# LYAPUNOV

## FOR STABILITY AND CONTROL OF

## NONLINEAR SYSTEMS

# Advanced and Multivariable Control

## *Lyapunov Stability*

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designing controllers, in SW testing

because f don't depend explicitly on time, there is "t" but not explicit!

(finite number of states)

Dynamic, finite dimensional, time invariant system:

generic NON LIN

$$\dot{x}(t) = f\left(x(t), u(t)\right), \quad x(t_0) = x_0 \quad x \in R^n \quad u \in R^m$$

NON LINEAR SYST

initial value

(STATE)

states number

input number

finite dimensional syst since limited dimension of I/O

$$\dot{x}(t) = f\left(x(t)\right)$$ autonomous

if f don't depend on u(t) → no outside forcing

$$\dot{x}(t) = Ax(t) + Bu(t)$$ linear → when f is linear on states and input

$$\left\{ \begin{array}{l} A \in \mathbb{R}^{n \times n} \text{ matrix} \\ B \in \mathbb{R}^{n \times m} \end{array} \right\}$$

equilibrium pair/point $(\bar{x}, \bar{u})$ → $f(\bar{x}, \bar{u}) = 0$

↳ $\dot{x} = 0$ → $x$ = constant ⇒ to find equilibrium we fix $\bar{u}$ and solve this equation to find $\bar{x}$

$\dot{x} = 0$ (PARTICULAR case)

$$\boxed{A\bar{x} = -B\bar{u}}$$

linear

set of eq to solve

otherwise fixing $\bar{x}$ and finding $\bar{u}$

n non lin eq in n+m unkn.

IF A non-singular → unique equilibrium for constant $\bar{u}$

$\bar{x} = -A^{-1}B\bar{u}$

A singular (null eig values)

infinite solutions ($\infty$ eq points)

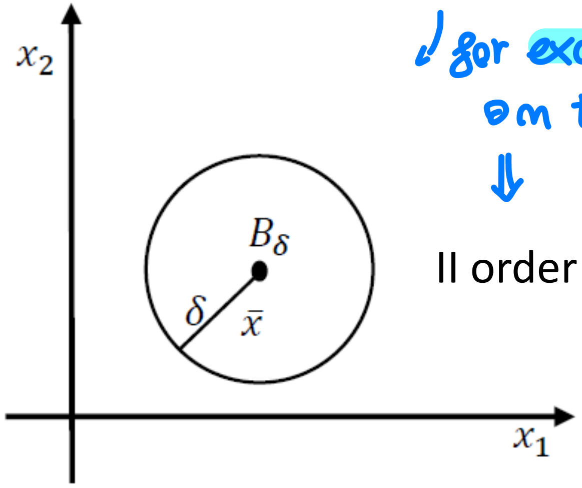
no solutions ($\emptyset$ eq. points)

The equilibrium $\bar{x}$ is *isolated* if there exists $\delta > 0$ such that there does not exist any other equilibrium in (No other equilibrium near $\bar{x}$)

$$B_\delta\left(\bar{x}, \delta\right) := \{x : \|x - \bar{x}\| \le \delta\}$$

for example in a 2x2 syst (2nd ORD) using 2-norm on the definition

$\Downarrow$




II order system, norm-2 $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

($\delta$ such that no equilibrium point in the neighbor of $\bar{x}$ with distance less than $\delta$)

fixed parameter of the syst.

real only variable

single variable function

For any fixed $\bar{u}$, the system takes the form $\dot{x}(t) = f\left(x(t), \bar{u}\right) = \varphi\left(x(t)\right)$

If the equilibrium is $\bar{x}$, set $x(t) = \bar{x} + \delta x(t)$ and write the system as $\delta\dot{x}(t) = \varphi\left(\bar{x} + \delta x(t)\right)$ with equilibrium at the origin (@ $\delta x = 0$)

$\dot{x} = f(x, \bar{u}) \qquad x = \bar{x} + \delta x \Rightarrow \delta\dot{x}(t) = \varphi(\bar{x} + \delta x(t)) = \tilde{\varphi}(\delta x)$

## Stability

$(\bar{x}, \bar{u})$ equilibrium point

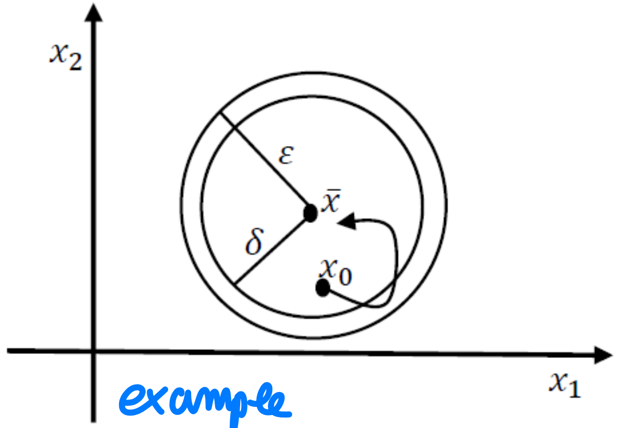
taking $\bar{x}$ and an $\varepsilon$ radius around $\bar{x}$, exist $\delta$ radius such that if $x_0$ inside $\delta$ the trajectory remain inside the $\varepsilon$ radius ball

The equilibrium $\boxed{\bar{x} \text{ is stable}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all the initial states $x_0$ satisfying

$$\|x_0 - \bar{x}\| \leq \delta$$

it holds that

$$\|x(t) - \bar{x}\| \leq \varepsilon, \quad \forall t \geq 0$$




example II ORD syst

{ starting in a neighbour $\delta$ of equilibrium point you remain inside $\varepsilon$ }

MORE !

## Asymptotic stability

holds previous request

If $\bar{x}$ is a stable equilibrium and, in addition,

( if asymp comes back to equilibrium $\bar{x}$ )

$$\lim_{t \to \infty} \|x(t) - \bar{x}\| = 0$$

then $\boxed{\bar{x} \text{ is an } \textit{asymptotically stable}}$ equilibrium.

NON Global result

(obs) ⇒ linear syst: stability of syst! / NON Lin syst: stability of equilibrium point, NOT all syst

Asymptotic stability is a *local property*, since $\delta$ can be very small. The equilibrium $\boxed{\bar{x} \text{ is } \textit{globally (asymptotically) stable}}$ if it is asymptotically stable for any $x_0 \in R^n$

"locally" when $\delta$ is small, $x_0$ must stay near $\bar{x}$

↳ global if asymp stable $\forall x_0$, you can move far from $\bar{x}$

- $\bar{x}$ can be locally stable when $\delta$ limited
- globally if valid $\forall \delta$ so $\forall$ initial perturbation near $\bar{x}$

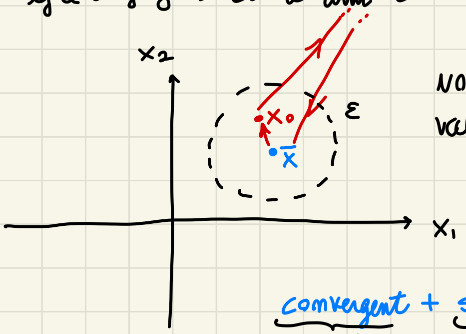
↓

When CONTROLLING syst. the equilibrium point to stay can have a small region $\delta$ of attraction → trajectory for if too perturb

⇓

the "size of stability", how far can I moove from the stability point is also important

- if a traj goes to $\infty$ and than comes back to equilibrium




NOT stable, the theory is NOT valid ⇒ CONVERGENCE to equilibrium but NOT stability

$\underbrace{\text{convergent}}_{\text{come back to eq.}}$ + $\underbrace{\text{stable}}_{\text{remain near equilibrium}}$ → asymp stable

When studying a NON LIN syst → design a controller around an equilibrium point ⇒ you want Asymp stability of control + possible motion near equilibrium

if locally asymp. stab

↓

"global stability" when you want a good behav.

↳ (equilibrium holds also for $x_0$ far from $\bar{x}$)

NOT only local but also find the region of attraction (≃ size of $\delta$)

**Linear systems** → stability is a property of the system

Just compute eig values to check stability

the system is asymptotically stable **if and only if** all the eigenvalues of A have negative real part

**Nonlinear systems** → stability is a property of the equilibrium → you can have different stability for different equilibrium points

to check stability, study LINEARIZED syst when possible

Consider $\dot{x}(t) = f\left(x(t), u(t)\right)$   $f \in C^1, \quad f(\bar{x}, \bar{u}) = 0$

Lyapunov theory possible if $f \in C^1$, sometimes you cannot use linearization!

⇒ you can conclude about stability But NOT about Region of stability! NO sufficent INFO about quality of equilibrium

and let
$$\begin{cases} x(t) = \bar{x} + \delta x(t) \\ u(t) = \bar{u} + \delta u(t) \end{cases}$$

perturbed variable around equilibrium

The linearized model is

$$\delta \dot{x}(t) = A\delta x(t) + B\delta u(t)$$

$$\delta\dot{x} = f(\bar{x}+\delta x, \bar{u}+\delta u) = \dots$$
$$\cong A\,\delta x + B\,\delta u$$
standard linearization

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}, u=\bar{u}}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x}, u=\bar{u}}$$

study linearized model ⇒

from linearized syst!

(while on linear case is necessary + sufficient)

**Theorem**

only sufficient!

- if all the eigenvalues of $A$ have negative real part, then the equilibrium $(\bar{x}, \bar{u})$ is asymptotically stable;
- if at least one eigenvalue of $A$ has positive real part, then the equilibrium $(\bar{x}, \bar{u})$ is unstable;
- if all the eigenvalues of $A$ have negative or null real part, no conclusion can be drawn on the stability of the equilibrium from the analysis of the linearized system. ← sufficient condition, anything can be concluded ⇒ different approaches needed!

(to conclude something about attraction region)

usefull pratically, but sometimes NOT enough

**Problems**

- What to do in the case of nondifferentiable functions $f$?
- No information about the stability region ($\delta$ in the definition of stability)

($\delta$ could be very small, from INFO about linearized system, you don't know anything about region of attraction)

**Example 1** (I ORD, NON lin, syst)

eq. $(\dot{x}=0)$

$\pm\bar{x}=0$

fixing

$\dot{x}(t)=x(t)u(t)$ , $u=\bar{u}=\mp 1$ , → in any case $\bar{x}=0$ is an equilibrium, but …

eq.

(already linear)

eig = -1

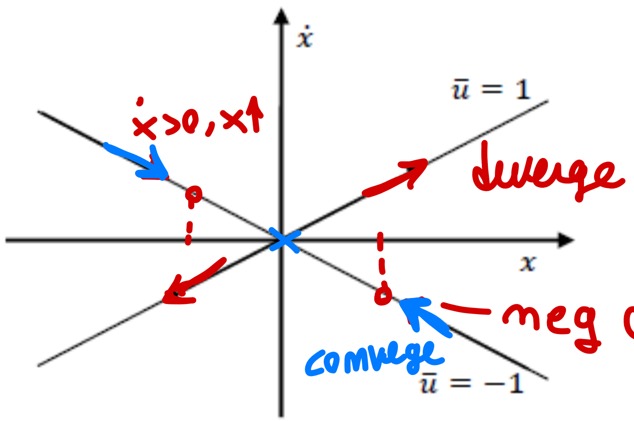
$\bar{u}=-1$ → $\dot{x}(t)=-x(t)$ → negative eigenvalue, asymptotically stable equilibrium

eig = +1

$\bar{u}=1$ → $\dot{x}(t)=x(t)$ → positive eigenvalue, unstable equilibrium

another stability check. only for I ORD! from syst equation, you draw $\dot{x}(x)$ with $\bar{u}=\pm 1$

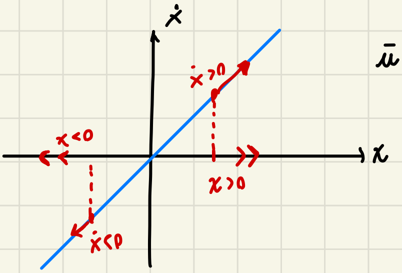
Another way to study the system (for first order systems)




Why for $\bar{u}=-1$ the origin is a **globally asymptotically stable** equilibrium?

neg derivative $\dot{x}$ tends to come back $x\downarrow$

look on $\dot{x}$ to understand where $x$ tends to go,




$\bar{u}=1 \rightarrow$ mooves away from equilibrium

**Example 2** ( NON LIN syst , automomus syst, $u=0$ ) (2 eq. points)

$$\dot{x}(t) = -x(t) + x^{2}(t) \quad \rightarrow \quad \bar{x}=0 \quad \text{is an equilibrium}$$

$\dot{x}=0 \Rightarrow -x+x^2=0 \Leftrightarrow x(-1+x)=0 \quad \begin{cases} \bar{x}=0 \\ \bar{x}=1 \end{cases}$

$\Downarrow \ \bar{x}=0$

can be Globally one? NO if more than 1 eq never globally eq!

**Linearized model**

$$\delta\dot{x}(t) = -\delta x(t) \quad \rightarrow \quad \text{asymptotically stable equilibrium}$$

eig = -1

$\delta\dot{x} = -\delta x + \cancel{2\bar{x}\delta x}$ , $\bar{x}=0$

you can study also $\bar{x}=1$ in this way or different approach

{ to understand also region of attraction }

*What about the* ***region of attraction****?*

↓ solution of linearized eq

Analytical solution

$$x(t) = \frac{e^{-t}x_0}{1-x_0+e^{-t}x_0}$$

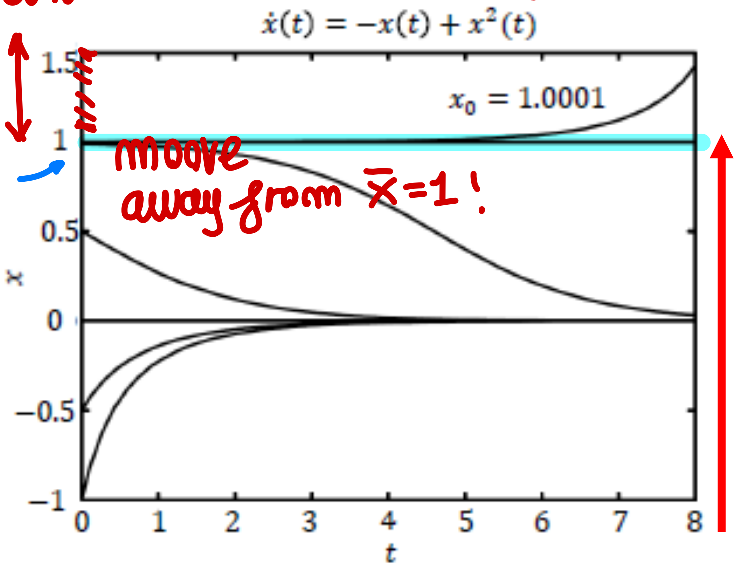
↳ trajectory of syst

(*)

while $\forall x_0>1$ goes away! unstable!

infer this region

equilibrium point when $x_0=1$ remain here

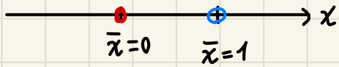



*Region of attraction*
*(-inf,1)*

if $x_0 \in (-\infty, 1)$ we have traj asymp moove to 0
↳ $\bar{x}=0$ is asymp stable equilibrium

remain here forever, never goes back to $0$

$x_0 = 1$




NO GLOBAL equilibrium stability when more than one!

(*) $\bar{x} = 1$ unstable does NOT means each perturbation goes to $\infty$, some comes back to $\bar{x} = 0$, JUST the stability on $\bar{x} = 1$ don't Hold !

NOT use the $\dot{x}(x)$ in more ord syst... solve by sw with trajectories

## II order systems – the phase (or state) plane

→ to study trajectory around equilibrium

Interesting to study the *trajectories* (evolution) of the system in the plane $(x_1,x_2)$

{ Pendulum trajectory }

∀ point ∈ $(x_1,x_2)$ plane compute & plot state evolution

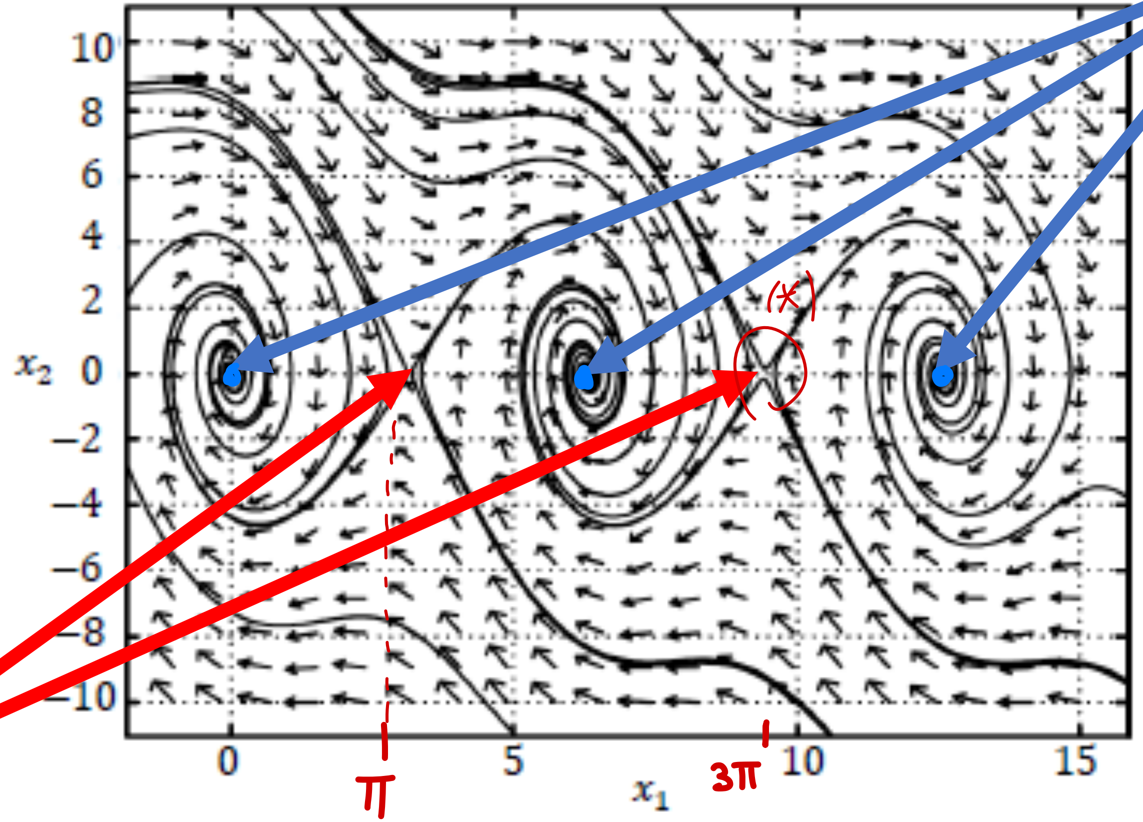
(also giving idea of region of attraction)

lot information here! about syst characteristics

because more equilibria points

***Asymptotically stable equilibria (but NOT globally stable)***

traj come back to initial equilibrium




locally traj on non lin syst is like the linear syst

↓

locally ≃ linear traj

This allows one to have an idea of the region of attraction

(*) unstable focus

***Unstable equilibria***

system goes away here

↑ move away ∀ perturbation near this eq point

Figure 1.10: Pendulum - phase portrait $(M = L = k = 1)$.

