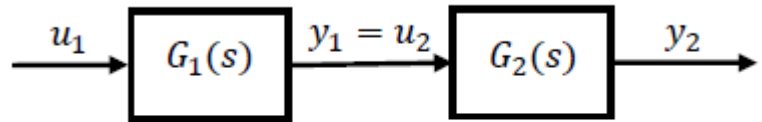# Advanced and Multivariable Control
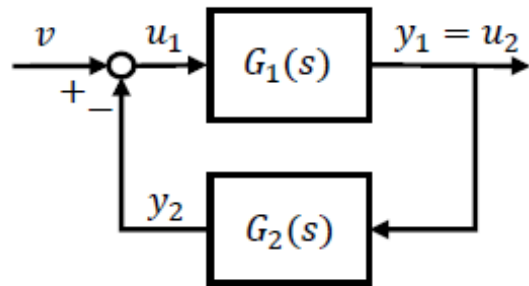
Multivariable systems analysis

Riccardo Scattolini

Block schemes

$$Y_2(s) = G_2(s)U_2(s) = G_2(s)G_1(s)U_1(s)$$

Figure 5.1: Systems in series configuration.

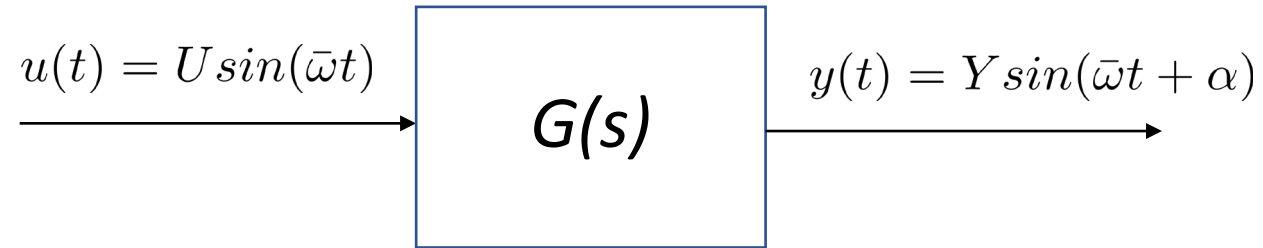$$Y_1(s) = (I_p + G_1(s)G_2(s))^{-1}G_1(s)V(s)$$

but also …

$$Y_1(s) = G_1(s)(I_m + G_2(s)G_1(s))^{-1}V(s)$$

Figure 5.2: Systems in feedback configuration.

Check the previous results!

Frequency response

$$u(t) = U sin(\bar{\omega} t) \qquad \boxed{G(s)} \qquad y(t) = Y sin(\bar{\omega} t + \alpha)$$

SISO

$$Y = U \, |G(j\bar{\omega})| \text{ and } \alpha = \arg(G(j\bar{\omega})) \qquad |G(j\bar{\omega})| = \frac{|Y(j\bar{\omega})|}{|U(j\bar{\omega})|}$$
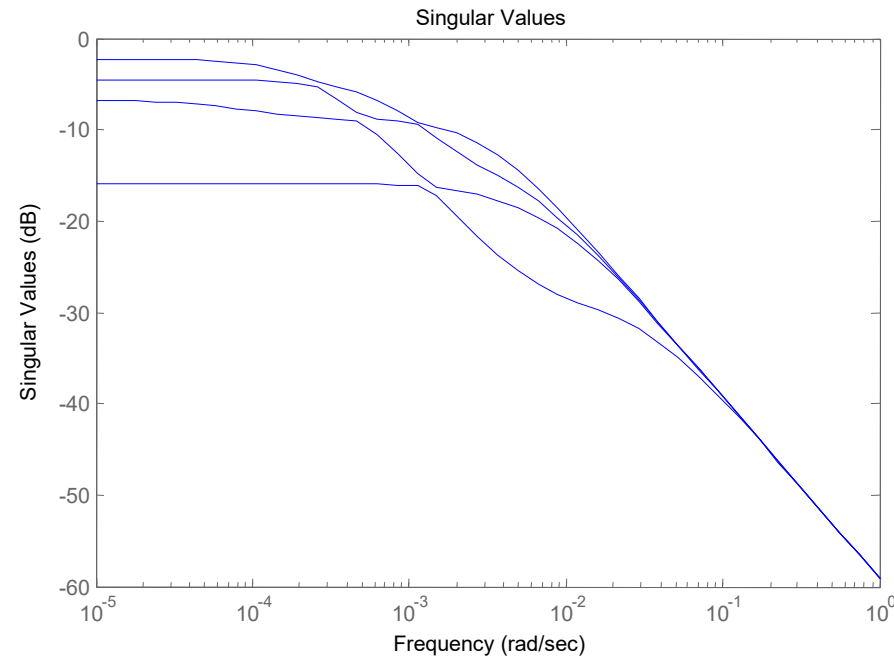
«gain»

MIMO

$$\frac{\|Y(j\omega)\|_2}{\|U(j\omega)\|_2} = \frac{\|G(j\omega)U(j\omega)\|_2}{\|U(j\omega)\|_2}$$

Gain and input directions in MIMO systems

$$\underline{\sigma}(G(j\omega)) \leq \frac{\|G(j\omega)U(j\omega)\|_2}{\|U(j\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

