Exercise session 1 - Structural Properties

Advanced and Multivariable Control

Lorenzo Nigro, Prof. Riccardo Scattolini
Dipartimento di Elettronica, Informatica e Bioingegneria - Politecnico di Milano

Some preliminary information

Professor: Riccardo Scattolini

Teaching assistants: Lorenzo Nigro

Lab assistants: Lorenzo Nigro, Matteo Luigi De Pasali

Slide credits: Fabio Bonassi

- Anyone is invited to raise questions during the exercise sessions.
- If you wish to further discuss the topic, drop an e-mail asking for an appointment.
- No online classroom, registrations of last year available on WeBeep.
- **Attending exercise sessions and laboratories is highly recommended!**

Contact information:

- Mail Address:
  - `lorenzo.nigro@polimi.it`
  - `matteoluigi.depascali@polimi.it` (LAB)
- Office: 016, Building 20, 2nd floor

Some organizational stuff

- Exercise sessions are on Tuesday, exceptions will be notified in advance. One class only.
- Six laboratories of 3 hours each, held on Tuesdays, with me and Matteo Luigi De Pascali (S.0.1), One class only.
- A schedule of lessons, excercise Sessions and laboratories will be shared on Beep.

Concerning the laboratories, the following software will be necessary:

- MATLAB (my version is R2022a)
- pplane (see WeBeep)
- Control System Toolbox
- CasADI Toolbox (web.casadi.org/get)
  Download the .zip and add it to your Matlab path

Reachability in continuous-time

POLITECNICO MILANO 1863

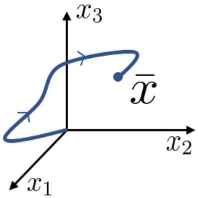
arbitrary state $\bar{x}$

Definition - Reachability (continuous-time systems)

Given the continuous-time system $\dot{x}(t) = A\,x(t) + B\,u(t)$, a state $\bar{x}$ is said to be **reachable** if there exists an arbitrary finite time $\bar{t}$ and an input realization $\bar{u}(\tau)$, $\tau \in [0, \bar{t}\,)$, such that starting from the origin (i.e. $x(0) = 0$), $x(\bar{t}\,) = \bar{x}$.

independently from traj taken it reach state $\bar{x}$

In other words, a state is reachable if it is possible to design an input sequence to **bring the state** $x(t)$ **from the origin to the desired value**.



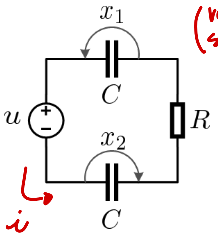
Definition

A system is said to be **fully reachable** if all its states are reachable.

Reachability - Example

two states = voltage V on capacitors

Consider a system made of two capacitors of same size, at equal initial charge.



control input ("voltage source")

$$\begin{cases} u = x_1 + R\,i + x_2 \\ C\dot{x}_1 = i \\ C\dot{x}_2 = i \end{cases} \rightarrow \begin{cases} i = -\frac{1}{R}\left[x_1 + x_2 - u\right] \\ \dot{x}_1 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \\ \dot{x}_2 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \end{cases}$$

Taking $x = [x_1, x_2]^T$ as state vector, being u the input:

{System matrix}

$$A = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{RC} \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{RC} \end{bmatrix}$$

NON Reachable ... one current, same voltage

Remark: The system is not fully reachable, since the two states x_1 and x_2 must be equal (capacitors with same size and initial states, fed by the same current i). The "target" states $\bar{x} = (\bar{x}_1, \bar{x}_2)$ are reachable if and only if $\bar{x}_1 = \bar{x}_2$.

