LQ CONTROL

FINITE/INFINITE HORIZON

How OPTIMAL CONTROL
can be used to design
classical Regulator

Advanced and Multivariable Control

Linear Quadratic Control

Riccardo Scattolini

→ to solve a specific kind of problems

System

(Linear systems)

@ $t_0 = 0 \rightarrow x_0$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \qquad (t_0 = 0)$$

initial state

QUADRATIC
Cost function

(LQ CONTROL) Linear Quadratic Control

→ m(x)

$$J(x_0, u(\cdot), 0) = \int_0^T \left(x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau)\right) d\tau + x'(T)Sx(T)$$

quadratic form on state

quadratic on u term!

quadratic on final state x(T)

Design parameters

symm. matrix!

$$Q = Q' \geq 0, \quad S = S' \geq 0, \quad R = R' > 0$$

strictly pos. def!

$R > 0$ all the components of u are weighted in J, no solutions based on impulsive u

- for now forget about constraints .. deal later
- our objective is to stabilize the system

↳ have stable equilibrium

IF we take

$Q \neq Q^T$ (non symm!) $\rightsquigarrow Q_{ns}$ we can proove that

$\Downarrow$

$x^T Q_{ns} x \longrightarrow (x^T Q_s x)$, $Q_s = \dfrac{Q_{ns} + Q^T_{ns}}{2}$

you can still obtain a symm form!

HJB equation

$$\min_u \left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\}$$

positive

$$\min_u \left\{ \underbrace{x'Qx + u'Ru}_{l(x,u)} + \frac{\partial J^o(x,t)}{\partial x} (Ax+Bu) \right\}$$

first, to find the min...

Just derive to obtain the sol and impose = 0

$$\boxed{2u'R + \frac{\partial J^o(x,t)}{\partial x} B = 0}$$

assuming R non singular allows to compute (R^{-1})

general form of optimal control law

$$u^o = \underbrace{-\frac{1}{2} R^{-1} B' \left(\frac{\partial J^o(x,t)}{\partial x} \right)'}_{K(x,t)}$$

(recall, ***R>0***)

taking that u^o and substitute...

$$\min_u \left\{ x'Qx + u'Ru + \frac{\partial J^o(x,t)}{\partial x}(Ax + Bu) \right\} \qquad u^o = -\frac{1}{2}R^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)'$$

$$\begin{aligned} \frac{\partial J^o(x,t)}{\partial t} &= -x'Qx - \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)' \\ &\quad - \left(\frac{\partial J^o(x,t)}{\partial x}\right)\left[Ax - \frac{1}{2}BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)'\right] \end{aligned}$$

$$\begin{aligned} -\frac{\partial J^o(x,t)}{\partial t} &= x'Qx - \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right)BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)' \\ &\quad + \left(\frac{\partial J^o(x,t)}{\partial x}\right)Ax \end{aligned}$$

$$J^o(x,T) = x'Sx = m(x)$$

Partial derivative eq.
... try to solve
by quadratic solution

$$-\frac{\partial J^o(x,t)}{\partial t} = x'Qx - \frac{1}{4}\left(\frac{\partial J^o(x,t)}{\partial x}\right) BR^{-1}B'\left(\frac{\partial J^o(x,t)}{\partial x}\right)' + \left(\frac{\partial J^o(x,t)}{\partial x}\right) Ax$$

Tentative solution

I can guess opt solution as a quadratic form!

QUADRATIC

$$J^o(x,t) = x'P(t)x \quad , \quad P(T) = S \longleftrightarrow J^o(x,T) = x'Sx = m(x)$$

{optimal function}

(equivalent equation)

$$-\frac{\partial J^o(x,t)}{\partial t} \qquad \qquad \frac{\partial J^o(x,t)}{\partial x}$$

$$\begin{aligned} -x'\dot{P}(t)x &= x'Qx - x'P(t)BR^{-1}B'P'(t)x + 2x'P(t)Ax \\ &= x'Qx - x'P(t)BR^{-1}B'P'(t)x + 2x'\underline{\boxed{\frac{P(t)A + A'P'(t)}{2}}}x \end{aligned}$$

here I would like to have a symm. term... I can obtain it by transform (P(t) A)

[I discover our guess is best solution if this equation is satisfied]

DIFFICULT to solve! can be considered only in simple cases

$$\boxed{\begin{gathered} \dot{P}(t) + Q - P(t)BR^{-1}B'P'(t) + P(t)A + A'P'(t) = 0 \\ P(T) = S \end{gathered}}$$

differential Riccati equation

(used for Kalmann Filter)

{NOT easy to solve!} typically solved numerically by software!

to solve you proceed backwards from T

Finite horizon optimal control law

time varing control law

$$u^o(t) = \underbrace{-R^{-1}B'P'(t)}_{K(t)}x(t) = -K(t)x(t)$$

similar to pole placement control law

for a continuous system control → {time variant control}

Comments

- $P(t) = P'(t)$ symm!
- $J^0(x,t) = x'P(t)x \geq 0$ optimal cost function @ t
- $K(t)$ time varying and defined over a finite interval [0,T] ⇝ outside that period? what to do?
- difficult to solve a matrix differential equation
- non easy to use in «standard» control problems (stabilization of time invariant systems)

Infinite horizon optimal control law

BASIC IDEA (before mathematics)

{ we wanna move from finite horizon to infinite horizon }

Idea

- $T \to \infty$ (overall time $[0, \infty]$ considered)
- properly weight all the state variables in the cost function ***J***

$\Downarrow$ you can obtain stabilizing solution

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, S = 0 \xrightarrow{\textit{the integrand is}} q_1 x_1^2 + q_2 x_2^2 + r_1 u_1^2 + r_2 u_2^2$$

Properly weight the terms on the cost function!

If ***q_1>0, q_2>0*** the only possibility is that asymptotically the state variables must tend to zero

$$x_1 \to 0, x_2 \to 0 \Rightarrow \text{asymptotic stability}$$

I ORD lin. syst $\dot{x} = ax + bu$

propurly weight state variable

$$J = \int_0^\infty (\dot{x}(\tau)\, q\, x(\tau) + \dot{u}(\tau) R\, u(\tau))\, d\tau$$

$\downarrow$

$q x^2(\tau)$, $q > 0$

$Q \geq 0$

$$\begin{vmatrix} x_1 & x_2 \end{vmatrix} \begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$\downarrow$

$$J = \int_0^\infty (x_1^2 q_1 + x_2^2 q_2)\, dt \qquad q_1, q_2 > 0$$

take ∞ horizom and propurly weight the states

IF $q_1 = 0$ no propurly weighted variable → issue

Infinite horizon LQ$_\infty$

Finite horizon LQ $\longrightarrow$ $T \to \infty, \quad S = 0$ $\longrightarrow$ $$J(x_0, u(\cdot), 0) = \int_0^\infty (x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau))\, d\tau$$

$$Q = Q' \geq 0, \quad R = R' > 0$$

important Results...

