NORM, GAINS, I/O STABILITY, SMALL GAIN THEOREM and CIRCLE CRITERION

Advanced and Multivariable Control

{signal and systems}

Norms, gains, small gain theorem

Riccardo Scattolini

Norms of vectors

of numbers

2 norm

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} \quad |e|_2 = \sqrt{e'e} = \sqrt{\sum_{i=1}^{m} e_i^2}$$

$$|e|_\infty = \max_i |e_i|$$

(max value on the vector)

inf norm

of signals

2-norm, *or* *H_2 norm*

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_m(t) \end{bmatrix} \quad \|e\|_2 = \sqrt{\int_{-\infty}^{+\infty} (e'(\tau)e(\tau))d\tau}$$

similar to number norm but with integral

$$\|e\|_\infty = \sup_t \left(\sup_i |e_i(t)| \right)$$

max component of vector over time (sperior:= max overall)

infinity norm, *or* *H_∞ norm*

Singular values

Φ : [p×m] complex number matrix

The *singular values* of the matrix $\Phi \in C^{p,m}$ are the $k = min(p, m)$ largest roots of the eigenvalues of $\underbrace{\Phi^* \Phi}$ or of $\Phi \Phi^*$

↓ # of sing values

(ϕ^* : complex conj matrix)

ϕ^* : • transpose of ϕ
• with complex conj of each ϕ element

$$\sigma_i(\Phi) \; : \; = \sqrt{\lambda_i(\Phi^* \Phi)} = \sqrt{\lambda_i(\Phi \Phi^*)}, \quad m = p$$

$$\sigma_i(\Phi) \; : \; = \sqrt{\lambda_i(\Phi^* \Phi)}, \quad m > p$$ ← more col than rows

$$\sigma_i(\Phi) \; : \; = \sqrt{\lambda_i(\Phi \Phi^*)}, \quad m < p$$ ← more rows

$(i = 1, \dots, k)$

↑ depending on matrix structure change the evaluation

Singular value decomposition

(SVD) → automatically evaluated by software

Any matrix $\Phi \in C^{p,m}$ can be partitioned with the singular value decomposition

$$\boxed{\Phi = U\Sigma V^*}$$

complex conj of V

where the matrices $U \in C^{p\times p}$ and $V \in C^{m\times m}$ are unitary, while the matrix Σ is defined by

↳ unitary matrix
- $U^* = U^{-1}$
- eig (U) = 1
- sing (U) = 1

(*)

depending on rows/column number

$$\begin{cases} \Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \in R^{p\times m}, & p \geq m \\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} \in R^{p\times m}, & p \leq m \end{cases}$$

where

$$\Sigma_1 = diag\underbrace{\{\sigma_1, \sigma_2, \ldots, \sigma_k\}}_{\text{with SVD ordered elements}}; \quad k = \min(p, m)$$

Σ_1 : [K×K] diag matrix real values

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k$$

structure of matrix

(*) **Unitary matrix** $\quad U^* = U^{-1} \quad , \quad |\lambda_i(U)| = 1, \forall i \quad , \quad \sigma_i(U) = 1, \forall i$

Minimum and maximum singular values

Letting

$$\Phi = U\Sigma V^*$$

with

$$\begin{aligned} \Sigma &= \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \in R^{p\times m}, \quad p \geq m \\ \Sigma &= \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} \in R^{p\times m}, \quad p \leq m \end{aligned}$$

where

$$\Sigma_1 = diag\left\{\sigma_1, \sigma_2, \ldots, \sigma_k\right\}; \quad k = \min(p, m)$$

(DEF) the maximum singular value is $\bar{\sigma} = \sigma_1$ and the minimum singular value is

$$\underline{\sigma} \equiv \begin{cases} \sigma_m & if \quad p \geq m \\ 0 & if \quad p < m \end{cases}$$

if more rows σ_m

0 otherwise

Induced *p-norm* of a matrix

norm of given matrix A

$$\|A\|_{ip} = \sup_{d \neq 0} \frac{\|Ad\|_p}{\|d\|_p}$$

A·d is a vector, (of proper dimension)
taking the superior value overall $p = 1,2,..,\infty$

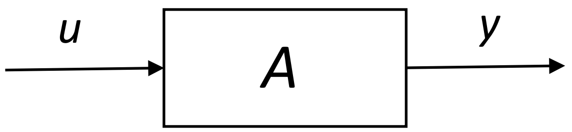
Induced 2-norm

$$\|A\|_{i2} = \sup_{d \neq 0} \frac{\|Ad\|_2}{\|d\|_2} = \bar{\sigma}(A)$$

the 2 norm is the max singular values of matrix A

Norm of a «map» *A*

A ≃ "static map of a system u→y"



(scalar system)

$$\sup_{u \neq 0} \frac{|y{=}Au|_2}{|u|_2} = \sqrt{\lambda_{\max}(A'A)} = \boxed{\bar{\sigma}(A)}$$

take sup respect u

gain of the map

relating I/O

Norm of systems → dynamical system $(u(t) \to y(t))$

$$u(t) \longrightarrow \boxed{S(u(t))} \longrightarrow y(t)$$

not necessarily linear

holds also for NON LIN syst

Let L_2 be the space of functions, null for $t<0$, and whose absolute value raised to the 2^{nd} power has finite integral, that is, if $u \in L_2$,

$$\|u\|_2 = \sqrt{\int_0^{+\infty} (u'(\tau)u(\tau))d\tau} < +\infty$$

↓(finite integral)

↑ here we assume $u \neq 0$ for $t > 0$ (for $t<0$ $u=0$)

for S we can define the

Gain γ of S

GAIN γ, H∞ GAIN

⇓

$y = S(u)$

$$\gamma = \|S\|_\infty = \sup_{u\in L_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u\in L_2} \frac{\|S(u)\|_2}{\|u\|_2} \longleftrightarrow \|y\|_2 \leq \|S\|_\infty \|u\|_2 \quad , \quad \forall u \in L_2$$

H∞ norm of S

↑ relate 2 norm of In to Out

Example of linear system – the integrator

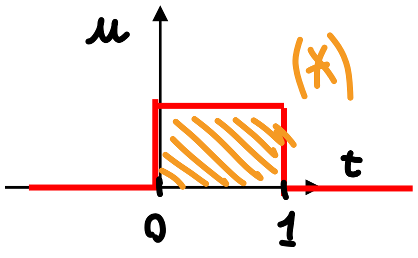
definitions interpretation

(INTEGRATOR)

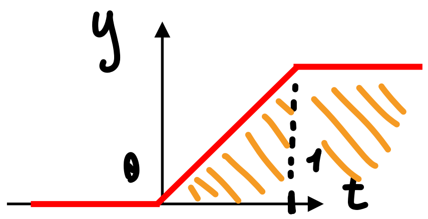
$$y(t)=\int_0^{+\infty} u(\tau)d\tau, \qquad Y(s)=\frac{1}{s}U(s)$$

T. F (in Laplace)

Input *u = sca (t)*



↓ through integrator



y = ram(t)