The singular values $\underline{\sigma}(G(j\omega))$ and $\bar{\sigma}(G(j\omega))$ of the system are called "principal gains"

4

The condition number is defined as

$$\gamma(G(j\omega)) = \frac{\bar{\sigma}(G(j\omega))}{\underline{\sigma}(G(j\omega))}$$

A system with a condition number close to 1 is "easy" to control, since it is not sensitive to the direction of the input

Example

Consider a static system with two inputs, two outputs, and described by the matrix

$$G = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the following inputs with norm equal to one

$$U_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, U_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, U_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, U_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, U_5 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$$

The corresponding outputs are

$$Y_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, Y_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, Y_3 = \begin{bmatrix} 2.12 \\ 4.95 \end{bmatrix}, Y_4 = \begin{bmatrix} -0.707 \\ -0.707 \end{bmatrix}, Y_5 = \begin{bmatrix} -0.4 \\ 0 \end{bmatrix}$$

with norms

$$\|Y_1\|_2 = 3.16, \|Y_2\|_2 = 4.47, \|Y_3\|_2 = 5.38, \|Y_4\|_2 = 1.00, \|Y_5\|_2 = 0.4$$

$$\bar{\sigma}(G) = 5.47; \quad \underline{\sigma}(G) = 0.37$$

Feedback systems

Loop transfer functions

$$L(s) = G(s)R(s), \quad L_u(s) = R(s)G(s)$$

Sensitivity functions

$$S(s) = (I + L(s))^{-1} \qquad\qquad sensitivity$$
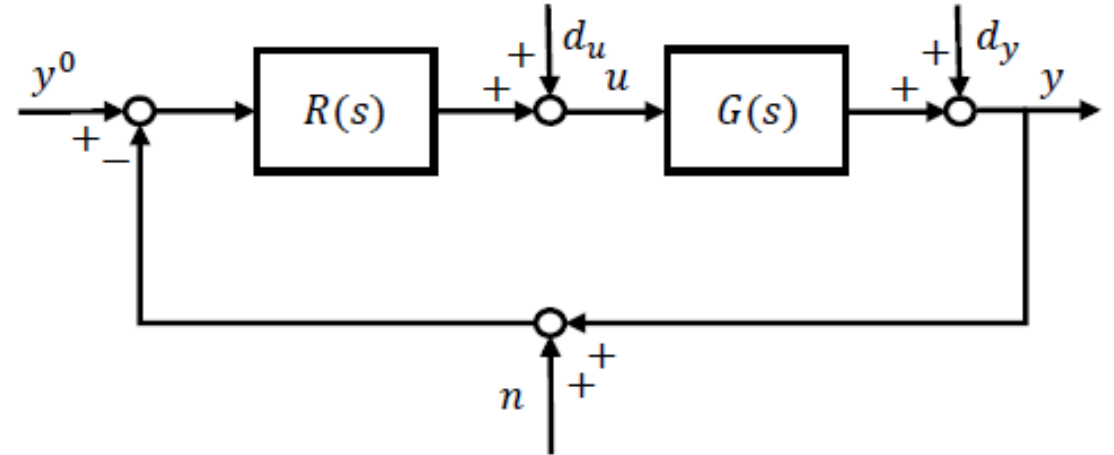$$T(s) = (I + L(s))^{-1} L(s) = L(s) (I + L(s))^{-1} \quad complementary\ sensitivity$$
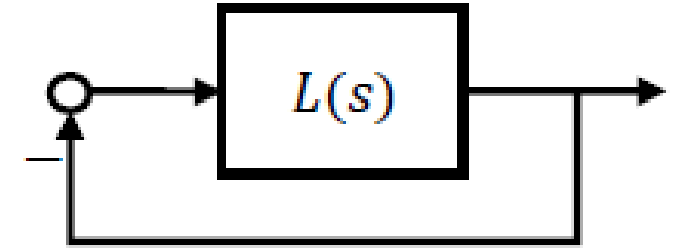
Closed-loop functions

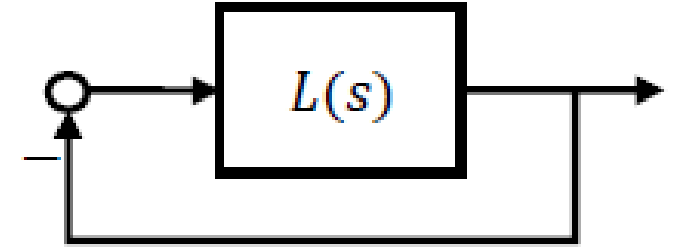$$Y(s) = T(s) (Y^o(s) - N(s)) + S(s)D_y(s) + S(s)G(s)D_u(s)$$

No conceptual differences with respect to the SISO case

Stability of feedback systems

A system composed by several blocks is asymptotically stable if: (a) there are no hidden cancellations of unstable modes, i.e. cancellations between poles and zeros with nonnegative real part inside its blocks, and (b) bounded inputs applied at any point of the system produce bounded outputs at any point of the system