(Matlab) pplane (≠ version MATLAB)

↓

SW to define the eq of the non lin system to plot the trajectory → obtaining ∀ initial point $\bar{x}$ the traj

⇓

\+ check for equilibrium → analyze eq. and region of attraction

qualitative conclusion about state traj

**Linear systems** → qualitative traj behaviours

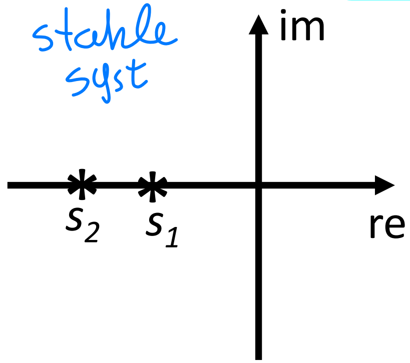
II ORD syst traj depends on eig values position, the behav changes after a trasformation $\tilde{A}$ but similar

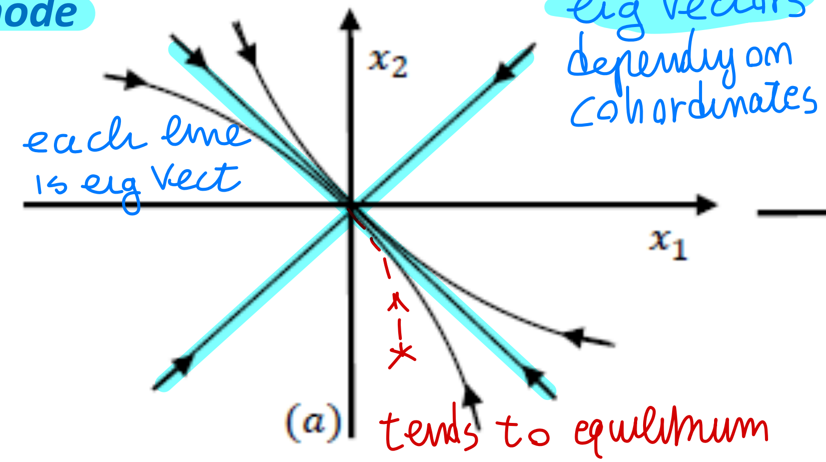
Easy to characterize the form of the state trajectories as functions of the eigenvalues

eig values depends only on the state matrix, NOT on equilibrium

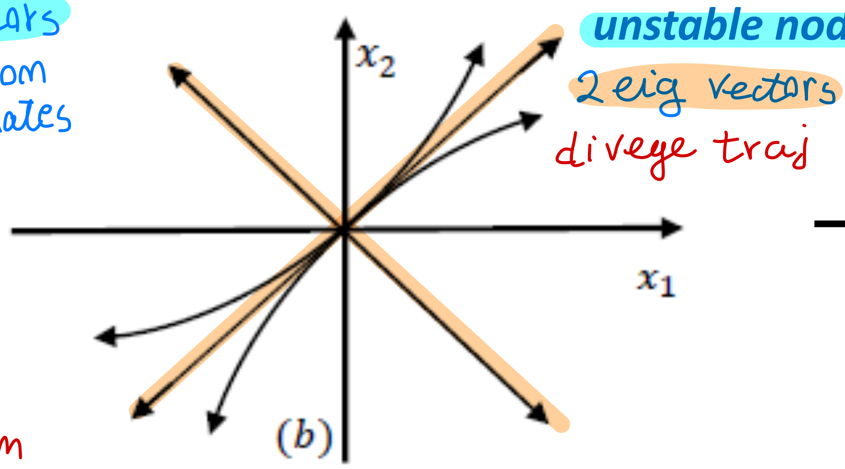
**stable node**

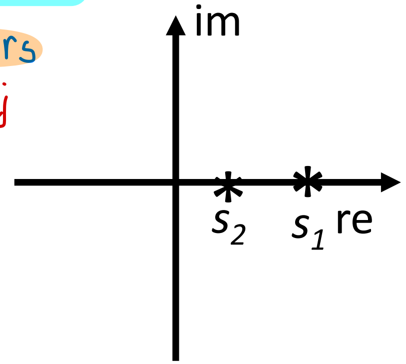
stable syst



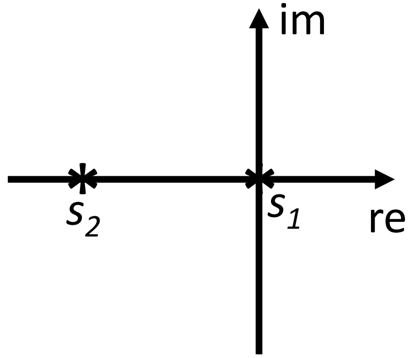



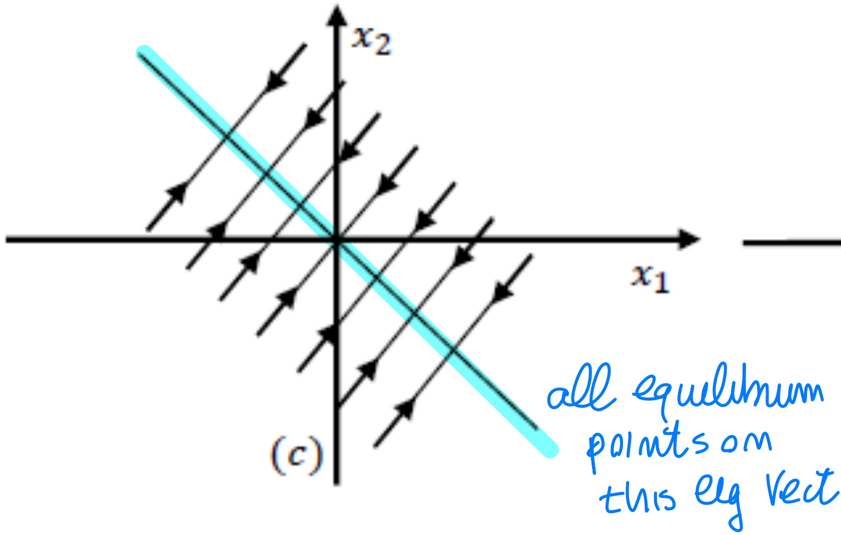

**unstable node**

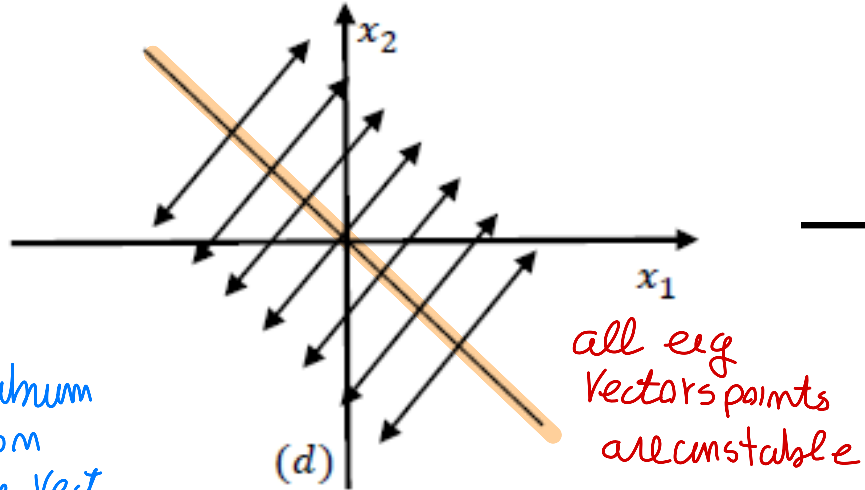


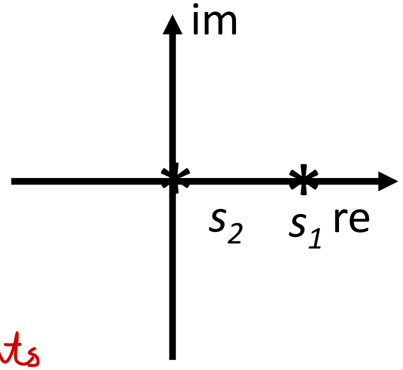




simply stable



$\dot{x} = A x$ $\Rightarrow$ trasformation $\tilde{x} = T x \Rightarrow x = T^{-1} \tilde{x}$

↓ eig depends on A

$$T^{-1} \dot{\tilde{x}} = A T^{-1} \tilde{x}$$

$$\dot{\tilde{x}} = T A T^{-1} \tilde{x}$$

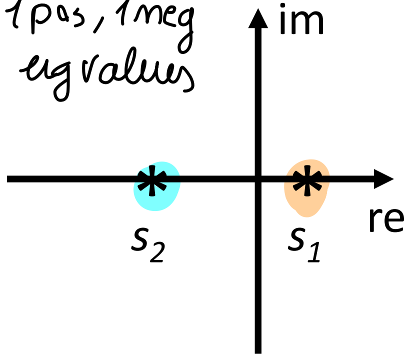
↳ $\tilde{A} = T A T^{-1}$

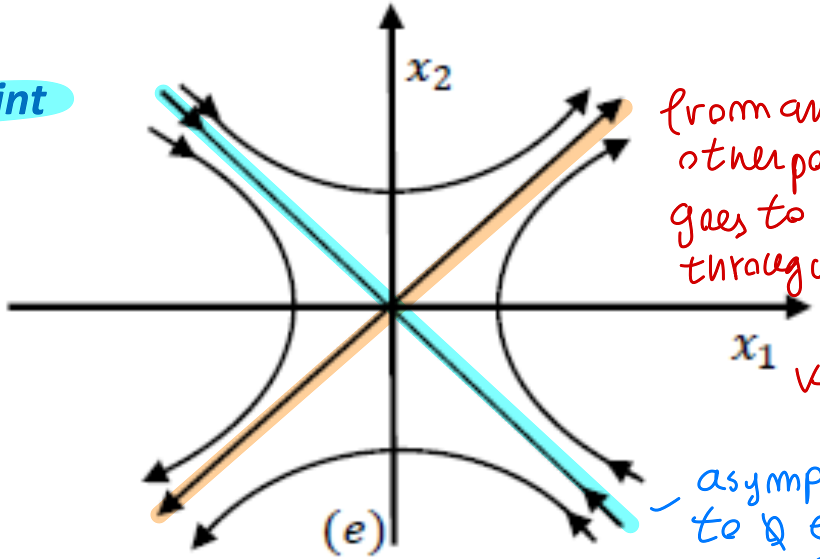
new representations, same I/O relation

same eig values but different eig vectors!

**saddle point**

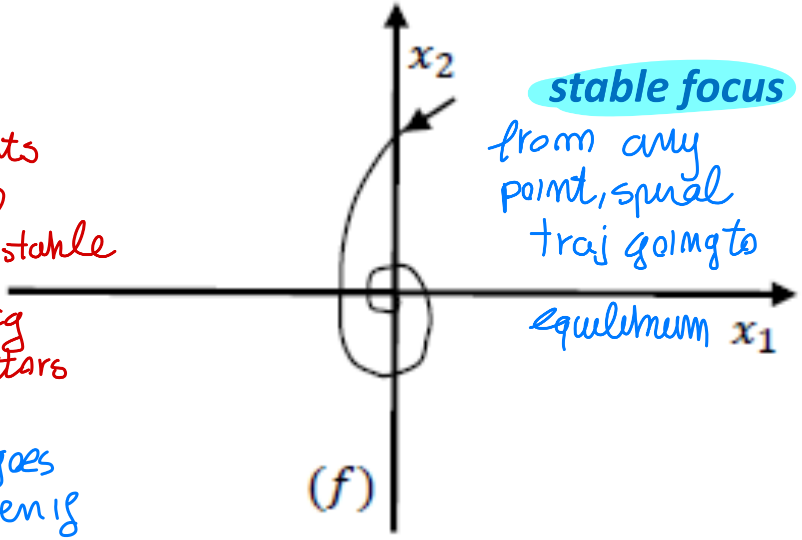
1 pos, 1 neg eig values




from any other points goes to ∞ through unstable eig vectors

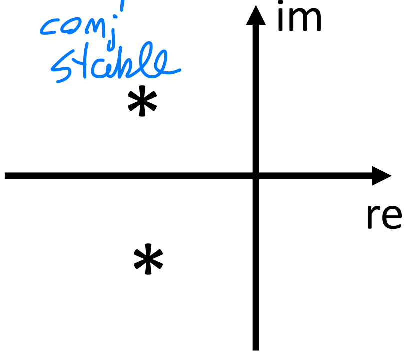
asymp goes to 0 even if unstable..




**stable focus**

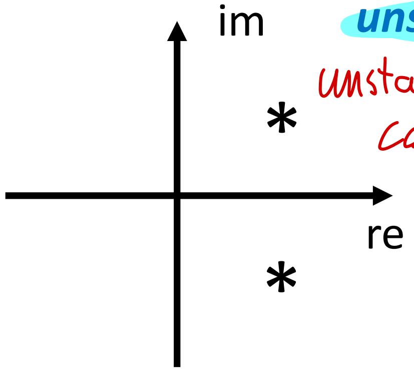
from any point, spiral traj going to equilibrium

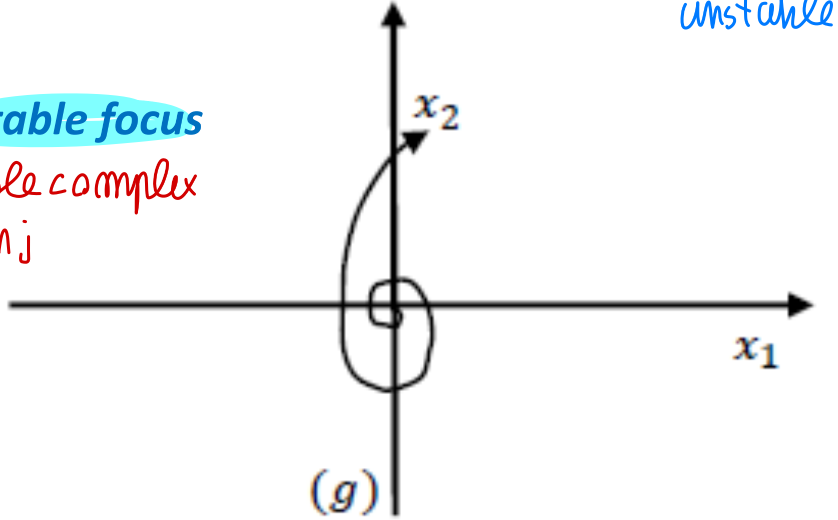
complex conj stable




**unstable focus**

unstable complex conj

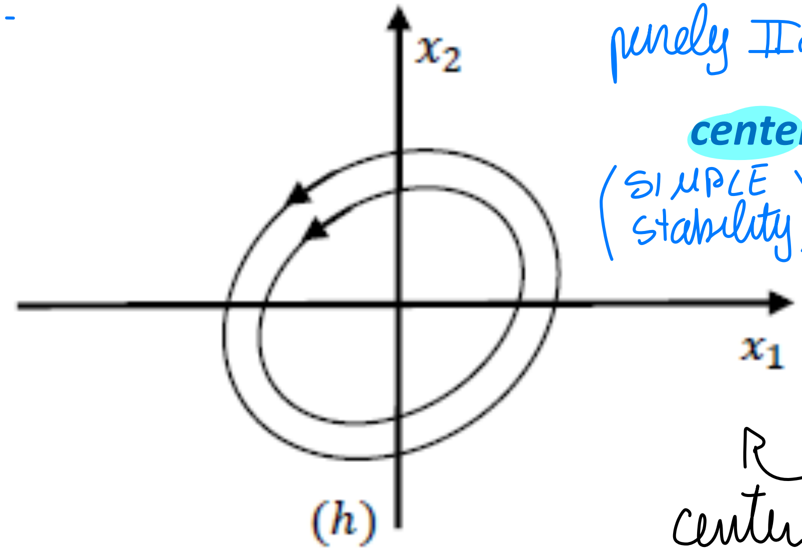


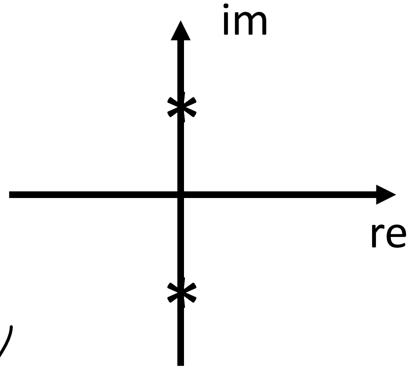




purely Im part eig values

**center**

(SIMPLE stability)




center traj and DIRECTION depends on eig. vectors

( plus other forms for repeated eigenvalues)

*Note that the eigenvectors depend on the basis you consider*

**Nonlinear systems** symple pendulum model of phase plane

If $f$ is $C_1$, in a neighbor of an equilibrium, the trajectories are the ones of the corresponding linearized system

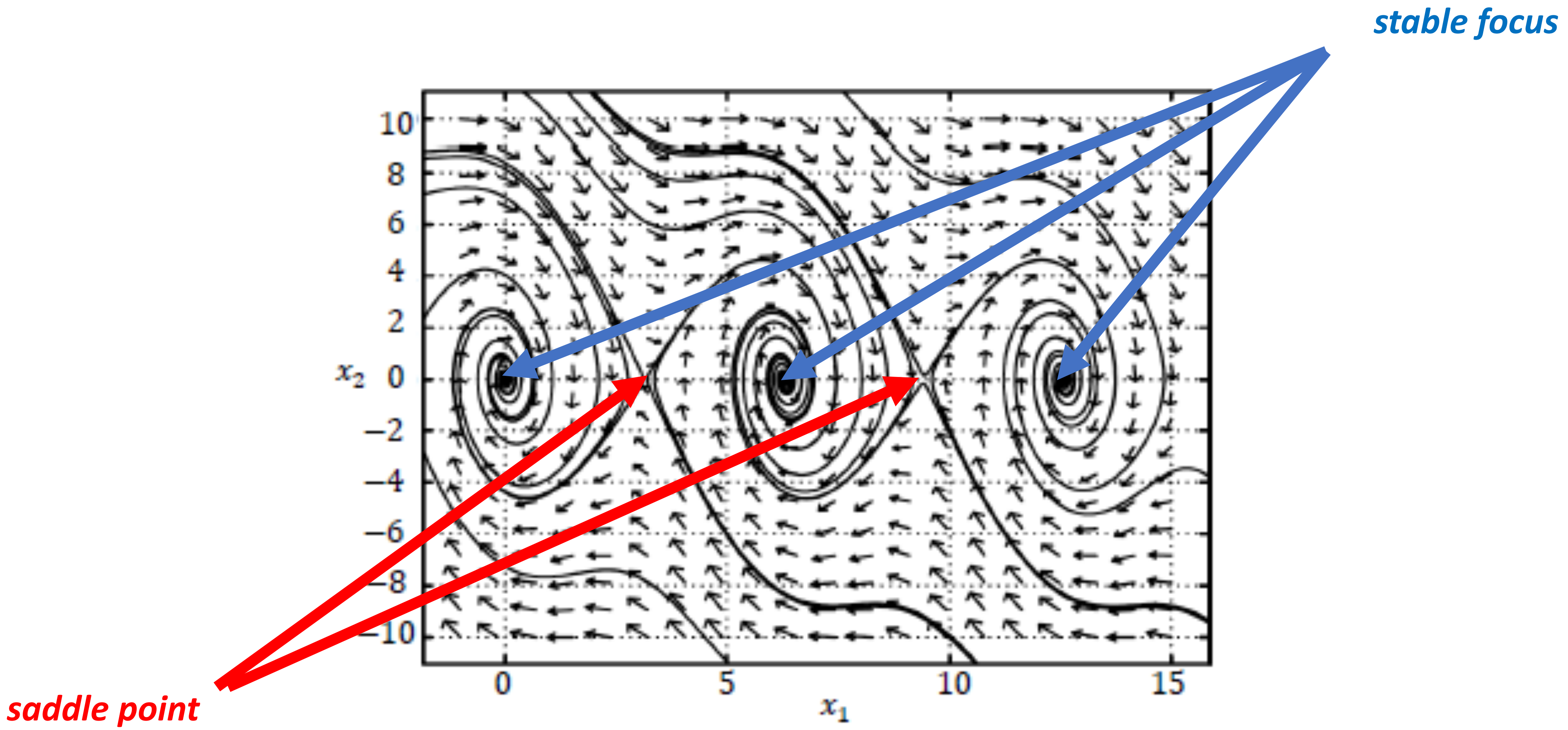
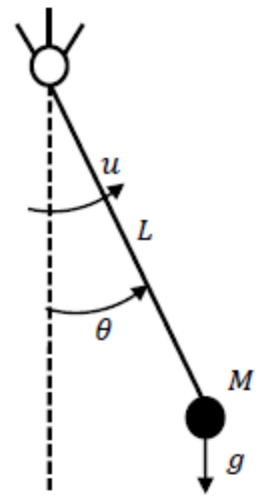



Figure 1.10: Pendulum - phase portrait ($M = L = k = 1$).

