Advanced and Multivariable Control

Norms, gains, small gain theorem

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Norms of vectors

of numbers

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}$$

2 norm

$$|e|_2 = \sqrt{e'e} = \sqrt{\sum_{i=1}^{m} e_i^2}$$

$$|e|_\infty = \max_i |e_i|$$

inf norm

of signals

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_m(t) \end{bmatrix}$$

2-norm, or H_2 norm

$$\|e\|_2 = \sqrt{\int_{-\infty}^{+\infty} (e'(\tau)e(\tau))d\tau}$$

$$\|e\|_\infty = \sup_t \left(\sup_i |e_i(t)| \right)$$

infinity norm, or H_∞ norm

Singular values

The *singular values* of the matrix $\Phi \in C^{p,m}$ are the $k = min(p,m)$ largest roots of the eigenvalues of $\Phi^*\Phi$ or of $\Phi\Phi^*$

$$\begin{aligned}
\sigma_i(\Phi) &: = \sqrt{\lambda_i(\Phi^*\Phi)} = \sqrt{\lambda_i(\Phi\Phi^*)}, \quad m = p \\
\sigma_i(\Phi) &: = \sqrt{\lambda_i(\Phi^*\Phi)}, \quad m > p \\
\sigma_i(\Phi) &: = \sqrt{\lambda_i(\Phi\Phi^*)}, \quad m < p
\end{aligned}$$

Singular value decomposition

Any matrix $\Phi \in C^{p,m}$ can be partitioned with the singular value decomposition

$$\Phi = U \Sigma V^*$$

where the matrices $U \in C^{p \times p}$ and $V \in C^{m \times m}$ are unitary, while the matrix Σ is defined by

$$\begin{aligned} \Sigma &= \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \in R^{p \times m}, \quad p \geq m \\ \Sigma &= \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} \in R^{p \times m}, \quad p \leq m \end{aligned}$$

where

$$\Sigma_1 = diag\left\{\sigma_1, \sigma_2, \ldots, \sigma_k\right\}; \quad k = \min(p, m)$$

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k$$

Unitary matrix $\quad U^* = U^{-1} \quad , \quad |\lambda_i(U)| = 1, \forall i \quad , \quad \sigma_i(U) = 1, \forall i$

Minimum and maximum singular values

Letting

$$\Phi = U \Sigma V^*$$

with

$$\begin{aligned} \Sigma &= \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \in R^{p \times m}, \quad p \geq m \\ \Sigma &= \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} \in R^{p \times m}, \quad p \leq m \end{aligned}$$

where

$$\Sigma_1 = diag\left\{\sigma_1, \sigma_2, \ldots, \sigma_k\right\}; \quad k = \min(p, m)$$

the maximum singular value is $\bar{\sigma} = \sigma_1$ and the minimum singular value is

$$\underline{\sigma} \equiv \begin{cases} \sigma_m & if \quad p \geq m \\ 0 & if \quad p < m \end{cases}$$

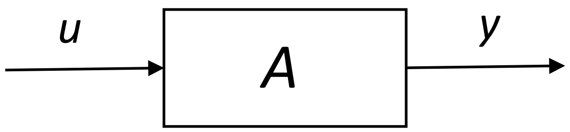
Induced *p-norm* of a matrix

$$\|A\|_{ip} = \sup_{d \neq 0} \frac{\|Ad\|_p}{\|d\|_p}$$

Induced 2-norm

$$\|A\|_{i2} = \sup_{d \neq 0} \frac{\|Ad\|_2}{\|d\|_2} = \bar{\sigma}(A)$$

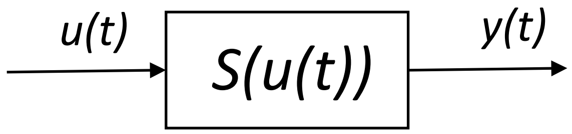
Norm of a «map» *A*



$$\sup_{u \neq 0} \frac{|y{=}Au|_2}{|u|_2} = \sqrt{\lambda_{\max}(A'A)} = \bar{\sigma}(A)$$

gain of the map

Norm of systems



not necessarily linear

Let L_2 be the space of functions, null for $t < 0$, and whose absolute value raised to the 2^{nd} power has finite integral, that is, if $u \in L_2$,

$$\|u\|_2 = \sqrt{\int_0^{+\infty} (u'(\tau)u(\tau))d\tau} < +\infty$$

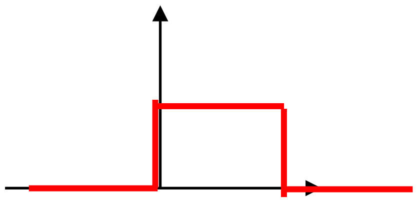
Gain γ of *S*

$$\gamma = \|S\|_\infty = \sup_{u \in L_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in L_2} \frac{\|S(u)\|_2}{\|u\|_2} \quad \longleftrightarrow \quad \|y\|_2 \leq \|S\|_\infty \|u\|_2 \quad , \quad \forall u \in L_2$$

