Autarky - Mathematical Formulation

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1 Introduction

The Autarky project focuses on developing advanced optimization frameworks for the reliable and efficient operation of mini-grids in rural or weakly connected regions. These mini-grids often integrate renewable energy sources, energy storage systems, and backup generators while facing significant challenges, such as unpredictable demand, variable renewable generation, and unreliable main-grid connections.

The Autarky model aims to address these challenges by employing robust mathematical techniques, focusing in particular on methodologies like Individual Chance Constraints (ICC) and Joint Chance Constraints (JCC). These approaches enable the system to handle uncertainties in solar generation, electricity demand, and grid outages while maintaining a high level of reliability and cost-effectiveness.

Implemented in Julia, Autarky utilizes state-of-the-art optimization packages, including JuMP, to model the economic dispatch and sizing of mini-grids. The framework incorporates probabilistic methods to optimize reserves for energy storage and backup generators, ensuring that local demand can be met autonomously during outage events with a specified reliability level.

Potential applications of the Autarky model span several domains, including rural electrification, energy transition planning, and disaster resilience in energy systems. By integrating forecasting methods, detailed cost modeling, and innovative optimization strategies, the project contributes to advancing decentralized energy solutions aligned with the United Nations' Sustainable Development Goal 7 (SDG-7): universal access to affordable, reliable, and sustainable energy.

This document builds upon the foundational joint chance-constrained dispatch framework introduced in [1], extending it with a full capacity sizing layer, unit commitment formulation, refined cost modeling, and modular seasonal time series integration tailored for practical mini-grid planning.

2 Deterministic Model - Sizing Formulation

This section presents the integrated mathematical formulation of the sizing and dispatch model for a mini-grid system, as implemented with a linear deterministic. The model optimizes both system capacities and operational decisions under deterministic assumptions, minimizing the Net Present Cost (NPC) while accounting for capital and operational expenditures, salvage value, and discounting logic.

2.1 Sets

The model operates over the following sets:

- T: Set of operational time steps (hours).
- Y: Set of years in the project lifetime.
- S: Set of seasons in a year.

2.2 Parameters

Project and Optimization Settings. The Autarky model simulates the economic and technical behavior of a mini-grid over a multi-year horizon. The system lifetime is set as a project-level parameter, and the simulation operates with hourly time steps. Seasonal variability can be enabled, allowing users to define representative periods (e.g., four seasons), and to model each with its respective load and renewable generation profiles. Optimization settings support constraints such as maximum investment cost, minimum renewable energy share, and allowable lost load percentage. Users may configure both grid-tied and off-grid scenarios, including grid import/export permissions and connection limits.

Solar Photovoltaic (PV) System. The solar PV component is characterized by an *inverter efficiency* (η_{inv}) and has fixed techno-economic parameters including *specific investment cost*, *O&M cost*, and *lifetime*. Users can optionally specify tilt, azimuth, and albedo coefficients. If data download is enabled, the PVGIS API provides high-resolution solar irradiance data; otherwise, users can provide their own time series inputs.

Wind Turbine System. If activated, the wind turbine system is modeled with a *nominal capacity*, *lifetime*, and efficiency parameters including *inverter* and *drivetrain efficiency*. The turbine can be either vertical or horizontal axis, and is characterized by its *rotor diameter* and *hub height*. Users can upload a custom wind power curve or allow integration with the PVGIS API to download wind speed data, which is then converted to energy using the turbine curve. Economic parameters include *specific CAPEX*, *annual OPEX*, and optional subsidies.

Battery System. Battery performance is defined by *charging and discharging efficiencies* ($\eta_{\text{charge}}, \eta_{\text{discharge}}$), and operational flexibility through *charge/discharge times*. It operates within a bounded *state of charge* range ($\text{SOC}_{\min}, \text{SOC}_{\max}$) and starts at a defined initial state ($\text{SOC}_{\text{initial}}$). The system has a defined *nominal energy capacity* and lifetime, and is associated with *capital expenditure* (battery_capex) and *operational expenditure* (battery_opex) per year.

Backup Generator. The generator operates at a specified *nominal efficiency* and can optionally include partial-load efficiency behavior if enabled, using a piecewise approximation. Capital and operational costs are specified per unit of installed capacity. The generator consumes fuel, modeled by a *lower heating value* and a per-liter *fuel cost*. An optional cap on total annual fuel consumption may be enforced to limit emissions or ensure logistical feasibility.

