^2-m^2]\,[(q+p)^2-m^2]} = p_\mu B_1(p^2;m,m)\,. \tag{6.88}$$

The expressions for the scalar functions defined above can be obtained by Lorentz contracting both members of the defining equations in all possible ways. We begin by multiplying by p^μ both members of Eq. (6.88):

$$\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{q\cdot p}{[q^2-m^2]\,[(q+p)^2-m^2]} = p^2B_1(p^2;m,m)\,. \tag{6.89}$$

Now we work out the left-hand side by completing the square at the numerator:

$$
\begin{aligned}
\int \frac{d^n q}{(2\pi)^n} \frac{q \cdot p}{[q^2 - m^2]\,[(q+p)^2 - m^2]} &= \frac{1}{2}\int \frac{d^n q}{(2\pi)^n} \frac{q^2 + 2(q\cdot p) + p^2 - m^2 - q^2 - p^2 + m^2}{[q^2 - m^2]\,[(q+p)^2 - m^2]} \\
&= \frac{1}{2}\int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2]} \\
&- \frac{1}{2}\int \frac{d^n q}{(2\pi)^n} \frac{1}{[(q+p)^2 - m^2]} \\
&- \frac{p^2}{2}\int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2]\,[(q+p)^2 - m^2]} \,. \qquad (6.90)
\end{aligned}
$$

The scalar integrals with one denominator have already been calculated in Eq. (5.68), using euclidean momenta. We can take the result, paying attention that we are now using minkowskian momenta:

$$
\begin{aligned}
\mu^{4-n}\int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2]} &= \mu^{4-n}\int \frac{d^n q}{(2\pi)^n} \frac{1}{[(q+p)^2 - m^2]} \\
&= i\mu^{4-n}\int \frac{d^n q_E}{(2\pi)^n} \frac{1}{[-q_E^2 - m^2]} \\
&= (+i)\frac{m^2}{16\pi^2}\left[\Delta_{UV} - \log\frac{m^2}{\mu^2} + 1\right] . \qquad (6.91)
\end{aligned}
$$

It is useful to give a name to this basic scalar integral:

$$
A_0(m) = \mu^{4-n}\int \frac{d^n q}{(2\pi)^n} \frac{1}{[q^2 - m^2]} = (+i)\frac{m^2}{16\pi^2}\left[\Delta_{UV} - \log\frac{m^2}{\mu^2} + 1\right] . \qquad (6.92)
$$

A similar argument applies to the scalar integral with two denominators, which has been calculated in Eq. (5.69):

$$
\begin{aligned}
B_0(p^2; m, m) &= \mu^{(4-n)}\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + m^2)\left((q+p)^2 + m^2\right)} \\
&= (+i)\frac{1}{16\pi^2}\left\{\Delta_{UV} - \int_0^1 dx\, \ln\left[\frac{p^2 x(1-x) + m^2}{\mu^2}\right]\right\} . \qquad (6.93)
\end{aligned}
$$

Substituting Eq. (6.93) in Eq. (6.89) we obtain

$$
B_1(p^2; m, m) = (-i)\frac{1}{32\pi^2}\left\{\Delta_{UV} - \int_0^1 dx\, \ln\left[\frac{p^2 x(1-x) + m^2}{\mu^2}\right]\right\} = -\frac{1}{2}B_0(p^2; m, m) \,. \qquad (6.94)
$$

The logarithm in Eq. (6.94) develops an imaginary part for $p^2 > 4m^2$.
The calculation of the scalar functions B_{00} and B_{11} is more involved. Lorentz contracting both members of Eq. (6.87) with $g^{\mu\nu}$ and $p^\mu p^\nu$ we get the following system of equations

for B_{00} and B_{11} [5]:

$$\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{q^2}{[q^2-m^2]\,[(q+p)^2-m^2]}=nB_{00}(p^2,m^2)+p^2B_{11}(p^2,m^2) \tag{6.95}$$
$$\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)^2}{[q^2-m^2]\,[(q+p)^2-m^2]}=p^2B_{00}(p^2,m^2)+\left(p^2\right)^2B_{11}(p^2,m^2) \tag{6.96}$$

It is now necessary to compute the integrals on the l.h.s. of the above equations:

$$\begin{aligned}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{q^2}{[q^2-m^2]\,[(q+p)^2-m^2]} &= \mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{q^2-m^2+m^2}{[q^2-m^2]\,[(q+p)^2-m^2]}\\ &= \mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{1}{[(q+p)^2-m^2]}\\ &+ \mu^{4-n}m^2\int\frac{d^nq}{(2\pi)^n}\frac{1}{[q^2-m^2]\,[(q+p)^2-m^2]}\\ &= A_0(m)+m^2B_0(p^2;m,m)\,,\end{aligned}$$

$$\begin{aligned}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)^2}{[q^2-m^2]\,[(q+p)^2-m^2]} &= \frac{1}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{\left[q^2+2(q\cdot p)+p^2-m^2\right](q\cdot p)}{[q^2-m^2]\,[(q+p)^2-m^2]}\\ &- \frac{1}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{\left[q^2+p^2-m^2\right](q\cdot p)}{[q^2-m^2]\,[(q+p)^2-m^2]}\\ &= \frac{1}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)}{[q^2-m^2]}\\ &- \frac{1}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)}{[(q+p)^2-m^2]}\\ &- \frac{p^2}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)}{[q^2-m^2]\,[(q+p)^2-m^2]}\\ &= \frac{1}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)}{[q^2-m^2]}\\ &- \frac{1}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)-p^2}{[q^2-m^2]}\\ &- \frac{p^2}{2}\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{(q\cdot p)}{[q^2-m^2]\,[(q+p)^2-m^2]}\\ &= +\frac{p^2}{2}A_0(m)-\frac{(p^2)^2}{2}B_1(p^2;m,m)\,,\end{aligned} \tag{6.97}$$

where we have put to zero the integrals with denominator even under the exchange $q_\mu \to -q_\mu$ and the numerator odd under the same exchange.

[5] Since B_{00} and B_{11} can contain terms proportional to $\frac{1}{\varepsilon}$, the coefficients must be worked out in n dimensions, otherwise we miss finite terms. In particular, $g^{\mu\nu}g_{\mu\nu}=nI$.

