1

Topics

model of the CAMERA DEVICE

- **Introduction and the Camera Optical System**
- Planar (2D) Projective Geometry
- Spatial (3D) Projective Geometry
- Camera Geometry (3D → 2D Projection)

ex: a cylinder is symmetric about an axis

A powerful tool: SYMMETRY

{many human environment has degrees of $\boxed{\text{simmetry}}$!}

⇓ important topic needed...

A SYSTEM IS SYMMETRIC WRT A CERTAIN TRANSFORMATION

(respect what?)

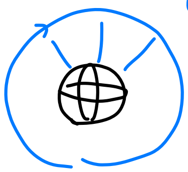
IF

APPLYING THAT TRANSFORMATION TO THE SYSTEM,

NO MODIFICATION CAN BE MEASURED

↑ by this transformation we don't notice any difference on the image!
no measure any difference. Even if there was a modification...

EXAMPLES



- EMPTY SPACE: is symmetric under any motion

you never notice it, SYMMETRICAL under respect any motion / transf.

- IF WE ADD AN ELEMENT TO A SYMMETRIC SYSTEM, SYMMETRY CAN POSSIBLY STILL BE ENJOYED, BUT UNDER A RESTRICED SET OF TRANSFORMATIONS

from an empty space, if we add an element to the system, symmetry reduced! by any new element

on EMPTY SPACE

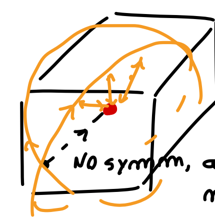
sphere centered on that point

e.g. A SINGLE POINT: is symmetric under any motion preserving the distance from the point → SPHERICAL SYMMETRY

surface • ‖spherical symmetry‖

motion don't change how the point is represented ⇒ no motion measure change!

any motion preserving distance

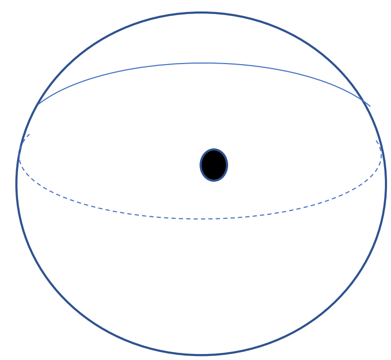


NOTE: the same symmetry is enjoyed by a SPHERE

EXAMPLES

e.g. A SINGLE POINT: is symmetric under any motion preserving the distance from the point → SPHERICAL SYMMETRY

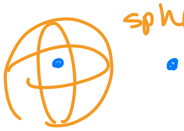
NOTE: the same symmetry is enjoyed by a SPHERE



EXAMPLES

LET'S ADD AN ELEMENT TO THE SYSTEM: a second point

when adding new point, spherical symmetry is lost! a new element reduce degree of simmetry! (less powerfull symmetry)



motion around sphere is no more symm motion

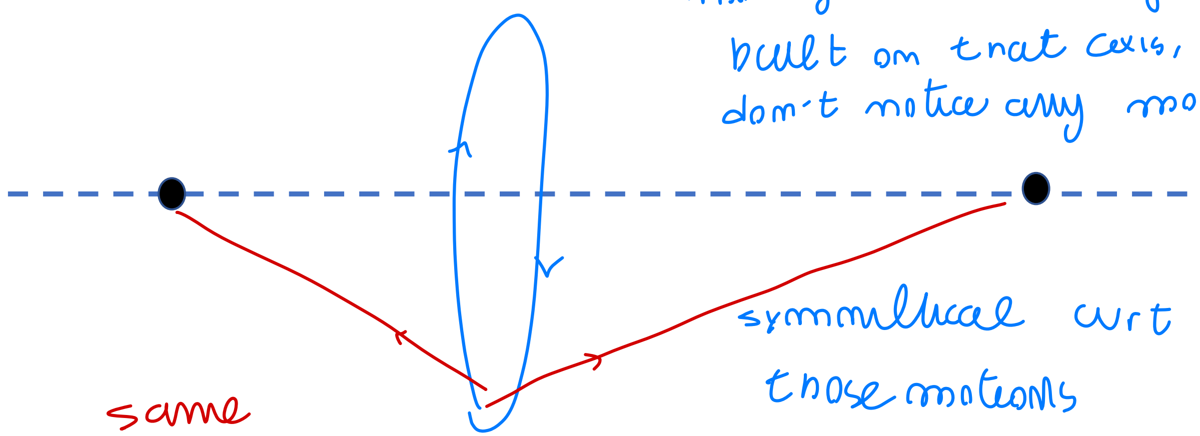
EXAMPLES

LET'S ADD AN ELEMENT TO THE SYSTEM: a second point

Let us call AXIS he straight line joining the two points

axial symmetry

moving on a circumference built on that axis, I don't notice any modification



symmetrical wrt those motions

same difference

so, new element reduce symmetry

EMPTY: motion in free dim, symm
SPHERICAL: motion along sphere
= 2D surface DOF moving
AXIAL: freedom in a single dim 1D symm motion

EXAMPLES

↑ symm
(sets are reduced !)

circumference with plane perpendicular to the axis

A TWO POINT system: is symmetric under any motion along a circumference orthogonal to the axis, centered on it

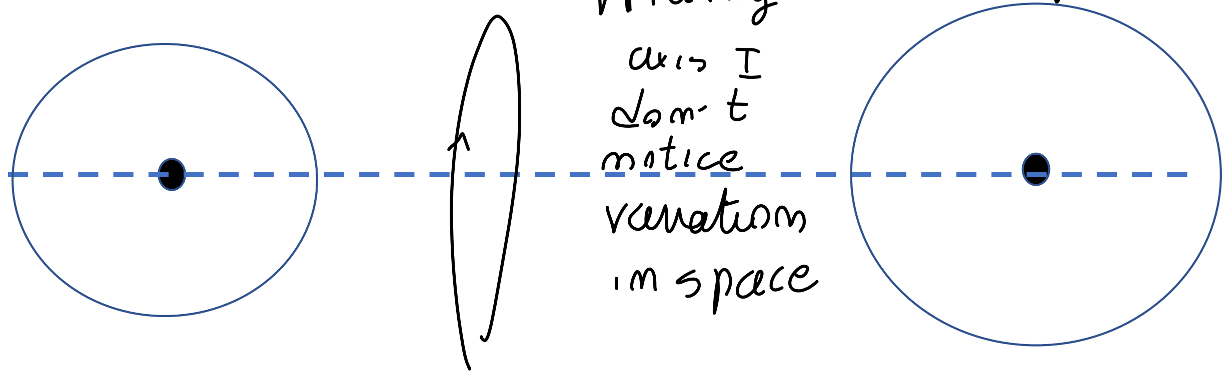
→ AXIAL SYMMETRY := symmetry about one axis

(rotate about the axis)

EXAMPLES

equivalent to two spheres in space

rotating on circumference center on axis I don't notice variation in space



A TWO POINT system: is symmetric under any motion along a circumference orthogonal to the axis, centered on it

→AXIAL SYMMETRY

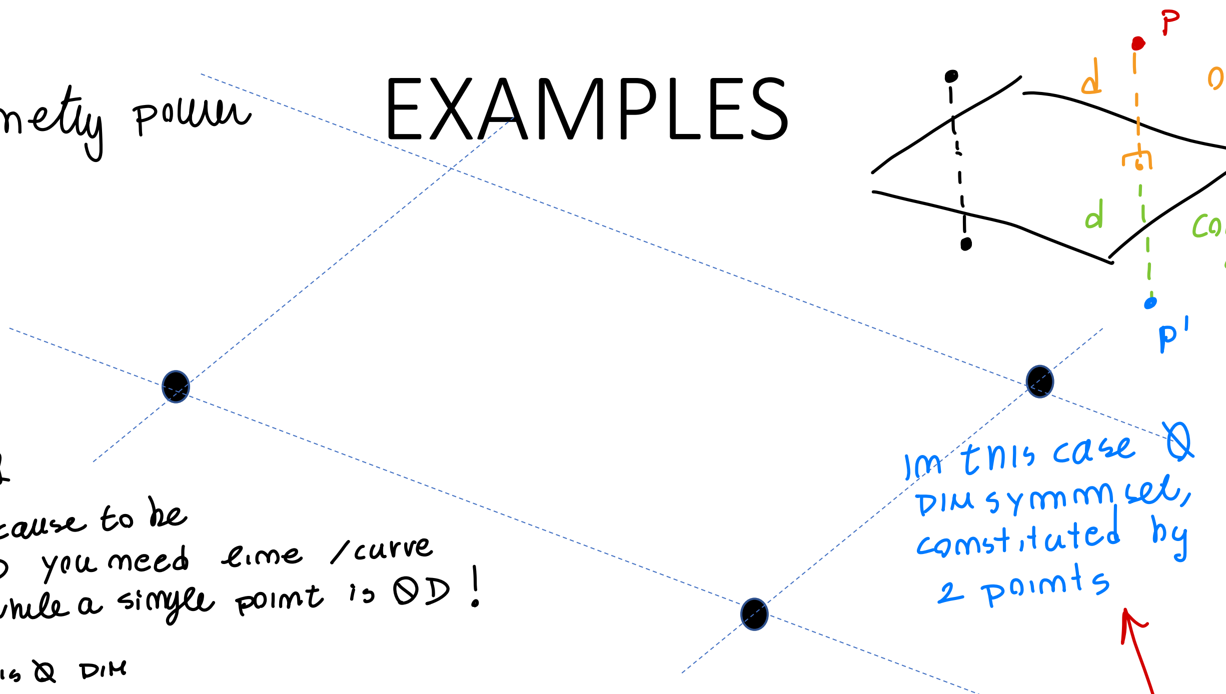
NOTE: the same symmetry is enjoyed by a TWO SHPERE system

EXAMPLES

LET’S ADD ONE MORE ELEMENT TO THE SYSTEM: a third point

EXAMPLES

we further reduce symmetry power



ORTHOGONAL PROJECTION

continue by same d

symm set

symm set := couple of points → 0 DIM, because to be 1D you need line /curve while a single point is 0D !

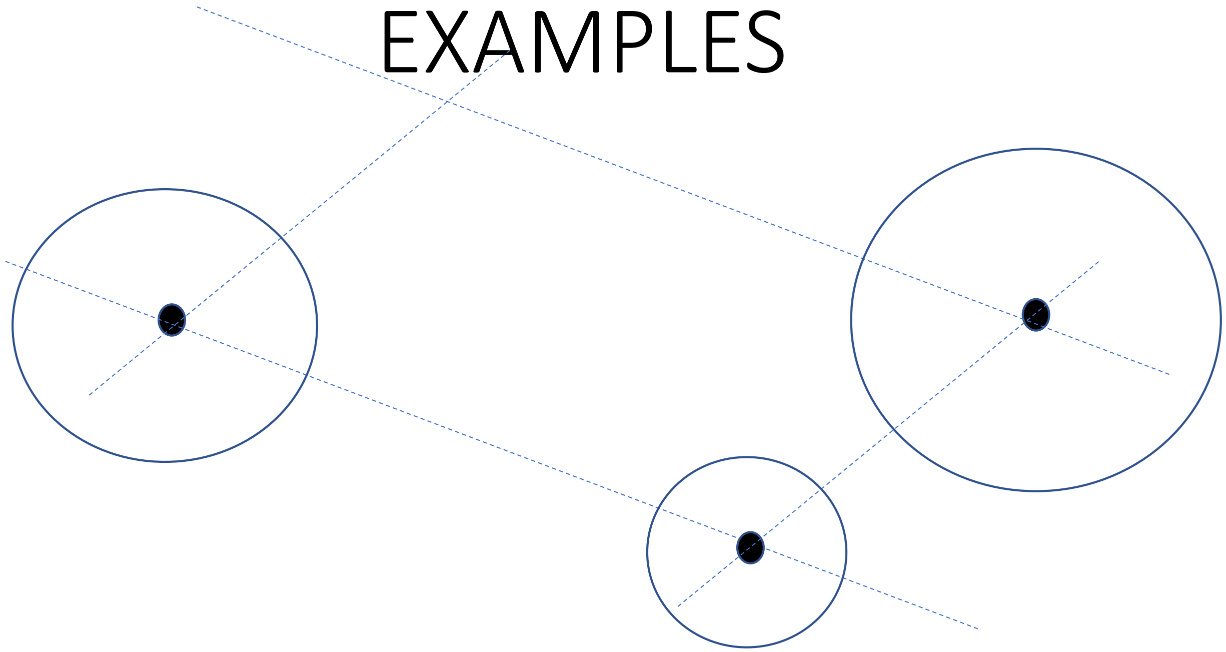
just 2 points is 0 DIM

in this case 0 DIM symm set, constituted by 2 points

A THREE POINT system: is symmetric under «mirroring» wrt the plane passing through the three points → PLANAR SYMMETRY

means that having a point, only one motion preserve symm empty space := 3D symm

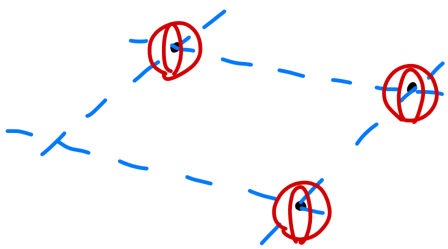
EXAMPLES



A THREE POINT system: is symmetric under «mirroring» wrt the plane passing through the three points → PLANAR SYMMETRY

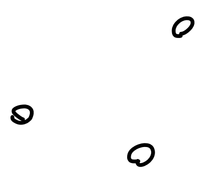
NOTE: the same symmetry is enjoyed by a THREE SHPERE system

SYMMETRY is important
↓
allow applications



PLANAR SYMMETRY
↓ when calibrating a CAMERA.
↓
to do it, you show to the camera a checker wall in several positions (this allow calibration!)

when Im observing basketball match, to CALIBRATE camera, you do it with what you see → basketball in many positions... to CALIBRATE camera
extract 3 images of basketball in many poses



you have PLANAR SYMMETRY that allow you to calibrate camera

SYMMETRY is important for our CAMERA SYSTEM

electronic screen device,

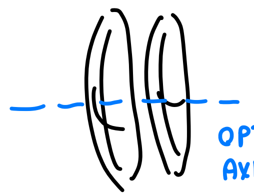
most important geometrically part is LENS

constituted by two sphere intersections

axial symmetry!

⇓

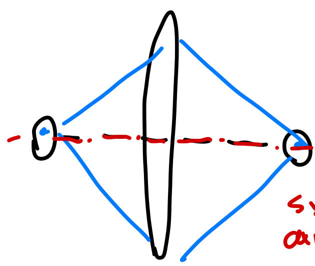
there is a symmetry axis



OPTICAL AXIS → everything is symmetric wrt to this axis

⇓

easy model of CAMERA behavior thanks to this symmetric property



Symm axis for lens

How optical sensor is constituted

more light = signal increase (with mapping)

OPTICAL SENSOR: CAMERA

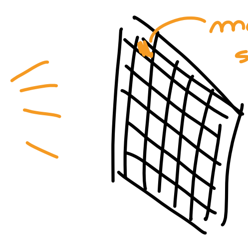
organized on a PLANAR SCREEN

respond with electrical signal to light signal

- Screen with (order of 10^6 or more) photosensitive elements called *pixels* (PIXEL = PICture Element) (elements sensitive to light) composition elements of picture
  - Optical → electric transducers
- **Optical system** to select direction of incoming light at each element
- Electric circuits that collect the signal generated at the pixels

- 30-60 frames per second (even more in some cameras)

organized as a MATRIX of PIXELS



map electrical signal, related to light coming to each pixel

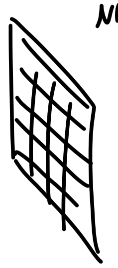
LIGHT! → more charge
DARK → less charge

OPTICAL SENSOR: CAMERA

- Screen with (order of 10^6 or more) photosensitive elements called *pixels* (PIXEL = PICture Element)
  - Optical → electric transducers
- **Optical system** to select direction of incoming light at each element
- Electric circuits that collect the signal generated at the pixels

- 30-60 frames per second (even more in some cameras)

signal high charge where much lighter object surface, DARK object less charge → reconstruct light profile and store image in a MATRIX



NOT enough to use JUST one screen of PIXEL MATRIX!

↳ in this simple configurations, all PIXELS receive same light!

because pixels very close!

⇒ uniform grey image :(

↓

NO collect true image

we want to project properly → map objects in the MAP of pixels → use each pixels for a single light direction

NOT for all directions...

each select only some light

⇓

done using an **OPTICAL SYSTEM!**

↓

selection system that allow each pixel to be interested by light coming from a single direction

---

↳ represent as MATRIX.

EX) **MATLAB** is perfect to work with **images ~ MATRIX**

= "MATRIX LABORATORY"

OPTICAL SYSTEM:

simplified camera model: lens = intersection of two spheres

take sphere radius enough

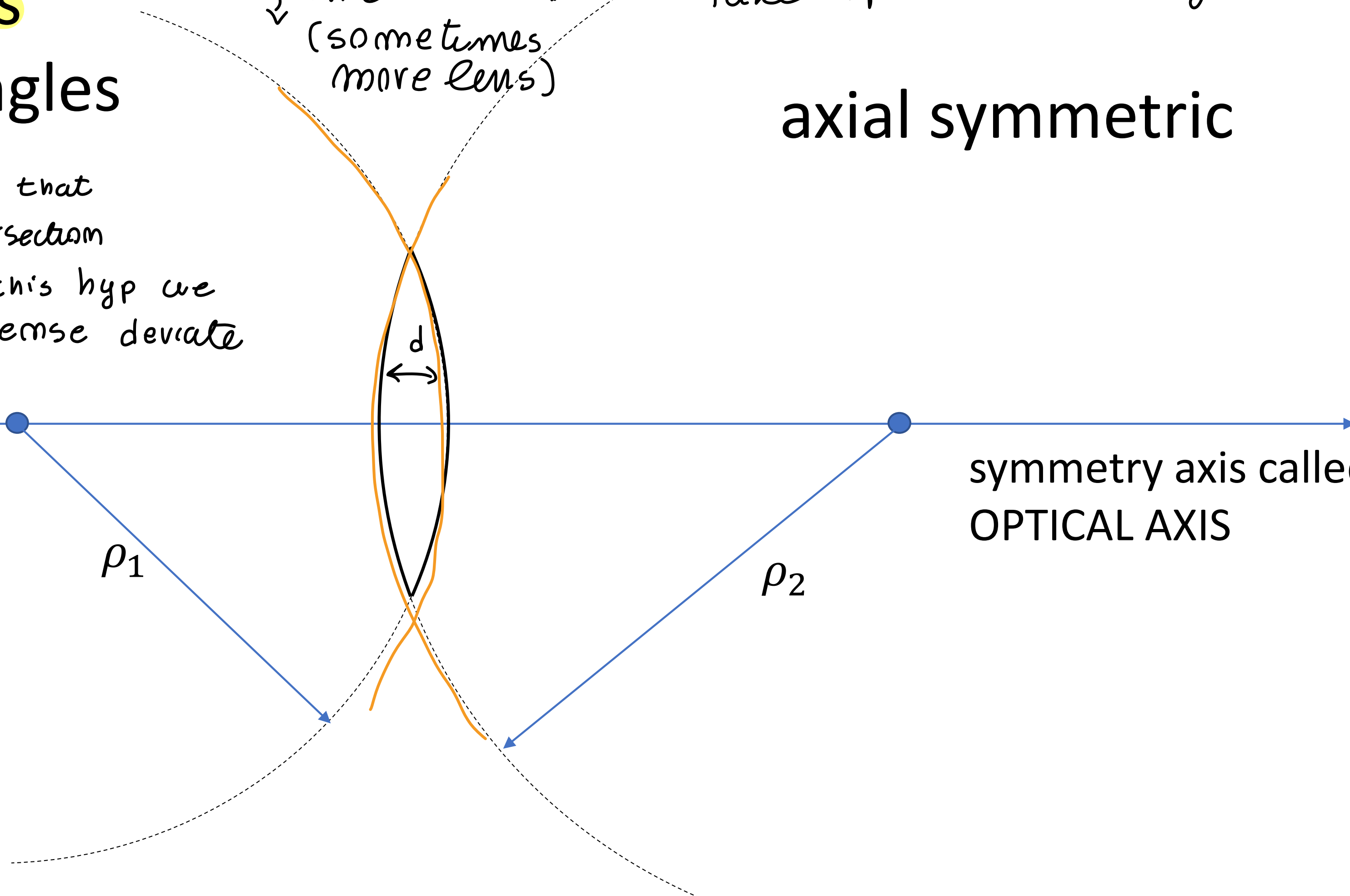
one lens! (sometimes more lens)

- Thin lens
- Small angles

take sphere such that d is small... intersection very thin ⇒ under this hyp we can study HOW lense deviate light

how does lense deviate straight of light, and how to collect it to build meaningfull image.

axial symmetric



symmetry axis called OPTICAL AXIS

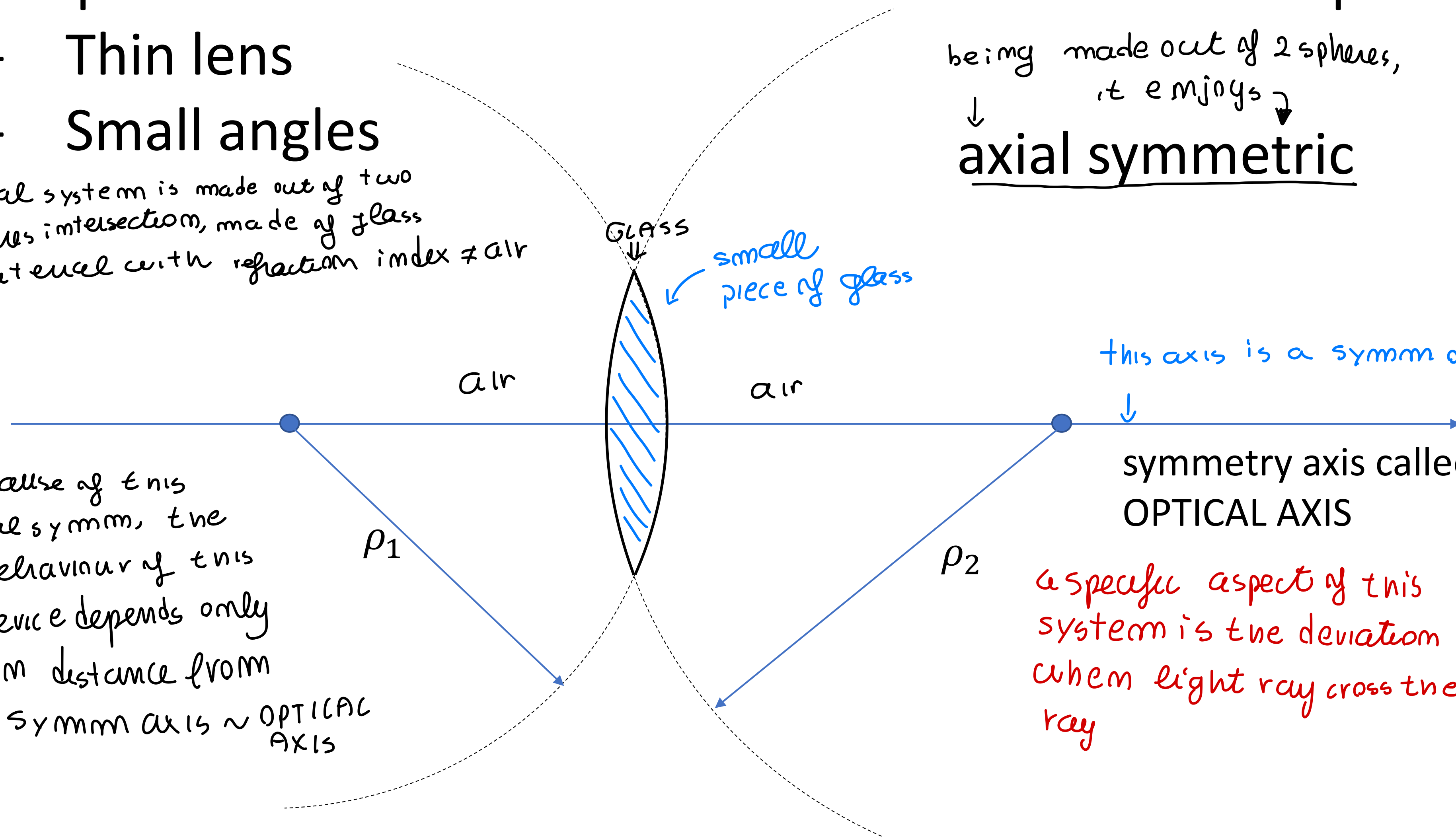
simplified camera model: lens = intersection of two spheres

- Thin lens
- Small angles

being made out of 2 spheres, it enjoys

axial symmetric

optical system is made out of two spheres intersection, made of glass material with refraction index ≠ air



symmetry axis called OPTICAL AXIS

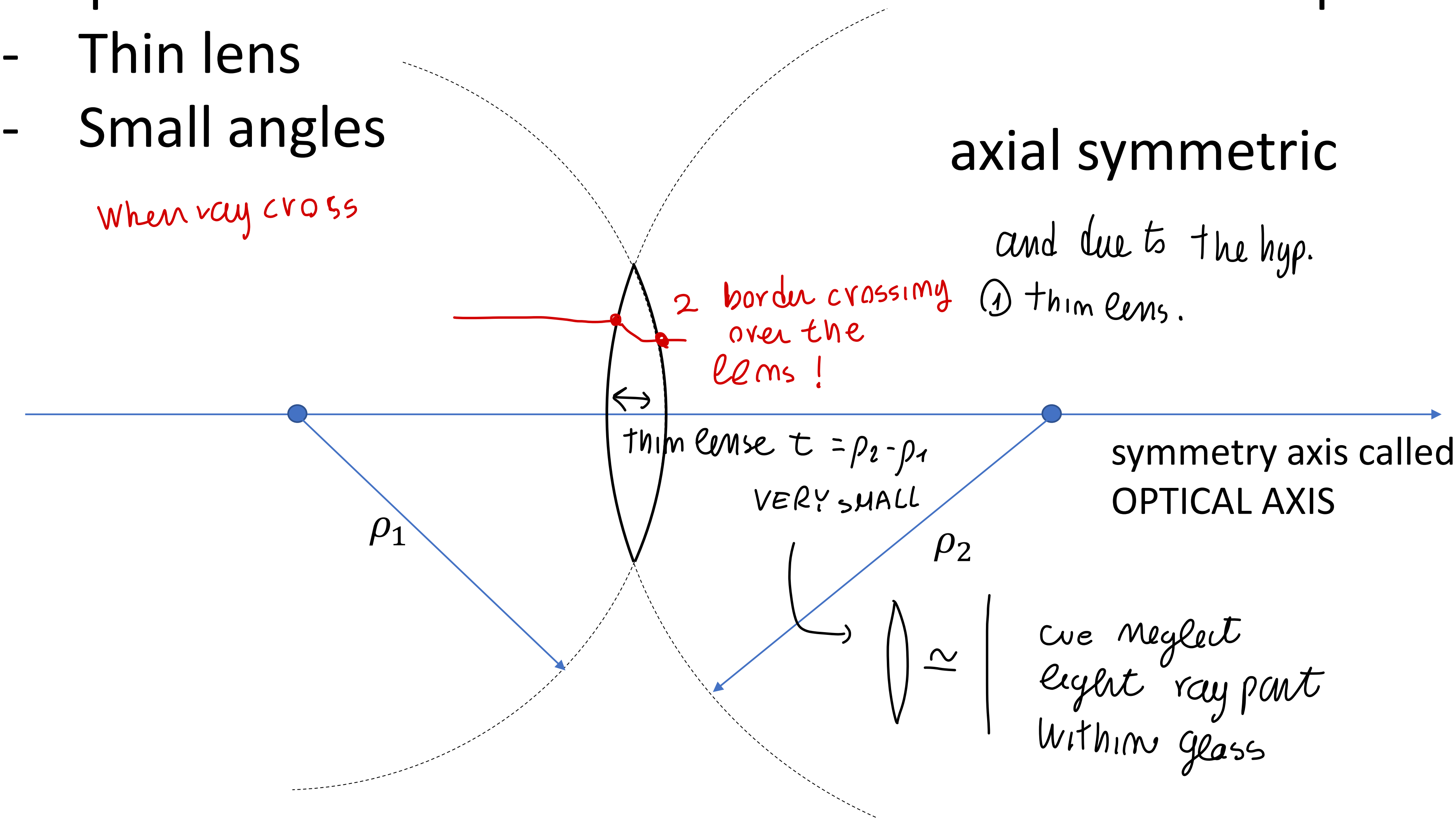
because of this axial symm, the behaviour of this device depends only on distance from symm axis ~ OPTICAL AXIS

a specific aspect of this system is the deviation when light ray cross the ray

simplified camera model: lens = intersection of two spheres

- Thin lens
- Small angles

axial symmetric



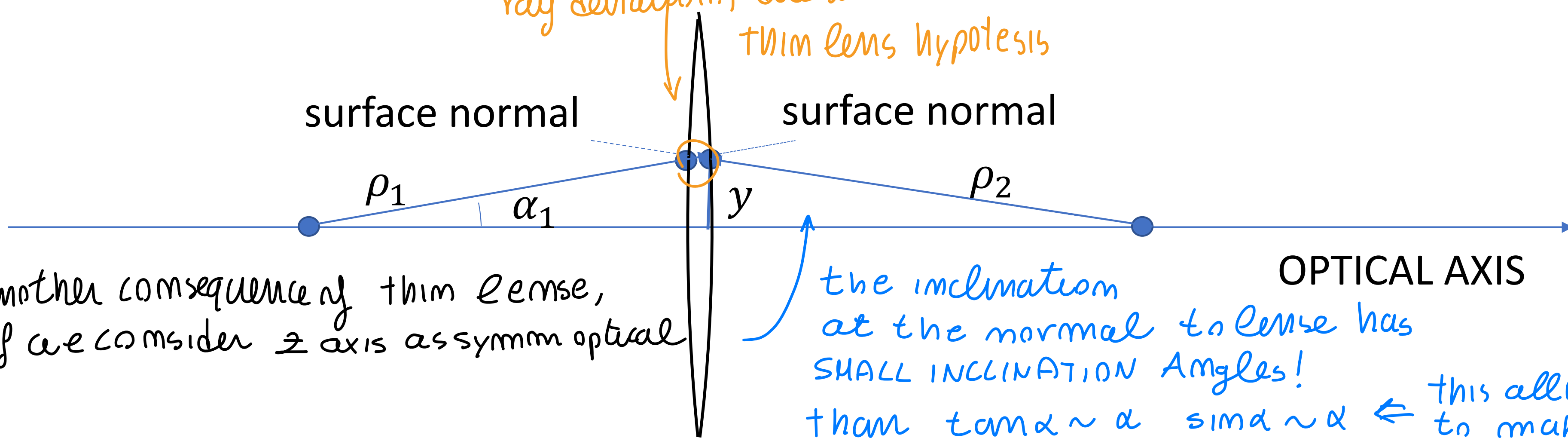
simplified camera model: lens = intersection of two spheres

- **Thin lens**
- **Small angles**

we neglect the internal light ray deviation, due to thin lens hypotesis

Hp: small angles → $\alpha_1 = y_1/\rho_1$

and also: $\alpha_2 = -y_2/\rho_2$



another consequence of thin lense, if we consider z axis as symm optical

the inclination at the normal to lense has SMALL INCLINATION Angles! than $\tan\alpha \sim \alpha$ $\sin\alpha \sim \alpha$ ⇐ this allow to make functions LINEAR!