If the pair (A, B) is reachable

($\bar{P}$ s.m.d matrix)
solution of integration

Solve by integration the PDE
{ integrate backwards from ∞ to the value (T) }

A *the solution of the differential Riccati equation with initial condition $P(T) = 0$ tends, for $T \to \infty$, to a constant matrix $\bar{P} \geq 0$ solution of the algebraic Riccati equation*

$$0 = A'\bar{P} + \bar{P}A + Q - \bar{P}BR^{-1}B'\bar{P}$$

↳ simpler to solve numerically!

(TEXT of EXAM, not to Remember!)
algebraic Riccati equation
you consider only $\bar{P} \geq 0$ solution!

B *the asymptotic control law is*

{ control law defined $\forall t$ }

from diff Riccati equation

$$u(t) = -R^{-1}B'\bar{P}x(t) = -\bar{K}x(t)$$

(1st problem solved..)
time invariant control law
↳ STABILITY?!

where $\bar{K} = R^{-1}B'\bar{P}$

We don't guarantee this contr law is stabilizing!

simpler and time invariant (like pole placement), not yet stabilizing
we must guarantee to weight the states

to guarantee that C.L is stabilizing
we should guarantee
the weight...

from the time invariant control law... → to properly define the requirements

How to weight the states?

as product of C_q matrix

Partition the matrix **Q**: $Q = C_q' C_q$ (not unique, for instance ***Q=1*** **->** ***C***$_q$***=1*** or C_q***=-1***)

Define the *fictitious output* $\tilde{y}(\tau) = C_q x(\tau)$ and write the cost function as

$$
\begin{aligned}
J(x_0, u(\cdot), 0) &= \int_0^\infty \left(x'(\tau)C_q'C_q x(\tau) + u'(\tau)Ru(\tau)\right) d\tau \\
&= \int_0^\infty \left(\tilde{y}'(\tau)\tilde{y}(\tau) + u'(\tau)Ru(\tau)\right) d\tau
\end{aligned}
$$

$$\Downarrow$$

Weighting the states -> guaranteeing that the state is «visible» from the fictitious output $\boldsymbol{y_{tilde}}$ that is that the state is ***observable*** from output $\boldsymbol{y_{tilde}}$

It is important to weight all the states ← { ask that states observable from fictitious output }

IF I want to weight a part of the states, NOT All

$\downarrow$

$$J = \int_0^{\infty} \left(x^T(\tau)\, Q\, x(\tau) + \dots \right) d\tau$$

$\Downarrow$

$$\underbrace{x^T(\tau)\, C_q^T}_{\tilde{y}^T} \overbrace{}^{Q} \underbrace{C_q\, x(\tau)}_{\tilde{y}} \leftarrow (A, C_q)$$

Two intermediate results (readability necessary to lead to spd solution on DRE)

$\Downarrow$

- *Result 1*

Let C_{q1} and C_{q2} be two partitions of Q, i.e.

$$Q = C'_{q1}C_{q1} = C'_{q2}C_{q2}$$

partitions of Q

Then, if (A, C_{q1}) is observable, also (A, C_{q2}) is observable

- *Result 2*

The asymptotic solution $\bar{P}$ of the Riccati equation is positive definite if and only if (A, C_q) is observable

Some comments how to choose Q

If Q p.d means that
A, Cq observable for sure !

Choosing **Q > 0** automatically solves the problem. In fact, ***Q>0 → C$_q$*** square and full rank

sometimes better to weight the output, properly select Q

If the system has an output $y(t) = Cx(t)$ and the pair **(*A,C*) is observable**, choosing ***Q=C'C*** satisfies the condition. In addition, it can be reasonable to consider the cost function

$$
\begin{aligned}
J(x_0, u(\cdot), 0) &= \int_0^\infty \left(x'(\tau)C'Cx(\tau) + u'(\tau)Ru(\tau)\right) d\tau \\
&= \int_0^\infty \left(y'(\tau)y(\tau) + u'(\tau)Ru(\tau)\right) d\tau
\end{aligned}
$$

Remember that the state depends on the state coordinate, that can change, and it is not always possible to interpret its meaning, while the output is uniquely and clearly definedd

Now I can take the integral as...

from a design parameter $\rightarrow$ good way to select the weighting matrix of the states

(good to take it ">0")

Main result

If

the pair (A, B) is reachable
the pair (A, C_q) is observable

then

A the optimal control law is given by

$$\boxed{u(t) = -\bar{K}x(t)}$$

with

$$\bar{K} = R^{-1}B'\bar{P}$$

same structure of pole placement but LQ control has more DOF

where $\bar{P}$ is the unique positive definite solution of the stationary Riccati equation

$$\left\{ 0 = \bar{P}A + A'\bar{P} + Q - \bar{P}BR^{-1}B'\bar{P} \right\}$$

→ the solution of this equation is obtained by "ARE" function

B the closed-loop system → $\boxed{\dot{x}(t) = (A - B\bar{K})x(t)}$

is asymptotically stable.

time invariant, stabilizing control law

which solves that equation

Proof of stability (only) in closed loop

(Ask on exam) !

Consider $J^o(x) = x'\bar{P}x$ as a Lyapunov function $\rightsquigarrow$ $J^o(x) = x'\bar{P}x \geq 0$

optimal cost function

Lyap funct!

(by definition)

this satisfy Lyap theory ? $\frac{\partial V}{\partial t} > 0$?

$$
\begin{aligned}
\frac{\partial J^o(x)}{\partial t} &= \dot{x}'\bar{P}x + x'\bar{P}\dot{x} \\
&= x'(A - B\bar{K})'\bar{P}x + x'\bar{P}\boxed{(A - B\bar{K})x} \\
&= x'\left\{A'\bar{P} - \bar{K}'B'\bar{P} + \bar{P}A - \bar{P}B\bar{K}\right\}x \\
&= x'\left\{A'\bar{P} - \bar{P}BR^{-1}B'\bar{P} + \bar{P}A - \bar{P}BR^{-1}B'\bar{P}\right\}x \\
&= -x'\left\{Q + \bar{P}BR^{-1}B'\bar{P}\right\}x \\
&= -x'\left\{Q + \bar{K}'R\bar{K}\right\}x
\end{aligned}
$$

$\dot{x}$ = close loop form...