Nyquist theorem for MIMO systems

Let P_{ol} be the number of poles of $L(s)$ with positive real part. Then, the closed-loop system with loop transfer function $L(s)$ and negative feedback is asymptotically stable if and only if the Nyquist plot of $\det(I + L(s))$: (i) does not pass through the origin, (ii) the number of its encirclements (positive in the anticlockwise direction) around the origin is P_{ol}.
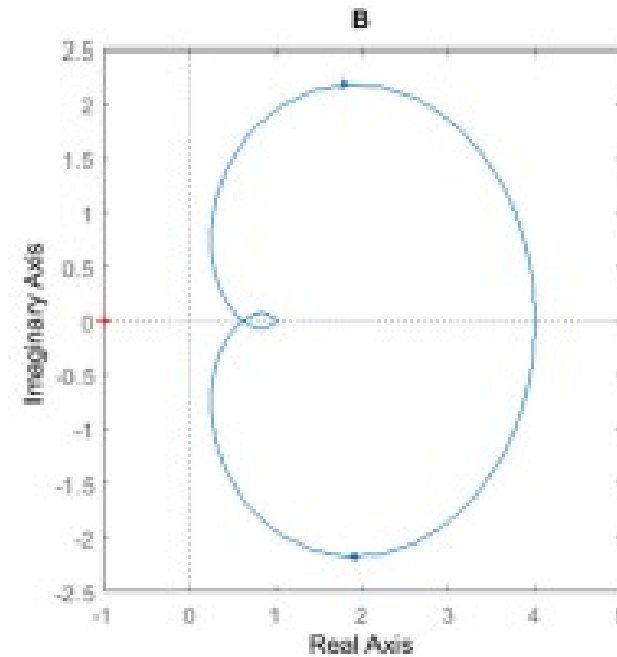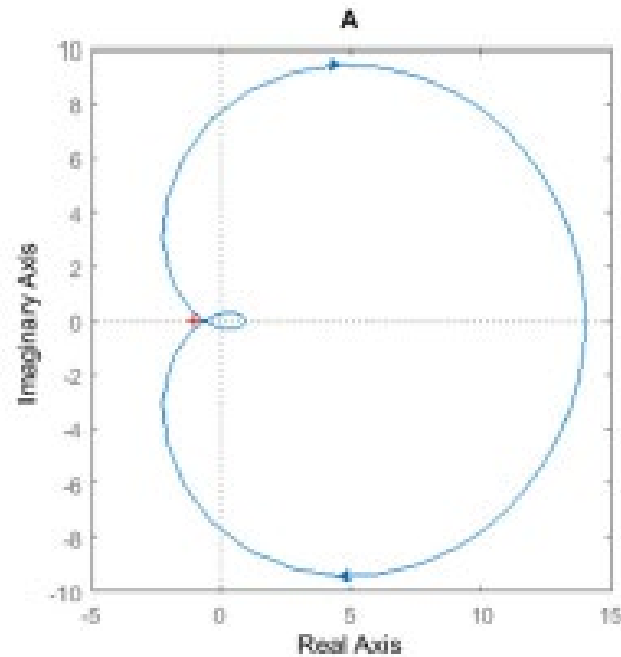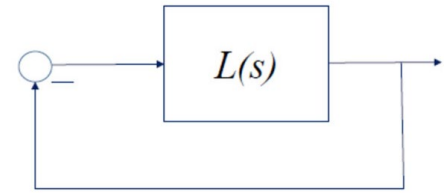
Not easy to understand how to modify L(s) to get stability

Bode criterion not available (remember the singular values!)

Exercise (exam)

L(s) is a 2x2 matrix with no unstable poles

Consider the following Nyquist plots of $det\ (I+L(s))$ and specify which case (A), (B) corresponds to asymptotically stable closed-loop systems. Motivate your answer.

Case B, since in view of the Nyquist criterion for MIMO systems, and in view of the fact that L(s) does not have unstable poles, the Nyquist plot must not encircle the origin.

Stability from the closed-loop transfer functions

$$Y(s) = T(s)\left(Y^o(s) - N(s)\right) + S(s)D_y(s) + S(s)G(s)D_u(s)$$

The feedback system is internally stable if and only if the transfer functions
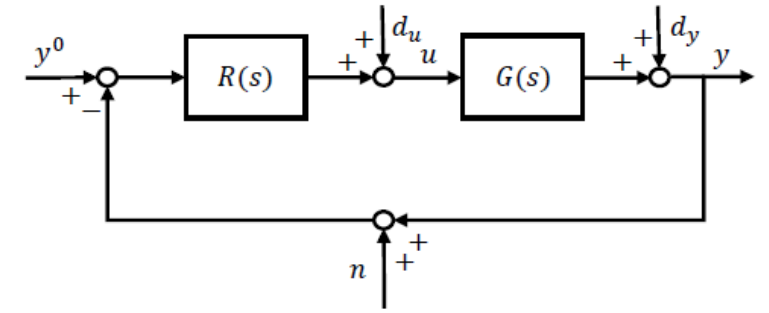
$$
\begin{aligned}
K_1(s) &= (I + L(s))^{-1} L(s) \\
K_2(s) &= (I + L(s))^{-1} \\
K_3(s) &= (I + L(s))^{-1} G(s) \\
K_4(s) &= (I + L_u(s))^{-1} R(s) \\
K_5(s) &= (I + L_u(s))^{-1}
\end{aligned}
$$

are asymptotically stable

Why do we need to look at all these transfer functions? Because in some of them forbidden cancellations could be hidden
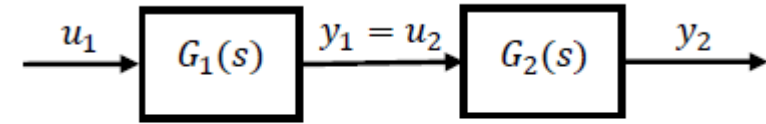
Cancellations and stability

