Advanced and Multivariable Control

Linear Quadratic Control

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System

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \qquad (t_0 = 0)$$

Cost function

$$J(x_0, u(\cdot), 0) = \int_0^T (x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau))\, d\tau + x'(T)Sx(T)$$

Design parameters

$$Q = Q' \geq 0, \quad S = S' \geq 0, \quad R = R' > 0$$

$R > 0$ all the components of u are weighted in J, no solutions based on impulsive u

HJB equation

$$\min_u \left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\}$$

$$\downarrow$$

$$\min_u \left\{ x'Qx + u'Ru + \frac{\partial J^o(x,t)}{\partial x} (Ax + Bu) \right\}$$

$$\downarrow$$

$$2u'R + \frac{\partial J^o(x,t)}{\partial x} B = 0$$

$$\downarrow$$

$$u^o = -\frac{1}{2} R^{-1} B' \left(\frac{\partial J^o(x,t)}{\partial x} \right)'$$

(recall, ***R>0***)

$$\min_u \left\{ x'Qx + u'Ru + \frac{\partial J^o(x,t)}{\partial x}(Ax+Bu) \right\}$$

$$u^o = -\frac{1}{2}R^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)'$$

$$\begin{aligned}\frac{\partial J^o(x,t)}{\partial t} &= -x'Qx - \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)' \\ &\quad - \left(\frac{\partial J^o(x,t)}{\partial x}\right)\left[Ax - \frac{1}{2}BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)'\right]\end{aligned}$$

$$\begin{aligned}-\frac{\partial J^o(x,t)}{\partial t} &= x'Qx - \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)' \\ &\quad + \left(\frac{\partial J^o(x,t)}{\partial x}\right)Ax\end{aligned}$$

$$J^o(x,T) = x'Sx = m(x)$$

$$-\frac{\partial J^o(x,t)}{\partial t} = x'Qx - \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)' + \left(\frac{\partial J^o(x,t)}{\partial x}\right)Ax$$

Tentative solution

$$J^o(x,t) = x'P(t)x \quad , \quad P(T) = S \longleftrightarrow J^o(x,T) = x'Sx = m(x)$$

$$-\frac{\partial J^o(x,t)}{\partial t} \qquad\qquad \frac{\partial J^o(x,t)}{\partial x}$$

$$\begin{aligned} -x'\dot{P}(t)x &= x'Qx - x'P(t)BR^{-1}B'P'(t)x + 2x'P(t)Ax \\ &= x'Qx - x'P(t)BR^{-1}B'P'(t)x + 2x'\frac{P(t)A + A'P'(t)}{2}x \end{aligned}$$

$$\boxed{\begin{gathered} \dot{P}(t) + Q - P(t)BR^{-1}B'P'(t) + P(t)A + A'P'(t) = 0 \\ P(T) = S \end{gathered}}$$

differential Riccati equation

Finite horizon optimal control law

$$u^o(t) = -\underbrace{R^{-1}B'P'(t)}_{K(t)}x(t) = -K(t)x(t)$$

Comments

- $P(t) = P'(t)$
- $J^0(x,t) = x'P(t)x \geq 0$
- $K(t)$ time varying and defined over a finite interval
- difficult to solve a matrix differential equation
- non easy to use in «standard» control problems (stabilization of time invariant systems)

Infinite horizon optimal control law

Idea

- $T \to \infty$
- properly weight all the state variables in the cost function ***J***

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, S = 0 \xrightarrow{\textit{the integrand is}} q_1 x_1^2 + q_2 x_2^2 + r_1 u_1^2 + r_2 u_2^2$$

If $q_1>0$, $q_2>0$ the only possibility is that asymptotically the state variables must tend to zero

$$x_1 \to 0, x_2 \to 0 \Rightarrow \text{asymptotic stability}$$

Infinite horizon LQ_∞

Finite horizon LQ $\longrightarrow$ $T \to \infty, \quad S = 0$ $\longrightarrow$

$$J(x_0, u(\cdot), 0) = \int_0^\infty (x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau))\, d\tau$$

$$Q = Q' \geq 0, \quad R = R' > 0$$

If the pair (A, B) is reachable

A *the solution of the differential Riccati equation with initial condition* $P(T) = 0$ *tends, for* $T \to \infty$, *to a* $\boxed{\textit{constant matrix } \bar{P} \geq 0}$ *solution of the algebraic Riccati equation*

$$0 = A'\bar{P} + \bar{P}A + Q - \bar{P}BR^{-1}B'\bar{P}$$

algebraic Riccati equation

B *the asymptotic control law is*

$$u(t) = -R^{-1}B'\bar{P}x(t) = -\bar{K}x(t)$$

time invariant control law

where $\bar{K} = R^{-1}B'\bar{P}$

simpler and time invariant (like pole placement), not yet stabilizing
we must guarantee to weight the states

How to weight the states?