Grid Connection. When enabled, the grid connection allows energy import and optionally export, up to a defined *maximum capacity*. The economic interaction is determined by the time-varying *import cost* and *export price*, which reflect tariffs or market-based mechanisms.

2.3 Time-Series Inputs

Time-series inputs are provided through CSV files, with one value per operational time step. These inputs are fundamental for enforcing energy balance constraints, computing operational expenditures (OPEX), and simulating grid availability and outages.

- load[t]: Electricity demand at time t [kWh]. **(Mandatory)**
- solar_unit[t]: Solar unit production per unit of installed capacity at time t [kWh/kW], representing the available solar resource and the maximum possible generation for 1 kW of installed PV. **(if enabled)**
- wind_unit[t]: Wind unit production per unit of installed capacity at time t [kWh/kW], used if wind turbines are enabled. **(if enabled)**
- grid_cost[t]: Cost of importing electricity from the grid at time t [USD/kWh]. **(if enabled)**
- grid_price[t]: Price received for exporting electricity to the grid at time t [USD/kWh]. **(if enabled)**
- grid_availability[t]: Binary availability of the grid connection at time t [0 = outage, 1 = available], used to model stochastic outages in grid-connected scenarios. **(if enabled)**

Note on Seasonality

If seasonality is enabled, each time series must contain one column per season (e.g., four columns for four seasons), and the number of rows must match the number of operational time steps in a representative period (e.g., 24 for a day, 168 for a week). Columns must appear in the same order as defined in the seasonal mapping.

2.4 Decision Variables

Sizing Variables

The optimization model defines sizing variables only for components that are enabled in the project configuration, allowing for a modular and flexible model structure. Moreover, if the unit commitment option is activated, these sizing variables are declared as integers, representing the number of standardized technology units (e.g., PV panels, wind turbines, battery packs, generators) to be installed.

The following variables determine the installed capacities of key technologies, establishing the upper bounds for their respective dispatch:

- solar_capacity: Installed solar PV capacity [kW], determining the maximum potential solar power generation.
- wind_capacity: Installed wind turbine capacity [kW], if enabled.
- battery_capacity: Total battery energy storage capacity [kWh], defining the system's ability to store electricity.
- gen_capacity: Diesel generator capacity [kW], defining the maximum output available from the backup generator.

Operational and Dispatch Variables

In addition to sizing, the model defines time-dependent operational variables for each enabled component. These variables are indexed over the simulation horizon and, when seasonal modeling is enabled, across representative periods. They are constrained by the sizing variables and other techno-economic parameters to maintain physical feasibility and ensure energy balance.

- solar_production$[t, s]$: Electricity produced by the solar PV system at time t in season s [kWh], limited by solar capacity and the time-varying unit production.
- wind_production$[t, s]$: Electricity produced by the wind turbine at time t in season s [kWh], if wind is enabled.
- battery_charge$[t, s]$, battery_discharge$[t, s]$: Battery charge and discharge power [kWh] at time t, subject to rate and efficiency constraints.
- SOC$[t, s]$: State of charge of the battery [kWh], evolving over time based on charging and discharging activity.
- gen_production$[t, s]$: Power output of the diesel generator [kWh] at time t, bounded by installed capacity.
- gen_fuel_consumption$[t, s]$: Fuel consumption of the generator [liters/hour] at time t, included only if partial load efficiency modeling is enabled.
- grid_import$[t, s]$: Electricity imported from the main grid [kWh] at time t, if the grid is enabled and available.
- grid_export$[t, s]$: Electricity exported to the main grid [kWh] at time t, included only if grid export is allowed.
- lost_load$[t, s]$: Unserved load or curtailed demand [kWh], included only if a nonzero share of lost load is allowed.

2.5 Constraints

Energy Balance The core operational constraint in the model is the energy balance equation, which ensures that the total power supply—including all enabled components—matches the electricity demand at every time step and in every representative period (season):

$$\begin{aligned}
&\text{solar_production}[t,s] + \text{wind_production}[t,s] + \text{generator_production}[t,s] \\
&+ \big(\text{battery_discharge}[t,s] - \text{battery_charge}[t,s]\big) \\
&+ \text{grid_import}[t,s] - \text{grid_export}[t,s] + \text{lost_load}[t,s] = \text{load}[t,s], \quad \forall t, s.
\end{aligned}$$

Only components that are enabled in the configuration contribute to the balance. This modular structure enables the model to flexibly handle different mini-grid configurations ranging from fully off-grid islanded systems to grid-connected hybrid solutions.