Light ray: 1. air-glass refraction at y_1 2. glass-air refraction at y_2

Hp: thin lens → very short light path within the lens → $y_1 = y_2 = y$

simplified camera model: lens = intersection of two spheres

- **Thin lens**
- **Small angles**

light rays considered forms small angles with vertical axis

Hp: small angles → $\alpha_1 = y_1/\rho_1$

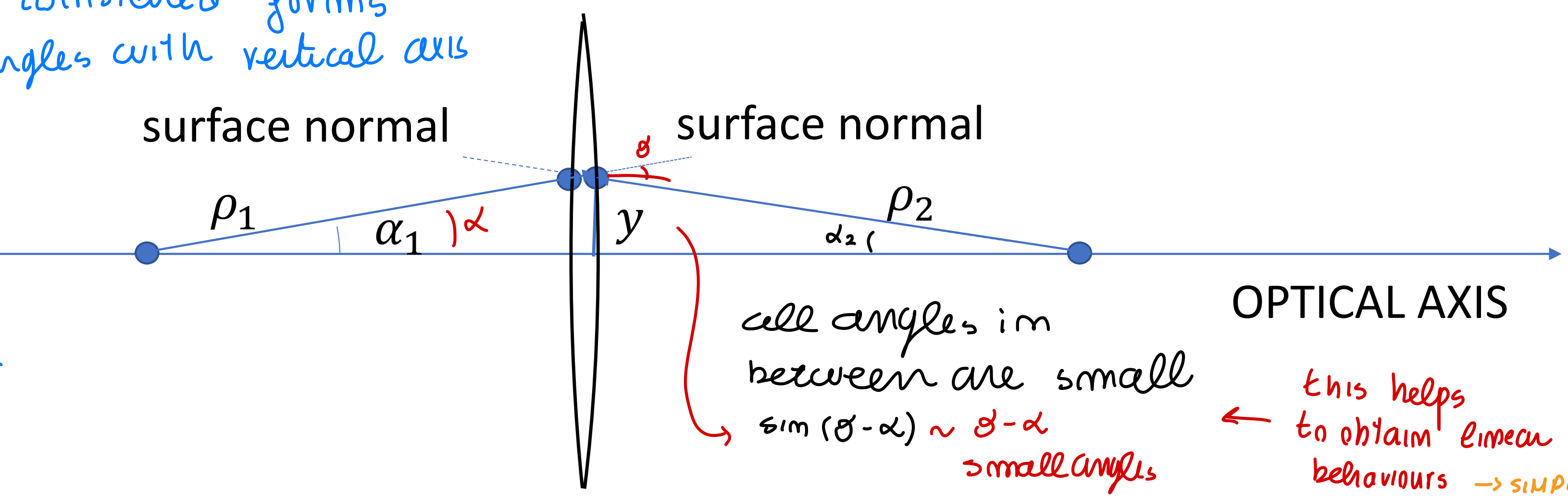
and also: $\alpha_2 = -y_2/\rho_2$

$\alpha_1 \sim$ SMALL

$\sin\alpha_1 \sim \alpha_1$

$\alpha_2 \sim$ SMALL

$\sin\alpha_2 \sim \alpha_2$



← this helps to obtain linear behaviours → SIMPLIFYED MODEL

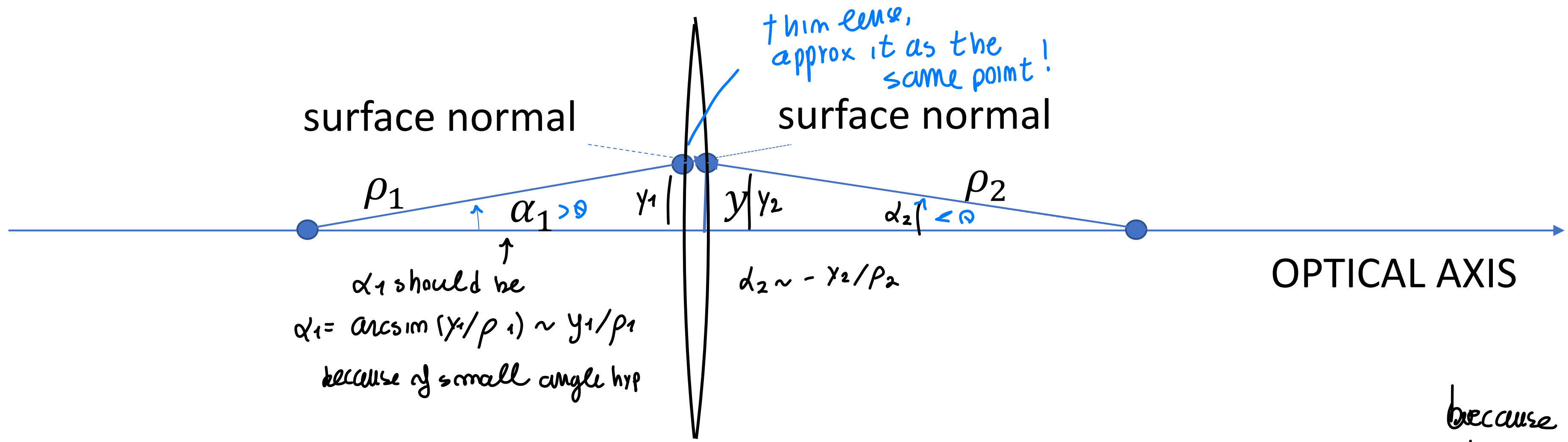
Light ray: 1. air-glass refraction at y_1 2. glass-air refraction at y_2

Hp: thin lens → very short light path within the lens → $y_1 = y_2 = y$

simplified camera model: lens = intersection of two spheres

- **Thin lens**
- **Small angles**

Hp: small angles → $\begin{cases} \alpha_1 = y_1/\rho_1 \\ \alpha_2 = -y_2/\rho_2 \end{cases}$ and also:



Light ray: 1. air-glass refraction at y_1 2. glass-air refraction at y_2

Hp: thin lens → very short light path within the lens → $\boxed{y_1 = y_2 = y}$

because thin lense,

$\begin{cases} \alpha_1 \sim y/\rho_1 \\ \alpha_2 \sim -y/\rho_2 \end{cases}$

← this hyp NOT works always, also maybe incoming light ray forms angle α big causing "LENSE ABERRATION" ⇒ causing distorsions in the image

(we focus on small angles / thin lense ⇒ NO ABERRATION)

The deviation of a light ray crossing the lens

DISTORSION to the image can occur

→ what happens to the ray continuation after the lense... ⟶ D

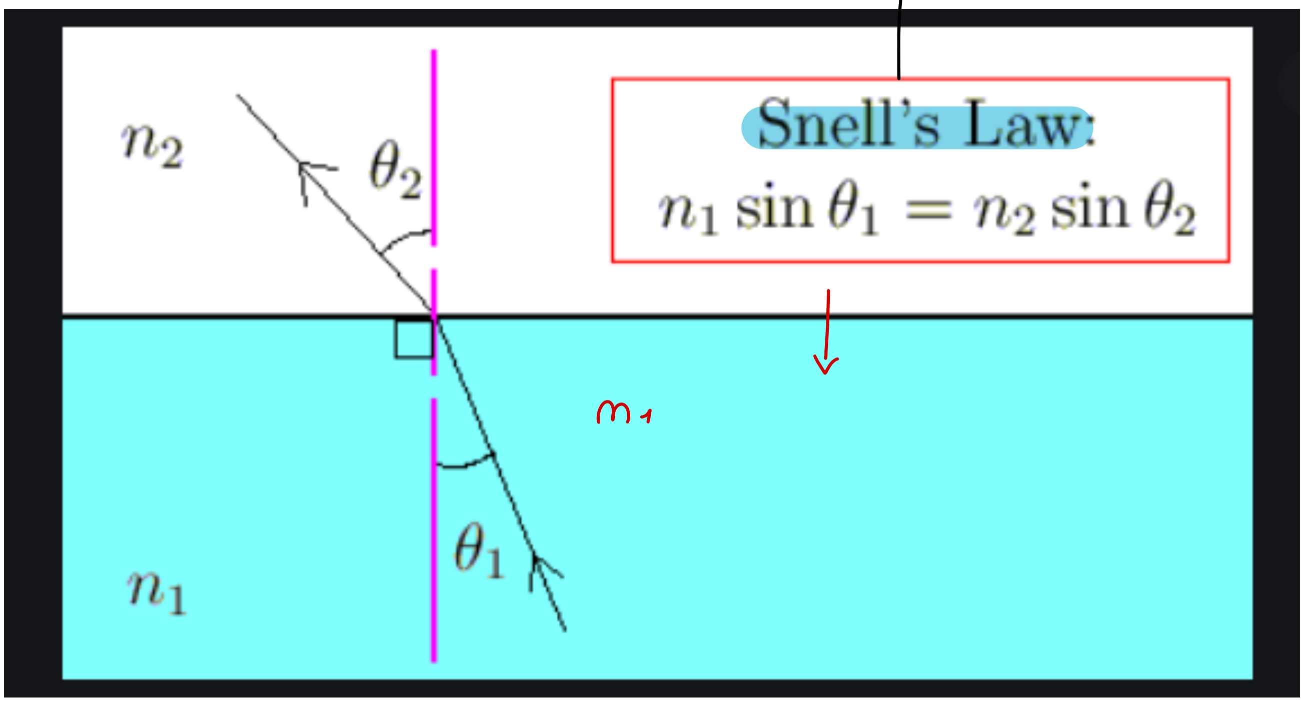
18/09

$\Rightarrow$ phenomenon: **refraction**

deviation when a light ray cross a border between two media which has different speed of light propagation ($\sim 3 \cdot 10^8$ m/s) Air

$\uparrow$ light propagation speed change in different media.

refraction of light

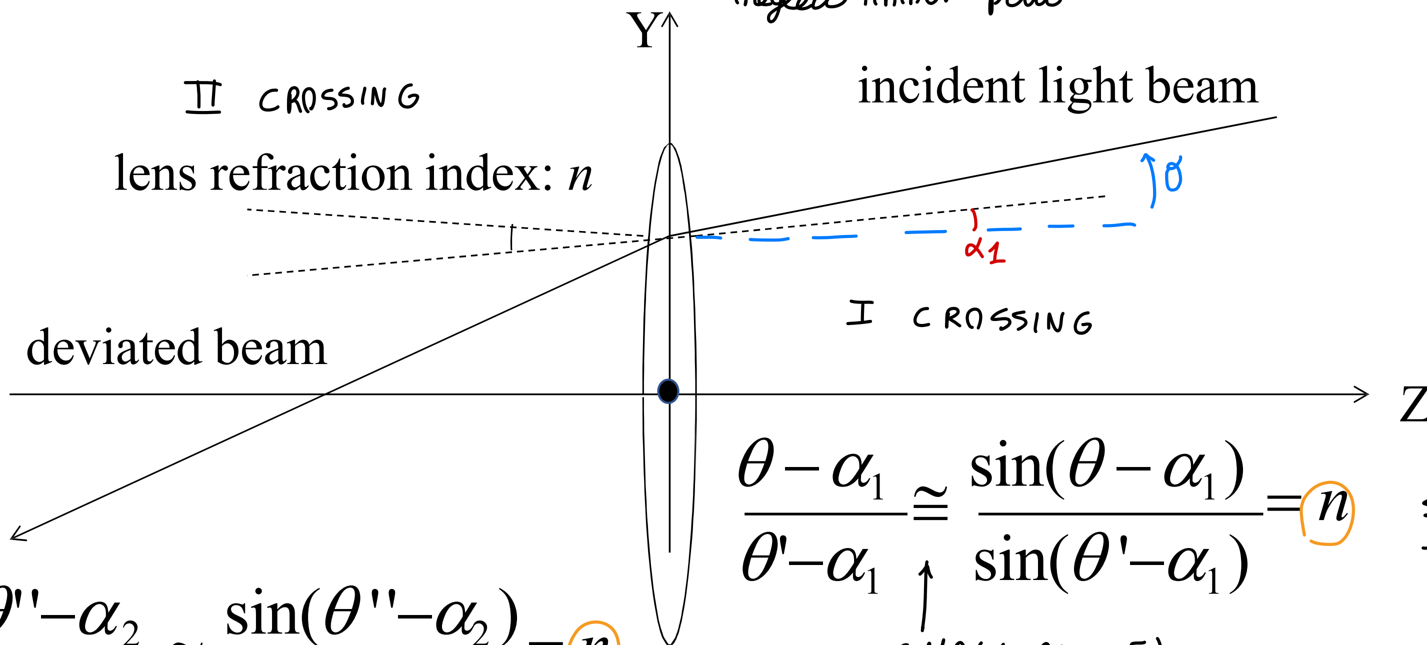


$n = \frac{n_1}{n_2}$ RELATIVE REFRACTION INDEX of a medium relative to (air/vacuum)

$\downarrow$ for any medium $n \geq 1$

because $\frac{v_{air}}{v_{material}} \geq 1$

ratio between light propagation speed between air and a media (water etc) is $n_i = C_{material} / C_{air} \Leftarrow$ light we have 2 CROSSING, but we neglect inner part



$$\frac{\theta-\alpha_1}{\theta'-\alpha_1} \cong \frac{\sin(\theta-\alpha_1)}{\sin(\theta'-\alpha_1)} = n$$

snell's law

(SMALL ANGLE)

$$\frac{\theta''-\alpha_2}{\theta'-\alpha_2} \cong \frac{\sin(\theta''-\alpha_2)}{\sin(\theta'-\alpha_2)} = n$$

deviation angle ? $\delta\theta = \theta''-\theta$

combining it...

$$\delta\theta = (n-1)Y\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)$$

$$\begin{cases} \vartheta - \alpha_1 = n(\vartheta' - \alpha_1) \\ \vartheta'' - \alpha_2 = n(\vartheta' - \alpha_2) \end{cases}$$

we care about the global deviation angle, so the overall $(\vartheta'' - \vartheta)$

overall deviation

$$\delta\vartheta = \vartheta'' - \vartheta$$

recall that:

due to small lens hyp:

$\alpha_1 = \dfrac{y}{\rho_1}$, $\alpha_2 = -\dfrac{y}{\rho_2}$ where same y (due to thin lense hyp.)

$$\vartheta'' - \vartheta - \alpha_2 + \alpha_1 = n\cancel{\vartheta'} - \cancel{n\vartheta'} - n\alpha_2 + n\alpha_1$$

$$\vartheta'' - \vartheta = \delta\vartheta = (n-1)(\alpha_1 - \alpha_2) = (n-1)\,y\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

$$\downarrow$$

$\delta\vartheta \sim y$ proportional to distance between optical axis and incident point (thanks to small angle hyp.) $\begin{cases} \sin\alpha \sim \alpha \\ \tan\alpha \sim \alpha \end{cases}$

also

$\delta\vartheta \sim \left.\begin{array}{l} (n-1) \\ \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} \end{array}\right\}$ construction parameter fixed!

- $n \sim$ medium of lense
- $\rho_1, \rho_2 \sim$ curvature of lense surface!

FIXED!

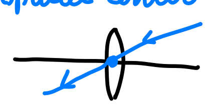
constants

$$\Downarrow$$

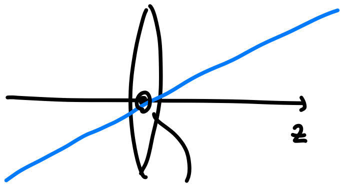
$$\delta\vartheta = \underset{\substack{\uparrow \\ \text{constant}}}{K}\,y$$

$K = (n-1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$

IF $\underline{y = 0}$: the light ray hit in optical center of lense



then $\delta\vartheta = 0$

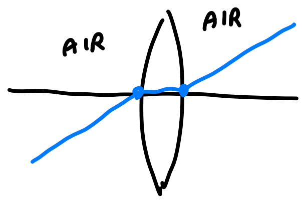


$\delta\vartheta = 0$!

$\Downarrow$

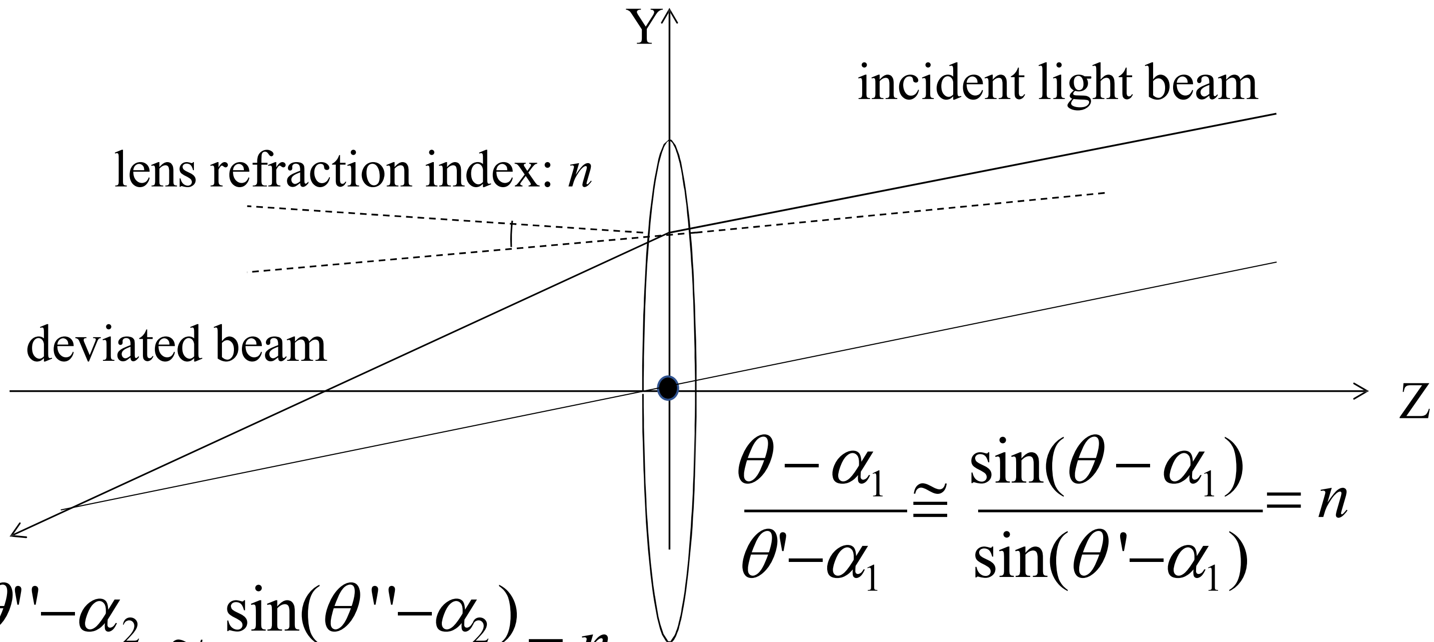
No deviation for any light beam hitting the lense on the optical axis

on that point, the two normals are // z axis → Double refraction with same deviation



$\delta\vartheta = 0$

while further from z axis y↑ than $\delta\vartheta$↑ more inclination angle → proportional deviation to y



$$\frac{\theta-\alpha_1}{\theta'-\alpha_1} \cong \frac{\sin(\theta-\alpha_1)}{\sin(\theta'-\alpha_1)} = n$$

$$\frac{\theta''-\alpha_2}{\theta'-\alpha_2} \cong \frac{\sin(\theta''-\alpha_2)}{\sin(\theta'-\alpha_2)} = n$$

deviation angle ? $\delta\theta = \theta'' - \theta$

$$\delta\theta = (n-1)Y\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

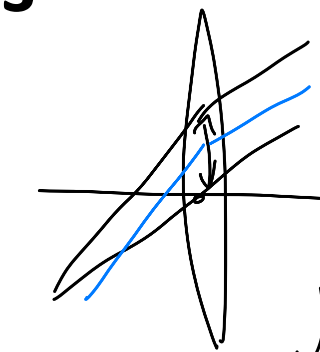
$Y = 0 \rightarrow \delta\theta = 0$ undeviated ray

Focalization (convergence) of parallel rays

↳ what when we have a family of parallel rays...

Focalization of parallel rays

$$f = \frac{1}{(n-1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}$$



half deviation

all converge to same point!

ALL PARALLEL RAYS CROSSING THE LENS

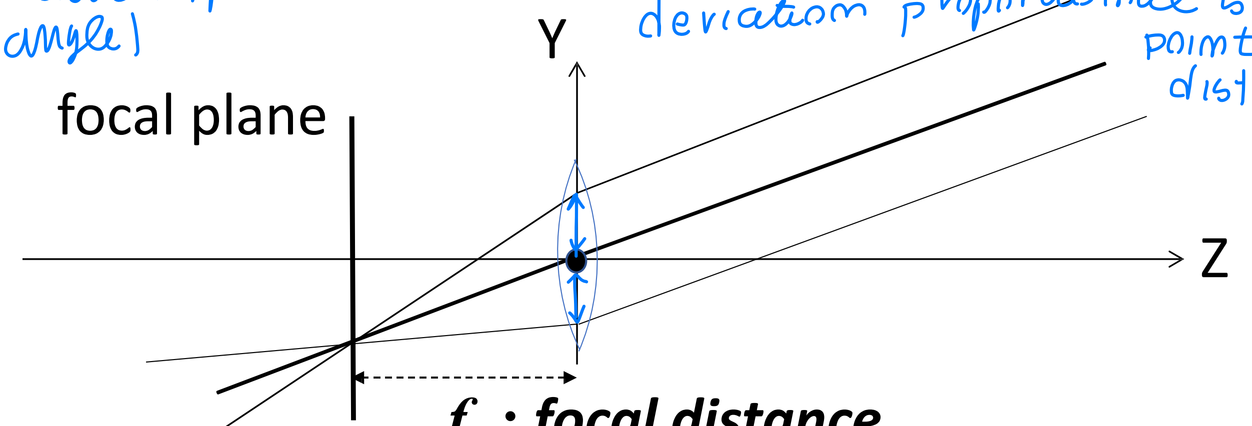
concur at a common point on the a *focal* plane $Z = -f$

↳ under linear hyp (small angle)

⇓

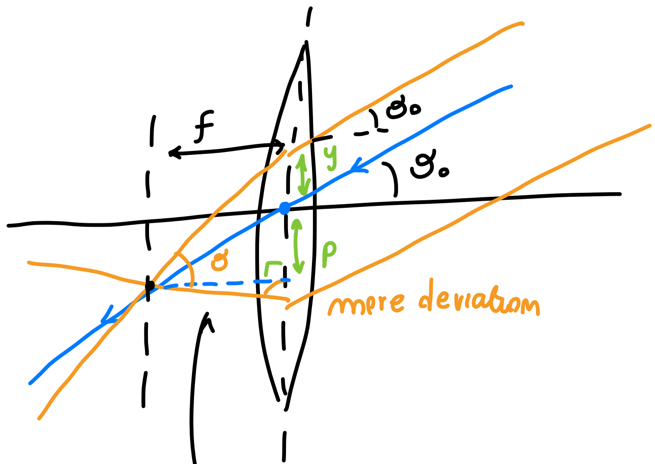
maybe all parallel beams cross at same point

deviation proportional to hitting point distance



***f* : focal distance**

θ_0: undeviated angle



$$\left\{\begin{array}{l} \frac{f}{p} = \tan\theta_0 \sim \theta_0\, p \\ \frac{f}{p+y} = \tan\theta \sim \theta\,(p+y) \end{array}\right.$$

$$\frac{f}{\theta} = p+y$$

$$y = \frac{f}{\theta} - p = \frac{f}{\theta} - \frac{f}{\theta_0}$$

$\theta - \theta_0 = \delta\theta$ deviation angle

$\delta\theta \sim y$ PROPORTIONAL to y

$f/\theta \to y = f\,\delta\theta \to \delta\theta = \frac{y}{f}$

$\delta\theta = Ky$

$f \sim \frac{1}{K}$

than

$f \sim n, \rho_1, \rho_2$ is a CONSTANT!!

$\Downarrow$

$$K = (n-1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

(from before computation)

this meeting point distance of lense plane is constant f indipendently of θ_0 inclination of parallel rays family $\leftarrow$ f depends on medium/glass = FOCAL DISTANCE of the lense

Focalization of parallel rays: same plane for any direction

$$f = \frac{1}{(n - 1)(\frac{1}{\rho_1} + \frac{1}{\rho_2})}$$

ALL PARALLEL RAYS CROSSING THE LENS

concur at a common point on the a *focal* plane $Z = -f$

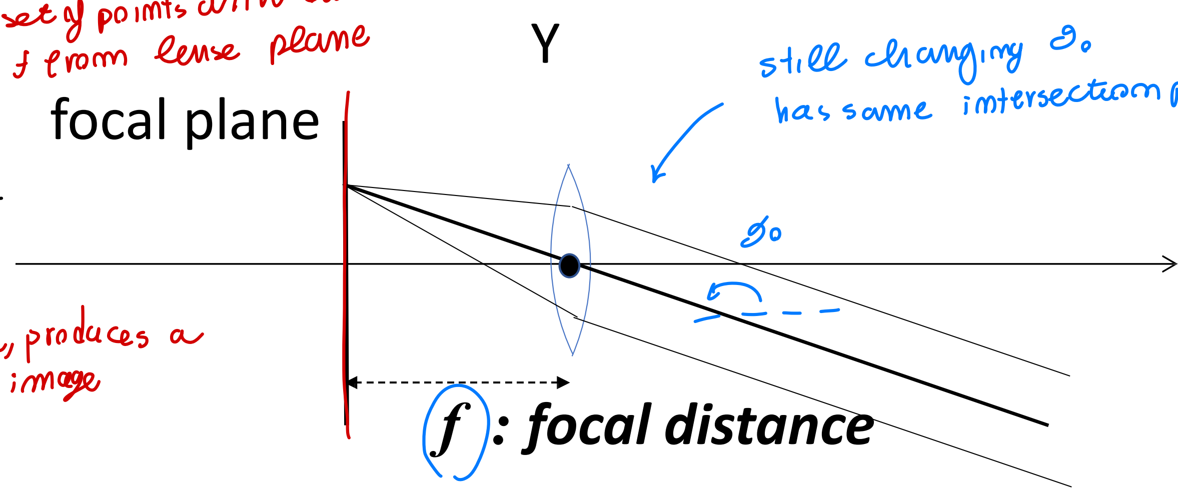
we define focal plane as set of points with distance f from lense plane

this is usefull! IF we want to build meaningfull image... I want part of picture to project in a specific point

⇓

any point in environment, produces a single point in the image

still changing ϑ_0 has same intersection point



Any point in environment ~ single point in image

⇓

requires to select direction of light ray...

light rays no scatter for large image ⇒ PROJECT to an exact point!