Partition the matrix ***Q***: $Q = C_q' C_q$ (not unique, for instance ***Q=1 -> C_q=1*** or ***C_q=-1***)

Define the *fictitious output* $\tilde{y}(\tau) = C_q x(\tau)$ and write the cost function as

$$
\begin{aligned}
J(x_0, u(\cdot), 0) &= \int_0^\infty \left(x'(\tau) C_q' C_q x(\tau) + u'(\tau) R u(\tau)\right) d\tau \\
&= \int_0^\infty \left(\tilde{y}'(\tau)\tilde{y}(\tau) + u'(\tau) R u(\tau)\right) d\tau
\end{aligned}
$$

Weighting the states -> guaranteeing that the state is «visible» from the fictitious output ***y_{tilde}*** that is that the state is ***observable*** from output ***y_{tilde}***

Two intermediate results

Result 1

Let C_{q1} and C_{q2} be two partitions of Q, i.e.

$$Q = C'_{q1} C_{q1} = C'_{q2} C_{q2}$$

Then, if (A, C_{q1}) is observable, also (A, C_{q2}) is observable

Result 2

The asymptotic solution $\bar{P}$ of the Riccati equation is positive definite if and only if (A, C_q) is observable

Some comments

Choosing ***Q > 0*** automatically solves the problem. In fact, ***Q>0*** → ***C_q*** square and full rank

If the system has an output $y(t) = Cx(t)$ and the pair (***A,C***) is observable, choosing ***Q=C'C*** satisfies the condition. In addition, it can be reasonable to consider the cost function

$$
\begin{aligned}
J(x_0, u(\cdot), 0) &= \int_0^{\infty} \left(x'(\tau)C'Cx(\tau) + u'(\tau)Ru(\tau)\right) d\tau \\
&= \int_0^{\infty} \left(y'(\tau)y(\tau) + u'(\tau)Ru(\tau)\right) d\tau
\end{aligned}
$$

Remember that the state depends on the state coordinate, that can change, and it is not always possible to interpret its meaning, while the output is uniquely and clearly definedd

Main result

If

the pair (A, B) is reachable
the pair (A, C_q) is observable

then

A the optimal control law is given by

$$\boxed{u(t) = -\bar{K}x(t)}$$

with

$$\bar{K} = R^{-1}B'\bar{P}$$

where $\bar{P}$ is the unique positive definite solution of the stationary Riccati equation

$$0 = \bar{P}A + A'\bar{P} + Q - \bar{P}BR^{-1}B'\bar{P}$$

B the closed-loop system

$$\boxed{\dot{x}(t) = (A - B\bar{K})x(t)}$$

is asymptotically stable.

time invariant, stabilizing control law

Proof of stability (only)

Consider $J^o(x) = x'\bar{P}x$ as a Lyapunov function

$$
\begin{aligned}
\frac{\partial J^o(x)}{\partial t} &= \dot{x}'\bar{P}x + x'\bar{P}\dot{x} \\
&= x'(A - B\bar{K})'\bar{P}x + x'\bar{P}(A - B\bar{K})x \\
&= x'\left\{A'\bar{P} - \bar{K}'B'\bar{P} + \bar{P}A - \bar{P}B\bar{K}\right\}x \\
&= x'\left\{A'\bar{P} - \bar{P}BR^{-1}B'\bar{P} + \bar{P}A - \bar{P}BR^{-1}B'\bar{P}\right\}x \\
&= -x'\left\{Q + \bar{P}BR^{-1}B'\bar{P}\right\}x \\
&= -x'\left\{Q + \bar{K}'R\bar{K}\right\}x
\end{aligned}
$$

If ***Q>0*** the proof is completed. If ***Q*** is only semidefinite positive … apply Krasowski La Salle

Proof of stability (continued)

$$Q \geq 0 \longrightarrow \text{assume by contradiction that } \frac{\partial J^o(\bar{x}(t))}{\partial t} = 0,$$

$$\downarrow$$

$$-\bar{x}'(t)\left\{Q + \bar{P}BR^{-1}B'\bar{P}\right\}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$\bar{x}'(t)Q\bar{x}(t) = 0, \quad \forall t \quad \text{and} \quad \bar{x}'(t)\bar{P}BR^{-1}B'\bar{P}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$B'\bar{P}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$u(t) = -R^{-1}B'\bar{P}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$\bar{x}(t) = e^{At}\bar{x}_0, \quad \bar{x}_0 \neq 0$$

$$\downarrow$$

$$\bar{x}'(t)Q\bar{x}(t) = \bar{x}_0'e^{A't}C_q'C_qe^{At}\bar{x}_0 = 0, \quad \forall t, \quad \bar{x}_0 \neq 0$$

It can be proven that this contradicts the observability of (A,C_q)

Example

system $\dot{x}(t) = x(t) + u(t)$ $\quad A = 1,\ B = 1$ $\quad$ Riccati eq. $2\bar{P} + Q - \dfrac{\bar{P}^2}{R} = 0$ $\quad \bar{P} = \dfrac{2 \pm \sqrt{4\left(1+\frac{Q}{R}\right)}}{2/R}$

$$Q = 3,\ R = 1\ \left(\tfrac{Q}{R} = 3\right) \qquad \bar{P} = \frac{2 \pm \sqrt{16}}{2} = \begin{cases} 3 & solution > 0 \\ -1 \end{cases}$$