The system of Eqs. (6.95) and (6.96) becomes

$$A_0(m) + m^2 B_0(p^2; m, m) = n B_{00}(p^2, m^2) + p^2 B_{11}(p^2, m^2) \tag{6.98}$$

$$\frac{p^2}{2} A_0(m) - \frac{(p^2)^2}{2} B_1(p^2; m, m) = p^2 B_{00}(p^2, m^2) + \left(p^2\right)^2 B_{11}(p^2, m^2)\,. \tag{6.99}$$

whose solution is

$$B_{00}(p^2, m^2) = \frac{1}{2(n-1)} \left[A_0(m) + 2m^2 B_0(p^2; m, m) + p^2 B_1(p^2; m, m)\right] \tag{6.100}$$

$$B_{11}(p^2, m^2) = \frac{1}{2(n-1)p^2} \left[(n-2) A_0(m) - 2m^2 B_0(p^2; m, m) - p^2 n B_1(p^2; m, m)\right]\,. \tag{6.101}$$

By means of the above expressions for the scalar functions $A_0(m)$, $B_{0/1}(p^2; m, m)$, $B_{00}(p^2, m)$ and $B_{11}(p^2, m)$, Eq. (6.86) can be rewritten as

$$\begin{aligned} \Sigma^A_{\mu\nu}(p) = & - 8ie^2 \left[g_{\mu\nu} B_{00}(p^2, m) + p_\mu p_\nu B_{11}(p^2, m)\right] \\ & - 8ie^2 p_\mu p_\nu B_1(p^2; m, m) + 4ie^2 g_{\mu\nu} \left(A_0(m) - \frac{p^2}{2} B_0(p^2; m, m)\right) . \end{aligned} \tag{6.102}$$

It is convenient to split $\Sigma^A_{\mu\nu}(p)$ in a transverse term, proportional to $g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$ and a longitudinal term, proportional to $p_\mu p_\nu$:

$$\begin{aligned} \Sigma^A_{\mu\nu}(p) = & - 2ie^2 \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \left[4B_{00}(p^2, m) - 2A_0(m) + p^2 B_0(p^2; m, m)\right] \\ & - 2ie^2 \frac{p_\mu p_\nu}{p^2} \left[4p^2 \left(B_{11}(p^2, m) + B_1(p^2; m, m)\right)\right. \\ & + \left.\left(4B_{00}(p^2, m) - 2A_0(m) + p^2 B_0(p^2, m)\right)\right]\,. \end{aligned} \tag{6.103}$$

Inserting the expressions for B_{00}, B_{11} and B_1 of Eqs. (6.94,6.100,6.101), it is straightforward to verify that the coefficient of $\frac{p_\mu p_\nu}{p^2}$ vanishes, in agreement with Eq. (6.66), derived on general grounds, according to which the photon self-energy can not have a longitudinal part because of gauge-invariance. Therefore,

$$\begin{aligned} \Sigma^A_{\mu\nu}(p) &= \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \Sigma^A_T(p^2)\,, \\ \Sigma^A_T(p^2) &= -2ie^2 \left[4B_{00}(p^2, m) - 2A_0(m) + p^2 B_0(p^2; m, m)\right]\,. \end{aligned} \tag{6.104}$$

Substituting Eqs. (6.94,6.98) in Eq. (6.104) we obtain the following expression for $\Sigma^A_T(p^2)$:

$$\Sigma^A_T(p^2) = -2ie^2 \left[\frac{(2-n)\left(2A_0(m) - p^2 B_0(p^2; m, m)\right) + 4m^2 B_0(p^2; m, m)}{(n-1)}\right]\,. \tag{6.105}$$

It is useful to work out further the expression of $\Sigma_T^A(p^2)$. In fact since the functions A_0, B_0 and B_1 contain the poles $\frac{1}{\varepsilon}$, we need to expand the coefficients around $n = 4$ up to terms of $\mathcal{O}(\varepsilon)$, in order to get explicitely the finite terms originating from products of the kind $\varepsilon\frac{1}{\varepsilon}$. To this aim it is convenient to observe the following identity:

$$A_0(m) = \frac{m^2 B_0(0, m, m)}{(1 - \varepsilon)}, \tag{6.106}$$

as can be verified. By means of Eq. (6.106) we can rewrite Eq. (6.105) as follows:

$$\begin{aligned}
\Sigma_T^A(p^2) &= -2i\frac{e^2}{3-2\varepsilon}\left[\frac{-4m^2(1-\varepsilon)}{(1-\varepsilon)}B_0(0) + 2(1-\varepsilon)p^2 B_0(p^2) + 4m^2 B_0(p^2)\right] && (6.107)\\
&= -\frac{2}{3}ie^2\left(1+\frac{2}{3}\varepsilon\right)\left\{2(1-\varepsilon)p^2 B_0(p^2;m,m) + 4m^2\left[B_0(p^2;m,m) - B_0(0;m,m)\right]\right\}\\
&= -\frac{2}{3}ie^2\left\{2p^2 B_0(p^2;m,m) - \frac{2p^2}{3}\left(\frac{i}{16\pi^2}\right) + 4m^2\left[B_0(p^2;m,m) - B_0(0;m,m)\right]\right\}\\
&+ \mathcal{O}(\varepsilon)\,. && (6.108)
\end{aligned}$$

Since the difference $B_0(p^2;m,m) - B_0(0;m,m)$ is proportional to p^2 (it can be seen by Taylor expansion around $p^2 = 0$), Eq. (6.108) says that $\Sigma_T^A(p^2) \sim p^2$ and, therefore, $\Sigma_T^A(0) = 0$, as required by Eq. (6.66) and the photon remains massless after one-loop corrections.

6.4.2 Electron self-energy

The two-point 1PI electron Green function $\Gamma^{(0,2)}$, in momentum space, can be decomposed into its free part and into the so called electron self-energy $\Sigma^\psi(p)$ as follows:

$$\begin{aligned}
\Gamma^{(0,2)}(p,-p) &= (\not{p} - m) + \Sigma^\psi(p)\\
&= (\not{p} - m) + \left[\not{p}\Sigma_V^\psi(p^2) + m\Sigma_S^\psi(p^2)\right]\,. && (6.109)
\end{aligned}$$

$\Sigma^\psi(p)$ can be calculated from the Feynman diagram of Fig. (6.2):

$$\Sigma^\psi(p) = ie^2\mu^{(4-n)}\int\frac{d^n q}{(2\pi)^n}\frac{\gamma_\mu(\not{q}+m)\gamma^\mu}{[q^2-m^2+i\varepsilon]\,[(q-p)^2-\lambda^2+i\varepsilon]}\,. \tag{6.110}$$

In Eq. (6.110) we have adopted the Feynman gauge for the photon propagator, for the sake of simplicity, and we have introduced a "small" photon mass, in order to avoid possible infrared divergences (to be discussed later on in Section 6.6). This procedure breaks gauge invariance. However, in the present case of QED, it can be shown that the gauge-breaking terms are proportional to λ and thus vanish for physical quantities, calculated in the limit $\lambda \to 0$ [6].

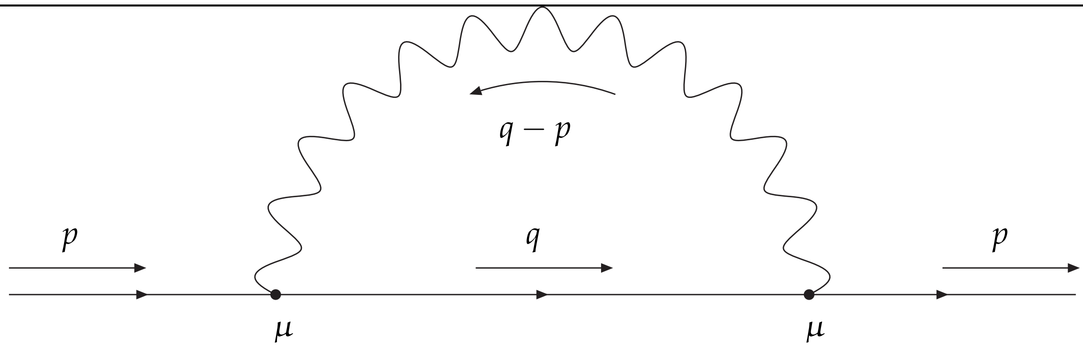

Figure 6.2: One-loop electron self-energy diagram.

