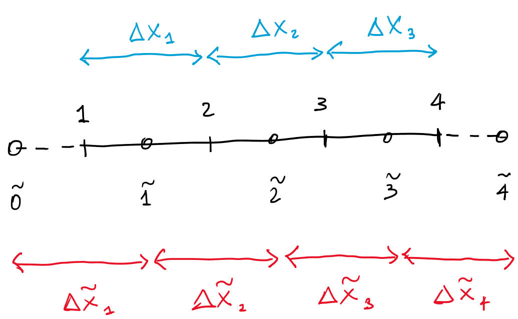
SOLUTION OF A POISSON EQUATION BY CENTERED FINITE DIFFERENCES

Why use staggered grid approach?

- conservative methods $\longrightarrow$ divergence with 2^{nd} ord. acc. at cell center
- bcs can be imposed without loss of accuracy with ghost cells

Consider a cell-centered approach with ghost cells. Assume 4 nodes:

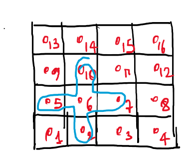


solve on interior nodes $\{\tilde{1}, \tilde{2}, \tilde{3}\}$

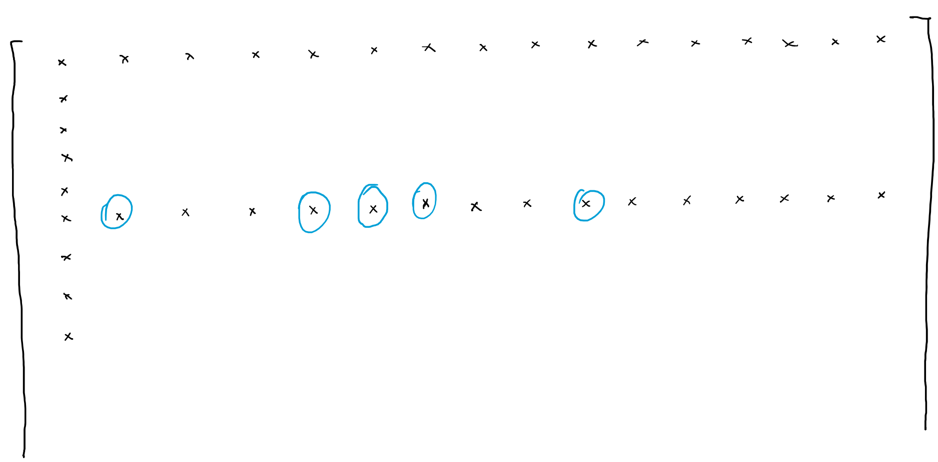
4 nodes
5 cell centers
3 dof

In the general case of non-uniform grid spacing:

$$\boxed{\left.\frac{\partial^2 u}{\partial x^2}\right|_i \cong \frac{1}{\Delta x_i \, \Delta \tilde{x}_{i+1}} \tilde{u}_{i+1} - \frac{1}{\Delta x_i \, \Delta \tilde{x}_{i+1}} \tilde{u}_i + \frac{1}{\Delta x_i \, \Delta \tilde{x}_i} \tilde{u}_{i-1} - \frac{1}{\Delta x_i \, \Delta \tilde{x}_i} \tilde{u}_i = \tilde{b}_i}$$



$A =$



pentadiagonal matrix

How to apply a generalized boundary condition: $\alpha u + \beta \frac{\partial u}{\partial x} = \gamma$
(it's easier to deal with corner nodes!)

LEFT BOUNDARY ($i = \tilde{1}$)

$$\frac{1}{\Delta x_1 \Delta \tilde{x}_2}\tilde{u}_2 - \frac{1}{\Delta x_1 \Delta \tilde{x}_2}\tilde{u}_1 + \frac{1}{\Delta x_1 \Delta \tilde{x}_1}\tilde{u}_0 - \frac{1}{\Delta x_1 \Delta \tilde{x}_1}\tilde{u}_1 = \tilde{b}_1$$

