Homogenous coordinates: lines, angles and vanishing points

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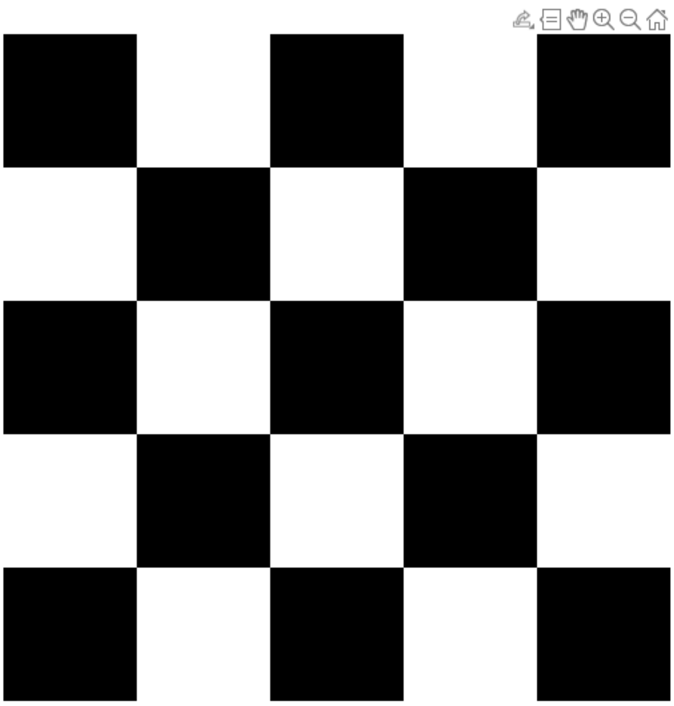
Goal: familiarize with homogeneous coordinates

```
clear
close all
clc
%font size
FNT_SZ = 28;
```

```
% load a checkerboard image
I = imread('E1_data/checkerboard.png');
```

manually select points

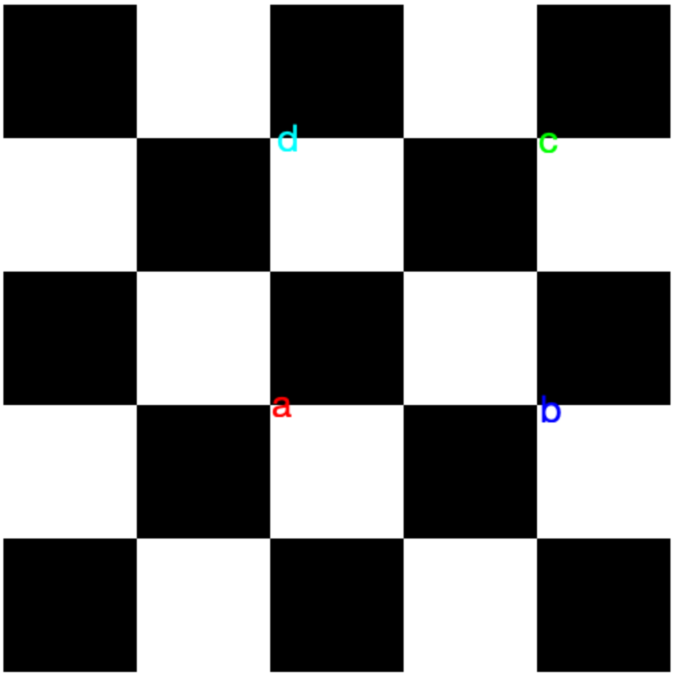
```
figure(1), imshow(I);
hold on;
[x, y] = getpts();
```



```
% save points in the homogeneous cooridnate. It is enough to set to 1 the third component
a = [x(1); y(1); 1];
b = [x(2); y(2); 1];
c = [x(3); y(3); 1];
d = [x(4); y(4); 1];
```