In the following we will assume, for simplicity of notation, that the infinitesimal imaginary part $-i\varepsilon$ is understood in the terms m^2 and λ^2. Performing the index contraction in th numerator, Eq. (6.110) becomes

$$\Sigma^{\psi}(p) = ie^2\mu^{(4-n)}\int \frac{d^nq}{(2\pi)^n}\,\frac{(2-n)\not{q}+nm}{[q^2-m^2]\,[(q-p)^2-\lambda^2]}\,. \tag{6.111}$$

The term proportional to $\not{q}$ gives a tensor integral of the same kind of Eq.(6.88), but with two different masses in the denominator:

$$\mu^{4-n}\int \frac{d^nq}{(2\pi)^n}\frac{q_\mu}{[q^2-m^2]\,[(q-p)^2-\lambda^2]} = p_\mu B_1(p^2;m,\lambda)\,. \tag{6.112}$$

Contracting both sides of Eq. (6.112) with p^μ we get

$$\begin{aligned} p^2B_1(p^2;m,\lambda) &= \mu^{4-n}\int \frac{d^nq}{(2\pi)^n}\frac{q\cdot p}{[q^2-m^2]\,[(q-p)^2-\lambda^2]} \\ &= \frac{1}{2}\mu^{4-n}\int \frac{d^nq}{(2\pi)^n}\frac{-\left[(q-p)^2-\lambda^2\right]+q^2-m^2+p^2-\lambda^2+m^2}{[q^2-m^2]\,[(q-p)^2-\lambda^2]} \\ &= -\frac{1}{2}A_0(m)+\frac{1}{2}A_0(\lambda)+\frac{1}{2}\left(p^2-\lambda^2+m^2\right)B_0(p^2;m,\lambda)\,. \end{aligned} \tag{6.113}$$

Neglecting terms of $\mathcal{O}(\lambda^2)$, we have

$$B_1(p^2;m,\lambda) = \frac{1}{2p^2}\left[-A_0(m)+\left(p^2+m^2\right)B_0(p^2;m,\lambda)\right]\,, \tag{6.114}$$

[6]In the discussion of infrared divergencies, we will see that they are cancelled by the interplay between virtual and real radiation, at variance with ultraviolet divergencies, which are cancelled by the renormalization procedure. An alternative, gauge invariant, way of regularize the theory from infrared divergencies is the dimensional regularization, which we will not treat here (it is treated, for instance, in the book of T. Muta, cited in the bibliography). Suffice it to say that for non-abelian gauge theories with massless gauge bosons, as is the case of Quantum Chromodynamics, it is much more convenient to regularize the infrared divergencies with dimensional regularization.

and Eq. (6.111) becomes

$$\begin{aligned}\Sigma^{\psi}(p) &= ie^2\mu^{(4-n)}\int\frac{d^nq}{(2\pi)^n}\,\frac{(2-n)\not{q}+nm}{[q^2-m^2]\,[(q-p)^2-\lambda^2]}\\ &= ie^2\Bigg\{(2-n)\frac{\not{p}}{2p^2}\left[-A_0(m)+(p^2+m^2)B_0(p^2;m,\lambda)\right]\\ &+\ nmIB_0(p^2;m,\lambda)\Bigg\}\,. \qquad (6.115)\end{aligned}$$

Substituting in Eq. (6.115) $n=4-2\varepsilon$, we can write

$$\Sigma^{\psi}(p) = \not{p}\Sigma^{\psi}_V(p^2)+\Sigma^{\psi}_S(p^2)\,, \qquad (6.116)$$

$$\Sigma^{\psi}_V(p^2) = ie^2\left\{(1-\varepsilon)\frac{\left[A_0(m)-(p^2+m^2)B_0(p^2;m,\lambda)\right]}{p^2}\right\}, \qquad (6.117)$$

$$\Sigma^{\psi}_S(p^2) = ie^2(4-2\varepsilon)B_0(p^2;m,\lambda)\,. \qquad (6.118)$$

6.4.3 Explicit calculation of the Ward identity at one-loop

In this section we verify explicitely at one-loop order the Ward identity of Eqs. (6.47) and (6.48), between the 1PI photon-electron vertex

$$\left(\hat{\Gamma}_b\right)^{(1,2)}_{\mu}(k,p_1,p_2) = e\gamma_\mu + e\Lambda_\mu(q,p_1,p_2)\,, \qquad (6.119)$$

where we have added the one-loop term $e\Lambda_\mu$ to the tree-level term $e\gamma_\mu$, and the electron self-energy. The vertex correction is given by the Feynman diagram of Fig. (6.3). Its expression can be calculated as follows:

$$\Lambda_\mu(q,p_1,p_2) = -ie^2\mu^{(4-n)}\int\frac{d^nq}{(2\pi)^n}\gamma_\rho\frac{(\not{q}-\not{p}_1+m)}{[(q-p_1)^2-m^2]}\gamma_\mu\frac{(\not{q}+\not{p}_1+m)}{[(q+p_1)^2-m^2]}\gamma^\rho\frac{1}{q^2-\lambda^2}\,. \qquad (6.120)$$

Upon contracting both sides of Eq. (6.119) with $k^\mu=-p_1^\mu-p_2^\mu$ [7], the term γ_μ on the r.h.s. gets replaced with $-\not{p}_1-\not{p}_2$. It is now useful to write the following identity (simply by addition and subtraction):

$$-\not{p}_1-\not{p}_2 = (-\not{p}_1+\not{q}-m)+(-\not{q}-\not{p}_2+m) \qquad (6.121)$$

[7] We remind that we assume all momenta as incoming into the vertex, *i.e.* momentum conservation requires $k+p_1+p_2=0$.

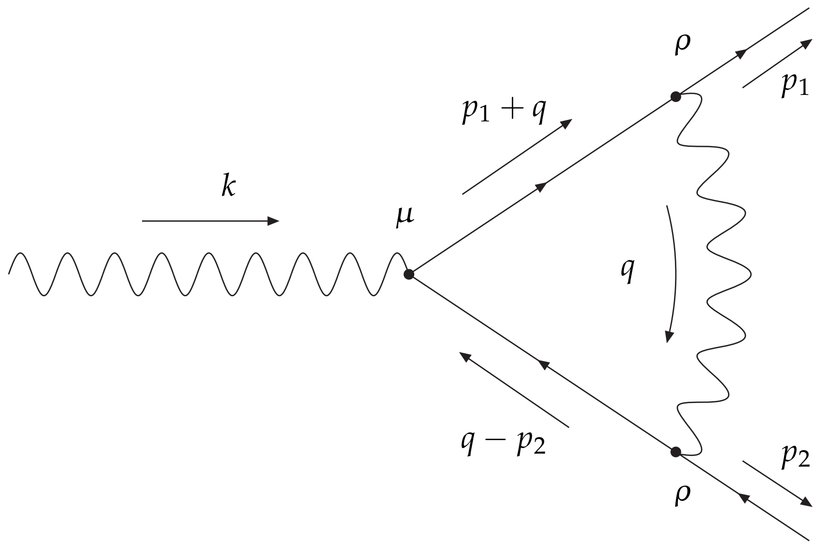

Figure 6.3: One-loop vertex correction diagram.