(by definition)

that area (*)

$$u(t)=\begin{cases}1 & , \quad 0<t<1\\ 0 & , \quad t\geq 1\end{cases} \longleftrightarrow \|u\|_2=\sqrt{\int_0^{+\infty} u'(\tau)u(\tau)d\tau}=1\in L_2 \rightarrow u\in L_2$$

funct family

$$\|y\|_2=+\infty$$

$\uparrow$ H_2 norm ...

$$\int_0^{+\infty} y'(\tau)y(\tau)d\tau=+\infty$$

output $y\notin L_2$

∞ GAIN syst $\Longrightarrow$

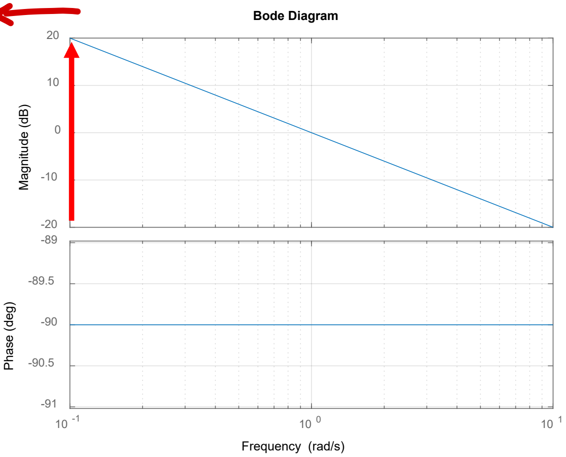
↓

infinite gain

on (BODE DIAG)

∞ gain @ $\omega=0$

$\omega=0$ at left



Example – SISO asymptotically stable linear system

$$Y(s) = G(s)U(s) \quad , \quad Y(j\omega) = G(j\omega)U(j\omega)$$

↳ poles of G(s) → eig (A) have neg Re part!

Parseval theorem (*)

↓ allows to move from time to freq and viceversa!

freq domain (*)

$$\|y\|_2^2 = \int_{-\infty}^{+\infty} y^2(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |Y(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 |U(j\omega)|^2 d\omega$$

(by definition)

if $\underbrace{|G(j\omega)| \leq K}_{\text{max Limit of } G}$ with $|G(j\omega)| = K$ for some ω

(assumption)

|G| ≤ K always... so overall limited!

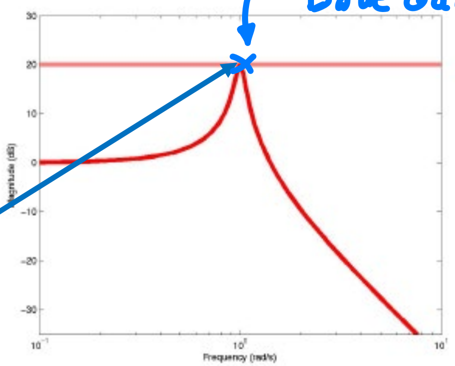
from freq to Time (*)

$$\|y\|_2^2 \leq K^2 \frac{1}{2\pi}\int_{-\infty}^{+\infty} |U(j\omega)|^2 d\omega = K^2 \int_{-\infty}^{+\infty} u^2(\tau)d\tau = K^2 \|u\|_2^2$$

($\|u\|_2$, $\|y\|_2$ relationship!)

∞ norm: max possible amplification

↑ H∞, ∞ Gain is the max of Bode Gain



new Gain definition, different from STATIC GAIN!

We can define H∞ Gain of G as:

$$\|G\|_\infty = \sup_\omega \underbrace{|G(j\omega)|}_{\text{from SISO T.F of asymp stable syst}} = K$$

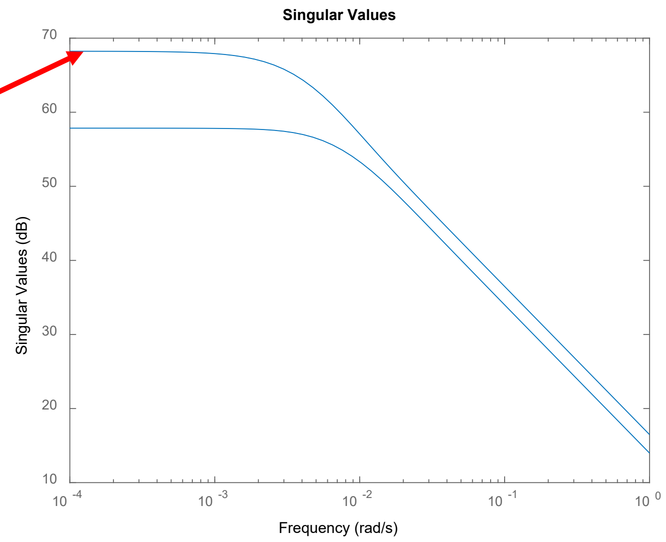
new H∞ Gain def

$\boxed{\text{gain:}}$ supremum of the modulus of the frequency response of $G(s)$

MULTI IN/OUT systems!

Extension to MIMO asymptotically stable linear system

$\sigma(\omega)$ evolution



for each σ we can draw bode diag

sup of max sing value

as defined previousely...

$$\gamma = \|G\|_\infty = \sup_\omega \bar{\sigma}\left(G(j\omega)\right)$$

non null sing values depends on I, O number... any σ depends on ω

(NOTE: MIMO syst → $G(j\omega)$ is a matrix of T.F)

≠ from SISO case... here you look to σ (sing. val)

Input – Output (I/O) stability

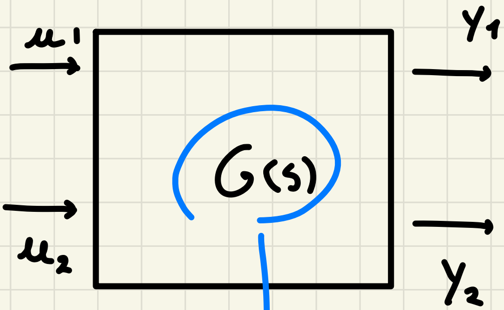
DEFINITION

A system $y = S(u)$ is input-output (I/O) stable if it has finite gain

($u \to y$ system)

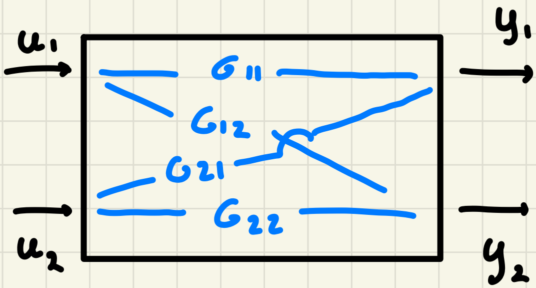
$\forall u \in L_2 \to y \in L_2$

MIMO syst.



the T.F is a matrix

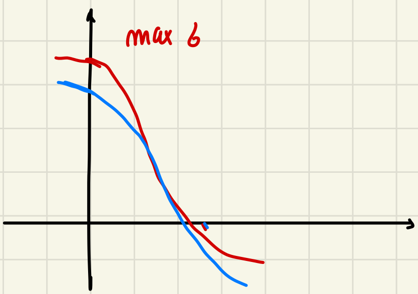
$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$



we will have 2 sing values

you can have strange behaviour...