$\bar{x} = [x_1 \neq x_2]$ non reachable

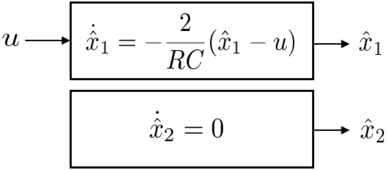
↳ changing the variables

To highlight this, let's make a change of variables: $\hat{x}_1 = x_1 + x_2$, $\hat{x}_2 = x_1 - x_2$. Then:

$$\begin{cases} \dot{\hat{x}}_1 = \dot{x}_1 + \dot{x}_2 = -\frac{2}{RC}\left[\hat{x}_1 - u\right] \\ \dot{\hat{x}}_2 = \dot{x}_1 - \dot{x}_2 = 0 \end{cases} \quad \rightarrow \quad \hat{A} = \begin{bmatrix} -\frac{2}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \frac{2}{RC} \\ 0 \end{bmatrix}$$

$\Downarrow$ In this way we can represent system:

(different set of variables matrix)



By means of this change of variables, the system is decomposed into:

- A reachable part, $\hat{x}_1$, function of the input u.
- An unreachable part, $\hat{x}_2$, onto which no control variable is acting.

u acts only on $\hat{x}_1$ while $\hat{x}_2$ is not reached by u $\rightarrow$ x_2 is unreachable!

$\hookrightarrow$ in discrete time $\Rightarrow$

Reachability in discrete-time

POLITECNICO MILANO 1863

Definition - Reachability (discrete-time systems)

Given the system $x(k+1) = A\,x(k) + B\,u(k)$, with $x(0) = 0$, a state $\bar{x}$ is said to be **reachable in $\bar{k}$ steps** if there exists an input sequence $u(0), \dots, u(\bar{k})$ such that $x(\bar{k}) = \bar{x}$.

Definition

A system whose states are reachable in $\bar{k}$ steps is **fully reachable in $\bar{k}$ steps**.

Reachability Matrix

POLITECNICO MILANO 1863

How to assess the reachability? We start from discrete-time system and then extend the results to continuous-time ones. Starting from $x(0) = 0$, one has that:

$$x(1) = \cancel{Ax(0)} + Bu(0) = Bu(0)$$
$$x(2) = Ax(1) + Bu(1) = ABu(0) + Bu(1)$$
$$x(3) = Ax(2) + Bu(2) = Ax(1) + .. = AABu(0) + ABu(1) + Bu(2)$$
$$\vdots$$
$$x(n) = A^{n-1}Bu(0) + A^{n-2}Bu(1) + ... + ABu(n-2) + Bu(n-1)$$

group it into a matrix M_R (in reversed order)

Definition - Reachability Matrix

$$\mathcal{M}_R = \begin{bmatrix} B & AB & ...A^{n-1}B \end{bmatrix} \tag{1}$$

Remark: $x(n) = \mathcal{M}_R \cdot \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$

redundant to use more than n-1

Importance... we can reach $x(n)$, state x in n steps with this input sequence of $u(k)$
$\Rightarrow$ analyze its properties...

Reachability Matrix (cont'd)

POLITECNICO MILANO 1863

full Rank: any input sequence leads us to a state vector $x \neq \underline{0}$

Theorem 1 - Necessary and sufficient condition for reachability of linear systems

A linear system is **fully reachable** iff $\text{rank}\,(\mathcal{M}_R) = n$, where n is the system's order.

use until term (n-1) ...

Remark: Why is $\mathcal{M}_R$ constructed using $x(n)$? Suppose to consider $x(n+1)$. Then $\mathcal{M}_R$ will contain $A^n B$ as well. But, in light of Cayley-Hamilton theorem, A^n can be written as a linear combination of $I, A^1, ..., A^{n-1}$, and thus this extra term does not affect the rank of $\mathcal{M}_R$.

other column will be linearly depend if using more than n-1 terms

Remark: In discrete-time linear systems, states are reachable **at most** in n steps.

Remark - Reachability of continuos-time systems

For **continuous-time** linear systems, $\mathcal{M}_R$ is computed as for discrete-time ones, i.e. by (1), and the same condition as Theorem 1 holds.

Reachability - Example 1 (cont'd)

POLITECNICO MILANO 1863

(circuit with 2 capacitors)

Considering the previous example, let's compute $\mathcal{M}_R$ (in the original coordinates). Being $n = 2$:

from A, B of syst.. $\longrightarrow$

$$\mathcal{M}_R = [B \quad AB] = \begin{bmatrix} \frac{1}{RC} & -\frac{2}{(RC)^2} \\ \frac{1}{RC} & -\frac{2}{(RC)^2} \end{bmatrix}$$

$\leftarrow$ same row... linear dependent

↳ not fully reach

rank=1

Since $\text{rank}(\mathcal{M}_R) = 1 < n$, the system is not fully reachable (as previously discussed).

