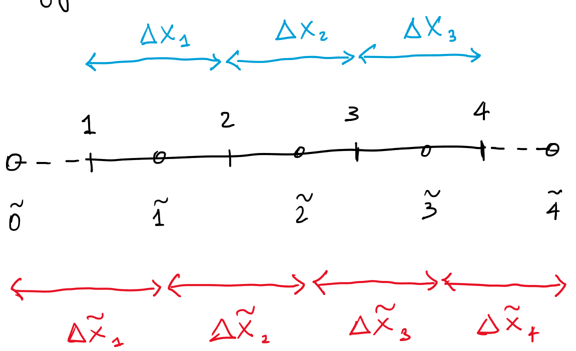
Staggered grid arrangement:



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = b$$

DISCRETE FORM $\{r, \theta, z\} \to \{i, j, k\}$

$$\frac{r_{i+1}}{\tilde{r}_i\,\Delta r_i\,\Delta\tilde{r}_{i+1}}\left(\tilde{u}_{i+1,j,k} - \tilde{u}_{i,j,k}\right) + \frac{r_i}{\tilde{r}_i\,\Delta r_i\,\Delta\tilde{r}_i}\left(\tilde{u}_{i-1,j,k} - \tilde{u}_{i,j,k}\right) +$$

$$+ \frac{1}{\Delta z_k\,\Delta\tilde{z}_{k+1}}\left(\tilde{u}_{i,j,k+1} - \tilde{u}_{i,j,k}\right) + \frac{1}{\Delta z_k\,\Delta\tilde{z}_k}\left(\tilde{u}_{i,j,k-1} - \tilde{u}_{i,j,k}\right) +$$

$$+ \frac{1}{\tilde{r}_i^2\,\Delta\theta^2}\left(\tilde{u}_{i,j+1,k} - 2\,\tilde{u}_{i,j,k} + \tilde{u}_{i,j-1,k}\right) = \tilde{b}_{i,j,k}$$

After applying DFT:

$$\lambda_{i,m,k} = \frac{2}{\tilde{r}_i^2\,\Delta\theta^2}\left(\cos\left(\frac{2m\pi}{N_\theta - 2}\right) - 1\right),$$ with $k = 0, \dots, N_\theta - 1$, N_θ number of nodes in the θ dir.

$$\left[-\frac{r_{i+1}}{\tilde{r}_i\,\Delta r_i\,\Delta\tilde{r}_{i+1}} - \frac{r_i}{\tilde{r}_i\,\Delta r_i\,\Delta\tilde{r}_i} - \frac{1}{\Delta z_k\,\Delta\tilde{z}_{k+1}} - \frac{1}{\Delta z_k\,\Delta\tilde{z}_k} + \lambda_{i,m,k}\right]\hat{u}_{i,m,k} +$$

$$\left[- \frac{r_{i+1}}{\tilde{r}_i \, \Delta r_i \, \Delta \tilde{r}_{i+1}} - \frac{r_i}{\tilde{r}_i \, \Delta r_i \, \Delta \tilde{r}_i} - \frac{1}{\Delta z_k \, \Delta \tilde{z}_{k+1}} - \frac{1}{\Delta z_k \, \Delta \tilde{z}_k} + \lambda_{i,m,k} \right] \hat{u}_{i,m,k} +$$

$$\frac{r_{i+1}}{\tilde{r}_i \, \Delta r_i \, \Delta \tilde{r}_{i+1}} \, \hat{u}_{i+1,m,k} +$$

$$\frac{r_i}{\tilde{r}_i \, \Delta r_i \, \Delta \tilde{r}_i} \, \hat{u}_{i-1,m,k} +$$

$$\frac{1}{\Delta z_k \, \Delta \tilde{z}_{k+1}} \, \hat{u}_{i,m,k+1} +$$

$$\frac{1}{\Delta z_k \, \Delta \tilde{z}_k} \, \hat{u}_{i,m,k-1} = \hat{b}_{i,m,k}$$

SOLUTION PROCEDURE

- compute DFT_θ of $b(r,\theta,z) \to \hat{b}(r,m,z)$
- for $m = 1, \dots, N_\theta - 1$
    - assemble the m^{th} pentadiagonal problem
    - solve for $\hat{u}_{i,m,k}$
- inverse trasform of $\hat{u}_{i,m,k} \to u_{i,j,k}$

ALTERNATIVE PROCEDURE :

- compute $DFT_z(DFT_\theta)$: $b(r,\theta,z) \to \hat{b}(r,m,\ell)$
- for $m = 1, \dots, N_\theta - 1$

    for $\ell = 1, \dots, N_z - 1$
    - assemble the $\{m,\ell\}$ tridiagonal problem
    - solve for $\hat{u}_{i,m,\ell}$

- assemble the $\{m,l\}$ tridiagonal problem
- solve for $\hat{u}_{i,m,l}$

- inverse trasform of $\hat{u}_{i,m,l} \rightarrow u_{i,j,k}$

TREATMENT OF THE AXIS

Variable staggering allows to cancel out the ghost unknown at the axis $(r=0)$. At the first node $(r = \tilde{r}_1)$

$$\frac{r_1}{\tilde{r}_1 \Delta r_1 \Delta \tilde{r}_2} \left(\tilde{u}_{2,j,k} - \tilde{u}_{1,j,k} \right) + \cancel{\frac{r_0}{\tilde{r}_1 \Delta r_1 \Delta \tilde{r}_1} \left(\tilde{u}_{0,j,k} - \tilde{u}_{1,j,k} \right)}$$

$r_0 = 0$