sing value Bode diag



example

$G(j\omega)$ matrix of complex values...
compute $\forall\omega$ the singular values (σ)

σ equals to min number of in(OR)out

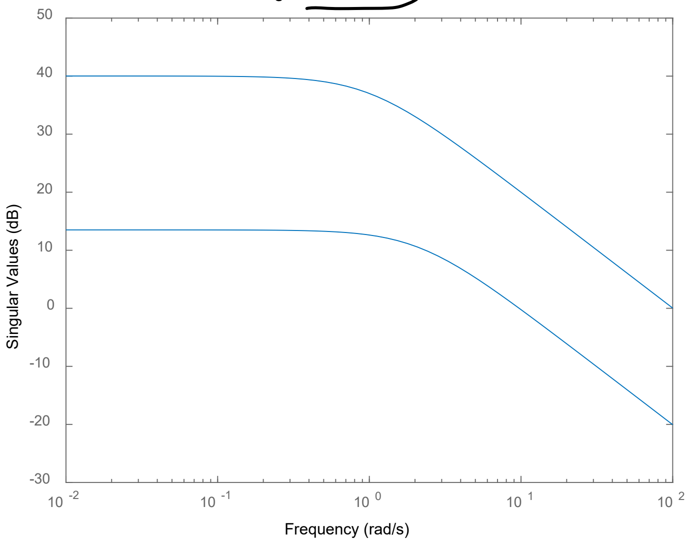
$$G(s) = \begin{bmatrix} \frac{100}{(s+1)} & \frac{10}{(s+1)(s+2)} \\ \frac{10}{(s+2)} & \frac{10}{(s+2)} \end{bmatrix}$$

[2 x 2] system

MATLAB

```
g11=tf(100,[1 1]);
g12=tf(10,conv([1 1],[1 2]));
g21=tf(10,[1 2]);
g22=tf(10,[1 2]);
G1=[g11 g12;g21 g22]
sigma(G1)
```

diag of σ



Different definitions of gain for asymptotically stable linear systems

- *gain*, or *infinite-norm gain*: $\gamma = \|G\|_\infty$
- *gain at a given frequency* ω

$$\frac{\|Y(j\omega)\|_2}{\|U(j\omega)\|_2} = \frac{\|G(j\omega)U(j\omega)\|_2}{\|U(j\omega)\|_2}$$

SISO systems: the gain at a given frequency ω is $|G(j\omega)|$

MIMO systems:

$$\underline{\sigma}(G(j\omega)) \leq \frac{\|G(j\omega)U(j\omega)\|_2}{\|U(j\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

always in between min, max sing value

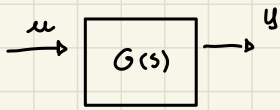
It depends on the applied input

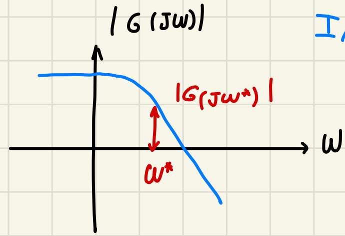
on MIMO syst is important to look also to the T.F Gain

- *static gain*: gain at $\omega = 0$. ← $|G(0)|$ typical gain seen until now

NOT equal ∀ output, depend on the INPUT

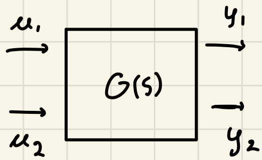
SISO systems

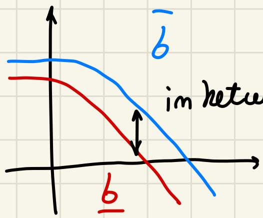




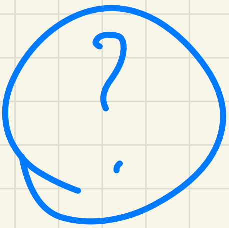
I/O Relation univoquely defined

MIMO





the work of syst depends also on I direction

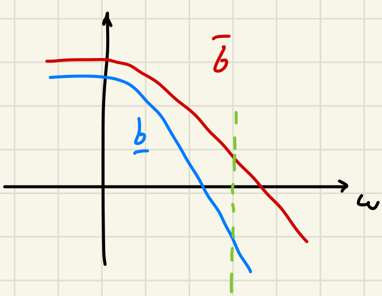


depended on how input is selected

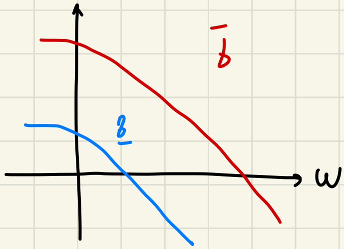
→ gain depends on input direction

this is why when design controllers (no BODE Theory)

different Gain to look at with strong influence



$\underline{\sigma}$, $\bar{\sigma}$ is relevant because in some case relevant

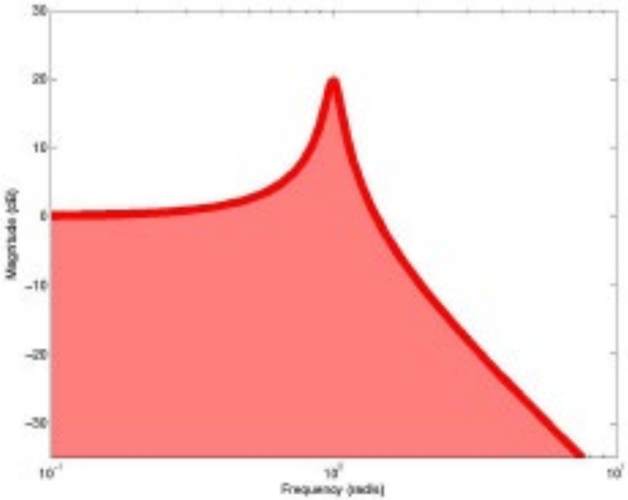


2-norm gain for asymptotically stable, strictly proper, linear systems

(H_2 GAIN) (important for Robust control H_∞)

SISO

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega}$$



{2 norm related to area!}

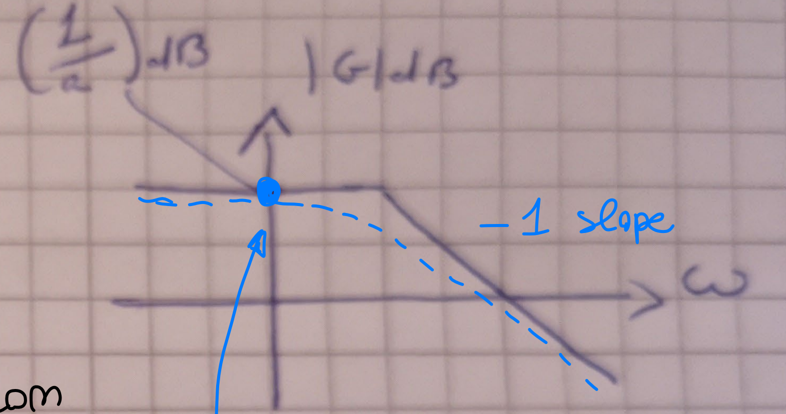
MIMO

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} tr(G(j\omega)G'(-j\omega))d\omega}$$

(consider the TRACE!)