We have also shown that, even if the system is not fully reachable, **by means of a suitable change of variables it is possible to decompose the system into its reachable and unreachable part**.

This leads to the following theorem, which is stated for continuous-time systems, but holds for discrete-time ones as well.

seen phisically that NOT Reachable...

is easy to check it also analytically

↳ always possible to split the system ⇒

Theorem - Reachability decomposition

Given the system $\dot{x} = Ax + Bu$, not fully reachable, there exists a non-unique change of variables $\hat{x} = T_R x$, where $\hat{x} = [\hat{x}_r^T, \hat{x}_{nr}^T]^T$, which allows to write the system as:

$$\begin{cases} \dot{\hat{x}}_r = \hat{A}_r \hat{x}_r + \hat{A}_x \hat{x}_{nr} + \hat{B}_r u \\ \dot{\hat{x}}_{nr} = \hat{A}_{nr} \hat{x}_{nr} \end{cases} \quad \Leftrightarrow \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$$

where $\hat{A} = \begin{bmatrix} \hat{A}_r & \hat{A}_x \\ \mathbf{0} & \hat{A}_{nr} \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} \hat{B}_r \\ \mathbf{0} \end{bmatrix}$.

$\hat{x}_r$ is called **reachable part** of the system, while $\hat{x}_{nr}$ is called **unreachable part**.

In particular, if $\text{rank}(\mathcal{M}_R) = n_r$, then $\hat{A}_r$ is $n_r \times n_r$, $\hat{B}_r$ is $n_r \times m$, and

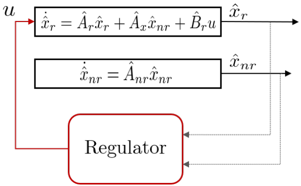
$$\text{rank}\left(\hat{\mathcal{M}}_R = \begin{bmatrix} \hat{B}_r & \hat{A}_r \hat{B}_r & \dots & \hat{A}_r^{n_r-1} \hat{B}_r \end{bmatrix}\right) = n_r$$

not trivial to find... it is possible to find which states are reach/unreach...

Stabilizability

more general concept than Reachability

usefull to know during design of Regulator
to know what we can control..



Remark: The regulator can act **only** on the reachable part.

- If the unreachable part is asymptotically stable, $\hat{x}_{nr}$ goes to zero and its effect on $\hat{x}_r$ vanishes.
- If the unreachable part is unstable, **nothing can be done to stabilize the system**.

Definition - Stabilizability

A system whose unreachable part is asymptotically stable is said to be **stabilizable**.

Stabilizability is a milder condition than reachability. If a system is not fully reachable, we must check that the dynamics of the unreachable part are asymptotically stable, i.e.:

- All the eigenvalues of $\hat{A}_{nr}$ have negative real part (continuous-time systems)
- All the eigenvalues of $\hat{A}_{nr}$ lie within the unit circle (discrete-time systems)

(to check stabilizability)

the unreachable part MUST be asymp stable OR it will tends to diverge and we cannot do anything when difficult to find change of variable

Checking stabilizability requires to find a change of variables that allows to decompose the system into its reachable and unreachable part, which is not straightforward.

PBH reachability test

The system is fully reachable if and only if

$$P_R(s) = [sI - A \quad B]$$

has rank n for any complex value s.

only dependency on s (sI - A) is how we find eig (A)

Note that the only values of s that *could* decrease the rank of $P_R(s)$ are the eigenvalues of A, since for those values it holds that $\det(sI - A) = 0$. only s = eig(A) can reduce rank of P_R

⇓ CONSEQUENSES

Remark

To assess the reachability of a system, it is enough to check that for any s_* eigenvalue of A:

$$\text{rank}(P_R(s_*)) = n$$

(see if parts of unreach parts)

PBH stabilizability condition

POLITECNICO MILANO 1863

The PBH reachability test allows to derive a condition for the stabilizability of the system.

PBH stabilizability condition

If the rank of $P_R(s)$ is decreased only in correspondence of asymptotically stable eigenvalues, the system is stabilizable.

