6/2014 Solutions

Ex1

$V(x) > 0$, $\dot V(x) = 2(x_1+x_2)(\dot x_1+\dot x_2) + 2x_1^3\dot x_1$

$$\dot V(x) = 2(x_1+x_2)(-\cancel{2x_1}+\cancel{x_2}+\cancel{2x_1}-x_1^3-\cancel{x_2}) + 2x_1^3(-2x_1+x_2)$$

$$= 2(x_1+x_2)(-x_1^3) + 2x_1^3(-2x_1+x_2)$$

$$= -2x_1^4 - \cancel{2x_1^3x_2} - 4x_1^4 + \cancel{2x_1^3x_2}$$

$$= -6x_1^4 \le 0$$ stability but not asymptotic stability

Lemma (La Salle - Krasovsky)

$\dot V(x) = 0$ if and only if $x_1(t) = 0$

from the state equations we have

$$x_2(t) = 0$$

So, the origin is the only asymptotically stable equilibrium.

Ex2

1) $\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = u \end{cases} \longrightarrow A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \ B = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$$J = \int_0^\infty (x'(\tau)Qx(\tau) + u'(\tau)Ru(\tau))\,d\tau$$

where

$R = 1$, $Q = \begin{vmatrix} 9 & 0 \\ 0 & 0 \end{vmatrix} \longrightarrow Q = C_q'C_q, \ C_q = \begin{vmatrix} \sqrt{9} & 0 \end{vmatrix}$

Observability of (A, C_q)

$$M_o = \begin{vmatrix} C_q \\ C_q A \end{vmatrix} = \begin{vmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q} \end{vmatrix} \text{, o.k. for } q \neq 0$$

reachability of (A, B)

$$M_r = \begin{vmatrix} B & AB \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \text{, o.k.}$$

The LQ_∞ control law is stabilizing.

2) For $q=1$ the stationary solution of the control Riccati equation is $P = \begin{vmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{vmatrix}$

Correspondingly $K = R^{-1}B'P$, since $R=1$, is

$$K = \begin{vmatrix} 0 & 1 \end{vmatrix} \begin{vmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{2} \end{vmatrix}$$

and the closed-loop eigenvalues are the solutions of

$$\det\left(sI - (A - BK)\right) = \det\left(\begin{vmatrix} s & 0 \\ 0 & s \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{vmatrix} \right)$$

$$\det \begin{vmatrix} s & -1 \\ 1 & s+\sqrt{2} \end{vmatrix} = s^2 + \sqrt{2}\, s + 1$$