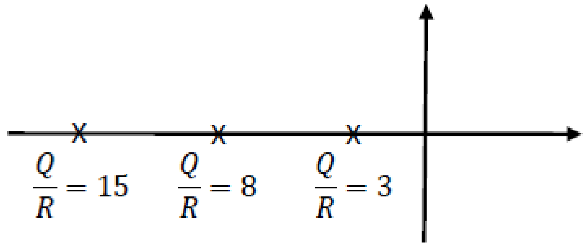
$$\bar{K} = R^{-1}B'\bar{P} = 3$$

$$A - B\bar{K} = 1 - 3 = -2$$

$$Q = 8,\ R = 1\ \left(\tfrac{Q}{R} = 8\right) \qquad \bar{P} = \frac{2 \pm \sqrt{36}}{2} = \begin{cases} 4 & solution > 0 \\ -2 \end{cases}$$

$$\bar{K} = R^{-1}B'\bar{P} = 4$$

$$A - B\bar{K} = 1 - 4 = -3$$



$$Q = 15,\ R = 1\ \left(\tfrac{Q}{R} = 15\right) \qquad \bar{P} = \frac{2 \pm \sqrt{64}}{2} = \begin{cases} 5 & solution > 0 \\ -3 \end{cases}$$

$$\bar{K} = R^{-1}B'\bar{P} = 5$$

$$A - B\bar{K} = 1 - 5 = -4$$

Example LQ_{inf}

system $\dot{x}(t) = ax(t) + b_1u_1(t) + b_2u_2(t) \qquad A = a,\ B = [b_1\ \ b_2]$

weights $Q = q, \quad R^{-1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}, \quad R = \begin{bmatrix} 1/\rho_1 & 0 \\ 0 & 1/\rho_2 \end{bmatrix}$

algebraic Riccati equation $0 = 2\bar{p}a + q - \bar{p}^2\left(b_1^2\rho_1 + b_2^2\rho_2\right)$

$$a = 0,\ b_1 = b_2 = 1 \qquad \bar{p} = \sqrt{\frac{q}{\rho_1 + \rho_2}}$$

$$\bar{K} = R^{-1}B'\bar{p} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sqrt{\frac{q}{\rho_1 + \rho_2}} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}\sqrt{\frac{q}{\rho_1 + \rho_2}} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\rho_1^2 q}{\rho_1+\rho_2}} \\ \sqrt{\frac{\rho_2^2 q}{\rho_1+\rho_2}} \end{bmatrix}$$

The less you weight one control variable with respect to the other (ρ_1 greater than ρ_2 for instance), the larger is the corresponding control gain with respect to the other

Exercise LQ_∞

Q=1

$$\dot{x} = ax + u, \quad A = a, \quad B = 1$$

Riccati eq. $\quad P^2 - 2aRP - R = 0 \rightarrow P = aR + \sqrt{(aR)^2 + R}$

$$u = -\frac{P}{R}x = -\left(a + \sqrt{a^2 + 1/R}\right)x$$

$$\dot{x} = \left(a - a - \sqrt{a^2 + 1/R}\right)x = -\sqrt{a^2 + 1/R}\;x$$

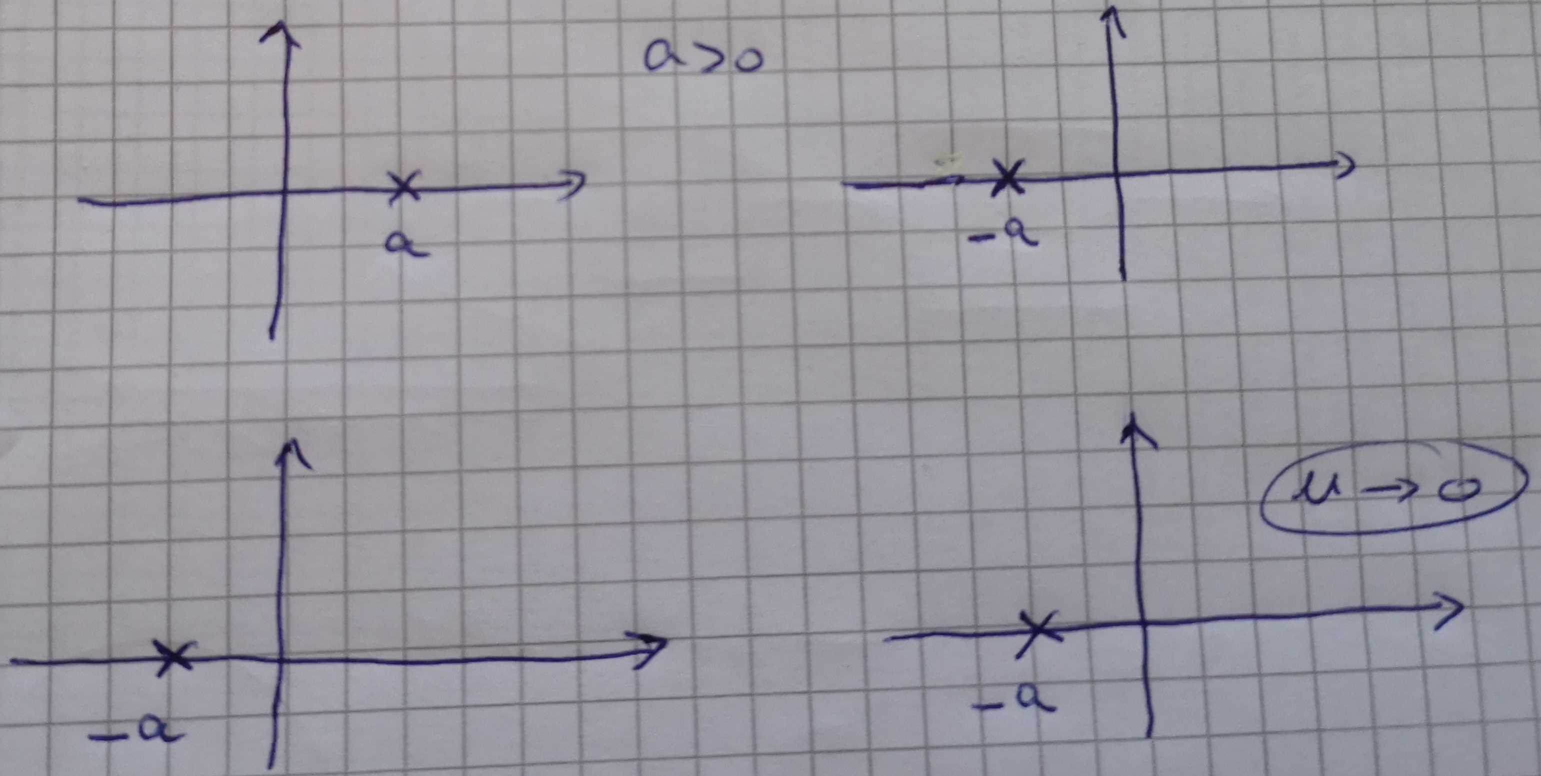
If $R \rightarrow \infty \quad \rightarrow \quad \dot{x} = -|a|\,x$