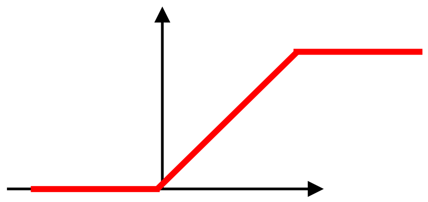
Example of linear system – the integrator

$$y(t) = \int_0^{+\infty} u(\tau) d\tau, \quad Y(s) = \frac{1}{s} U(s)$$

Input *u = sca (t)*

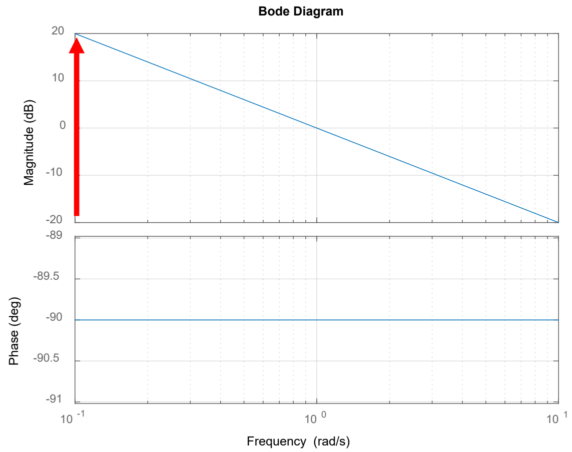


$$u(t) = \begin{cases} 1 & , \quad 0 < t < 1 \\ 0 & , \quad t \geq 1 \end{cases} \qquad \|u\|_2 = \sqrt{\int_0^{+\infty} u'(\tau) u(\tau) d\tau} = 1 \in L_2$$



$$\|y\|_2 = +\infty$$

infinite gain



Example – SISO asymptotically stable linear system

$$Y(s) = G(s)U(s) \quad , \quad Y(j\omega) = G(j\omega)U(j\omega)$$

Parseval theorem

$$\|y\|_2^2 = \int_{-\infty}^{+\infty} y^2(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |Y(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 |U(j\omega)|^2 d\omega$$

$$\text{if } |G(j\omega)| \leq K \text{ with } |G(j\omega)| = K \text{ for some } \omega$$

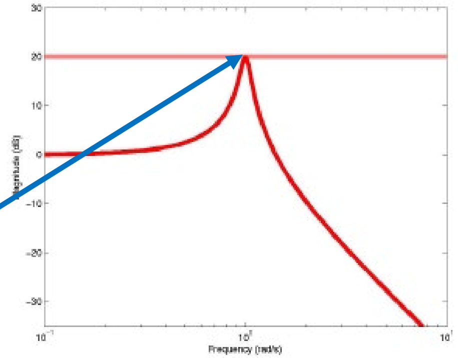
$$\downarrow$$

$$\|y\|_2^2 \leq K^2 \frac{1}{2\pi}\int_{-\infty}^{+\infty} |U(j\omega)|^2 d\omega = K^2 \int_{-\infty}^{+\infty} u^2(\tau)d\tau \leq K^2 \|u\|_2^2$$

$$\downarrow$$

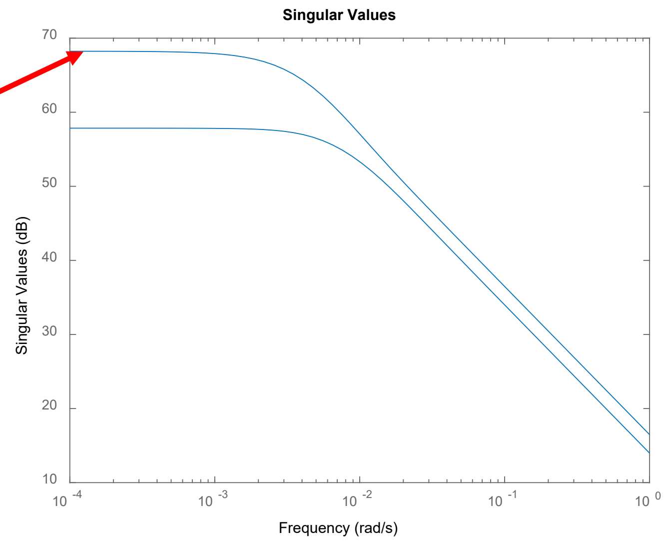
$$\|G\|_\infty = \sup_\omega |G(j\omega)| = K$$