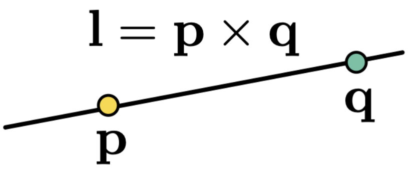
Draw the points over the image

```
text(a(1), a(2), 'a', 'FontSize', FNT_SZ, 'Color', 'r')
text(b(1), b(2), 'b', 'FontSize', FNT_SZ, 'Color', 'b')
text(c(1), c(2), 'c', 'FontSize', FNT_SZ, 'Color', 'g')
text(d(1), d(2), 'd', 'FontSize', FNT_SZ, 'Color', 'c')
```



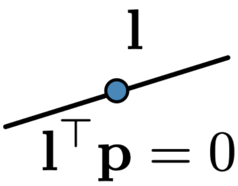
Compute the parameters of a few lines equations

lines are represented as 3D homogeneous vector collecting the coefficients of the line equation. Lines are points in the "dual plane".




```
lab = cross(a, b); % this is the reference
lad = cross(a, d); % orthogonal to lab
lac = cross(a, c); % 45 degrees with lac
lcd = cross(c, d); % parallel to lab
```

Check some incidence relations



they should be zero as $a \in \ell_{\mathrm{ab}}$ and $c \in \ell_{\mathrm{cd}}$

the incidence relation correspond to lcd(1)*c(1) + lcd(2)*c(2) + lcd(3)*c(3) = 0

```
a' * lab % this should be zero when a \in lab
```

```
ans = -7.2760e-12
```

```
c' * lcd %
```

```
ans = 3.6948e-12
```

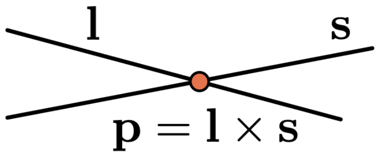
intersect the lines with the image borders

The general equation of a line is $\ell\ :\mathrm{ax}+\mathrm{by}+c=0$. The equation of the "first column" $\mathrm{c1}\,:x=1 \Rightarrow a=1, b=0, c=-1$ (remember that the first coordinate in matlab is the row, the second the column.

```
c1 = [1; 0; -1]; % parameters of left-most column
c500 = [1; 0; -500]; % parameters of right-most column

r1 = [0; 1; -1]; % parameters of top-most row
r500 = [0; 1; -500]; % parameters of bottom row
```

The intersection can be easily computed as the dot product between the representations of lines




```
% compute the intersection between lab and the first column
x1 = cross(c1, lab) % point in homogeneous coordinates
```

```
x1 = 3×1
10^4 ×
    0.0201
    5.9098
    0.0201
```

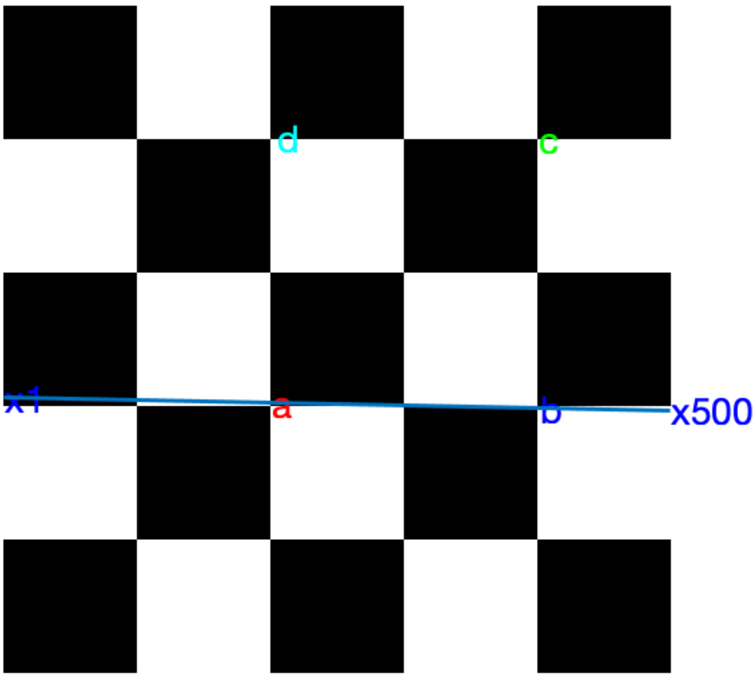
In order to plot x1 it is necessary to get to non homogenous (cartesian) coordinates.