Example H_2 - H_∞

approx Bode diag



$$\underbrace{G(s)}_{} = \frac{1}{s+a}, \quad a>0$$

compute

H_2, H_∞ of simple SISO syst by definition

∞ Gain $\rightarrow$ $\boxed{\|G\|_\infty = \frac{1}{a}}$ (max of bode diag)

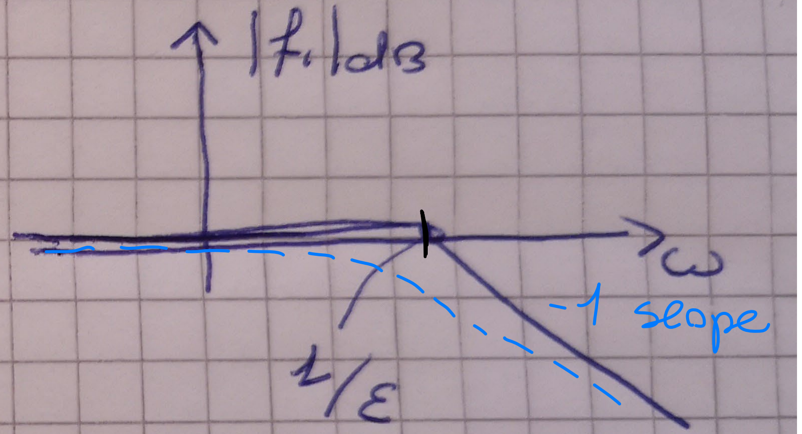
NO RELATION!

2 Gain $\rightarrow$ $\|G\|_2 = .. = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega} = \sqrt{\frac{1}{2a}}$

Example $H_2 - H_\infty$

$$f(s) = \frac{1}{\varepsilon s + 1}$$

approx Bode diag



$\varepsilon \to 0$

max Gain

$\|f_1\|_\infty = 1$, $\|f_1\|_2 = \infty$

∞ 2-norm!

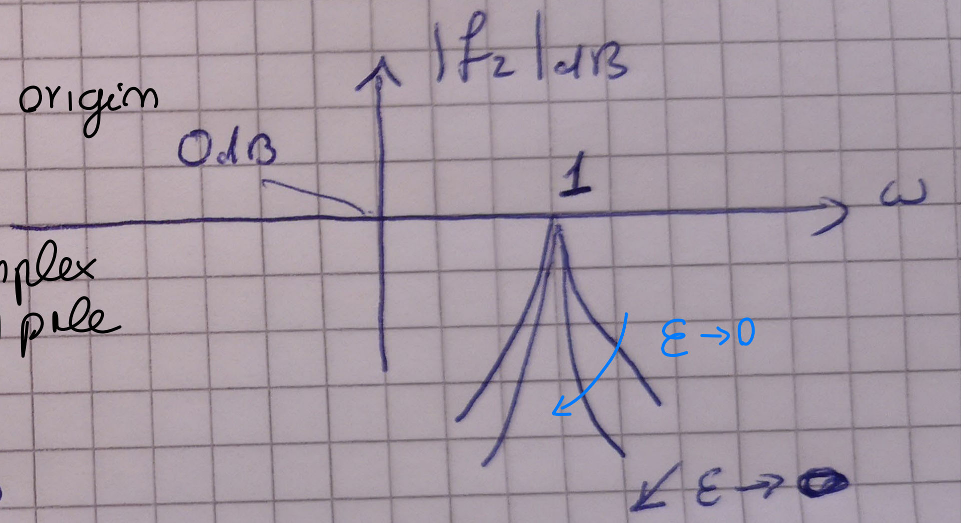
$\boxed{\text{Example } H_2 - H_\infty}$

zero on origin

$$f_2(s) = \frac{\varepsilon s}{s^2 + \varepsilon s + 1}$$

← complex conj pole

II ORD system

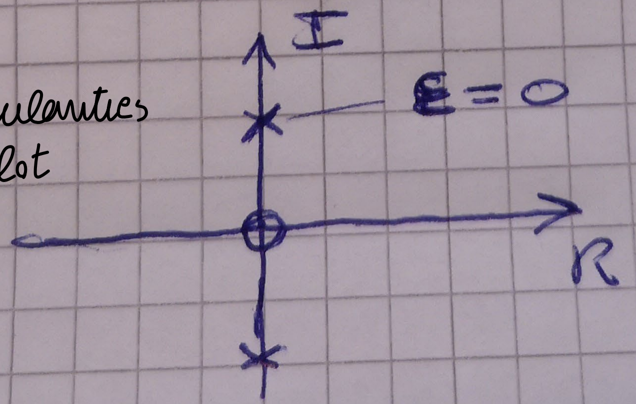


$\|f_2\|_\infty = 1$, $\|f_2\|_2 = 0$

max value

Smaller ε, thinner spike of Bode plot

singularities plot

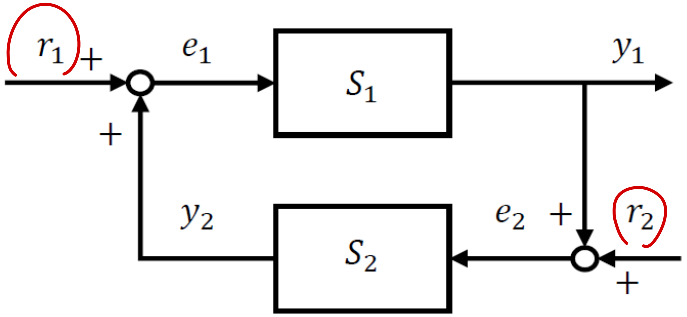


Small gain theorem *(one of the most useful tools for the analysis of nonlinear feedback systems)*

(to study syst stability)

(for Nyquist you look on loop T.F)

study STABILITY



exogenous signals

→ finite Gain S_1, S_2

Assume that S_1 and S_2 are I/O stable systems. Then

$$\boxed{\text{the feedback system is I/O stable if } \|S_1\|_\infty \|S_2\|_\infty < 1}$$

important also for synthesis of Regulator modifying S_1, S_2 feedback good syst

- If S_1 and S_2 are linear the condition is

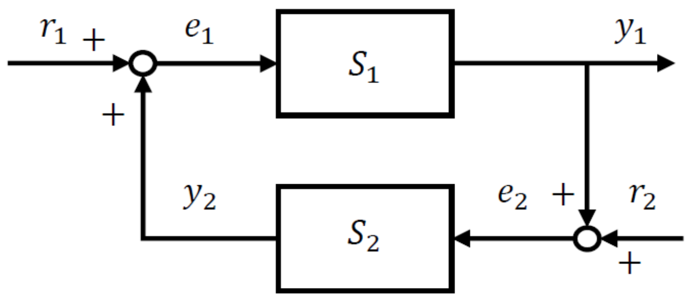
$$\|S_1 S_2\|_\infty < 1$$

less restrictive condition

only sufficient conditions

Proof dimonstration of the theorem

⇓ express signals as function of exog signals



$$e_1 = r_1 + S_2(r_2 + y_1), \quad y_1 = S_1(e_1)$$

Therefore,

↳ 2 norm

$$\|e_1\|_2 \le \|r_1\|_2 + \|S_2\|_\infty \left(\|r_2\|_2 + \|S_1\|_\infty \|e_1\|_2\right)$$