gain: supremum of the modulus of the frequency response of $G(s)$



Extension to MIMO asymptotically stable linear system

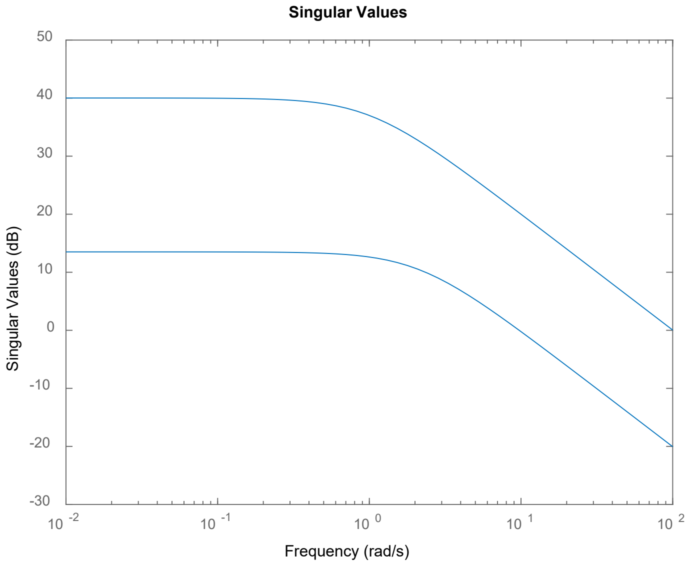
$$\gamma = \|G\|_{\infty} = \sup_{\omega} \bar{\sigma}\left(G(j\omega)\right)$$



Input – Output (I/O) stability

A system $y = S(u)$ is input-output (I/O) stable if it has finite gain

$$G(s) = \begin{bmatrix} \frac{100}{(s+1)} & \frac{10}{(s+1)(s+2)} \\ \frac{10}{(s+2)} & \frac{10}{(s+2)} \end{bmatrix}$$




```
g11=tf(100,[1 1]);
g12=tf(10,conv([1 1],[1 2]));
g21=tf(10,[1 2]);
g22=tf(10,[1 2]);
G1=[g11 g12;g21 g22]
sigma(G1)
```

Different definitions of gain for asymptotically stable linear systems

- *gain*, or *infinite-norm gain*: $\gamma = \|G\|_\infty$
- *gain at a given frequency* ω

$$\frac{\|Y(j\omega)\|_2}{\|U(j\omega)\|_2} = \frac{\|G(j\omega)U(j\omega)\|_2}{\|U(j\omega)\|_2}$$

$SISO$ systems: the gain at a given frequency ω is $|G(j\omega)|$
$MIMO$ systems:

$$\underline{\sigma}(G(j\omega)) \leq \frac{\|G(j\omega)U(j\omega)\|_2}{\|U(j\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

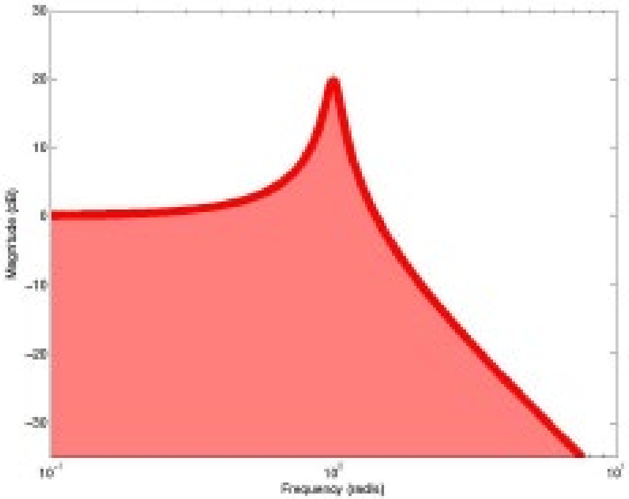
It depends on the applied input

- *static gain*: gain at $\omega = 0$.

2-norm gain for asymptotically stable, strictly proper, linear systems

SISO

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega}$$

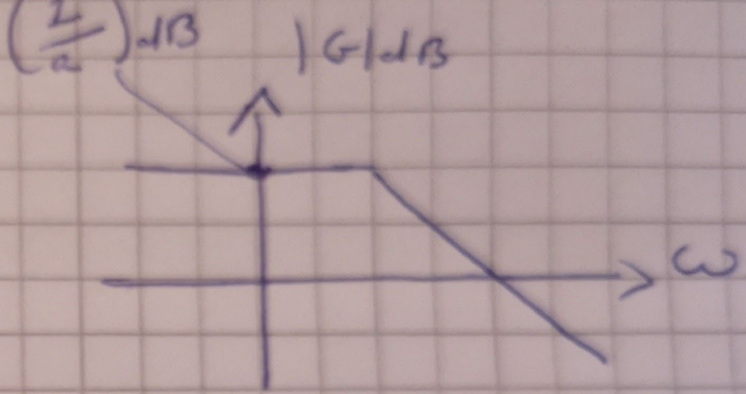


MIMO

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} tr(G(j\omega)G'(-j\omega))d\omega}$$