⇓

another assumption needed

as if lense has controllable exposed part

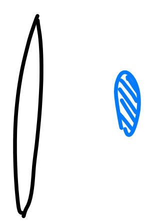
↳ ≃ APERTURE

exposed part that can be controlled by myself : APERTURE

done using a $\underbrace{\text{DIAPHRAM device}}_{}$

=

opaque surface of variable size



I can control device such that, diameter change, Rotating the device

← cover parts of the lense.. depends on how DIAPHRACT ODERATE

→ I control the diameter of the exposed part of the lense

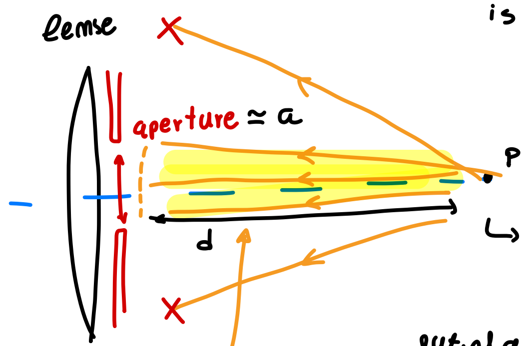
↳ JUST small circle open...

modify exposed part of lense

Place the screen where parallel rays converge

IF all the objects of interests are at a distance (which is as aperture diameter a (camera parameter))
↑ diaphram value

↓ if distance object - lemse is d



↳ if $\underbrace{d >> a}$
depth >> aperture

out of all light rays only a small part reach lense

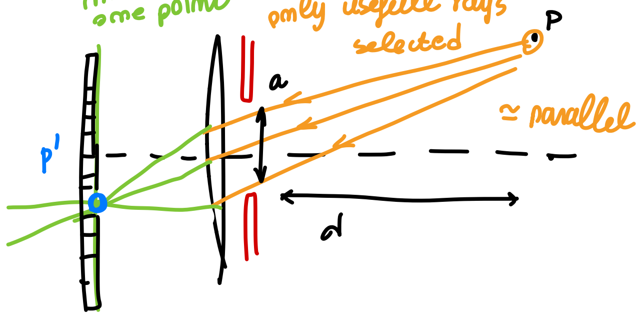
and we can assume those beams almost parallel

↳ so we are in the situation described before...

we can use focal plane property, the parallel rays will all meet at focal plane point

the light produced by one point in the world will be all collected at single point on screen

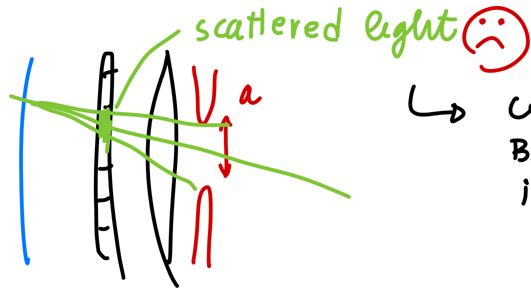
⇓ then: if we place the screen of photosensitive elements on the focal plane... all points P far enough $d >> a$ the situation approx as if rays // parallel



good property!

they meet on same plane

if instead plane of photosensitive sensor is NOT on focal plane



↳ we have BLURRED / DEFOCUS image, NOT point → point'

so OVERLAPPING!

photograph without proper set-up of camera → out of focus!

↓

all further than $d \sim 100\,a$ on focus,

we have the screen to be on focal plane

↑

our hypotesis! ⇒ where all parallel beams converge!

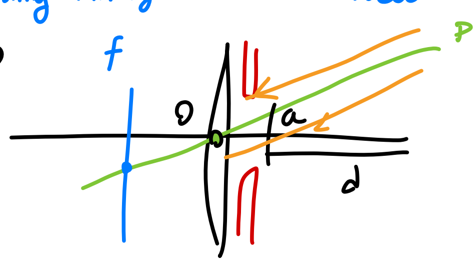
at distance f

focal length

$$f = \left((n-1)\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\right)^{-1}$$

doing so, the relationship between p in space and P' in focal plane is easy, characterized by just the ray passing through lense center

just consider PO



we only consider that one

intersection lense plane ~ optical axis = "o" image center

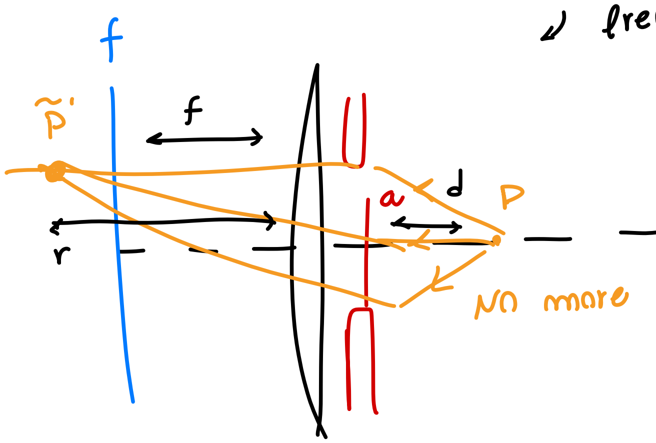
than PO continue until p' at intersection of f plane

IF object is too close

d > a but no more d >> a !

$\underbrace{}$ frennel law holds...

you should approach points to lense plane



they meet again on $\tilde{p}$ NOT on focal plane... $r > f$

frenel law $\frac{1}{f} = \frac{1}{d} + \frac{1}{r}$

BUT you can't have convergence @ one point always if r vary

only when $d \gg a$ ($d \sim 100a$) → than focal plane always valid as intersecting point

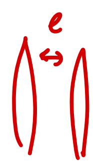
NOTE: real system may have more than one len.

→ focusing device ....

f NOT change...
you can build optical system with more
lense changing focus

ex. phone camera $a \sim 1$ mm
than $d \sim 10$ cm is OK to stay in focus
$d \sim 2$ cm bad... blurring

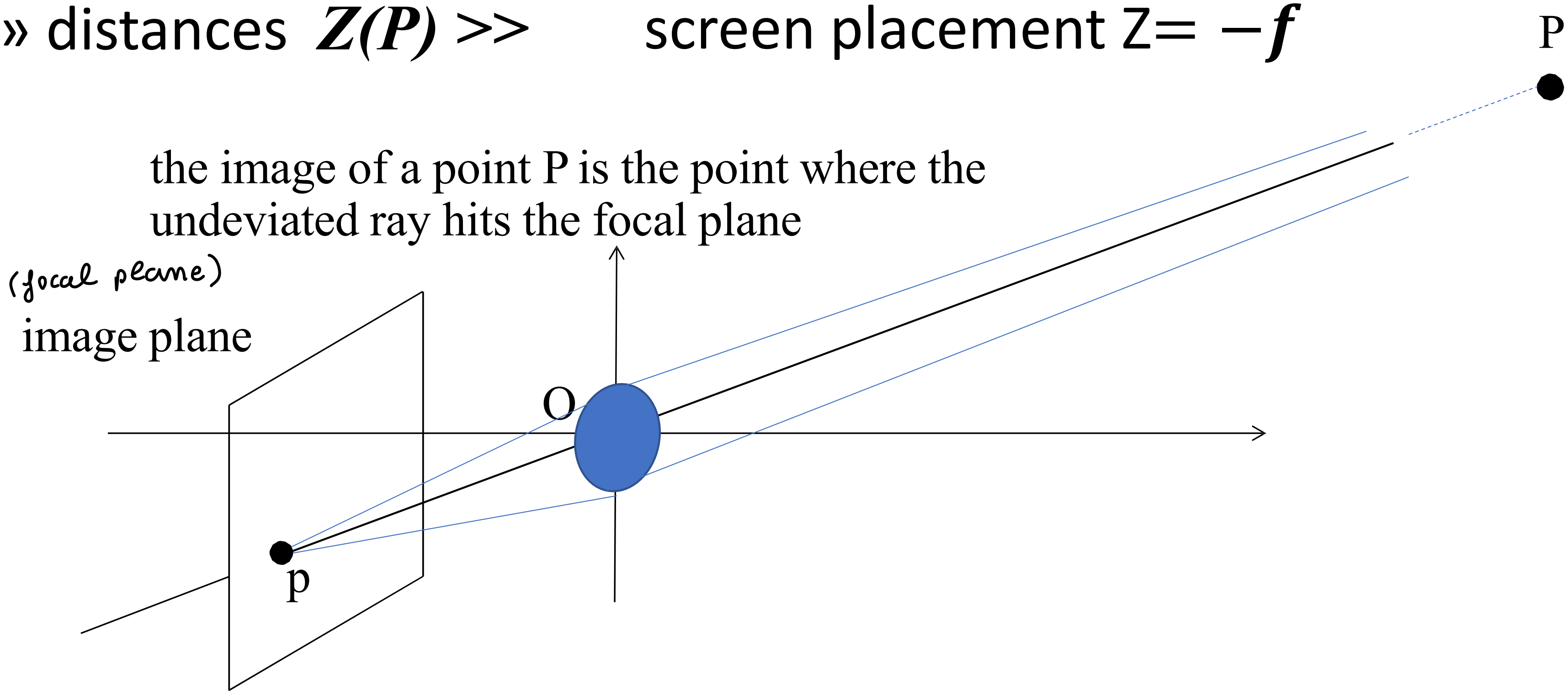
in 2 lenses camera → change



e to make
on focus
the image

Hp: «large» distances ***Z(P)* >> *aperture***   screen placement Z= $-f$

the image of a point P is the point where the undeviated ray hits the focal plane



p = image of P is on the line(P,O) and on the image plane

p = image of P = image plane ∩ line(O,P)

Model: the pin-hole camera model

Our simplified model:

(i) thin spherical lens, (ii) *small angles*, (iii) $\boldsymbol{Z(P)} >> \boldsymbol{a}$, (iv) Z= $\boldsymbol{f}$

→ PIN-HOLE CAMERA

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gir·hoc ieift,fii:ncaJGfuptrio.rpm t.bquiu.p:1rian1?J1i
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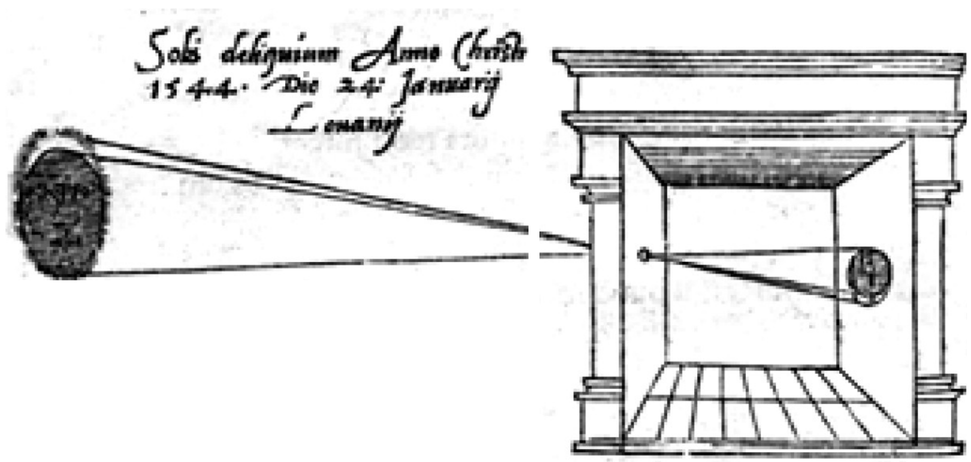

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The image of a point P in the scene

↳ How would you describe mathematically point P and its image p relationship?

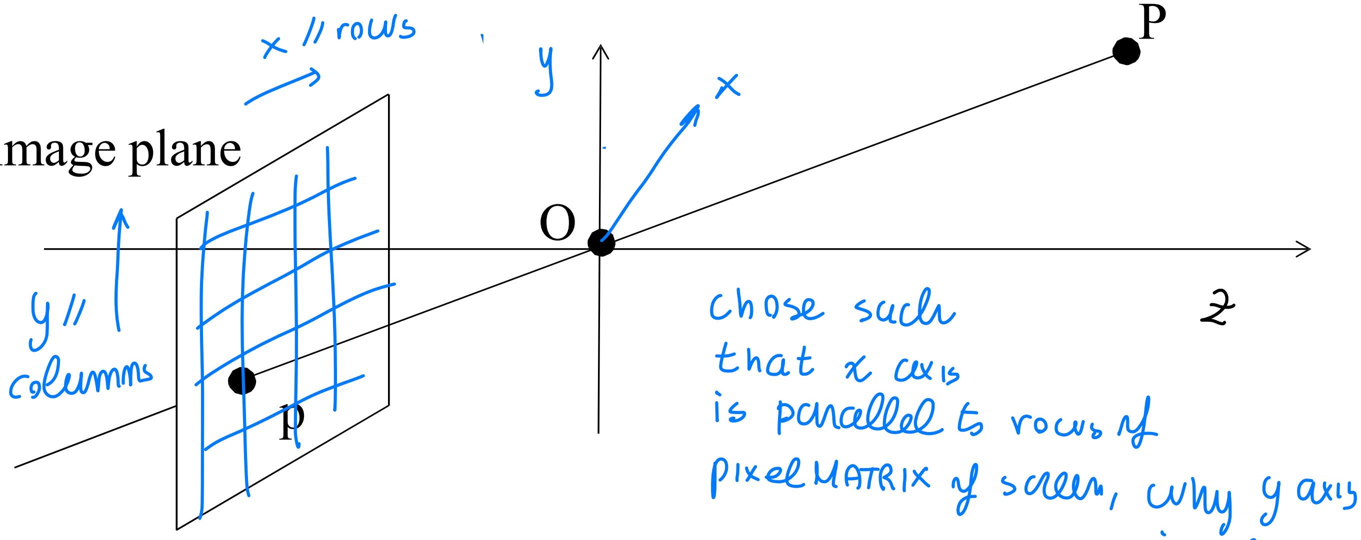
↓

all depends on the coordinate system (use the most simple... ⟶)

Hp: «large» distances $\boldsymbol{Z(P) \gg f}$ screen placement $Z = -\boldsymbol{f}$

the image of a point P belongs to the line (P,O)

With this reference all is more comfortable!



p = image of P = is on the image plane and on the line(O,P) → p = image plane ∩ line(O,P)

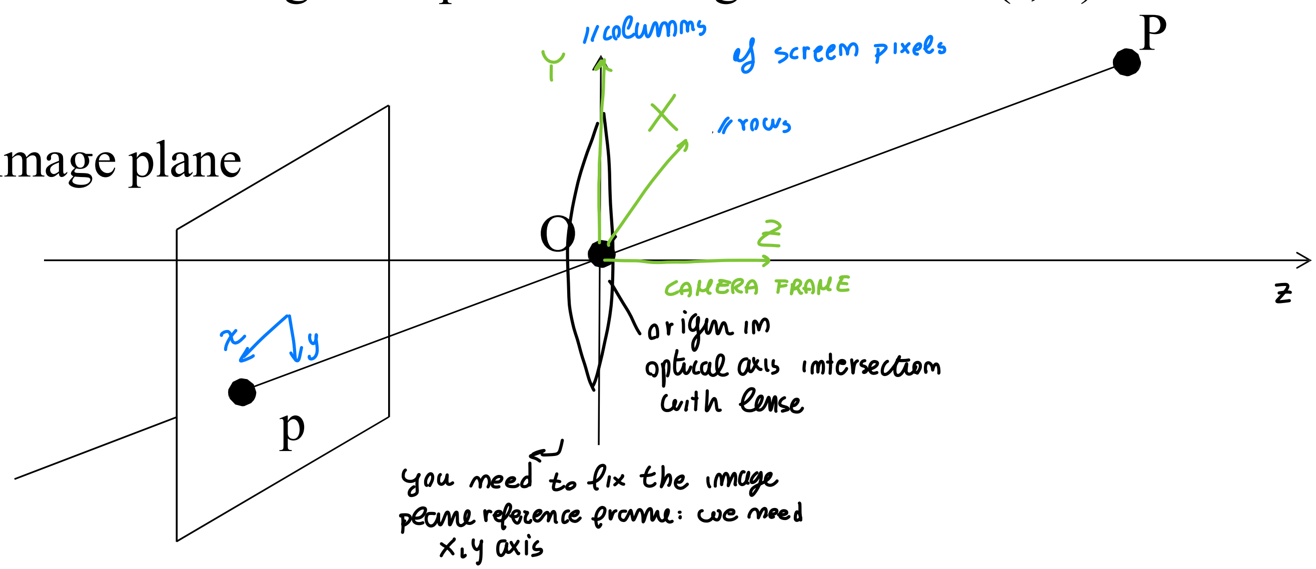
viewing ray of p: line(O,p) =
locus of the scene points projecting onto image point p

Hp: «large» distances $Z(P) >> f$ screen placement $Z = -f$

more comfortable reference frame

the image of a point P belongs to the line (P,O)

x,y of point p: p_x, p_y with this reference are
$p_x = \frac{f}{Z} X$



x,y image axis

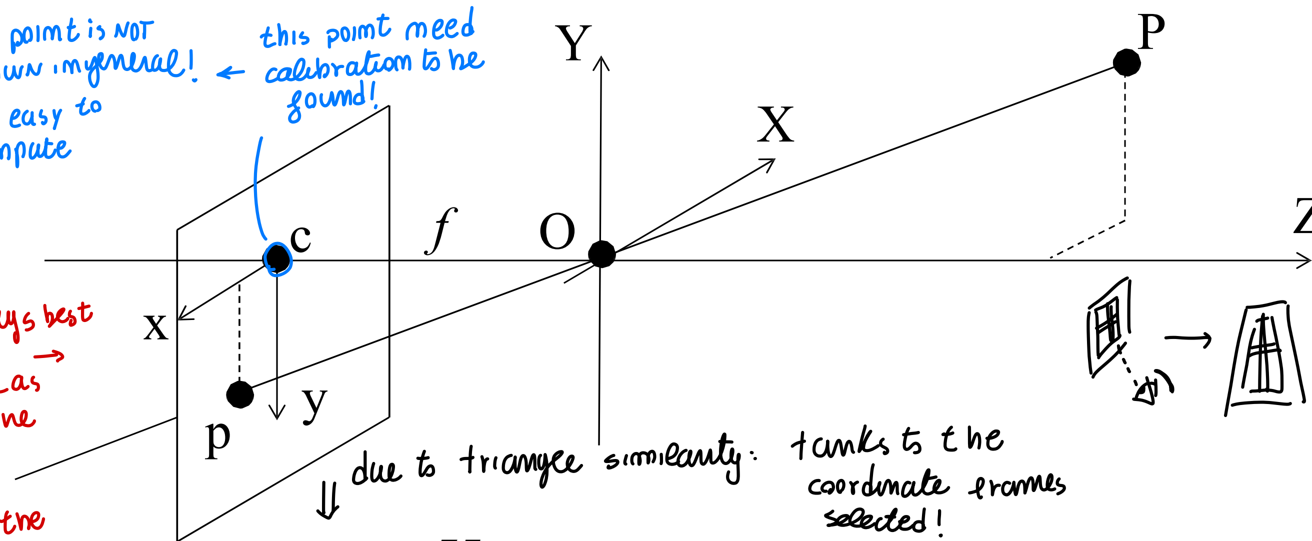
x // X
y // Y
along opposite direction
↓
make similar triangle possible to build, simplify relationship!

p = image of P = is on the image plane and on the line(O,P) → p = image plane ∩ line(O,P)

viewing ray of p: line(O,p) =
locus of the scene points projecting onto image point p

this point is NOT known in general! ← this point need calibration to be found!

NOT easy to compute



NOT always best origin... → if image as MATRIX, the origin O should stay at the angle of matrix

⇓ due to triangle similarity: tanks to the coordinate frames selected!

What you see is NOT what you get...

with this comfortable reference ⇒

$$\begin{cases} x = f\,\dfrac{X}{Z} \\[2ex] y = f\,\dfrac{Y}{Z} \end{cases}$$

central projection

Nice relationship BUT

- -nonlinear

(SINCE NON LINEAR)

- -not shape-preserving
- -not length-ratio preserving

Taking image from different point, you don't see in same way... appears with NO PRESERVED SHAPE

Bad news:
what you see IS NOT what you get

even after simplification + linear assumption...

the fact that $x = f\dfrac{X}{Z}$ divide by Z

due to this,

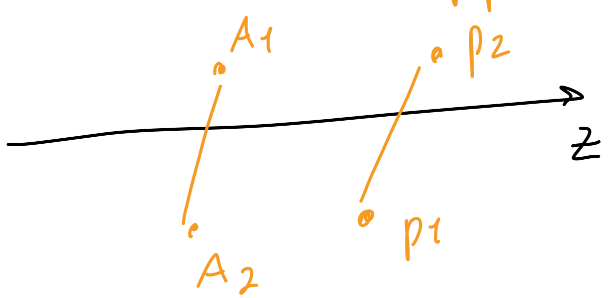
2 points at Z big seems closer

↓

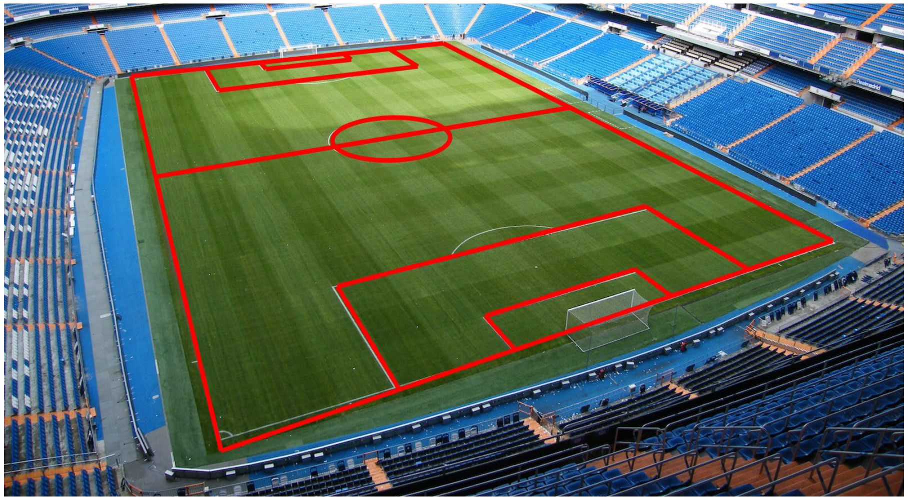
we perceive smaller distance for further object → PERSPECTIVE

even same distances appear different

A_1A_2 seems bigger than p_1p_2 because of $\frac{1}{z}$



what you see **IS NOT** what you get ! =>

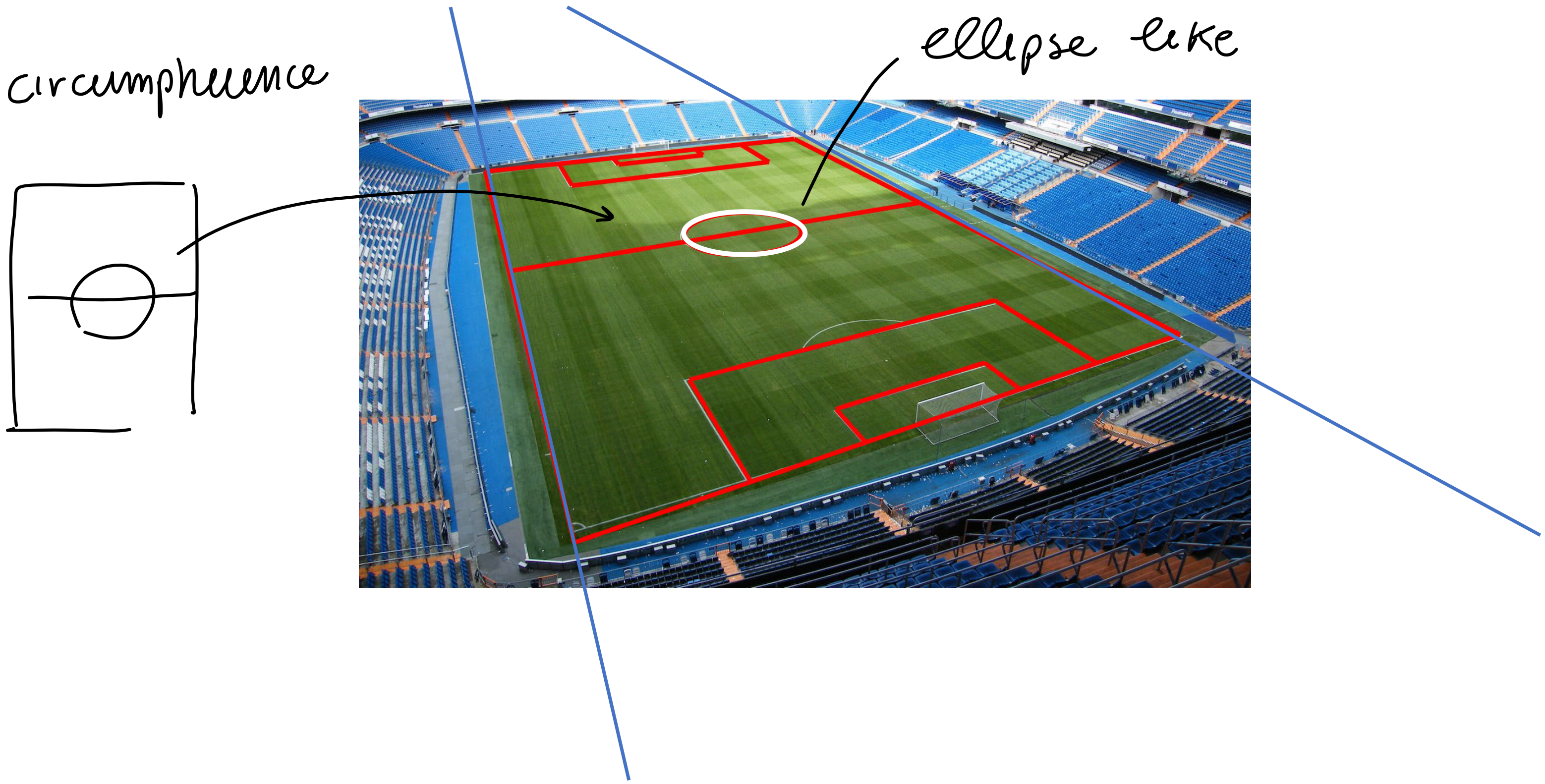


Parallel lines **DO NOT PROJECT ONTO** parallel lines

you know those lines are parallel...