Find $\tilde{u}_0$ such that at the first node (x_1): $\alpha_1 u_1 + \beta_1 \left.\frac{\partial u}{\partial x}\right|_1 = \gamma_1$

$$\alpha_1 u_1 + \beta_1 \frac{\tilde{u}_1\left(\Delta\tilde{x}_1 - \frac{\Delta x_1}{2}\right)^2 - \tilde{u}_0\left(\frac{\Delta x_1}{2}\right)^2 + u_1\left[\left(\frac{\Delta x_1}{2}\right)^2 - \left(\Delta\tilde{x}_1 - \frac{\Delta x_1}{2}\right)^2\right]}{\left(\Delta\tilde{x}_1 - \frac{\Delta x_1}{2}\right)\left(\frac{\Delta x_1}{2}\right)\Delta\tilde{x}_1} = \gamma_1$$

$$\alpha_1 u_1 + \beta_1 d^+ \tilde{u}_1 - \beta_1 d^- \tilde{u}_0 + \beta_1 d\, u_1 = \gamma_1$$

$$u_1 = \frac{\gamma_1}{\alpha_1 + \beta_1 d} - \frac{\beta_1 d^+}{\alpha_1 + \beta_1 d}\tilde{u}_1 + \frac{\beta_1 d^-}{\alpha_1 + \beta_1 d}\tilde{u}_0$$

↳ 2^{ND} order accurate at u_1

Interpolate at the boundary and set the equivalence:

$$\frac{u_1 - \tilde{u}_0}{\Delta\tilde{x}_1 - \Delta x_1/2} = \frac{\tilde{u}_1 - \tilde{u}_0}{\Delta\tilde{x}_1} \Rightarrow u_1 = \left(1 - \frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\right)\tilde{u}_0 + \frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\tilde{u}_1$$

$$\frac{\gamma_1}{\alpha_1 + \beta_1 d} - \frac{\beta_1 d^+}{\alpha_1 + \beta_1 d}\tilde{u}_1 + \frac{\beta_1 d^-}{\alpha_1 + \beta_1 d}\tilde{u}_0 = \left(1 - \frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\right)\tilde{u}_0 + \frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\tilde{u}_1$$

$$\left(\frac{\beta_1 d^-}{\alpha_1 + \beta_1 d} + \frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1} - 1\right)\tilde{u}_0 = \left(\frac{\beta_1 d^+}{\alpha_1 + \beta_1 d} + \frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\right)\tilde{u}_1 - \frac{\gamma_1}{\alpha_1 + \beta_1 d}$$

$$\left[\beta_1 d^- + \left(\frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1} - 1\right)\left(\alpha_1 + \beta_1 d\right)\right]\tilde{u}_0 = \left[\beta_1 d^+ + \left(\frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\right)\left(\alpha_1 + \beta_1 d\right)\right]\tilde{u}_1 - \gamma_1$$

$$\tilde{u}_0 = C_1 \tilde{u}_1 + C_2$$

$$C_1 = \frac{\beta_1 d^+ + \left(\frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1}\right)\left(\alpha_1 + \beta_1 d\right)}{\beta_1 d^- + \left(\frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1} - 1\right)\left(\alpha_1 + \beta_1 d\right)}$$

$$C_2 = \frac{-\gamma_1}{\beta_1 d^- + \left(\frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1} - 1\right)\left(\alpha_1 + \beta_1 d\right)}$$

$$\rightarrow \quad c_2 = \frac{\gamma_1}{\beta_1 d^- + \left(\frac{2\Delta\tilde{x}_1 - \Delta x_1}{2\Delta\tilde{x}_1} - 1\right)\left(\alpha_1 + \beta_1 d\right)}$$

Dirichelet : $\Delta\tilde{x}_1 = \frac{\Delta x_1}{2}$, $\alpha = 1$, $\beta = 0 \longrightarrow c_1 = -1$, $c_2 = 2\gamma_1$

Neumann : $\Delta\tilde{x}_1 = \frac{\Delta x_1}{2}$, $\alpha = 0$, $\beta = 1 \longrightarrow c_1 = 1$, $c_2 = -\gamma_1 \Delta\tilde{x}_1$

Plug into node equation :

$$\frac{1}{\Delta x_1 \Delta\tilde{x}_2}\tilde{u}_2 - \frac{1}{\Delta x_1 \Delta\tilde{x}_2}\tilde{u}_1 + \frac{c_1}{\Delta x_1 \Delta\tilde{x}_1}\tilde{u}_1 + \frac{c_2}{\Delta x_1 \Delta\tilde{x}_1} - \frac{1}{\Delta x_1 \Delta\tilde{x}_1}\tilde{u}_1 = \tilde{b}_1$$

$$\frac{1}{\Delta x_1 \Delta\tilde{x}_2}\tilde{u}_2 + \left(-\frac{1}{\Delta x_1 \Delta\tilde{x}_2} + \frac{c_1 - 1}{\Delta x_1 \Delta\tilde{x}_1}\right)\tilde{u}_1 = \tilde{b}_1 - \frac{c_2}{\Delta x_1 \Delta\tilde{x}_1}$$