Bear in mind that x_1 and λx_1 are equivalent in P^n for all $\lambda \neq 0$.

To draw the point in the image plane, we need to take the point corresponding to x1 having third component equal to 1.

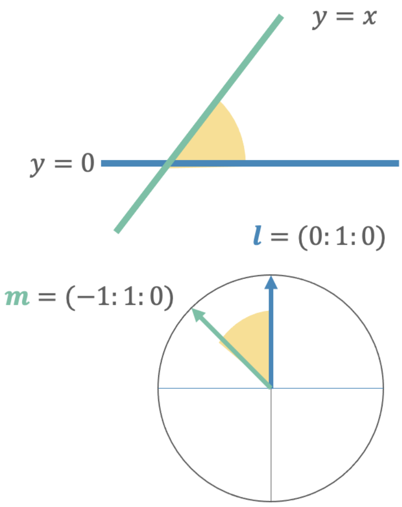
```
x1 = x1/x1(3);
text(x1(1), x1(2), 'x1', 'FontSize', FNT_SZ, 'Color', 'b')

% do the same with the right most column
x500 = cross(c500, lab);
x500 = x500 / x500(3);
text(x500(1), x500(2), 'x500', 'FontSize', FNT_SZ, 'Color', 'b')
plot([x1(1), x500(1)],[x1(2), x500(2)], 'LineWidth', 3)
```



Compute the angles on these images

that is given by the formula $\cos(\theta) = \dfrac{l_1 m_1 + l_2 m_2}{\sqrt{||l_{1:2}||^2 * ||m_{1:2}||^2}}$




```
l = lab;
m = lac
```

```
m = 3×1
10^4 ×
    0.0198
    0.0200
   -9.9398
```

```
cosTheta = (l(1) * m(1) + l(2) * m(2))/sqrt(sum(l(1:2).^2)*sum(m(1:2).^2));
theta = acosd(cosTheta)
```

```
theta = 45.8521
```

Verify the property that any linear combination of a,b belongs to lab

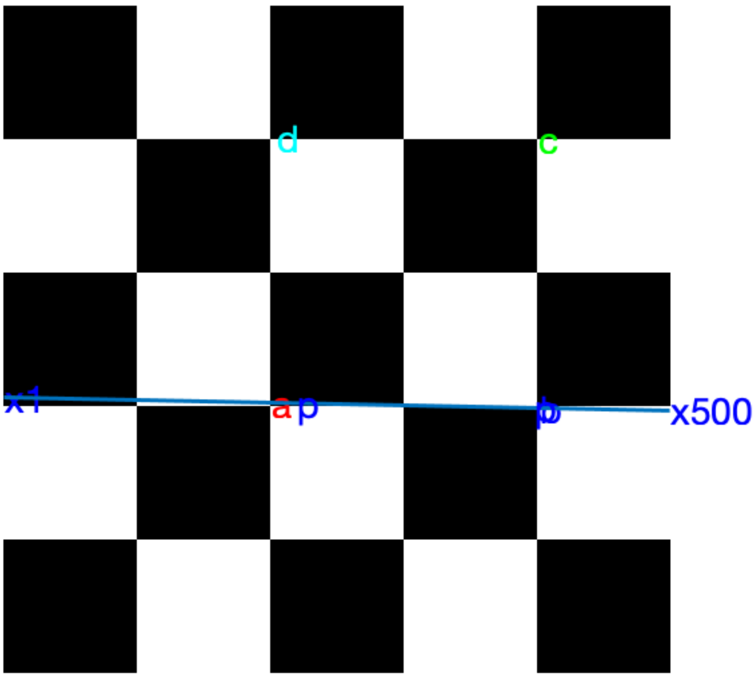
```
lambda = rand(1);
mu = 1- lambda; % this is a convex combination. So points will be in between the two. The result holds for any combination

p = lambda * a + mu * b;
p' * lab
```