If $R \to \infty$ $\to$ $\dot{x} = -|a|\,x$

Open loop

Closed-loop

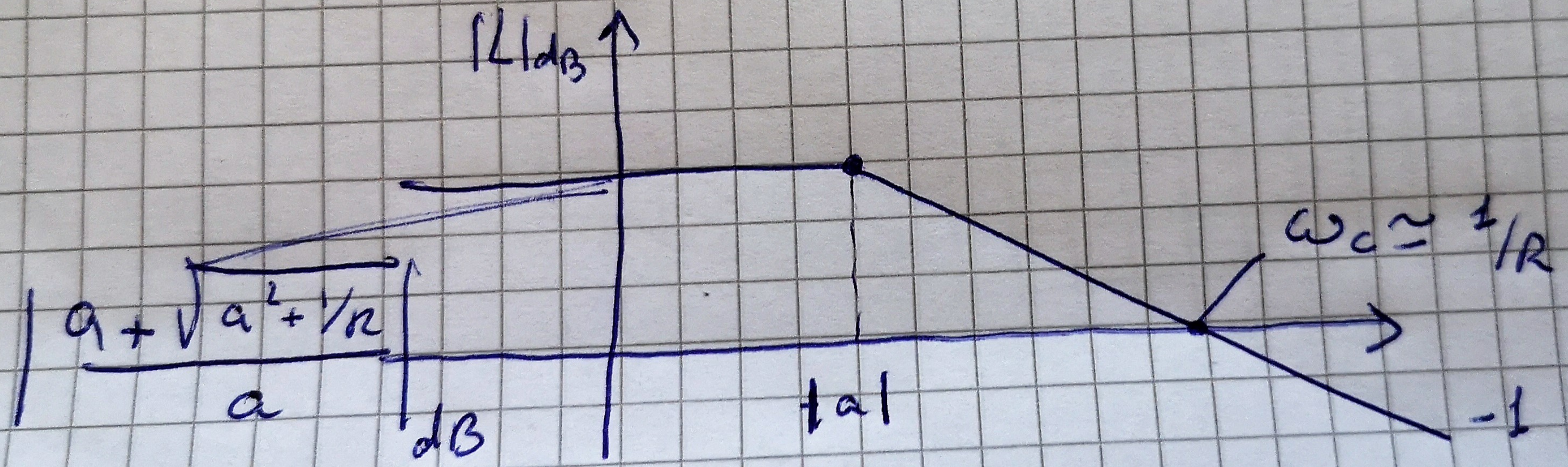
$a > 0$



$-a$ $-a$

If $R \to 0 \longrightarrow \dot{x} = -\alpha x, \quad \alpha \to \infty$

$$L(s) = G(s) \cdot K = \frac{1}{s-a}\left(a + \sqrt{a^2 + 1/R}\right)$$



The really difficul task (as always) is to tune the design parameters Q an R

Choice of the design parameters - normalization

$$Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & q_n \end{bmatrix}, \quad q_i \geq 0, \quad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_m \end{bmatrix}, \quad r_i > 0$$

but state and control variables can differ of orders of magnitude

$$J = \int_0^\infty \left(q_1 x_1^2(\tau) + \ldots + q_n x_n^2(\tau) + r_1 u_1^2(\tau) + \ldots + r_m u_m^2(\tau) \right) d\tau$$

$$\downarrow$$

$$\text{If } |x_i| < x_{\max_i},\ i = 1, \ldots, n,\ |u_i| < u_{\max_i},\ i = 1, \ldots, m$$

$$\downarrow$$

$$q_i = \frac{\tilde{q}_i}{x_{\max_i}^2}, i = 1, \ldots, n, \quad r_i = \frac{\tilde{r}_i}{u_{\max_i}^2}, i = 1, \ldots, m, \quad \tilde{q}_i \geq 0, \quad \tilde{r}_i > 0$$

$$\downarrow$$

$$J = \int_0^\infty \left(\tilde{q}_1 \frac{x_1^2(\tau)}{x_{\max_1}^2} + \ldots + \tilde{q}_n \frac{x_n^2(\tau)}{x_{\max_n}^2} + \tilde{r}_1 \frac{u_1^2(\tau)}{u_{\max_1}^2} + \ldots + \tilde{r}_m \frac{u_m^2(\tau)}{u_{\max_m}^2} \right) d\tau$$

now they weight quantities in (0,1)