Substituting Eq. (6.121) in Eq. (6.120) we obtain:

$$
\begin{aligned}
k^\mu \Lambda_\mu &= -ie^2\mu^{(4-n)} \int \frac{d^nq}{(2\pi)^n} \frac{\gamma_\rho(\not{q} - \not{p}_1 + m)\left[(\not{q} - \not{p}_1 - m) + (-(\not{q} + \not{p}_2) + m)\right](\not{q} + \not{p}_2 + m)\gamma^\rho}{(q^2-\lambda^2)((q-p_1)^2-m^2)((q+p_2)^2-m^2)} \\
&= -ie^2\mu^{(4-n)} \int \frac{d^nq}{(2\pi)^n} \frac{\gamma_\rho((q-p_1)^2-m^2)(\not{q} + \not{p}_2 + m)\gamma^\rho}{(q^2-\lambda^2)((q-p_1)^2-m^2)((q+p_2)^2-m^2)} \\
&\quad +ie^2\mu^{(4-n)} \int \frac{d^nq}{(2\pi)^n} \frac{\gamma_\rho(\not{q} - \not{p}_1 + m)((q+p_2)^2-m^2)\gamma^\rho}{(q^2-\lambda^2)((q-p_1)^2-m^2)((q+p_2)^2-m^2)} \\
&= -ie^2\mu^{(4-n)} \int \frac{d^nq}{(2\pi)^n} \frac{\gamma_\rho(\not{q} + \not{p}_2 + m)\gamma^\rho}{(q^2-\lambda^2)((q+p_2)^2-m^2)} \\
&\quad +ie^2\mu^{(4-n)} \int \frac{d^nq}{(2\pi)^n} \frac{\gamma_\rho(\not{q} - \not{p}_1 + m)\gamma^\rho}{(q^2-\lambda^2)((q-p_1)^2-m^2)} \\
&= -\Sigma^\psi(p_2) + \Sigma^\psi(-p_1) \\
&= \Sigma^\psi(p_2 + k) - \Sigma^\psi(p_2)\,.
\end{aligned}
\tag{6.122}
$$

6.4.4 QED counterterms in the on-shell renormalization scheme

In this section we will specify to the on-loop case the renormalization conditions described in Section (6.3.2), deriving explicit expressions for the counterterms, at one-loop. As we have seen, due to gauge invariance and the related the Ward identity, we need to determine only three counterterms, δZ_A, δZ_m and δZ_ψ, from the renormalization conditions involving the 1PI two-point Green function $\Gamma^{(2,0)}_{R,\mu\nu}$ and the 1PI two-point Green function $\Gamma^{(0,2)}_R$. First of all we have to derive the renormalized expressions, containing the dependence on the counterterms, of the scalar functions contained in the 1PI two-point Green functions. Then we put these expressions in the equations imposed by the on-shell renormalization conditions. The resultant equations are solved in order to find the explicit form of the one-loop counterterms.

From the QED Lagrangian expression of Eqs (6.53), (6.54) and Eq. (6.60) we can write

$$\begin{aligned}\Gamma^{(2,0)}_{R,\mu\nu}(q,-q) &= \Gamma^{(2,0)}_{\mu\nu}(q,-q) - \delta Z_A\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \quad (6.123)\\ \Gamma^{(0,2)}_{R}(p,-p) &= \Gamma(0,2)(p,-p) - \delta Z_m m + \delta Z_\psi\left(\not{p} - m\right), \quad (6.124)\end{aligned}$$

where we have used the symbol Γ_R to denote the renormalized, finite Green functions.

Expression of δZ_A

From Eq. (6.84) and Eq. (6.104) we have

$$\begin{aligned}\Gamma^{(2,0)}_{\mu\nu}(q,-q) &= -\left[g_{\mu\nu}q^2 - q_\mu q_\nu(1-\xi)\right] - \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Sigma^A_T(p^2)\\ &= -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left[q^2 + \Sigma^A_T(q^2)\right] - \xi q_\mu q_\nu\,. \quad (6.125)\end{aligned}$$

From Eq. (6.125) and according to the notation of Eq. (6.62), we can express the renormalized scalar function multiplying the transverse Lorentz tensor as follows:

$$\Gamma^A_{R,T}(q^2) = -\left(q^2 + \Sigma^A_T(q^2) + q^2\delta Z_A\right). \quad (6.126)$$

The condition of Eq. (6.68) gives

$$\frac{\partial}{\partial q^2}\left[-\left(q^2 + \Sigma^A_T(q^2) + q^2\delta Z_A\right)\right]\Big|_{q^2=0} = -1\,, \quad (6.127)$$

i.e.

$$\delta Z_A = -\frac{\partial\Sigma^A_T(q^2)}{\partial q^2}\Big|_{q^2=0}. \quad (6.128)$$

Taking the expression of Eq. (6.107) for $\Sigma^A_T(q^2)$ we get

$$\delta Z_A = +4ie^2\frac{(1-\varepsilon)B_0(0;m,m) + 2m^2B_0'(0;m,m)}{3-2\varepsilon}, \quad (6.129)$$

where we have assumed that $B_0'(p^2;m,m)$ does not contain terms of the form $\frac{1}{p^2}$, as we will see below. In order to evaluate $\frac{\partial B_0(p^2;m,m)}{\partial p^2}$, it is convenient to express the function $B_0(p^2;m,m)$ in terms of Feynman parameters, as already done in Eq. (5.69):

$$\begin{aligned}B_0(p^2;m,m) &= \mu^{(4-n)}\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-m^2)((q+p)^2-m^2)}\\ &= i\mu^{(4-n)}\int\frac{d^nq_E}{(2\pi)^n}\frac{1}{(q_E^2+m^2)((q_E+p_E)^2+m^2)}\\ &= i\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}\mu^{2\varepsilon}\int_0^1 dx\,(p_E^2x(1-x)+m^2)^{-\varepsilon}\\ &= \frac{i}{16\pi^2}(4\pi)^\varepsilon\Gamma(\varepsilon)\int_0^1 dx\left(\frac{p^2x^2-p^2x+m^2}{\mu^2}\right)^{-\varepsilon}, \quad (6.130)\end{aligned}$$

where we have used euclidean momenta in the intermediate steps and

$$\frac{\pi^{\frac{n}{2}}}{(2\pi)^n} = \frac{\pi^2\pi^{-\varepsilon}}{(2\pi)^4(4\pi^2)^{-\varepsilon}} = \frac{1}{16\pi^2}(4\pi)^{\varepsilon}\,. \tag{6.131}$$

From Eq. (6.130), setting $p^2 = 0$ in the integrand, we derive straightforwardly

$$B_0(0; m, m) = \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\Gamma(\varepsilon)\,. \tag{6.132}$$

Taking the derivative under the integral in Eq. (6.130), we have

$$\frac{\partial}{\partial p^2}B_0(p^2; m, m) = \frac{i}{16\pi^2}(4\pi)^{\varepsilon}\Gamma(\varepsilon)(-\varepsilon)\int_0^1 dx\,\left(\frac{p^2x^2 - p^2x + m^2}{\mu^2}\right)^{(-1-\varepsilon)}\frac{1}{\mu^2}(x^2 - x)\,. \tag{6.133}$$