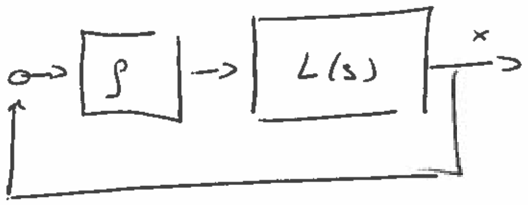
$$\downarrow$$

$$s_{1,2} = -0.707 \mp j0.707$$

3) Loop transfer function

$$L(s) = K(sI - A)^{-1}B = \frac{\sqrt{2}\, s + 1}{s^2}$$

gain margin

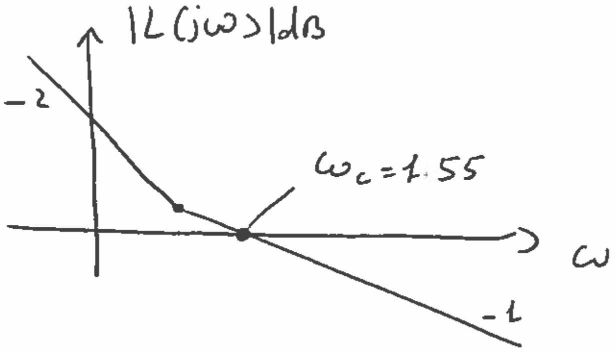


characteristic equation $\quad s^2 + \rho\sqrt{2}\, s + \rho = 0$

with all the roots with $Re(\,) < 0$ for $\rho > 0$

therefore the gain margin is $(0, \infty)$

phase margin



$\varphi_m \simeq 65$

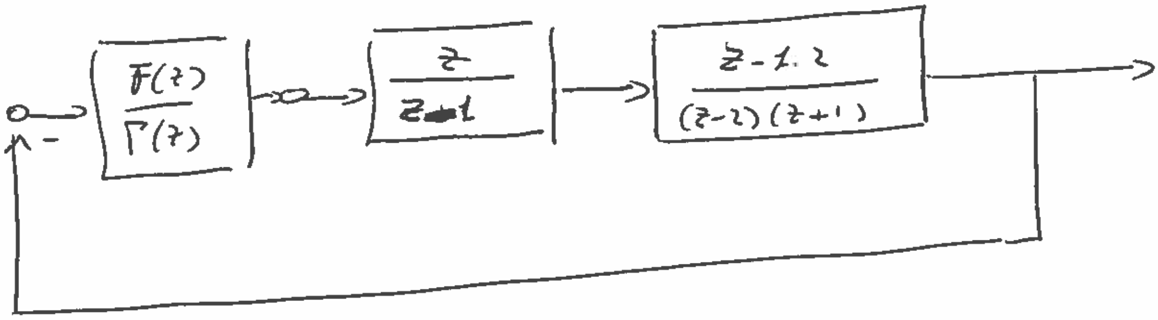
larger that the one guaranteed by the robustness properties of MPC

Ex 3

Condition 1 $\rightarrow$ $m \geq p$

Condition 2 $\rightarrow$ no invariant zeros at the origin

Ex 4



$$A(z) = (z-1)(z+1)(z-2) = z^3 - 2z^2 - z + 2$$

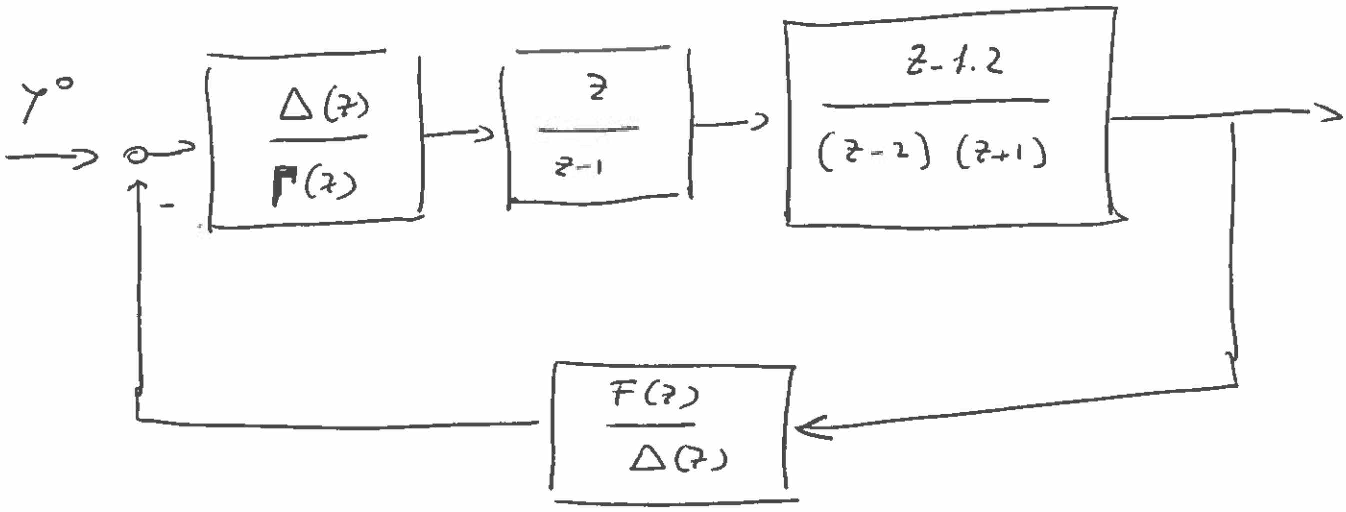
$$B(z) = z(z-1.2) = z^2 - 1.2z$$

$$F(z) = f_2 z^2 + f_1 z + f_0$$

$$\Gamma(z) = \gamma_2 z^2 + \gamma_1 z + \gamma_0$$

$$P(z) = (z-0.5)^5 = z^5 + p_4 z^4 + p_3 z^3 + p_2 z^2 + p_1 z + p_0$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & -1.2 & 1 & 0 \\ 2 & -1 & -2 & 0 & -1.2 & 1 \\ 0 & 2 & -1 & 0 & 0 & -1.2 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} \gamma_2 \\ \gamma_1 \\ \gamma_0 \\ f_2 \\ f_1 \\ f_0 \end{vmatrix} = \begin{vmatrix} 1 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0 \end{vmatrix}$$



$$\Delta(z) = \delta_2 z^2 + \delta_1 z + \delta_0 z^0$$

must be chosen with stable roots $(|\ \ | < 1)$

and with $\Delta(1) = F(1)$ for static

conditions

$\boxed{\text{Ex 5}}$

See the notes