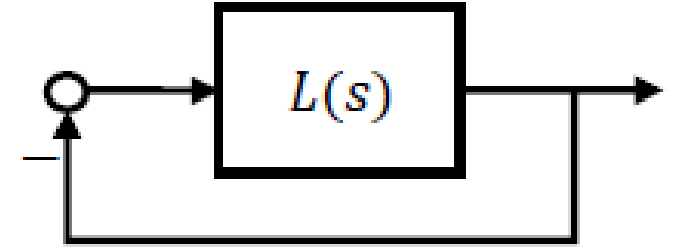Figure 5.1: Systems in series configuration.

If $G_1(s)$ and $G_2(s)$ have a common pole/zero, it can happen that the cancellation does not occur

A fake MIMO system: poles and zeros are the ones of the single transfer functions

$$G_1(s) = \begin{bmatrix} \dfrac{4}{s+3} & 0 \\ 0 & \dfrac{-4(s-1)}{s} \end{bmatrix}, \quad G_2(s) = \begin{bmatrix} \dfrac{4}{s-1} & 0 \\ 0 & \dfrac{1}{s+8} \end{bmatrix} \longrightarrow G_2(s)G_1(s) = \begin{bmatrix} \dfrac{16}{(s+3)(s-1)} & 0 \\ 0 & \dfrac{-4(s-1)}{s(s+8)} \end{bmatrix}$$

A cancellation between $G_1(s)$ and $G_2(s)$ exists only if the poles of $G_1(s)$ and/or of $G_2(s)$ are not included in the poles of $G_1(s)G_2(s)$ (or $G_2(s)G_1(s)$)

Small gain theorem for MIMO systems

A closed-loop system made by asymptotically stable linear systems and with loop transfer function $L(s)$ is asymptotically stable if the loop gain is less than 1, that is if
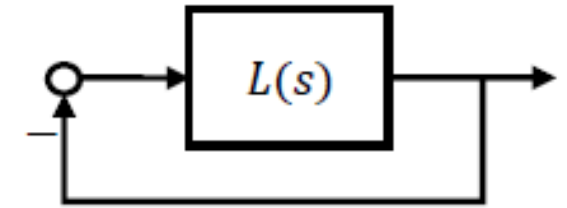
$$\|L\|_\infty < 1$$

Also in this case:

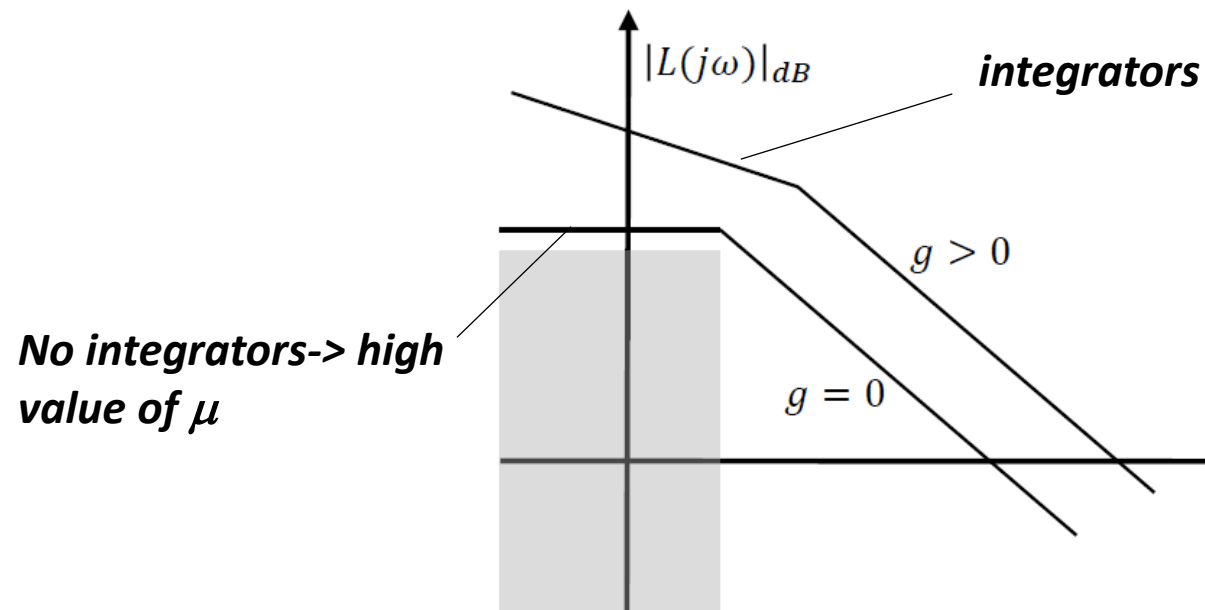- only a sufficient condition
- very conservative