Choice of the design parameters – weights on the outputs

System with outputs
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

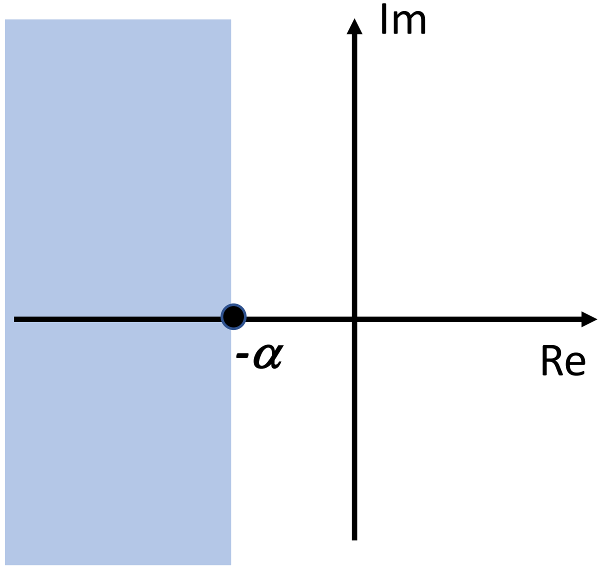
Choice of the weights $\quad Q = C'\bar{Q}C, \quad \bar{Q} \geq 0, \quad \bar{Q} \in R^{p,p}$

$$J = \int_0^\infty \left(x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau)\right) d\tau = \int_0^\infty \left(x'(\tau)C'\bar{Q}Cx(\tau) + u'(\tau)Ru(\tau)\right) d\tau = \int_0^\infty \left(y'(\tau)\bar{Q}y(\tau) + u'(\tau)Ru(\tau)\right) d\tau$$

Usually, it is much simpler to weight the ouputs, always with a clear meaning, than the states, whose meaning depends on the state space representation

Choice of the design parameters – prescribed rate of convergence

Can we use ***LQ**$_{inf}$* so that the closed-loop poles have negative real part smaller than *-α ? (α>0)*



Idea

this tends to infinity as an exponential

these must tend to zero faster

$$J(x_0, u(\cdot), 0) = \int_0^\infty e^{2\alpha\tau} \left(x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau)\right) d\tau$$

$$J(x_0, u(\cdot), 0) = \int_0^{\infty} e^{2\alpha\tau} \left(x'(\tau) Q x(\tau) + u'(\tau) R u(\tau) \right) d\tau$$

define
$$\begin{aligned} \tilde{x}(t) &= e^{\alpha t} x(t) \\ \tilde{u}(t) &= e^{\alpha t} u(t) \end{aligned}$$

$$\tilde{A} = \alpha I + A$$

$$\begin{aligned} \dot{\tilde{x}}(t) &= \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t) = \alpha \tilde{x}(t) + e^{\alpha t} \left(A x(t) + B u(t) \right) \\ &= \alpha \tilde{x}(t) + A \tilde{x}(t) + B \tilde{u}(t) = (\alpha I + A) \tilde{x}(t) + B \tilde{u}(t) \\ &= \tilde{A} \tilde{x}(t) + B \tilde{u}(t) \end{aligned}$$

$$J(x_0, u(\cdot), 0) = \int_0^{\infty} \left(\tilde{x}'(\tau) Q \tilde{x}(\tau) + \tilde{u}'(\tau) R \tilde{u}(\tau) \right) d\tau$$

this is a standard LQ_{inf} control problem in the tilda variables

$$\tilde{u}(t) = -R^{-1} B' \bar{P}_\alpha \tilde{x}(t) = -\bar{K}_\alpha \tilde{x}(t)$$

$$0 = \bar{P}_\alpha \tilde{A} + \tilde{A}' \bar{P}_\alpha - \bar{P}_\alpha B R^{-1} B' \bar{P}_\alpha + Q$$

closed-loop eigenvalues

$$\tilde{A} - B\bar{K}_\alpha = A + \alpha I - B\bar{K}_\alpha$$

with negative real part

eigenvalues of

$$A - B\bar{K}_\alpha$$

with real part smaller than **-α**

Summary of the method

1. Build the matrix $\tilde{A} = A + \alpha I$.
2. Compute the matrix $\bar{K}_\alpha$, solution of the LQ problem, for the system described by the matrices $\tilde{A}$ and B.
3. Implement the control law

$$u(t) = -\bar{K}_\alpha x(t)$$

Example

$$\dot{x}(t) = x(t) + u(t) \qquad A = B = 1 \qquad \tilde{A} = A + \alpha = 1 + \alpha$$

$$0 = 2\bar{P}\tilde{A} + Q - \bar{P}^2 R^{-1} \qquad \frac{\bar{P}^2}{R} - 2\,(1+\alpha)\,\bar{P} - Q = 0 \qquad \bar{P} = \frac{(1+\alpha) + \sqrt{(1+\alpha)^2 + \frac{Q}{R}}}{1/R}$$

$$\bar{K}_\alpha = R^{-1}B\bar{P} = (1+\alpha) + \sqrt{(1+\alpha)^2 + \frac{Q}{R}}$$

$$A - B\bar{K}_\alpha = -\alpha - \sqrt{(1+\alpha)^2 + \frac{Q}{R}} \leq -\alpha$$

12/7/2018

Consider the system

$$\dot{x}_1(t) = bu(t)$$
$$\dot{x}_2(t) = x_1(t) + u(t)$$

and assume that you want to design an infinite horizon LQ control with $Q=diag(q_1,q_2)$, $R=1$.

A. Compute the conditions guaranteeing that the solution of the infinite horizon LQ control is stabilizing.
B. With $Q=I$, $b=1$, the solution of the steady-state Riccati equation is $P=I$. Check the stability of the closed-loop system a) computing the closed-loop eigenvalues, b) by using a suitable Lyapunov function (the one used for the stability analysis of LQ control).
C. Assume now to implement the feedback control law $u(t) = -\rho K x(t)$ (K is again the solution of the LQ problem), specify the set of values of ρ guaranteed by LQ control so that the closed-loop system remains asymptotically stable.