Example H_2 - H_∞

$$G(s) = \frac{1}{s+a}, \quad a>0$$

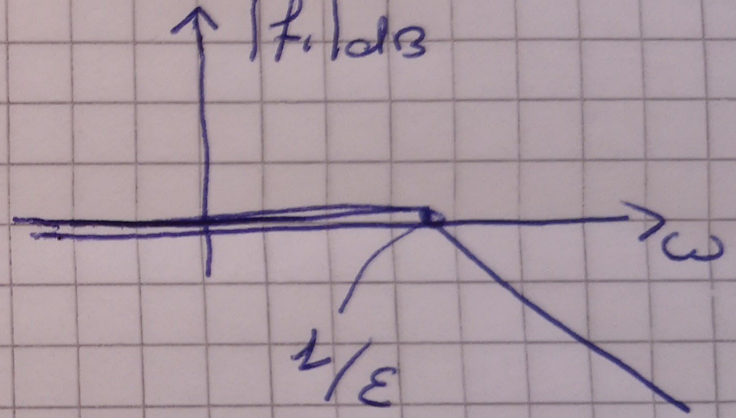


$$\|G\|_\infty = \frac{1}{a}$$

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega} = \sqrt{\frac{1}{2a}}$$

$\boxed{\text{Example } H_2 - H_\infty}$

$$f(s) = \frac{1}{\varepsilon s + 1}$$

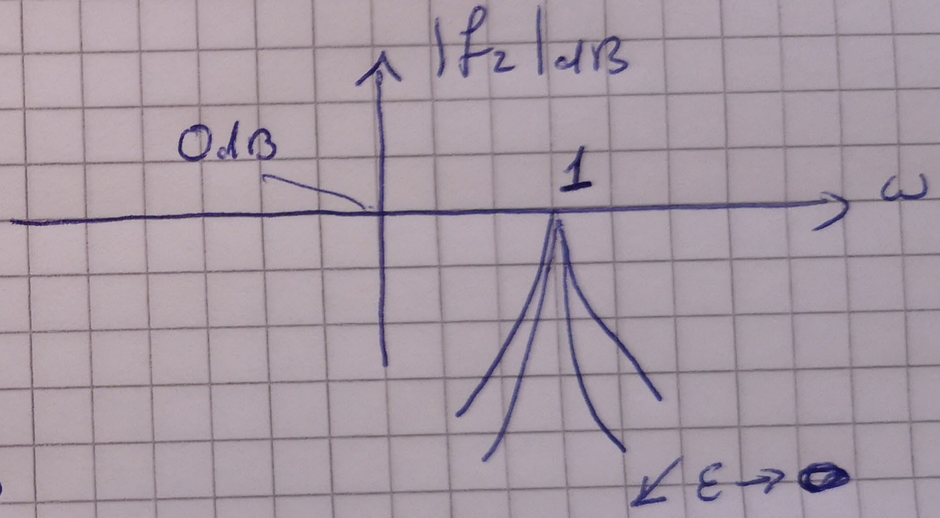


$\varepsilon \to 0$

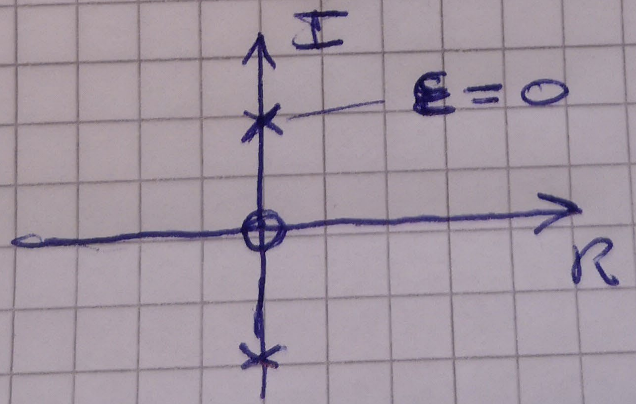
$$\| f_1 \|_\infty = 1 \quad , \quad \| f_1 \|_2 = \infty$$

Example $H_2 - H_\infty$

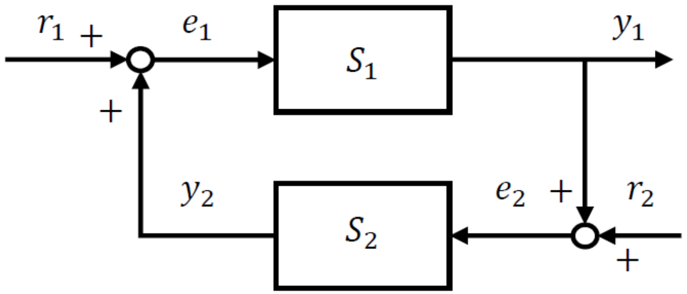
$$f_2(s) = \frac{\varepsilon s}{s^2 + \varepsilon s + 1}$$



$$\|f_2\|_\infty = 1 \;, \quad \|f_2\|_2 = 0$$



Small gain theorem *(one of the most useful tools for the analysis of nonlinear feedback systems)*