Capacity Constraints Each operational variable is constrained by its associated technology capacity, time resolution, and technical parameters such as charge/discharge durations or partial load behavior.

$$0 \leq \text{solar\_production}[t,s] \leq \text{solar\_units} \cdot \text{solar\_unit}[t,s], \quad \forall t,s, \tag{1}$$
$$0 \leq \text{wind\_production}[t,s] \leq \text{wind\_units} \cdot \text{wind\_unit}[t,s], \quad \forall t,s, \tag{2}$$
$$0 \leq \text{battery\_charge}[t,s] \leq \frac{\text{battery\_units} \cdot \text{battery\_capacity}}{t_{\text{charge}}} \cdot \Delta t, \quad \forall t,s, \tag{3}$$
$$0 \leq \text{battery\_discharge}[t,s] \leq \frac{\text{battery\_units} \cdot \text{battery\_capacity}}{t_{\text{discharge}}} \cdot \Delta t, \quad \forall t,s, \tag{4}$$
$$0 \leq \text{generator\_production}[t,s] \leq \text{generator\_units} \cdot \text{gen\_capacity} \cdot \Delta t, \quad \forall t,s, \tag{5}$$
$$\text{grid\_import}[t,s] \leq \text{grid\_availability}[t,s] \cdot \text{max\_line\_capacity} \cdot \Delta t, \quad \forall t,s, \tag{6}$$
$$\text{grid\_export}[t,s] \leq \text{grid\_availability}[t,s] \cdot \text{max\_line\_capacity} \cdot \Delta t, \quad \forall t,s. \tag{7}$$

If partial load modeling is enabled for the generator, fuel consumption is modeled with a piecewise linear approximation based on sampled generator efficiencies at different output levels.

State of Charge (SOC) Battery dynamics are explicitly modeled through state-of-charge constraints, which ensure consistent energy accounting over time while respecting minimum and maximum SOC bounds:

$$\text{SOC}_{\text{min}} \cdot \text{battery\_units} \cdot \text{battery\_capacity} \leq \text{SOC}[t,s] \leq \text{SOC}_{\text{max}} \cdot \text{battery\_units} \cdot \text{battery\_capacity}, \quad \forall t,s, \tag{8}$$
$$\text{SOC}[1,s] = \text{SOC}_0 \cdot \text{battery\_units} \cdot \text{battery\_capacity} + (\eta_{\text{charge}} \cdot \text{battery\_charge}[1,s] - \eta_{\text{discharge}} \cdot \text{battery\_discharge}[1,s]), \tag{9}$$
$$\text{SOC}[t,s] = \text{SOC}[t-1,s] + (\eta_{\text{charge}} \cdot \text{battery\_charge}[t,s] - \eta_{\text{discharge}} \cdot \text{battery\_discharge}[t,s]), \quad \forall t > 1, s, \tag{10}$$
$$\text{SOC}[T,s] = \text{SOC}_0 \cdot \text{battery\_units} \cdot \text{battery\_capacity}, \quad \forall s. \tag{11}$$

The final constraint enforces a cyclic end-of-horizon condition, resetting SOC to its initial level to ensure comparability between seasonal scenarios.

2.6 Objective Function

The objective of the optimization model is to minimize the Net Present Cost (NPC) of the energy system over its entire lifetime. The NPC aggregates multiple cost components:

- Capital Expenditures (CAPEX): upfront investments in technologies.
- Replacement Costs: discounted costs of future technology replacements based on their lifetimes.
- Subsidies: investment subsidies applied to eligible technologies.
- Operational Expenditures (OPEX): including both fixed and variable annual operating costs.
- Salvage Value: the discounted value of unused component lifetime at the end of the project horizon.