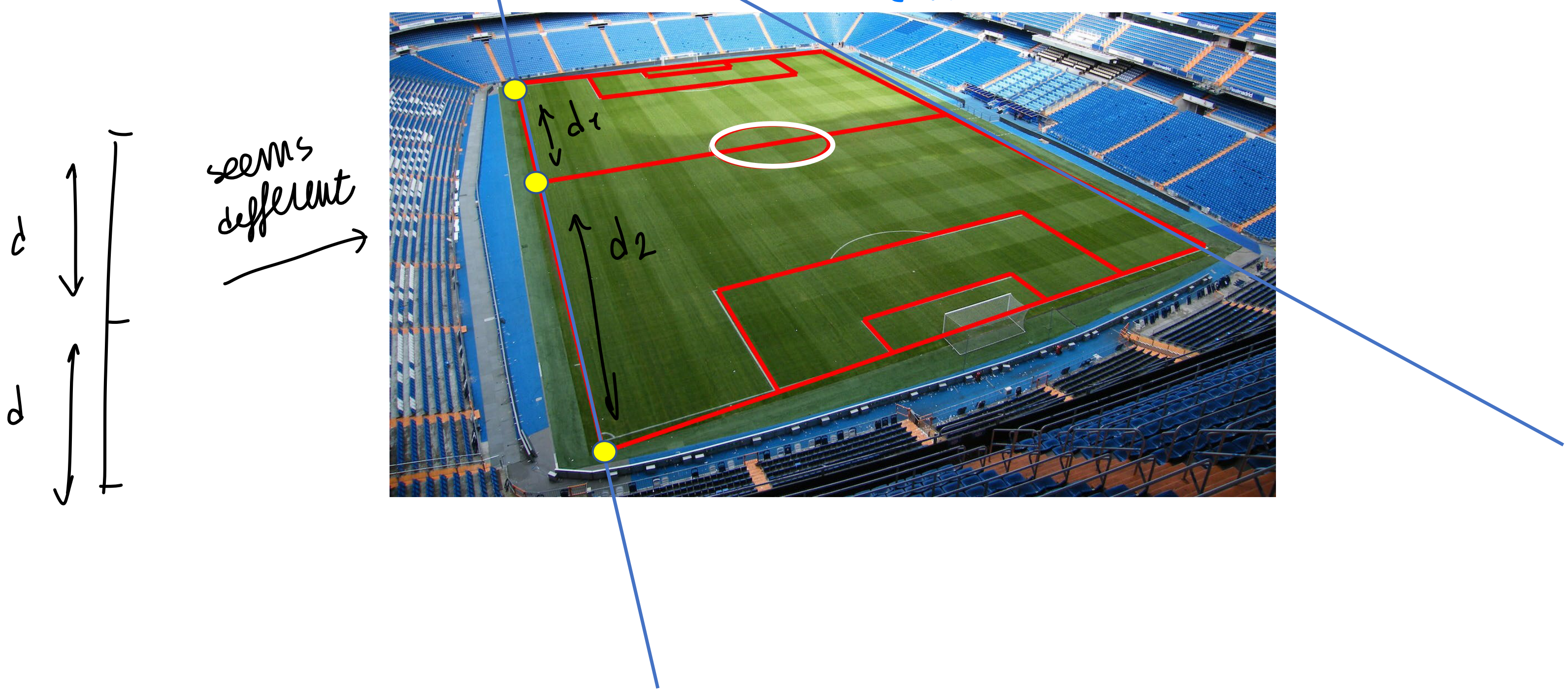
BUT on image view you lose it!

circumpherences **DO NOT PROJECT ONTO** circumferences



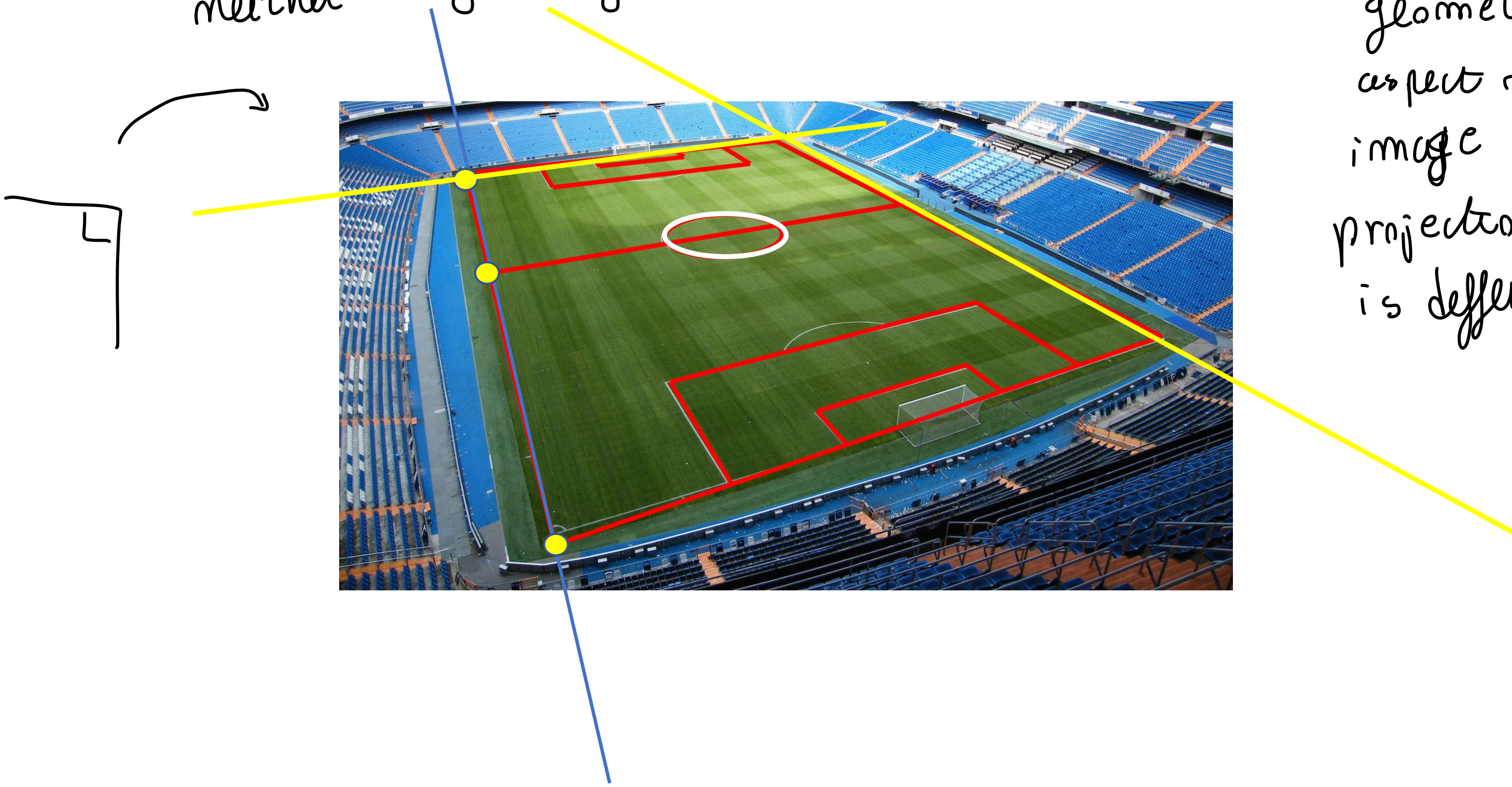
midpoints **DO NOT PROJECT ONTO** midpoints

midpoints DON'T PROJECT onto midpoints!
is an assumption that
cannot be done...



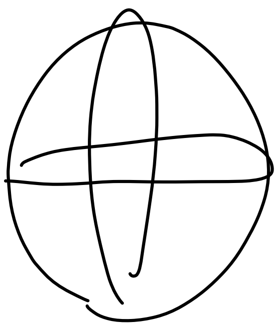
Orthogonal lines **DO NOT PROJECT ONTO** orthogonal lines

neither orthogonality is maintained...

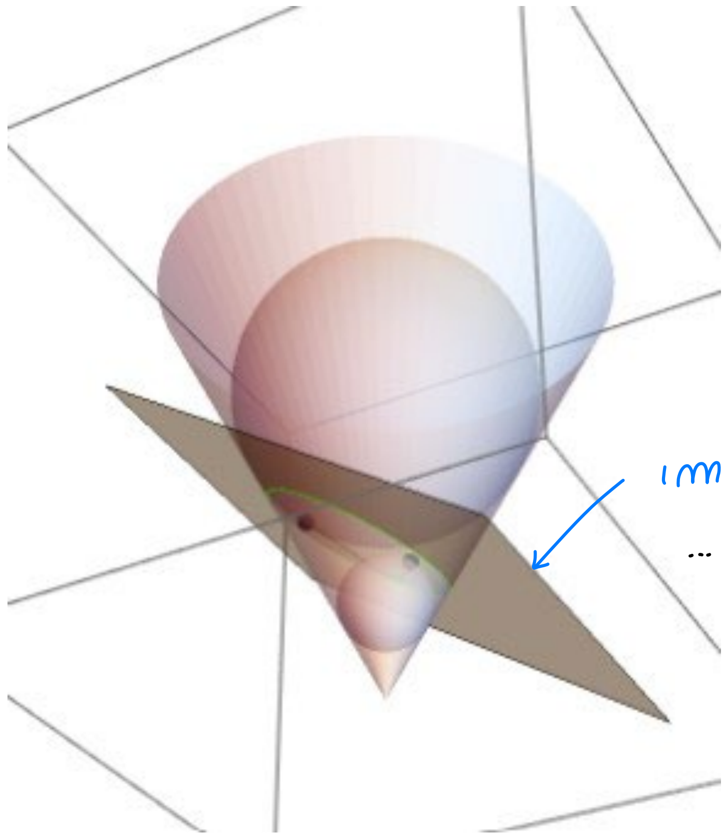


geometry aspect of image projection is different

spheres **DO NOT PROJECT ONTO** circles



even image of a sphere is an ellipses!



become an ellipses

image plane

... the sphere become circle only if axial symmm... when img plane is orthogonal to cone axis

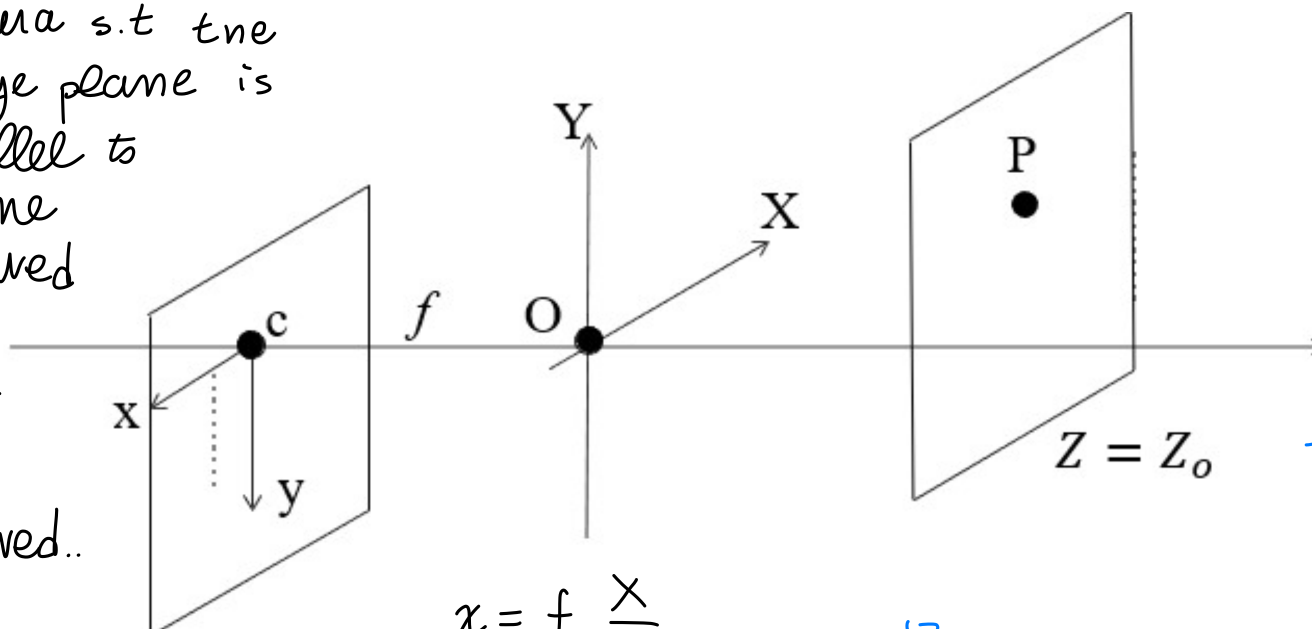
When observing a PLANAR SCENE

Exception:

planar scene **parallel** to image plane

$$Z = Z_o = constant$$

placing the camera s.t the image plane is parallel to plane observed



Z = constant among all 2D plane observed.. + then SHAPE is PRESERVED

preserve shape!

same than $\begin{cases} X \sim x \\ Y \sim y \end{cases}$

$$x = f\,\frac{X}{Z}$$

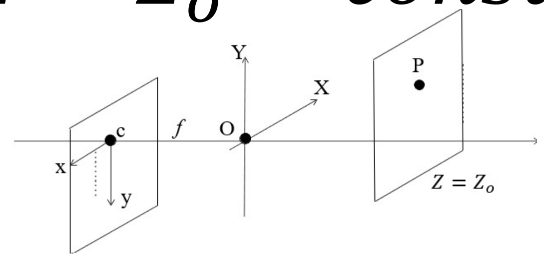
$$y = f\,\frac{Y}{Z}$$

same Z everywhere $Z = Z_0$

Exception:
planar scene **parallel** to image plane

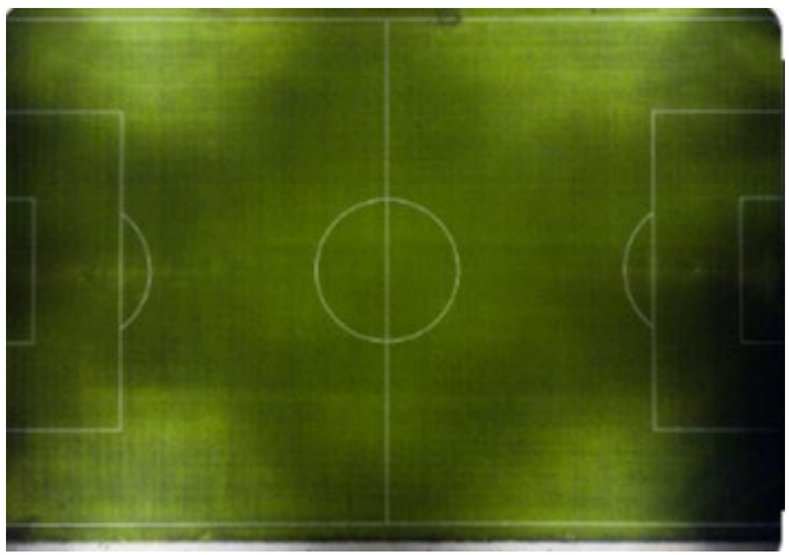
$$Z = Z_o = constant$$

when drone look down
to the football
scene from the top,
image plane // plane,
all is correct shape



$$x = f\frac{X}{Z_o} = kX$$

$$y = f\frac{Y}{Z_o} = kY$$



The image is a **scaled** version of the scene:

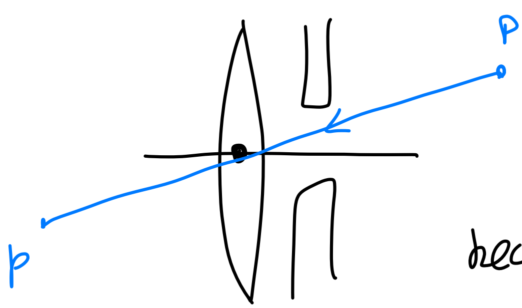
→ **same shape**

↓

angles preserved

TO UNDERSTAND WHAT WE GET FROM WHAT WE SEE,
WE NEED TO STUDY THE GEOMETRY OF CENTRAL PROJECTION

this kind of camera projection is called "CENTRAL" PROJECTION



because all lines meet on the center

Preview of Geometry:
useful properties

1. The vanishing point of a direction

≃ punto di fuga (introduced on painting art Piero della Francesca)

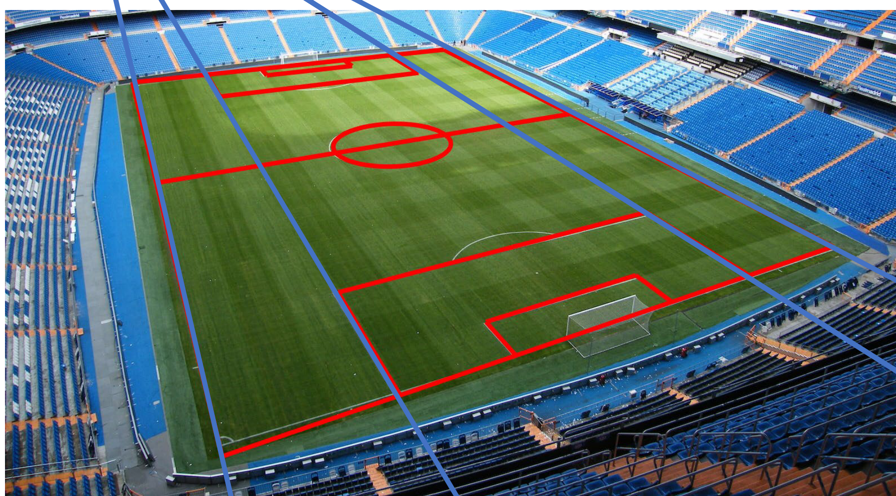
PERSPECTIVE

Parallel lines project onto **concurrent** lines

vanishing point of a direction

associated to a direction (any direction of parallel lines has its own vanishing point)

if no aberration (hyp. of small angles) those points meet on common POINT
= vanishing point



Preview of Geometry: useful properties

2. The cross ratio of four colinear points

any time we have 4 co-linear points, there is SOMETHING which remain invariant even in the perspective image

→ EXAMPLE..

ratio of lengths: $\underbrace{\text{NOT INVARIANT}}$

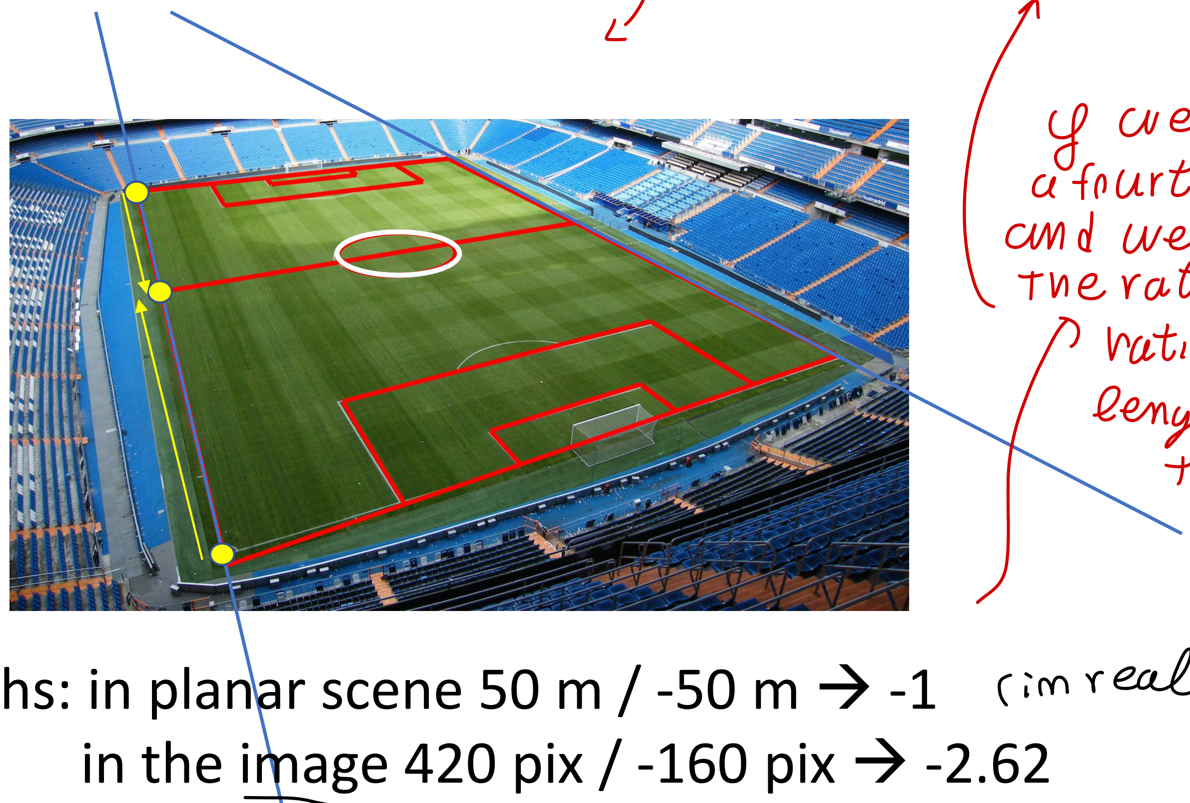
≃ CROSS-RATIO.

"double ratio"

(BIRAPPORTO)

if we introduce a fourth point and we take the ratio between ratios of lenghts...

this is INVARIANT



ratio of lengths: in planar scene 50 m / -50 m → -1 (in real world)

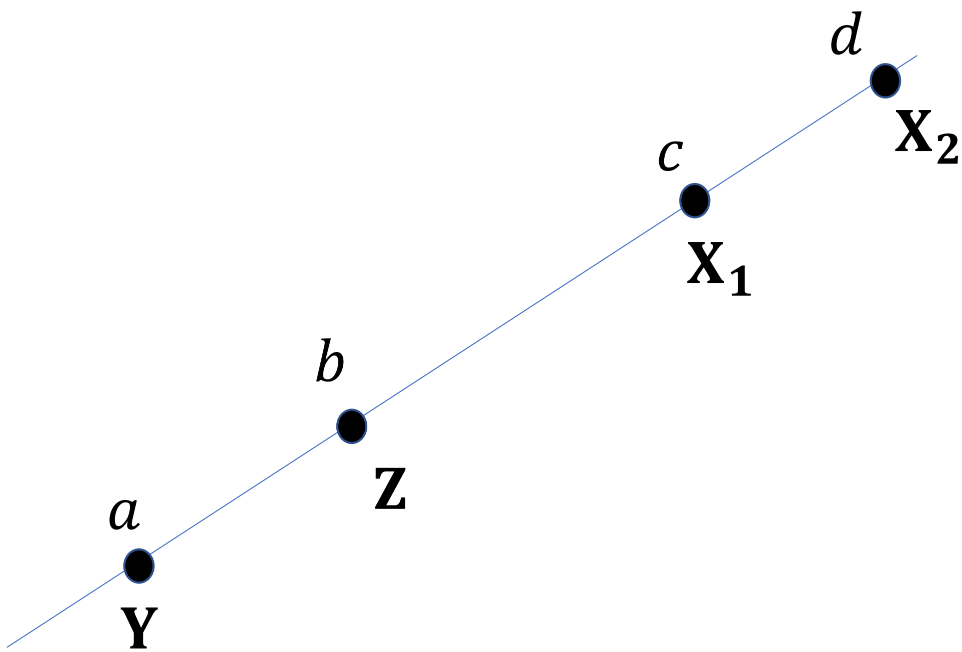
in the image 420 pix / -160 pix → -2.62

Cross ratio of a 4-tuple of colinear points

$$CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

double ratio,
this is
invariant

taking 4 points



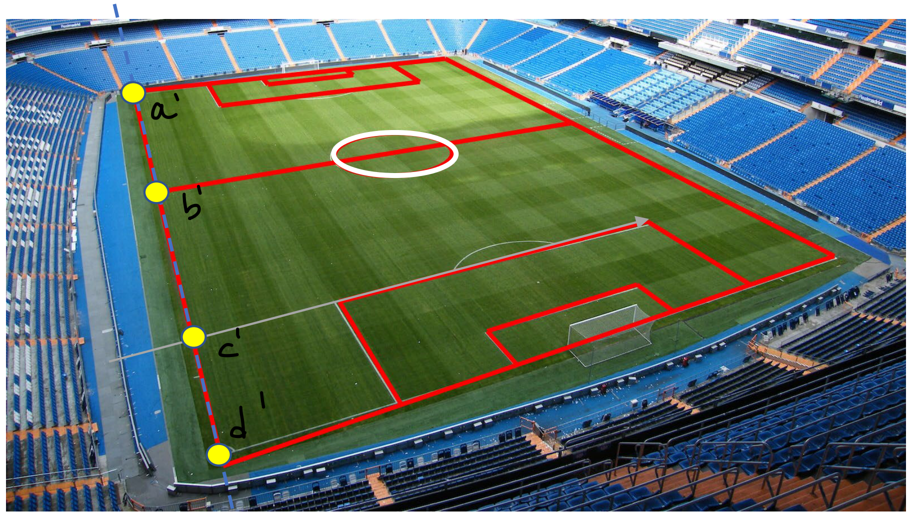
the cross ratio, i.e., the ratio between the two ratios, INVARIANT: it is preserved from real word to image

a

b

c

d



$$\frac{\left(\frac{c-a}{c-b}\right)}{\left(\frac{d-a}{d-b}\right)} = \frac{\left(\frac{c'-a'}{c'-b'}\right)}{\left(\frac{d'-a'}{d'-b'}\right)}$$

real wored　　image plane

under thin lens hyp..

this allow some power..

taking 4 points each ratio of length is preserved!

this is a property preserved...

- Introduction and the Camera Optical System
- **Planar (2D) Projective Geometry**
- Spatial (3D) Projective Geometry
- Camera Geometry (3D → 2D Projection)

geometry of the combination
of the two space & image plane!

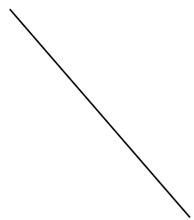
Planar (2D) Projective Geometry

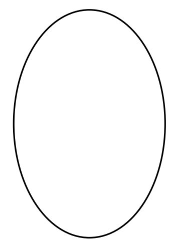
Planar Projective Geometry

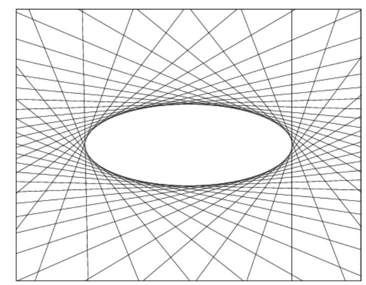
1 FIRST we consider elements of geometry

- **Elements**
  - **Points**
  - Lines
  - Conics
  - Dual conics

then



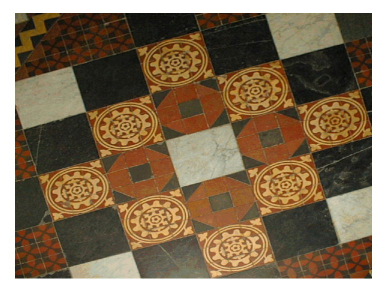


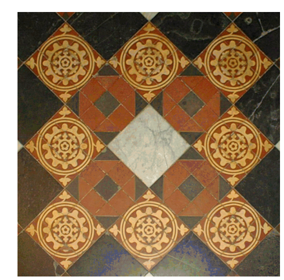


4 elements must be known

2

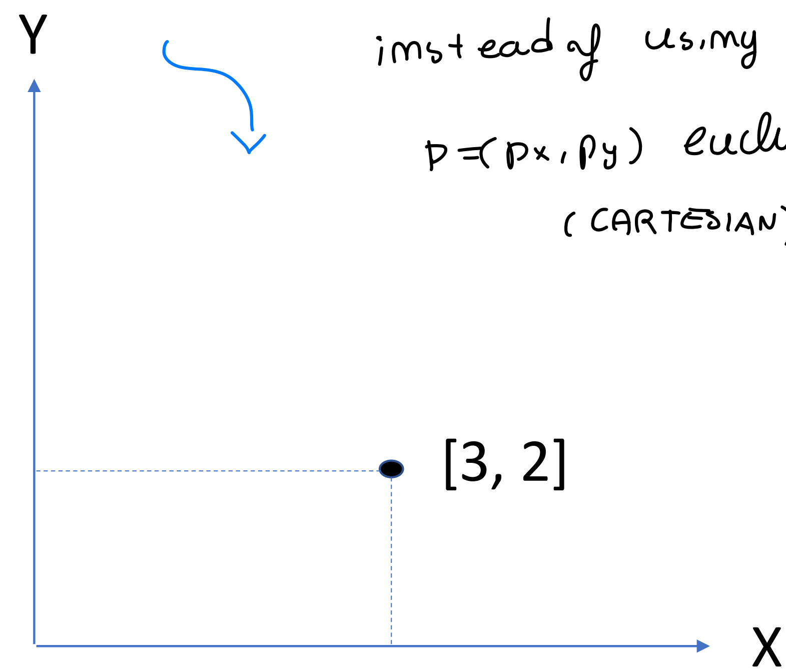
- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities





Points in 2D Projective Geometry

Euclidean plane – cartesian coordinates



instead of using

$p=(p_x, p_y)$ euclidean plane

(CARTESIAN)

↳ we need a different system of cohordinates

"

|| homogeneus cohordinates $\tilde{p}$ ||

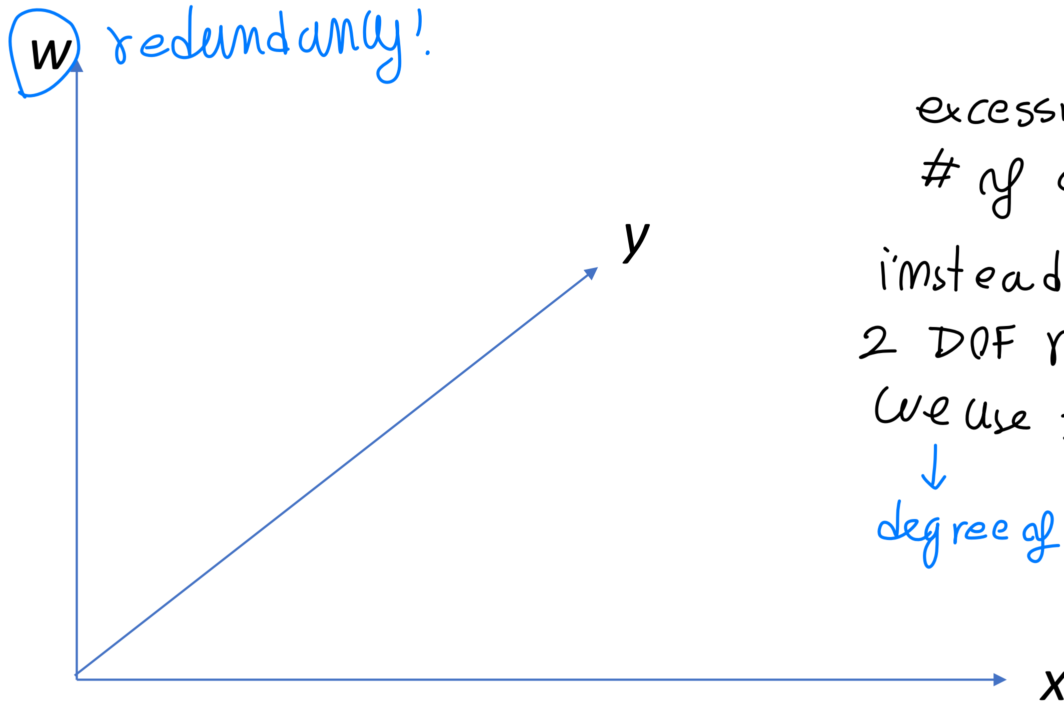
this give you lots of property respect cartesian

Homogeneous coordinates

"fake" → they use 3D space
to describe plane!