Steady-state Riccati equation: $A'P + PA + Q - PBR^{-1}B'P = 0$

Solution

A.

$$A=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad B=\begin{bmatrix}b\\ 1\end{bmatrix}\rightarrow \text{reachability matrix } M_r=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}b & 0\\ 1 & b\end{bmatrix}\rightarrow \text{ condition } b\neq 0$$

$$Q=\begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix}\rightarrow Q^{1/2}=\begin{bmatrix}\sqrt{q_1} & 0\\ 0 & \sqrt{q_2}\end{bmatrix}\rightarrow \text{observability matrix } M_o=\begin{bmatrix}Q^{1/2}\\ Q^{1/2}A\end{bmatrix}=\begin{bmatrix}\sqrt{q_1} & 0\\ 0 & \sqrt{q_2}\\ 0 & 0\\ \sqrt{q_2} & 0\end{bmatrix}\rightarrow q_2>0$$

B.

$$K=R^{-1}B'P=\begin{bmatrix}1 & 1\end{bmatrix}\rightarrow A-BK=\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix}\rightarrow \text{eigenvalues } -1,-1$$

$$J^o=x'Px=x'x>0\rightarrow \frac{\partial J^o}{\partial t}=x'(A-BK)'x+x'(A-BK)x=-x'\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}x<0$$

C.

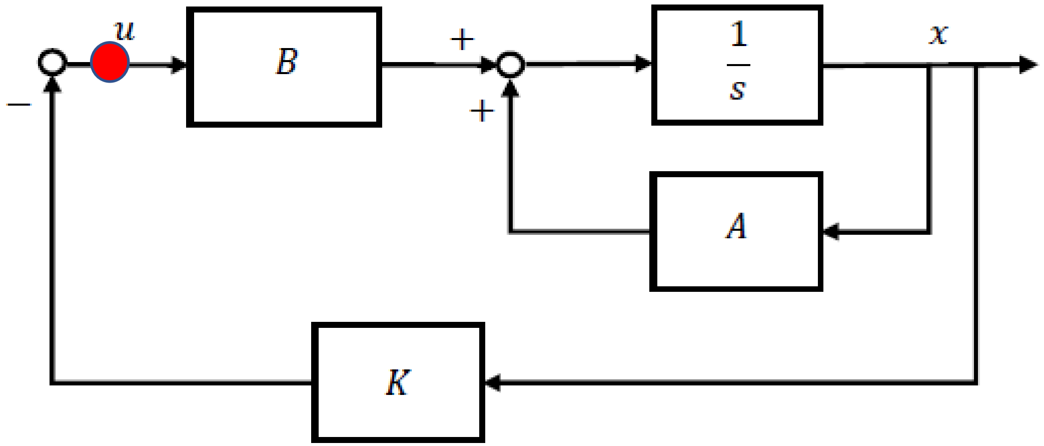
$$A = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$A - \rho BK = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \rho \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{BK} = \begin{vmatrix} -\rho & -\rho \\ 1-\rho & -\rho \end{vmatrix}$$

$$\det\left(sI - (A - \rho BK)\right) = \begin{vmatrix} s+\rho & \rho \\ \rho-1 & s+\rho \end{vmatrix} = s^2 + 2\rho s + \rho$$

$$\begin{cases} \rho > 0 \\ 2\rho > 0 \end{cases} \qquad \rho > 0$$

Robustness of *LQ$_{inf}$* control with respect to uncertainties *at the plant input*



*It is **stabilizing**, it is **optimal** (with respect to the selected design parameters), but is it also **robust** with respect to uncertainties at the plant input?*

With simple manipulations of the Riccati equation, it is possible to obtain the following relationship

$$G_c'(-s)QG_c(s) + R = \Gamma'(-s)R\Gamma(s)$$

$$G_c(s) = (sI - A)^{-1}B$$ transfer function from ***u*** to ***x***

$$\Gamma(s) = I + \underbrace{K(sI - A)^{-1}B}_{\text{Loop transfer function } L(s)}$$ Inverse of the *sensitivity function* (*return difference*)

$$G_c'(-s)QG_c(s) + R = \Gamma'(-s)R\Gamma(s) \longrightarrow \Gamma^*(j\omega)R\Gamma(j\omega) \geq R$$