**Pendulum** model

(external torque)

$$ML^2\ddot{\vartheta}(t) = -k\dot{\vartheta}(t) - MLg\sin\vartheta(t) + u(t), \quad M,\ L,\ k > 0$$




$$\Big\downarrow \quad x_1 = \vartheta,\ x_2 = \dot{\vartheta}$$

(in normal form)
II ORD syst (2 states)

state syst model
$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{g}{L}\sin(x_1(t)) - \cancel{\dfrac{k}{ML^2}x_2(t)} + \cancel{\dfrac{1}{ML^2}u(t)} \end{cases}$$

$\Downarrow$

equilibria $\quad \Big\downarrow \quad \bar{u} = 0$

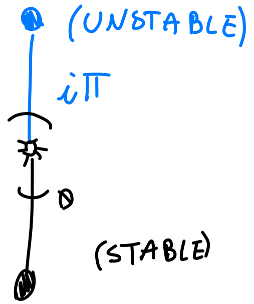
$$\boxed{\bar{x}_2 = 0}$$

$\dot{\vartheta} = 0$ (at eq. obviously No speed)

$$\frac{g}{L}\sin(\bar{x}_1) = 0 \quad \rightarrow \quad \boxed{\bar{x}_1 = i\pi,\ i = 0, 1, \ldots}$$

equilibria

(from II equations)




$\Big\downarrow$ (linearization)

(linear system)
$$\begin{cases} \delta\dot{x}_1(t) = \delta x_2(t) \\ \delta\dot{x}_2(t) = -\dfrac{g}{L}\cos(\bar{x}_1)\,\delta x_1(t) - \dfrac{k}{ML^2}\delta x_2(t) + \dfrac{1}{ML^2}\delta u(t) \end{cases}$$

## Pendulum – linearized model

$\theta$ in $x_1$, $I$ in $\delta x_2$

$$\begin{cases} \delta \dot{x}_1(t) = \delta x_2(t) \\ \delta \dot{x}_2(t) = -\frac{g}{L}\cos(\bar{x}_1)\,\delta x_1(t) - \frac{k}{ML^2}\delta x_2(t) + \frac{1}{ML^2}\delta u(t) \end{cases}$$

linearized state space matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(\bar{x}_1) & -\frac{k}{ML^2} \end{bmatrix}$$

$\Downarrow$

For $M = 1,\ k = 1,\ L = 1,\ g = 9.8$ the eigenvalues are

$\Downarrow$

$s = -0.5 \mp j3$ ($\bar{x}_1 = 2\pi i$, stable focus)

$s_1 = 2.67,\ s_2 = -3.67$ ($\bar{x}_1 = (2i+1)\pi$, saddle point)

p plane

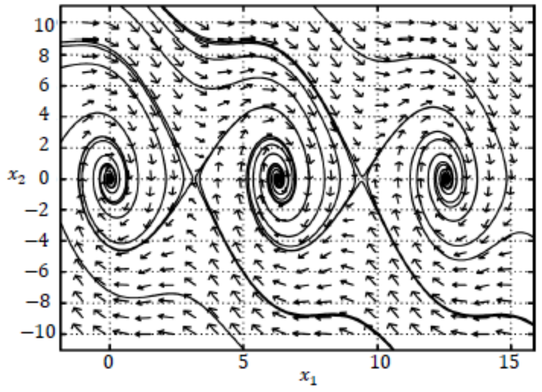



Figure 1.10: Pendulum - phase portrait ($M = L = k = 1$).

• on reachability... $\Rightarrow$ from [A] in canonical form

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \longrightarrow (sI - A) = \begin{bmatrix} s & -1 \\ \alpha & s+\beta \end{bmatrix}$$

$$\det(sI - A) = s^2 + \beta s + \alpha$$

N.S.C is that all coefficents $\alpha > 0,\ \beta > 0 \Rightarrow$ asym stable

N.C for higher order syst $\quad s^n + \alpha_1 s^{n-1} + \dots + \alpha_n = 0$

$\alpha_i > 0 \ \forall i$

**How to compute the phase plane?**

Use the SW *pplaneM.m* (free download), where *M* is the release of Matlab

*Example: van der Pol oscillator*

*But what to do for non differentiable and/or higher order systems?*

The **Lyapunov theory** can provide a satisfactory answer

(and it is the most popular and useful method for the analysis of nonlinear systems and for the nonlinear control synthesis)

A gentle introduction through examples…

## Another example – van der Pol oscillator

(II ORD syst)

$$m\ddot{x}(t) + 2c\left(x^2(t) - 1\right)\dot{x}(t) + kx(t) = 0, \qquad m,\ c,\ k > 0$$

normal form

$$x_1 = x,\ x_2 = \dot{x}$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \dfrac{2c}{m}\left(1 - x_1^2(t)\right)x_2(t) - \dfrac{k}{m}x_1(t) \end{cases}$$

*Linearization at the origin*

stability of equilibrium point $(\bar{x}_1 = 0, \bar{x}_2 = 0)$

$$\begin{cases} \delta\dot{x}_1(t) = \delta x_2(t) \\ \delta\dot{x}_2(t) = -\dfrac{k}{m}\delta x_1(t) + \dfrac{2c}{m}\delta x_2(t) \end{cases}$$

*eigenvalues*

Re>0

$$s = \frac{c}{m} \mp \sqrt{\frac{c^2}{m^2} - \frac{k}{m}}$$

*c=0.1,m=1,k=1*

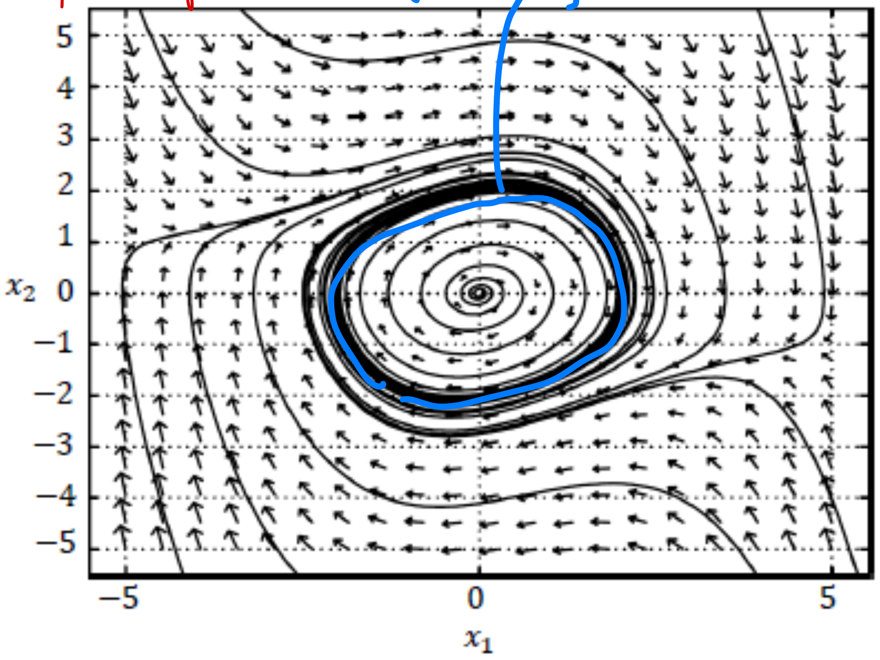
***unstable focus***

closed traj (NON LIN syst) from any point traj goes to a limit cycle

couple of complex conj eig values with Re > 0

phase plane

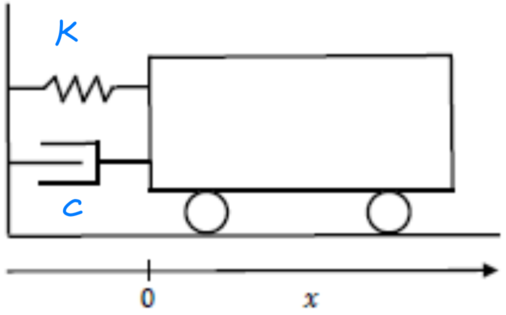
(limit cycle)




move away from the equilibrium!
⇓
definition of stability don't hold, just go away, not sure if it goes to ∞

## The cart example (to define Lyapunov)

dissipative syst → stability




(model)

$$m\ddot{x} = -h(\dot{x}) - k(x)$$

in this case $h(\dot{x}) \notin C^1$ not linearizable around $x=0$

empirical approach to relate stability to energy!

$$\begin{cases} k(x) = k_0 x + k_1 x^3 & \text{spring} \\ h(\dot{x}) = b\dot{x}|\dot{x}| & \text{damper} \end{cases}$$

(depending on speed)

absolute value don't allow standard linearization

$$b, k_0, k_1 > 0$$

$$m\ddot{x} + b\dot{x}\,|\dot{x}| + k_0 x + k_1 x^3 = 0$$

Skip the dependence on time (if not necessary)

(STATES)

$$\boxed{x_1 = x}\ ,\ \boxed{x_2 = \dot{x}}$$

in normal form

state space form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m}\{-bx_2|x_2| - k_0 x_1 - k_1 x_1^3\} \end{cases}$$

we cannot compute linearized syst

so we introduce [Lyapunov theory]

NOT!

*Linerization method not useful*

overall energy of the system

- $T = \frac{1}{2}mx_2^2$ kynetic energy
- $U = k_o\frac{x_1^2}{2} + k_1\frac{x_1^4}{4}$ potential energy

[to study equilibrium stability of this syst we analyze the energy]

$$V(x_1, x_2) = T(x_2) + U(x_1) \quad \text{total energy}$$

$$V(x_1, x_2) = k_o\frac{x_1^2}{2} + k_1\frac{x_1^4}{4} + \frac{1}{2}mx_2^2$$

(overall energy function)

null overall energy

$V(x_1, x_2) = 0$ for $x_1 = 0$ and $x_2 = 0$ (the origin)

$V(x_1, x_2) > 0$ for $x_1 \neq 0$ and/or $x_2 \neq 0$

so all outside the origin

if the equilibrium position is asymp stable → you should have continuous energy reduction until you reach the equilibrium

for example eq. on ORIGIN where
$U = 0, T = 0 \Rightarrow V = 0$

look energy evolution to see if it goes to eq.

looking on V, to check its behav. ↑↓ we analyze $\frac{dV}{dt}$ along system trajectory x

to study how evolve

$$V(x_1, x_2) = k_o\frac{x_1^2}{2} + k_1\frac{x_1^4}{4} + \frac{1}{2}mx_2^2$$

from a given initial point, if $\frac{\partial V}{\partial t} < 0$ ∀ traj, the cart dissipate energy until reach equilibrium

Derivative of *V(x)*, *x=[x₁ x₂]'*, with respect to time and along the trajectories

derivative along a traj ...(to see how syst evolve)

according to syst equation of $\dot{x}_1, \dot{x}_2$

(replacing $\dot{x}_1, \dot{x}_2$ expression)

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{dV(x)}{dx}\frac{dx}{dt} = k_0x_1\dot{x}_1 + k_1x_1^3\dot{x}_1 + mx_2\dot{x}_2 = -bx_2^2|x_2| \boxed{\leq 0}$$

$(\dot{x}_1, \dot{x}_2)$ (any point) $x_1 \neq 0$ ← (steady state)

>0 >0 >0

energy function decrease until $x_2 = 0$

$\dot{V} \leq 0$

only $x_2$ contribution, V↓ until $x_2 \to 0$, nothing known about $x_1$

This means that we have dissipation of energy until $x_2=0$ (null velocity)

↳ diss.

Why $\dot{V}(x) \leq 0$ and not $\dot{V}(x) < 0$? Because it can be null also for $x_1 \neq 0$

$\dot{V}$ could be = 0 for $x_1 \neq 0$ → the cart moove and remains far from 0

In any case, this means that the system will tend to an equilibrium condition with null velocity (but maybe not null position).

(I can say cart remain on constant position)

**General idea**: try to study the equilibria of a system by analyzing a suitable «*energy function*»

V↓ until $x_2 \to 0$, anything about $x_1$, IT could STOP in $x_1 \neq 0$

half of result, main idea is to find V(x) and understand how it evolves over time

⇒ study an energy function to conclude som about eq

Non lin → linearization

| IF NOT possible

↳ study of energy function to check until dissipation on origin

main idea of Lyapunov

$V(x) \downarrow$ until we reach $x_2 = 0$ $\forall x_1 \neq 0$

in principle we could reach any point with $x_1 \neq 0$ and $x_2 = 0$

(even if phisically we know that this is NOT possible, in principle it could happen looking to the equation of $\dot{V}(x)$)

[even if the only eq is $(0,0)$ !

⇓ find a function that ↓↓ until reach equilibrium

dissipation of energy

← here $\dot{V}(x) \leq 0$

L $\leq$ because we can have $\dot{V}(x) = 0$ for $x_1 \neq 0$

this is not true that $V(x) < 0$ $\forall$ point near origin

⇓

equilibrium with null velocity BUT non null position

Lyapunov

↓

select a mathem. function with this behaviour

$\Longrightarrow$ to study we should describe better $\leq 0$, function sign definition

 Needed def

## Positive definite functions

A *scalar* function $V(x)$, *continuous* with its first derivatives, is locally:

"LOCALLY"

nothing asked about neighb. size

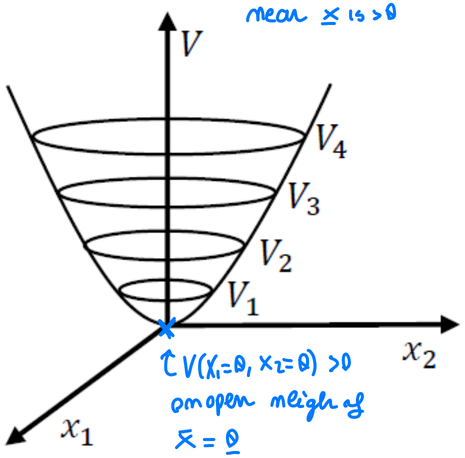
- *positive definite* in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) > 0$ for any $x$ belonging to an open neighbor of $\bar{x}$ (see the figure where $\bar{x} = 0$)

- *semidefinite positive* in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) \geq 0$ for any $x$ belonging to an open neighbor of $\bar{x}$

- *definite negative* in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) < 0$ for any $x$ belonging to a neighbor of $\bar{x}$

- *semidefinite negative* in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) \leq 0$ for any $x$ belonging to an open neighbor of $\bar{x}$

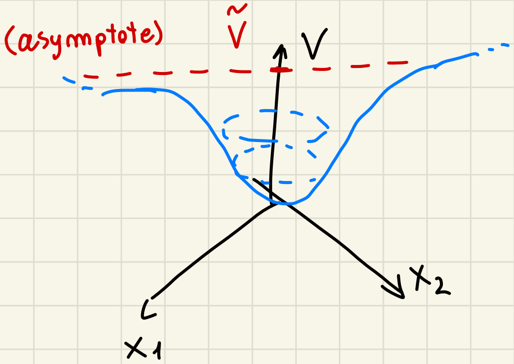
Pos def if any direction local mean $x$ is $>0$




condition GLOBAL PROPERTIES IF (everywhere)

All these definitions are **global** if the corresponding conditions are fulfilled for any $x \neq \bar{x}$

If $V(x)$ is positive definite in $\bar{x}$ and $\lim_{\|x\| \to \infty} V(x) = +\infty$, then $V(x)$ is *radially unbounded* → you could have function NON RADIALLY UNBOUNDED !



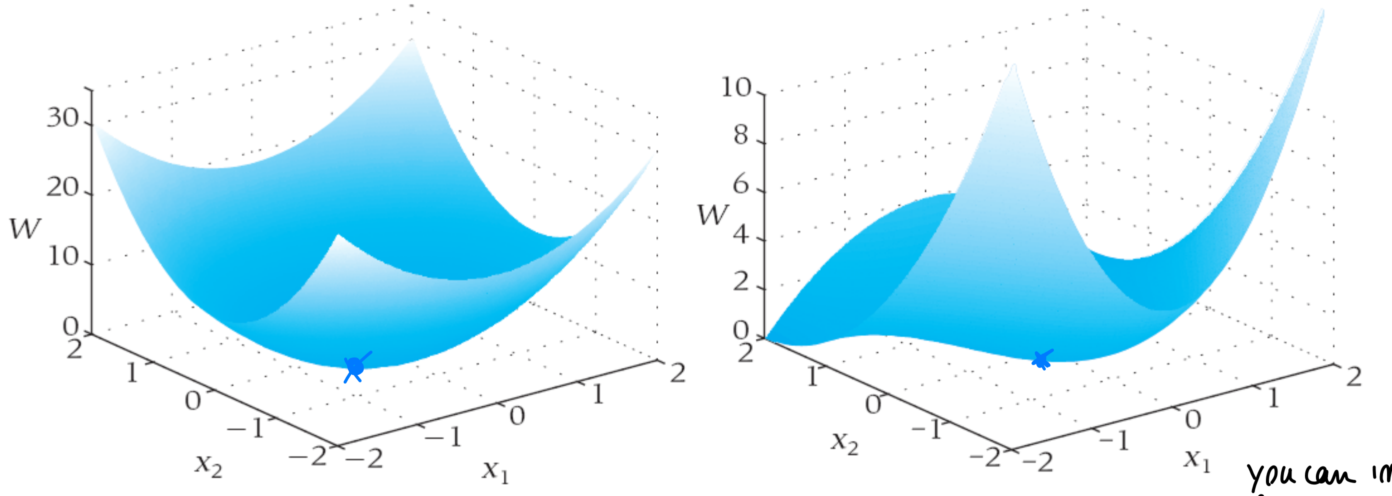

for $\|x\| \to +\infty$

$V(x) \to \tilde{V}$ so

this is NOT Radially unbounded

Which is NOT a good result → limit value of $V(x)$

$$\boxed{\bar{x} = 0}$$




a) b)

paraboloid
$>0$ $\forall x$
everywhere, not only on neighbour

$$V(x) = 3x_1^2 + 5x_2^2 \Big|_{0,0} = 0$$

Globally positive definite and radially unbounded

you can infer it from expression

$V(x)>0$ near $\underline{x}=0$ but NOT $\forall x$!

$x \simeq 0$

$$V(x) = x_1^2 + x_2^2 - x_1^2 x_2 \Big|_{0,0} = 0$$

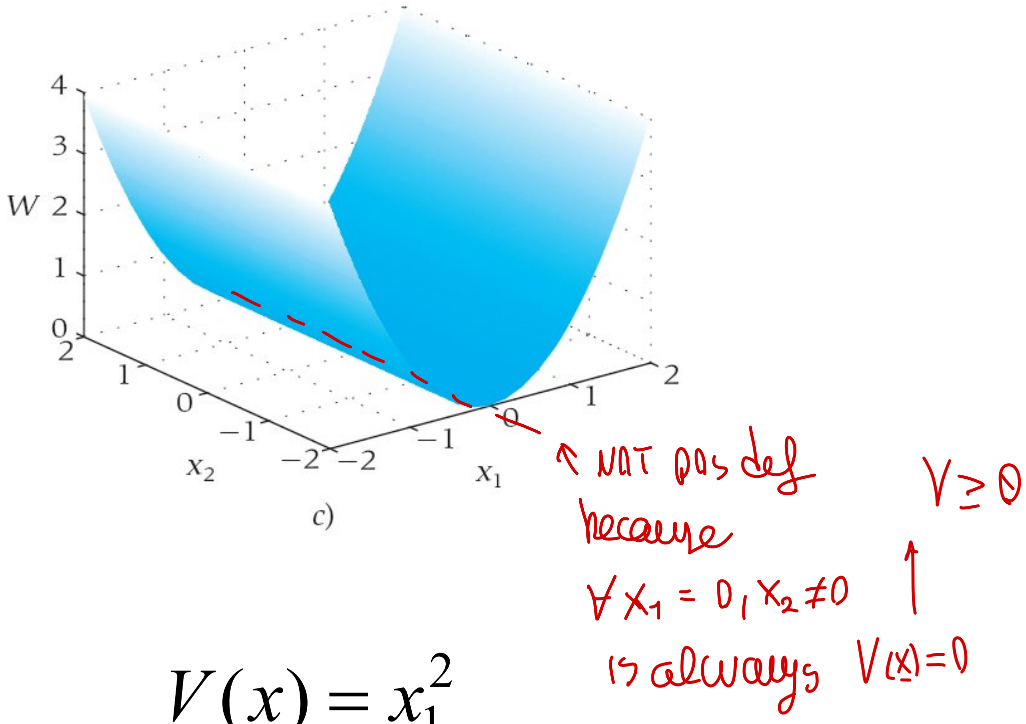
for $\underline{x} \simeq 0$

predomination $x_1^2 + x_2^2 >> x_1^2 x_2$ ← III ORD

(locally) positive definite

*In a neighbor of the origin the second order terms dominate*

$$\boxed{\bar{x} = 0}$$



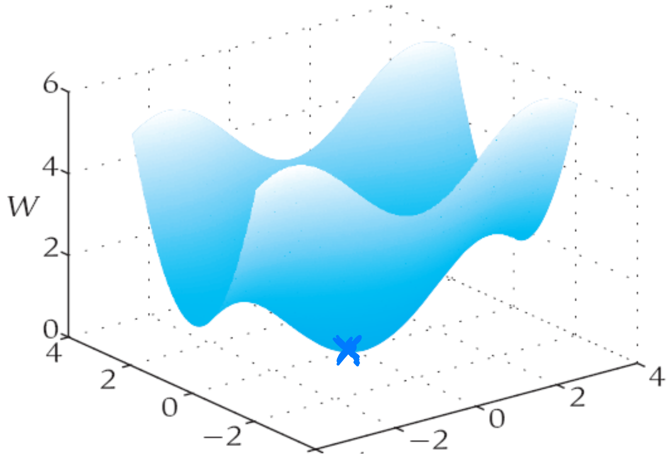
c)