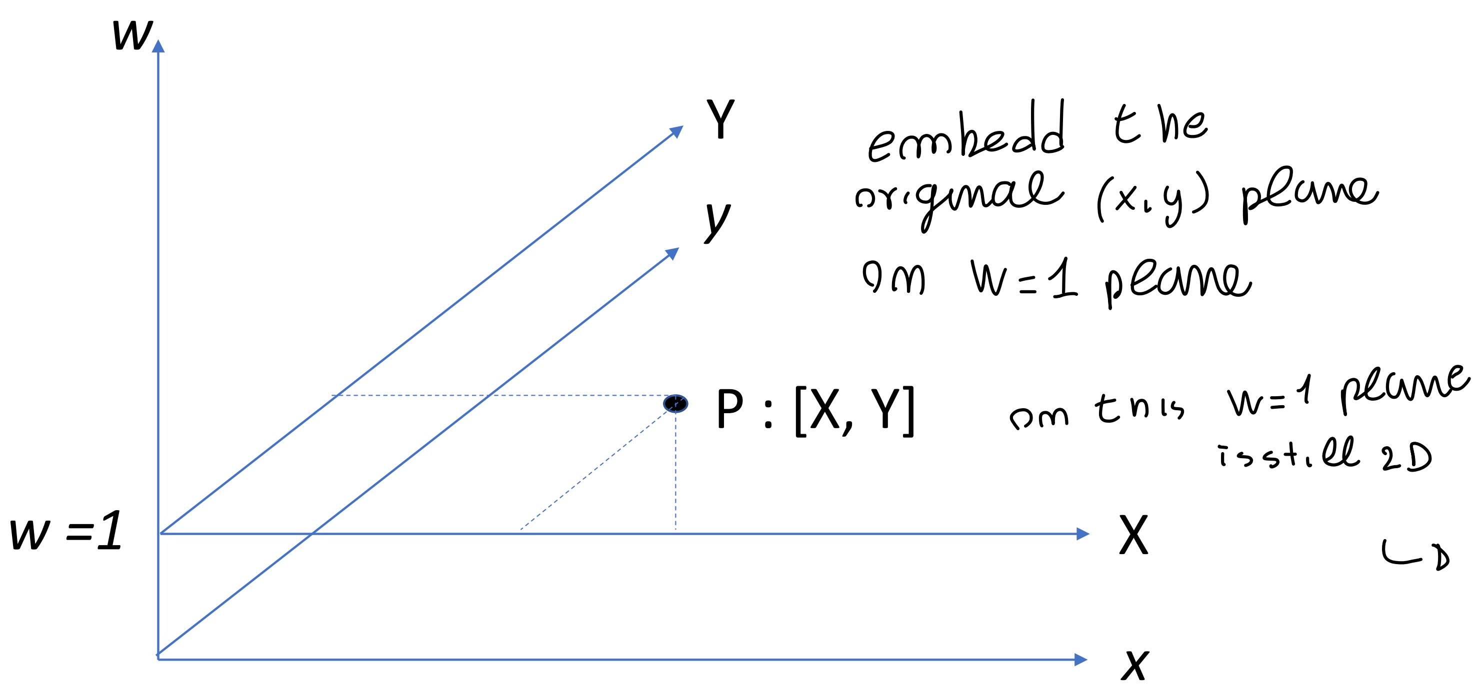
(study plane 2D
using 3D space) $\Longrightarrow$

Consider a (3D) space of coordinate vectors

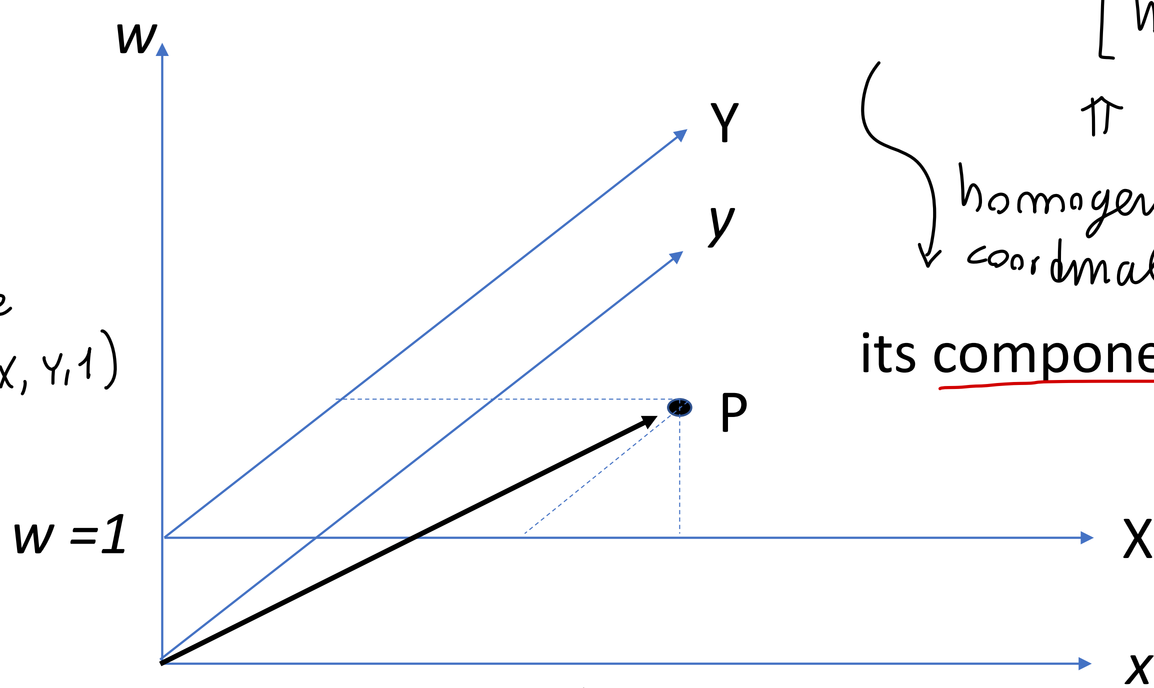


excessive
# of coordinates
instead of the
2 DOF representation,
we use 3 coord for 2 DOF
↓
degree of redundancy

Embed the Euclidean plane into the (3D) space of coordinate vectors as the plane $w = 1$



Consider the vector from the origin of the space to the point P



homogeneus coordinates

↓

considering the components (X, Y, 1) then, relaxing the condition, any random vector with $w \neq 1$ goes in another place $\Rightarrow w\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$

[higher dimensional space]

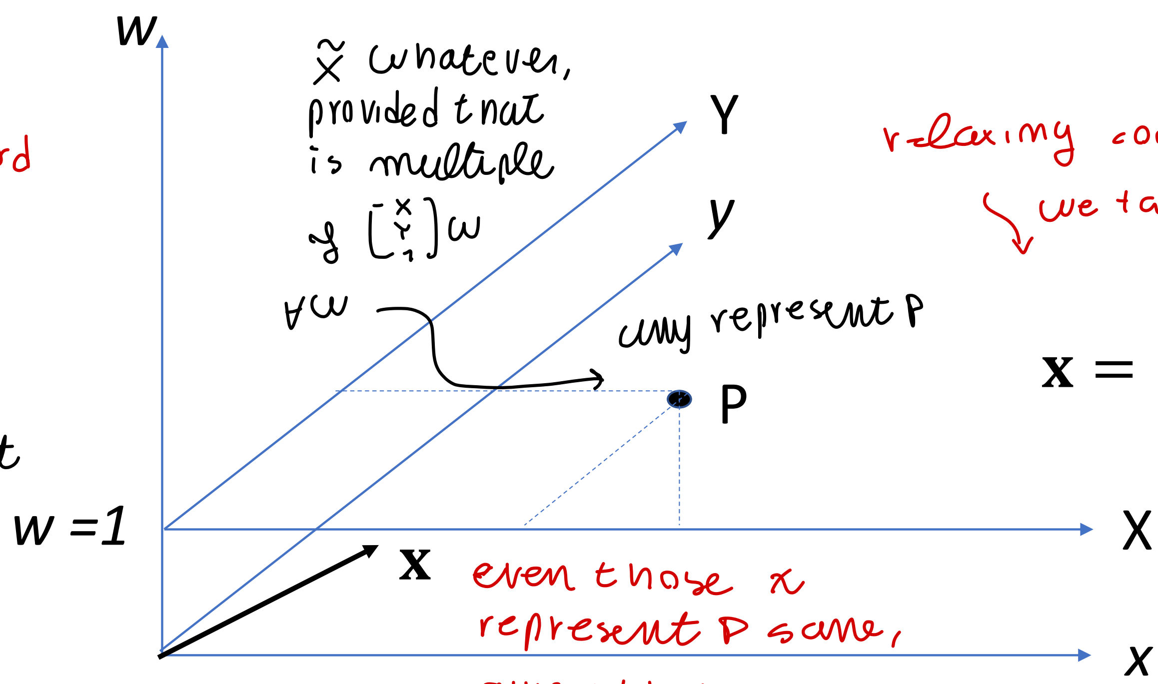
⇑

homogeneus coordinate

its components are $\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$ ← w

the point P is represented by any vector **x** , that is a nonzero multiple of it

homogeneus coord
"
whatever, provided that
$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} \omega$ represent the point



$\tilde{x}$ whatever, provided that is multiple of $\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \omega$

$\forall \omega$

any represent P

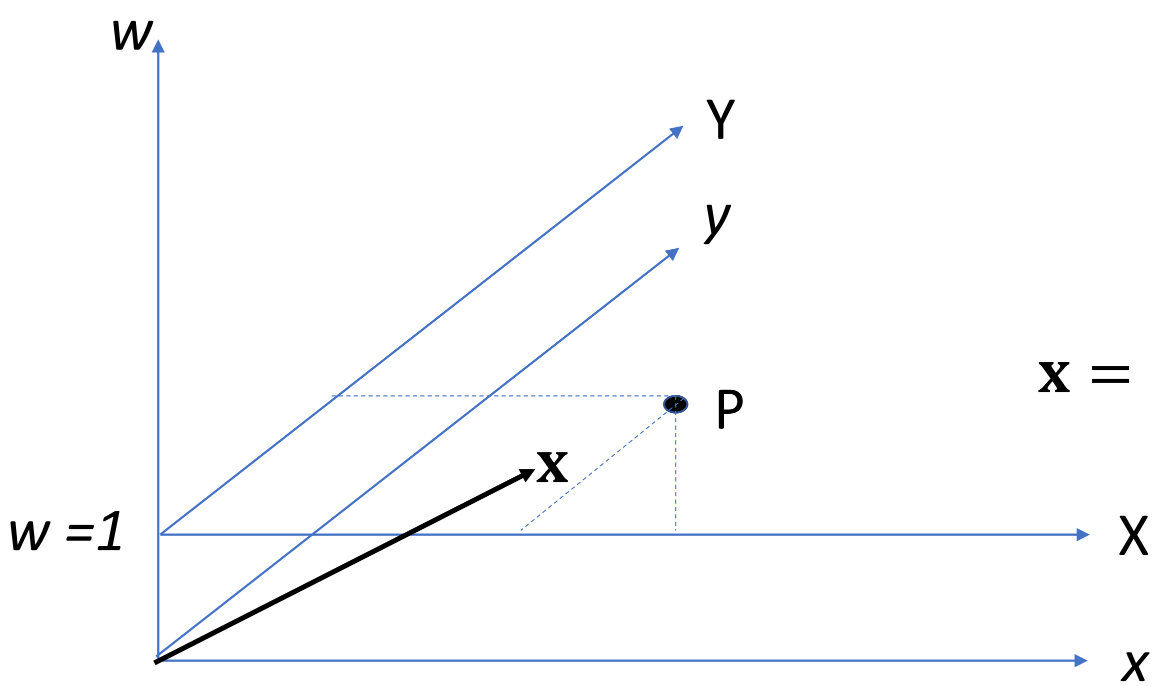
even those x represent P same, given that $[x\ y\ \omega]$ is multiple of $[X\ Y\ 1]\ \omega$

relaxing conditions
we take any w

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

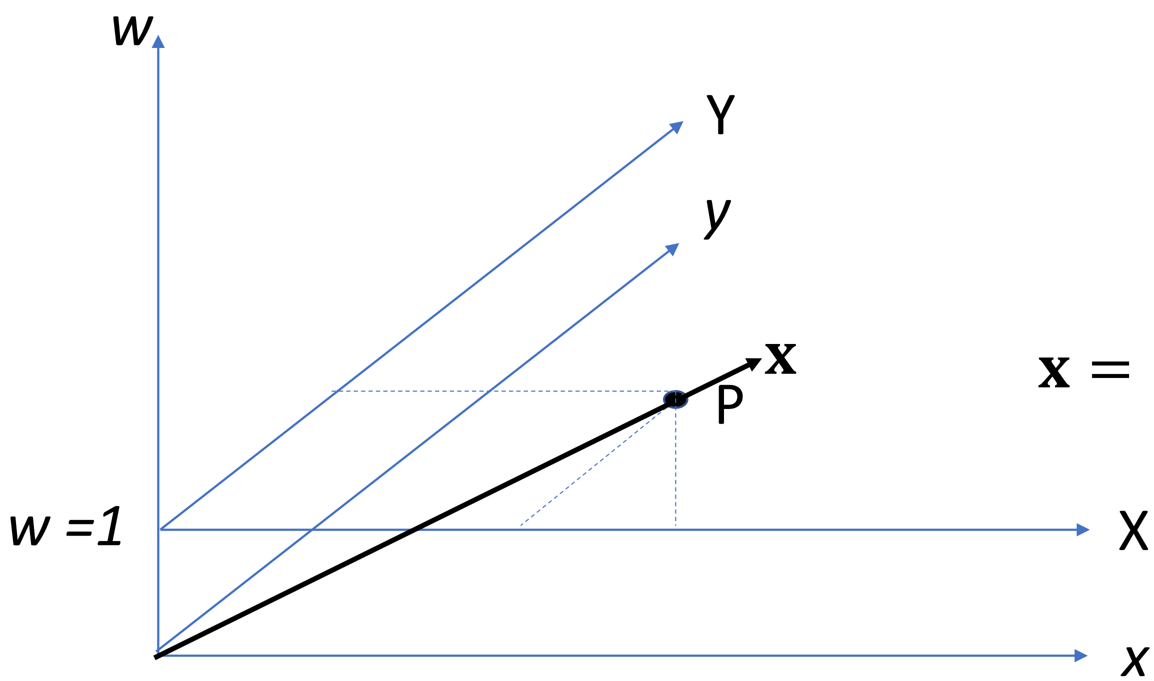
this is a parallel vector, but with a multiple difference

the point P is represented by any vector **x** , that is a nonzero multiple of it



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

the point P is represented by any vector **x** , that is a nonzero multiple of it



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

the point P is represented by any vector **x** , that is a nonzero multiple of it

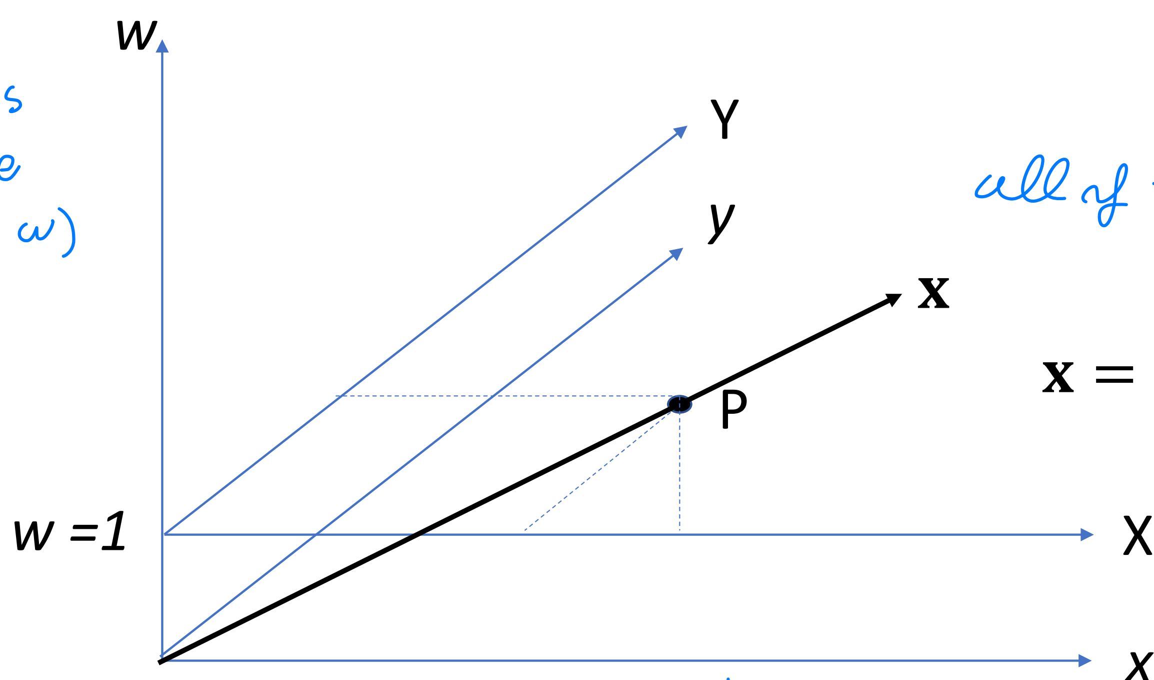
MANY VECTORS represent the same. (x y w)

↓

any vector multiple of (X y 1)·w represent same point

↑

REDUNDANCY!



same point!

all of them

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

|| 3 coordinates for an entity of 2DOF ||

the point P is represented by any vector **x** , that is a nonzero multiple of it

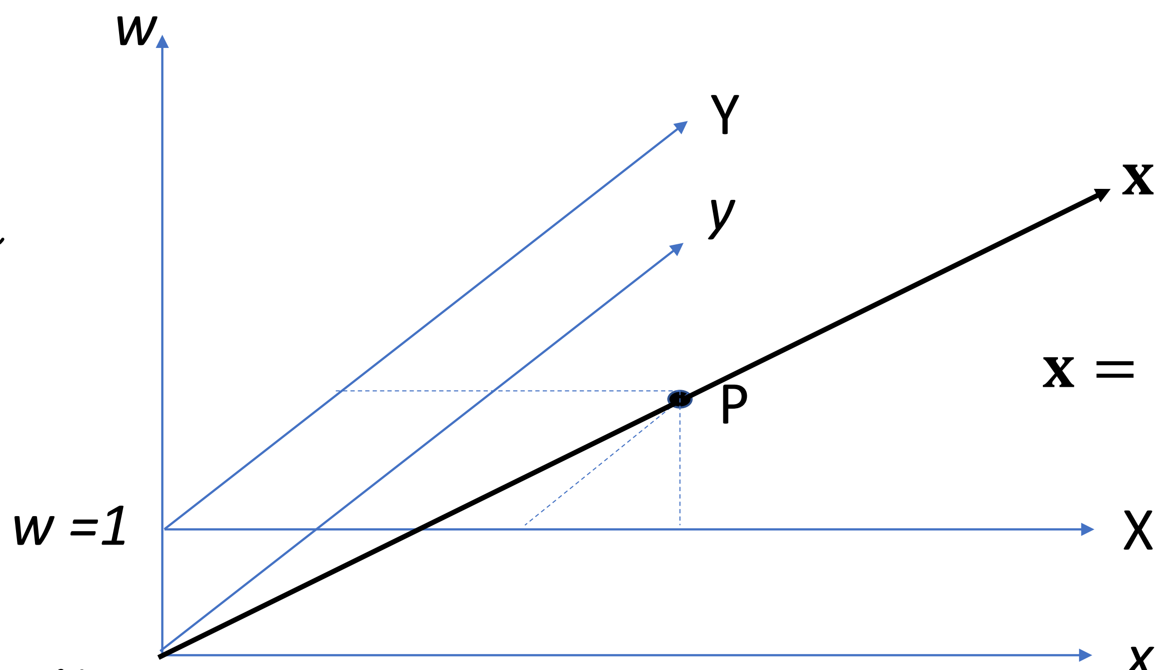
all of them are multiple of each other...

all of those vectors

$\begin{pmatrix} x \\ y \\ w \end{pmatrix}$

represent same P

of all parallel to eachother!



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

REDUNDANCY! 3 coord. for 3 DOF → ∞ vectors homogeneous represent same point

← all of them are multiple of each other...

$(\lambda \neq 0)$

cartesian

A vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and all its nonzero multiples $\lambda \begin{bmatrix} x \\ y \\ w \end{bmatrix}$, including $\begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$,

represent the point of cartesian coordinates $[X \quad Y] = [x/w \quad y/w]$ on the Euclidean plane

(just divide by w is enough to get cartesian one)

when last is w=1, first two are CARTESIAN x, y coord

→ homogeneity: any vector $\mathbf{x}$ is equivalent to all its nonzero multiples $\lambda\mathbf{x}$, $\lambda \neq \mathbf{0}$ since they represent the same point

taking $\lambda = 1/w$ allow you to derive cartesian coordinate → location of the point in plane

→ $[x \quad y \quad w]$ are **homogeneous** coordinates of the point on the plane

A vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and all its nonzero multiples $\lambda \begin{bmatrix} x \\ y \\ w \end{bmatrix}$, including $\begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$, represent the point of cartesian coordinates $[X \quad Y] = [x/w \quad y/w]$ on the Euclidean plane

→ homogeneity: any vector $\mathbf{x}$ is equivalent to all its nonzero multiples $\lambda\mathbf{x}$, $\lambda \neq \mathbf{0}$

since they represent the same point

same point

anytime same point in space! A point is equal to another $x = x' = \lambda x$

→ $[x \quad y \quad w]$ are **homogeneous** coordinates of the point on the plane

$\boxed{\text{redundancy}}$

3 homogeneous coordinates to represent points in the 2D plane (2 dof)

an infinite number of equivalent representations for a single point, namely all nonzero multiples of the vector $[X \quad Y \quad 1]^T$

null vector give you NO direction!
→ undetermine

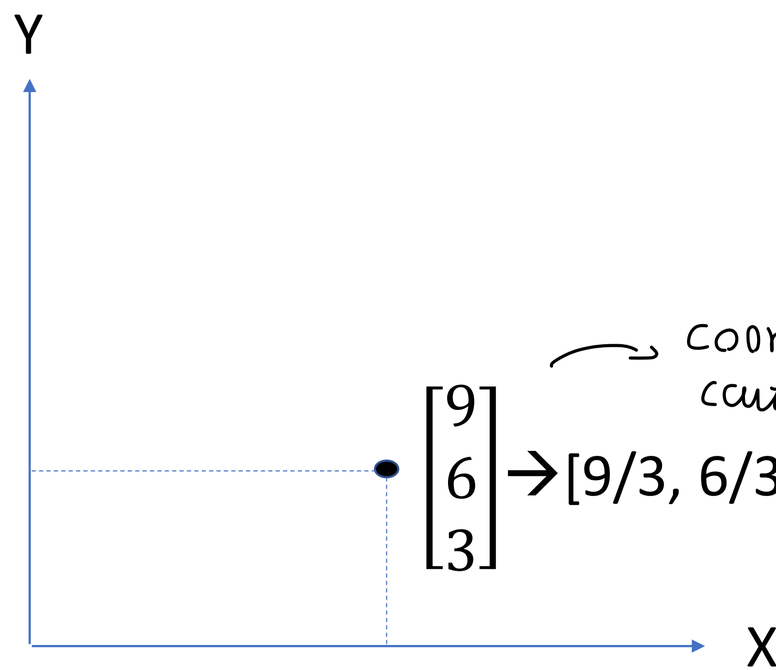
the null vector $[0 \quad 0 \quad 0]^T$ **does not** represent any point

all triples x y w are good points except for $0 \in \mathbb{R}^3$! because $w \neq 0$!