$\bar{K} = R^{-1}B'\bar{P}$ substitute

$\rightsquigarrow$ so we can conclude that closed loop syst is asymp. stable

If ***Q>0*** the proof is completed. If ***Q*** is only semidefinite positive … apply Krasowski La Salle

Proof of stability (continued)

(NOT ask on exam...)

When: $Q \geq 0$ $\longrightarrow$ assume by contradiction that $\frac{\partial J^o(\bar{x}(t))}{\partial t} = 0,$

$$\downarrow$$

$$-\bar{x}'(t)\left\{Q + \bar{P}BR^{-1}B'\bar{P}\right\}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$\bar{x}'(t)Q\bar{x}(t) = 0, \quad \forall t \quad \text{and} \quad \bar{x}'(t)\bar{P}BR^{-1}B'\bar{P}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$B'\bar{P}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$u(t) = -R^{-1}B'\bar{P}\bar{x}(t) = 0, \quad \forall t$$

$$\downarrow$$

$$\bar{x}(t) = e^{At}\bar{x}_0, \quad \bar{x}_0 \neq 0$$

$$\downarrow$$

$$\bar{x}'(t)Q\bar{x}(t) = \bar{x}_0'e^{A't}C_q'C_qe^{At}\bar{x}_0 = 0, \quad \forall t, \quad \bar{x}_0 \neq 0$$

It can be proven that this contradicts the observability of (A,C_q)

OBJECTIVE:

(control the system by LQ CONTROL)

Example (IORD system) unstable

system $\dot{x}(t) = x(t) + u(t)$ $\rightsquigarrow A = 1,\ B = 1$

B ≠ 0 guarantee Reachability in IORD syst.

$\boxed{\text{Riccati eq.}}$ $2\bar{P} + Q - \dfrac{\bar{P}^2}{R} = 0$ $\quad \bar{P} = \dfrac{2 \pm \sqrt{4\left(1 + \frac{Q}{R}\right)}}{2/R}$

solve this respect P

(for)

- $Q = 3,\ R = 1\ \left(\frac{Q}{R} = 3\right)$ $\quad \bar{P} = \dfrac{2 \pm \sqrt{16}}{2} = \begin{cases} 3 & \textit{solution} > 0 \\ -1 \end{cases}$

  take the sol p.d! (with (+))

  $\bar{K} = R^{-1} B' \bar{P} = 3$

  $A - B\bar{K} = 1 - 3 = -2$ ← close loop eig value... system stabilized as expected!

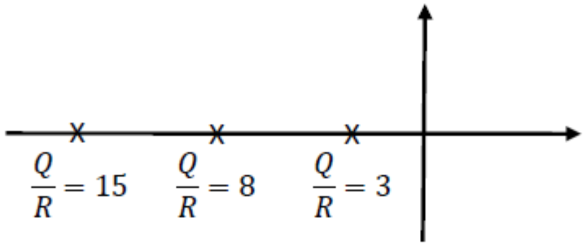
- $Q = 8,\ R = 1\ \left(\frac{Q}{R} = 8\right)$ $\quad \bar{P} = \dfrac{2 \pm \sqrt{36}}{2} = \begin{cases} 4 & \textit{solution} > 0 \\ -2 \end{cases}$

  weight more state evolution than input

  ↳ FASTER CLOSED LOOP syst ⇒

  $\bar{K} = R^{-1} B' \bar{P} = 4$

  $A - B\bar{K} = 1 - 4 = -3$



- $Q = 15,\ R = 1\ \left(\frac{Q}{R} = 15\right)$ $\quad \bar{P} = \dfrac{2 \pm \sqrt{64}}{2} = \begin{cases} 5 & \textit{solution} > 0 \\ -3 \end{cases}$

  $\bar{K} = R^{-1} B' \bar{P} = 5$

  $A - B\bar{K} = 1 - 5 = -4$ ← eig value move to -4

with Q↑ FASTER system!

Example LQ_{inf}

check OBS + REACH...

system $\dot{x}(t) = ax(t) + b_1u_1(t) + b_2u_2(t)$ $\quad A = a,\ B = [b_1\ b_2]$

IF q>0 enough for OBS.

weights $Q = q, \quad R^{-1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}, \quad R = \begin{bmatrix} 1/\rho_1 & 0 \\ 0 & 1/\rho_2 \end{bmatrix}$

(as a scalar)

diag matrix easy to compute R^{-1}

algebraic Riccati equation $0 = 2\bar{p}a + q - \bar{p}^2\left(b_1^2\rho_1 + b_2^2\rho_2\right)$ $\Leftarrow$ {IF $\rho_2 >> \rho_1$ I want to use a lot value of u_1 less u_2! → this weight!}

$a = 0,\ b_1 = b_2 = 1 \qquad \bar{p} = \sqrt{\dfrac{q}{\rho_1 + \rho_2}}$ ← positive solution!

$$\bar{K} = R^{-1}B'\bar{p} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sqrt{\frac{q}{\rho_1 + \rho_2}} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}\sqrt{\frac{q}{\rho_1 + \rho_2}} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\rho_1^2 q}{\rho_1+\rho_2}} \\ \sqrt{\frac{\rho_2^2 q}{\rho_1+\rho_2}} \end{bmatrix}$$

The less you weight one control variable with respect to the other (ρ_1 greater than ρ_2 for instance), the larger is the corresponding control gain with respect to the other

IF $\rho_1 >> \rho_2$ $\bar{K}_1 >> \bar{K}_2$

so you want to use more u_1 than u_2 → reflected on R choice

Exercise LQ_∞

(I ORD System)

$\dot{x} = ax + u$, $A = a$, $B = 1$

Q=1

Riccati eq. $P^2 - 2aRP - R = 0 \rightarrow \boxed{P = aR + \sqrt{(aR)^2 + R}}$

$$u = -\frac{P}{R}x = -\left(a + \sqrt{a^2 + 1/R}\right)x$$

from the formula

$$\dot{x} = \left(a - a - \sqrt{a^2 + 1/R}\right)x = \boxed{-\sqrt{a^2 + 1/R}}\; x$$

α

$= K$

on FREQ. Domain

If $R \rightarrow \infty \rightarrow \boxed{\dot{x} = -|a|x}$

more weight on control var!