All time-dependent cash flows are discounted to their present value using an annual discount factor. The complete objective function minimized by the model is:

$$
\begin{aligned}
\min\ \text{NPC} := \quad & \left(\sum_{i\in\mathcal{T}} x_i \cdot C_{\text{capex},i}\right) \\
& + \left(\sum_{i\in\mathcal{T}} \sum_{y\in R_i} x_i \cdot C_{\text{capex},i} \cdot \text{discount\_factor}[y]\right) \\
& - \left(\sum_{i\in\mathcal{T}_{\text{sub}}} x_i \cdot C_{\text{capex},i} \cdot \text{subsidy}_i\right) \\
& + \left(\sum_{y=1}^{L} \left[\sum_{s=1}^{S} w_s \sum_{t=1}^{T} \text{OPEX}_{\text{variable}}[t,s] + \sum_{i\in\mathcal{T}} x_i \cdot C_{\text{capex},i} \cdot \text{OPEX}_i\right] \cdot \text{discount\_factor}[y]\right) \\
& - \left(\sum_{i\in\mathcal{T}} x_i \cdot C_{\text{capex},i} \cdot f_i^{\text{salvage}} \cdot \text{discount\_factor}[L]\right)
\end{aligned}
$$

Annual Cost Scaling In **Autarky**, the operational model simulates a typical representative period for each season (e.g., one day or one week). These periods are scaled to represent their share of the full year using *seasonal weights.*

The seasonal scaling process includes:

- **Granularity:** Each representative period has a fixed number of time steps depending on whether the simulation is daily, weekly, or yearly.
- **Year Scale Factor:**
$$
\text{year\_scale\_factor} = \frac{8760}{\text{operation\_time\_steps}}
$$
- **Season Weights:** If seasonal modeling is active, each season s is assigned a weight w_s based on how many months it represents:
$$
w_s = \left(\frac{\#\text{ months in season } s}{12}\right) \cdot \text{year\_scale\_factor}
$$
- These weights are used to scale variable OPEX for each season before applying discounting over the lifetime of the project.

Variable Operational Expenditure (OPEX) Variable operating costs include generator fuel costs and grid-related flows (imports and exports). These are calculated per time step t and season s as:

$$
\begin{aligned}
\text{OPEX}_{\text{variable}}[t,s] = c_{\text{fuel}} \cdot & \begin{cases} \text{fuel\_consumption}[t,s], & \text{if partial load is enabled} \\ \frac{\text{gen\_production}[t,s]}{\text{LHV}}, & \text{otherwise} \end{cases} \\
& + c_{\text{grid}}[t,s] \cdot \text{grid\_import}[t,s] \ - \ p_{\text{grid}}[t,s] \cdot \text{grid\_export}[t,s]
\end{aligned}
$$

These seasonal OPEX values are scaled using the seasonal weights and aggregated into the full-year operational cost, which is then discounted using the project-specific discount rate.

Note on Actualization and Discounting

CAPEX is not discounted because it is assumed to be paid as an upfront investment in year 1. All other time-dependent cash flows—such as replacements, annual operational costs, and salvage—are discounted to their present value to reflect the time value of money. The discount factor for each year y is defined as:

$$\text{discount\_factor}[y] = \frac{1}{(1+r)^y}$$

where r is the annual discount rate and L is the project lifetime in years.
Component replacements are scheduled based on each technology's lifetime (e.g., every 10 or 20 years). Their cost is included only for the years within the project horizon and is discounted accordingly.
The salvage value accounts for the residual value of components at the end of the project. To compute it, the model:

- Identifies the last installation year for each component,
- Computes the unused portion of its lifetime after year L,
- Applies a proportional salvage fraction:

$$f_i^{\text{salvage}} = \max\left(0, \frac{\text{unused lifetime}}{\text{component lifetime}}\right)$$

- Multiplies it by the component's investment cost and discounts it using the factor for year L.

This ensures that only truly residual value is recovered at the end of the simulation horizon and that the economic lifespan of each component is respected.

3 Handling Uncertainties – Expected Value Sizing Model

The Expected Value Sizing Model builds directly on top of the Regular Sizing Model by incorporating the statistical characteristics of forecasting errors. While the deterministic model assumes perfect foresight of demand and renewable production, this model introduces uncertainty by modeling the expected impacts of prediction errors for load and solar generation.

All the fundamental parameters, variables, and constraints from the Regular Model remain valid unless explicitly extended or modified. The key enhancements in the Expected Value Model include: (i) additional error-driven cost components, (ii) outage-related parameters, and (iii) new nonlinear expressions derived from probabilistic expectations.