Example: Consider the system $\dot{x} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}}_{[A]} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{[B]} u$. Its eigenvalues are $s_1 = -3$, $s_2 = 1$.

If you compute $\mathcal{M}_R$, rank$(\mathcal{M}_R) < n$. The system is not fully reachable. Is it at least stabilizable?

$$P_R(s) = [sI - A \quad B] = \begin{bmatrix} s+1 & -2 & 1 \\ -2 & s+1 & 1 \end{bmatrix}$$

PBH TEST

check the rank in correspondence of eig values: to determine which belongs to reach/unreach

- rank $(P_R(s_1))$ = rank $\left(\begin{bmatrix} -2 & -2 & 1 \\ -2 & -2 & 1 \end{bmatrix} \right) = 1 < n$ same row
- rank $(P_R(s_2))$ = rank $\left(\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 1 \end{bmatrix} \right) = 2 = n$ OK, REACH PART

$\Rightarrow$ The system is stabilizable, because the eigenvalue causing the loss of rank is asymptotically stable ($s_1 = -3$). belongs to → NON Reach part

Controllability

opposite from Reach... starting from $\bar{x}$ can we reach the origin?
$\bar{x} \longrightarrow 0$

POLITECNICO MILANO 1863

- Reachability: bring the state from the origin to any $\bar{x}$.
- **Controllability**: bring the states from any initial $\bar{x}_0$ to the origin.

Definition - Controllability (continuous-time systems)

Given a continuous-time system $\dot{x}(t) = Ax(t) + Bu(t)$, a state $x(0) = \bar{x}_0$ is said **controllable** if there exists a finite arbitrary $\bar{t} > 0$, and an input profile $u(\tau)$, $\tau \in [0, \bar{t})$, such that $x(\bar{t}) = 0$.

Definition - Controllability (discrete-time systems)

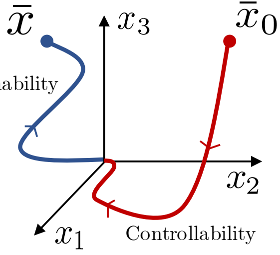
Given a discrete-time system $x(k+1) = Ax(k) + Bu(k)$, a state $x(0) = \bar{x}_0$ is said **controllable** in $\bar{k}$ steps if there exists an input sequence $u(0), ..., u(\bar{k})$ such that $x(\bar{k}) = 0$.

Definition - Full controllability

A system is said to be fully controllable if all its states are controllable.

Controllability vs reachability

start from origin and go to arbitrary state



from any $\bar{x}_0$ go back to origin

Remark

- For continuous systems, the set of reachable states matches the set of controllable states.
- For discrete systems, if a state $\bar{x}$ is reachable it is also controllable, **but not viceversa!**

$\Downarrow$

Example: Given the system $x(k+1) = 0$:

- All the states are controllable, since they are brough to 0 at $\bar{k} = 1$;
- Only the origin is reachable. → no input sequence to reach a state $\bar{x}$

Cancellations and unreachability

Where does the unreachability of the system come from?

- From a problem of the model (e.g. the two-capacitors circuit)
- From a zero-pole cancellation

Example

a system sequence

$$\xrightarrow{u} \boxed{G_1(s) = \frac{s-1}{s}} \xrightarrow{y_1} \boxed{G_2(s) = \frac{1}{s-1}} \xrightarrow{y_2}$$

{ state space }

$$G_1 : \begin{cases} \dot{x}_1 = -u \\ y_1 = x_1 + u \end{cases} \qquad G_2 : \begin{cases} \dot{x}_2 = x_2 + y_1 = x_1 + x_2 + u \\ y_2 = x_2 \end{cases}$$

The state-space equation of the system is hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

Cancellations and unreachability (cont'd)

POLITECNICO MILANO 1863

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Let's now check the reachability of the system: by reachability matrix

$$\mathcal{M}_R = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\,(\mathcal{M}_R) = 1 < n$$

NOT FULLY Reachable

Due to the zero/pole cancellation, an unreachable part is created. Being such unreachable part associated to the unstable pole ($s = 1$), the system is not stabilizable!

critical, because on $s=1$ ($\mathrm{Re}(s)>0$)