→ $\underbrace{\text{Projective plane}}_{\text{extension of euclidean plane}}$ $\mathbb{P}^2 = \overbrace{\{[x \quad y \quad w]^T \in \mathbb{R}^3\}}^{\text{set of triples}} \underbrace{- \{[0 \quad 0 \quad 0]^T\}}_{\text{NOT ADMITTED!}}$

→ its two degrees of freedom are the two independent ratios between the three coordinates $x : y : w$

Example: cartesian coordinates vs homogeneous coordinates

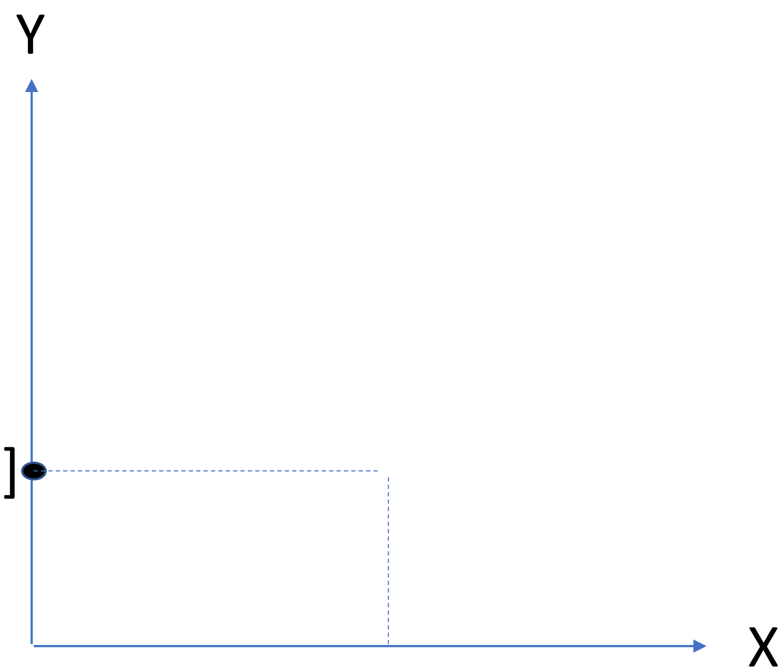


coordinate in 2D cartesian plane

$$\begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \rightarrow [9/3, 6/3] = [3, 2]$$

Example: cartesian coordinates vs homogeneous coordinates

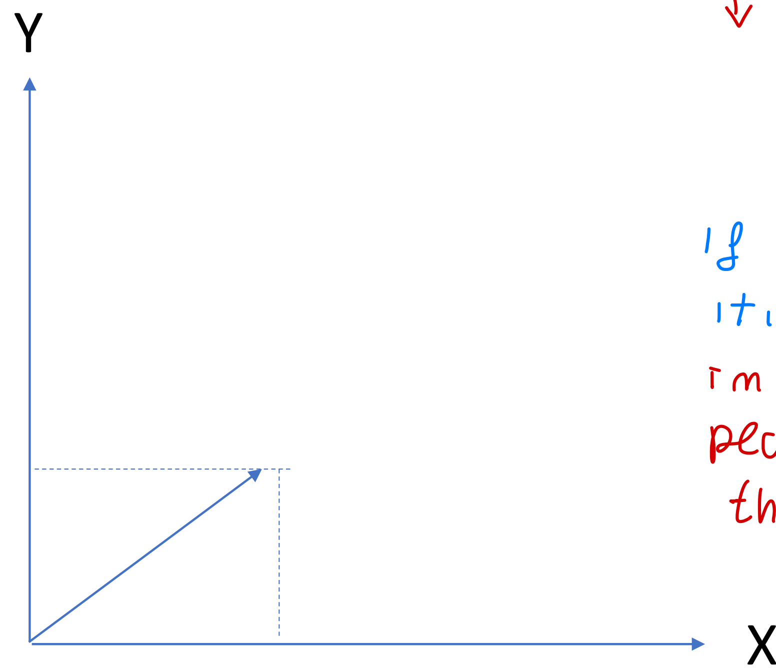
$$\begin{bmatrix} 0 \\ -8 \\ -4 \end{bmatrix} \rightarrow [0,\ 2]$$



$$\frac{1}{-4}\begin{pmatrix} 0 \\ -8 \end{pmatrix}$$

Points at the infinity

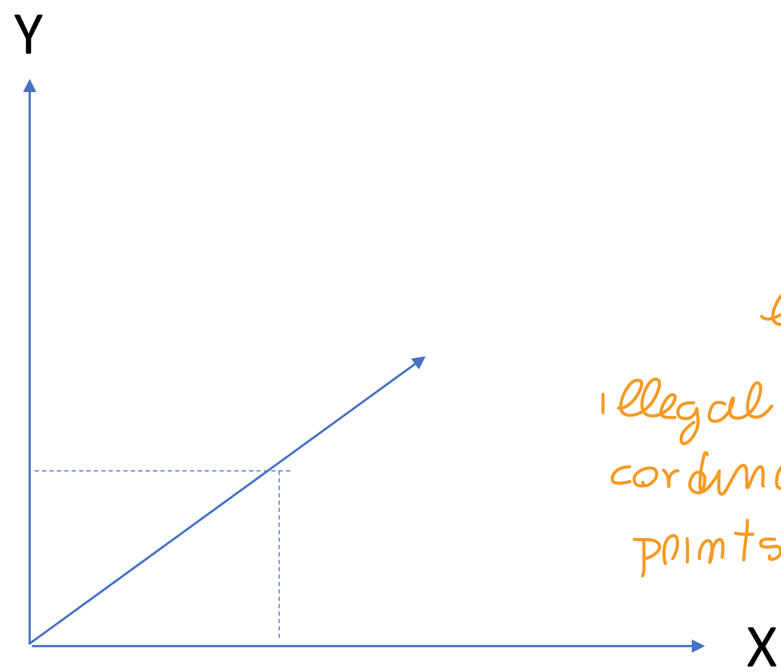
points "at the infinity"



$[3 \quad 2 \quad 0]^T \rightarrow ?$

If W = 0 is it illegal? is OK in the projective plane... this is the limit

points at the infinity



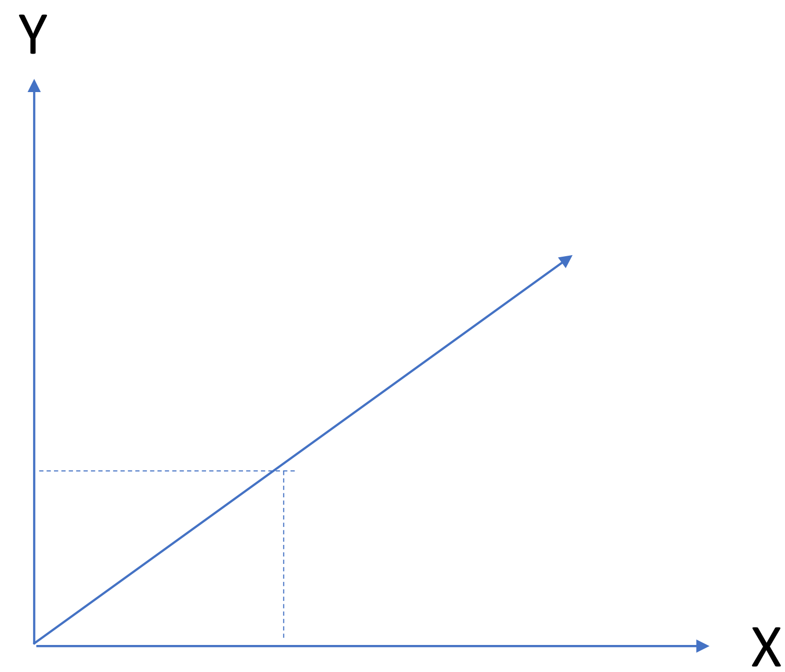
$$[3 \quad 2 \quad 0]^T$$
$$\rightarrow \lim_{w \to 0} [3/w \quad 2/w]$$

limiting point

illegal in cartesian coordinates (where set of points whose x, y are real, but $\infty \notin \mathbb{R}$)

it is illegal in cartesian, OK in projective plane

points at the infinity



$$[3 \quad 2 \quad 0]^T$$

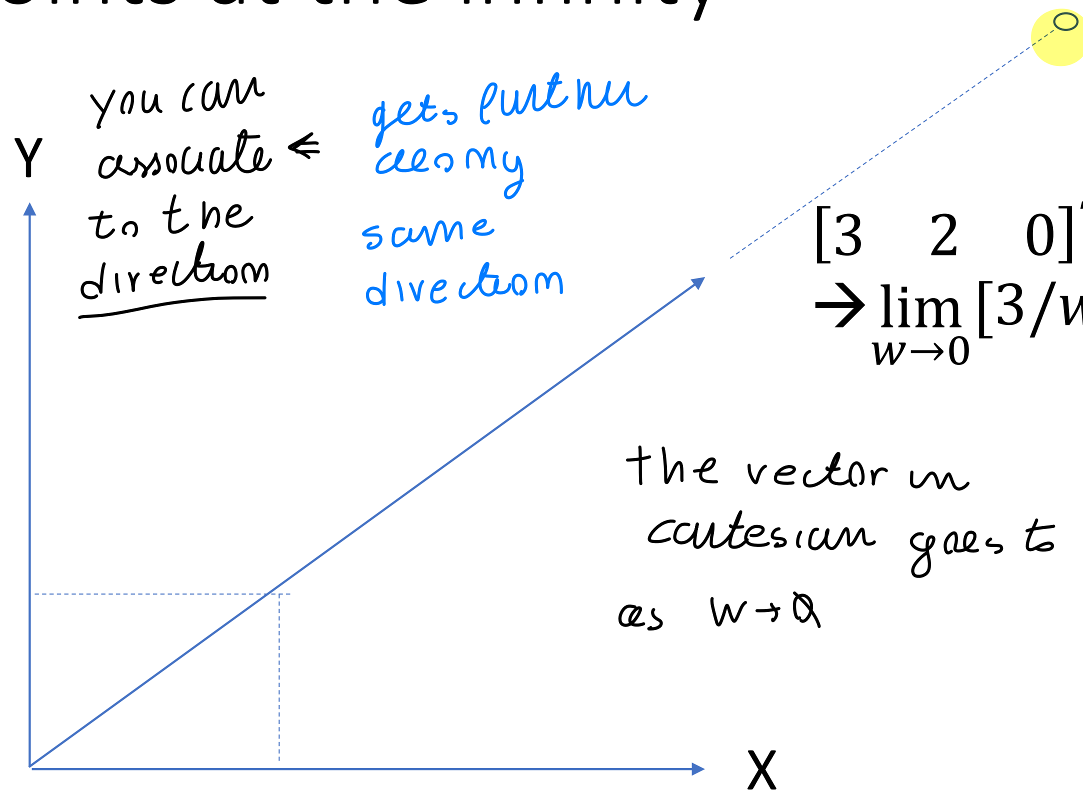
$$\rightarrow \lim_{w \to 0} [3/w \quad 2/w]$$

points at the infinity

goes further away along a certain DIRECTION

you can associate to the direction ⇐

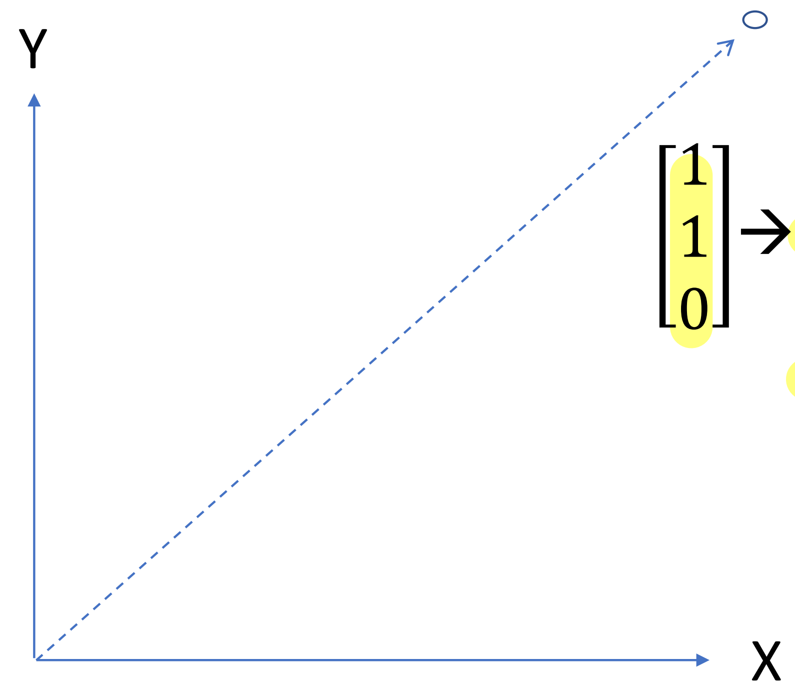
gets further along same direction



$$[3 \quad 2 \quad 0]^T \rightarrow \lim_{w \to 0} [3/w \quad 2/w]$$

the vector in cartesian goes to ∞ as $w \to 0$

Points at the infinity, who represent directions, are not part of the Euclidean plane: they are extra points, well defined within the Projective plane.



$\begin{bmatrix}1\\1\\0\end{bmatrix} \rightarrow$ point at the infinity along direction [1, 1]

Euclidean plane and Projective plane

extension of Euclidean plane

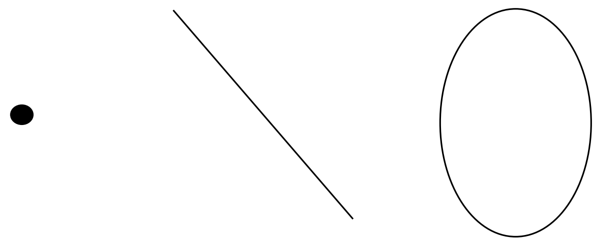
Projective plane $\mathbb{P}^2 = \{[x \quad y \quad w]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

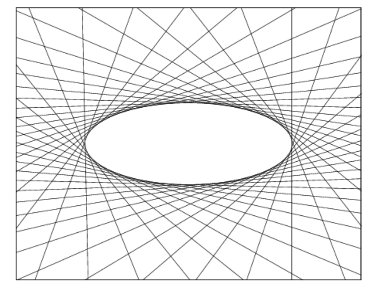
$=$ $\underbrace{\text{Euclidean plane} \cup \text{set of the points at the infinity}}$

include also points at ∞
as part of projective plane!
(EXTENSION!)

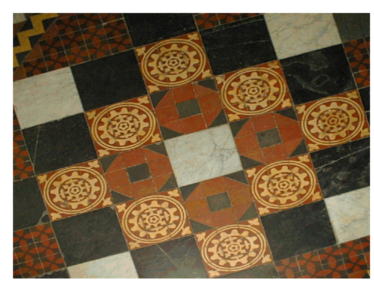
Planar Projective Geometry

- **Elements**
  - Points
  - **Lines**
  - Conics
  - Dual conics



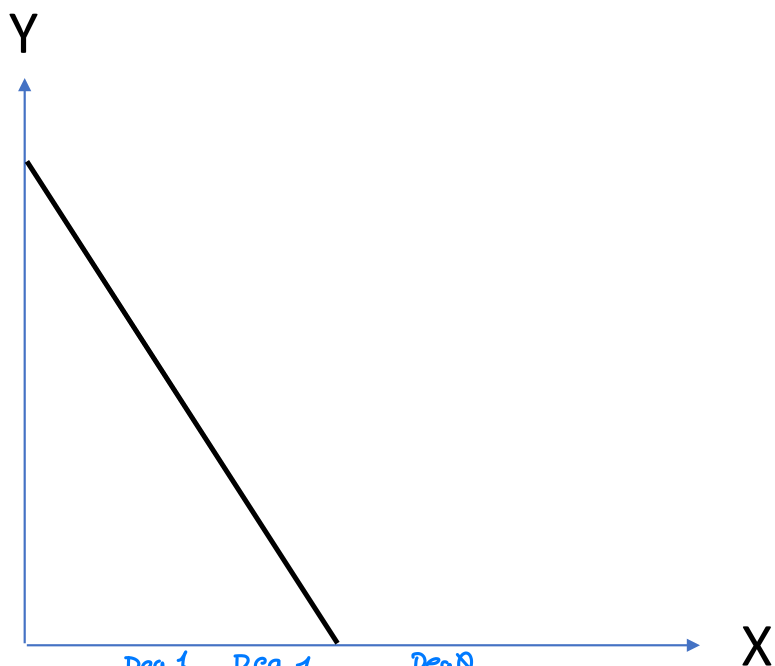


- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities



Lines in 2D Projective Geometry

Consider a line on the Euclidean plane



any time, represent line with parameters (a, b, c)

Deg 1 Deg 1 Deg 0

NOT homogeneus ← $aX + bY + c = 0$

$$a\frac{x}{w} + b\frac{y}{w} + c = 0$$

$X = \frac{x}{\omega}$ $Y = \frac{y}{\omega}$

homogeneus coordinate

|| linear homogeneus equation ||

all deg 1 term : homogeneus ← $ax + by + cw = 0$

Deg 1 Deg 1 Deg 1

easy to write in MATRIX FORM

the point (x y w) is on line (a b c) correspond to scalar product

$$ax + by + cw = 0 \rightarrow \underbrace{[a \quad b \quad c]}_{\text{line param}} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

point coordinates homogeneus

homogeneous, linear equation in $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$: $\boxed{\boldsymbol{l}^T\mathbf{x} = 0}$, where the vector $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

x is on the line

and all its nonzero multiples $\lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ represent a line

this is the same line

also this vector of parameter is homogeneus

$\lambda \ell^T x = 0$ same line!

$\ell^T, \lambda \ell^T$ are same line ← homogeneus vector

→ homogeneity: any vector $\boldsymbol{l}$ is equivalent to all its nonzero multiples $\lambda \boldsymbol{l}, \lambda \neq \mathbf{0}$

and they represent the same line

→ $[a \quad b \quad c]$ are **homogeneous** parameters of the line

[x y w] are homogeneus coordinates of the point

line redundancy are the same as points redundancy! in fact 3 homogeneous line param. in 2D plane even if there are 2 DOF ?

redundancy

3 homogeneous parameters to represent lines in the 2D plane (2 dof)

{ orientation
+translation

an infinite number of equivalent representations for a single line, namely all nonzero multiples of the vector $[a \quad b \quad c]^T$

the null vector $[0 \quad 0 \quad 0]^T$ **does not** represent any line

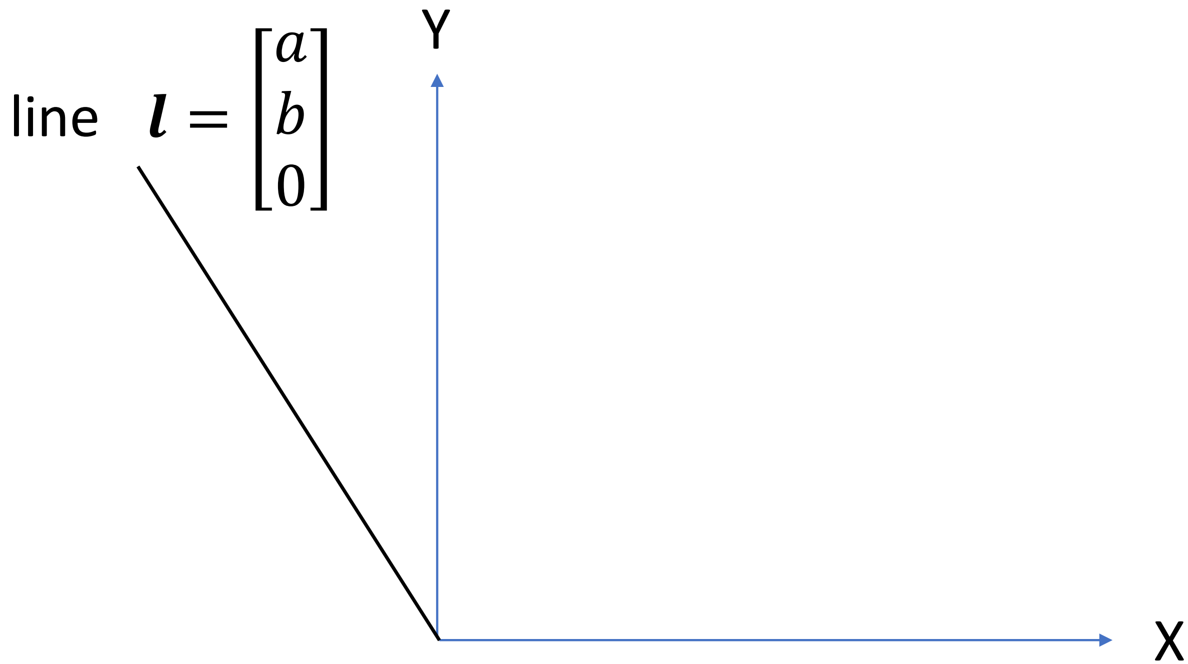
$0x + 0y + 0z = 0$ nothing! useless

→ Projective «dual» plane $\mathbb{P}^2 = \{[a \quad b \quad c]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

→ its two degrees of freedom are the two independent ratios between the three parameters $a : b : c$

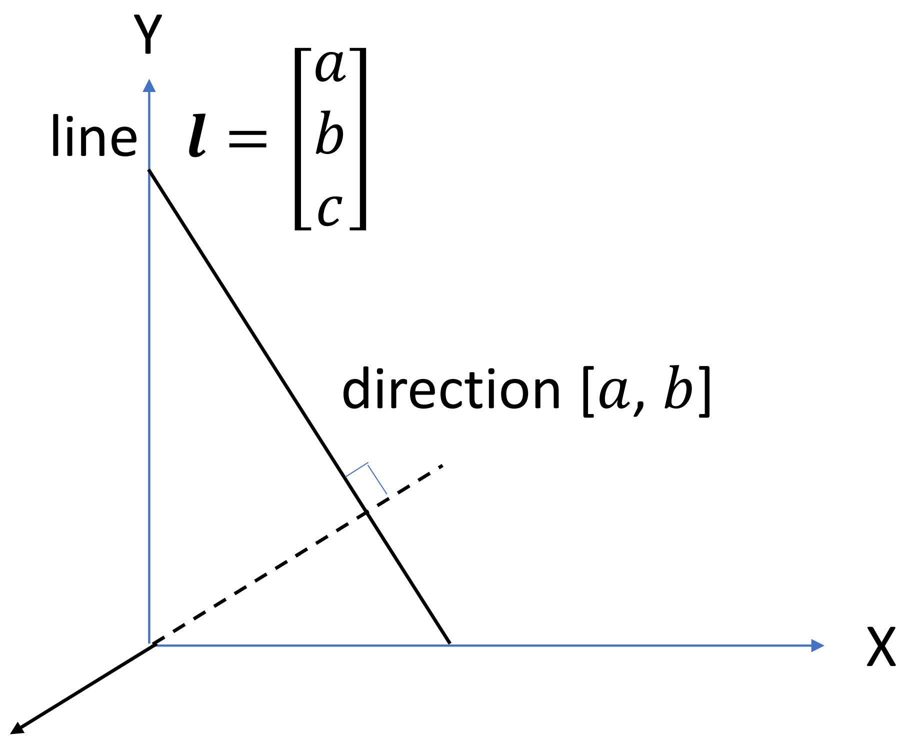
Three remarks

1. If the third parameter is null, $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, then $\boldsymbol{l}$ goes through point $[0,0]$



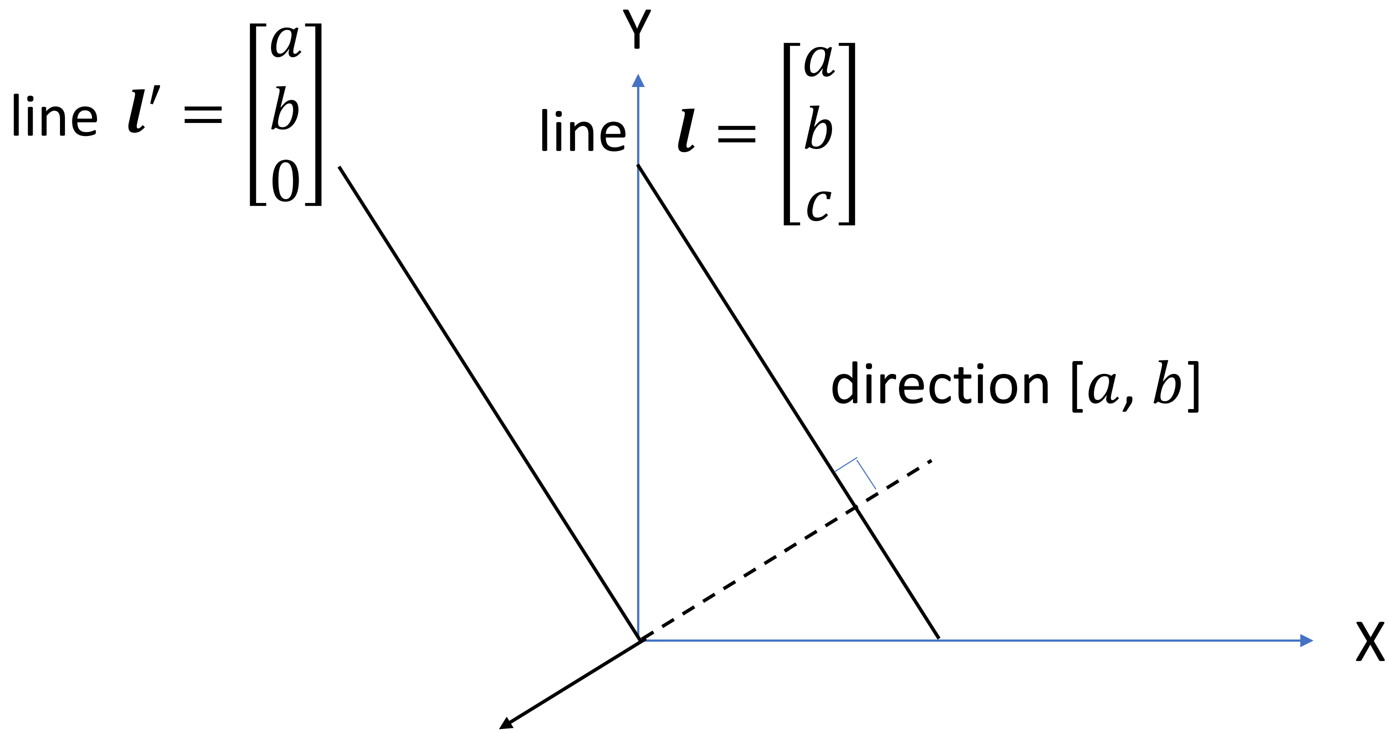
A line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{0}]^T$ whose third parameter is zero, goes through the origin of the plane

2. the direction $[a, b]$ is normal to the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,



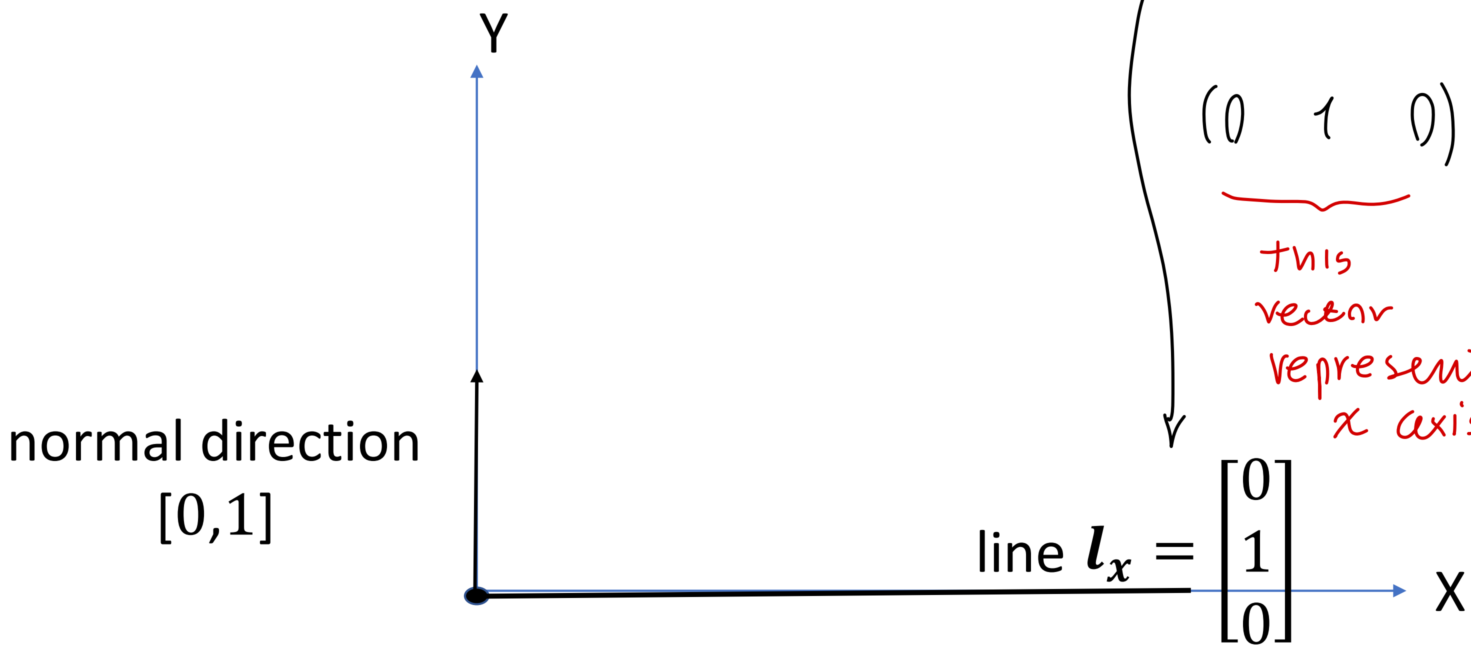
direction $[a, b]$ is normal to the line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]^{\boldsymbol{T}}$

3. the lines $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} a \\ b \\ c' \end{bmatrix}$ are parallel: their common direction is $[b$,-$a]$



direction $[a, b]$ is normal both to the line $\boldsymbol{l} = [a \quad b \quad c]^{\boldsymbol{T}}$
and to the line $\boldsymbol{l} = [a \quad b \quad 0]^{\boldsymbol{T}}$

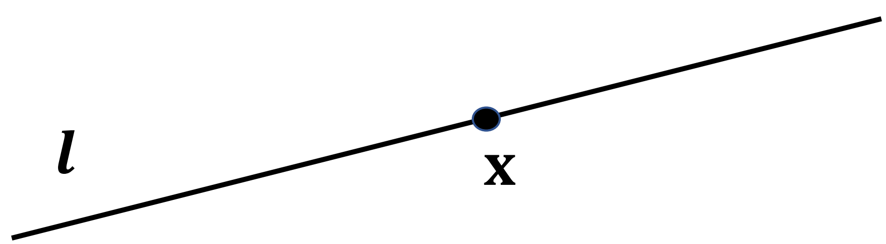
Example: the X-axis



line $\boldsymbol{l_x} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$$\underbrace{(0 \quad 1 \quad 0)}_{\text{this vector represent } x \text{ axis}} \begin{pmatrix}x\\y\\t\end{pmatrix} = \underbrace{y = 0}_{x \text{ axis}}$$

The incidence relation
a point is on a line, or a line goes through a point

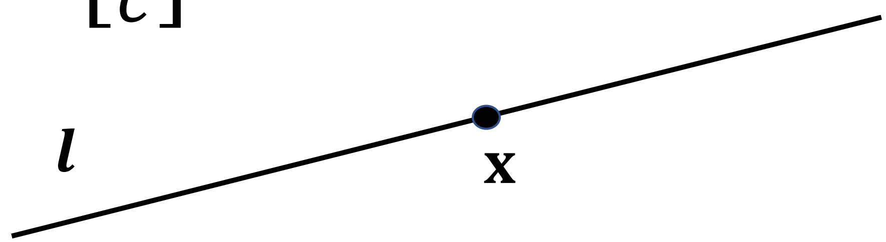


Incidence relation: $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \boldsymbol{l}^T \mathbf{x} = 0$

the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ is on the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

or

the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ goes through the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$



The line at the infinity:
the locus of the points at the infinity

points at the infinity are those represented by $[x \quad y \quad 0]$

$\forall x, y$ we vary the direction

set of points at the infinity changing $(x, y, 0)$

may be circle at infinity...

locus of the set of points at ∞ is a line!

straight line

not a curve, but a line!
which goes all points at ∞

← describe the "line at the infinity"

all co-linear

The «locus» of the points at the infinity

As there are infinite points at the infinity (one for each direction), what is the aspect of the set of these points at the infinity?