We can now set directly $p^2 = 0$ in Eq. (6.133):

$$\begin{aligned}
\left.\frac{\partial B_0(p^2; m, m)}{\partial p^2}\right|_{p^2=0} &= \frac{i}{16\pi^2}(4\pi)^{\varepsilon}\Gamma(\varepsilon)(-\varepsilon)\left(\frac{m^2}{\mu^2}\right)^{(-1-\varepsilon)}\frac{1}{\mu^2}\int_0^1 dx\,(x^2 - x) \\
&= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\frac{\varepsilon\Gamma(\varepsilon)}{6m^2}\,.
\end{aligned} \tag{6.134}$$

Inserting Eq. (6.132) and Eq. (6.134) into Eq. (6.129), we get

$$\begin{aligned}
\delta Z_A &= +4ie^2\left(\frac{i}{16\pi^2}\right)\left(\frac{1}{3 - 2\varepsilon}\right)\left[(1 - \varepsilon)\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\Gamma(\varepsilon) + 2m^2\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\frac{\varepsilon\Gamma(\varepsilon)}{6m^2}\right] \\
&= -\frac{e^2}{4\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\frac{1 - \varepsilon + \frac{1}{3}\varepsilon}{3 - 2\varepsilon}\Gamma(\varepsilon) \\
&= -\frac{\alpha}{3}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\Gamma(\varepsilon)\,.
\end{aligned} \tag{6.135}$$

Using in the above equation the relation $e^2 = 4\pi\alpha$, relating the electromagnetic coupling constant at the fine structure constant, and the Laurent expansions around the pole $\varepsilon = 0$ as in Section 5.5, we find the final expression for the counterterm δZ_A:

$$\delta Z_A = -\frac{\alpha}{3}\left[\Delta_{UV} + \log\left(\frac{\mu^2}{m^2}\right)\right] + \mathcal{O}(\varepsilon)\,. \tag{6.136}$$

Expression of δZ_m and δZ_ψ

Inserting Eq. (6.109) into Eq. (6.124), we get

$$\Gamma_R^{(0,2)}(p, -p) = (\not{p} - m) + \left[\not{p}\Sigma_V^{\psi}(p^2) + m\Sigma_S^{\psi}(p^2)\right] - \delta Z_m m + \delta Z_\psi\,(\not{p} - m)\,. \tag{6.137}$$

From Eq. (6.71), which we report here, we also had

$$\Gamma_R^{(0,2)}(p,-p) = (\not{p} - m)\Gamma_{R,V}^{(0,2)}(p^2) + m\left(\Gamma_{R,V}^{(0,2)}(p^2) + \Gamma_{R,S}^{(0,2)}(p^2)\right) . \tag{6.138}$$

Equating Eq. (6.137) and Eq. (6.138), we find the following relations [8]

$$\begin{aligned} \Gamma_{R,V}^{(0,2)} &= 1 + \Sigma_V^{\psi}(p^2) + \delta Z_{\psi} && (6.139)\\ \Gamma_{R,S}^{(0,2)} &= -1 + \Sigma_S^{\psi}(p^2) - \delta Z_m - \delta Z_{\psi} . && (6.140) \end{aligned}$$

$\Gamma_{R,V}^{(0,2)}$ and $\Gamma_{R,S}^{(0,2)}$ have to satisfy the on-shell renormalization conditions of Eq. (6.75) and Eq. (6.76):

$$\begin{aligned} &1 + \Sigma_V^{\psi}(m^2) + \delta Z_{\psi} - 1 + \Sigma_S^{\psi}(m^2) - \delta Z_m - \delta Z_{\psi} = 0\\ &1 + \Sigma_V^{\psi}(m^2) + \delta Z_{\psi} + 2m^2\left[\Sigma_V^{\psi'}(m^2) + \Sigma_S^{\psi'}(m^2)\right] = 1 , \end{aligned} \tag{6.141}$$

i.e.

$$\begin{aligned} \delta Z_m &= \Sigma_V^{\psi}(m^2) + \Sigma_S^{\psi}(m^2) && (6.142)\\ \delta Z_{\psi} &= -\Sigma_V^{\psi}(m^2) - 2m^2\left[\Sigma_V^{\psi'}(m^2) + \Sigma_S^{\psi'}(m^2)\right] . && (6.143) \end{aligned}$$

Taking from Eqs. (6.117) and (6.118) the expressions of $\Sigma_V^{\psi}(p^2)$ and $\Sigma_S^{\psi}(p^2)$, respectively, we find

$$\delta Z_m = ie^2\left[(1-\varepsilon)\frac{A_0(m) - 2m^2 B_0(m^2, m, \lambda)}{m^2} + (4 - 2\varepsilon)B_0(m^2, m, \lambda)\right] . \tag{6.144}$$

We can write the function $B_0(p^2; m, \lambda)$ in the same way as in Eq. (6.130), considering the general expression of the argument of the logarithm for two different masses, as in Eq. (5.63):

$$B_0(p^2, m, \lambda) = \frac{i}{16\pi^2}(4\pi)^{\varepsilon}\Gamma(\varepsilon)\int_0^1 dx\left(\frac{-p^2 x(1-x) + \lambda^2 + (m^2 - \lambda^2)x}{\mu^2}\right)^{-\varepsilon} . \tag{6.145}$$

Setting $p^2 = m^2$ and $\lambda^2 = 0$ (the limit $\lambda \to 0$ of $B_0(p^2, m, \lambda)$ is not singular), we have

$$\begin{aligned} B_0(m^2, m, 0) &= \frac{i}{16\pi^2}(4\pi)^{\varepsilon}\Gamma(\varepsilon)\int_0^1 dx\left(\frac{m^2 x^2}{\mu^2}\right)^{-\varepsilon}\\ &= \frac{i}{16\pi^2}(4\pi)^{\varepsilon}\frac{\Gamma(\varepsilon)}{1-2\varepsilon}\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} . \end{aligned} \tag{6.146}$$

[8] $\Gamma_{R,V}(p^2)$ is the coefficient of $\not{p}$, while $\Gamma_{S,V}(p^2)$ is the coefficient of m.

