Solutions

ADVANCED AND MULTIVARIABLE CONTROL

9/6/2023

Surname and name ....

Signature ....

Exercise 1 (3 marks)

Consider a continuous-time linear system (**select the correct answer**):

$$\dot{x}(t) = Ax(t) + Bu(t) + Md$$
$$y(t) = Cx(t) + Nd$$

where *d* is a constant, but unknown disturbance and the number of states is *n*.

With reference to this system, it is possible to estimate the value of *d* if and only if

- o The number *r* of disturbances is greater than the number *p* of outputs,
- ✗ The pair (*A*,*C*) is observable and $rank \begin{bmatrix} A & M \\ C & N \end{bmatrix} = n + r,$
- o The pair (*A*,*C*) is observable and $rank \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + m,$
- o There are no invariant zeros from *u* to *y* at the origin,
- o no answer

Exercise 2 (3 marks)

The separation principle states that **(select the correct answer)**:

- o for any linear or nonlinear dynamic system it is possible to independently design a state feedback control law and an observer such that the resulting closed-loop system has the eigenvalues of the system with state feedback and the eigenvalues of the observer.
- X for any linear dynamic system it is possible to independently design a state feedback control law and an observer such that the resulting closed-loop system has the eigenvalues of the system with state feedback and the eigenvalues of the observer.
- o Only for linear discrete time dynamic systems it is possible to independently design a state feedback control law and an observer such that the resulting closed-loop system has the eigenvalues of the system with state feedback and the eigenvalues of the observer.
- o for any linear dynamic system it is possible to independently design a state feedback control law and an observer such that the resulting closed-loop system has the eigenvalues of the system with state feedback and the eigenvalues of the observer provided that the two blocks have been designed with pole-placement.
- o no answer

Exercise 3 (3 marks)

LQ control for discrete time systems guarantees **(select the correct answer)**:

- o both gain and phase margins smaller than for continuous time systems
- o nothing about gain and/or phase margins
- o gain margin which can be improved with Loop Transfer Recovery procedures
- X gain margin smaller than in continuous time
- o no answer

Exercise 4 (3 marks)

In model reduction techniques (**select the correct answer**)

- o it is possible to select the order of the reduced order model by looking at an upper bound of the H2 norm of the difference between the original and the approximate transfer functions
- X it is possible to select the order of the reduced order model by looking at an upper bound of the Hinf norm of the difference between the original and the approximate transfer functions
- o It is not possible to guarantee that the static gains of the original and the approximate transfer functions are the same
- o It is not required to transform the system in balanced realization form
- o No answer

Exercise 5 (3 marks)

In MPC for linear systems, it is possible to guarantee asymptotic zero error between a constant set point y^o and the output y provided that (**select the correct answer**)

- o In the cost function the difference (y^o-y) is suitably weighted,
- o The resulting closed-loop system is asymptotically stable
- X An integrator is added to the system by reformulating it in velocity form
- o The prediction horizon is sufficiently long
- o None of the previous answers
- o No answer

Exercise 6 (6 marks)

Consider the discrete time system

$$x(k+1) = \frac{-ax(k)}{(1+x^2)} \quad , \quad a > 0$$

a) Compute the equilibrium point

$$\bar{x} = \frac{-a\bar{x}}{1+\bar{x}^2} \longrightarrow \bar{x}\left(1+\bar{x}^2+a\right)=0$$

$\bar{x}=0$ is the only one equilibrium point

b) Considering the linearized model at the equilibrium, compute the condition on *a* such that the equilibrium is asymptotically stable

Linearized system $\quad \delta x(k+1) = -a\,\delta x(k)$

Stability condition $\quad |a|<1 \longrightarrow 0<a<1$

(considering that $a>0$)

c) The solution of the Lyapunov equation *A'PA-P+Q=0, Q=1*, of the linearized system is

$$P = \frac{1}{1-a^2}$$

With $a \in (0,1)$, $P>0$ so that $V(x) = x'Px$ is a candidate Lyap.

With this solution, define a proper Lyapunov function and check the stability of the equilibrium of the original nonlinear system.

$V(x) = Px^2$

$$\Delta V(x) = \underset{>0}{P}\underbrace{\left[\frac{a^2}{(1+x^2)^2} - 1\right]}_{<0}\underset{>0}{x^2} < 0$$

for $0<a<1$

equilibrium asymptotically stable

Exercise 7 (7 marks)

Consider the first order system $\dot{x}(t) = x(t) + u(t) + v_x(t)$
$y(t) = x(t) + v_y(t)$

Where $v_x = WN(0,1)$, $v_y = WN(0,1)$

1. Enlarge the system with an integrator acting on the error variable.

$$\begin{vmatrix} \dot{x} \\ \dot{v} \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}}_{\bar{A}} \begin{vmatrix} x \\ v \end{vmatrix} + \underbrace{\begin{vmatrix} 1 \\ 0 \end{vmatrix}}_{\bar{B}} u + \begin{vmatrix} 1 \\ 0 \end{vmatrix} v_x$$

$$y = |1 \quad 0| \begin{vmatrix} x \\ v \end{vmatrix}$$

2. Given Q=I, R=1, for the enlarged system, compute the infinite horizon LQ control gain and the corresponding closed-loop eigenvalues. Note that the steady-state Riccati equation and its solutions are

$$0 = A'\bar{P} + \bar{P}A + Q - \bar{P}BR^{-1}B'\bar{P} \quad , \quad \bar{P} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$K = R^{-1}\bar{B}'P = [3 \quad -1]$$

$$\bar{A} - \bar{B}K = \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix} \rightarrow \det(sI - (A - BK)) = (s+1)^2$$

3. Design a Kalman Filter to estimate the state x.

$$KF: \quad A = 1, C = 1, \tilde{Q} = 1, \tilde{R} = 1 \rightarrow \tilde{P} = 1 + \sqrt{2}, \; L = \tilde{P}$$

$$\dot{\hat{x}} = \hat{x} + u + L[y - \hat{x}] \rightarrow \dot{\hat{x}} = -\sqrt{2}\,\hat{x} + u + Ly \; (KF)$$

4. Specify the control law obtained with the solution of the two previous questions and the eigenvalues of the corresponding closed-loop system.

Control Law
$$\begin{cases} \dot{\hat{x}} = -\sqrt{2}\,\hat{x} + u + L(y) & KF \\ u = -K \begin{vmatrix} \hat{x} \\ v \end{vmatrix} \end{cases}$$

Eigenvalues $(\bar{A} - \bar{B}K) \rightarrow s = -1, \; s = -1$

$(A - LC) \rightarrow s = -\sqrt{2}$

Exercise 8 (5 marks)

Consider the system described by the transfer function

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{bmatrix}$$

Compute the zeros and the poles.

$$M_{12} = \frac{3(s-2)}{(s+1)^2(s+2)}$$

$$M_{13} = \frac{3(s-2)}{(s+1)^2(s+2)}$$

$$M_{23} = \frac{2(s-2)(-s^2-s+2)}{(s+1)^2(s+2)}$$

Poles $s=-1$ (double), $s=-2$

Zeros $s=2$

(Additional cancellation in $s=-2$ not included)