Simply:

constraint of all points at the infinity
(linear + homogeneus)

$w = 0$

$\Downarrow$

This set is a line: $[\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{w}\end{bmatrix} = 0$, actually $[\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{1}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{w}\end{bmatrix} = \boldsymbol{w} = 0$

namely, **the line at the infinity** $\boldsymbol{l}_\infty = \begin{bmatrix}0\\0\\1\end{bmatrix}$

same $\forall \lambda$: $l_\infty = \lambda \begin{pmatrix}0\\0\\1\end{pmatrix}$
is a line!

for example:

considering point at the infinity

you have point at the infinity for different lines in an image..

↳ is represented as a line of points at the infinity (not as circumference or other shape)

↑

easy to perceive in transformation between elements

01/10

The duality principle between points and lines

↳ lines / points correspondence ~ DUALITY property !

⇓

in planar geometry the point-lines duality derive by the fact that dot product is commutative ⟹

Since dot product is commutative

→ incidence relation is commutative

COMMUTATIVE PRODUCT

$$\boldsymbol{l}^T\mathbf{x} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{w}\end{bmatrix} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]\begin{bmatrix}\boldsymbol{a}\\\boldsymbol{b}\\\boldsymbol{c}\end{bmatrix} = \mathbf{x}^T\boldsymbol{l} = \mathbf{0}$$

point $\mathbf{x}$ is on line $\boldsymbol{l}$ ⇔ **point** $\boldsymbol{l}$ is on **line** $\mathbf{x}$

line

point

we can consider

another line ℓ

ℓ, x different elements... → lines, points on vector $\mathbb{R}^3$, with

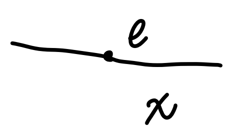
incidence $\ell^T x = x^T \ell = 0$ equivalent considering a new line ℓ and point x $\Big\}$ $\ell \neq x$

$\downarrow$

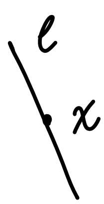
when we know that point x and line ℓ are incident : $x \in \ell$, ℓ intersect x

DUALITY!

also another equivalent relationship line x, point ℓ are such that $\ell \in x$, x intersect ℓ



$x^T \ell = 0$



$\ell^T x = 0$

SAME PROPERTY

duality!

point $\mathbf{x}$ is on line $\boldsymbol{l}$ (i.e. line $\boldsymbol{l}$ goes through point $\mathbf{x}$)

$\leftrightarrow$

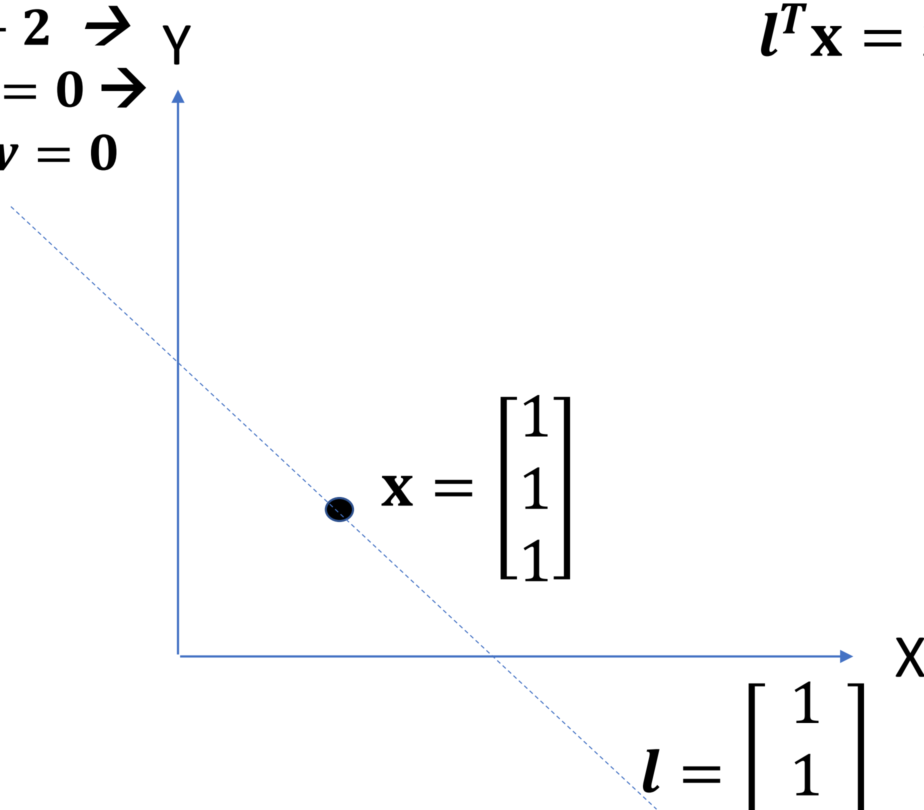
point $\boldsymbol{l}$ is on **line** $\mathbf{x}$ (i.e. line $\mathbf{x}$ goes through point $\boldsymbol{l}$)

Principle of **duality** between points and lines
(50% discount principle)

point $\mathbf{x}$ is on line $\boldsymbol{l}$ (i.e. line $\boldsymbol{l}$ goes through point $\mathbf{x}$)

$$\boldsymbol{Y} = -\boldsymbol{X} + \mathbf{2} \rightarrow$$
$$\boldsymbol{X} + \boldsymbol{Y} - \mathbf{2} = \mathbf{0} \rightarrow$$
$$\mathbf{x}+\boldsymbol{y} - \mathbf{2}\boldsymbol{w} = \mathbf{0}$$

$$\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$$



$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

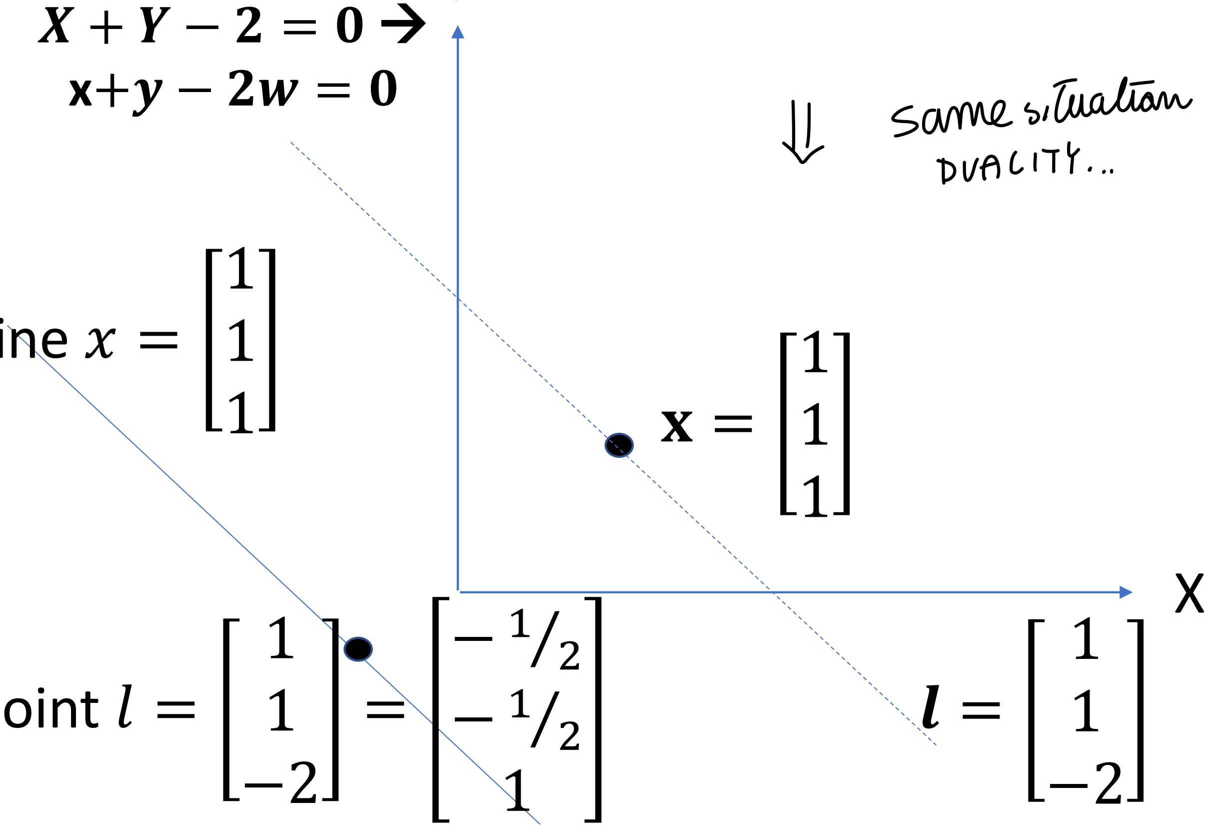
$$\boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

point $\boldsymbol{l}$ is on line $\boldsymbol{x}$ (i.e. line $\boldsymbol{x}$ goes through point $\boldsymbol{l}$)

$\boldsymbol{Y = -X + 2} \rightarrow$

$\boldsymbol{X + Y - 2 = 0} \rightarrow$

$\mathbf{x} + \boldsymbol{y} - \mathbf{2}\boldsymbol{w} = \mathbf{0}$



line $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

point $l = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -{}^{1}/_{2} \\ -{}^{1}/_{2} \\ 1 \end{bmatrix}$

$\boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$$\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$$

$\Downarrow$ same situation
DUALITY...

if we proof
a theorem
about points - lines
a DUAL THEOREM
where roles of
lines and points
are exchanged holds!

For any true sentence containing the items

- point
- line
- is on
- goes through

every theoreme proove the dual thanks to DUALITY, which is guaranteed by commutativeness of product

there is a DUAL sentence -also true- obtained by substituting, in the previous one, each occurence of

- point	by	- line
- line	by	- point
- is on	by	- goes through
- goes through	by	- is on

The point on two lines

homogeneus coordinates → we are in the plane → with 2 coordinates
im cartesian representation

$(x, y) \Rightarrow (x, y, w)$

to add also
points at infinity
as projective plane,
when $w = 0$ you
deal with point at ∞
while if $w \neq 0$ is
an ordinary point

←

if $w \neq 0$ and
$(x\ y\ w)$ is
same $(x\ y)$ point

homogeneus → cartesian
$(x\quad y\quad w) \longrightarrow (x/w\quad y/w)$
if $w \neq 0$

the third coordinate is powerfull because allow us to find simmetry between points and lines

$\Downarrow$

DUALITY

if given two points x_A, x_B → for two points always exist a line passing through those points

the equation of line is obtain solving a linear system

line require 2 number $y = mx + q$ all lines on plane BUT NOT VERTICAL

$\Downarrow$

we use $\ell = [a \ \ b \ \ c]$ triplets of coefficent to represent line

$\ell^T x = 0$ line, equal to any λ $\lambda \ell^T x = 0$

$x = (x \ \ y \ \ w)^T$ $\lambda \neq 0$ scalar ↑ same line

ℓ as $\underbrace{\text{triplets of coordinate}}$ ~ homogeneus coordinate

multiplicative factor defining point in projective plane and lines in a proper projective plane

given two lines

ℓ_1, ℓ_2 defined by triplets of numbers...

↳ the intersection is found solving a linear problem...

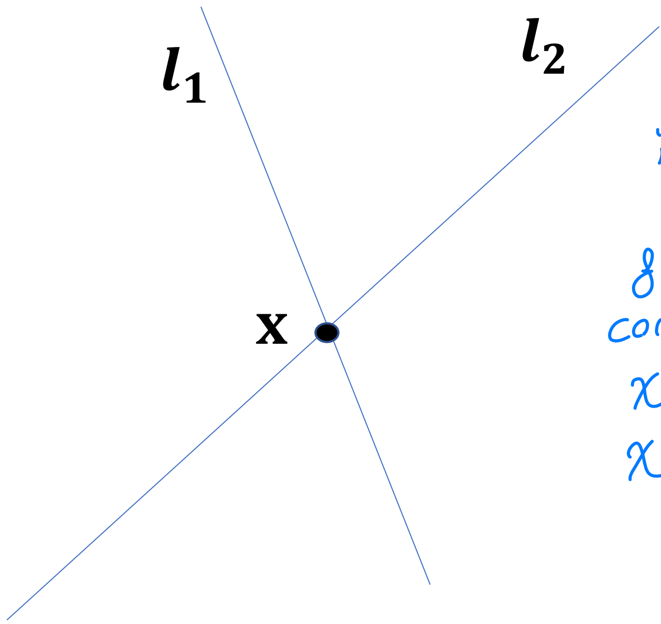
$$\begin{bmatrix} \ell_1^T \\ \ell_2^T \end{bmatrix} x = 0 \qquad 2\left\{ \underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}}_{3} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0 \right.$$

↳ solve it by the Null space

the point on two lines

↳ to find intersection point... is found considering that

$x \in l_1$
$x \in l_2$



$$\begin{cases} \boldsymbol{l}_1{}^T\mathbf{x} = 0 \\ \boldsymbol{l}_2{}^T\mathbf{x} = 0 \end{cases}$$

$x \in l_1$
$x \in l_2$

MATRIX FORM

$$\begin{bmatrix} \boldsymbol{l}_1{}^T \\ \boldsymbol{l}_2{}^T \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↓ SOLUTION

$$\mathbf{x} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{l}_1{}^T \\ \boldsymbol{l}_2{}^T \end{bmatrix}\right)$$

vectors as x

Right Null Space of matrix $\begin{bmatrix} l_1^T \\ l_2^T \end{bmatrix}$

$\mathbf{x}$ is a vector orthogonal to both $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ vectors

→

$\mathbf{x}$ is (a multiple of) the cross product of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$: $\mathbf{x} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$

$\begin{bmatrix} \ell_1^T \\ \ell_2^T \end{bmatrix}$ is a 2x3 matrix $\rightarrow$ RNS is 1D vector space... HOW MANY POINTS?

$\Downarrow$ Just ONE! in fact the ∞ solutions for x are all multiples of a common x

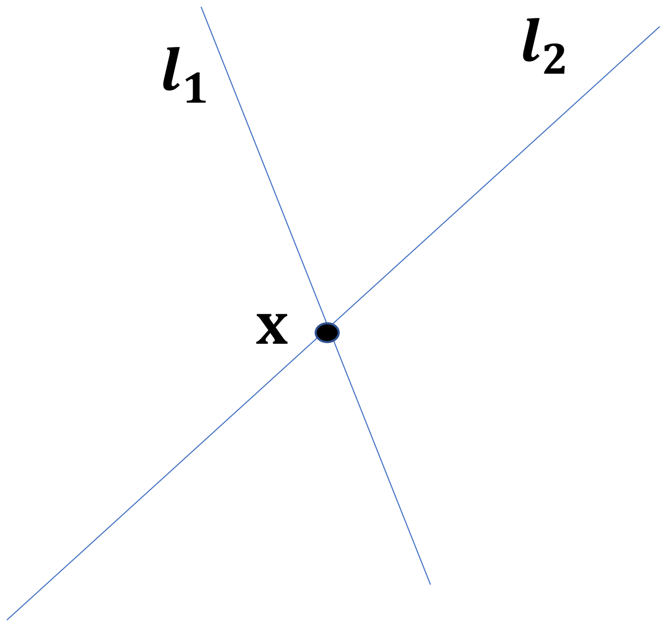
$\rightarrow$ BUT they represent just ONE point

you are solving linear system in 3 unknowns (x, y, w) using only 2 equations! $\Rightarrow$ the solution space has dimension of single point...
(Kernel has dimension one)

$\rightarrow$ the Null space define same point... multiplying by $\lambda \neq 0$ you get same representation in homogeneus coordinates..

it is NORMAL that the system has dimension 1, all solutions represent same point in homogeneus coord..

the point on two lines



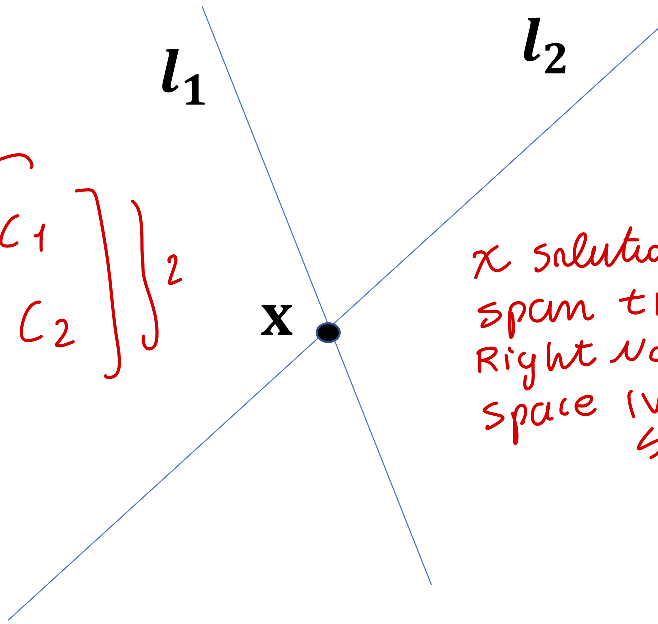
$$\begin{cases} \boldsymbol{l_1}^T\mathbf{x} = 0 \\ \boldsymbol{l_2}^T\mathbf{x} = 0 \end{cases}$$

$$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{RNS}(\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix})$$

the point on two lines

$$\left.\overbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}}^{3}\right\}2$$



x solution span the Right Null space (vector space)

$$\begin{cases} \boldsymbol{l_1}^T\mathbf{x} = 0 \\ \boldsymbol{l_2}^T\mathbf{x} = 0 \end{cases}$$

$$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\right)$$

$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}$ is a 2x3 matrix → RNS is a 1D vector space: how many points?

Just ONE: in fact the ∞ solutions for x are all multiples of a common x

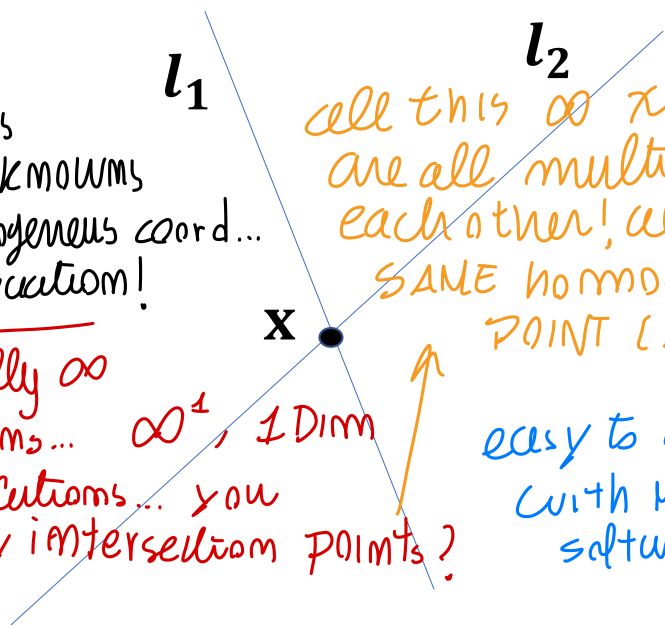
→ but they represent just ONE point !!

the point on two lines

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \emptyset$$

2×3, NOT invertible,
2 equations in 3 unknowns
are homogeneus coord...
MISSING equation!

$\underbrace{}$
algebraically ∞ x solutions... ∞^1, 1 Dim
set of solutions... you find many intersection points?



all this ∞ x sol are all multiple to each other! all SAME homogeneus POINT [X Y]

RIGHT NULL SPACE
vector space
right multiplication

$$\begin{cases} \boldsymbol{l_1}^T\mathbf{x} = 0 \\ \boldsymbol{l_2}^T\mathbf{x} = 0 \end{cases}$$

x on right

$$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solution of this eq is the set of vectors x making $\begin{pmatrix} l_1^T \\ l_2^T \end{pmatrix} x = \emptyset$, so its Nullspace, make ∅ by right multiplic.

easy to derive with MATLAB software.. ⇐

$$\mathbf{x} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\right)$$

$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}$ is a 2x3 matrix → RNS is a 1D vector space: how many points?

Just ONE: in fact the ∞ solutions for x are all multiples of a common x

→ but they represent just ONE point !!

← algebraically ∞ solutions, but all s.t $\lambda x \sim x$ equivalent, SAME POINT! we have ONE POINT in cartesian space! more ∞ in homogeneus cordinate

Observation:

as ORTHOGONAL to both!

for 3 component vectors, the RNS of 2 rows matrix can be solved taky rows CROSS PRODUCT

in

$$\begin{cases} \boldsymbol{l_1}^T \mathbf{x} = 0 \\ \boldsymbol{l_2}^T \mathbf{x} = 0 \end{cases}$$

to get two lines intersection, it is possible to get it in a simple way by CROSS PRODUCT

$\mathbf{x}$ is a vector orthogonal to both $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ vectors

→

in the 2D Projective Geometry

$\mathbf{x}$ is (a multiple of) the cross product of $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$

x to belong to l_1 and l_2 must be orthogonal to both in 3D space! → then that is the CROSS PRODUCT

$$\mathbf{x} = \boldsymbol{l_1} \times \boldsymbol{l_2}$$

x both ORTHOGONAL to e_1 and e_2

$\downarrow$

this guarantee incident relationship

special only for planar geometry in 2D!

(NOT equivalent in 3D)

$$\begin{cases} l_1^T x = l_2^T x = 0 \\ x = l_1 \times l_2 \end{cases}$$

ORTHOGONAL vector when point product $= 0$

ortogonal to both

by computing $\begin{cases} \ell_1^T \cdot (\ell_1 \times \ell_2) = 0 \\ \ell_2^T (\ell_2 \times \ell_1) = 0 \end{cases}$
and

$\ell_1 \times \ell_2$ is a POINT that pass through both the lines

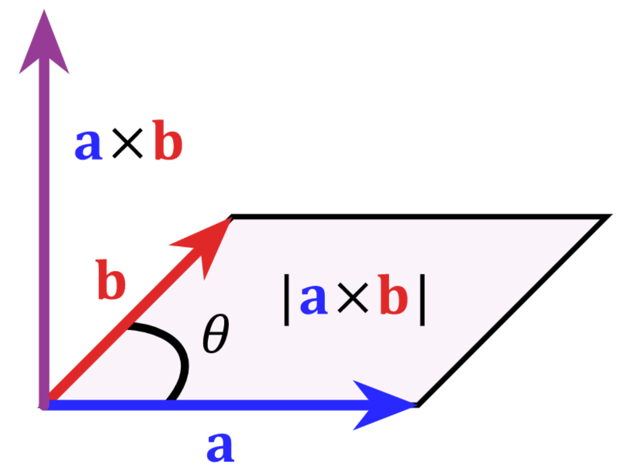
then, given 2 lines coefficents,
close form solution is given by CROSS PRODUCT

Cross Product

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ be two vectors, their cross product is a vector $\boldsymbol{a} \times \boldsymbol{b} \in \mathbb{R}^3$

- That is perpendicular to the plane $< \boldsymbol{a}, \boldsymbol{b} >$
- Has orientation of the right-hand rule
- Has lenght proportional to the area of the parallelogram spanned by the vectors, $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin \boldsymbol{\theta}$

lines as vectors in 3D space.
And CROSS PRODUCT is a 3D vector
satisfying that relation...



Giacomo Boracchi

Cross Product

Rmk: the cross product can be also computed as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

same in homogeneus coord

being $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ the versors of $\mathbb{R}^3$ and $|\cdot|$ the determinant

Rmk: the cross product is anti-commutative

just change in multiplicative factor

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$$

But this is not an issue when we want to intersect two lines, since the result in the same point of $\mathbb{P}^2$ (equivalence up to a multiplication by -1)

Example: intersection of two parallel lines

Suppose that lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ are parallel: this means that

$(l_1 \parallel l_2)$

$$\begin{cases} l_1 = [a \quad b \quad c_1]^T \text{ and} \\ l_2 = [a \quad b \quad c_2]^T \end{cases}$$

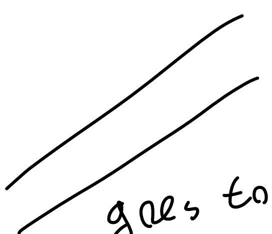
parallel lines as homogeneus have same a b coefficents

The point $\mathbf{x} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]^{\boldsymbol{T}}$ common two these two lines satisfies both

point of intersection (don't exist)

↓

goes towards infinity! it should have $w=0$



$$\begin{cases} ax + by + c_1 w = 0 \\ \text{and} \\ ax + by + c_2 w = 0 \end{cases}$$

$$\downarrow$$

$$\mathbf{x} = [b \quad -a \quad 0]^T$$

as expected $w=0$

Namely, the point at the infinty along the direction of both lines

(remember: [b, -a] is the direction of both lines)

PARTICULAR CASE

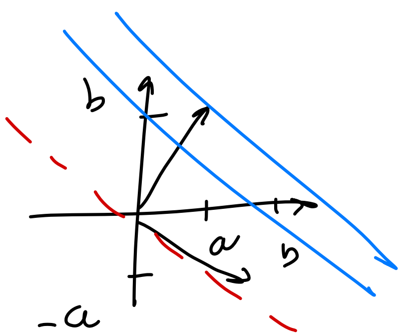
Example: intersection of two parallel lines

Suppose that lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ are parallel: this means that

$$l_1 = [a \quad b \quad c_1]^T \text{ and}$$
$$l_2 = [a \quad b \quad c_2]^T$$

parallel lines have same a, b while different c_i

The point $\mathbf{x} = [x \quad y \quad w]^T$ common two these two lines satisfies both



$$\begin{cases} ax + by + c_1 w = 0 \\ \text{and} \\ ax + by + c_2 w = 0 \end{cases}$$

subtract the two equations...

$$\downarrow$$

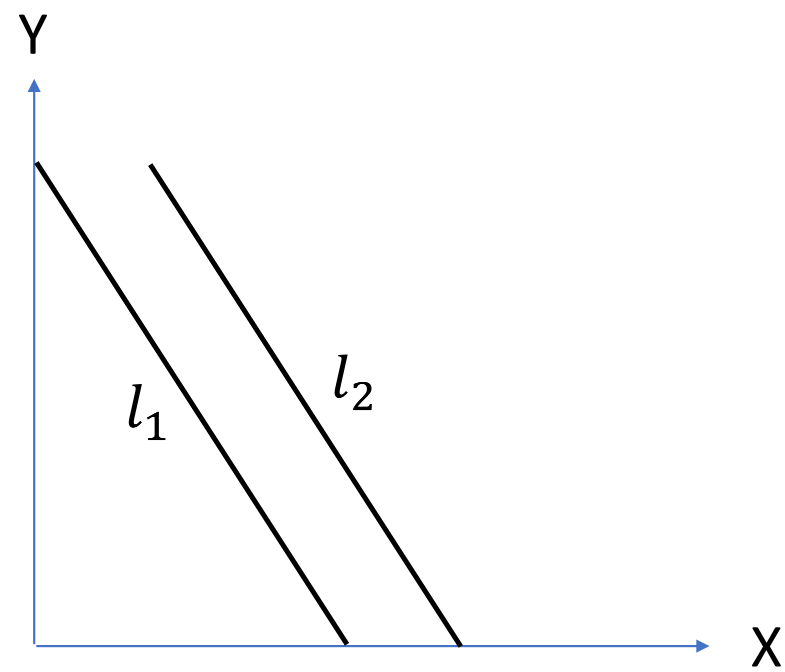
$$\mathbf{x} = [b \quad -a \quad 0]^T$$

$w = 0 \Rightarrow$ POINT at the infinity!