Static properties – let's recall some results for SISO systems

Assuming that the feedback system is asymptotically stable, given a constant set-point with Laplace transform $Y^0(s) = \frac{A}{s}$, the output asymptotically tends to
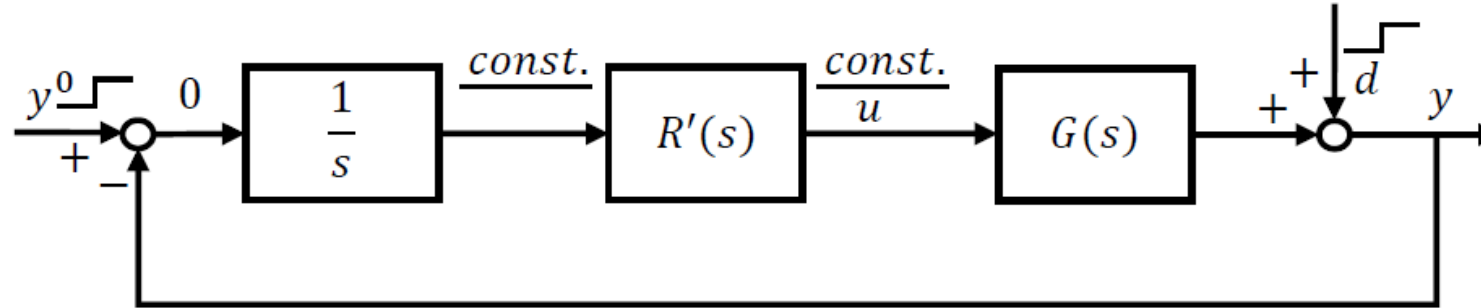
$$L(s) = \frac{\mu}{s^g} \frac{\prod (1 + \tau_i s)}{\prod (1 + T_j s)}$$

$$y_\infty = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s \frac{L(s)}{1 + L(s)} \frac{A}{s} = \begin{cases} \frac{\mu}{1+\mu} A & g = 0 \\ A & g > 0 \end{cases}$$

$|L(j\omega)|_{dB}$ *integrators*

$g > 0$

No integrators-> high value of μ

$g = 0$

ω

Interpretation – how the integrator works

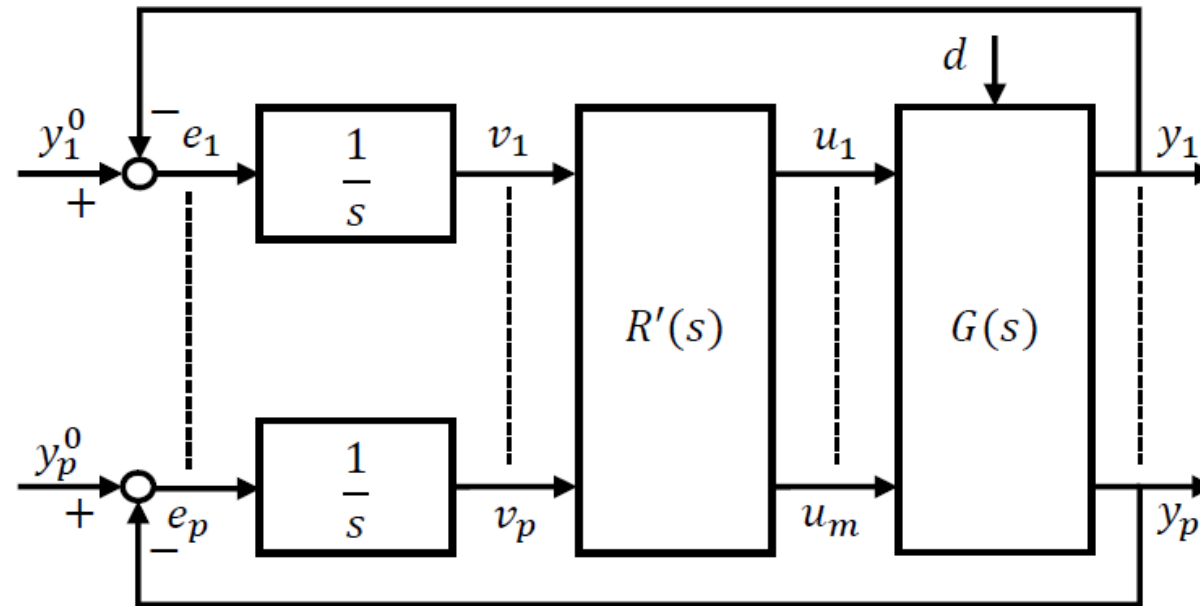If the closed-loop system is asymptotically stable, for constant exogenous signals, the input of the integrator must be asymptotically zero

Classical design approach: first put an integrator to guarantee static performance, then choose *R'(s)* to stabilize *(1/s)G(s).* The overall regulator is *(1/s)R'(s)*

When it works? *G(s)* must not have derivative actions

The same approach can be used for MIMO systems

Static performance for MIMO systems – basic idea

One integrator for each error

R'(s) stabilizes *G(s)* + integrators

Required conditions for MIMO system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Md \\
y(t) &= Cx(t) + Nd
\end{aligned}
$$

$$x \in R^n,\ u \in R^m,\ y \in R^p,\ \text{and}\ d \in R^r$$

If we want that, for constant *d*, the output reaches a constant reference value *yº,* it must hold that at the steady state

$$
\begin{aligned}
0 &= A\bar{x} + B\bar{u} + Md \\
y^0 &= C\bar{x} + Nd
\end{aligned}
\longrightarrow
\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}
\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}
=
\begin{bmatrix} 0 & -M \\ I & -N \end{bmatrix}
\begin{bmatrix} y^0 \\ d \end{bmatrix}
$$