It follows that

$$\|e_1\|_2 \le \frac{\|r_1\|_2 + \|S_2\|_\infty \|r_2\|_2}{1 - \|S_1\|_\infty \|S_2\|_\infty}$$

bounded signal of $\|e_1\|$

↑ avoid den ≠ 0 $\|S_1\|_\infty \|S_2\|_\infty < 1$! to be limited pos.

and, with similar developments,

$$\|e_2\|_2 \le \frac{\|r_2\|_2 + \|S_1\|_\infty \|r_1\|_2}{1 - \|S_1\|_\infty \|S_2\|_\infty}$$

Then, in view of the previous assumptions, the gain is finite

⇒ good to study I/O stability of non lin syst.

assume we work with...

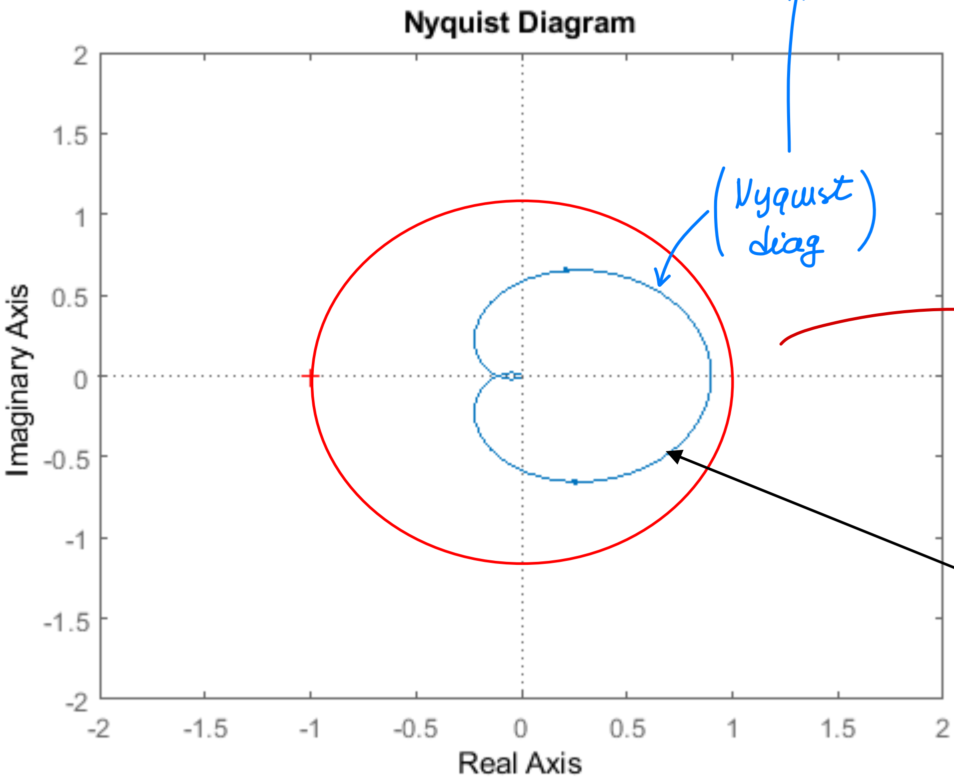
SISO Linear systems

Nyquist criterion:

$$\boxed{\begin{array}{l} N=0 \\ P=0 \end{array}}$$

to guarantee stability... during design phase this leads to conservative conditions...

NO circle around $-1+j0$



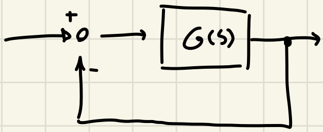
(Nyquist diag)

all diag of Nyquist inside the circle of radius 1

$S_1(j\omega)S_2(j\omega)$

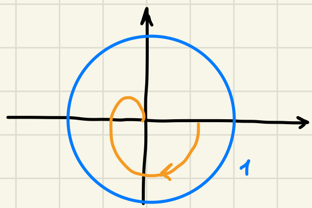
(conservativeness of small Gain Theorem)

I/O stable syst, Asymp. stable



(Nyquist th) $P = 0$

polar diag of G



polar diag

$$\boxed{\|S_1 S_2\|_\infty < 1}$$

(condition to satisfy)

$$\left\{ \begin{array}{l} S_1 = G \\ S_2 = 1 \end{array} \right\}$$

$\Downarrow$

condition verified: $\|S_1 \underset{\overset{1}{\shortparallel}}{S_2}\|_\infty < 1$

↳ $\|S_1\|_\infty < 1$ to apply th

$\|G\|_\infty < 1$

strictly inside unit circle and $\# N = 0$ of round around $-1 + j0$

sufficient condition!

↓

NOT the only possibility!

(conservative condition)

funct cont, NONLIN

linear part asymp. stable

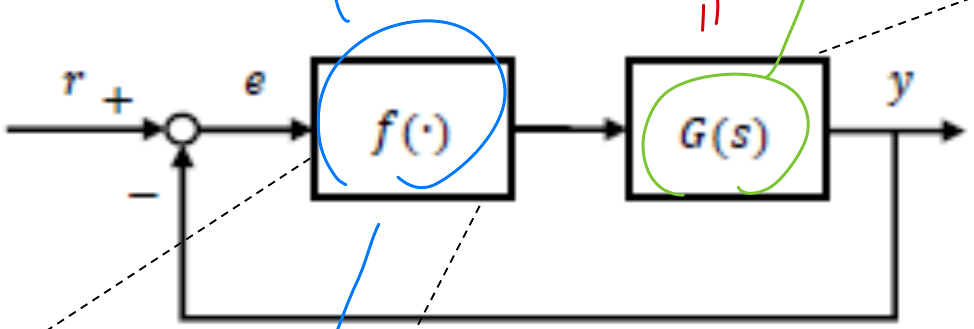
Stability of feedback systems with static sector nonlinearity

small Gain theorem
usefull for stability
of particular
system class

asymptotically stable SISO system
↓
we can apply
small gain th

S_2 =

S_1 =

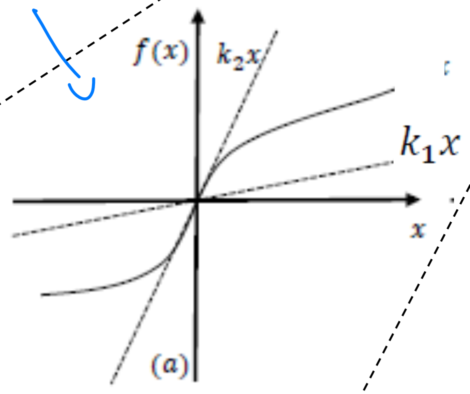


like saturation

limit zone of nonlinearity →

sector

non
linearity



nonlinear, continuous function
uniquely defined for any input

$f(0) = 0$ and $k_1 e \leq f(e) \leq k_2 e$

↑ inside a certain sector!