RIGHT BOUNDARY $(i = \tilde{3})$

$$\frac{1}{\Delta x_3 \Delta\tilde{x}_4}\tilde{u}_4 - \frac{1}{\Delta x_3 \Delta\tilde{x}_4}\tilde{u}_3 + \frac{1}{\Delta x_3 \Delta\tilde{x}_3}\tilde{u}_2 - \frac{1}{\Delta x_3 \Delta\tilde{x}_3}\tilde{u}_3 = \tilde{b}_3$$

Find $\tilde{u}_4$ such that at the first node (x_4) : $\alpha_4 u_4 + \beta_4 \left.\frac{\partial u}{\partial x}\right|_4 = \gamma_4$

$$\alpha_4 u_4 + \beta_4 \frac{\tilde{u}_4 (\Delta x_3/2)^2 - \tilde{u}_3\left(\Delta\tilde{x}_4 - \frac{\Delta x_3}{2}\right)^2 + u_4\left[\left(\Delta\tilde{x}_4 - \frac{\Delta x_3}{2}\right)^2 - \left(\frac{\Delta x_3}{2}\right)^2\right]}{\left(\Delta\tilde{x}_4 - \frac{\Delta x_3}{2}\right)\left(\frac{\Delta x_3}{2}\right)\Delta\tilde{x}_4} = \gamma_4$$

$$\alpha_4 u_4 + \beta_4 d^+ \tilde{u}_4 - \beta_4 d^- \tilde{u}_3 + \beta_4 d\, u_4 = \gamma_4$$

$$u_4 = \frac{\gamma_4}{\alpha_4 + \beta_4 d} - \frac{\beta_4 d^+}{\alpha_4 + \beta_4 d}\tilde{u}_4 + \frac{\beta_4 d^-}{\alpha_4 + \beta_4 d}\tilde{w}_3$$

$\hookrightarrow$ 2ND order accurate at u_4

Linear interpolation at node 4 and set the equivalence:

$$\frac{u_4 - \tilde{u}_3}{\Delta x_3/2} = \frac{\tilde{u}_4 - \tilde{u}_3}{\Delta \tilde{x}_4} \implies u_4 = \left(1 - \frac{\Delta x_3}{2\Delta \tilde{x}_4}\right)\tilde{u}_3 + \frac{\Delta x_3}{2\Delta \tilde{x}_4}\tilde{u}_4$$

$$\frac{\gamma_4}{\alpha_4 + \beta_4 d} - \frac{\beta_4 d^+}{\alpha_4 + \beta_4 d}\tilde{u}_4 + \frac{\beta_4 d^-}{\alpha_4 + \beta_4 d}\tilde{u}_3 = \left(1 - \frac{\Delta x_3}{2\Delta \tilde{x}_4}\right)\tilde{u}_3 + \frac{\Delta x_3}{2\Delta \tilde{x}_4}\tilde{u}_4$$

$$\left(\frac{\beta_4 d^+}{\alpha_4 + \beta_4 d} + \frac{\Delta x_3}{2\Delta \tilde{x}_4}\right)\tilde{u}_4 = \left(\frac{\beta_4 d^-}{\alpha_4 + \beta_4 d} + \frac{\Delta x_3}{2\Delta \tilde{x}_4} - 1\right)\tilde{u}_3 + \frac{\gamma_4}{\alpha_4 + \beta_4 d}$$

$$\left[\beta_4 d^+ + \frac{\Delta x_3}{2\Delta \tilde{x}_4}\left(\alpha_4 + \beta_4 d\right)\right]\tilde{u}_4 = \left[\beta_4 d^- + \left(\frac{\Delta x_3}{2\Delta \tilde{x}_4} - 1\right)\left(\alpha_4 + \beta_4 d\right)\right]\tilde{u}_3 + \gamma_4$$

$$\tilde{u}_4 = C_3\tilde{u}_3 + C_4$$

$$\longrightarrow C_3 = \frac{\beta_4 d^- + \left(\frac{\Delta x_3}{2\Delta \tilde{x}_4} - 1\right)\left(\alpha_4 + \beta_4 d\right)}{\beta_4 d^+ + \frac{\Delta x_3}{2\Delta \tilde{x}_4}\left(\alpha_4 + \beta_4 d\right)}$$

$$\longrightarrow C_4 = \frac{\gamma_4}{\beta_4 d^+ + \frac{\Delta x_3}{2\Delta \tilde{x}_4}\left(\alpha_4 + \beta_4 d\right)}$$

Dirichelet : $\Delta \tilde{x}_4 = \frac{\Delta x_3}{2}$, $\alpha_4 = 1$, $\beta_4 = 0$ $\longrightarrow$ $C_3 = -1$, $C_4 = 2\gamma_4$