Final comments

POLITECNICO MILANO 1863

Comment

Reachability is a fundamental property describing systems' structure and the possibility to regulate them. However, **it does not "describe" how** the system reaches the target state $\bar{x}$, i.e. if it exhibits overshoots, oscillations, etc.

only guarantee we can reach that state

Comment

Typically, systems are characterized by the saturation of control variables. In these cases, all the previous results are **not valid**. In these cases, one shall resort to the concept of *constrained reachability*.

reach in n steps only in teory, BUT if saturation it does not holds

Observability in continuous-time

POLITECNICO MILANO 1863

Definition - Observability (continuous-time system)

Consider a continuous-time autonomous linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

A non-null state $x(0) = \bar{x}_0$ is **non-observable** if, $\forall \bar{t} > 0$ finite, the corresponding free movement due to $\bar{x}_0$, denoted by $\bar{y}(t)$, is constantly zero, i.e. $\bar{y}(\tau) = 0 \quad \forall \tau \in [0, \bar{t}\,]$.

A system **without non observable** states is said to be **fully observable**.

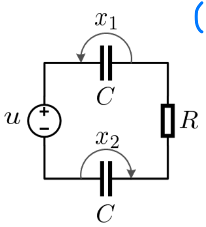
- For observable systems, there do not exist non-null states $\bar{x}$ causing null outputs.
- If the system is not observable, there exist states $\bar{x}$ not "showing up" on the output. **Measuring the outputs is not sufficient to know what's going on with the system.**

Observability - Example

POLITECNICO
MILANO 1863

Consider the previous system made of two capacitors of same size, at equal initial charge.

($x_1 = x_2$ bounded equals)



$$\begin{cases} i = -\frac{1}{R}\left[x_1 + x_2 - u\right] \\ \dot{x}_1 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \\ \dot{x}_2 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \end{cases}$$

It is reminded that $x_1(t) = x_2(t)$.

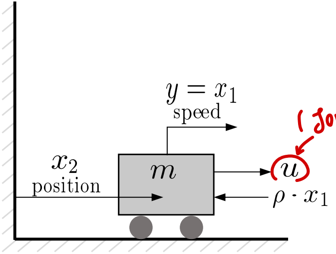
(∀t we know our state)

- Assume to measure $y_1 = x_1 + x_2$: ∀x_1, x_2 combination → output non 0
  The states are **observable** (when x_1 and x_2 are non-null, y_1 is not-null)

- Assume instead to measure $y_2 = x_1 - x_2$: (measuring voltage)
  The states are **non-observable** (y_2 is null even when x_1 and x_2 are non-null)

  NO information on the syst dyn

Observability - Example



(force acting on the cart)

$$\begin{cases} \dot{x}_1 = \frac{1}{m}(u - \rho x_1) \\ \dot{x}_2 = x_1 \\ y(k) = x_1 \end{cases}$$

[state space model] $\Rightarrow$ $A = \begin{bmatrix} -\frac{\rho}{m} & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Since we measure the speed, we don't have any information on the position!

- If we initialize the system in $\bar{x}_0 = [0, 1]^T$, it stays still, since $y(t) = x_1(t) = 0$. The position does not affect the output. so it is NON OBS... meas output we cannot reconstruct position
- Measuring the output (the speed) we cannot reconstruct the position, unless we know the exact initial position).

The system is not fully observable.

POLITECNICO MILANO 1863

Definition - Observability (discrete-time system)

Consider a discrete-time autonomous linear system:

$$\begin{cases} x(k+1) = Ax(k) \\ y(k) = Cx(k) \end{cases}$$

A non-null state $x(0) = \bar{x}_0$ is **non-observable** if, $\forall \bar{k} \in \mathbb{N}$ finite, the corresponding free movement due to $\bar{x}_0$, denoted by $\bar{y}(k)$, is constantly zero, i.e. $\bar{y}(k) = 0 \quad \forall k \in \{0, \bar{k}\}$.

A system **without non observable** states is said to be **fully observable**.

