Generalized_bcs

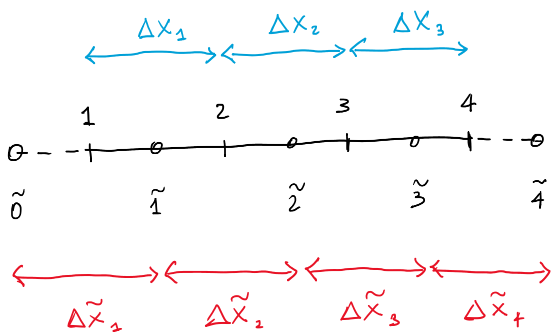
Thursday, 2 May 2024 17:07

SOLUTION OF A POISSON EQUATION BY CENTERED FINITE DIFFERENCES

Why use staggered grid approach ?

- conservative methods → divergence with 2^{nd} ord. acc. at cell center
- bcs can be imposed without loss of accuracy with ghost cells

Consider a cell-centered approach with ghost cells. Assume 4 nodes:



solve on interior nodes $\{\tilde{1}, \tilde{2}, \tilde{3}\}$

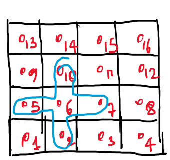
4 nodes

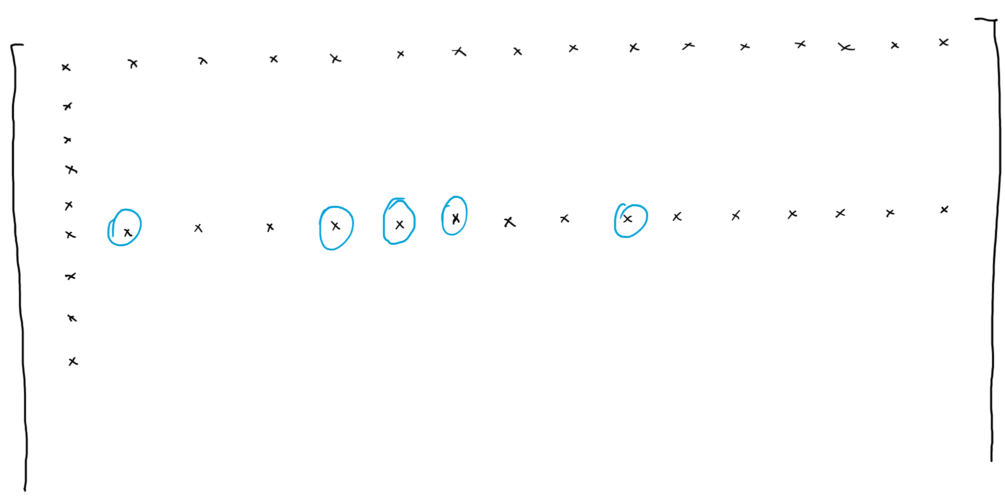
5 cell centers

3 dof

In the general case of non-uniform grid spacing :

$$\boxed{\left.\frac{\partial^2 u}{\partial x^2}\right|_i \cong \frac{1}{\Delta x_i \Delta\tilde{x}_{i+1}}\tilde{u}_{i+1} - \frac{1}{\Delta x_i \Delta\tilde{x}_{i+1}}\tilde{u}_i + \frac{1}{\Delta x_i \Delta\tilde{x}_i}\tilde{u}_{i-1} - \frac{1}{\Delta x_i \Delta\tilde{x}_i}\tilde{u}_i = \tilde{b}_i}$$



$A =$ 

pentadiagonal matrix

How to apply a generalized boundary condition : $\alpha u + \beta \frac{\partial u}{\partial x} = \gamma$

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(it's easier to deal with corner nodes!)

LEFT BOUNDARY ($i = \tilde{1}$)

$$\frac{1}{\Delta x_1 \Delta \tilde{x}_2} \tilde{u}_2 - \frac{1}{\Delta x_1 \Delta \tilde{x}_2} \tilde{u}_1 + \frac{1}{\Delta x_1 \Delta \tilde{x}_1} \tilde{u}_0 - \frac{1}{\Delta x_1 \Delta \tilde{x}_1} \tilde{u}_1 = \tilde{b}_1$$

Find $\tilde{u}_0$ such that at the first node (x_1): $\alpha_1 u_1 + \beta_1 \frac{\tilde{u}_1 - \tilde{u}_0}{\Delta \tilde{x}_1} = \gamma_1$

$$u_1 = \frac{\gamma_1}{\alpha_1} - \frac{\beta_1}{\Delta \tilde{x}_1 \alpha_1} \left(\tilde{u}_1 - \tilde{u}_0 \right)$$

Interpolate at the boundary and substitute:

$$\frac{u_1 - \tilde{u}_0}{\cancel{\Delta \tilde{x}_1}/2} = \frac{\tilde{u}_1 - \tilde{u}_0}{\cancel{\Delta \tilde{x}_1}} \implies 2 u_1 - 2 \tilde{u}_0 = \tilde{u}_1 - \tilde{u}_0$$

$$2 \frac{\gamma_1}{\alpha_1} - \frac{2 \beta_1}{\Delta \tilde{x}_1 \alpha_1} \tilde{u}_1 + \frac{2 \beta}{\Delta \tilde{x}_1 \alpha_1} \tilde{u}_0 - 2 \tilde{u}_0 = \tilde{u}_1 - \tilde{u}_0$$

$$\left(\frac{2 \beta_1}{\Delta \tilde{x}_1} - \alpha_1 \right) \tilde{u}_0 = \left(\frac{2 \beta_1}{\Delta \tilde{x}_1} + \alpha_1 \right) \tilde{u}_1 - 2 \gamma_1$$

$$\tilde{u}_0 = C_1 \tilde{u}_1 + C_2 \longrightarrow C_1 = \frac{2 \beta_1 + \alpha_1 \Delta \tilde{x}_1}{2 \beta_1 - \alpha_1 \Delta \tilde{x}_1}$$

$$C_2 = \frac{-2 \gamma_1 \Delta \tilde{x}_1}{2 \beta_1 - \alpha_1 \Delta \tilde{x}_1}$$