3.1 Additional Input Data and Parameters

To implement this uncertainty-aware formulation, the model requires the following additional inputs:

- **Forecasting Error Time Series:** For each season (or for the full year if seasonality is disabled), the model expects CSV files containing historical or simulated errors:
  - `load_errors_$s.csv`
  - `solar_errors_$s.csv`

  Each file contains one row per operational time step and one column per day of historical forecast error. These are used to compute the seasonal covariance matrices.
- **Error Covariance Matrices:** The load and solar error matrices are assumed to be uncorrelated and are added to obtain the combined forecasting error covariance matrix Σ_s for each season s. This matrix is used to compute the standard deviation σ_t for each time step.
- **Outage Parameters:** Defined under the `uncertainty_settings` block of the parameter file:

– `outage_duration` κ: Average duration (in hours) of grid outages.
– `outage_probability` $\omega \in [0, 1]$: Likelihood of one outage event per simulation horizon.

Assumptions for Forecasting Errors

- **Independence:** Load and solar forecasting errors are assumed uncorrelated.
- **Normality:** The joint distribution of errors is modeled as multivariate normal.
- **Unbiased:** The expected value of each forecasting error is zero, $\mu_t = 0$.

The overall error covariance matrix is constructed as:

$$\Sigma_s = \text{cov}(\text{load}) + \text{cov}(\text{solar}), \quad \sigma_t = \sqrt{\Sigma_s[t, t]}$$

These statistics are computed seasonally, enabling the model to account for time- and season-specific uncertainty levels.

3.2 Reserve Variables and Constraints

To enhance system robustness against uncertainty and grid outages, the Expected Value Sizing Model introduces additional reserve variables:

- battery_reserve$[t, s]$: Extra battery discharge capacity reserved to respond to unexpected demand or production shortfalls.
- generator_reserve$[t, s]$: Additional generator capacity committed for unexpected needs, particularly during outages or forecast deviations.

These reserves ensure reliable service under uncertainty by allowing the system to dynamically allocate additional dispatch capability without violating physical or operational constraints.

Battery Operation Constraints Battery operation is extended to incorporate reserve dispatch:

$$\text{Charging Limit:} \qquad \text{battery\_charge}[t, s] \leq \frac{\text{battery\_capacity}}{t_{\text{charge}}} \cdot \Delta t \tag{12}$$

$$\text{Discharging + Reserve:} \qquad \text{battery\_discharge}[t, s] + \text{battery\_reserve}[t, s] \leq \frac{\text{battery\_capacity}}{t_{\text{discharge}}} \cdot \Delta t \tag{13}$$

Battery SOC Constraints Battery state of charge evolves with charging and discharging, and must remain feasible even under reserve discharge during outages:

$$\text{SOC}_{\text{min}} \cdot \text{battery\_capacity} \leq \text{SOC}[t, s] \leq \text{SOC}_{\text{max}} \cdot \text{battery\_capacity} \tag{14}$$

$$\text{SOC}[1, s] = \text{SOC}_0 \cdot \text{battery\_capacity} + (\eta_{\text{charge}} \cdot \text{battery\_charge}[1, s] - \eta_{\text{discharge}} \cdot \text{battery\_discharge}[1, s]) \tag{15}$$

$$\text{SOC}[t, s] = \text{SOC}[t-1, s] + (\eta_{\text{charge}} \cdot \text{battery\_charge}[t, s] - \eta_{\text{discharge}} \cdot \text{battery\_discharge}[t, s]), \quad t > 1 \tag{16}$$

$$\text{SOC}[T, s] = \text{SOC}_0 \cdot \text{battery\_capacity} \tag{17}$$

To ensure that enough energy is preserved during outages, the following condition is enforced for any potential outage window:

$$\text{SOC}[t, s] - \sum_{s'=\max(1, t-\kappa)}^{t} \eta_{\text{discharge}} \cdot \text{battery\_reserve}[s', s] \geq \text{SOC}_{\text{min}} \cdot \text{battery\_capacity}$$

Generator Operation with Reserve Generator reserve is added to its production but bounded by capacity:

$$\text{generator\_production}[t,s] + \text{generator\_reserve}[t,s] \leq \text{generator\_capacity} \cdot \Delta t$$