(Theorem)

sufficient condition for I/O stability
of system

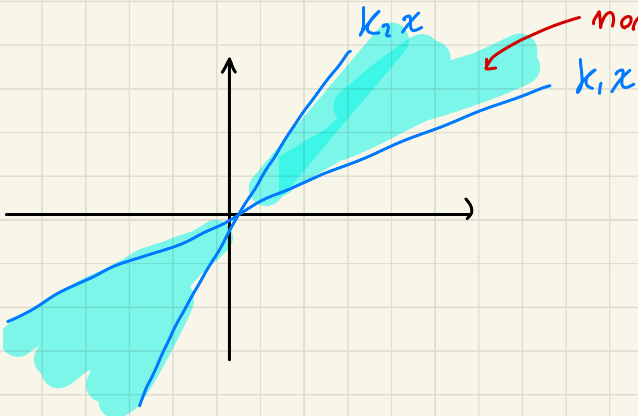
In view of the small gain theorem, I/O stability of the feedback system is guaranteed if

H∞ Gain
of G(s)

$$k_2 \sup_{\omega} |G(j\omega)| < 1$$

condition imposed by small Gain

↳ (restructive condition)

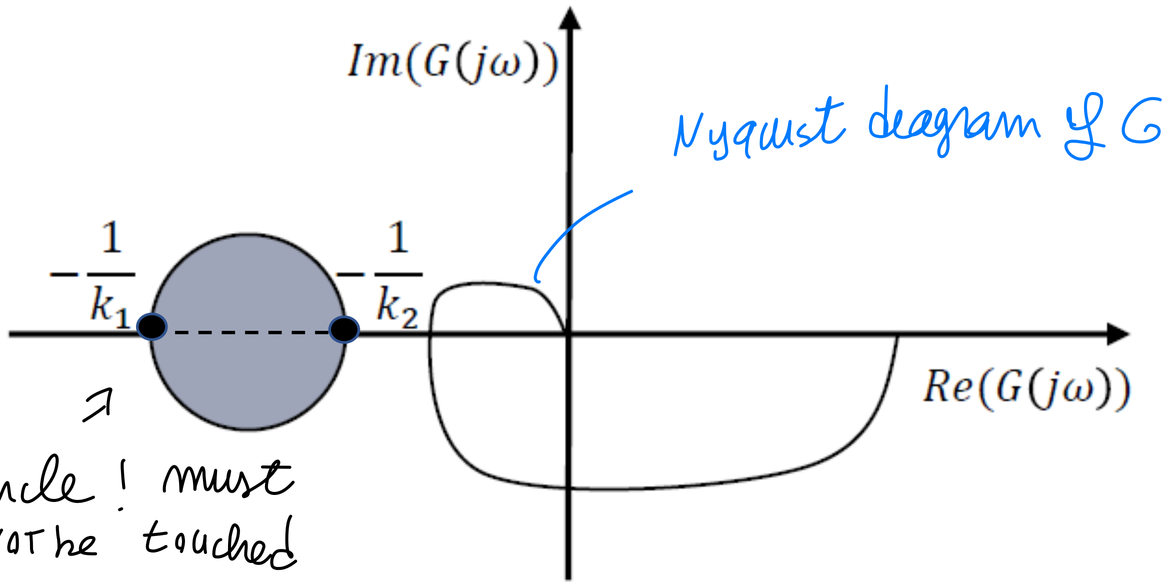


non linearity inside this sector!

⇒ { RELAXING condition }

A less stringent condition : THE CIRCLE CRITERION (*proof in the textbook*)

(Simpler condition to check)



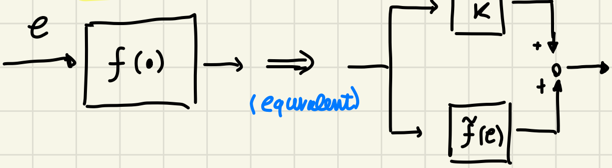
CIRCLE CRITERION

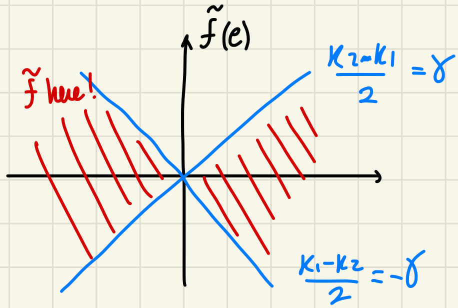
→ PROOF ⇒

The closed-loop system is $\boxed{\text{I/O stable}}$ if the Nyquist diagram of $G(s)$ does not encompass, intersect, or touch the circle with diameter given by the segment $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ and located on the x axis

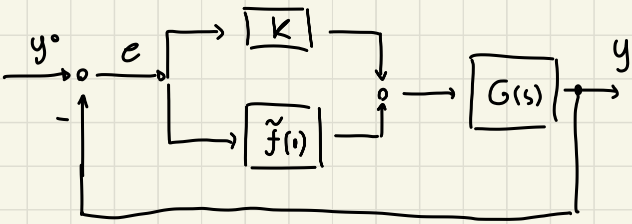
$K_1 < K_2$

CIRCLE CRITERION





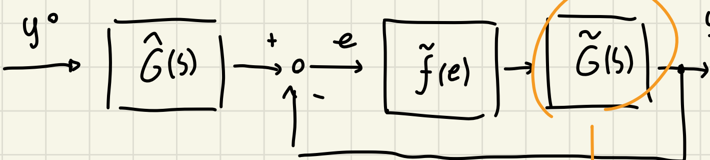
$$\begin{cases} K = \dfrac{k_1+k_2}{2} \\ \tilde{f}(e) + K = fe \end{cases}$$



$$e = \underbrace{\frac{1}{1+KG(s)}}_{\hat{G}(s)} y^\circ - \underbrace{\frac{G(s)}{1+KG(s)}}_{\tilde{G}(s)} \tilde{f}(e)$$

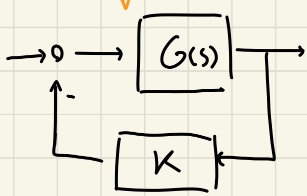
same stability property! (same denominator)

$\equiv$



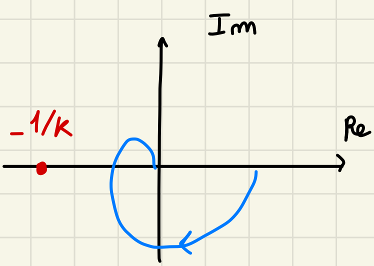
assume

$\tilde{G}, \hat{G}$ asymp. stable



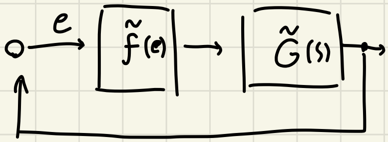
$\equiv \tilde{G}(s)$

Nyqust theorem:



No circles around $-1/K$ point for Nyq th

Consider:



small Gain Th

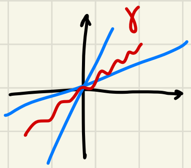
syst I/O stable if

$$\gamma \, |\tilde G(j\omega)| < 1 \quad \forall \omega$$

⇒ from this expression

$$\frac{1}{|\tilde G(j\omega)|} > \gamma \quad \forall \omega$$

remember (?)