System matrix *P(0)*

$$
\Sigma = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in R^{n+p,n+m}
\longrightarrow
$$

Conditions:
1. $p \le m$
2. $rank(\Sigma) = n + p$

17

Conditions:

1. $p \leq m$ ⟶   *At least as many inputs as outputs*
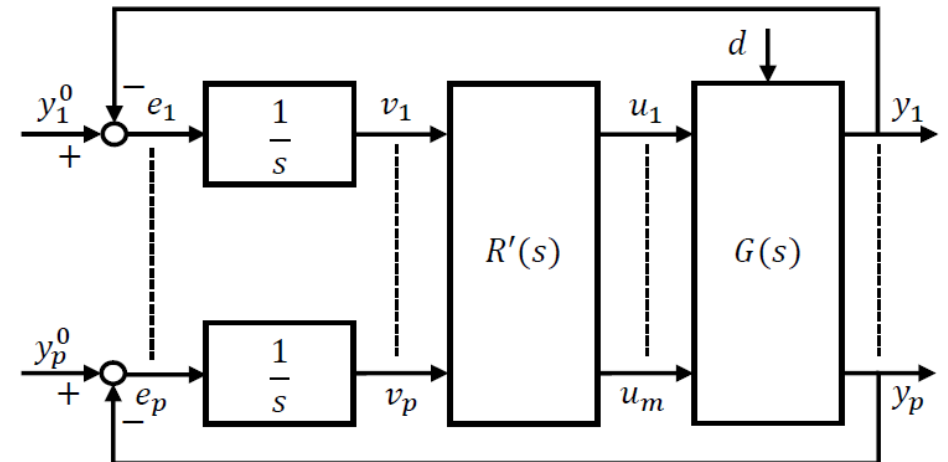
2. $rank(\Sigma) = n + p$ ⟶   *No invariant zeros s=0, i.e. no derivative actions*

R'(s) must stabilize the plant + the integrators

Integrators

$$
\begin{aligned}
\dot{v}(t) &= e(t) \\
&= y^0 - y(t) \\
&= y^0 - Cx(t) - Nd
\end{aligned}
$$

Plant + integrators

$$
\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & M \\ I & -N \end{bmatrix} \begin{bmatrix} y^0 \\ d \end{bmatrix}
$$

$$
v(t) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}
$$

This is the system that must be considered in the design of the stabilizing regulator (with pole-placement, LQ,…)

Letting
$$
\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & I \end{bmatrix}
$$

$(\bar{A}, \bar{B})$ must be reachable

$(\bar{A}, \bar{C})$ must be observable

observability of $(\bar{A}, \bar{C})$

The pair is observable *iff* the original pair is observable (see the textbook)

reachability of $(\bar{A}, \bar{B})$

The pair is reachable *iff* the original pair is reachable *and* the system under control does not have invariant zeros in *s=0,* i.e. no derivative actions (see the textbook)

This simply means that the added integrators must not cancel with zeros of the system in *s=0*

Dynamic performance (*sketch, see the textbook*)
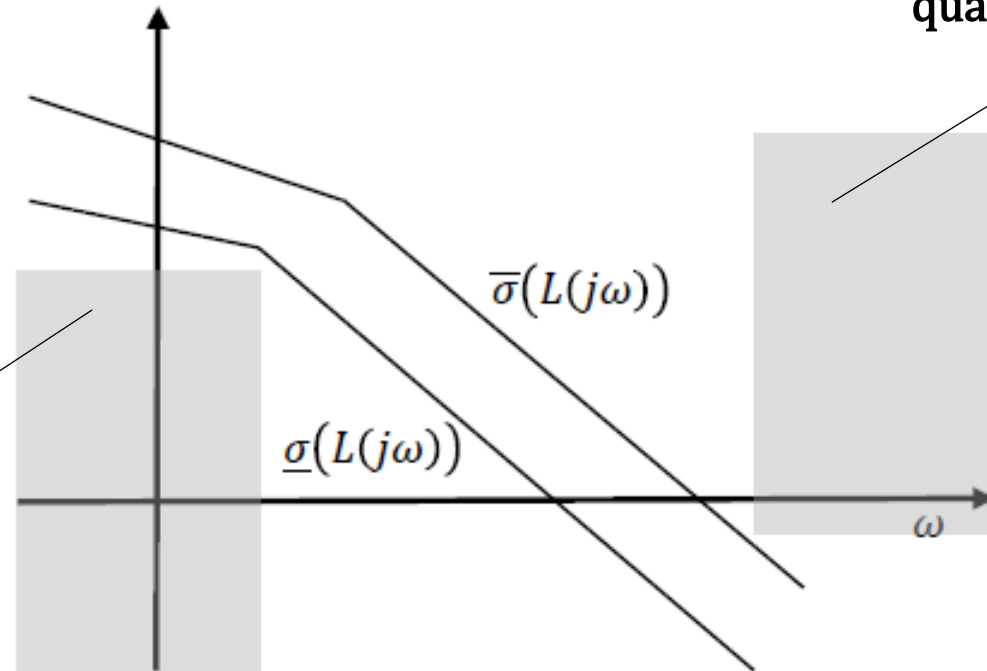
$$Y(s) = T(s)\left(Y^o(s) - N(s)\right) + S(s)D_y(s)$$

$$U(s) = (I + R(s)G(s))^{-1} R(s)\left(Y^o(s) - D_y(s) - N(s)\right)$$

T small at high frequency

$\downarrow$

Maximum s.v. of *L(s)* smaller than a given quantity

Loop singular values of L(s)

