# ADVANCED AND MULTIVARIABLE CONTROL

# February 14, 2020

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**Exercise 1**

Consider the system

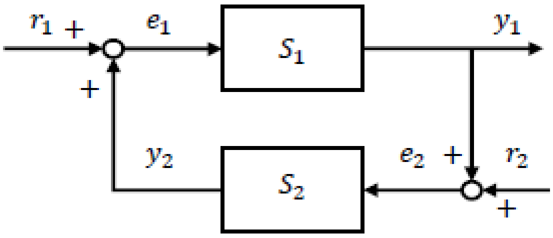
$$\dot{x}_1(\mathrm{t}) = -x_1(\mathrm{t}) + x_1(t)^3 x_2(t)$$
$$\dot{x}_2(\mathrm{t}) = -2x_2(\mathrm{t}) + x_1^2(\mathrm{t}) - x_2^2(\mathrm{t})$$

a. Analyze the stability of the origin with the corresponding linearized system by looking at the corresponding eigenvalues.
b. Analyze the stability of the origin with the corresponding linearized system by solving the corresponding Lyapunov equation *A'P+PA=-Q* (with a proper choice of the matrix Q).
c. With the solution of the Lyapunov equation, define a Lyapunov function for the original nonlinear system and show that the equilibrium is asymptotically stable. Discuss if it is possible to conclude something about the global stability of the origin.

**Exercise 2**

a. For a generic open loop system *y(t)=S(u(t))*, define its gain and the condition for Input/Output stability.

b. For the generic feedback system reported in the following figure, state the small gain theorem.



**Exercise 3**

Consider the following discrete-time system, where *d* is an unknown constant,

$$
\begin{aligned}
x_1(k+1) &= x_2(k) + d \\
x_2(k+1) &= x_1(k) + x_2(k) + u(k) \\
y(k) &= x_1(k)
\end{aligned}
$$

A. Design a reduced order observer for this system in order to estimate the state $x_2$ and the disturbance *d*.
B. Based on the estimated state, compute a state feedback control law which places all closed-loop the eigenvalues in z=0.5.

**Exercise 4**

Given a discrete-time linear system with input *u*, state *x*, output *y*, formulate an MPC control problem where the goal is to track a given constant reference signal $y^o$ subject to constraints on the minimum and maximum values of the input variable. Discuss at least one method to include an integral action in the control law.

***Solution Exercise 1***

a. The linearized model at the origin is

$$
\begin{aligned}
\dot{\delta x_1}(\mathrm{t}) &= -\delta x_1(\mathrm{t}) \\
\dot{\delta x_2}(\mathrm{t}) &= -2\delta x_2(\mathrm{t})
\end{aligned}
$$

With eigenvalues *s=-1,s=-2.*

b. The dynamic matrix of the linearize system is *diag(-1,-2)*. By taking a matrix *Q=diag(-2,-4)* the solution to the Lyapunov equation is *P=I.* Being *P>0* the stability of the origini s (obviously) verified also in this case.
c. Taking *V(x)=0.5x'Px=* $0.5(x_1^2 + x_2^2) > 0$ one has

$$\dot{V}(x) = -x_1^2 + x_1^4 x_2 - x_2 x_1^2 - x_2^3 - 2x_2^2$$

Which is locally <0, since the second order terms dominate. However, the result is only local, since higher order terms have been neglected.

**Solution exercise 2**

See the notes.

***Solution Exercise 3***

By defining the fictitious dynamics *d(k+1)=d(k)* to *d*, the system equations can be written as

$$
\begin{aligned}
x_2(k+1) &= x_2(k) + y(k) + u(k) \\
d(k+1) &= d(k) \\
y(k+1) &= x_2(k) + d(k)
\end{aligned}
$$

Or

$$\begin{bmatrix} x_2(k+1) \\ d(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(k) \\ u(k) \end{bmatrix}$$

output transformation

$$y(k+1) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_2(k) \\ d(k) \end{bmatrix}$$

The observer is

$$\hat{x}_2(k+1) = \hat{x}_2(k) + y(k) + u(k) + l_1\left(y(k+1) - \hat{x}_2(k) - \hat{d}(k)\right)$$

$$\hat{d}(k+1) = \hat{d}(k) + l_2\left(y(k+1) - \hat{x}_2(k) - \hat{d}(k)\right)$$

The gain $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ must be chosen to assign the eigenvalues of $A - LC,\ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

As for the control law, consider the following matrices of the original system

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The feedback matrix

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

And the control law

$$u(k) = -K\hat{x}(k),\quad \hat{x}(k) = \begin{bmatrix} x_1(k) \\ \hat{x}_2(k) \end{bmatrix}\ ,\ \text{eig}(A - BK) \text{ in } 0.5 \rightarrow K = \begin{bmatrix} 1.25 & 0 \end{bmatrix}$$

**Solution exercise 4**

See the notes