→ IF neg eig value in open loop remain on $-a$

If $R \to \infty$ $\longrightarrow$ $\dot{x} = -|a|\,x$

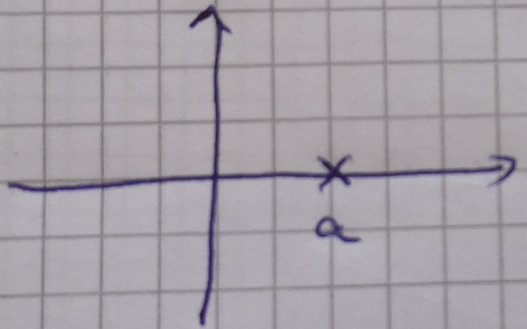
asymp stable in CL

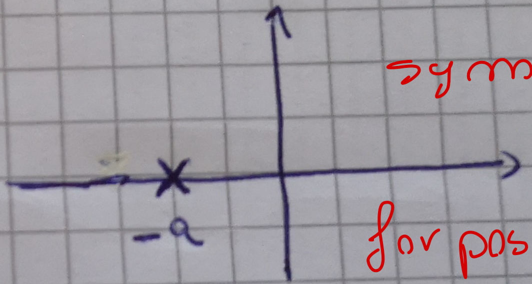
Open loop

Closed-loop

remain as it is.. like NOT closing the loop!

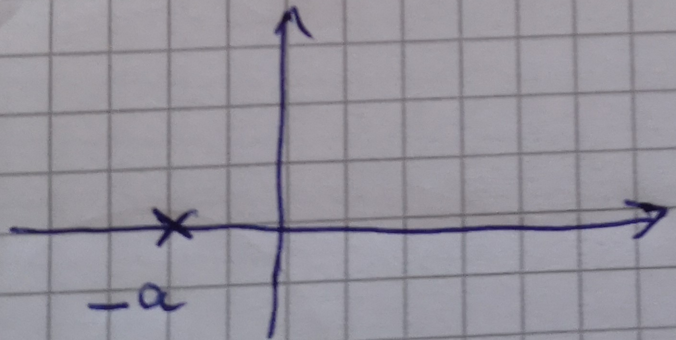
$a > 0$

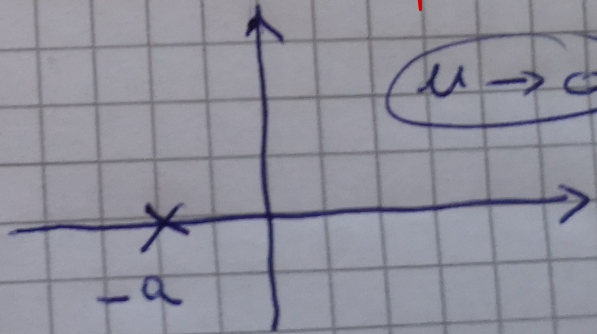




symm eig value

for pos eig value in open loop





$u \to 0$

$-a$ $-a$

"Cheap control"

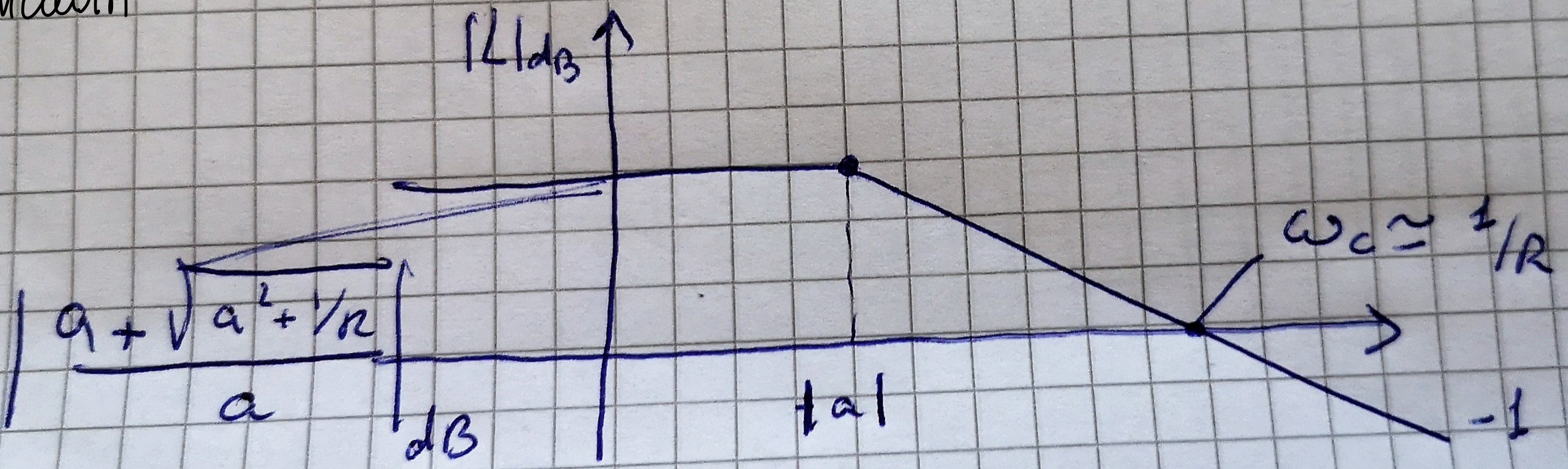
If $R \to 0$ $\longrightarrow$ $\boxed{\dot{x} = -\alpha x}$ $(\alpha \to \infty)$

when $R \to 0$

$\uparrow \dot{x} \to \infty$

Interpretation on freq. domain, by its transfer function

$$L(s) = G(s) \cdot K = \frac{1}{s-a}\left(a + \sqrt{a^2 + 1/R}\right)$$



The really difficul task (as always) is to tune the design parameters Q an R

HARD to choice!
done according to some criterion

Choice of the design parameters - normalization

positive s.d matrix

$$Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & q_n \end{bmatrix}, \quad q_i \geq 0, \quad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_m \end{bmatrix}, \quad r_i > 0$$

but state and control variables can differ of orders of magnitude

properly chosen q_i, r_i?