Assume that S_1 and S_2 are I/O stable systems. Then

$$\boxed{\text{the feedback system is I/O stable if } \|S_1\|_\infty \|S_2\|_\infty < 1}$$

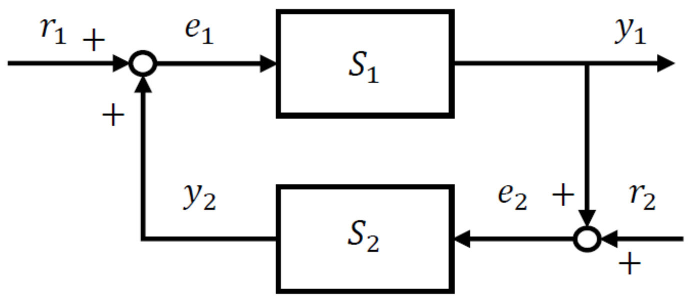
If S_1 and S_2 are linear the condition is

$$\|S_1 S_2\|_\infty < 1$$

less restrictive

only sufficient conditions

Proof



$$e_1 = r_1 + S_2(r_2 + y_1), \quad y_1 = S_1(e_1)$$

Therefore,

$$\|e_1\|_2 \le \|r_1\|_2 + \|S_2\|_\infty \left(\|r_2\|_2 + \|S_1\|_\infty \|e_1\|_2\right)$$

It follows that

$$\|e_1\|_2 \le \frac{\|r_1\|_2 + \|S_2\|_\infty \|r_2\|_2}{1 - \|S_1\|_\infty \|S_2\|_\infty}$$

and, with similar developments,

$$\|e_2\|_2 \le \frac{\|r_2\|_2 + \|S_1\|_\infty \|r_1\|_2}{1 - \|S_1\|_\infty \|S_2\|_\infty}$$

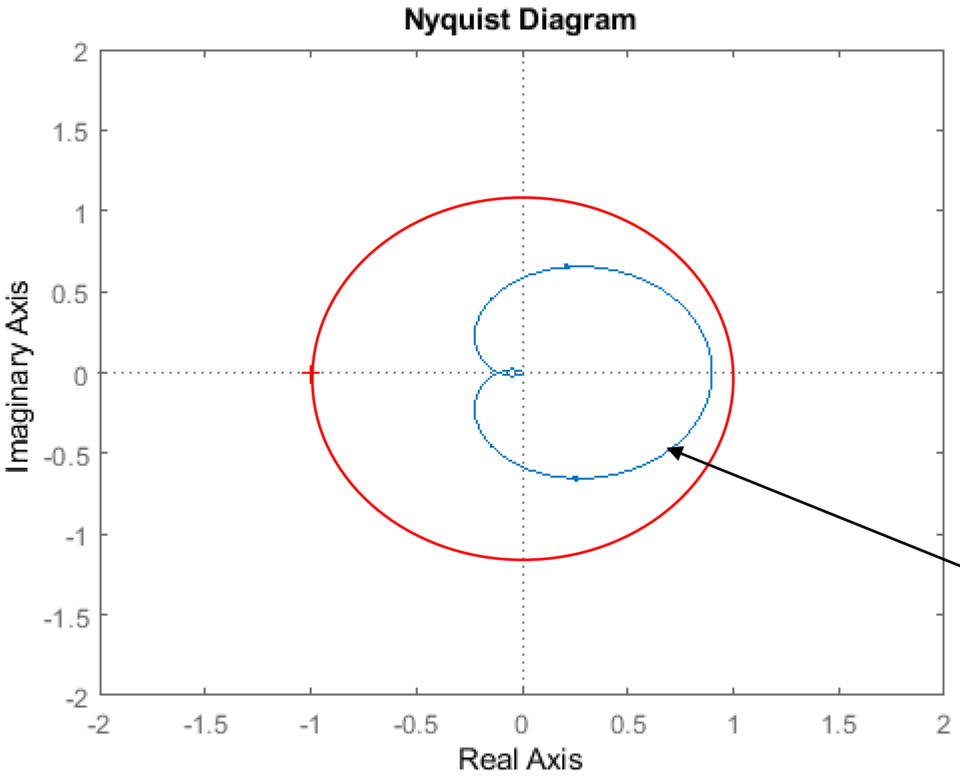
Then, in view of the previous assumptions, the gain is finite

SISO Linear systems

Nyquist criterion:

$N=0$

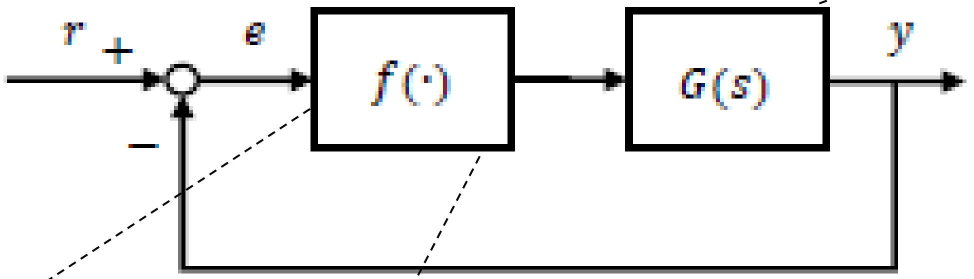
$P=0$

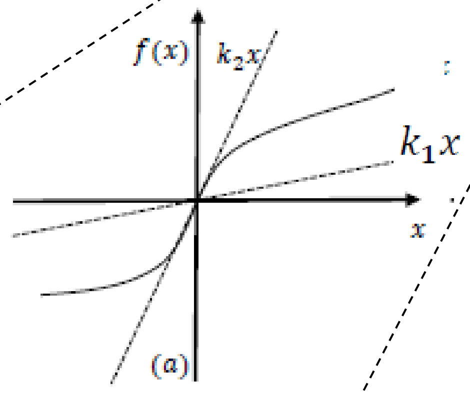


$S_1(j\omega)S_2(j\omega)$

Stability of feedback systems with static sector nonlinearity

asymptotically stable SISO system





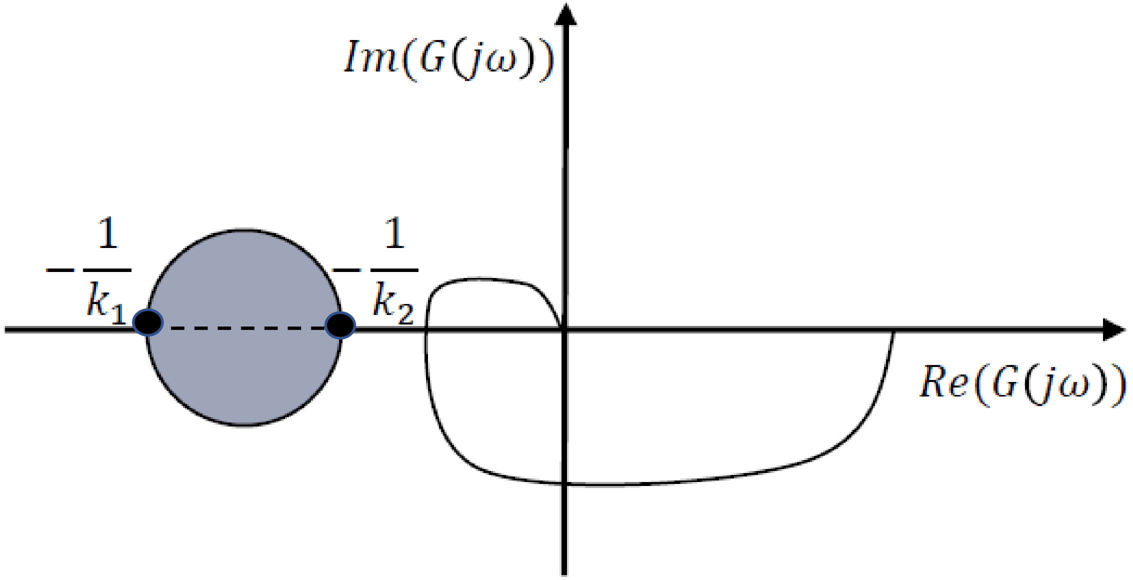
nonlinear, continuous function uniquely defined for any input