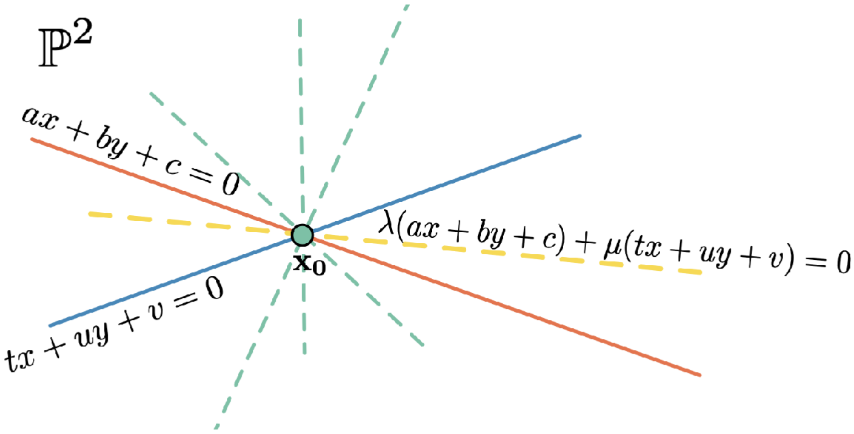
```
ans = -1.2847e-11
```

```

p = p/p(3);
text(p(1), p(2), 'p', 'FontSize', FNT_SZ, 'Color', 'b')
```



Dual property: any linear combination of two lines l1 and l2 passes through their intersection




```
lambda = rand(1);
mu = 1- lambda; % this is a convex combination. So points will be in between the two. The result holds for any combination

x0 = cross(lab, lad)
```

```
x0 = 3×1
10^7 ×
   -0.8043
   -1.1924
   -0.0040
```

```
% linear combination of lines
l = lambda * lab + mu * lad
```

```
l = 3×1
10^4 ×
    0.0123
    0.0078
   -4.7904
```

```
% check aInt belongs to l
x0'*l
```

```
ans = -3.5763e-07
```

What you get if you intersect parallel lines?

```
vac = cross(lab, lcd)
```

```
vac = 3×1
10^6 ×
   -7.7234
    0.0177
    0.0006
```

```
vac = vac / vac(3) % look at this point, isn't it strange? It is a point at the infinity!
```

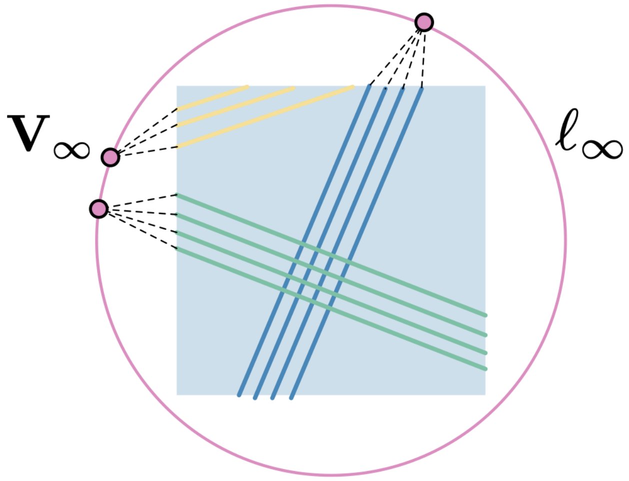
```
vac = 3×1
10^4 ×
   -1.3248
    0.0030
    0.0001
```

```
vab = cross(r1, r500)
```

```
vab = 3×1
  -499
     0
     0
```

```
vad = cross(c1, c500)
```

```
vad = 3×1
     0
   499
     0
```



$$\mathbb{P}^2 = \mathbb{E}^2 \cup \ell_\infty$$