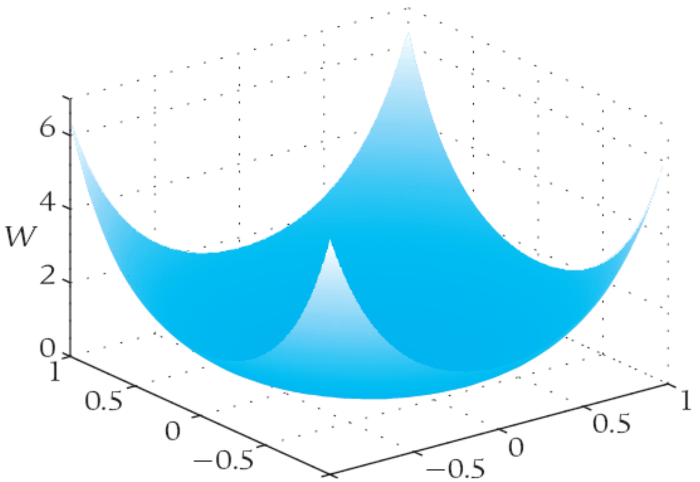
$V \geq 0$

$$V(x) = x_1^2$$

Positive semidefinite

$$\boxed{\bar{x} = 0}$$




>0 mean x≈0

positive contribute

$$V(x) = 1 - \cos x_1 + x_2^2$$

Positive definite

$$V(x) > 0$$

$$\text{for} -2\pi < x_1 < 2\pi \ and\ for\ any\ x_2$$

$$V(x) = e^{x_1^2 + x_2^2} - 1$$

Globally positive definite and radially unbounded

c)

d)

$$\boxed{\bar{x} = 0}$$

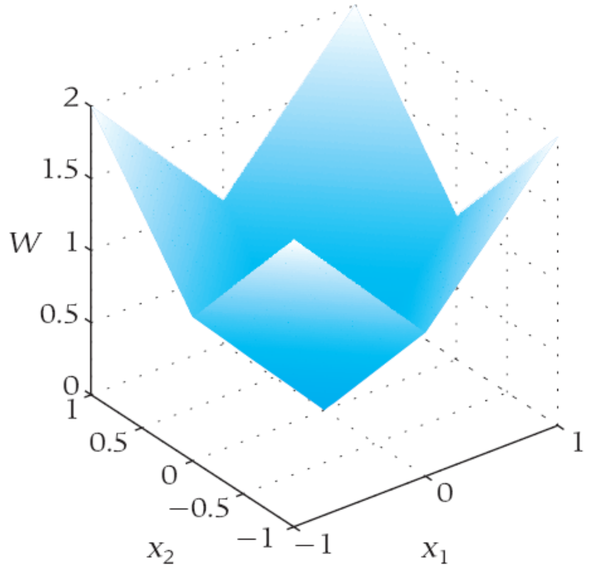
$$V(x) = \sum_{i=1}^{n} |x_i|$$

Globally positive definite and radially unbounded

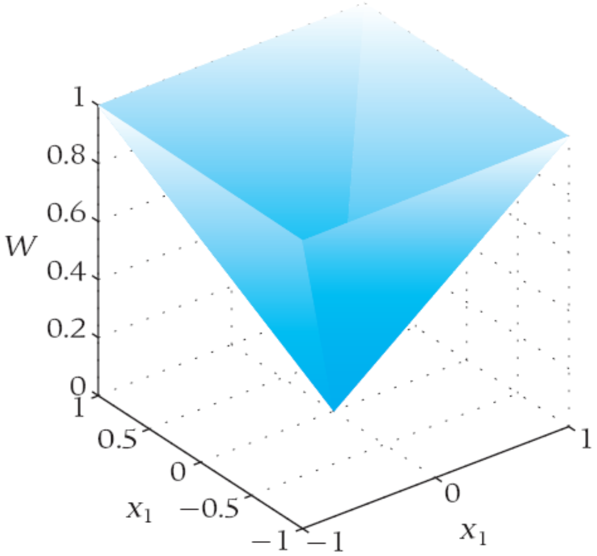
goes to $\infty$ $\forall x \to \infty$ continuously grow

$$V(x) = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Globally positive definite and radially unbounded




c)




d)

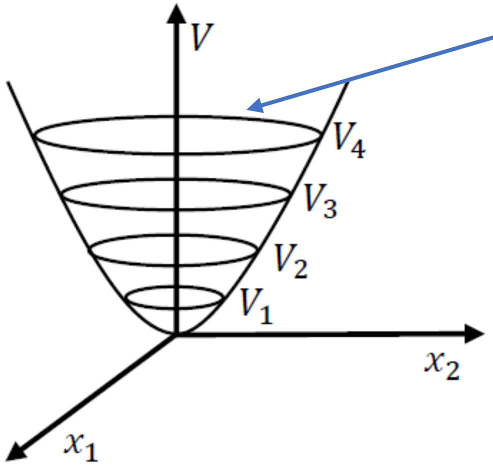
the Theory is necessary for Lyapunov

Given a function $V(x)$ positive definite in $\bar{x}$, the values of $x$ such that $V(x) = \bar{V}$, where $\bar{V}$ is a positive value, define a *level surface*, or a *level line* in the case of second order systems

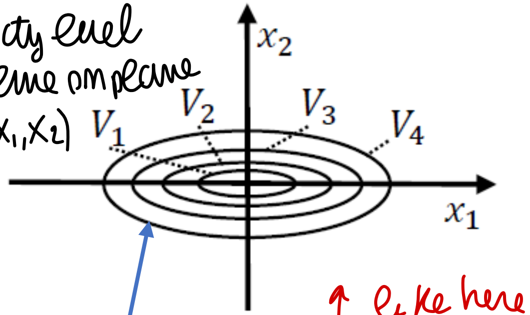
a line were $V(x) = \bar{V}$ (cut V with a plane)

these closed lines are level surface / line

$\boxed{\bar{x} = 0}$




projecty level line on plane $(x_1, x_2)$ $\Rightarrow$




like here, each line $\forall \bar{V}$ different is around $\bar{x}$

(important) result!

Given $\bar{V} > 0$ sufficiently small, the corresponding level line is a closed curve that includes $\bar{x}$ (relevant to analyze p-plane)

and Given $V_2 > V_1 > 0$ the set of values of $x$ such that $V(x) \leq V_1$ is contained into the set of values of $x$ such that $V(x) \leq V_2$

(main stability Theorem)

TO STUDY STABILITY you have to be able to find this function V(x) BUT...

- only sufficent (you cannot be able to find V(x))
- IT can be difficult to find V(x)!

## Stability - Lyapunov theorem

Consider

NON LIN given the syst: syst. AUTONOMUS, considering u(t)= const for example

{study $\bar{x}$ STABILITY}

$$\dot{x}(t) = \varphi(x(t)) \quad , \quad \varphi \in C^1 \quad , \quad \varphi(\bar{x}) = 0$$

($\bar{x}$ equilibrium point)

If there exists a function $V(x)$, continuous with its derivative, positive definite in $\bar{x}$ and such that its derivative, along the state trajectories, $\dot{V}(x)$ is semidefinite negative in $\bar{x}$, i.e. such that for any $x$ belonging to a neighbor of $\bar{x}$

on physical syst use energetic idea

PRO but looking on V(x) you get clue about attraction region! (more than lineariz approach)

along traj! ⇒ including syst equations

$$\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}\varphi(x) \leq 0$$

only sufficent

you can have a syst in eq. BUT cannot find the V(x) semi neg def

then $\bar{x}$ is a stable equilibrium

{We will extend for asymp. st.} ⇒ extensions!

{IF you can find V(x)>0 and such that $\dot{V}(x) \leq 0$ near $\bar{x}$ then $\bar{x}$ is a STABLE EQ}

problem is to find it, associate to energy → NO AUTOMATIC procedure easily

*V(x)* is called a **Lyapunov function**

energy function

Why LYAPUNOV THEORY WORKS?
(proof of theorem)

**Recall the definition of stability**

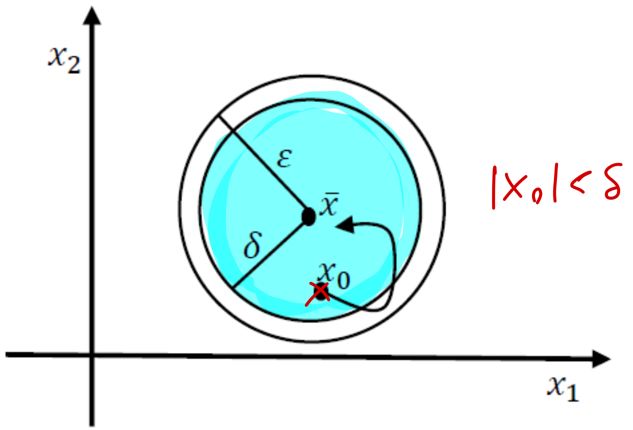
The equilibrium $\boxed{\bar{x} \text{ is stable}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all the initial states $x_0$ satisfying

(mean eq.)

$$\|x_0 - \bar{x}\| \leq \delta$$

it holds that

$$\|x(t) - \bar{x}\| \leq \varepsilon, \quad \forall t \geq 0$$



2 states syst assumpt. and $\bar{x}$ equilibrium, you take $\varepsilon$ an a ball of radius $\varepsilon$ ($B_\varepsilon$)

## Sketch of the proof

(apply stability theory)

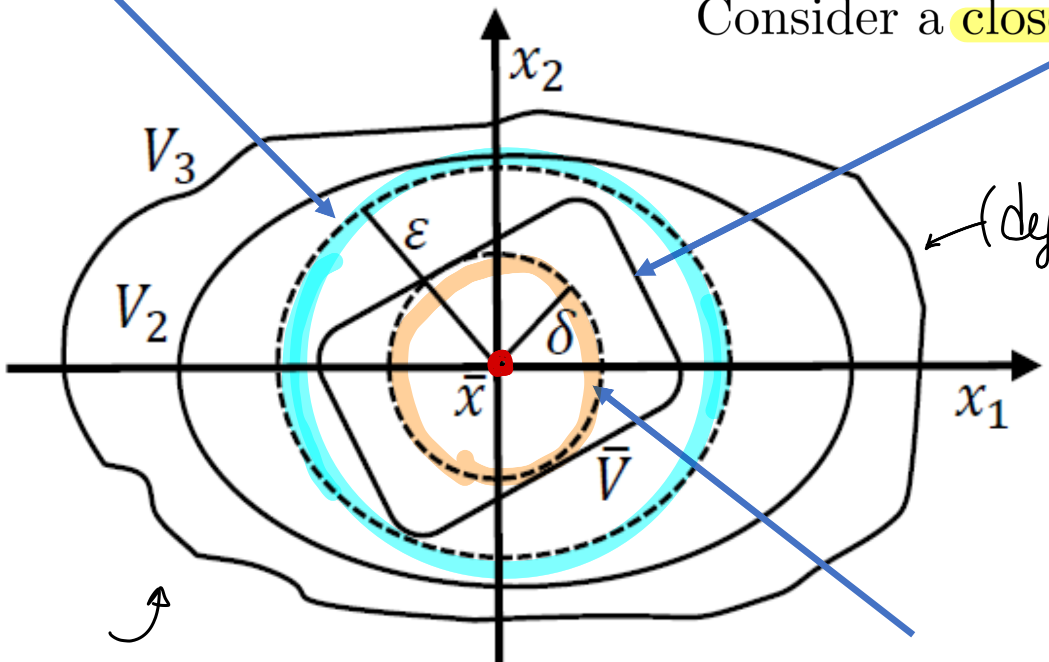
Given $\varepsilon > 0$, consider the set $B_\varepsilon := \{x : \|x - \bar{x}\| < \varepsilon\}$

Consider a closed level line $\bar{V}$ contained in $B_\varepsilon$

(nest closed level lines)

from eq. I find $V > 0$ mean $\bar{x}$ such that $\dot{V} \leq 0$, so all points try to go to $\bar{x} = 0$ or remain on same lev line

↓ sufficient for SIMPLE STABILITY of eq.




(different level lines)

if $\dot{V} \leq 0$, you cannot go outside level line, only decrease

remain always inside ball of radius $\varepsilon$

you can take also $\dot{V} = 0$, remain on same lev line

Define by $\delta$ the minimal distance of this curve from $\bar{x}$

$B_\delta$

↳ eq. proven

Since by assumption $\dot{V} \leq 0$, for any initial state $x_0 \in B_\delta$, the state trajectory is fully contained in $\bar{V}$, i.e. it is inside $B_\varepsilon$

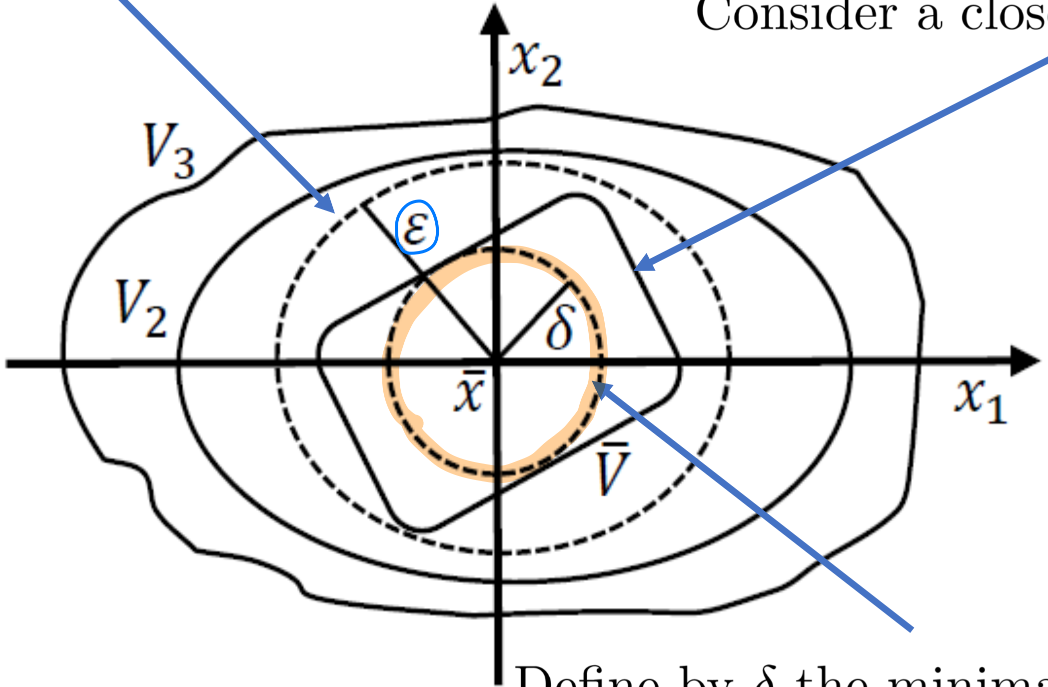
taking $\bar{V}$ small, one inside the other if we move to $V > \bar{V}$

## Sketch of the proof

Given $\varepsilon > 0$, consider the set $B_\varepsilon := \{x : \|x - \bar{x}\| \ < \varepsilon\}$

Consider a closed level line $\bar{V}$ contained in $B_\varepsilon$

taking $\varepsilon > 0$, we define $B_\varepsilon$ where I study stability
↓
level line correspondy to $B_\varepsilon$ ⇒ we define a new $\bar{V}$ level line all contained on $B_\varepsilon$...
taking the $B_\delta$ included into $\bar{V}$



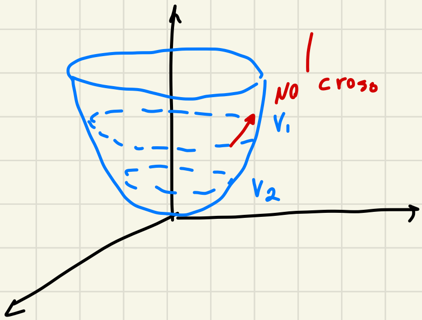

{you remain inside $\bar{V}$, so inside $B_\varepsilon$!}
so $\forall x_0 \in B_\delta$ ...
(STABILITY)

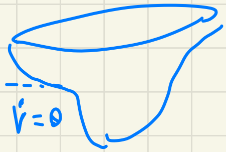
Define by $\delta$ the minimal distance of this curve from $\bar{x}$

↓
Clearly $\forall x_0 \in B_\delta$ you remain inside $\bar{V}$
⇒ you cannot have trajectory mooving outside level line... $\dot{V} \leq 0$ ⇒ IT cannot happen that $V\uparrow$ along traj!

Since by assumption $\dot{V} \leq 0$, for any initial state $x_0 \in B_\delta$, the state trajectory is fully contained in $\bar{V}$, i.e. it is inside $B_\varepsilon$

$\dot{V} \le 0$ IT cannot increase V along traj



- **Asymptotic stability**

{IF $\dot{V} = 0$ you remain in a level line, no motion if syst to the equilibrium!}

$\dot{V} < 0$

If the assumptions of the previous Theorem hold and $\dot{V}$ is negative definite in $\bar{x}$, then $\bar{x}$ is an asymptotically stable equilibrium

you cannot remain on a level line! ⇒ continuous decrease so come back to equilibrium for sure

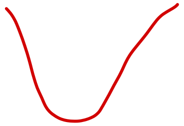
- **Global asymptotic stability**

If there exists a function $V(x)$, continuous with its derivative, globally positive definite in $\bar{x}$, radially unbounded, and such that its derivative, along the state trajectories, $\dot{V}(x)$ is globally definite negative in $\bar{x}$, then $\bar{x}$ is the unique globally asymptotically stable equilibrium of the system

if globally, it can have only one asymp. stable equilibrium

(*) ⇒

↳ globally because



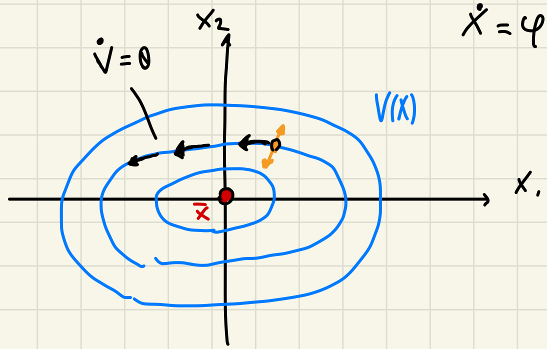
and unique cause ONLY one globally st.

**Instability**

If there exists a function $V(x)$ continuous with its derivative, positive definite in $\bar{x}$ and such that $\dot{V}(x)$ is positive definite in $\bar{x}$, then $\bar{x}$ is an unstable equilibrium

↳ (GROWING)

?




$\dot{X} = \varphi(X)$

"stability"
=
(near equilibrium)

$\left\{ \begin{array}{l} \text{move along} \\ \text{level line !} \\ \text{near equilibrium} \end{array} \right\}$ $\hookrightarrow$ this lead to next Th...

$\left\{ \begin{array}{l} \text{if from a II ORD syst} \\ \quad \downarrow \\ \text{you have } \bar{X}_1, \bar{X}_2 \rightarrow \text{global stability of } \bar{X}_1 \end{array} \right.$

from any $x_0 \in \mathbb{R}^2$ come back to $\bar{X}_1$

(*)
"UNIQUE"

NOT possible if you have more than 1 eq points!

if $x_0 = \bar{X}_2$ it remain here!

(When $\dot V = 0$, we can conclude something more?)

What if the Lyapunov function is only *semidefinite negative*?