Neumann : $\Delta \tilde{x}_4 = \frac{\Delta x_3}{2}$, $\alpha_4 = 0$, $\beta_4 = 1$ $\longrightarrow$ $C_3 = 1$, $C_4 = \gamma_4 \Delta \tilde{x}_4$

$$\frac{C_3}{\Delta x_3 \Delta \tilde{x}_4}\tilde{u}_3 + \frac{C_4}{\Delta x_3 \Delta \tilde{x}_4} - \frac{1}{\Delta x_3 \Delta \tilde{x}_4}\tilde{u}_3 + \frac{1}{\Delta x_3 \Delta \tilde{x}_3}\tilde{u}_2 - \frac{1}{\Delta x_3 \Delta \tilde{x}_3}\tilde{u}_3 = \tilde{b}_3$$

$$\left(\frac{C_3 - 1}{\Delta x_3 \Delta \tilde{x}_4} - \frac{1}{\Delta x_3 \Delta \tilde{x}_3}\right)\tilde{u}_3 + \frac{1}{\Delta x_3 \Delta \tilde{x}_3}\tilde{u}_2 = \tilde{b}_3 - \frac{C_4}{\Delta x_3 \Delta \tilde{x}_4}$$

ESTIMATION OF DERIVATIVES AT THE BOUNDARY

When it comes to evaluating derivatives at the outermost points (last center point), a linear extrapolation of the ghost value may compromise the accuracy of the derivative. Thus, an higher order extrapolation is performed.

Consider the 3rd order Lagrange polynomial interpolating the boundary node (it needs 4 points)

$$u_4 = b_1 \tilde{u}_1 + b_2 \tilde{u}_2 + b_3 \tilde{u}_3 + b_4 \tilde{u}_4$$

The boundary condition is discretized as:

$$\alpha_4 u_4 + \beta_4 \left.\frac{\partial u}{\partial x}\right|_4 = \gamma_4$$

$$\alpha_4 u_4 + \beta_4 \frac{\tilde{u}_4 (\Delta x_3/2)^2 - \tilde{u}_3 \left(\Delta\tilde{x}_4 - \frac{\Delta x_3}{2}\right)^2 + u_4 \left[\left(\Delta\tilde{x}_4 - \frac{\Delta x_3}{2}\right)^2 - \left(\frac{\Delta x_3}{2}\right)^2\right]}{\left(\Delta\tilde{x}_4 - \frac{\Delta x_3}{2}\right)\left(\frac{\Delta x_3}{2}\right)\Delta\tilde{x}_4} = \gamma_4$$

$$\alpha_4 u_4 + \beta_4 d^+ \tilde{u}_4 - \beta_4 d^- \tilde{u}_3 + \beta_4 d\, u_4 = \gamma_4$$

$$u_4 = \frac{\gamma_4}{\alpha_4 + \beta_4 d} - \frac{\beta_4 d^+}{\alpha_4 + \beta_4 d} \tilde{u}_4 + \frac{\beta_4 d^-}{\alpha_4 + \beta_4 d} \tilde{u}_3$$

Set the equivalence:

$$\frac{\gamma_4}{\alpha_4 + \beta_4 d} - \frac{\beta_4 d^+}{\alpha_4 + \beta_4 d} \tilde{u}_4 + \frac{\beta_4 d^-}{\alpha_4 + \beta_4 d} \tilde{u}_3 = b_1 \tilde{u}_1 + b_2 \tilde{u}_2 + b_3 \tilde{u}_3 + b_4 \tilde{u}_4$$

$$\left(-b_4 - \frac{\beta_4 d^+}{\alpha_4 + \beta_4 d}\right)\tilde{u}_4 = b_1 \tilde{u}_1 + b_2 \tilde{u}_2 + \left(b_3 - \frac{\beta_4 d^-}{\alpha_4 + \beta_4 d}\right)\tilde{u}_3 - \frac{\gamma_4}{\alpha_4 + \beta_4 d}$$

$$\left[-b_4 \left(\alpha_4 + \beta_4 d\right) - \beta_4 d^+\right]\tilde{u}_4 = \left(b_1 \tilde{u}_1 + b_2 \tilde{u}_2\right)\left(\alpha_4 + \beta_4 d\right) +$$

$$\left[- b_4 \left(\alpha_4 + \beta_4 d \right) - \beta_4 d^+ \right] \tilde{u}_4 = \left(b_1 \tilde{u}_1 + b_2 \tilde{u}_2 \right) \left(\alpha_4 + \beta_4 d \right) +$$

$$+ \left[b_3 \left(\alpha_4 + \beta_4 d \right) - \beta_4 d^- \right] \tilde{u}_3 - \gamma_4$$