Analogously, for the $A_0(m)$ function we have, taking the results of Eqs.(5.50,5.51):

$$\begin{aligned} A_0(m) &= \mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\frac{1}{q^2-m^2} = -i\mu^{4-n}\int\frac{d^nq_E}{(2\pi)^n}\frac{1}{q_E^2+m^2} \\ &= (-i)\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma\left(1-\frac{n}{2}\right)}{\Gamma(1)}\frac{(m^2)^{\frac{n}{2}-1}}{\mu^{-2\varepsilon}} = +i\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma(\varepsilon)}{1-\varepsilon}\frac{(m^2)^{\frac{n-4+2}{2}}}{\mu^{-2\varepsilon}} \\ &= \frac{i}{16\pi^2}m^2(4\pi)^\varepsilon\frac{\Gamma(\varepsilon)}{1-\varepsilon}\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon}, \end{aligned} \tag{6.147}$$

where we have used the expression of Eq. (6.131). Using Eqs. (6.147) and (6.146) in Eq. (6.144), we get

$$\delta Z_m = ie^2\left(\frac{i}{16\pi^2}\right)(4\pi)^\varepsilon\Gamma(\varepsilon)m^2\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon}\frac{3-2\varepsilon}{1-2\varepsilon}. \tag{6.148}$$

Using in the above equation the relation $e^2 = 4\pi\alpha$, relating the electromagnetic coupling constant at the fine structure constant, and the Laurent expansions around the pole $\varepsilon = 0$ as in Section 5.5, we find the final expression for the counterterm δZ_m:

$$\delta Z_m = -\frac{3\alpha}{4\pi}\left[\Delta_{UV} - \ln\left(\frac{m^2}{\mu^2}\right) + \frac{4}{3}\right] + \mathcal{O}(\varepsilon). \tag{6.149}$$

In order to determine δZ_ψ, we have to take the derivative, w.r.t. p^2, of $\Sigma_V^\psi(p^2)$ and $\Sigma_S^\psi(p^2)$ of Eqs. (6.117) and (6.118):

$$\begin{aligned} \Sigma_V^{\psi\prime}(p^2) &= ie^2\Big\{-\frac{1}{(p^2)^2}(1-\varepsilon)\Big[A_0(m)-(p^2+m^2)B_0(p^2;m,\lambda)\Big] \\ &- \frac{(1-\varepsilon)}{p^2}\Big[B_0(p^2;m,\lambda)+(p^2+m^2)B_0'(p^2;m,\lambda)\Big]\Big\} \end{aligned} \tag{6.150}$$

$$\Sigma_S^{\psi\prime}(p^2) = ie^2(4-2\varepsilon)B_0'(p^2;m,\lambda). \tag{6.151}$$

To evaluate $B_0'(p^2;m,\lambda)$, we take the derivative w.r.t. p^2 of both sides of Eq. (6.145):

$$B_0'(p^2;m,\lambda) = \frac{i}{16\pi^2}(4\pi)^\varepsilon\Gamma(\varepsilon)(-\varepsilon)\int_0^1 dx\left(\frac{\mu^2}{-p^2x(1-x)+\lambda^2+(m^2-\lambda^2)x}\right)^{1+\varepsilon}\left[\frac{-x(1-x)}{\mu^2}\right]$$

Setting $p^2 = m^2$ in the above equation we have

$$\begin{aligned} B_0'(m^2;m,\lambda) &= \frac{i}{16\pi^2}(4\pi)^\varepsilon\Gamma(\varepsilon)\varepsilon\int_0^1 dx\,\frac{\mu^{2\varepsilon}x(1-x)}{(m^2x^2+\lambda^2-\lambda^2x)^{1+\varepsilon}} \\ &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon\frac{\varepsilon\Gamma(\varepsilon)}{m^2}\int_0^1 dx\,\frac{x(1-x)}{\left[x^2+\frac{\lambda^2}{m^2}(1-x)\right]^{1+\varepsilon}}. \end{aligned} \tag{6.152}$$

Remark 1: the above expression contains the term $\varepsilon\Gamma(\varepsilon)$, which is **not UV divergent**. This is due to the fact that the coefficient of the ultraviolet term Δ_{UV} is independent of the momentum. As a consequence, *we can set $\varepsilon = 0$ in Eq. (6.152)*.
Remark 2: the integral contained in Eq. (6.152) is **infrared divergent**: the integral is divergent if we set $\lambda = 0$. This is the reason why we introduced a photon mass in Section 6.4.2. Furthermore when $\lambda \neq 0$, the term $x^2 + \frac{\lambda^2}{m^2}(1-x)$ is always positive for $\lambda \ll m$. The small photon mass λ regularizes the theory for the infrared (IR) divergences. The IR divergencies are typical of theories with massless particles. Contrary to the UV divergencies, the IR ones are due to the low energy regime of the theory (*soft limit*). **The UV divergencies are not removed by the renormalization procedure but by the interplay between virtual and real radiation**. We will discuss in some more detail how this interplay works in Section 6.6.
According to the above remarks, we can rewrite Eq. (6.152) with $\varepsilon = 0$:

$$B_0'(m^2; m, \lambda) = \frac{i}{16\pi^2}\frac{1}{m^2}\int_0^1 dx\, \frac{x(1-x)}{\left[x^2 + \frac{\lambda^2}{m^2}(1-x)\right]}\,. \tag{6.153}$$

We now perform the x integration in Eq. (6.153) neglecting terms of $\mathcal{O}(\lambda)$:

$$\begin{aligned}
\int_0^1 dx\, \frac{x(1-x)}{\left[x^2 + \frac{\lambda^2}{m^2}(1-x)\right]} &= -1 + \int_0^1 dx\, \frac{x}{\left[x^2 + \frac{\lambda^2}{m^2}(1-x)\right]} + \frac{\lambda^2}{m^2}\int_0^1 dx\, \frac{1-x}{\left[x^2 + \frac{\lambda^2}{m^2}(1-x)\right]} \\
&= -1 + \frac{1}{2}\int_0^1 dx\, \frac{2x - \frac{\lambda^2}{m^2}}{\left[x^2 + \frac{\lambda^2}{m^2}(1-x)\right]} + \frac{1}{2}\frac{\lambda^2}{m^2}\int_0^1 dx\, \frac{1}{\left[x^2 + \frac{\lambda^2}{m^2}(1-x)\right]} \\
&= -1 - \frac{1}{2}\ln\frac{\lambda^2}{m^2} + \mathcal{O}(\lambda^2)\,. \qquad (6.154)
\end{aligned}$$

By means of the above result we can write

$$B_0'(m^2; m, \lambda) = \frac{i}{16\pi^2}\left(-1 - \frac{1}{2}\ln\frac{\lambda^2}{m^2}\right) + \mathcal{O}(\lambda^2)\,. \tag{6.155}$$

Substituting Eq. (6.117), together with Eqs. (6.150), (6.151), (6.147), (6.146) and (6.155) in Eq. (6.143), we obtain

$$\delta Z_\psi = \frac{e^2}{16\pi^2}\left[-(4\pi)^\varepsilon\Gamma(\varepsilon)\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} - 4\left(1 + \ln\frac{\lambda^2}{m^2}\right)\right] + \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda^2)\,. \tag{6.156}$$

Using in the above equation the relation $e^2 = 4\pi\alpha$, relating the electromagnetic coupling constant at the fine structure constant, and the Laurent expansions around the pole $\varepsilon = 0$ as in Section 5.5, we find the final expression for the counterterm δZ_ψ:

$$\delta Z_\psi = -\frac{\alpha}{4\pi}\left[\Delta_{UV} - \ln\frac{m^2}{\mu^2} + 2\left(\ln\frac{\lambda^2}{m^2} + 2\right)\right] + \mathcal{O}(\varepsilon) + \mathcal{O}(\lambda^2)\,. \tag{6.157}$$

The IR divergence is displayed in the logarithmic term containig the mass of the photon. Eqs. (6.136), (6.149) and (6.157) are the complete expressions for QED renormalization at one-loop order.