**Krasowski – La Salle theorem**

(STABILITY)

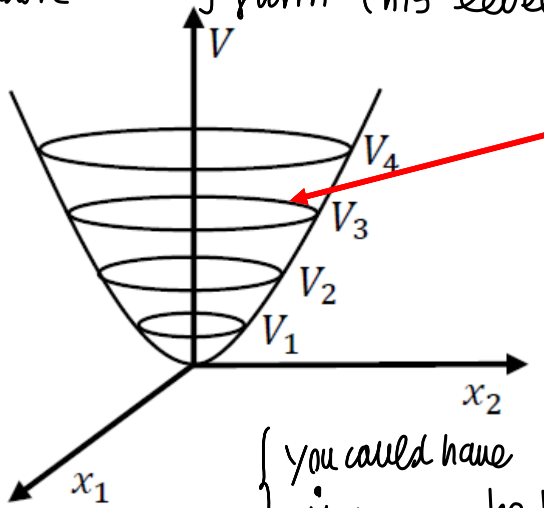
if moove from $\bar x$ compatible with syst

$V(x) > 0$

If there exists a function $V(x)$ continuous with its derivative, positive definite in $\bar{x}$, such that $\dot{V}(x)$ is semidefinite negative in $\bar{x}$, and the set $S := \left\{ x : \dot{V}(x) = 0 \right\}$ does not contain perturbed (with respect to $\bar{x}$) trajectories compatible with the system, then $\bar{x}$ is an asymptotically stable equilibrium point

could we remain fixed on $\dot V = 0$, on a certain level line? if looking to syst traj, it tells you moove away from this level line

you can conclude asymp. st. because syst prevent you to remain on traj



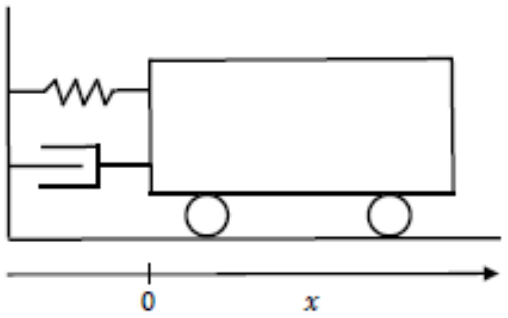

In principle, since *Vdot* is only *semidefinite negative*, the state could remain along this level line. However, this behavior should be allowed by the system's dynamics (state equations). If it is not possible, this means that the «*active*» condition is *Vdot<0* and the state converges to the origin

$\dot V = 0$ NOT ALLOWED

behaviour of remain on level line NOT POSSIBLE!

{ You could have $\dot V \le 0$, can be $\dot V = 0$, remain on a level line! ⇒ if dyn such that you cannot remain on a level line → asymp. }

## The cart example




$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m}\{-bx_2|x_2| - k_0x_1 - k_1x_1^3\} \end{cases}$$

(syst equations)

LYAPUNOV FUNCTION

$$V(x_1,x_2) = T(x_2) + U(x_1) = \frac{1}{2}mx_2^2 + k_o\frac{x_1^2}{2} + k_1\frac{x_1^4}{4} > 0 \quad \forall (x_1,x_2) \neq 0$$

$$\dot{V}(x) = -bx_2^2|x_2| \leq 0$$

only (simple) stability can be concluded with the Lyapunov theorem, but is it possible to have a trajectory of the system with *Vdot=0*?

IT could have any $x_1$ values, @ end get a $x_2=0$

Lyap. "origing" JUST stable

Asymp? → KRASOWSKI if traj of system such that $\dot{V}=0$ is NOT possible stop for $x_1 \neq 0$, so asymp

$$\dot{V}(x) = 0 \rightarrow x_2 = 0 \rightarrow \dot{x}_1 = 0 \rightarrow x_1 \text{ constant } \bar{x}_1$$

possible to remain along level line

$$-(k_0 + k_1\bar{x}_1^2)\bar{x}_1 = 0 \longrightarrow \bar{x}_1 = 0$$

$\boxed{\dot{X}_2 = 0}$

$x_1, x_2$ cannot moove along traj

Lyapunov ↔ Krasowski

(look in detail for traj info)

So, the only possible trajectory compatible with *Vdot=0* is the origin sometimes unable to find V(x)

{ many theory to find V(x) automatically } ⇐ to study an eq point if you are NOT able to find V(x)
NOTHING IS KNOWN

## Comments

most important tool to study non lin syst

power result → BUT sufficient

The previous results are ***only sufficient conditions***, if you are unable to find a suitable Lyapunov function satisfying one of the above theorems *you cannot conclude anything* about the stability of the equilibrium

⇓

The really difficult task is to find a suitable Lyapunov function. If possible, one can resort to the idea of «energy», like in the cart example. Otherwise, it is possible to try with a quadratic function of the form

physically

mathematically

$$\boxed{V(x) = (x-\bar{x})' P (x-\bar{x})}$$

use $(x-\bar{x})$: (because $V(\bar{x}) = 0$ on $\bar{x}$)

(Typical way of solving)

(STANDARD APPROACH)

where *P* is a positive definite matrix *(P>0)*, i.e. $v'Pv > 0$ for any $v \neq 0$. More details on the choice of *P* will be discussed next .

↳ symm matrix such that

## Exercise 1

Consider the system $\dot{x} = -x^3$ and study the stability of the origin. →(only eq. point)

First of all, note that the origin is an equilibrium (the only one). Moreover, the linearized system is

from linearized syst $\delta\dot{x} = -3\cancel{\bar{x}^2}\delta x = 0$
$\bar{x} = 0$

$$\delta\dot{x}(t) = 0$$

No conclusion from lin. model analysis

with zero eigenvalue. Therefore, nothing can be concluded from the analysis of the linearized system.

Let's try with a quadratic Lyapunov function

$$V(x) = x^2 > 0$$

try with quadratic function

GLOBALLY POS. DEF

Its derivative is

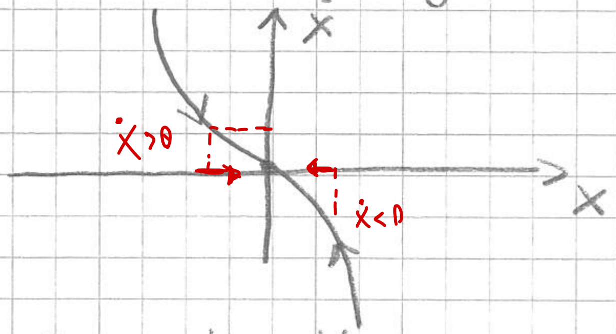
$$\dot{V}(x) = 2x\dot{x} = -2x^4 < 0$$

$$\begin{cases} V > 0 \\ \dot{V} < 0 \quad \forall x \neq 0 \end{cases}$$

So the stability is an asymptotically stable equilibrium. Moreover, since $V(x)$ is radially unbounded and $\dot{V}(x) < 0$, $\forall x \neq 0$, the equilibrium is globally asymptotically stable.

Note that, since the system is first order, one can look at the function $x - \dot{x}$

simple graphical analysis

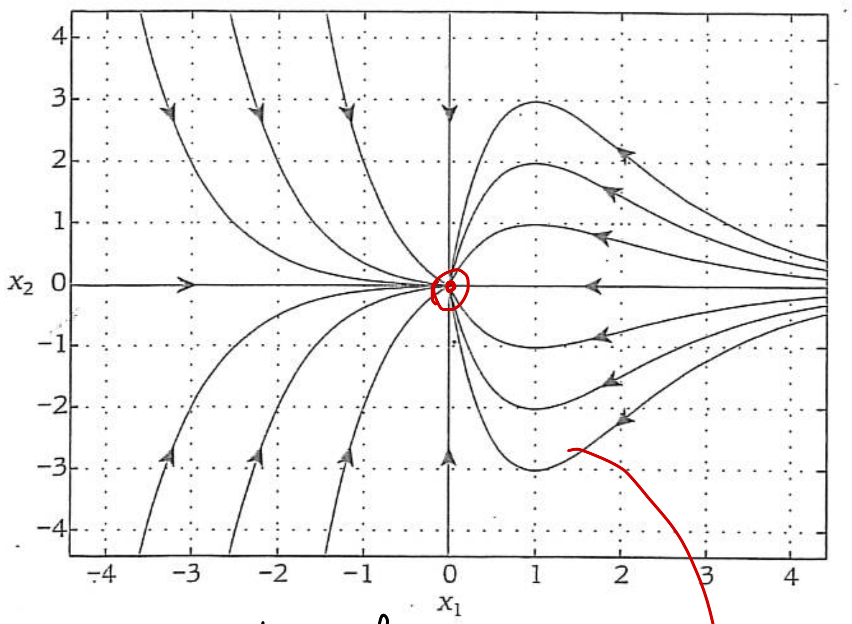



which leads to the same conclusions.

## Exercise 2

Consider the system

$$\begin{cases} \dot{x}_1(t) &= -x_1(t) \\ \dot{x}_2(t) &= x_2(t)(x_1(t)-1) \end{cases}$$




It is apparent that the origin is an equilibrium $\bar{x} = \underline{0}$

Moreover, looking at the state trajectories (computed with pplane), One can easily conclude that the equilibrium is unique and globally asymptotically stable — from any point reach origine → asymp st from pplane ispection

- Can we obtain the same result with the Lyapunov theory? Consider

way to solve it: {quadratic function}

$$V(x) = 0.5(a_1x_1^2 + a_2x_2^2) \quad , \quad \boxed{a_1 > 0} \quad , \quad \boxed{a_2 > 0}$$

Just random to simplify 2 on d/dx

← pos values such that $V(x) > 0$ for $x \neq 0$ (POS. DEF)

Correspondingly

$$\hookrightarrow \quad \dot{V}(x) = a_1x_1\dot{x}_1 + a_2x_2\dot{x}_2 = \underbrace{-a_1x_1^2 - a_2x_2^2}_{\text{dominating terms!}} + \underbrace{\cancel{a_2x_1x_2^2}}_{\text{negligible for } x \simeq 0}$$

II ORD

use $\dot{x}_1, \dot{x}_2$ ↳ from syst eq. along Traj

same conclusion with Lyapunov?

⇓ looking @ $\dot{V}$.. STABILITY?

mean origin $x=0$ for sure $\dot V \le 0$, NO GLOBAL result...

Therefore ↓ ↓ LOCALLY

$$\dot V(x) = -a_1x_1^2 - a_2x_2^2(1-x_1) < 0 \boxed{\text{for } x_1 < 1}$$

limit attraction region → result by Lyapunov only for $x_1<1$

Sufficient cond → can exist a better Lyapunov

and we can conclude that the origin is a *locally* (*not globally!*) asymptotically stable equilibrium

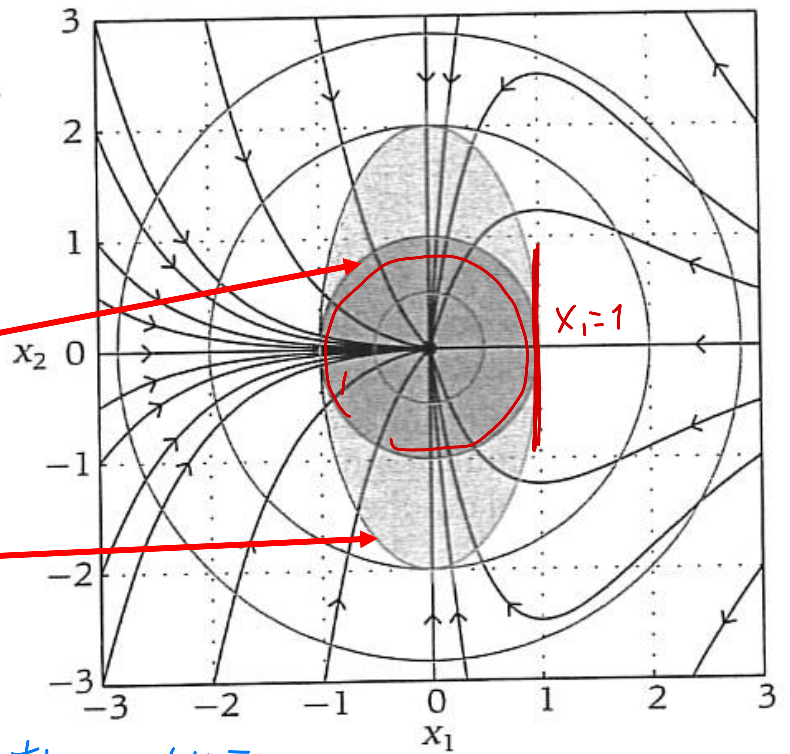
↓

Can we estimate the region of attraction? (where *Vdot<0*)

(REGION of ATTRACTION by LYAPUNOV)

- If we take $a_1=a_2$ → the estimated maximum region of attraction is a circle tangent to $x_1=1$
  V(x) represent a circle centered in 0 with radius 1 where for sure $\dot V(x)<0$
- If we take $a_1 \neq a_2$ → the estimated maximum region of attraction is an ellipsoid tangent to $x_1=1$, which can be made larger always for $x_1<1$
  ↑ larger region respect $a_1=a_2$...

max region of attract



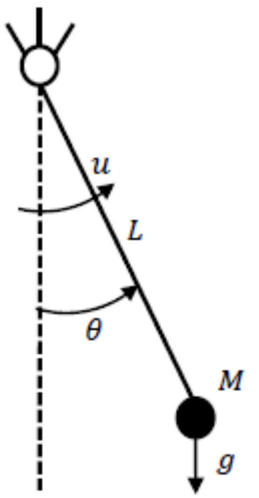

anyway ⇒ GLOBAL cannot be found with Lyapunov here... But better than LINEARIZED SYST

In any case, we are unable to conclude the global stability, unless we find a different Lyapunov function

↑ NO conclusion on global stab → But I can try to enlarge region

## Pendulum

physical meaning example




$x_1 = \vartheta,\ x_2 = \dot{\vartheta}$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\dfrac{g}{L}\sin(x_1(t)) - \dfrac{k}{ML^2}x_2(t) + \dfrac{1}{ML^2}u(t) \end{cases}$$

$$\downarrow \quad u = 0$$

$$V(x) = U(x) + T(x) = \frac{g}{L}(1 - cos(x_1)) + \frac{1}{2}x_2^2$$

overall energy (POT + KIN)

$$\dot{V}(x) = \frac{g}{L}\dot{x}_1 sin(x_1) + x_2\dot{x}_2 = -\frac{k}{ML^2}x_2^2 \leq 0 \longrightarrow$$ Simple stability

nothing about $x_1$

as on cart I can reach $x_2 = 0$ but $x_1$?

Krasowski - La Salle asymptotic stability of the origin

so you use Krasowski $x_2 = 0$

$$\dot{V}(x) = 0 \rightarrow \boxed{x_2 = 0} \rightarrow \dot{x}_1 = 0 \rightarrow x_1 \text{ constant } \bar{x}_1$$

$$\frac{g}{L}sin(\bar{x}_1) = 0 \longrightarrow \boxed{\bar{x}_1 = 0}$$

PARTICULAR cases

## Lyapunov stability for linear systems

⇓ usefull because RESULTS used for other ends

↓ (reformulation of Lyapunov)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

sufficient to study eq of A!

stability of syst all (not only of eq.)

⇑ Lyapunov can be usefull

**Theorem**

The necessary and sufficient condition for the asymptotic stability of the system is that, given any matrix $Q = Q' > 0$, there exists a matrix $\underbrace{P = P' > 0}_{\text{symm, pos. def}}$ verifying the following Lyapunov equation

as you like, but pos def!

$$A'P + PA = -Q$$

**Lyapunov equation**

You can take any Q matrix [mxm] symm. pos def, solve Lyap eq → P and check P symm pos def to be asymp. st.

↳ if P is pos def, so the syst is asymp. stable!

What does it mean? How to use it? Take any *Q>0* and solve the Lyapunov eq. with respect to *P*. If (and only if ) *P>0*, then the system is asymptotically stable

Is it easier than computing the eigenvalues of *A*? No, but it is very important in many control design methods

result of Lyapunov eq usefull NOT only for stability → more usage

take any Q and solve for P → P>0: asymp stable

Lyapunov eq can be solved by MATLAB

↓ "lyap.m" function to compute P MATRIX

(P, Q usefull for CONTROL design theory) proof Lyapunov theory

**Proof – sufficiency (a sort of exercise)**

for LIN. SYST, stability is property of syst we can fix it

In the case of linear systems, in order to study the stability we can set **u**=0. Then the system becomes $\dot{x}(t) = Ax(t)$

Given any matrix $Q > 0$, assume that there exists $P$ satisfying the Lyapunov equation and set $\boxed{V(x) = x'Px}$. It follows that $V(x) > 0$ and

use P as nucleus of quadratic funct of Lyapunov function

$$
\begin{aligned}
\dot{V} &= \dot{x}'Px + x'P\dot{x} = x'A'Px + x'PAx \\
&= x'(A'P + PA)x = -x'Qx < 0
\end{aligned}
$$

→ origin is asymp stable eq point and V is Lyap. funct

Hence, $V$ is a Lyapunov function for the system

⇓

asymp stable for LYAPUNOV TH.

**Proof – necessity:** ***see the textbook***

as seen from pendulum p-plane, near equilibria non lin syst took the traj of linearized syst

**How to find a Lyapunov function for a system linearizable at the equilibrium?**

to compute region of attraction

IF you know an equilibrium, if syst can be linearized near equilibrium

Assume you want to consider the origin as equilibrium point

1. • Linearize the system around equilibrium → find $A$ (IF system can be linearized)
2. • Solve the Lyapunov equation for the linearized system with a $Q>0$ → allows compute the matrix $\boxed{P>0}$
3. • Use the Lyapunov function $V(x)=x'Px$ for the analysis of the equilibrium of the original nonlinear system. A least locally it must work

for original non lin syst.

WORKS because NON LIN system can be approx by the LINEARIZED one near eq.

This can be useful to estimate the region of convergence

(NOT only to check for asymp. stability)

## Example

Analyse the stability of the origin of the system II ORD syst

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \end{cases}$$

origin equilibrium for sure

$x=(0,0) \rightarrow \dot{x}=(0,0)$

***Solution 1***

Just guarantee "STABLE" eq

Consider the quadratic Lyapunov function $V(x) = 0.5(x_1^2 + x_2^2) > 0$

$$\dot{V}(x) = \cancel{-x_1 x_2} + \cancel{x_1 x_2} + (x_1^2 - 1)x_2^2 \leq 0 \text{ locally } (x_1^2 < 1)$$

guarantee only in a certain range (LOCALLY)

$\dot{V} \leq 0$

(mooving $x_1$ but $x_2=0 \rightarrow \dot{V}=0$ so semi def neg!)