$$J = \int_0^\infty \left(q_1 x_1^2(\tau) + \ldots + q_n x_n^2(\tau) + r_1 u_1^2(\tau) + \ldots + r_m u_m^2(\tau) \right) d\tau$$

$$\downarrow$$

$$\text{If } |x_i| < x_{\max_i},\ i = 1, \ldots, n,\ |u_i| < u_{\max_i},\ i = 1, \ldots, m$$

$$\downarrow$$

when using LQ control you can use normalized value properly chosen

$$q_i = \frac{\tilde{q}_i}{x_{\max_i}^2}, i = 1, \ldots, n, \quad r_i = \frac{\tilde{r}_i}{u_{\max_i}^2}, i = 1, \ldots, m, \quad \tilde{q}_i \geq 0, \quad \tilde{r}_i > 0$$

$$\downarrow$$

$$J = \int_0^\infty \left(\tilde{q}_1 \frac{x_1^2(\tau)}{x_{\max_1}^2} + \ldots + \tilde{q}_n \frac{x_n^2(\tau)}{x_{\max_n}^2} + \tilde{r}_1 \frac{u_1^2(\tau)}{u_{\max_1}^2} + \ldots + \tilde{r}_m \frac{u_m^2(\tau)}{u_{\max_m}^2} \right) d\tau$$

now they weight quantities in (0,1)

easy tuning of $\tilde{q}_i$

Choice of the design parameters – weights on the outputs

II choice possibility considering to weight properly outputs, not states

System with outputs
$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$$

When weighting output you can take $Q = C^T \bar{Q} C$

Choice of the weights $\quad Q = C'\bar{Q}C, \quad \bar{Q} \geq 0, \quad \bar{Q} \in R^{p,p}$

$$J = \int_0^\infty \left(x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau)\right) d\tau = \int_0^\infty \left(x'(\tau)C'\bar{Q}Cx(\tau) + u'(\tau)Ru(\tau)\right) d\tau = \int_0^\infty \left(y'(\tau)\bar{Q}y(\tau) + u'(\tau)Ru(\tau)\right) d\tau$$

Usually, it is much simpler to weight the ouputs, always with a clear meaning, than the states, whose meaning depends on the state space representation

>> "LQR" MATLAB function solve that problem for you ! define A, B, Q, R, (Γ) and solve by SOFTWARE ⇝ obtaining

Choice of the design parameters – prescribed rate of convergence

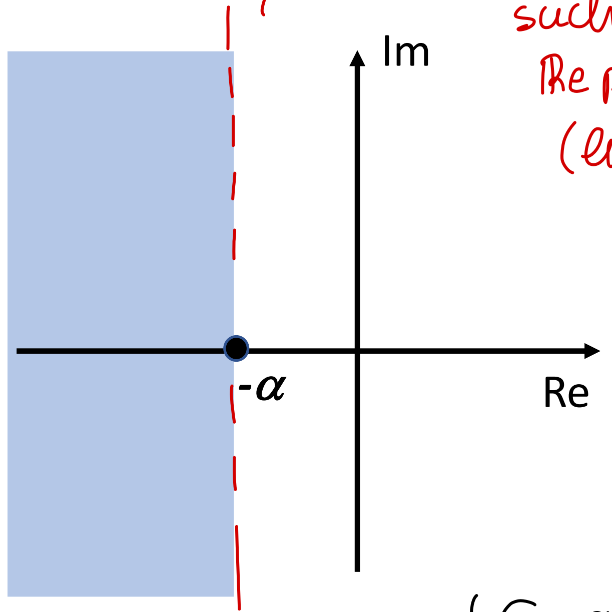
some additional considerations...

basic algorithms characteristics..

how to modify the alg such that Neg Re part < $-\alpha$! (eint neg.)

Can we use LQ_{inf} so that the closed-loop poles have negative real part smaller than $-\alpha$? (α>0)

α: design parameter to project the control



Guarantee eig value on left hand side of $-\alpha$ vertical line

Idea

Use different performance index! ⟹

this tends to infinity as an exponential

these must tend to zero faster

$$J(x_0,u(\cdot),0)=\int_0^{\infty} e^{2\alpha\tau}\left(x'(\tau)Qx(\tau)+u'(\tau)Ru(\tau)\right)d\tau$$

to guarantee finite cost function...

this goes to 0 FASTER!

$$J(x_0, u(\cdot), 0) = \int_0^{\infty} e^{2\alpha\tau} \left(x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau)\right) d\tau$$

define
$$\begin{cases} \tilde{x}(t) = e^{\alpha t}x(t) \\ \tilde{u}(t) = e^{\alpha t}u(t) \end{cases}$$

$\tilde{A} = \alpha I + A$

$$\begin{aligned}\dot{\tilde{x}}(t) &= \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t) = \alpha\tilde{x}(t) + e^{\alpha t}\left(Ax(t) + Bu(t)\right) \\ &= \alpha\tilde{x}(t) + A\tilde{x}(t) + B\tilde{u}(t) = (\alpha I + A)\,\tilde{x}(t) + B\tilde{u}(t) \\ &= \underbrace{\tilde{A}\tilde{x}(t) + B\tilde{u}(t)}_{\text{different system}}\end{aligned}$$

taking the defined cost function, written as...

$$J(x_0, u(\cdot), 0) = \int_0^{\infty} \left(\tilde{x}'(\tau)Q\tilde{x}(\tau) + \tilde{u}'(\tau)R\tilde{u}(\tau)\right) d\tau$$

new cost function

this is a standard LQ$_{inf}$ control problem in the tilda variables

↓ LQ CONTROL...

$$\tilde{u}(t) = -R^{-1}B'\bar{P}_\alpha\tilde{x}(t) = -\bar{K}_\alpha\tilde{x}(t) \quad \Longleftarrow \quad 0 = \bar{P}_\alpha\tilde{A} + \tilde{A}'\bar{P}_\alpha - \bar{P}_\alpha BR^{-1}B'\bar{P}_\alpha + Q$$

(SOLUTION)

Solution of ARE

closed-loop eigenvalues

$$\tilde{A} - B\bar{K}_\alpha = \underbrace{A + \alpha I}_{\tilde{A}} - B\bar{K}_\alpha$$

with negative real part

must be...

eigenvalues of

$$A - B\bar{K}_\alpha$$

with real part smaller than **-α**

Summary of the method method simple modification...

1. Build the matrix $\tilde{A} = A + \alpha I$.

2. Compute the matrix $\bar{K}_\alpha$, solution of the LQ problem, for the system described by the matrices $\tilde{A}$ and B.

3. Implement the control law

$$u(t) = -\bar{K}_\alpha x(t)$$

obtaining eig values
with Re part < -α as desired !

additional
design constraints

Example

like mooving Im axis on the left!

pos Re eig val UNSTABLE

Redefine to have

$$\dot{x}(t) = x(t) + u(t) \qquad A = B = 1 \qquad \tilde{A} = A + \alpha = 1 + \alpha$$

eig with Re < −α

solve with $\tilde{A}$

$$0 = 2\bar{P}\tilde{A} + Q - \bar{P}^2 R^{-1} \qquad \frac{\bar{P}^2}{R} - 2(1+\alpha)\bar{P} - Q = 0 \qquad \bar{P} = \frac{(1+\alpha) + \sqrt{(1+\alpha)^2 + \frac{Q}{R}}}{1/R}$$

standard formula to compute the gain...

$$\bar{K}_\alpha = R^{-1} B \bar{P} = (1+\alpha) + \sqrt{(1+\alpha)^2 + \frac{Q}{R}}$$

in closed loop

$$A - B\bar{K}_\alpha = \underbrace{-\alpha - \sqrt{\underbrace{(1+\alpha)^2 + \frac{Q}{R}}_{>0}}}_{\leq \alpha\,!} \leq -\alpha$$

as desired

12/7/2018

Consider the system

II ORD system

$$\begin{cases} \dot{x}_1(t) = bu(t) \\ \dot{x}_2(t) = x_1(t) + u(t) \end{cases}$$

and assume that you want to design an infinite horizon LQ control with *Q=diag(q₁,q₂), R=1*.