Energy Balance Constraint The energy balance is written as an inequality to allow slack when needed, and includes all reserve contributions:

$$\begin{aligned}\text{solar}[t,s] + \text{wind}[t,s] + \text{generator}[t,s] + \text{battery\_discharge}[t,s] - \text{battery\_charge}[t,s] \\ + \text{grid\_import}[t,s] - \text{grid\_export}[t,s] \geq \text{load}[t,s], \quad \forall t,s\end{aligned} \tag{18}$$

3.3 Expected Shortfall Penalty

Forecasting errors can still lead to unmet demand or unexpected surplus, even with reserves. The Expected Value model quantifies this through a nonlinear penalty function representing the expected shortfall cost:

$$\mathbb{E}[c_{\gamma,t}(\gamma_t)] = c_t \cdot \sigma_t \cdot \phi\left(\frac{\mu_t}{\sigma_t}\right) + c_t \cdot \mu_t \cdot \Phi\left(\frac{\mu_t}{\sigma_t}\right)$$

- ϕ is the standard normal PDF.
- Φ is the standard normal CDF.
- σ_t is the standard deviation of net imbalance at time t, derived from forecast error covariance.
- $\mu_t \approx 0$ under the assumption of unbiased errors.
- c_t is a composite cost that includes both grid import cost and penalty for exchange mismatches.

To evaluate this penalty, the model introduces mismatch variables $y[t,s]$, defined as the net difference between load and available supply for each time step and season:

$$y[t,s] = \text{load}[t,s] - \text{net\_supply}[t,s]$$

The expected penalty is then evaluated using a nonlinear function $\psi(y)$, and integrated into the total OPEX. Gradients of this function are also registered to support efficient nonlinear optimization.

This mechanism allows the Expected Value Model to penalize unbalanced operations realistically without requiring full scenario enumeration.

3.4 Objective Function – Expected Value Sizing Model

The objective of the Expected Value Sizing Model is to minimize the total Net Present Cost (NPC) of the system while explicitly accounting for forecast uncertainties and outage-related risks. The cost structure remains consistent with the deterministic model and includes:

- Capital Expenditures (CAPEX)
- Discounted Replacement Costs
- Subsidies (as a negative cost)
- Salvage Value (discounted)
- Discounted Operational Expenditures (OPEX), which now include:
  - **Core Operational Costs:** Mainly fuel costs under regular conditions.
  - **Outage Costs:** Grid use and penalty from expected shortfall during outages.
  - **Reserve Costs:** Fuel and battery use dedicated to cover outages.
  - **Non-Outage Costs:** Grid use and penalty during normal operation.

The total NPC expression is:

$$\begin{aligned}
\min\ \text{NPC} := \quad & \left(\sum_{i\in\mathcal{T}} x_i \cdot C_{\text{capex},i}\right) \\
& + \left(\sum_{i\in\mathcal{T}}\sum_{y\in R_i} x_i \cdot C_{\text{capex},i} \cdot \text{discount_factor}[y]\right) \\
& - \left(\sum_{i\in\mathcal{T}_{\text{sub}}} x_i \cdot C_{\text{capex},i} \cdot \text{subsidy}_i\right) \\
& + \left(\sum_{y=1}^{L} [\text{OPEX}_{\text{fixed}} + \text{Annual_OPEX}(y)] \cdot \text{discount_factor}[y]\right) \\
& - \left(\sum_{i\in\mathcal{T}} x_i \cdot C_{\text{capex},i} \cdot f_i^{\text{salvage}} \cdot \text{discount_factor}[L]\right)
\end{aligned}$$

Annual Operational Expenditure (OPEX) – Detailed Breakdown The Expected Value Sizing Model includes four main components in its annual operational cost, aggregated across all seasons and scaled to reflect the probability and duration of grid outages. Specifically:

- **Core Operational Costs** represent generator fuel use under regular conditions.
- **Outage Costs** capture grid import/export costs and expected energy shortfall penalties during simulated outages. They are weighted by the probability of an outage ω, and the fact that it can occur at any time step in the horizon T, with duration κ.
- **Reserve Costs** represent the additional use of battery and generator capacity during outage periods.
- **Non-Outage Costs** account for grid exchange costs and shortfall penalties when the system is operating without outages (with probability $1-\omega$).