Why only semidefinite negative? Because in a neighbor of the origin $x_2=0$ and $x_1$ «small» leads to *Vdot(x)=0*

We cannot guarantee $\dot{V}_2<0$ ... we only have simple stability looking $V(x)$

***Try to use the Krasowski La Salle theorem to prove that the origin is an asymptotically stable equilibrium***

to proof ORIGIN is A.S

find the V(X) with the procedure seen by Lyap equation
using linearized system

***Solution 2***

Compute the linearized model

$$\begin{cases} \delta\dot{x}_1 = -\delta x_2 \\ \delta\dot{x}_2 = \delta x_1 - \delta x_2 \end{cases} \xrightarrow[\text{system state matrix}]{\text{linear}} A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Solve the Lyapunov equation with $\boxed{Q=I}$ (for ex. take) $\rightarrow$ $\boxed{P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}} > 0$

find P through linearized system, then from here obtain V(x) of syst

$$V(x) = x'Px = 1.5x_1^2 - x_1x_2 + x_2^2$$

once defined V(x)
you come back to non lin syst.

$$\dot{V}(x) = \underbrace{-x_1^2 - x_2^2}_{\text{II ORD}} - x_1^3x_2 + 2x_1^2x_2^2 < 0 \text{ locally}$$

II ORD dominating near origin!

higher order terms
(IV ORD)

with this I can conclude "ASYMPTOTIC STABILITY"

Impossible to conclude something about global stability

With *pplane* it can be verified that the origin **is not** a globally stable equilibrium

## Exercise

Given the system (II ORD) → with lots equilibria

$$\begin{cases} \dot{x}_1(t) = (x_1(t) - x_2(t))(x_2^2(t) - 1) \\ \dot{x}_2(t) = (x_1^2(t) + x_2(t))(x_1^2(t) - 1) \end{cases}$$

**Question 1**

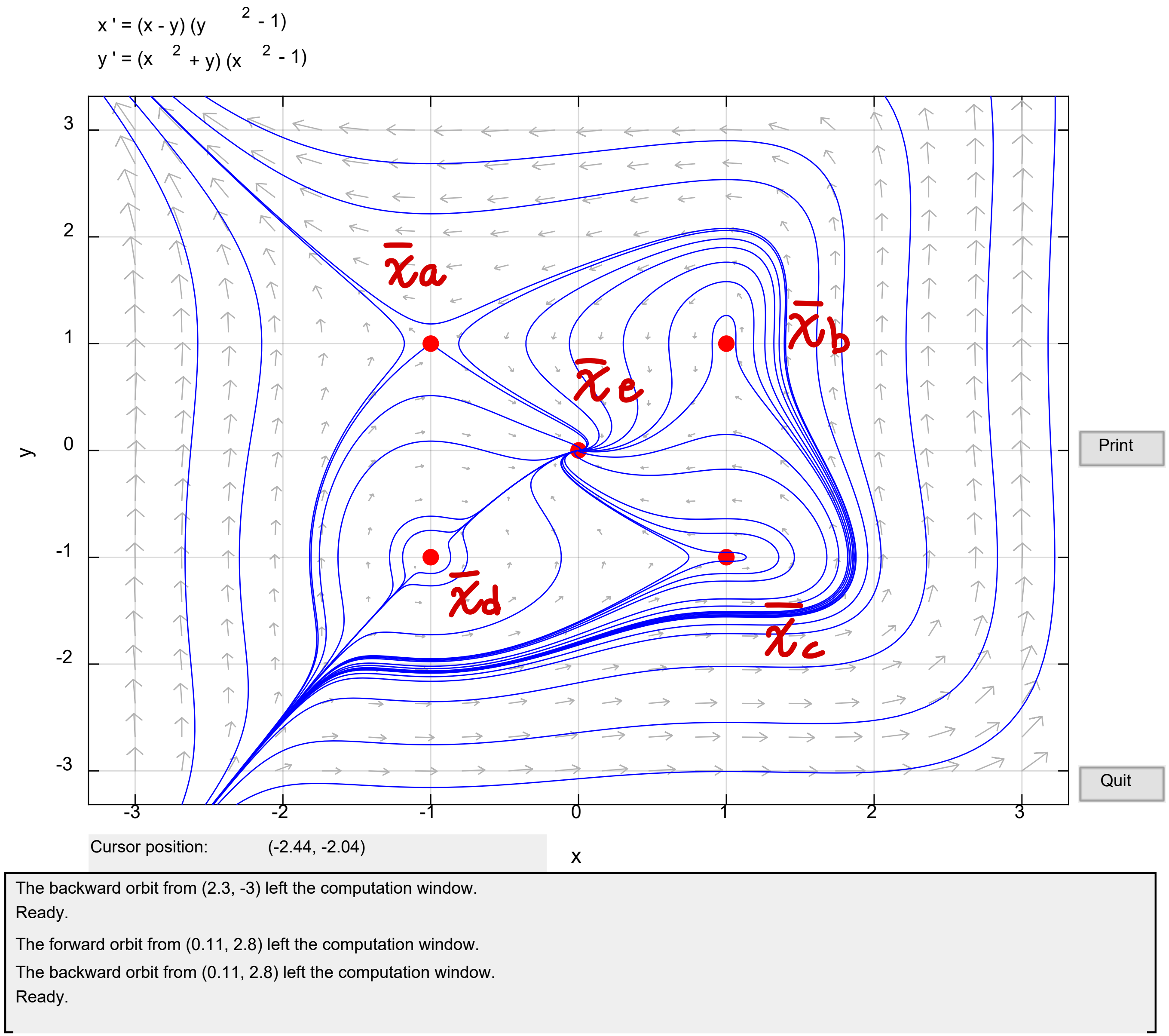
Looking at the phase plane, find the equilibria and judge their stability

↳(Rest points)

**Answer question 1**

$$\bar{x}_a = (-1, 1), \bar{x}_b = (1, 1), \bar{x}_c = (1, -1), \bar{x}_d = (-1, -1), \bar{x}_e = (0, 0)$$

By looking at the trajectories, it seems that the only asymptotically stable equibrium point is (0,0)



**Question 2**

Check if it is possible to conclude something about the stability of $\bar{x}_a, \bar{x}_b, \bar{x}_e$ by looking at the linearized system

**Question 3**

Concerning the equilibrium at the origin, check again the result by $\underbrace{\text{using the Lyapunov function}}$ (in a slightly inproper way) considering

corresponding to that P

$$P = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

(The proper way is: given a *Q>0*, check if the solution *P* of the Lyapunov equation *A'P+PA=-Q)*

to check stability you have A, chose $Q>0 \rightarrow$ and see if P satisfy Lyap $P>0$

$\Rightarrow$ HERE P is given $\rightarrow$ see if $Q>0$ (opposite use of classical approach)

[To avoid matrix equation solution]

**Answer question 2** can be checked by linearized model, linearize the system and analyze each equilibrium point

linearized model

$$\begin{cases} \delta\dot{x}_1 = (\bar{x}_2^2 - 1)\delta x_1 + (2\bar{x}_1\bar{x}_2 + 1 - 3\bar{x}_2^2)\delta x_2 \\ \delta\dot{x}_2 = (4\bar{x}_1^3 + 2\bar{x}_1\bar{x}_2 - 2\bar{x}_1)\delta x_1 + (\bar{x}_1^2 - 1)\delta x_2 \end{cases}$$

$$\bar{x}_a \to A = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \to \text{eigenvalues } s = \mp 4 \text{ unstable (saddle point)}$$

$$\bar{x}_b \to A = \begin{bmatrix} 0 & 0 \\ -4 & 0 \end{bmatrix} \to \text{eigenvalues } s = 0 \text{ no conclusion}$$

$$\bar{x}_c \to A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \to \text{eigenvalues } s = 1, -1 \text{ unstable}$$

**Answer question 3**

The solution to the Lyapunov equation is

inverse approach.. compute Q from P

$$Q = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} > 0$$

Extend result to DISCRETE time systems

{ LYAPUNOV Theory extended ! }

## Design of digital regulators

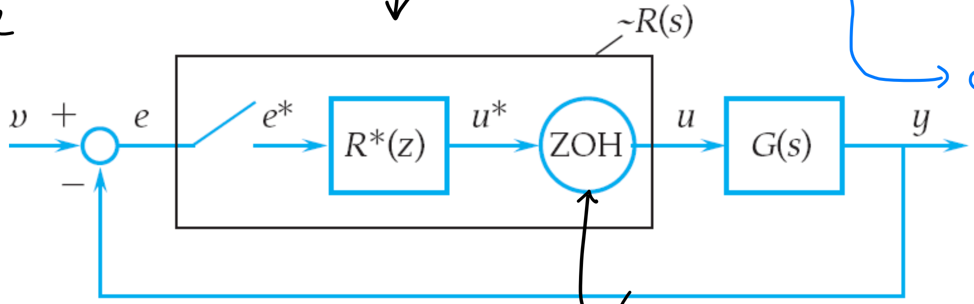
Working on DISCRETE TIME.. 2 STRATEGIES

moove to discrete time systems... NOWDAY most control syst are on discrete-time

**Strategy 1 – discretise a continuous-time regulator**

design R(s) on cont. time and discretize it R*(z)... using also a ZOH and a sampler

from R(s) to R*(z) DISCRETIZATION

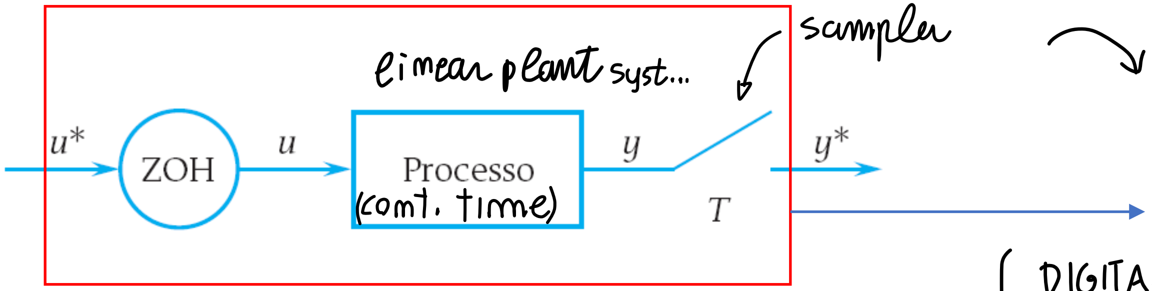



approx discrete by foreward euler, Tustin approx ...

(ZERO ORDER HOLD) ⇒ (with a phase shift effect !)

**Strategy 2 – design a discrete time regulator for a discrete time system**

discretize the plant, find G*(z), and consider discrete syst




overall syst on discrete time

discrete time syst

= G(z)

(T.F)

and syntesis of Reg directly on discrete time

T.F from u* to y*

(DISCRETE TIME SYST)

$$G(z) = C(zI - A)^{-1}B + D$$

↓ a priori or discretized

↳ STABILITY?

DIGITAL state space form

$$\begin{bmatrix} x(k+1) & = Ax(k) + Bu(k) \\ y(k) & = Cx(k) + Du(k) \end{bmatrix}$$

On MIMO Syst is better to work directly on descrete time

On complex Regulators is better to work on discretized system

## Discrete time systems

(discrete time index)

$$\boxed{x(k+1) = f\left(x(k), u(k)\right)}$$ Non linear time invariant (NOT k explicit) system

- $x(k+1) = f(x(k))$ autonomous systems → no input OR constant input
- $x(k+1) = Ax(k) + Bu(k)$ linear systems → when f linear on u, x

**Equilibrium** (constant state) POINT

$$\boxed{\bar{x} = f(\bar{x}, \bar{u})}$$

on discrete time, an equilibrium $\bar{x}$ must be constant... remain on $\bar{x}, \bar{u}$ given syst dym

For linear systems

$$\bar{x} = A\bar{x} + B\bar{u}$$

one solution only if eig ≠ 1 (otherwise ∞ solutions or ∅ solution)

If the eigenvalues of $A$ are different from 1

$$\bar{x} = (I - A)^{-1}B\bar{u}$$

(same definitions on DISCRETE TIME)

## Stability (nothing changes)

The equilibrium $\bar{x}$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all the initial states $x_0$ satisfying

(like on continuous time)

$$\|x_0 - \bar{x}\| \leq \delta$$

one has

$$\|x(k) - \bar{x}\| \leq \varepsilon \quad , \quad \forall k \geq 0$$

The equilibrium is asymptotically stable if, in addition,

$$\lim_{k\to\infty} \|x(k) - \bar{x}\| = 0$$

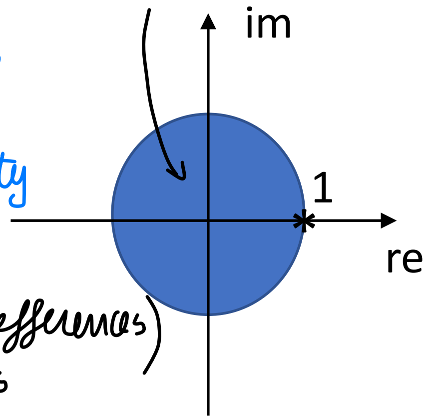
(come back to $\bar{x}$ for $k \to +\infty$)

STABILITY CONDITION for

## Linear systems

Necessary and sufficient condition for the asymptotic stability is that all the eigenvalues of *A* have modulus smaller than 1.

*Stability region*



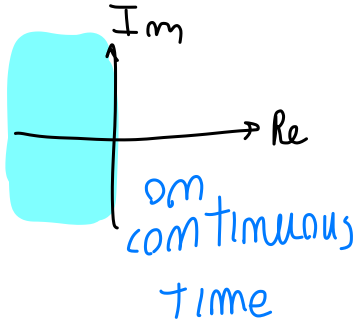

strictly inside stability region

(this makes a lot differences on regulator syntesis)

## Nonlinear systems

same result of continuous time derived...

$$x(k+1) = f\left(x(k), u(k)\right) \xrightarrow[\text{to study stability}]{\textit{Linearize (if possible)}} \delta x(k+1) = A\delta x(k) + B\delta u(k)$$




- if all the eigenvalues of $A$ have modulus smaller than 1, then the equilibrium $(\bar{x}, \bar{u})$ is asymptotically stable;
- if at least one eigenvalue of $A$ has modulus greater than 1, then the equilibrium $(\bar{x}, \bar{u})$ is unstable;
- if all the eigenvalues of $A$ have modulus smaller or equal to 1, nothing can be concluded on the stability properties of the equilibrium from the analysis of the linearized system.

ONLY SUFFICENT conditions

**Lyapunov method** extension on DISCRETE TIME

non lin, autonomous syst.

$$x(k+1) = \overbrace{\varphi(x(k))} \xrightarrow[\text{POINT}]{\text{equilibrium}} \bar{x} = \varphi(\bar{x})$$

$V(\bar{x})=0$, $V(x)>0$ near $\bar{x}$

- If there exists a function $V(x)$ continuous and *positive definite* in $\bar{x}$ and such that on discrete time we look at the increment (derivative does NOT make sense)

$$\Delta V(x) = V(\varphi(x)) - V(x) \leq 0$$

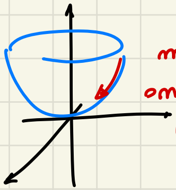
in a neighbor of $\bar{x}$, that is if ($\Delta V(x)$ is semidefinite negative in $\bar{x}$) then the equilibrium is stable. Moreover, if

- $$\Delta V(x) < 0$$

in a neighbor of $\bar{x}$, then $\bar{x}$ is an asymptotically stable equilibrium

## CONT. TIME

$\dot{V} < 0$



moving on this direction

## DISCRETE time

$$\Delta V = V(\underbrace{x(k+1)}_{\varphi(x)}) - V(x(k))$$

( V function decreases )

@ any step $V \downarrow$ so you moove thoghaurds equilibrium

same condition!

STABILITY guaranteed

↳ but is it possible to remain in $V=0$ for $x \neq 0$ ? asymp OR simply ?

**Krasowski - La Salle**

what if $\Delta V(x) \leq 0$

If there exists a function $V(x)$ *positive definite* in $\bar{x}$, with $\Delta V(x)$ negative semidefinite in $\bar{x}$ and the set

$$S := \{x : \Delta V(x) = 0\}$$

does not contain perturbed (with respect to $\bar{x}$) trajectories compatible with the system, then $\bar{x}$ is an asymptotically stable equilibrium

even if $\Delta V(x) = 0 \rightarrow$ system does not end up OR remain where $\Delta V = 0$

## Linear systems

Necessary and sufficient condition for the asymptotic stability of the linear system

(solution by Lyapunov)

$$x(k+1) = Ax(k)$$

is that for any matrix $Q = Q' > 0$ (symmm, pos def) there exists a matrix $P = P' > 0$ solving the Lyapunov equation

same idea of cont. time but diff eq

$$\Rightarrow \quad \| A'PA - P = -Q \|$$

≠ continuous time: $A^T P + PA = -Q$

- **Proof of sufficiency**

(check stability + more property by this equation)

Given any $Q > 0$, assume that there exists $P > 0$ solving the Lyapunov equation. Now consider the Lyapunov function $\underbrace{V(x) = x'Px}_{\text{pos def @ORIGIN}}$. It follows that

$$\begin{aligned}
\Delta V(x) &= x'(k+1)Px(k+1) - x'(k)Px(k) \\
\dots &= x'(k)A'PAx(k) - x'(k)Px(k) \\
&= x'(k)(A'PA - P)x(k) \\
&= -x'(k)Qx(k) < 0
\end{aligned}$$

**Proof of necessity →** see the textbook

**Example** II ORD parametric syst. on $\alpha, \gamma \in \mathbb{R}$

$$\begin{cases} x_1(k+1) = x_2(k)\left(\alpha x_1(k) + \gamma\right) \\ x_2(k+1) = x_1(k)\left(\alpha x_2(k) - 1\right) \end{cases}, \quad \gamma \neq -1$$

find and study equilibrium

**Equilibrium**

$$\begin{cases} \bar{x}_1 = \alpha \bar{x}_1 \bar{x}_2 + \gamma \bar{x}_2 \\ \bar{x}_2 = \alpha \bar{x}_1 \bar{x}_2 - \bar{x}_1 \end{cases} \rightarrow (\gamma+1)\bar{x}_2 = 0 \rightarrow \bar{x}_2 = 0, \ \bar{x}_1 = 0$$

origin as only equilibrium point

Question 1) Compute the values of $\alpha, \gamma$ such that the origin is asymptotically stable

$\Downarrow$

Linearized model

$$\begin{cases} \delta x_1(k+1) = \gamma \, \delta x_2(k) \\ \delta x_2(k+1) = -\delta x_1(k) \end{cases} \rightarrow A = \begin{vmatrix} 0 & \gamma \\ -1 & 0 \end{vmatrix}$$

$$\det(zI - A) = z^2 + \gamma = 0 \rightarrow z = \mp\sqrt{-\gamma}$$

eig values strictly inside unitary circle

Condition $|\gamma| < 1, \ \forall \alpha$

Question 2) For $\gamma = 0.5$ check the asymptotic stability of the linearized system with the Lyapunov equation

Take $Q = \begin{vmatrix} 0.5 & 0 \\ 0 & 1 \end{vmatrix}$, then diag solution

$A'PA - P = -Q \rightarrow P = \begin{vmatrix} 2 & 0 \\ 0 & 1.5 \end{vmatrix} > 0$ !

Lyap. equation DISCRETE TIME

OK, Asymp stability

use the solution of (2) to find V(X) for NONLIN Syst.

Question 3) Use the previous result to prove the asymptotic stability of the origin of the original system

$V(x) = x'Px = 2x_1^2 + 1.5x_2^2$ $\quad V(\varphi(X))$

$\Delta V(x) = 2\Big[\underbrace{x_2(\alpha x_1 + 0.5)}_{x_1(k+1)}\Big]^2 + 1.5\Big[x_1(\alpha x_2 - 1)\Big]^2$

$-2x_1^2 - 1.5x_2^2$ $\quad V(X)$

all function of k

$= 2\left[x_2^2(\alpha^2 x_1^2 + \alpha x_1 + 0.25\right] + \left[x_1^2(\alpha^2 x_2^2 - 2\alpha x_2 + 1\right]1.5$

$-2x_1^2 - 1.5x_2^2$

according to Lyapunov, asympt stability

$= -0.5x_1^2 - x_2^2 + \text{higher order terms} < 0$

LOCALLY

## Control synthesis with the Lyapunov theory (some simple ideas)

design of regulator by using Lyapunov

The Lyapunov theory is the basis for the development of many control synthesis methods for nonlinear systems. Let'a have a look at a couple of examples. Then, we'll introduce a systematic approach for a very specific class of nonlinear systems

(guarantee stability)

(to study NON LIN SYST LYAPUNOV is still used !)
⇓ can be used NOT only for analysis but also for NON LIN SYNTESIS Techniques... of Regulator

### A simple example

Given the system

NON LIN SYST ⇒ NON LIN CONTROL LAW

$$\begin{cases} \dot{x}_1 = -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 = -x_2^3 - x_2 \end{cases}$$

(design control law...)

(idea)

Problem: design a control law *u=k(x)* (*note:* ***nonlinear*** *and* ***state-feedback***) such that the origin is an asymptotically stable equilibrium point of the corresponding closed-loop system → by using Lyapunov

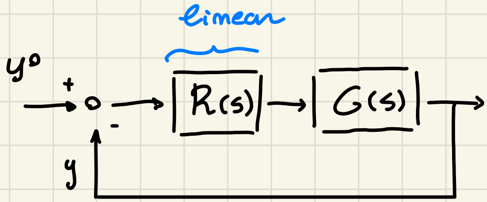
One could use the linearized model, but here we would like to obtain global asymptotic stability

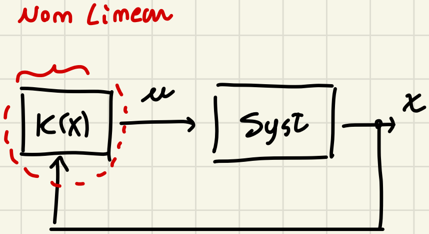
↑ better results!

Restrict analysis to consider state-feedback control law!