Playing with singular values, requirements on *T(s), S(s)* etc. can be transformed into requirements on the minimum and maximum singular values

$$\overline{\sigma}(L(j\omega))$$

T ≃I at low frequency
S ≃0 at low frequency

$$\underline{\sigma}(L(j\omega))$$

ω

$\downarrow$

Minimum s.v. of *L(s)* greater than a given quantity

21

Summary of the main problems and facts related to the design of regulators for MIMO systems

There is not a unique loop transfer function $L(s)$, it is not at all clear how to modify the loop transfer function (matrix) by a proper selection of the regulator

The Bode criterio does not exist, multivariable Root Locus is an extremely complex approach, and synthesis in the frequency domain turns out to be difficult

These are good motivations for the use of automatic synthesis procedures (pole placement, optimal contrl...)
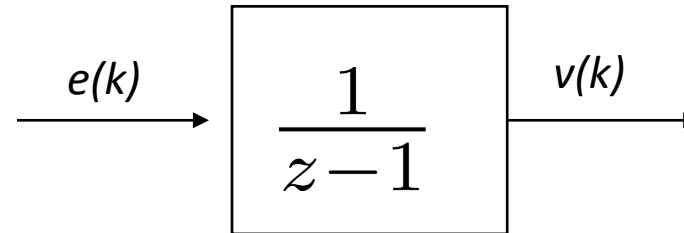
Anyway, choosing a priori the proper controller structure allows one to solve the static problem (asymptotic tracking of constant reference signals, asymptotic rejection of constant disturbances)

What about discrete-time systems? Most of the arguments should be reconsidered. However the structure with integrators can be used as well (next slide)

Integrators of discrete time systems
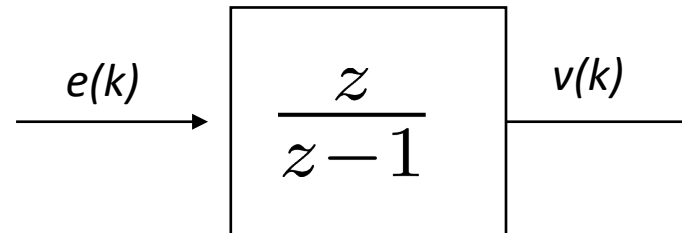
Form 1

$$v(k+1) = v(k) + e(k)$$

$e(k)$ → $\dfrac{1}{z-1}$ → $v(k)$

We'll use this form only because is slightly easier to manage

Form 2

$$\varphi(k+1) = \varphi(k) + e(k)$$
$$v(k) = \varphi(k) + e(k)$$

$e(k)$ → $\dfrac{z}{z-1}$ → $v(k)$

The zero in *z=0* guarantees **faster response** (anticipative term), but it can cause **implementation problems**. In zero time you should sample the error with A/D converter, compute *v*, and write it on the D/A port

Plant + integrators

$$
\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 & M \\ I & -N \end{bmatrix} \begin{bmatrix} y^0 \\ d \end{bmatrix}
$$

$$
v(k) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}
$$

Enlarged system

$$
\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 & M \\ I & -N \end{bmatrix} \begin{bmatrix} y^0 \\ d \end{bmatrix}
$$

$$
v(k) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}
$$

The system is reachable and observable iff the original system is reachable and observable and does not have invariant zeros in *z=1*

Static performance are guaranteed for constant *yᵒ, d.* Stabilization algorithms will be based on pole placement, optimal control, Model Predictive Control, …

… some (exam) exercises …

Consider an asymptotically stable system with static gain

$$G(0) = \begin{bmatrix} 1 & -0.5 \\ 2 & 1 \end{bmatrix}$$

and the constant inputs with 2-norm equal to 1

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Correspondingly $\|y_1\|_2 = 2.2361$, $\|y_2\|_2 = 2.1503$, $\|y_3\|_2 = 1.1180$. Then, the singular values of $G(0)$ are

○ 1.2180 , 2.2361

○ 1.1180 , 2.1121

○ 0.8508 , 2.3508

○ 0.8508 , 2.2120
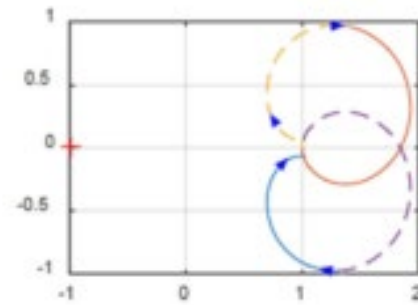
Consider a standard feedback system with negative feedbak, transfer functions G(s) and R(s) of the system and the regulator, and loop transfer function L(s)=G(s)R(s). Is it possible to study the stability of the closed loop system by looking at the sensitivity function S(s)=inv(I+L(s)? *
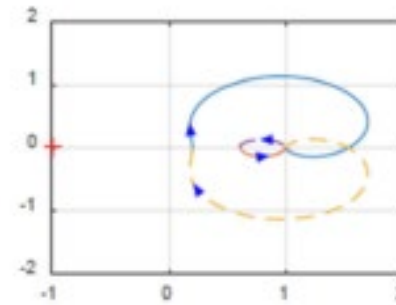(3 punti)