The full expression for the annual OPEX becomes:

$$\begin{aligned}
\text{Annual_OPEX} = \quad & \sum_{s=1}^{S} w_s \sum_{t=1}^{T} \left(\frac{\text{generator_production}[t,s]}{\text{LHV}} \cdot c_{\text{fuel}}\right) \\
& + \frac{\omega}{T} \cdot (\text{outage_costs} + \text{reserve_costs}) \\
& + (1-\omega) \cdot \text{non_outage_costs}
\end{aligned} \tag{19}$$

Outage Costs:

$$\begin{aligned}
\text{outage_costs} = \sum_{s=1}^{S} w_s \sum_{\tau=1}^{T} \sum_{t\notin\tau:\tau+\kappa} \Big(& c_{\text{grid}}[t,s] \cdot \text{grid_import}[t,s] \\
& - p_{\text{grid}}[t,s] \cdot \text{grid_export}[t,s] \\
& + \text{expected_shortfall}[t,s] \cdot c_{\text{exchange}}\Big)
\end{aligned} \tag{20}$$

Reserve Costs:

$$\begin{aligned}
\text{reserve_costs} = \sum_{s=1}^{S} w_s \sum_{\tau=1}^{T} \sum_{t\in\tau:\tau+\kappa} \Big(& \frac{\text{generator_reserve}[t,s]}{\text{LHV}} \cdot c_{\text{fuel}} \\
& + \text{battery_reserve}[t,s] \cdot c_{\text{battery,opex}}\Big)
\end{aligned} \tag{21}$$

(If partial load modeling is enabled, the generator reserve term is replaced by generator fuel reserve.)

Non-Outage Costs:

$$
\begin{aligned}
\text{non\_outage\_costs} = \sum_{s=1}^{S} w_s \sum_{t=1}^{T} \Big(& c_{\text{grid}}[t,s] \cdot \text{grid\_import}[t,s] \\
& - p_{\text{grid}}[t,s] \cdot \text{grid\_export}[t,s] \\
& + \text{expected\_shortfall}[t,s] \cdot c_{\text{exchange}} \Big)
\end{aligned} \tag{22}
$$

Each cost component is scaled with the seasonal weight w_s, then combined into an annual cost by incorporating the outage probability ω, and finally discounted over the project lifetime as part of the total Net Present Cost.

Note on Grid-Connected vs. Off-Grid Systems

The outage-related cost terms described above are only relevant in the case of a **weakly connected mini-grid** where the main grid is present but unreliable.
If the system is fully **off-grid** (i.e., `allow_grid_connection` = `false`), then:

- Grid-related terms (import/export, prices, costs) are excluded from the model.
- The **only uncertainty** considered in the operational cost is the **expected shortfall**, caused by forecasting errors in load and renewable generation.
- The terms outage_costs and non_outage_costs reduce to:

$$
\sum_{s=1}^{S} w_s \sum_{t=1}^{T} \text{expected\_shortfall}[t,s] \cdot c_{\text{exchange}}
$$

This distinction ensures that the model remains flexible and realistic for both fully islanded and weak-grid contexts.

4 Individual Chance Constraint (ICC) Model

The Individual Chance Constraint (ICC) model is a refinement of the Expected Value Sizing Model. While the Expected Value model incorporates forecasting uncertainty through expected costs, the ICC model explicitly enforces a reliability level for energy balance using probability theory.

The key novelty in the ICC formulation is the modification of the energy balance constraint to include a probabilistic lower bound. This ensures that the energy balance is satisfied at every time step with a given confidence level p, also called the *islanding probability.*

4.1 Quantile-Based Energy Balance Constraint

For each time step t and season s, the energy balance must satisfy:

$$
\begin{aligned}
\text{solar}[t,s] + \text{wind}[t,s] + \text{generator}[t,s] + \text{battery\_discharge}[t,s] \\
+ \text{generator\_reserve}[t,s] + \text{battery\_reserve}[t,s] \\
+ \text{grid\_import}[t,s] - \text{grid\_export}[t,s] \\
\geq d[t,s] + Q_t[t,s]
\end{aligned} \tag{23}
$$

where:

- $Q_t[t,s]$: the quantile of the forecast error distribution at time t, season s, corresponding to the reliability level p.
- $d[t,s]$: load demand at time t, season s.

This inequality ensures that the total available supply not only meets the nominal demand, but also has a sufficient buffer to cover stochastic deviations (e.g., underforecasted load or overforecasted renewable production).