Observability Matrix

How to assess the observability? We start from discrete-time systems, and then extend the results to continuous-time ones. Considering an initial state $x(0) = \bar{x}_0 \neq \mathbf{0}$, and no input:

similarly ↳ to reachability, from output sequence of evolution...

$$
\begin{aligned}
\bar{y}(0) &= Cx(0) = C\bar{x}_0 \\
\bar{y}(1) &= C\left(Ax(0)\right) = CA\bar{x}_0 \\
&\vdots \\
\bar{y}(n-1) &= CA^{n-1}\bar{x}_0
\end{aligned}
$$

collecting the coefficients

Definition - Observability Matrix

$$
\mathcal{M}_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2}
$$

used to understand what happens on its kernel

if full rank, no vector multiplyed by M_O give 0 →

Remark: $\bar{y} = \mathcal{M}_O \cdot \bar{x}_0$ ⟷

Since $\bar{x}_0 \neq 0$, the only way in which $\bar{y}(k)$ can be always zero, is that $\bar{x}_0 \in \text{Ker}\left(\mathcal{M}_O\right)$.

Observability Matrix (cont'd)

POLITECNICO MILANO 1863

Theorem 2 - Necessary and sufficient condition for observability of linear systems

A linear system is fully observable iff $\text{rank}\left(\mathcal{M}_O\right) = n$, where n is system's order.

FULL rank

Remark - Observability of continuous-time systems

For continuous-time linear systems, $\mathcal{M}_O$ is computed as for discrete-time ones, i.e. by (2), and the same condition as Theorem 2 holds.

Example: In the cart example, the observability matrix is not full rank, therefore the system is not fully observable:

$$\mathcal{M}_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\rho}{m} & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\left(\mathcal{M}_O\right) = 1 < n$$

not fully obs

Observable and non-observable parts

POLITECNICO MILANO 1863

as on Reachability ... OBS allows to change var. and split the system into obs/non obs part

Theorem - observability decomposition

If a system is not fully observable, there exists a non-unique change of variables, $\hat{x} = T_o x$, where $\hat{x} = \left[\hat{x}_o^T, \hat{x}_{no}^T\right]^T$, which allows to write the system as:

$$\begin{cases} \dot{\hat{x}}_o = \hat{A}_o \hat{x}_o + \hat{B}_o u \\ \dot{\hat{x}}_{no} = \hat{A}_x \hat{x}_o + \hat{A}_{no} \hat{x}_{no} + \hat{B}_{no} u \\ \hat{y} = \hat{C}_o \hat{x}_o \end{cases} \Leftrightarrow \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{y} = \hat{C}\hat{x} \end{cases}$$

where $\hat{A} = \begin{bmatrix} \hat{A}_o & \mathbf{0} \\ \hat{A}_x & \hat{A}_{no} \end{bmatrix}$, $\hat{B} = \begin{bmatrix} \hat{B}_o \\ \hat{B}_{no} \end{bmatrix}$, and $\hat{C} = \begin{bmatrix} \hat{C}_o & \mathbf{0} \end{bmatrix}$.

$\hat{x}_o$ is called **observable part** of the system, while $\hat{x}_{no}$ is called **non-observable part**.

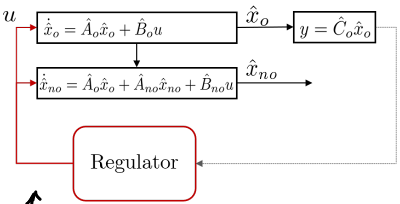
Being $\text{rank}(\mathcal{M}_O) = \text{rank}(\hat{\mathcal{M}}_O) = n_o$, then $\hat{A}_o$ is $n_o \times n_o$, $\hat{B}_o$ is $n_o \times m$, $\hat{C}_o$ is $p \times n_o$.

$$\begin{cases} \text{rank}(M_o) = \text{dimension of obs part} \\ \text{while } n - \text{rank}(M_o) = \text{non obs part} \end{cases}$$

Detectability

POLITECNICO MILANO 1863

Obs: to choose measurements of syst → we must deal with non obs part so we define.. Detectability



split the system into OBS/NONOBS if non obs is asymp stable → we can assume it not diverge

Remark: When we close the control loop, only the observable states are "accounted" by the regulator

- If the non-observable part is asymptotically stable, $\hat{x}_{no}$ vanishes.
- If the non-observable part is unstable, **diverging states cannot be detected from outputs**. The system cannot be controlled.