Single input systems, m=1

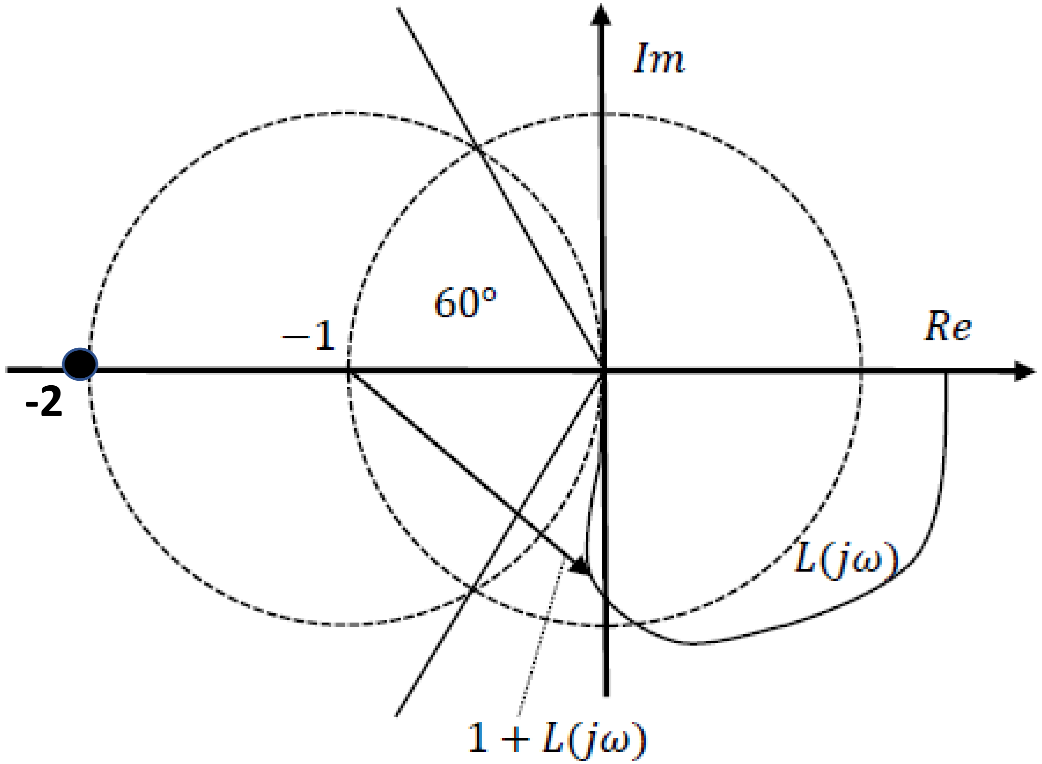
$$(1 + L(j\omega))^* R(1 + L(j\omega)) \geq R \longrightarrow |1 + L(j\omega)|^2 \geq 1$$

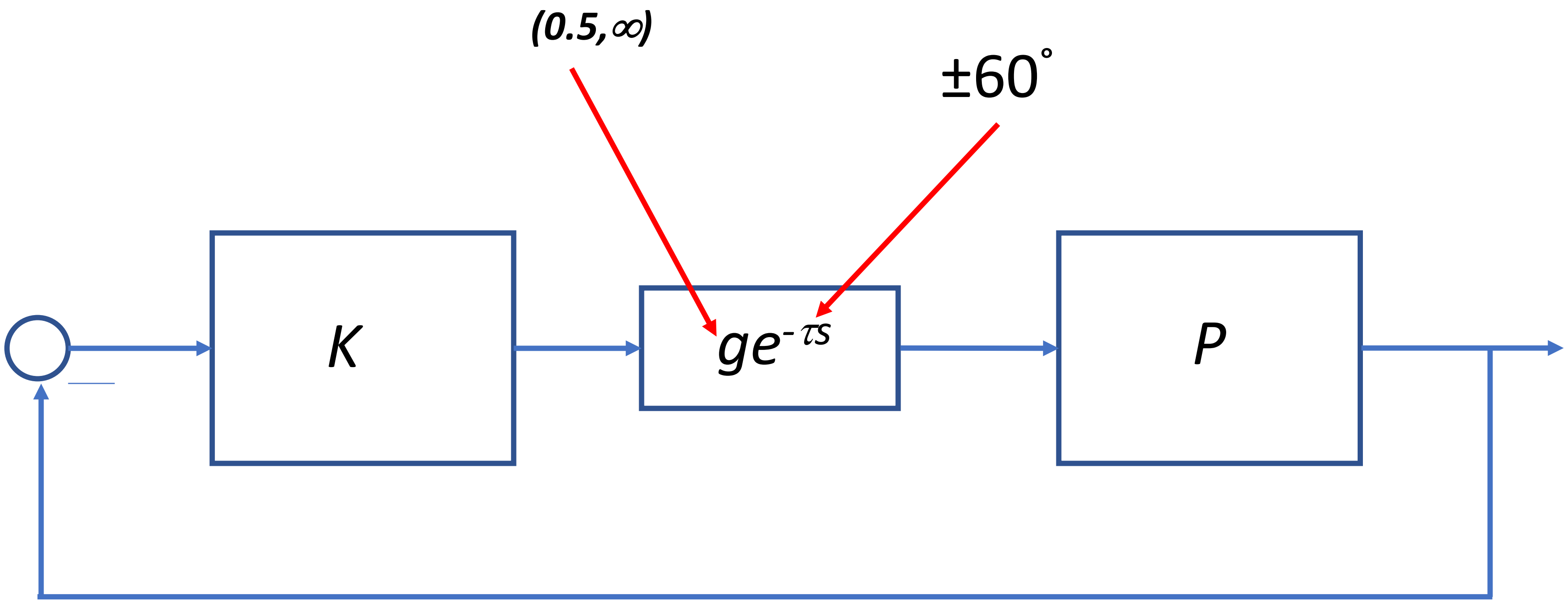
Feedback system robust with respect to:

- phase variations of ±60˚(***phase margin ±60***˚)
- gain variations (0.5,∞) (***gain margin (0.5,∞))***