6.5 The anomalous magnetic moment of the electron

See the minutes

6.6 Infrared divergencies in QED

6.6.1 Soft bremsstrahlung

See the minutes

6.6.2 Kinoshita-Lee-Nauenberg theorem at $\mathcal{O}(\alpha)$: example of cancellation of infrared divergencies

See the minutes

7

The renormalization group

We consider now the n-point (regularized, bare) Green function of Eq. (5.111), in momentum representation:

$$\hat{G}_E^{b\,(n)}(p_1, \ldots, p_n; \lambda_b(\lambda, m, \mu, \varepsilon), m_b(\lambda, m, \mu, \varepsilon), \varepsilon)\,. \tag{7.1}$$

The key observation is that, **in $\hat{G}_E^{b\,(n)}$, the two parameters λ_b and m_b of the bare theory depend on three parameters λ, m and μ, characterizing the renormalized theory**. As a consequence, the derivatives of $\hat{G}_E^{b\,(n)}$ w.r.t. λ, m and μ are not independent but must fullfil a constraint of the kind

$$\left(\mu\frac{\partial}{\partial\mu} + (\gamma_m)_\varepsilon m\frac{\partial}{\partial m} + \beta_\varepsilon\frac{\partial}{\partial\lambda}\right)\hat{G}_E^{b\,(n)}(p_1, \ldots, p_n; \lambda_b(\lambda, m, \mu, \varepsilon), m_b(\lambda, m, \mu, \varepsilon), \varepsilon) = 0\,. \tag{7.2}$$

We kept the suffix ε in γ_m and β in order to remind that we are working with the regularized bare Green function. We remind also the relations of Eqs. (5.91), (5.92) and (5.93), which we report here:

$$\begin{aligned} m_b &= Z_m^{\frac{1}{2}} m & (7.3)\\ \lambda_b &= \mu^{2\varepsilon}\frac{Z_\lambda}{Z_\varphi^2}\lambda & (7.4)\\ Z_i &= Z_i\left(\lambda, \varepsilon\right)\,, & (7.5) \end{aligned}$$

where, for the sake of simplicity, in Eq. (7.5) we assume that the factors Z_i are mass independent. Through the above relations we can calculate the derivatives necessary to the determination of the constants γ_m and β introduced in Eq. (7.2):

$$\begin{aligned} \frac{\partial \lambda_b}{\partial m} &= 0 \\ \frac{\partial m_b}{\partial m} &= \frac{m_b}{m} \\ \frac{\partial \lambda_b}{\partial \mu} &= \frac{2\varepsilon}{\mu}\lambda_b \\ \frac{\partial m_b}{\partial \mu} &= 0\,. \end{aligned} \tag{7.6}$$

Denoting with ξ the generic parameter μ, λ or m, we can write

$$\frac{\partial \hat{G}_E^{b\,(n)}}{\partial \xi} = \left(\frac{\partial \lambda_b}{\xi} \frac{\partial}{\partial \lambda_b} + \frac{\partial m_b}{\partial \xi} \frac{\partial}{\partial m_b} \right) \hat{G}_E^{b\,(n)}\,. \tag{7.7}$$

Explicitating the values of ξ and using the derivatives of Eq. (7.6), we can write

$$\begin{aligned} \frac{\partial \hat{G}_E^{b\,(n)}}{\partial \lambda} &= \left(\frac{\partial \lambda_b}{\partial \lambda} \frac{\partial}{\partial \lambda_b} + \frac{\partial m_b}{\partial \lambda} \frac{\partial}{\partial m_b} \right) \hat{G}_E^{b\,(n)} \\ m\frac{\partial \hat{G}_E^{b\,(n)}}{\partial m} &= m_b \frac{\partial}{\partial m_b} \hat{G}_E^{b\,(n)} \\ \mu\frac{\partial \hat{G}_E^{b\,(n)}}{\partial \mu} &= 2\varepsilon\lambda_b \frac{\partial}{\partial \lambda_b} \hat{G}_E^{b\,(n)}\,. \end{aligned} \tag{7.8}$$

Inserting Eq. (7.6) in Eq. (7.2) we get

$$\left\{ 2\varepsilon\lambda_b \frac{\partial}{\partial \lambda_b} + (\gamma_m)_\varepsilon m_b \frac{\partial}{\partial m_b} + \beta_\varepsilon \left(\frac{\partial \lambda_b}{\partial \lambda} \frac{\partial}{\partial \lambda_b} + \frac{\partial m_b}{\partial \lambda} \frac{\partial}{\partial m_b} \right) \right\} \hat{G}_E^{b\,(n)} = 0\,, \tag{7.9}$$

which becomes, grouping the coefficients of the two independent derivatives w.r.t λ_b and m_b:

$$\left\{ \left[2\varepsilon\lambda_b + \beta_\varepsilon \frac{\partial \lambda_b}{\partial \lambda} \right] \frac{\partial}{\partial \lambda_b} + \left[(\gamma_m)_\varepsilon m_b + \beta_\varepsilon \frac{\partial m_b}{\partial \lambda} \right] \frac{\partial}{\partial m_b} \right\} \hat{G}_E^{b\,(n)} = 0\,. \tag{7.10}$$

Eq. (7.10) is an identity if the coefficients of $\frac{\partial}{\partial \lambda_b}$ and $\frac{\partial}{\partial m_b}$ are separately equal to zero:

$$\frac{\partial \lambda_b}{\partial \lambda} \beta_\varepsilon = -2\varepsilon\lambda_b \tag{7.11}$$

$$\frac{\partial m_b}{\partial \lambda} \beta_\varepsilon = -m_b (\gamma_m)_\varepsilon\,. \tag{7.12}$$

We can solve the system w.r.t. β_ε and $(\gamma_m)_\varepsilon$:

$$\beta_\varepsilon = -2\varepsilon\lambda_b\left(\frac{\partial\lambda_b}{\partial\lambda}\right)^{-1} \tag{7.13}$$

$$(\gamma_m)_\varepsilon = -\beta_\varepsilon\frac{1}{m_b}\frac{\partial m_b}{\partial\lambda} = \frac{2\varepsilon\lambda_b}{m_b}\frac{\partial m_b}{\partial\lambda}\left(\frac{\partial\lambda_b}{\partial\lambda}\right)^{-1}. \tag{7.14}$$

Remark: according to Eq. (7.11), a finite theory, without divergencies, β_ε would be zero.