Dirichelet: $\alpha_1 = 1$, $\beta_1 = 0 \longrightarrow C_1 = -1$, $C_2 = 2 \gamma_1$

Neumann: $\alpha_1 = 0$, $\beta_1 = 1 \longrightarrow C_1 = 1$, $C_2 = -\gamma_1 \Delta \tilde{x}_1$

Plug into node equation:

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$$\frac{1}{\Delta x_1 \Delta \tilde{x}_2} \tilde{u}_2 - \frac{1}{\Delta x_1 \Delta \tilde{x}_2} \tilde{u}_1 + \frac{c_1}{\Delta x_1 \Delta \tilde{x}_1} \tilde{u}_1 + \frac{c_2}{\Delta x_1 \Delta \tilde{x}_1} - \frac{1}{\Delta x_1 \Delta \tilde{x}_1} \tilde{u}_1 = \tilde{b}_1$$

$$\frac{1}{\Delta x_1 \Delta \tilde{x}_2} \tilde{u}_2 + \left(- \frac{1}{\Delta x_1 \Delta \tilde{x}_2} + \frac{c_1 - 1}{\Delta x_1 \Delta \tilde{x}_1} \right) \tilde{u}_1 = \tilde{b}_1 - \frac{c_2}{\Delta x_1 \Delta \tilde{x}_1}$$

RIGHT BOUNDARY $(i = \tilde{3})$

$$\frac{1}{\Delta x_3 \Delta \tilde{x}_4} \tilde{u}_4 - \frac{1}{\Delta x_3 \Delta \tilde{x}_4} \tilde{u}_3 + \frac{1}{\Delta x_3 \Delta \tilde{x}_3} \tilde{u}_2 - \frac{1}{\Delta x_3 \Delta \tilde{x}_3} \tilde{u}_3 = \tilde{b}_3$$

Find $\tilde{u}_4$ such that at the first node (x_4): $\alpha_4 u_4 + \beta_4 \frac{\tilde{u}_4 - \tilde{u}_3}{\Delta \tilde{x}_4} = \gamma_4$

$$u_4 = \frac{\gamma_4}{\alpha_4} - \frac{\beta_4}{\Delta \tilde{x}_4 \alpha_4} (\tilde{u}_4 - \tilde{u}_3)$$

Interpolate:

$$\frac{u_4 - \tilde{u}_3}{\cancel{\Delta \tilde{x}_4} / 2} = \frac{\tilde{u}_4 - \tilde{u}_3}{\cancel{\Delta \tilde{x}_4}} \implies 2 u_4 - 2 \tilde{u}_3 = \tilde{u}_4 - \tilde{u}_3$$

$$2 \frac{\gamma_4}{\alpha_4} - \frac{2 \beta_4}{\alpha_4 \Delta \tilde{x}_4} \tilde{u}_4 + \frac{2 \beta_4}{\alpha_4 \Delta \tilde{x}_4} \tilde{u}_3 - 2 \tilde{u}_3 = \tilde{u}_4 - \tilde{u}_3$$

$$\left(- \frac{2 \beta_4}{\Delta \tilde{x}_4} - \alpha_4 \right) \tilde{u}_4 = \left(- \frac{2 \beta}{\Delta \tilde{x}_4} + \alpha_4 \right) \tilde{u}_3 - 2 \gamma_4$$

$$\tilde{u}_4 = c_3 \tilde{u}_3 + c_4$$

$$c_3 = \frac{-2 \beta_4 + \alpha_4 \Delta \tilde{x}_4}{-2 \beta_4 - \alpha_4 \Delta \tilde{x}_4}$$

$$c_4 = \frac{-2 \gamma_4 \Delta \tilde{x}_4}{-2 \beta_4 - \alpha_4 \Delta \tilde{x}_4}$$

Dirichelet : $\alpha_4 = 1, \ \beta_4 = 0 \longrightarrow c_3 = -1, \ c_4 = 2\gamma_4$

Neumann : $\alpha_4 = 0, \ \beta_4 = 1 \longrightarrow c_3 = 1, \ c_4 = \gamma_4 \Delta\tilde{x}_4$

$$\frac{c_3}{\Delta x_3 \Delta\tilde{x}_4}\tilde{u}_3 + \frac{c_4}{\Delta x_3 \Delta\tilde{x}_4} - \frac{1}{\Delta x_3 \Delta\tilde{x}_4}\tilde{u}_3 + \frac{1}{\Delta x_3 \Delta\tilde{x}_3}\tilde{u}_2 - \frac{1}{\Delta x_3 \Delta\tilde{x}_3}\tilde{u}_3 = \tilde{b}_3$$

$$\left(\frac{c_3 - 1}{\Delta x_3 \Delta\tilde{x}_4} - \frac{1}{\Delta x_3 \Delta\tilde{x}_3}\right)\tilde{u}_3 + \frac{1}{\Delta x_3 \Delta\tilde{x}_3}\tilde{u}_2 = \tilde{b}_3 - \frac{c_4}{\Delta x_3 \Delta\tilde{x}_4}$$