A. Compute the conditions guaranteeing that the solution of the infinite horizon LQ control is stabilizing.
B. With *Q=I, b=1*, the solution of the steady-state Riccati equation is *P=I*. Check the stability of the closed-loop system a) computing the closed-loop eigenvalues, b) by using a suitable Lyapunov function (the one used for the stability analysis of LQ control).
C. Assume now to implement the feedback control law $u(t) = -\rho K x(t)$ (K is again the solution of the LQ problem), specify the set of values of ρ guaranteed by LQ control so that the closed-loop system remains asymptotically stable.

Steady-state Riccati equation: $A'P + PA + Q - PBR^{-1}B'P = 0$

Solution

A.

$$A=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ B=\begin{bmatrix}b\\ 1\end{bmatrix}\rightarrow \text{reachability matrix } M_r=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}b & 0\\ 1 & b\end{bmatrix}\rightarrow \text{condition } b\neq 0$$

$$Q=\begin{bmatrix}q_1 & 0\\ 0 & q_2\end{bmatrix}\rightarrow Q^{1/2}=\begin{bmatrix}\sqrt{q_1} & 0\\ 0 & \sqrt{q_2}\end{bmatrix}\rightarrow \text{observability matrix } M_o=\begin{bmatrix}Q^{1/2}\\ Q^{1/2}A\end{bmatrix}=\begin{bmatrix}\sqrt{q_1} & 0\\ 0 & \sqrt{q_2}\\ 0 & 0\\ \sqrt{q_2} & 0\end{bmatrix}\rightarrow q_2>0$$

(minimal condition)

B.

stable, good result

$$K=R^{-1}B'P=\begin{bmatrix}1 & 1\end{bmatrix}\rightarrow A-BK=\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix}\rightarrow \text{eigenvalues } -1,-1$$

$$J^o=x'Px=x'x>0\rightarrow \frac{\partial J^o}{\partial t}=x'(A-BK)'x+x'(A-BK)x=-x'\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}x<0$$

Lyap function!

proof of stability

satisfied condition!

C.

$A = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$A - \rho BK = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \rho \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{BK} = \begin{vmatrix} -\rho & -\rho \\ 1-\rho & -\rho \end{vmatrix}$

for $\rho = 1$ as seen, eig $= -1, -1$

$\det\left(sI - (A - \rho BK)\right) = \begin{vmatrix} s+\rho & \rho \\ \rho-1 & s+\rho \end{vmatrix} = s^2 + 2\rho s + \rho$

$\begin{cases} \rho > 0 \\ 2\rho > 0 \end{cases} \quad \rho > 0$

conditions such that neg real part eig value

all coefficents are > 0 for cartesio, to have all eig < 0

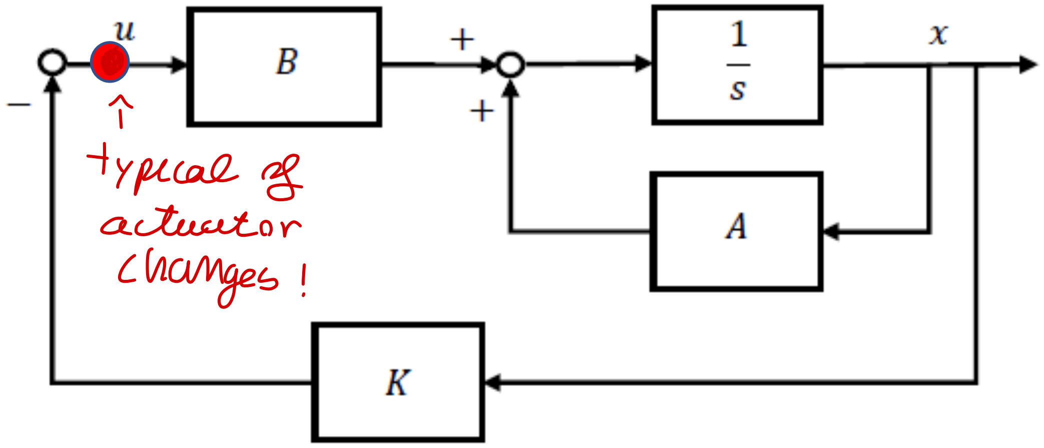
$\forall \rho > 0 \rightsquigarrow$ perturbed system is asymp stable!

LQ control has robustness properties.. guarantee asymp stable also for gain / phase variability

Feedback system

Robustness of LQ_{inf} control with respect to uncertainties *at the plant input*

under proper condition LQ stabilize a lot, can be modified for all request → and has ROBUSTNESS PROPERTIES



*It is **stabilizing**, it is **optimal** (with respect to the selected design parameters), but is it also **robust** with respect to uncertainties at the plant input?*

With simple manipulations of the Riccati equation, it is possible to obtain the following relationship

I can accept gain/phase variation

$$G_c'(-s)QG_c(s) + R = \Gamma'(-s)R\Gamma(s)$$

RELATED!

$$G_c(s) = (sI - A)^{-1}B$$ transfer function from ***u*** to ***x***

$$\Gamma(s) = I + \underbrace{K(sI - A)^{-1}B}_{\text{Loop transfer function } L(s)}$$ Inverse of the *sensitivity function* (*return difference*)

Loop transfer function ***L(s)***

from that relationship

$$G_c'(-s)QG_c(s) + R = \underbrace{\Gamma'(-s)}_{=\Gamma^*}R\Gamma(s) \longrightarrow \boxed{\Gamma^*(j\omega)R\Gamma(j\omega) \geq R}$$

for sure!