↘ Regulator uses the STATE of the system

{ Output feedback }




{ state feedback }

**NONLINEAR CONTROL !**

IF using a linearization approach (with analysis of the linearized system..) ↓ Restricted to work near equilibrium

↓

IF I want a Global stabilization

↳

**System** (Lyapunov theory to guarantee stability)

$\dot V(u)$

$$\begin{cases} \dot x_1 = -3x_1 + 2x_1x_2^2 + u \\ \dot x_2 = -x_2^3 - x_2 \end{cases}$$

(syst equation)

parameter! ⇓ still unknown

taking **Lyapunov function and its derivative**

find lyap function, using a quadratic one

CRITICAL term

$$V(x) = 0.5(x_1^2 + x_2^2) \longrightarrow \dot V(x) = x_1\dot x_1 + x_2\dot x_2 = -3x_1^2 + 2x_1^2x_2^2 + x_1 u - x_2^4 - x_2^2$$

$V(X) > 0$, $V(X) = 0$ near $\underline{X} = \underline{0}$ + Radially unbounded

$u = -2x_1x_2^2$ to cancel critical term

**Control law** state feedback CONTROL LAW

GLOBALLY NEG. DEF

$$\boxed{u = -2x_1x_2^2} \longrightarrow \dot V(x) = -3x_1^2 - x_2^4 - x_2^2 < 0 \longrightarrow \text{origin \textit{globally asympotically} stable}$$

state feedback non lin control law which globally stabilize the origin (There can exist others possible control law)

Idea to set $u$ such that, $u(\underline{x})$ guarantee $\dot V < 0$ $\forall x$ → GLOBAL ASYMP STABILITY

- not the only possible solution
- not a systematic approach

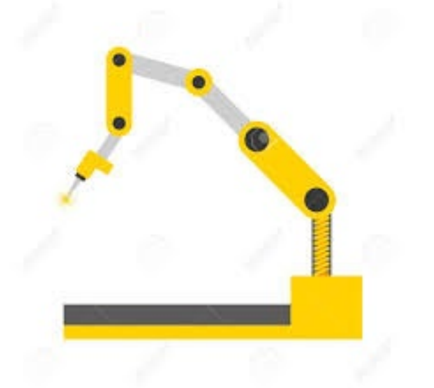
we want systematic alghoritms

↳ here only stability property → NOT performance guarance

PRATICAL Case

INDUSTRIAL

## An industrial problem – control of a manipulator – Lyapunov based solution



position / angle ... objective

Given the coordinates $q$, the goal is to drive the manipulator to the equilibrium defined by $q_d$

RIGID manipulator model ⇒ coord matrix

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v(q)\dot{q} + g(q) = \tau$$

*torque to the joints*
***control variable***

*inertial* (terms of syst.) — *centrifugal & Coriolis* (forces) — *friction* (force depending on $\dot{q}$) — *gravitational* (term)

summ is equal to torque applyed to the joins

change of coord so ORIGIN as equilibrium

Define $\tilde{q} = q_d - q$ and

ENERGY

$$V(\tilde{q},\dot{q}) = \frac{1}{2}\dot{q}' B(q)\dot{q} + \frac{1}{2}\tilde{q}' K_p \tilde{q}, \quad K_p > 0$$

(MATRIX pos.def)

*kinetic energy* — *potential energy*

force our system to reach $q_d$ by working on $\tau$

⇒ (smart choice) of V definition ⇒

Lyap function depending on $q, \dot{q}$ state variable ⇒ selected such that take account of energetic (V) quantities

highly NONLIN SYSTEM → try to solve by Lyapunov theory

$\tilde{q} = q_d - q$ when deriving, we derive $(-\dot{q})$

$$\dot{V} = \dot{q}' B(q)\ddot{q} + \frac{1}{2}\dot{q}'\dot{B}(q)\dot{q} - \dot{q}' K_p \tilde{q}$$

$$= \dot{q}'\left[-C(q,\dot{q})\dot{q} - F_v(q)\dot{q} - g(q) + \tau\right] + \frac{1}{2}\dot{q}'\dot{B}(q)\dot{q} - \dot{q}' K_p \tilde{q}$$

$$= \frac{1}{2}\dot{q}'\left[\dot{B}(q) - 2C(q,\dot{q})\right]\dot{q} - \dot{q}' F_v(q)\dot{q} + \dot{q}'\left[\tau - \cancel{g(q)} - \cancel{K_p \tilde{q}}\right]$$

always structurally equal to zero (Lagrange equations)

to guarantee $\dot{V} < 0$ and properly selecting $\tau$ such that cancel

for $\tau$ choice

Consider the control law

select $\tau$ such that $\boxed{\dot{V} < 0}$ to guarantee asymp stability

compensating gravitational speed

NON LIN control law state feedback

$$\boxed{\tau = g(q) + K_p \tilde{q} - K_d \dot{q}, \quad K_d > 0}$$

compensation of gravitational force

proportional to the error

additional derivative term proportional to derivative of $q$ → PD CONTROL MULTIVARIABLE

(like pendulum)

$$\dot{V} = -\dot{q}' F_v(q)\dot{q} - \dot{q}' K_d \dot{q} \le 0 \longrightarrow \dot{V} < 0 \text{ when } \dot{q} \neq 0,\ \dot{V} = 0 \text{ for } \dot{q} = 0 \text{ and for any } \tilde{q}$$

(with this $\tau$ control law)

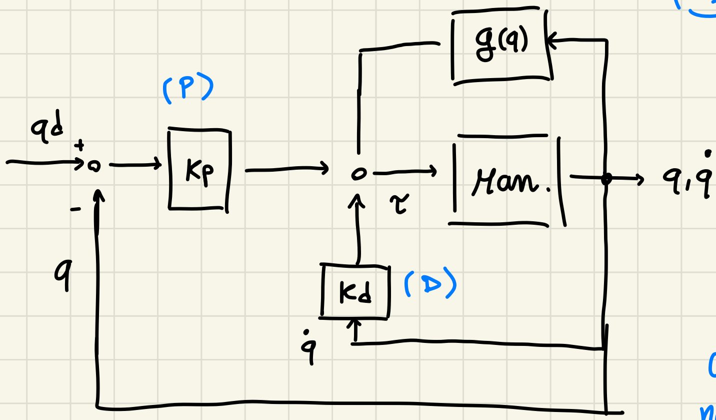
$(\dot{\tilde{q}} = -\dot{q})$ derivative term

velocity tends to 0

$\Downarrow$ $\dot{V}\downarrow\downarrow$ until $\dot{q} = 0$

BUT valid $\forall \tilde{q}$... NOT enough to say we reach origin ⇒ STEADY STATE

BUT is possible to have $\dot{q} = 0$ with $\tilde{q} \neq 0$? LESELLE Theorem (NOT POSSIBLE!) ← $\dot{q} = 0$ reached like a PD controller

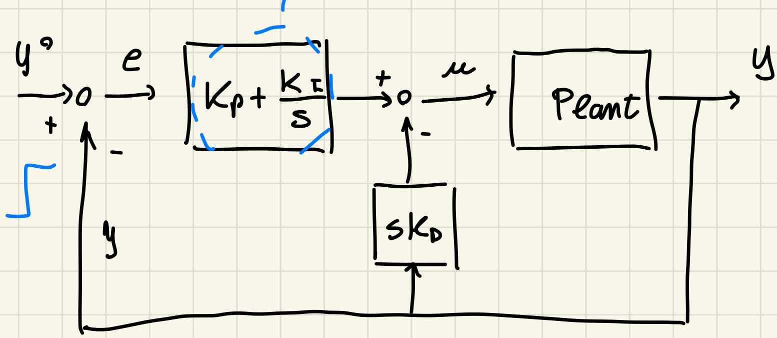



$$K_p = \begin{bmatrix} K_{p1} & & 0 \\ & \ddots & \\ 0 & & K_{pn} \end{bmatrix} > 0$$

$$K_d = \begin{bmatrix} K_{d1} & & 0 \\ & \ddots & \\ 0 & & K_{dn} \end{bmatrix} > 0$$

We can modify performance selecting well the terms $K_p, K_d$ properly!

to remove impulsive responces



back to syst motion... it is possible to have $\dot{q}=0$?

From the model equation with the selected control law

$\dot{q}=0,\ \tilde{q}\neq 0$? NOT possible

$K_p>0$ by choice pos. def.

consider the $\tilde{\tau}$ fixed by us...

$\dot{q}=0,\ \ddot{q}=0$

$$\cancel{B(q)\ddot{q}} + \cancel{C(q,\dot{q})\dot{q}} + \cancel{F_v(q)\dot{q}} = K_p\tilde{q} - \cancel{K_d\dot{q}} \qquad \text{for } \dot{q}=0$$

$\boxed{K_p\tilde{q}=0}$ for $\tilde{q}\neq 0$? $\boxed{NO}$ → only for $K_p\tilde{q}=0 \Leftrightarrow \tilde{q}=0 \rightarrow \boxed{q=q_d}$

only asymp stable eq. point

defining $S=\left\{\xi=\begin{bmatrix}\tilde{q}\\ \dot{q}\end{bmatrix} : \dot{q}=0\right\}$, one has $K_p\tilde{q}=0$, $\forall\xi\in S$ and, since $K_p>0$, $\tilde{q}=0$. In conclusion, $q=q_d$ and $\dot{q}=0$ is the only asymptotically stable equilibrium

interesting result, WIDELY used in ROBOTICS

Syntesis problem by LYAPUNOV theory → defintion of that CONTROL LAW

(Non linear control small hints)

**Notice** $K_p, K_d$ taken $> 0$

$\Rightarrow$ obviously syst behav. depends on our parameters

LYAPUNOV : related to STABILITY

BUT we are interested also on Properties

($\alpha$ choice is an open issue)

↳ $\alpha > 0$ how? it has a significant role on adaptation speed !

**Example of application of the Lyapunov theory – noise cancellation (from Astrom, Murray: «*Feedback systems*» WIKIBOOK)**

in headphones

↳ Lyapunov theory
to solve a particular problem

Internal microphone, picks up the signal e, combination of the desired signal ***s*** and of the external noise

↑
remove external noise
and have only s!

External microphone, picks up the external noise

what you want to cancel

Internal microphone, picks up the signal ***e***, combination of the desired signal ***s*** and of the external noise

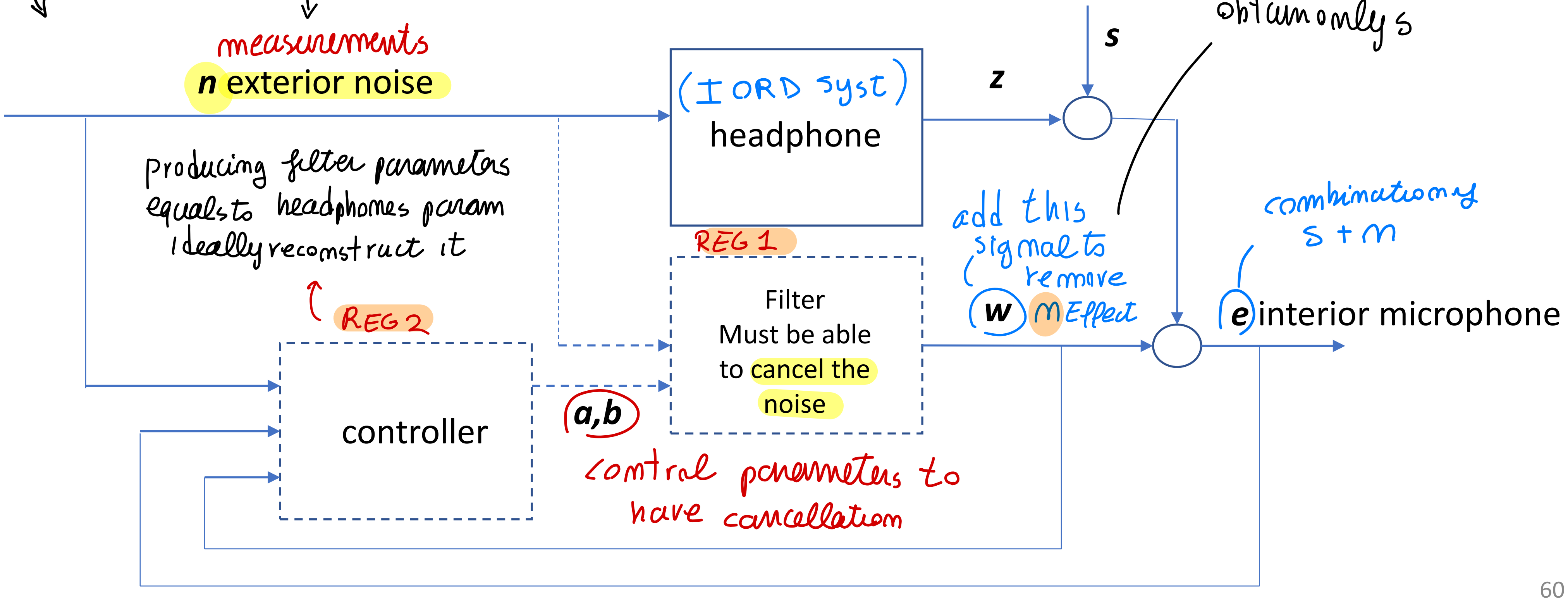
External microphone, picks up the external noise

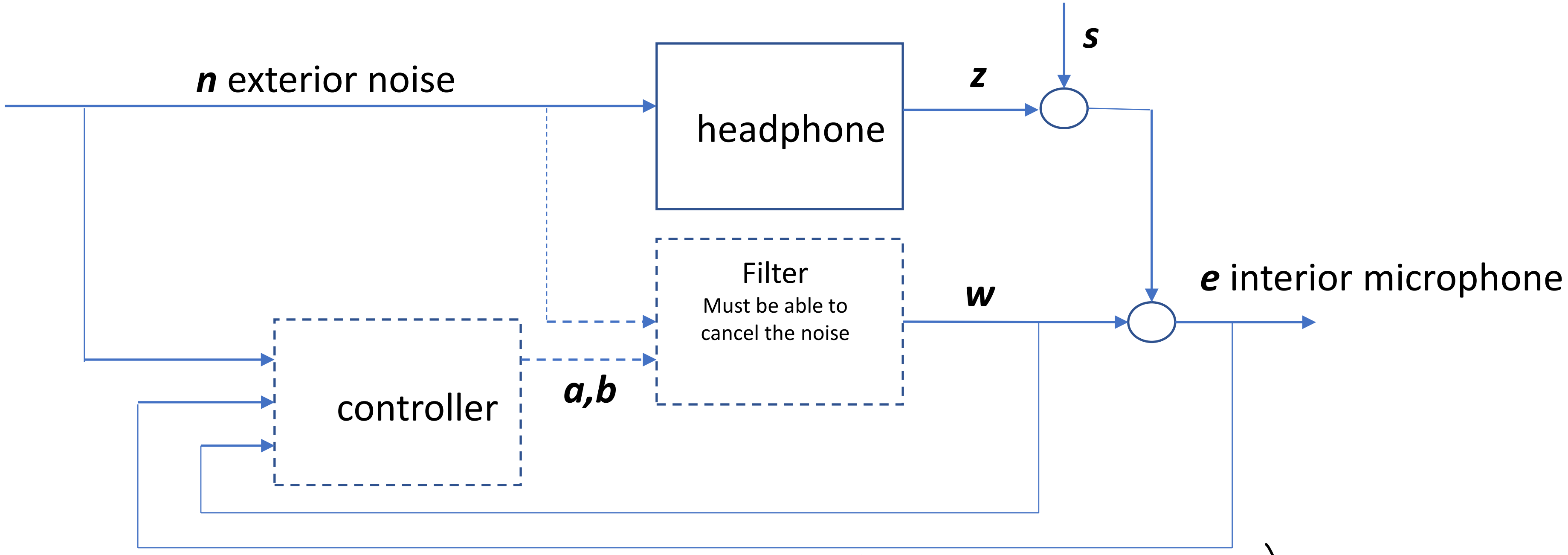
**Control scheme**

{ external microphone }

measurements

2 Block Regulator






I ORD syst headphones (asymp stable)

Assumption $\Rightarrow \left\{ \dot{z}(t) = a_o z(t) + b_o n(t) \right\}$, $a_0 < 0$, where $a_o, b_o$ are unknown

to reconstruct for cancelling the noise

I ORD syst which must be asymp st.

Filter $\dot{w}(t) = aw(t) + bn(t)$

I ORD FILTER, tuned a,b such that we cancel the noise ($a_o \simeq a$, $b_o \simeq b$) ideally

Consider $s = 0$ (we focus on noise)

Real signal to use (music) only noise remove

Define (skip the dependence on time) $x_1 = w - z = e$, $x_2 = a - a_o$, $x_3 = b - b_o$

$(s = 0)$

to study the system

We would like to have (at least) $x_1 \to 0$ $(e \to 0)$ $(w \simeq z)$ ideally

$$\dot z(t) = a_o z(t) + b_o n(t) \qquad \dot w(t) = aw(t) + bn(t) \qquad x_1 = e = w - z,\ x_2 = a - a_o,\ x_3 = b - b_o$$

By my new definitions

$$\dot x_1 = \dot w - \dot z = aw + bn - a_o z - b_o n = (a - a_o)w + a_o(w - z) + (b - b_o)n =$$
$$x_2 w + a_o x_1 + x_3 n$$

parameter such that $V(x)>0$ pos def ...

Take the Lyapunov function $V(x) = 0.5(\alpha x_1^2 + x_2^2 + x_3^2),\quad \alpha > 0$

$\Downarrow$

$$\dot V(x) = \alpha x_1 \dot x_1 + x_2 \dot x_2 + x_3 \dot x_3 = \alpha a_o x_1^2 + x_2(\cancel{\alpha x_1 w} + \dot x_2) + x_3(\cancel{\alpha x_1 n} + \dot x_3)$$

(TO obtain $\dot V < 0$) choose properly $\dot x$

select such that

Set $\dot x_2 = -\alpha x_1 w = -\alpha w e \qquad \dot x_3 = -\alpha x_1 n = -\alpha n e$

Then $\dot V(x) = \underbrace{\alpha}_{>0} \overbrace{a_o}^{<0} x_1^2 = \alpha a_o e^2 \le 0$

$\ne 0$ until $x_1 = 0$

(filter noise $-z$)

$\dot V \downarrow\downarrow$ until $x_1 = 0$ so $e \simeq 0$ $\longrightarrow$ $\boxed{e \to 0}$ $w - z \to 0$

($\dot V \downarrow\downarrow$ until reach $x_1 \simeq 0$)

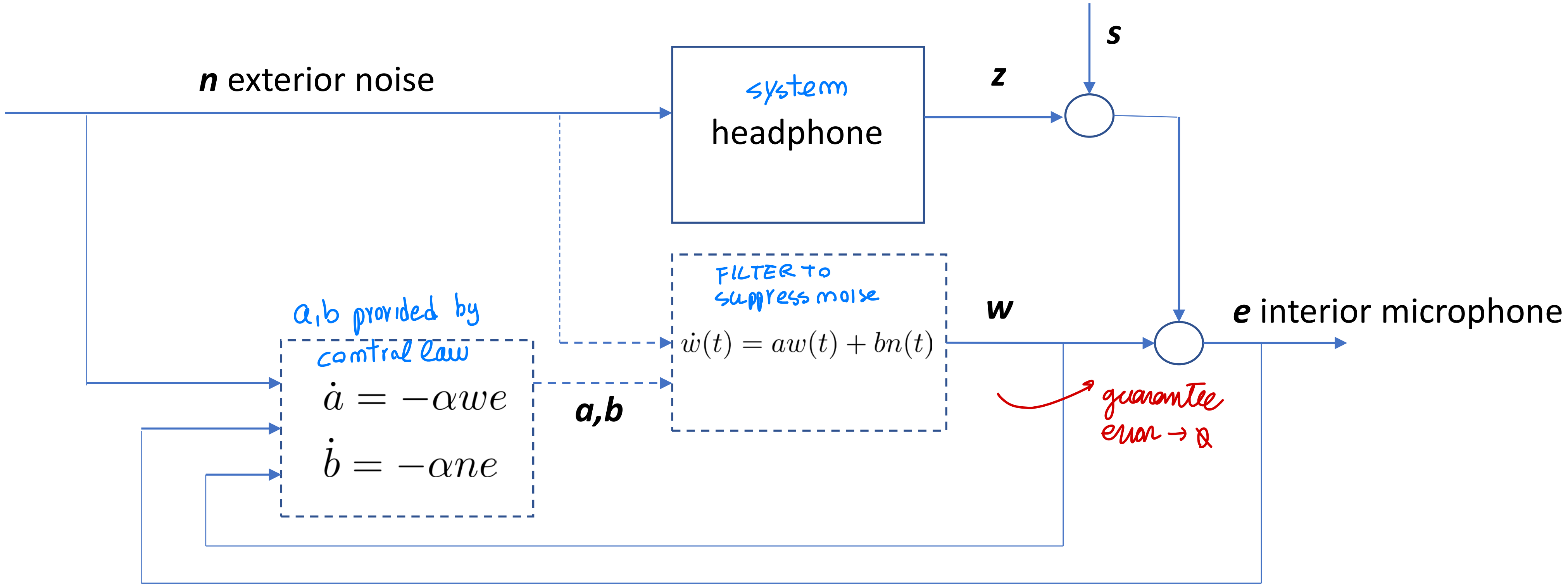
Note that $\dot x_2 = \dot a \qquad \dot x_3 = \dot b$ $\longrightarrow$ $\begin{cases} \dot a = -\alpha w e \\ \dot b = -\alpha n e \end{cases}$

a,b evolution:

states of I ORD syst

***adaptive control law***

↓ modify param based on syst




designing controllers
by Lyapunov theory tools

## Overall signal: sinusoid (*s*) + broad band noise (*n*)

a(t), b(t) evolution...