Definition - Detectability

A system whose non-observable part is asymptotically stable is said to be **detectable**.

Detectability is a milder condition than observability. If a system is not fully observable, we must check that the dynamics of the non-observable part are asymptotically stable.

PBH observability test

to check observability detectability of syst..

PBH observability test

The system is fully observable if and only if

$$P_O(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix}$$

has rank n for any complex value s.

checked $\forall s$ in theory... but the only s that could decrease P_O rank are eig(A)

As for reachability, the only values of s that *could* decrease the rank of $P_O(s)$ are the eigenvalues of A.

we check rank only for s = eig(A)

Remark

To assess the observability of a system, it is enough to check that for any s_*, eigenvalue of A:

$$\text{rank}(P_O(s_*)) = n$$

full rank ⇒ fully obs

$< n$: can be Detectable in corrispondence of asymp. st eig val

POLITECNICO MILANO 1863

PBH detectability condition

If the rank of $P_O(s)$ is decreased only in correspondence of asymptotically stable eigenvalues, the system is detectable.

Example: Consider the system $\begin{cases} \dot{x} = \overbrace{\begin{bmatrix} -1 & -2 \\ -4 & 1 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} u \\ y = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{C} x \end{cases}$, with eigenvalues $s_1 = -3$, $s_2 = 3$.

asymp st

unstable

If you compute $\mathcal{M}_O$, rank$(\mathcal{M}_O) < n$. The system is not fully observable. Is it at least detectable?

PBH test for detectability

Check for $s = s_1, s_2$ for which it can lose rank

$$P_O(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} s+1 & 2 \\ 4 & s-1 \\ 1 & -1 \end{bmatrix}$$

column lin dep.

$n=2 \Leftarrow s_1$ belongs to non OBS part

- rank $(P_O(s_1))$ = rank $\left(\begin{bmatrix} -2 & 2 \\ 4 & -4 \\ 1 & -1 \end{bmatrix}\right) = 1 < n$
- rank $(P_O(s_2))$ = rank $\left(\begin{bmatrix} 4 & 2 \\ 4 & 2 \\ 1 & -1 \end{bmatrix}\right) = 2 = n$

belongs to OBS PART

$\Rightarrow$ The system is detectable, because the eigenvalue causing the loss of rank is asymptotically stable ($s_1 = -3$).

not fully OBS but detectable

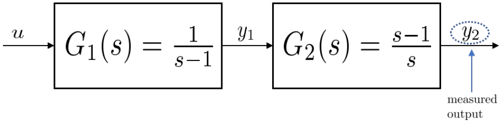
Cancellations and non-observability

⇓ what causes loss of obs? → measurement issues... from ZERO/POLE cancell.

Where does the non-observability of the system come from?

- From a problem of the model (e.g. the cart with speed measurement)
- From a zero-pole cancellation

Example



{state space}

$$G_1 : \begin{cases} \dot{x}_1 = x_1 + u \\ y_1 = x_1 \end{cases} \qquad G_2 : \begin{cases} \dot{x}_2 = -x_1 \\ y_2 = y_1 + x_2 = x_1 + x_2 \end{cases}$$

The state-space equation of the system is hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Cancellations and non-observability (cont'd)

POLITECNICO MILANO 1863

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Let's now check both the reachability and the observability of the system:

$$\mathcal{M}_R = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\,(\mathcal{M}_R) = 2 = n$$

fully Reachable

$$\mathcal{M}_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\,(\mathcal{M}_O) = 1 < n$$

not fully observable

Due to the zero/pole cancellation, a non-observable part is created. Being such non-observable part associated to the unstable pole ($s = 1$), the system is not detectable!

cancellation on unstable part!

PROPERTIES

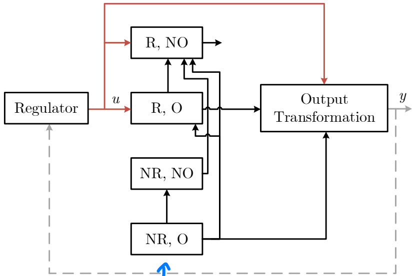
- Reachability → decomposable in Reach/unReach
- stabilizability
- controllability
- observability
- detectability

$\Downarrow$

multiple possible decomposition...