The physical meaning of Eq. (7.2) is not transparent because it refers to the "bare" Green functions. However, we remind Eq. (5.111), which relates "bare" and renormalized Green functions (which we report here in momentum representation):

$$\hat{G_E^b}^{(n)}(p_1,\ldots,p_n;\lambda_b(\lambda,m,\mu,\varepsilon),m_b(\lambda,m,\mu,\varepsilon),\varepsilon) = Z_\varphi^{\frac{n}{2}}\hat{G}_E^{(n)}(p_1,\ldots,p_n;\lambda,m,\mu,\varepsilon)\,. \tag{7.15}$$

We can use Eq. (7.15) into Eq. (7.2) and take the limit $\varepsilon \to 0$, obtaining

$$\left(\mu\frac{\partial}{\partial\mu} + \gamma_m m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + n\gamma_d\right)\hat{G_E}^{(n)}(p_1,\ldots,p_n;\lambda,m,\mu,\varepsilon) = 0\,, \tag{7.16}$$

where we have defined the **anomalous dimension** γ_d as

$$\gamma_d = \lim_{\varepsilon\to+0}\left\{\frac{1}{2}\beta_\varepsilon\frac{d}{d\lambda}\ln Z_{\varphi\varepsilon}(\lambda)\right\}. \tag{7.17}$$

Eq. (7.16) is the renormalization group equation. This form of the equation is useful in statistical mechanics, while in quantum field theory it is more useful to express it in another form, exploiting the fact that the Green function $\hat{G}_E^{(n)}$ is, dimensionally, an homogeneus function of degree d_n of the mass unit, *i.e.*

$$\left(\sum_i p_i\frac{\partial}{\partial p_i} + \mu\frac{\partial}{\partial\mu} + m\frac{\partial}{\partial m} - d_n\right)\hat{G}_E^{(n)}(p_1,\ldots,p_n;\lambda,m,\mu) = 0\,. \tag{7.18}$$

Subtracting Eq. (7.16) from the identity of Eq. (7.18), in order to cancel the term $\mu\frac{\partial}{\partial\mu}$, we get

$$\left\{\sum_i p_i\frac{\partial}{\partial p_i} + m[1-\gamma_m(\lambda)]\frac{\partial}{\partial m} - \beta(\lambda)\frac{\partial}{\partial\lambda} - [d_n + n\gamma_d(\lambda)]\right\}\hat{G}_E^{(n)}(p_1,\ldots,p_n;\lambda,m,\mu) = 0\,. \tag{7.19}$$

Eq. (7.19) is the standard form of the renormalization group equation. It can be rewritten in the following form

$$\left\{-s\frac{\partial}{\partial s} + m[1-\gamma_m(\lambda)]\frac{\partial}{\partial m} - \beta(\lambda)\frac{\partial}{\partial\lambda} - [d_n + n\gamma_d(\lambda)]\right\}\hat{G}_E^{(n)}\left(\frac{p_1}{s},\ldots,\frac{p_n}{s};\lambda,m,\mu\right) = 0\,. \tag{7.20}$$

Eq. (7.20) is a linear, partial differential equation of first order which can be solved by the method of characteristic curves of Cauchy. This method consists of choosing, in the space of variables s, m and λ, two curves $\lambda = \bar{\lambda}(s)$ and $m = \bar{m}(s)$ in such a way that Eq. 7.20 becomes an ordinary differential equation of the kind

$$\left\{ s\frac{d}{ds} + \left[d_n + n\gamma_d\left(\bar{\lambda}(s)\right)\right] \right\} \hat{G}_E^{(n)}(s) = 0\,. \tag{7.21}$$

Eq. (7.20) is equivalent to Eq. (7.21) if the following equations are satisfied

$$s\frac{d\bar{\lambda}(s)}{ds} = \beta\left(\bar{\lambda}(s)\right)\,, \tag{7.22}$$

$$s\frac{d\bar{m}(s)}{ds} = \bar{m}(s)\left[\gamma_m\left(\bar{\lambda}(s)\right) - 1\right]\,, \tag{7.23}$$

with the initial conditions

$$\bar{\lambda}(1) = \lambda\,, \tag{7.24}$$

$$\bar{m}(1) = m\,. \tag{7.25}$$

Eq. (7.21) can be solved by separation of variables:

$$\frac{d\hat{G}_E^{(n)}}{\hat{G}_E^{(n)}} = -\left[d_n + n\gamma_d\left(\bar{\lambda}(s)\right)\right]\frac{ds}{s}\,, \tag{7.26}$$

which gives, after integration:

$$\hat{G}_E^{(n)}\left(\frac{p_1}{s}, \ldots, \frac{p_n}{s}; \bar{\lambda}(s), \bar{m}(s)\right) = s^{-d_n} e^{-n\int_1^s \frac{ds'}{s'}\gamma_d(\bar{\lambda}(s'))} \hat{G}_E^{(n)}\left(p_1, \ldots, p_n; \lambda, m\right)\,. \tag{7.27}$$

The above solution respects the initial condition

$$\hat{G}_E^{(n)}\left(\frac{p_1}{s}, \ldots, \frac{p_n}{s}; \bar{\lambda}(s), \bar{m}(s)\right)_{|s=1} = \hat{G}_E^{(n)}\left(p_1, \ldots, p_n; \lambda, m\right)\,. \tag{7.28}$$

If we perform the substitution $p_i \to sp_i$ in Eq. (7.27), we obtain

$$\hat{G}_E^{(n)}\left(sp_1, \ldots, sp_n; \lambda, m\right) = s^{d_n} e^{n\int_1^s \frac{ds'}{s'}\gamma_d(\bar{\lambda}(s'))} \hat{G}_E^{(n)}\left(p_1, \ldots, p_n; \bar{\lambda}(s), \bar{m}(s)\right)\,. \tag{7.29}$$

The above Eq. (7.29) shows explicitely the meaning of the renormalization group: **a Green function calculated with rescaled momenta is equivalent to the same Green function calculated with fixed momenta but with scaled coupling constant and mass**. *Moreover*, **the Green function does not scale in a trivial way with the dimension d_n but with an additional contribution of the "anomalous dimension", given by the integral containing γ_d.**

Asymptotic and infrared freedom

The β function introduced previously determines the asymptotic behaviour of the theory; in particular whether the theory is free in the ultraviolet regime, the one of large external momenta, (*asymptotic freedom*) or in the infrared regime, the one of small external momenta, (*infrared freedom*). If we perform the following change of variables in Eq. (7.22)

$$s = e^t \,, \tag{7.30}$$

we see that $t = -\infty$ corresponds to null external momenta, while $t = +\infty$ corresponds to asymptotically large external momenta. From Eq. (7.30) we get $ds = sdt$, so Eq. (7.22) can be written as

$$\int_{\lambda}^{\bar{\lambda}(t)} \frac{d\lambda'}{\beta\,(\lambda')} = t \,. \tag{7.31}$$

By inspection of Eq. (7.31) we can see that $t = \pm\infty$ correspond to the values of λ for which $\beta(\lambda) = 0$. Since we are working in perturbation theory, we are moving in a region of small values of λ. Hence the function β is an analytical function of λ around $\lambda = 0$ and $\beta(0) = 0$. What is important for the asymptotic behaviour is the sign of $\beta(\lambda)$ close to $\lambda = 0$. We can have two cases:

1. $\beta(\lambda) > 0$ and $\frac{d\beta(\lambda)}{d\lambda} > 0$. In this case we have

$$\begin{aligned} t \to -\infty &\Longrightarrow \bar{\lambda} = 0 \\ t \to +\infty &\Longrightarrow \bar{\lambda} = \lambda^* \,. \end{aligned} \tag{7.32}$$

   In this case the theory is free in the infrared region.

2. $\beta(\lambda) < 0$ and $\frac{d\beta(\lambda)}{d\lambda} < 0$. In this case we have

$$\begin{aligned} t \to -\infty &\Longrightarrow \bar{\lambda} = \lambda^* \\ t \to +\infty &\Longrightarrow \bar{\lambda} = 0 \,. \end{aligned} \tag{7.33}$$

   In this case the theory is asymptotically free.

The calculation of the function β at one loop shows that the interacting theory $\lambda\varphi^4$ is free in the infrared.

Section to be completed

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