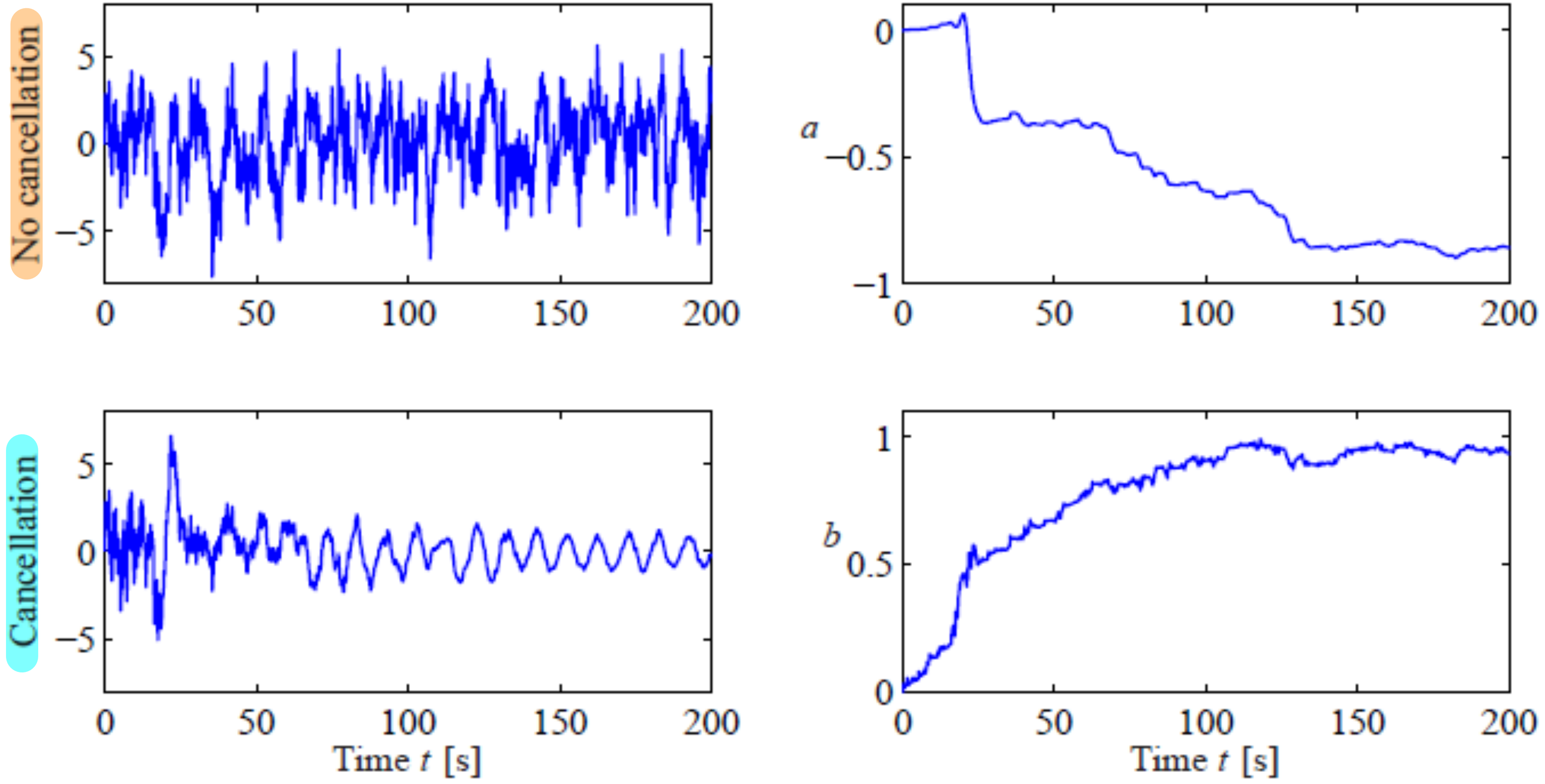



**Figure 4.20:** Simulation of noise cancellation. The top left figure shows the headphone signal without noise cancellation, and the bottom left figure shows the signal with noise cancellation. The right figures show the parameters *a* and *b* of the filter.

METHOD to design NONLIN CONTROLLER

stabilizing state feedback controllers

**A systematic approach – the backstepping method (based on Lyapunov theory)** → limitation of usage only for some systems

with LYAPUNOV Theory

for Design of NON LIN CONTROLLER

Consider the system (particular class of system!) → control law which stabilize system origin?

$$\begin{cases} \dot{x}_1(t) = f(x_1(t)) + g(x_1(t))x_2(t) \\ \dot{x}_2(t) = u(t) \end{cases}, \quad x_1 \in R^n, \quad x_2 \in R^1$$

⇓

restrict the usage! Limiting assumption to simplify description

where $f$ and $g$ are continuous and differentiable in a set $D \subset R^n$ and $f(0) = 0$

(this will be used for magnetic levitation syst.)

Assume to know a «*fictitious* control law»

$$x_2 = \phi_1(x_1), \ \phi_1(0) = 0$$

***Note: $x_2$ is a state, not a control variable***

such that the origin of the fictitious closed-loop system

focus on I equation

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1)$$

is an asymptotically stable equilibrium !

$\phi_1$ such that asymp. stability

such that origin is an asymp stable, $x_2$ as control variable

## A systematic approach – the backstepping method (based on Lyapunov theory)

for EXAM you should know!

Consider the system

system structure

2 equations describing overall system

$$\begin{cases} \dot{x}_1(t) = f(x_1(t)) + g(x_1(t))x_2(t) \\ \dot{x}_2(t) = u(t) \end{cases}, \quad x_1 \in R^n, \quad x_2 \in R^1$$

(can be generalized ...)

steps required

where $f$ and $g$ are continuous and differentiable in a set $D \subset R^n$ and $f(0) = 0$

try to use backstepping? → only writing the system on that form and find f, g

Assume to know a «*fictitious* control law»

assume $x_2$ not really a STATE, but a fictitious CONTROL VARIABLE

$$x_2 = \phi_1(x_1), \ \phi_1(0) = 0$$

↑ find this which produced $x_2$

**Note: $x_2$ is a state, not a control variable**

such that the origin of the fictitious closed-loop system

which stabilizes the origin of I eq.

FICTITIOUS CONTROL SYST...

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1)$$

(TRICK) because by default $x_2$ is a state

is an asymptotically stable equilibrium

$\phi_1$ such that ORIGIN is an asymp stable equilibrium with Lyapunov

Assume that for the fictitious closed-loop system

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1)$$

for this we can find a lyap function of $x_1$ which is such that!

you know a Lyapunov function $V_1(x_1) > 0$ such that

$$\dot{V}_1(x_1) = \frac{dV_1}{dx_1}\left(f(x_1) + g(x_1)\phi_1(x_1)\right) < 0$$

ORIGIN asymp. stable eq point for system with $\phi_1$ instead $x_2$

{Knowing $\phi_1$, $V_1$}

NOT needed to remember, provided if need to be used!

we should find $V_1$ and Lyapunov function

Then, the control law

unk design parameter

$$u = -\frac{dV_1(x_1)}{dx_1}g(x_1) - k(x_2 - \phi_1(x_1)) + \frac{d\phi_1(x_1)}{dx_1}\left(f(x_1) + g(x_1)x_2\right)$$

(influence also performance)

k>0 is a design parameter

stabilization! NOTHING discuss about performance

asymptotically stabilizes the original system with Lyapunov function

taking this formula of $u$ (with almost all terms known)

$$V_2(x_1, x_2) = V_1(x_1) + \frac{1}{2}\left(x_2 - \phi_1(x_1)\right)^2$$

by computation we can prove that by this Lyap funct... we check that ORIGIN is asympt. on all syst. by $V_2$

***Proof: nice exercise of the Lyapunov theory***

## Summary and example

II ORD syst.

$$\begin{cases} \dot{x}_1 = \underbrace{\boxed{x_1^2 - x_1^3}}_{f(x_1)} + x_2 \quad g(x_1) = 1 \\ \dot{x}_2 = u \end{cases}$$

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1) \quad , \quad \dot{x}_2 = u$$

$$u = -\frac{dV_1(x_1)}{dx_1}g(x_1) - k\left(x_2 - \phi_1(x_1)\right) + \frac{d\phi_1(x_1)}{dx_1}\left(f(x_1) + g(x_1)x_2\right)$$

**Step 1**: find a «fictitious control law» $x_2 = \phi_1(x_1) = -x_1^2 - x_1$

we should define $\phi_1$ such that substituted to $x_2$ stabilize the first set of equation

$x_2$ as function $x_1$

its hard to find $\phi_1$ and $V_1$ looking on syst

such that

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1) = -x_1^3 - \cancel{x_1}$$

Lyap. function

is asymptotically stable with $V_1(x_1) = 0.5x_1^2$ $\longrightarrow \dot{V} = x_1\dot{x}_1 = -x_1^4 - x_1^2 < 0$

(QUADRATIC)

HARD part is to find $\phi_1, V_1$

**Step 2**: apply the formula of the control law with with known terms

$$-\frac{dV_1(x_1)}{dx_1}g(x_1) = -x_1$$

$$-k\left(x_2 - \phi_1(x_1)\right) = -(x_2 + x_1 + x_1^2)$$

$$\frac{d\phi_1(x_1)}{dx_1}\left(f(x_1) + g(x_1)x_2\right) = -(2x_1 + 1)\left(x_1^2 - x_1^3 + x_2\right)$$

CONTROL LAW found automatically

$$\boxed{u = -x_1 - (x_2 + x_1 + x_1^2) - (2x_1 + 1)\left(x_1^2 - x_1^3 + x_2\right)}$$

than from $V_2$ we can check global asymp st.

$$\dot{x}_1 = f(x_1) + g(x_1) \cdot x_2$$

← to identity

↓

$$\dot{x}_1 = \underbrace{x_1^2 - x_1^3}_{f(x_1)} + x_2$$

↑ $g(x_1) = 1$

$\phi_1$ chosen such that $\dot{x}_1 = x_1^2 - x_1^3 + \overset{x_2}{\overset{\|}{\phi_1(x_1)}}$

has the ORIGIN as asymp stable equilibrium

→ difficult to find this $\phi_1(x_1)$

on the EXAMPLE $\phi_1(x_1)$ is simply defined by polynomial

such that $\dot{x}_1 = \underbrace{-x_1^3 - x_1}$

odd power of $x$ such that later when evaluate $V$ I get even power of $x$

!

NOT strictly necessary... I can take different $\phi$, BUT I prefer a simple solution! → DOF on the choice!

## A generalization

(simple extension)

(generalization Respect the case when $f_2=0$ $g_2=1$)

simple trick...

Consider the system

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t))x_2(t) \quad , \quad x_1 \in R^n, \ \ x_2 \in R^1 \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t))u(t) \end{cases}$$

assume a general $\dot{x}_2$

where $\underbrace{f_2 \in C^1}_{\text{scalar function}}$, $g_2 \in C^1$ and $g_2(x_1,x_2) \neq 0$ in the domain of interest (on state space where we work).

$\dot{x}_2 = \cancel{f_2} + u_a - \cancel{f_2}$

Set

we cancel out the new part and go back to previous form

$$u = \frac{1}{\boxed{g_2(x_1,x_2)}} \{u_a - \overbrace{f_2(x_1,x_2)}^{\text{cancel out } f_2}\}$$

$\neq 0$! where it works

remember the procedure!

where $u_a$ is a fictitious input to be properly selected. It then follows that

with this simplification... wise choise of $u$

$$\begin{cases} \dot{x}_1(t) = f_1(x_1) + g_1(x_1)x_2(t) \\ \dot{x}_2(t) = u_a \end{cases}$$

go back to previous standard form

This system has the standard form previously considered. Then, we can apply the corresponding theory

↳ previously theory to FIND $u_a$ and than invert the formula of $u_a(u)$ expressing $u$ ⇒

It turns out that

once found u for formula, we solve respect u!

$$
\begin{aligned}
u &= \frac{1}{g_2(x_1,x_2)}\left\{\frac{d\phi_1(x_1)}{dx_1}\left(f_1(x_1)+g_1(x_1)x_2\right)-k\left(x_2-\phi_1(x_1)\right)\right. \\
&- \left.\frac{dV_1(x_1)}{dx_1}g_1(x_1)-f_2(x_1,x_2)\right\}
\end{aligned}
$$

and the Lyapunov function for the closed-loop system is

and then you can define the lyap funct

$$V_2(x_1,x_2)=V_1(x_1)+\frac{1}{2}\left(x_2-\phi_1(x_1)\right)^2$$

**A systematic approach: *feedback linearization* (not strictly related to Lyapunov theory)**

A simple idea is to algebraically transform the dynamics of the nonlinear system into a linear one, and then to apply linear control techniques to stabilize the transformed linear system

Consider the system

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned}$$

and assume that $g_2(x_1, x_2) \neq 0$. Then consider the control law

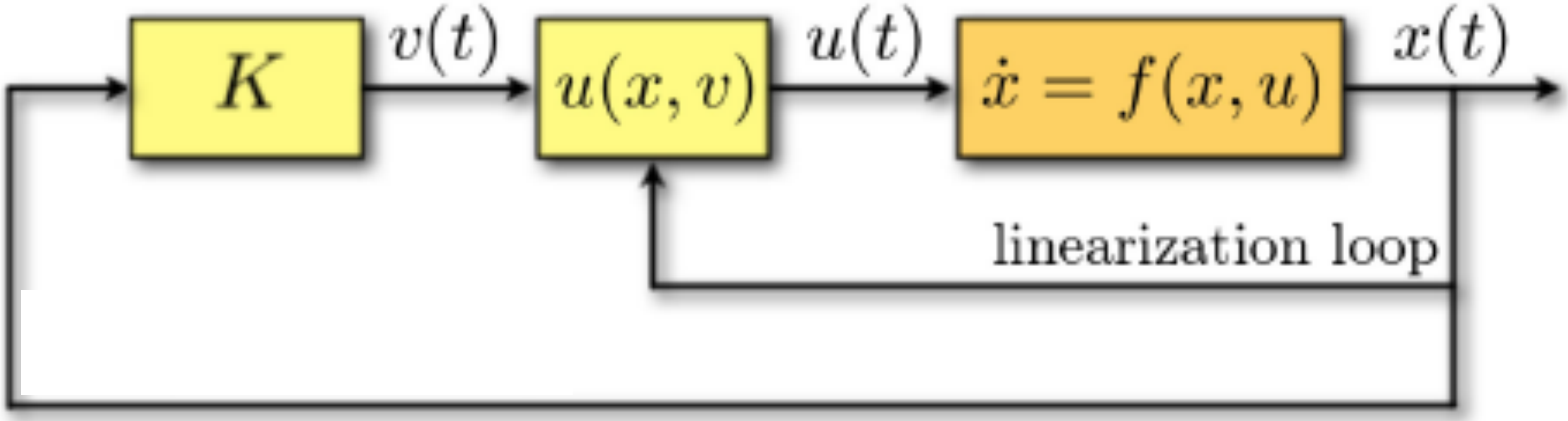
$$u = \frac{v - f_2(x_1, x_2)}{g_2(x_1, x_2)}$$

The system becomes linear

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= v \end{aligned}$$

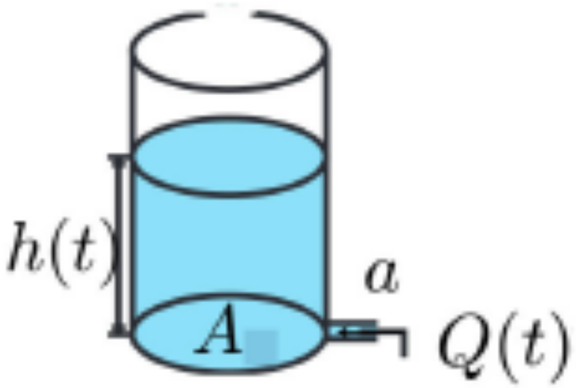
and with linear control theory one can seach for a stabilizing control law, for example $v = Kx$

A much more general theory has been developed. However, the basic idea is simply summarized in the following control scheme




*Courtesy of Prof. A. Bemporad*

**An example**




Model of the system $A\dot{h} = -a\sqrt{2gh} + u$

Reference level $h^o$

Feedback linearization law $u = -a\sqrt{2gh} + Av$

Resulting system $\dot{h} = v$ (integrator)

Proportional control law $v = ke, \quad e = h^o - h, \quad k > 0$

$$\dot{e} = -\dot{h} = -ke, \quad e \to 0, \quad h \to h^o$$

**Exercise** Lyapunov – Continuous time

$$\begin{cases} \dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2 \\ \dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2) \end{cases}$$

Study the stability of $\bar{x} = 0$

$$V(x) = 1/2\,(x_1^2 + x_2^2)$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2(x_1^2 + x_2^2 - 2) - \cancel{4x_1^2x_2^2}$$

$$+ \cancel{4x_1^2x_2^2} + x_2^2(x_1^2 + x_2^2 - 2)$$

$$= (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) < 0$$

$< 0$ in $x_1^2 + x_2^2 < 2$

Locally neg. def.

## Exercise Lyapunov-discrete time

$$x(k+1) = \frac{x(k)}{\sqrt{x^2(k)+1}}$$ study the stability of the origin

Linearised system $$\delta x(k+1) = \left.\frac{\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}}{x^2+1}\right|_{x=0} \delta x(k)$$

$\delta x(k+1) = \delta x(k) \rightarrow$ eigenvalue $= 1$, no conclusion

With Lyapunov $V(x) = x^2$

$$\Delta V(x) = x^2(k+1) - x^2(k) = \left(\frac{x}{\sqrt{x^2+1}}\right)^2 - x^2 = \frac{-x^4}{x^2+1} < 0$$

asymptotically stable

## EXERCISE

Consider the system

$$\dot{x}_1(t) = -x_1(t) + x_2(t)$$
$$\dot{x}_2(t) = -x_1^3(t) - x_2^3(t)$$

and the Lyapunov function $V(x) = bx_1^4 + ax_2^2(t)$. Select proper values of $a$, $b$ such that $V(x)$ can be used to prove the stability of the origin.

a. a=1, =b=1
b. a=2, b=1
c. a=0,b=1

**SOLUTION**

Consider the system

$$\dot{x}_1(t) = -x_1(t) + x_2(t)$$
$$\dot{x}_2(t) = -x_1^3(t) - x_2^3(t)$$

and the Lyapunov function $V(x) = bx_1^4 + ax_2^2(t)$. Select proper values of $a$, $b$ such that $V(x)$ can be used to prove the stability of the origin.

*Vdot=-4b$x_1^4$+4b$x_1^3x_2$-2a$x_1^3x_2$-2a$x_2^4$*

a. *a=1, =b=1 (Vdot=-4$x_1^4$+2$x_1^3x_2$-2$x_2^4$)*
b. *a=2, b=1 (Vdot=-4$x_1^2$-4$x_2^4$<0)*
c. *a=0,b=1 (V is not a Lyapunov function)*

**SOLUTION**

discrete time, so I have to check $\Delta V(x)$

Consider the discrete time system

$$\begin{cases} x_1(k+1) = 0.5x_1(k) + 1.5x_1^2(k)x_2(k) \\ x_2(k+1) = 0.5x_2(k) \end{cases}$$

By means of the quadratic Lyapunov function $V(x) = x_1^2 + x_2^2$, discuss the stability of the origin.

- ○ The origin is a locally asymptotically stable equilibrium
- ○ The origin is a globally asymptotically stable equilibrium
- ○ The origin is an unstable equilibrium
- ○ Nothing can be concluded with that Lyapunof function

**EXERCISE**

Consider the discrete time system

$$x_1(k+1) = 0.5x_1(k) + 1.5x_1^2(k)x_2(k)$$
$$x_2(k+1) = 0.5x_2(k)$$

By means of the quadratic Lyapunov function $V(x) = x_1^2 + x_2^2$, discuss the stability of the origin.

- ○ The origin is a locally asymptotically stable equilibrium ←
- ○ The origin is a globally asymptotically stable equilibrium
- ○ The origin is an unstable equilibrium
- ○ Nothing can be concluded with that Lyapunof function

$$\Delta V(x) = \underbrace{\left(0.5x_1 + 1.5x_1^2x_2\right)^2}_{x_1^2(k+1)} + \underbrace{(0.5x_2)^2}_{x_2^2(k+1)} - x_1^2 - x_2^2$$

$$= -0.75x_1^2 - 0.75x_2^2 + 1.5x_1^3x_2 + (1.5)^2x_1^4x_2^2$$

$\leq 0$ locally

Consider a second order system with dynamic matrix $A$, and a matrix $Q$ equal to the 2x2 identity, the solution of the Lyapunov equation is $P = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$. Then, the system is

$\Downarrow$

$P > 0$

$\hookrightarrow (s-0.5)(s-1.5) - 0.25$

$= s^2 - 2s - 1$

$s = \dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ $\nearrow >0$ $\searrow <0$

- asymptotically stable ←
- nothing can be concluded
- unstable
- simply stable