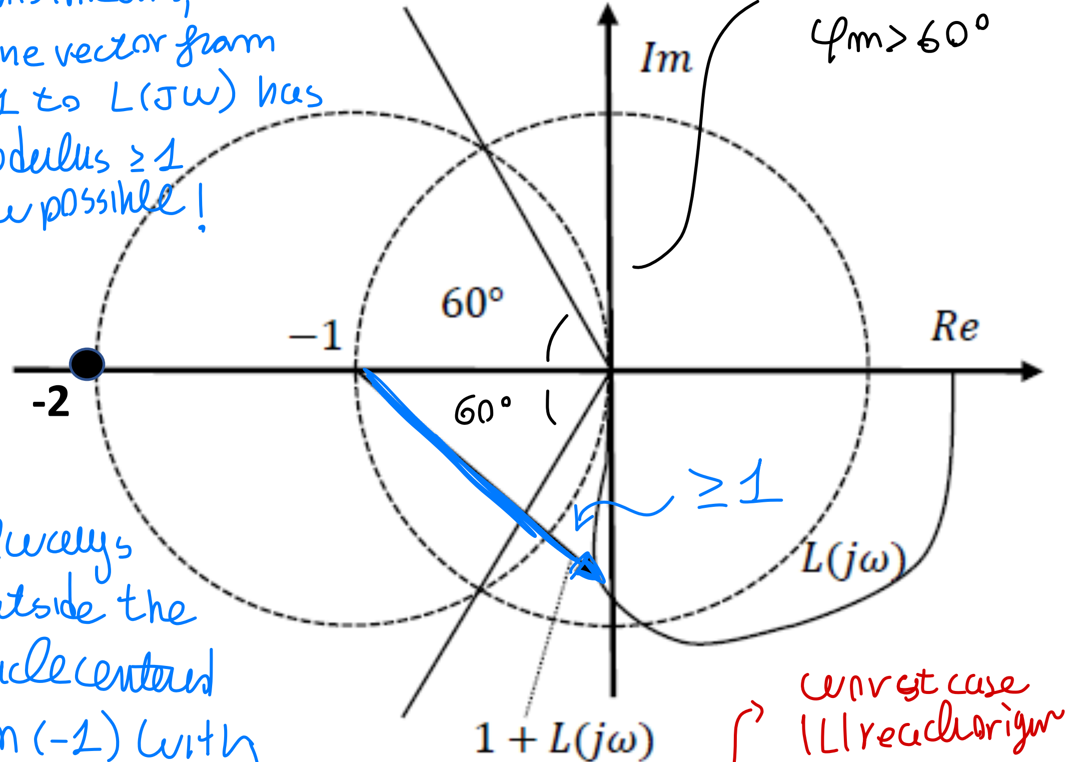
complex conjugate

Single input systems, m=1 (I ORD syst)

$$\overbrace{\boxed{(1+L(j\omega))^*}}^{\Gamma^*}R(1+L(j\omega)) \geq R \longrightarrow |1+L(j\omega)|^2 \geq 1$$

always at with $\varphi_m > 60°$

this means, the vector from -1 to $L(j\omega)$ has modulus ≥ 1 $\forall\omega$ possible!



always outside the circle centered in (−1) with radius 1 → OK gain variations ∞!

worst case |L| reach origin if ρ↑↑

Feedback system robust with respect to:

still stability!!

- phase variations of ±60° (***phase margin ±60°***)
- gain variations (0.5,∞) (***gain margin (0.5,∞)***)

good.

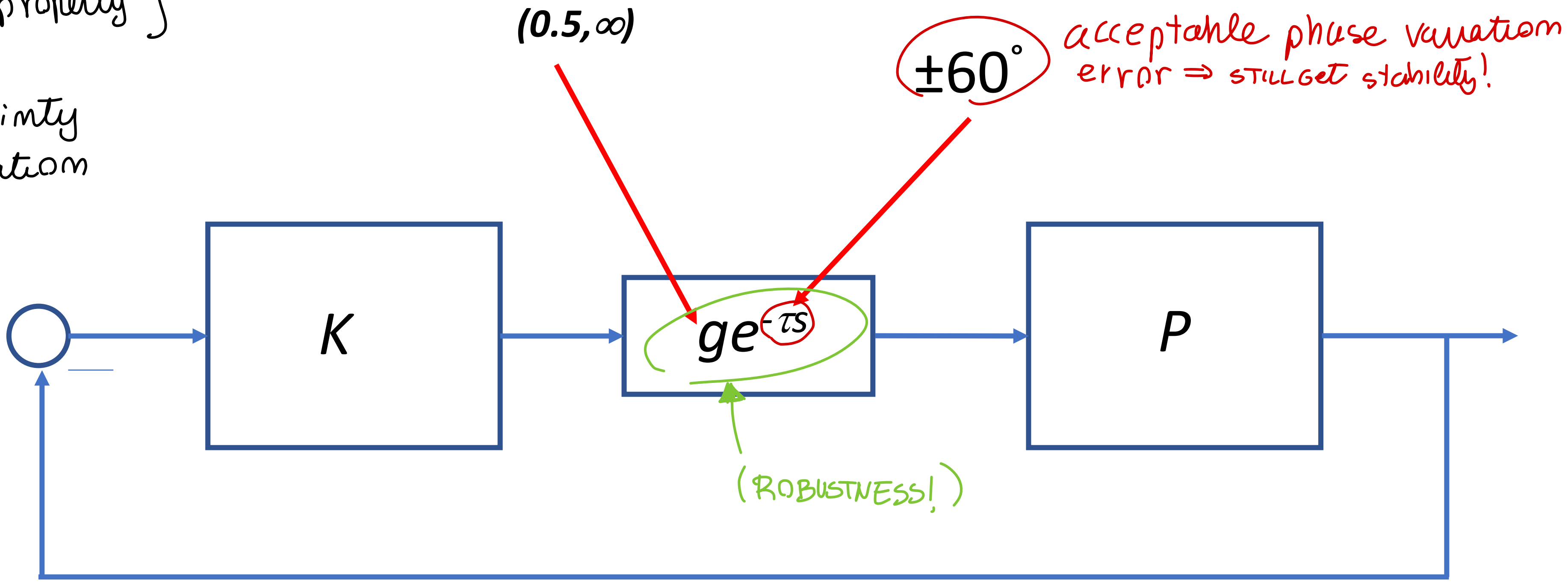
↓

but not guaranteed at the same time

so LQ control guarantees... $\Rightarrow$

property of SISO
when lin quadratic control
↓
we can check
ROBUSTNESS property
↓
gain uncertainty
and phase variation
↓
you can
maintain
stability
even for
uncertainty!