○ yes, if and only if G(s) and R(s) have stable poles

○ no never

○ yes, always

○ yes, if there are no cancellations of unstable poles of G(s) with (invariant) zeros of R(s) and viceversa

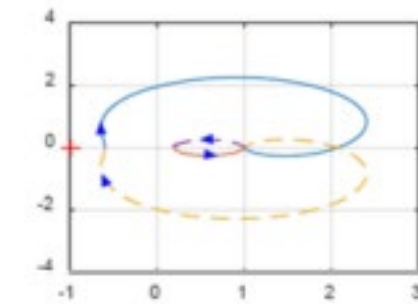Consider a feedback system with loop transfer function $L(s)$ without unstable poles.

Consider the following Nyquist plots of $det\ (I+L(s))$ and specify which cases (a), (b), (c), (d) correspond to asymptotically stable closed-loop systems.
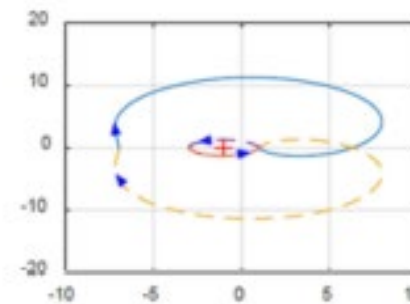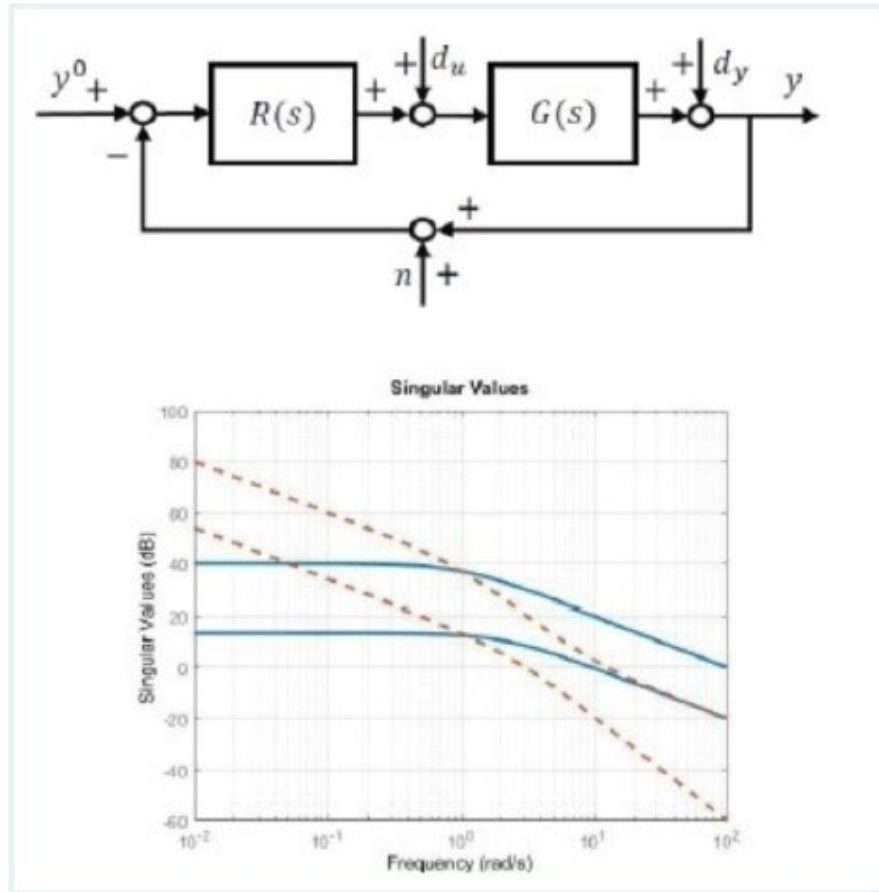

(a)

(b)

(c)

(d)

○ b, c, d

○ a, b

○ a, c, d

○ a, b, d

Singular Values

Consider the feedback system reported in the block diagram and two possible loop transfer functions L1(s)=R1(s)G1(s) and L2(s)=R2(s)G2(s) with the principal gains reported in the figure. Let L1(s) be associated to the dashed lines and L2(s) to the continuous ones. Select the true answer among the following ones:
(3 punti)

○ Assuming that at low and high frequency all the singular values diagrams have the same slope shown in the figure, it is likely that the two loop transfer functions do not have poles at the origin.

○ L1(s) and L2(s) have roughly the same crossover frequency.

○ At high frequency ($\omega > 20$) the guaranteed attenuation of the noise disturbance n provided by L2(s) is always greater than the one guaranteed by L1(s).

○ At low frequency ($\omega < 1$) the guaranteed attenuation of the disturbance dy provided by L1(s) is always greater than the one guaranteed by L2(s).