4.2 Probability Quantile Computation

Quantiles $Q_t[t, s]$ are precomputed from the forecast error distributions using the following procedure:

- Forecasting errors for solar and load are assumed to be normally distributed and unbiased (i.e., mean zero).
- The total error variance at each time step is the sum of the solar and load covariance matrices:

$$\sigma_t = \sqrt{\Sigma_{t,t}^{\text{load}} + \Sigma_{t,t}^{\text{solar}}}$$

- A normal distribution is constructed for each time step:

$$\xi_t \sim \mathcal{N}(0, \sigma_t^2)$$

- The quantile Q_t is then obtained using:

$$Q_t[t, s] = \Phi^{-1}(p) \cdot \sigma_t$$

  where $\Phi^{-1}(p)$ is the inverse cumulative distribution function (CDF) of the standard normal distribution at probability p.

These quantiles are computed once per time step and per season (if seasonality is enabled), and then included directly into the constraint matrix during model initialization.

Note on ICC Interpretation

The ICC model guarantees that the system is able to satisfy the load not just on average, but with high reliability (e.g., 95%, 99%) in every hour of operation.
It does this by augmenting the load requirement with a buffer Q_t, derived from statistical quantiles of the combined forecast error. This provides a robust but computationally efficient alternative to full scenario-based stochastic optimization.

5 Joint Chance Constraint (JCC) Model

The Joint Chance Constraint (JCC) model extends the ICC approach by enforcing system-wide reliability over a *window of consecutive hours*, rather than at individual time steps. This model is particularly well-suited to grid-connected systems with variable outage duration, where reliability must be ensured across the entire duration of a potential grid failure event. In contrast to ICC, which ensures that energy balance holds with high probability at each hour independently, the JCC model guarantees that supply meets demand *simultaneously across all hours in a potential outage window* with probability at least p. This reduces the risk of cumulative failure across multiple time steps during extended outages.

5.1 Multivariate Formulation

For each potential outage start time $\tau \in \{1, \ldots, T-\kappa\}$ and season $s \in \{1, \ldots, S\}$, the following constraint must hold:

$$P\left(\bigcap_{t=\tau}^{\tau+\kappa} \{\text{reserve\_mismatch}[t, s] \geq 0\}\right) \geq p \tag{24}$$

Here, the reserve mismatch at time t is defined as:

$$\begin{aligned} \text{reserve\_mismatch}[t, s] := {} & \text{generator\_reserve}[t, s] + \text{battery\_reserve}[t, s] \\ & + \text{grid\_import}[t, s] - \text{grid\_export}[t, s] - d[t, s] \end{aligned} \tag{25}$$

This expression represents the margin between total reserve supply and expected load during an outage.

5.2 Implementation via Multivariate Normal CDF

Forecast errors are assumed jointly normally distributed across the outage window. For each outage start time τ, the JCC constraint evaluates the cumulative probability:

$$\Phi_{\Sigma}(\mathbf{x}_{\tau}) \geq p$$

where Φ_{Σ} is the multivariate cumulative distribution function (CDF) of a normal distribution with zero mean and covariance matrix Σ, and $\mathbf{x}_{\tau}$ is the vector of reserve mismatches from time τ to $\tau + \kappa$.

The model uses numerical integration (e.g., quasi-Monte Carlo methods) to approximate this CDF and compute gradients for optimization.

5.3 JCC vs. ICC

- **ICC** controls the reliability at each time step independently:

$$P\left(\text{supply}[t, s] \geq d[t, s] + Q_t[t, s]\right) \geq p$$

- **JCC** controls the probability that all time steps in a window simultaneously satisfy the demand:

$$P\left(\forall t \in \tau : \tau + \kappa,\ \text{supply}[t, s] \geq d[t, s]\right) \geq p$$

JCC is more conservative and computationally demanding than ICC but provides stronger system-wide guarantees during sustained outage events.

Interpretation

The JCC model ensures that during a potential grid outage of duration κ, the system has sufficient reserves to cover demand across all affected hours with a joint reliability of at least p. This makes it particularly relevant for critical infrastructure, resilience analysis, and microgrids with unreliable grid connections.

References

[1] N. Ouanes, T. G. Grandón, H. Heitsch, et al. Optimizing the economic dispatch of weakly-connected mini-grids under uncertainty using joint chance constraints. *Annals of Operations Research*, 344:499–531, 2025.