$f(0) = 0$ and $k_1 e \leq f(e) \leq k_2 e$

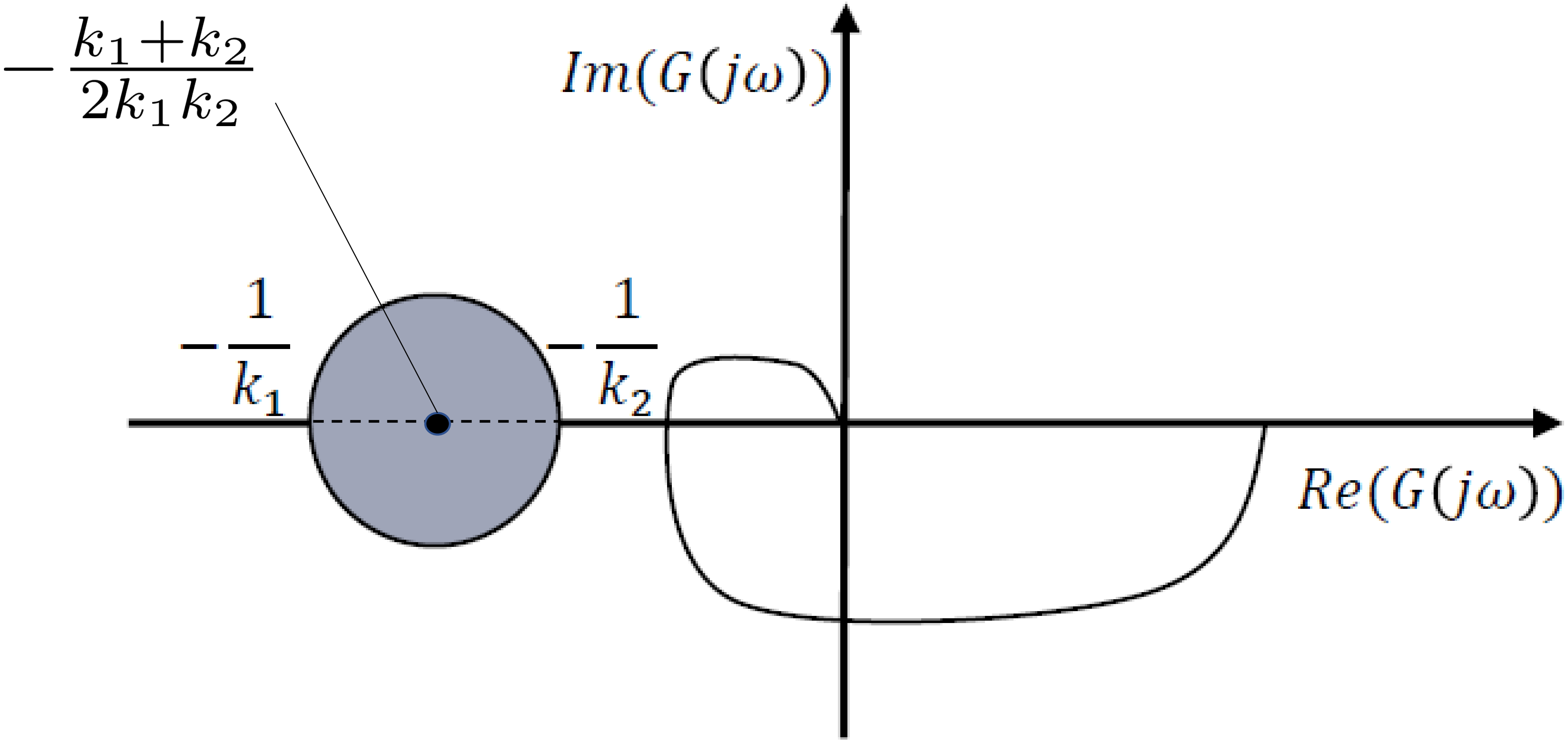
In view of the small gain theorem, I/O stability of the feedback system is guaranteed if

$$k_2 \sup_{\omega} |G(j\omega)| < 1$$

A less stringent condition : THE CIRCLE CRITERION (*proof in the textbook*)



The closed-loop system is I/O stable if the Nyquist diagram of $G(s)$ does not encompass, intersect, or touch the circle with diameter given by the segment $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ and located on the x axis



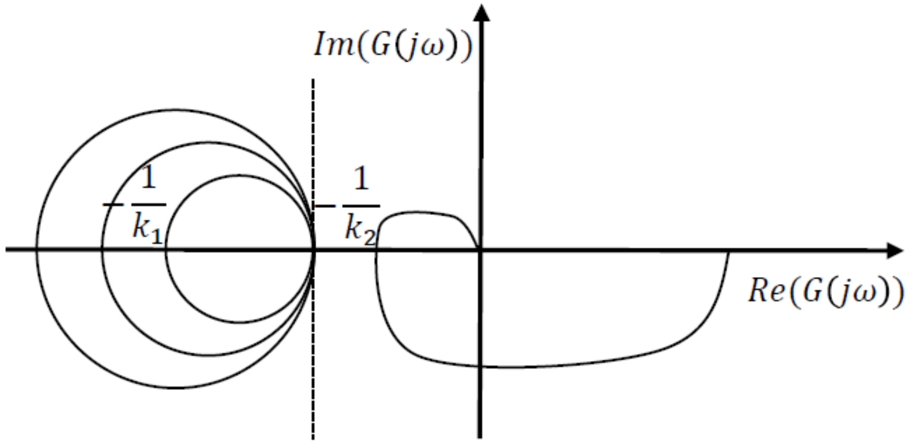
$k_1 = k_2 = 1 - - >$Nyquist criterion (If condition)