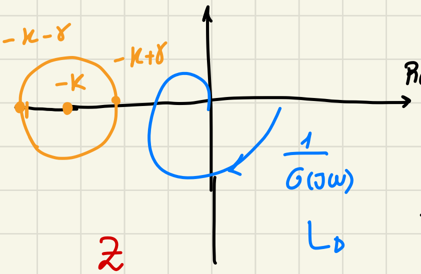
from $\tilde G$ definition:

$$\tilde G(s) = \frac{G(s)}{1 + K G(s)}$$

$$\Downarrow$$

$$\frac{1}{|\tilde G(s)|} = \left| \frac{1 + K G(s)}{G(s)} \right| \Rightarrow \left| \frac{1}{G(j\omega)} + K \right| > \gamma \quad \forall \omega$$

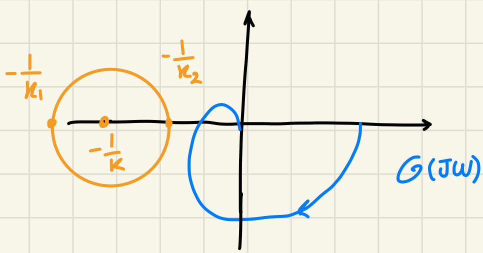
meaning that $\frac{1}{G(j\omega)}$ polar diag has always distance from K bigger than γ



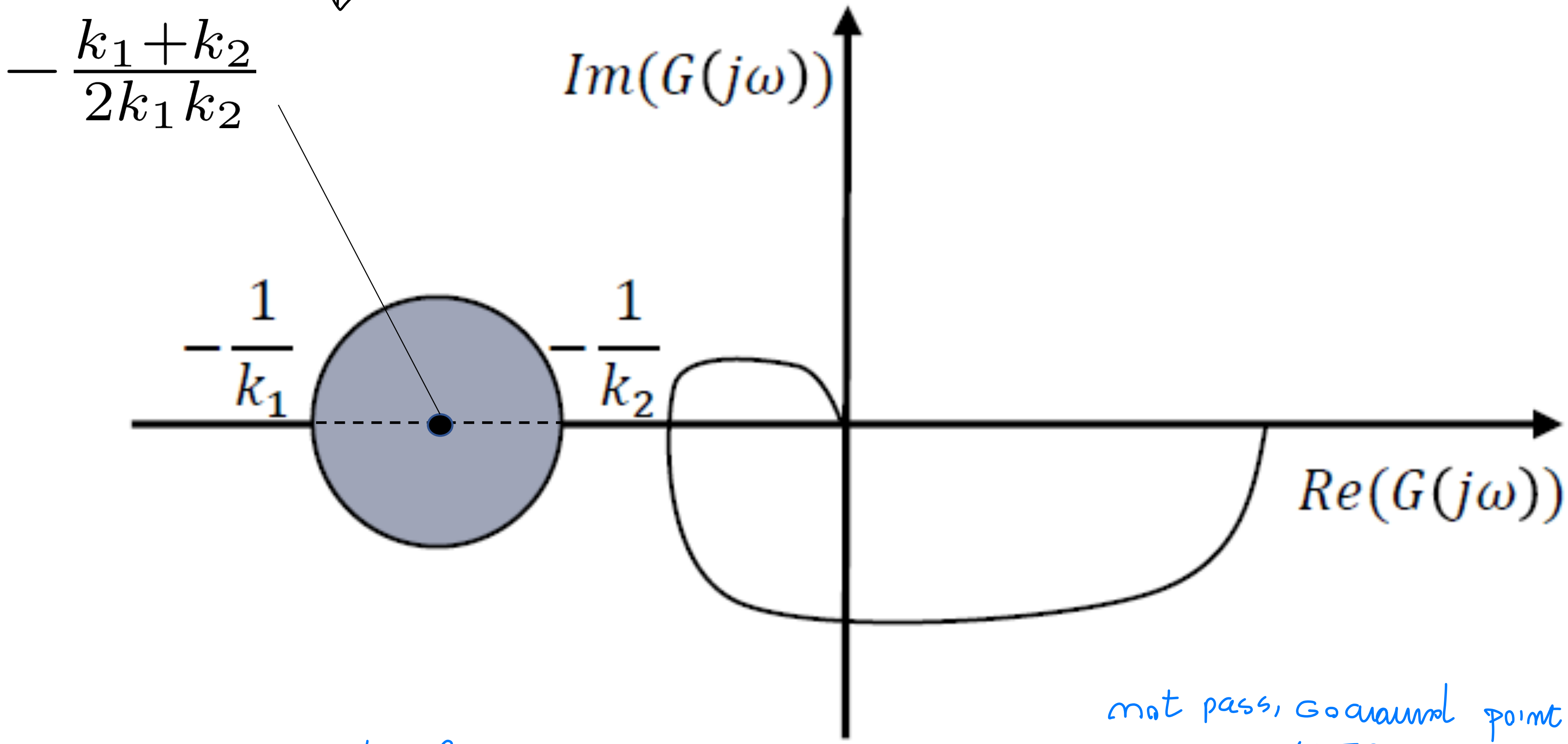
↳ to express on $G(j\omega)$

z domain

↳ transf: $\frac{1}{z}$



(equivalent to previous slide)



not pass, go around point $-1+J0$

Just a line, NOT a sector

yet known

IF) $\boxed{k_1 = k_2 = 1}$ – – >Nyquist criterion (If condition)

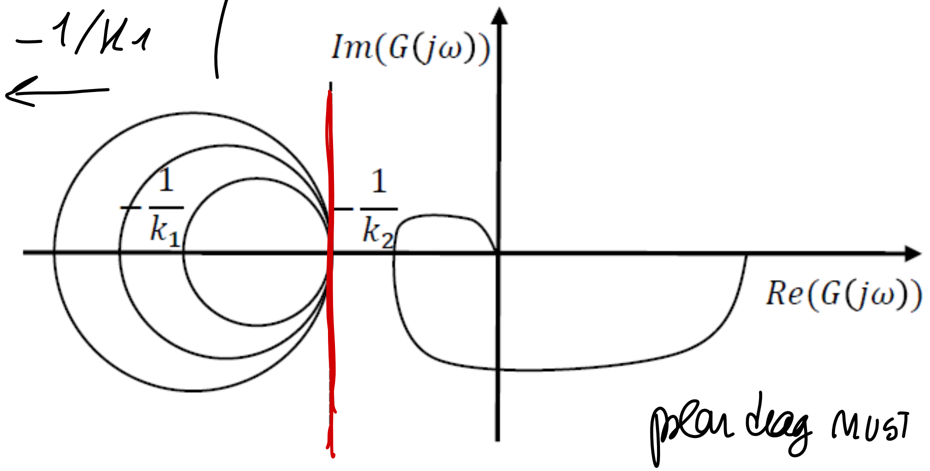
THE CIRCLE CRITERION *comments and interpretations*

If $k_1 \to 0$
becomes @ limit a circle
with ∞ radius, so a vertical
line!