Guarantee stability also if you have
$\rho \in (0.5, \infty)$
and a phase error (time shift)



… however, it can be proven that

for SISO system

sensitivity function

$$\begin{aligned} T(j\omega) &= 1 - S(j\omega) \\ &= 1 - \left[1 + K(j\omega I - A)^{-1}B\right]^{-1} \\ &= K(j\omega I - A + BK)^{-1}B \end{aligned}$$

$$\lim_{\omega \to \infty} j\omega T(j\omega) = KB = -R^{-1}B'PB$$

constant value → @ high freq (-1) slope of Bode diagram

high sensitive to noise! BAD

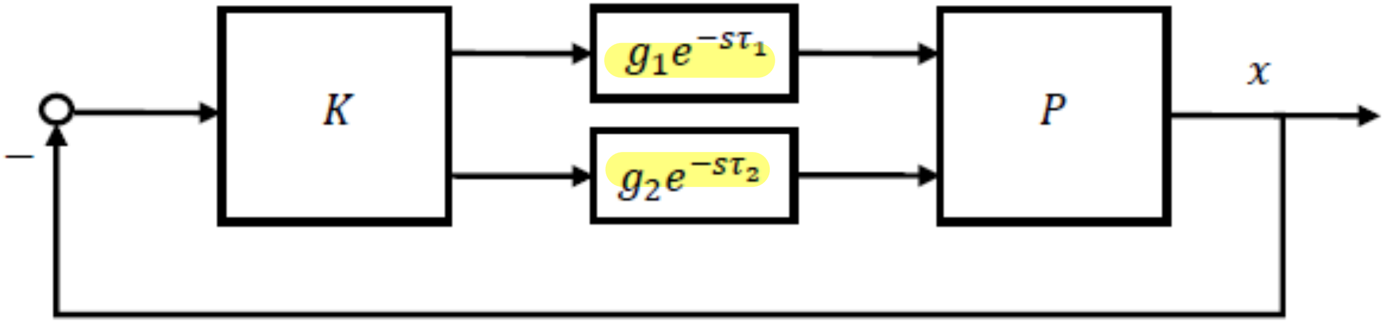
$|T(j\omega)|$ decreases at high frequency with slope -1

small attenuation of measurement noise and/or unmodelled dynamics at high frequency we can extend for MIMO systems this property!

Multi Input systems *m>1*

choosing $R = diag\{r_1, r_2, \ldots, r_m\}, \; r_i > 0,$

The closed-loop system



remains stable in front of

1. phase variations of magnitude up to 60^0;
2. gain variations $(1/2, \infty)$.

but not at the same time on the same channel

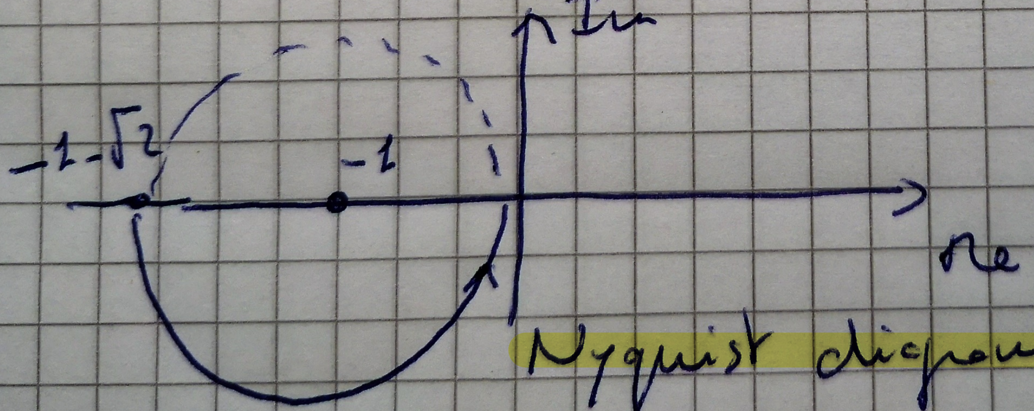
Example robustness LQ

$$\dot{x} = x + u \longrightarrow G(s) = \frac{1}{s-1}$$

UNSTABLE pole in $s=+1$

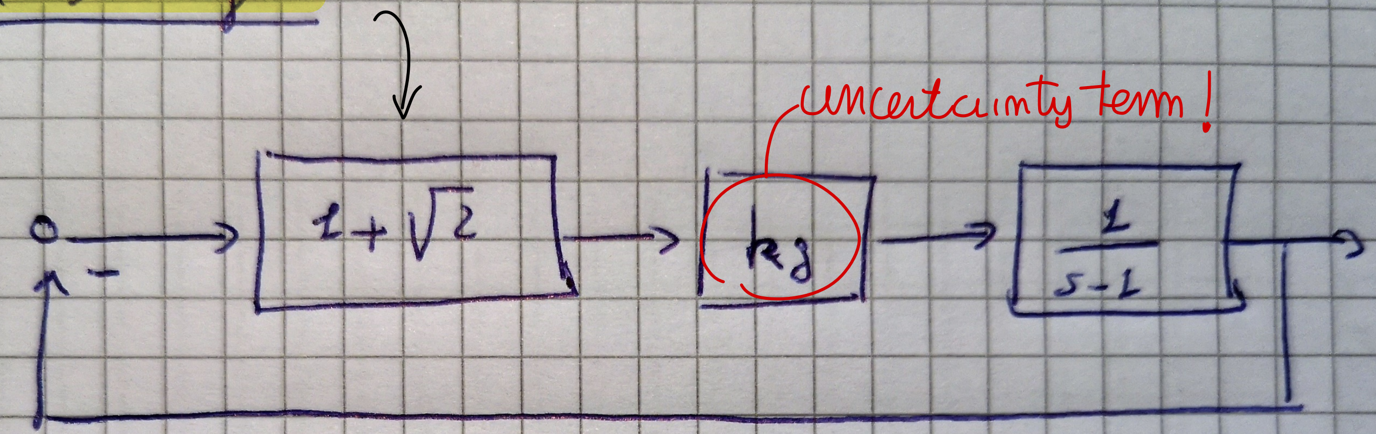
$$LQ_\infty \text{ with } R = Q = 1 \longrightarrow K = 1+\sqrt{2}$$

$$L(s) = K(sI-A)^{-1}B = \frac{1+\sqrt{2}}{s-1}$$



Nyquist diagram $\longrightarrow$ $P = N = 1$

Gain margin



⇓ to check allow K_g such Asymp.st

Characteristic equation

$$s - 1 + k_g\left(1+\sqrt{2}\right) = 0$$

$$s = 1 - k_g\left(1+\sqrt{2}\right) < 0$$

$$k_g > \frac{1}{1+\sqrt{2}} = 0.41$$

Gain margin nominal $(0.5, \infty)$

(here more ROBUST than the nominal $(0.5,\infty)$)

$K_g > 0.41$ to be asymp. stable...