THE CIRCLE CRITERION *comments and interpretations*

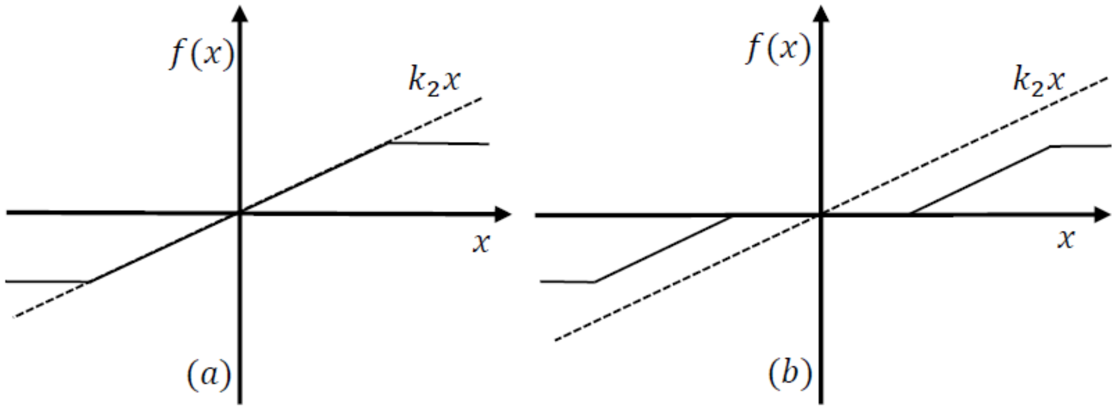
Only a sufficient condition

Can be generalized to non asymptotically stable *G(s)*

When $k_1 \rightarrow 0$ the circle becomes a vertical line passing through $-1/k_2$



Interesting cases, widely used in practical applications

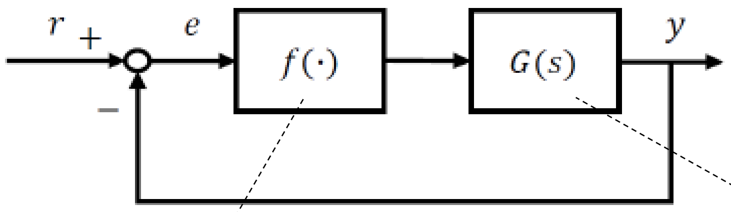


saturation *dead zone*

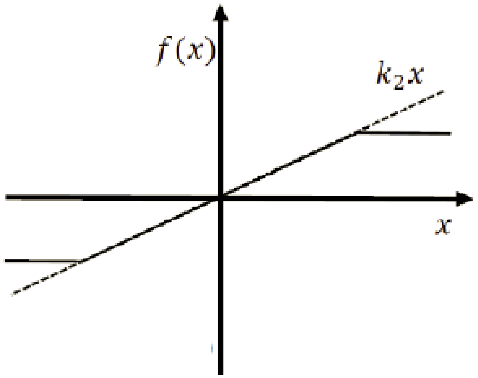
… some exercises …

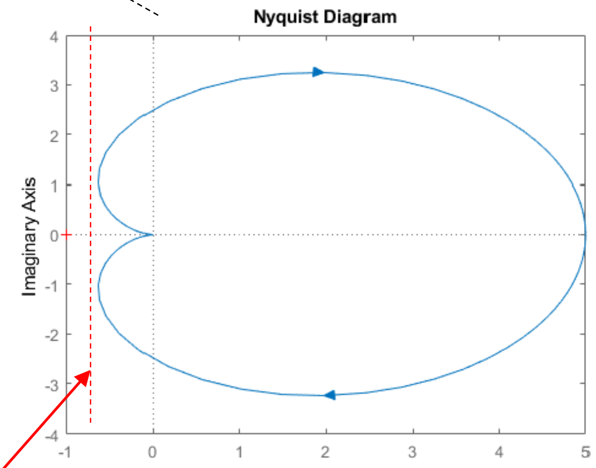
EXAM June 2019

B. Consider the feedback system



Where G(s) is the transfer function of an asymptotically stable system with the Nyquist diagram reported below together with the form of the saturation *f(.)*.

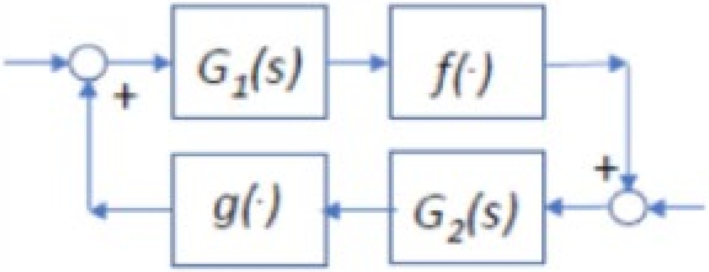




determine (qualitatively) the maximum value of *k2* guaranteeing the Input/Output stability of the system.

solution $\frac{-1}{k_2} \simeq -0.7$

Consider the system



where
$$G_1(s) = \frac{a}{Ts+1}, a > 0, T > 0,\ G_2(s) = \frac{1}{s+1}$$

and *f*, *g* are sector nonlinearities uniquely defined for any input with

$$k_1 x \le f(x) \le k_2 x$$
$$h_1 x \le g(x) \le h_2 x \qquad k_1, k_2, h_1, h_2 \text{ positive values}$$

What condition guarantees Input/Output stability?

○ k2 h2<1

○ T k2 h2 <1

○ k1 h1<1

○ a k2 h2<1