- Only a sufficient condition (from small gain Th)
- Can be generalized to non asymptotically stable *G(s)*
- When $k_1 \to 0$ the circle becomes a vertical line passing through $-1/k_2$

how to modify properly G(s)

analysis + syntesis tool



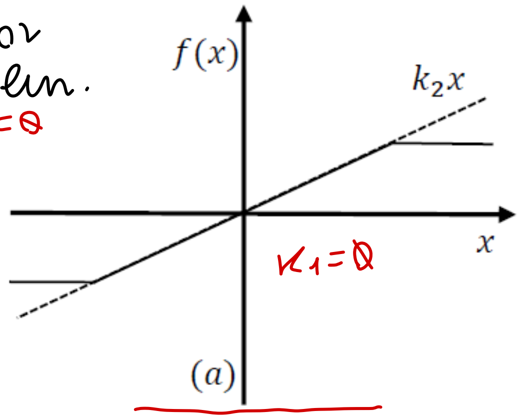
polar diag MUST remain on the right of $-1/k_2$

Interesting cases, widely used in practical applications

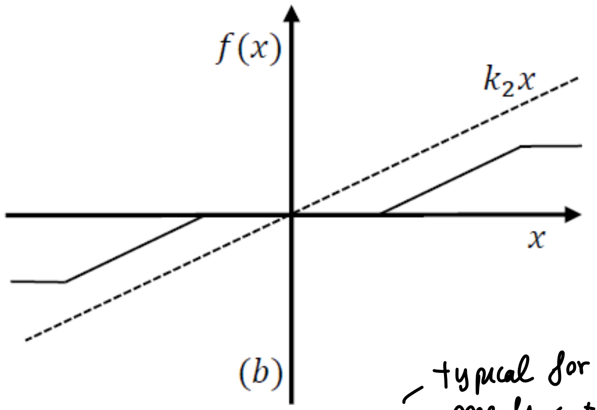
sector NON LINEARITY

sector non lin. for $k_1=0$

for example due to actuator (you can study I/O of the system)

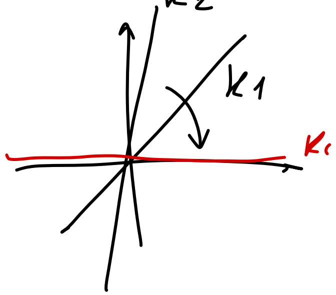


saturation



dead zone

typical for mech syst



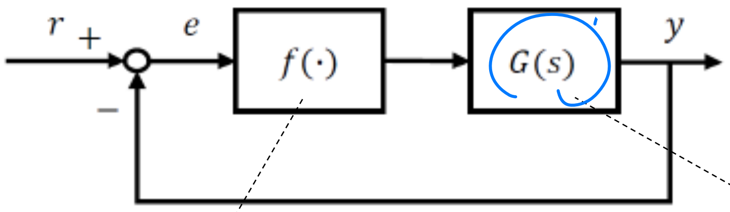
(Saturation with dead zone)

… some exercises …

EXAM June 2019

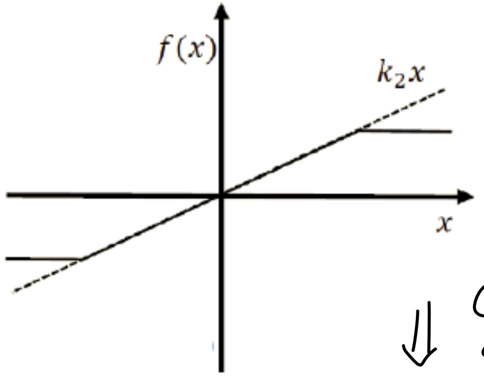
B. Consider the feedback system

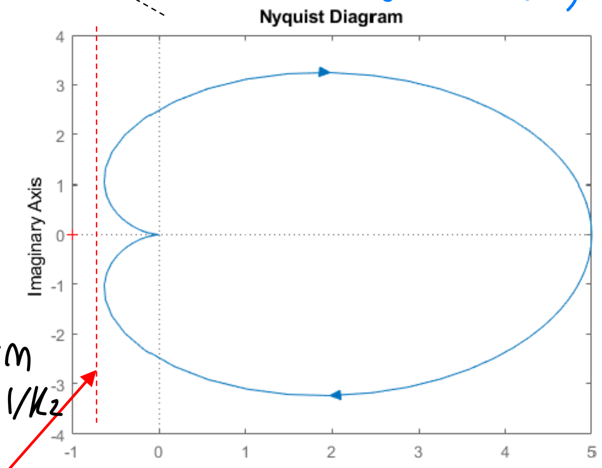
⇓



Where G(s) is the transfer function of an asymptotically stable system with the Nyquist diagram reported below together with the form of the saturation *f(.)*.

⇓ G(s)





⇓ case $k_1 = 0$, we must guarantee polar diag of G remain on the right of vertical line $-1/k_2$

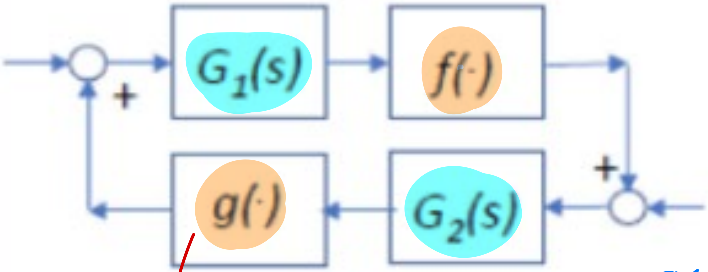
determine (qualitatively) the maximum value of *k2* guaranteeing the Input/Output stability of the system.

solution $\frac{-1}{k_2} \simeq -0.7$

apply the theory properly!

Consider the system feedback system

If G_1 is a Regulator you can use this analysis result for the syntesisy that Regulator



sector non lin

→ asymp stable T.F

where

$$G_1(s) = \frac{a}{Ts+1}, a > 0, T > 0, \; G_2(s) = \frac{1}{s+1}$$

and f, g are sector nonlinearities uniquely defined for any input with

$$k_1 x \le f(x) \le k_2 x$$

limits

$$h_1 x \le g(x) \le h_2 x \qquad k_1, k_2, h_1, h_2 \text{ positive values}$$

What condition guarantees Input/Output stability?

- ○ k2 h2<1
- ○ T k2 h2 <1
- ○ k1 h1<1
- ✗ a k2 h2<1

we can apply small gain Th and Guarantee H_∞ gain of overall loop < 1 $\quad |G_2|_{H_\infty} = 1 \quad |G_1|_{H_\infty} = a$

⇑

{ h_2, k_2 specify sector non lin + look to the H_∞ Gain of G_1, G_2 }