$\downarrow$

but not guaranteed at the same time





(0.5,∞)
±60°
K
$ge^{-\tau s}$
P

… however, it can be proven that

$$
\begin{aligned}
T(j\omega) &= 1 - S(j\omega) \\
&= 1 - \left[1 + K(j\omega I - A)^{-1}B\right]^{-1} \\
&= K(j\omega I - A + BK)^{-1}B
\end{aligned}
$$

$$\Big\downarrow$$

$$\lim_{\omega\to\infty} j\omega T(j\omega) = KB = -R^{-1}B'PB$$

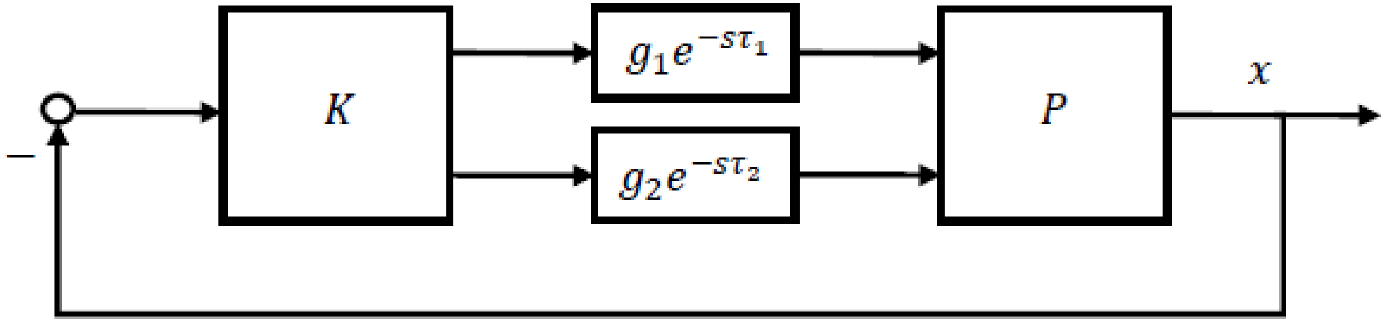
$|T(j\omega)|$ decreases at high frequency with slope -1

small attenuation of measurement noise and/or unmodelled dynamics at high frequency

Multi Input systems *m>1*

choosing $R = diag\{r_1, r_2, \ldots, r_m\}$, $r_i > 0$,

The closed-loop system



remains stable in front of

1. phase variations of magnitude up to 60^0;
2. gain variations $(1/2, \infty)$.

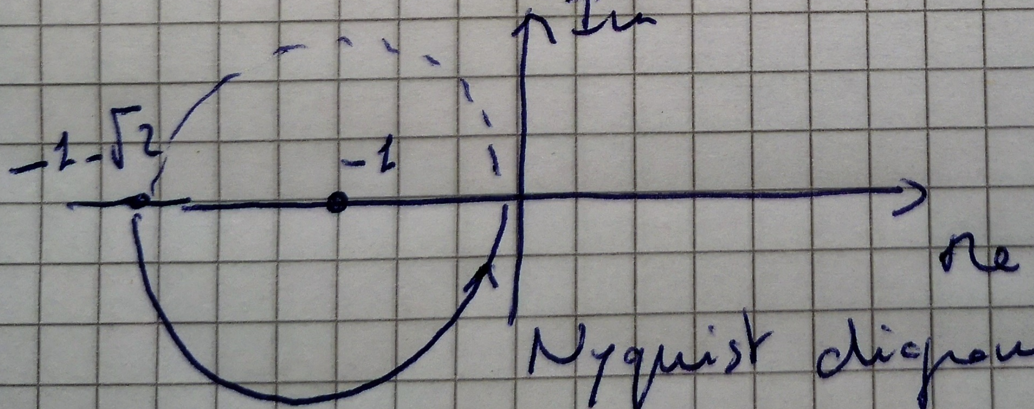
but not at the same time on the same channel

Example robustness LQ

$$\dot{x} = x + u \longrightarrow G(s) = \frac{1}{s-1}$$

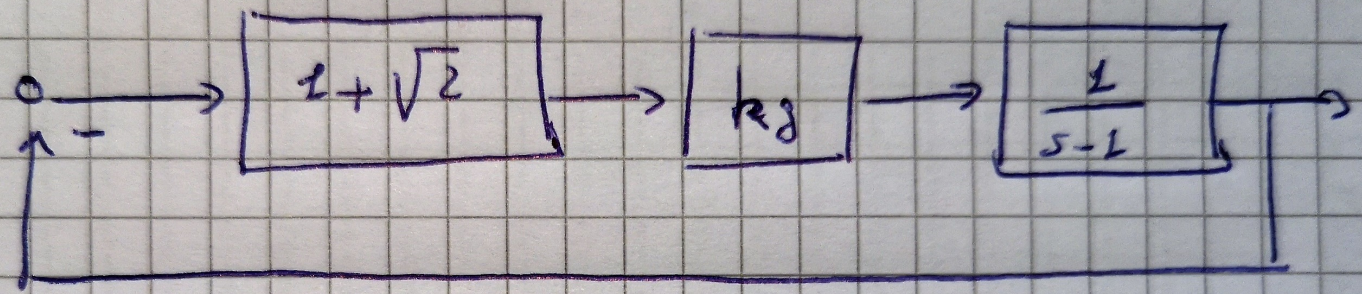
$$LQ_\infty \text{ with } R = Q = 1 \longrightarrow K = 1+\sqrt{2}$$

$$L(s) = K(sI-A)^{-1}B = \frac{1+\sqrt{2}}{s-1}$$



Nyquist diagram $\longrightarrow$ $P = N = 1$

Gain margin



Characteristic equation

$$s-1+k_g(1+\sqrt{2})=0$$

$$s=1-k_g(1+\sqrt{2})<0$$

$$k_g>\frac{1}{1+\sqrt{2}}=0.41$$