Kalman Canonical Decomposition

O) participate on OUTPUT
R) participate on CONTROL



during Regulator design, consider that properties (CRITICAL cancellation) → unstable parts hided

Kalman Canonical Decomposition

For any linear system there exists a change of variable that allows to decompose the system into four parts:

- (R, O): Reachable and observable part
- (NR, NO): Unreachable and non-observable part
- (R, NO): Reachable and non-observable part
- (NR, O): Unreachable and observable part

Observations:

- **Any transfer function represents the reachable and observable (R, O) part only.**
- If the other parts are asymptotically stable, this is not a problem. Otherwise, it is not possible to regulate the system.
- **Remember that the cancellations of unstable poles and zeros are forbidden**, because they create unstable unreachable/non-observable parts. ↑ to avoid

process that allows us to go from T.F to S.S form

POLITECNICO MILANO 1863

Given the transfer function of a SISO linear system, we want to **find the underlying state-space model** in its **minimal form.** (minimal ss)

↳ ss model using least number of space to represent our system

Consider a generic transfer function:

$$G(s) = \frac{\beta_n s^n + \beta_{n-1}s^{n-1} + ... + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_1 s + \alpha_0} = \hat{\beta}_n + \underbrace{\frac{\hat{\beta}_{n-1}s^{n-1} + ... + \hat{\beta}_1 s + \hat{\beta}_0}{s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_1 s + \alpha_0}}_{\text{Strictly proper}}$$

where $\hat{\beta}_n = \beta_n$ and $\hat{\beta}_i = \beta_i - \alpha_i\beta_n$, for $i = 0, ..., n-1$.

on SISO: min → can form to represent T.F G(s)

from one canonical rep... we apply long division

↳

Realization (SISO) (cont'd)

2 different possibility of s.s representation

POLITECNICO MILANO 1863

starting from can form... s.s representation by this MATRIX

Definition - reachability canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \dots & \hat{\beta}_{n-1} \end{bmatrix} \quad D = \hat{\beta}_n$$

Definition - observability canonical form

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & \dots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{n-1} \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad D = \hat{\beta}_n$$

Realization (MIMO)

POLITECNICO MILANO 1863

The realization problem is significantly more complex for MIMO systems.

Given $G(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{p1}(s) & \dots & G_{pm}(s) \end{bmatrix}$ we want to find the state-space model, such that

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

no minimal form easy to find...

In case of MIMO system, an extensive use of canonical forms is required, which is out of scope.

Example

POLITECNICO
MILANO 1863

Consider the system $\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$, which can be re-written as:

$$\begin{cases} y_1(s) = \tilde{y}_{11} + \tilde{y}_{12} = G_{11}(s)\, u_1(s) + G_{12}(s)\, u_2(s) \\ y_2(s) = \tilde{y}_{21} + \tilde{y}_{22} = G_{21}(s)\, u_1(s) + G_{22}(s)\, u_2(s) \end{cases}$$

To find a **non-minimal** realization of the system, we can find the state-space realization of each SISO element of the transfer matrix $G_{ij}(s)$ separately:

↳ apply separately $\forall$ SISO system

$$\begin{cases} \dot{\tilde{x}}_{ij} = A_{ij}\, \tilde{x}_{ij} + B_{ij}\, u_j \\ \tilde{y}_i = C_{ij}\, \tilde{x}_{ij} + D_{ij}\, u_j \end{cases}$$

Example (cont'd)

POLITECNICO MILANO 1863

The state-space model of the entire system is therefore:

NON minimal
but usable in MIMO syst.

$$
\begin{bmatrix} \dot{\tilde{x}}_{11} \\ \dot{\tilde{x}}_{12} \\ \dot{\tilde{x}}_{21} \\ \dot{\tilde{x}}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & & & \\ & A_{12} & & \\ & & A_{21} & \\ & & & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{12} \\ \tilde{x}_{21} \\ \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & \\ & B_{12} \\ B_{21} & \\ & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & & \\ & & C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{12} \\ \tilde{x}_{21} \\ \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

Remark

In general, this is a non-minimal realization.