Namely, the point at the infinty along the direction of both lines

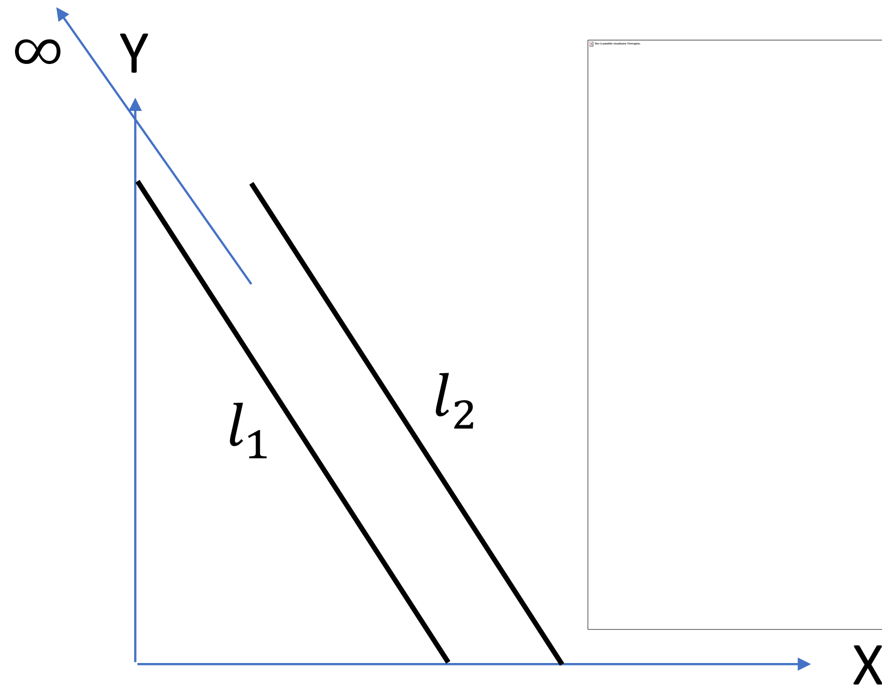
$l_i^T x = 0 \ (i = 1,2)$ (remember: [b, -a] is the direction of both lines)

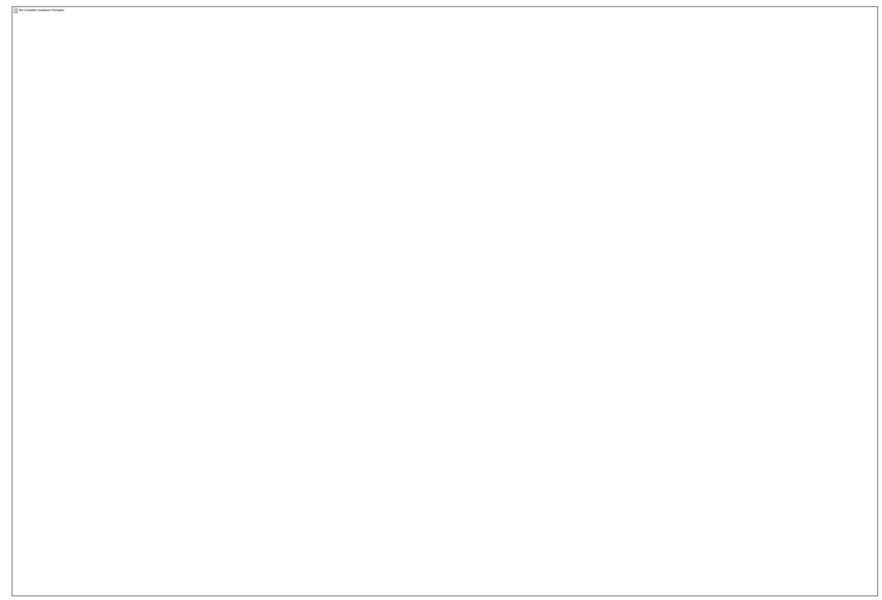
Example: intersection of two parallel lines



The intersection of two parallel line is the point at the infinity along their common direction

[b -a]
such that
a b is orthogonal..





Preview:

The vanishing point is the image of point at the infinity, i.e., the point where parallel lines intersect

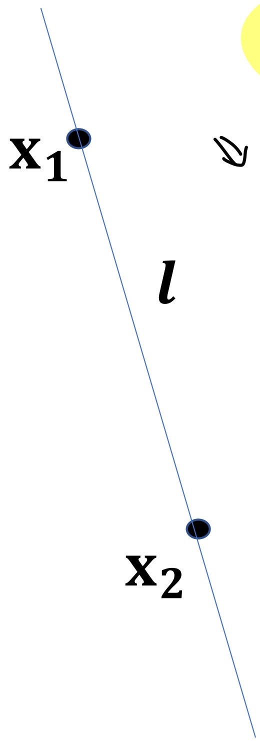
the line through two points

dual of the point on two lines! same procedure ⇒

we transform dual vocabulary!

Previous: point on two lines
DUAL: line through two points

all PROJECTIVE aspect seen for two lines can be symmetrically considered for 2 points



given x_1, x_2 points always exist a line l passing through it!

AS BEFORE! just solve

$$\boldsymbol{l} = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{x_1}^T \\ \mathbf{x_2}^T \end{bmatrix}\right) \leftarrow \begin{cases} x_1^T l = 0 \\ x_2^T l = 0 \end{cases}$$

with respect to points

or, just in 2D Proj. Geo.

$$\boldsymbol{l} = \mathbf{x_1} \times \mathbf{x_2}$$

line through two points is just vector of points!

symmetrically theorem proof also points holds

in euclidean plane, 2 lines can have 0 intersection when parallel, While in projective plane you always have one intersection point

A useful property and its dual

POINT ⟷ LINE
IS ON ⟷ GO THROUGH
TO ⟷ TO

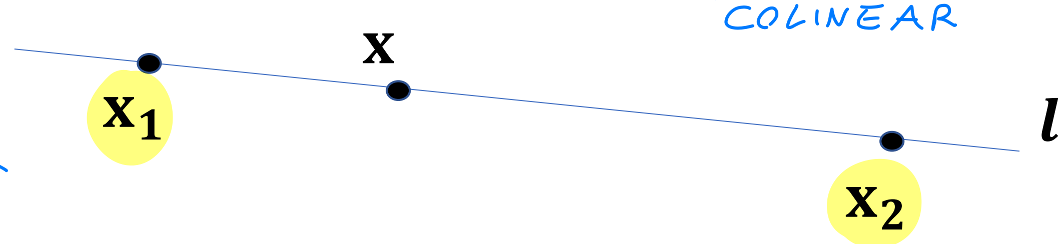
Example: linear combination of two points

Property: the point $\mathbf{x}$ given by the linear combination $\boxed{\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2}$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$ (i.e. on the line joining $\mathbf{x}_1$ and $\mathbf{x}_2$) . In less words: $\mathbf{x}$ is COLINEAR to $\mathbf{x}_1$ and $\mathbf{x}_2$

linear combination of x_1, x_2 belongs to same line going through $x_1, x_2 \rightarrow$ COLINEAR

easy to proof

two points x_1, x_2 are COLINEAR



Proof: the line $\boldsymbol{l}$ through both points satisfies $\boldsymbol{l}^T\mathbf{x_1} = 0$ and $\boldsymbol{l}^T\mathbf{x_2} = 0$.
By adding α times the first eqn to β times the second one, we obtain

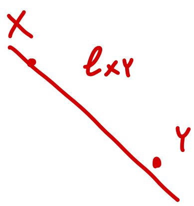
$$0 = \boldsymbol{l}^T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \boldsymbol{l}^T\mathbf{x} = 0$$

this still means $x \in \ell$

i.e. $\mathbf{x}$ is on the same line joining $\mathbf{x}_1$ and $\mathbf{x}_2$.

← that is NOT TRUE
in Euclidean geometry!

two points X, Y in cartesian coordinate



$\alpha X + (1-\alpha) Y \in \ell_{XY}$ only when CONVEX combination!

while

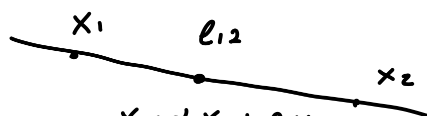
$\alpha X + \beta Y \notin \ell_{XY}$

in General taking any α, β!

↓

while in homogeneus coordinates it holds for any α, β!

in PROJECTIVE geometry NOT only for CONVEX combination



$X = \alpha X_1 + \beta X_2 \in \ell_{12}$ same line, COLINEAR → easy to proof with algebraic relation

Valid only in homogeneus!

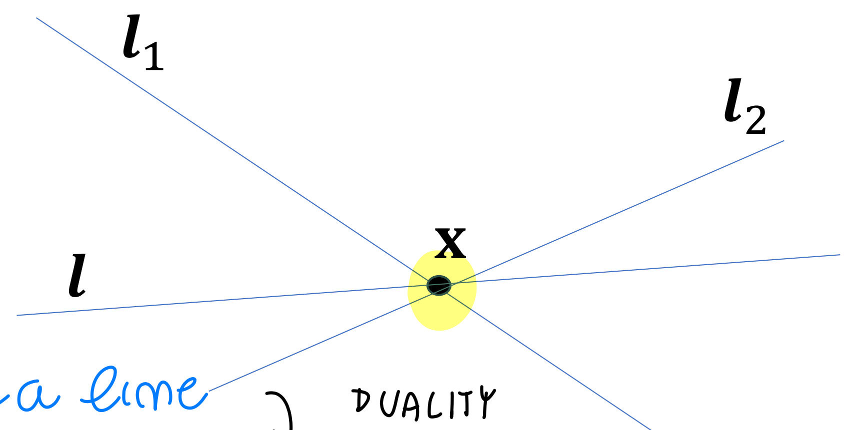
Duality POINT ⟷ LINE same property hold and can be proved in simple way

DUAL: linear combination of two lines

for DUALITY, considering lines: (NOT INTUITIVE graphically, BUT holds!)

Property: the point $\mathbf{x}$, given by the linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$, is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$

taking l_1, l_2 two lines... they meet on a point x by definition (projective plane..) and any linear combination of the two lines is still a line



linear combination of two lines is a line in the pencil belonging to intersection point

DUALITY

DUAL!

Dual: the line $\boldsymbol{l}$, given by the linear combination $\boldsymbol{l} = \alpha\boldsymbol{l}_1 + \beta\boldsymbol{l}_2$ of two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, goes through the point $\mathbf{x}$ on $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$.

- In less words: line $\boldsymbol{l} = \alpha\boldsymbol{l}_1 + \beta\boldsymbol{l}_2$ is CONCURRENT to lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

third line going through same point x shared by 2 line... CONCURRENT LINE!

in a PLANE with
- POINTS
- LINES ← to do stuff on lines I can image Dual plane where line are points and viceversa (SWAP the role!)

"CONCURRENT" line ~ goes through same point!

this is exactly the behaviour of light ray after crossing lens if they derive from d >> a points... parallel rays

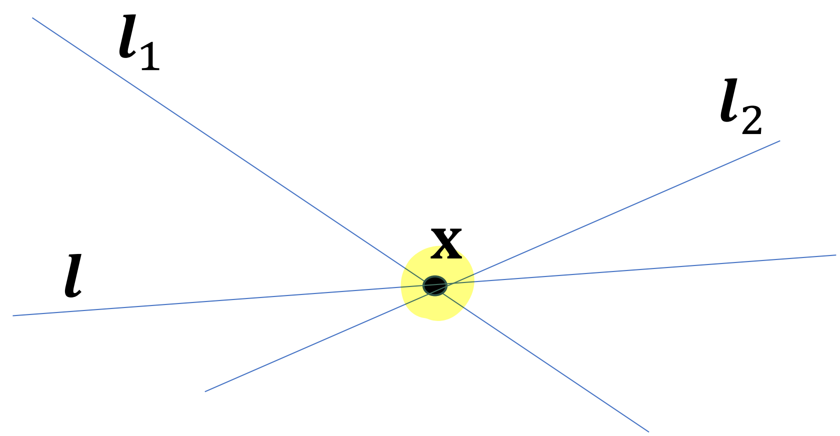
↓

all rays are concurrent at image

DUAL: linear combination of two lines

Dual property: the line $\boldsymbol{l}$, given by the linear combination $\boldsymbol{l} = \alpha \boldsymbol{l}_1 + \beta \boldsymbol{l}_2$ of two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, goes through the point $\mathbf{x}$ on $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

the POINT is colinear,
l line is CONCURRENT to l_1, l_2



new $\boxed{\text{expression: «concurrent»}}$

the line $\boldsymbol{l} = \alpha \boldsymbol{l}_1 + \beta \boldsymbol{l}_2$ is **concurrent** to lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

Duality is a mechanism ⇐ to prove theorem...

convenient when working with lines as points

in PROJECTIVE GEOMETRY, given a theorem you can replace those and holds

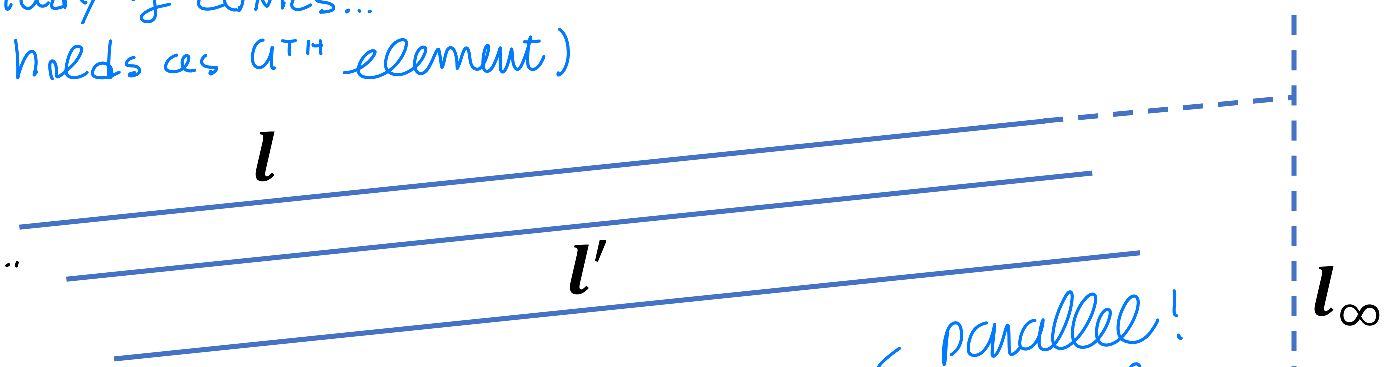
New dual pair

- point	by	- line
- line	by	- point
- is on	by	- goes through
- goes through	by	- is on
- colinear	by	- concurrent
- concurrent	by	- colinear

⇒ idea of Duality! and simple intersection by CROSS-PRODUCT

A special case: linear combinations of a line $\boldsymbol{l}$ and the line $\boldsymbol{l}_\infty$

IMPORTANT in the study of conics...
(dual conics also holds as 4TH element)



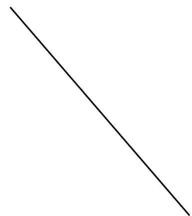
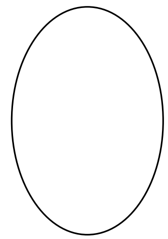
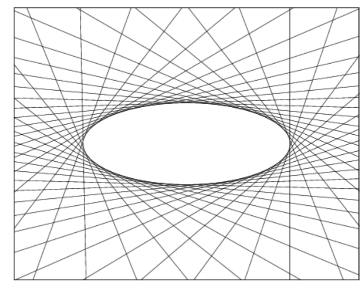
combining l line and l∞ line @ ∞...
you get ALL POSSIBLE lines parallel to the FIRST line!

parallel! same direction!

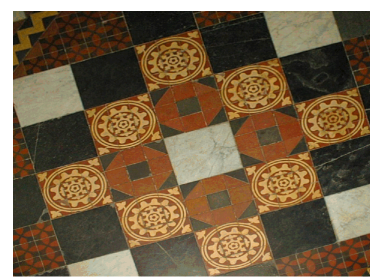
$$\boldsymbol{l}' = \boldsymbol{l} + \lambda \boldsymbol{l}_\infty = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c}' \end{bmatrix} \parallel \boldsymbol{l}$$

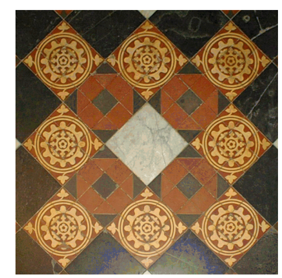
→ the set of all the lines $\boldsymbol{l}'$ parallel to $\boldsymbol{l}$

Planar Projective Geometry

- **Elements**
- Points 
- Lines 
- **Conics**  
- Dual conics ← obtain this definitions by duality

- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities





CONICS

derive from a CONE

Conics name derives from the intersection of cones with planes

comics as the intersection of a cone by a plane:

shape depends on relative position between plane and cone

union set of many lines going through a vertex

taking a line parallel to one of line creating the cone

plane parallel to cone lines

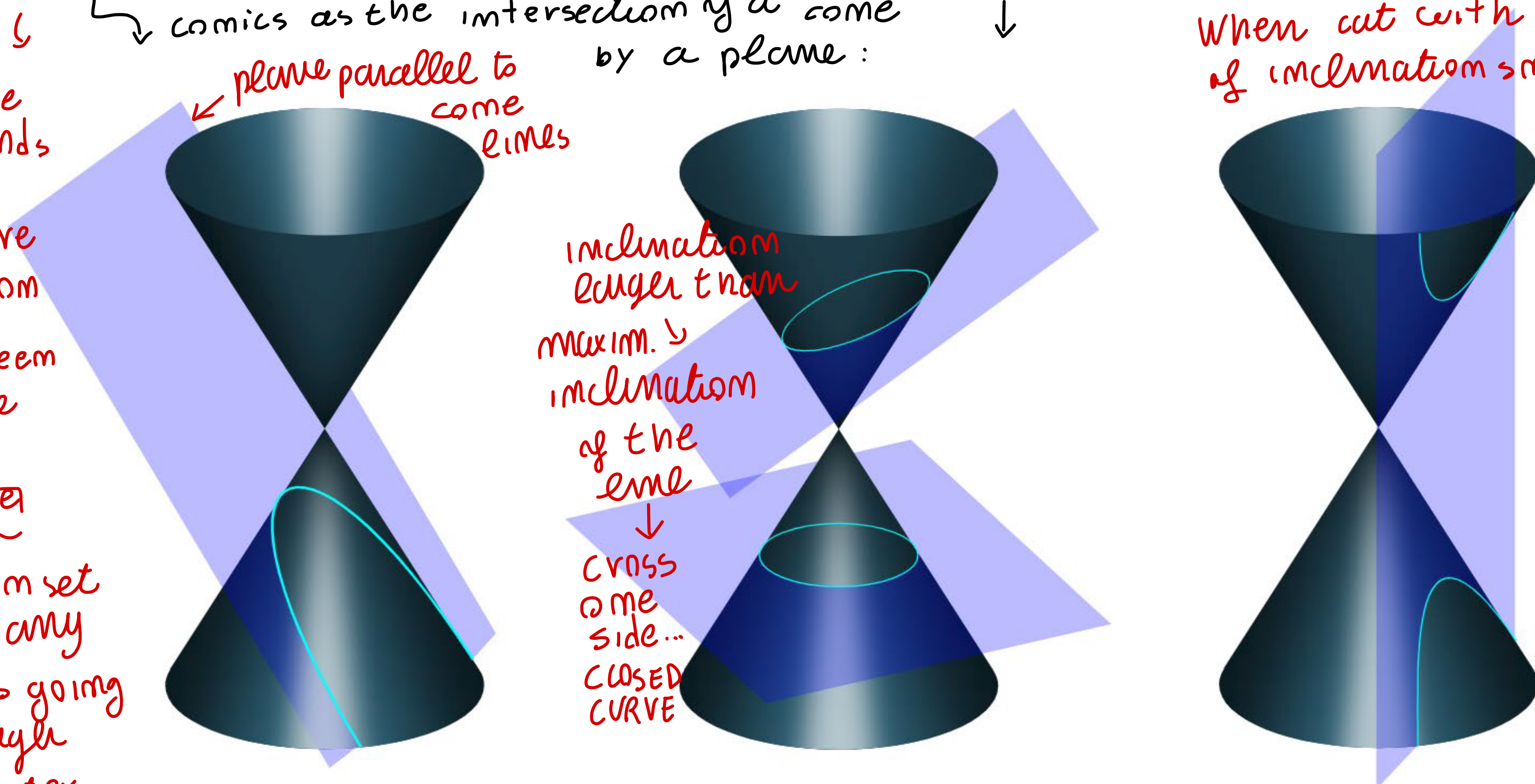
inclination larger than maxim. inclination of the line

cross one side... CLOSED CURVE

When cut with line of inclination smaller than largest inclination

you cat both sides of cone!

hyperbolae



(only one part is crossed)

CLOSED CURVE

PARABOLAE

ELLIPSE

HYPERBOLAE

Giacomo Boracchi

→ in ALGEBRIC (matrix) form, CONICS:

a line is a set of point which satisfy linear homogeneous equation ←

- **Lines**: a point $\mathbf{x}$ is on a line $\boldsymbol{l}$ if it satisfies a homogeneous ***linear*** equation, namely

$$\boldsymbol{l}^T \mathbf{x} = \mathbf{0}$$

↓ towards complexity but still homogeneus LINEAR → QUADRATIC !

- **Conics**: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}$ if it satisfies a homogeneous ***quadratic*** equation, namely

$$\mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{0}$$

where $\boldsymbol{C}$ is a 3x3 symmetric matrix.

you don't lose generality even with this request

↑ this is equation of a curve, NO more a line shape...

most general aspect $x^T C x = 0$ is quadratic equation

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

Conics in $\mathbb{P}^2$

consider a generic quadratic homogeneous $x^T C x = 0$
you want ANY COMBINATION, all terms...

- A conic is a curve described by a second-degree equation in the plane.
- In Euclidean coordinates a conic becomes
- $aX^2 + bXY + cY^2 + dX + eY + f = 0$
- i.e. a polynomial of degree 2. "Homogenizing" this by the replacements:
- X→ x/w, Y→ y/w gives
- $ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$
- or in matrix form

symmetrically distribute elements on matrix C

$$\mathbf{x}^\top \boldsymbol{C}\mathbf{x} = 0$$

- where the conic coefficient matrix $\boldsymbol{C}$ is given by $\boldsymbol{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$

you can always bring conics equation in this form

Conics in $\mathbb{P}^2$

This enjoy homogeneity property? ⇒ YES

- $\mathbf{x}^{\mathsf{T}} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \mathbf{x} = 0$

$\lambda C \to x^T(\cancel{\lambda} C)x = 0$ same point!

same solution

same represented curve...

6 indipendent elements

BUT being homogeneus, dividing all by f → you get 5 DOF!

CONIC has 5 dof, determined by 5 indipendent equations

- **Rmk** The conic coefficient matrix is symmetric,
- **Rmk** multiplying $\boldsymbol{C}$ by a non-zero scalar does not change. Only the ratios of the elements in $\boldsymbol{C}$ are important, as for homogeneous points and for lines. → $\boldsymbol{C}$ is a homogeneous matrix

point / line ≃ 2 DOF using 3D vector

$\frac{x}{w}$ $\frac{y}{w}$ only 2 indipendent ratios as DoF

- **Rmk** The conic has five degrees of freedom :
  - the ratios $\{a : b : c : d : e : f\}$ or equivalently
  - the six elements of a symmetric matrix minus one for scale.

to define a comics → 5 DOF
⇓
5 equations

1) comic goes through a certain point P ← you need P_i $i=1,..,5$
one scalar equation for each point

$$x_{p_i}^T C \, x_{p_i} = 0 \quad i=1,..,5$$

through 5 points → only one conics (chosing wisely points no dependent/colinear...)

only ratios $\frac{a}{b}$ $\frac{b}{c}$ $\frac{c}{d}$ $\frac{d}{e}$ $\frac{e}{f}$ are indipendent...

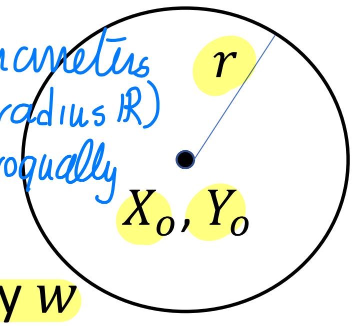
Example: the circumference

⇓ specific type of ellipses... (CLOSED CURVE)

First in cartesian coordinates:

$$(X - X_o)^2 + (Y - Y_o)^2 - r^2 = 0$$

⇒ 3 DOF parameters (center $\mathbb{R}^2$, radius $\mathbb{R}$) univoqually



X_o, Y_o are the center coordinates, r is the radius.

then in homogeneous coordinates:

replace X and Y by, respectively, x/w and y/w and multiply by w

$$x^2 - 2X_o w x + X_o^2 w^2 + y^2 - 2Y_o w y + Y_o^2 w^2 - r^2 w^2 = 0$$

- equal coefficents

reorder the coefficients

in MATRIX FORM:

0 coefficients

you can recognize if it is circumference from C shape...

$$\begin{bmatrix} x & y & w \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix}}_{\mathbf{x}^\top C \mathbf{x} = 0} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

in conics you want to derive the matrix C and any λC is conics

Intersection of a line and a conic

Intersection of a line and a conic

deg

Conic: quadratic equation on $\mathbf{x}$, namely $\mathbf{x}^\top \boldsymbol{C}\ \mathbf{x} = 0$ (2)

Line: linear equation on $\mathbf{x}$, namely $\boldsymbol{l}^T\mathbf{x} = \mathbf{0}$ (1)

$\Downarrow$ $\downarrow$ 2×1

Line-conic $\underbrace{\text{intersection}}$ leads to a degree 2 equation on $\mathbf{x}$

the space intersection has degree as product of degrees := 2

$\rightarrow$

there are always TWO intersection points between a line and a conic:

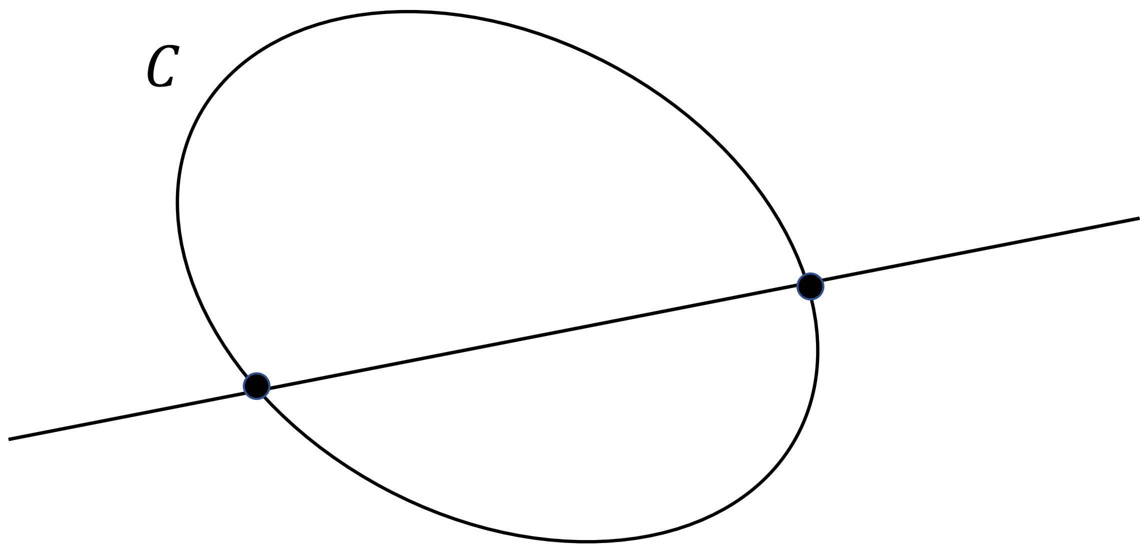
they can be real, or complex conjugate, distinct or coincident

[Foundamental Theorem of Algebra]

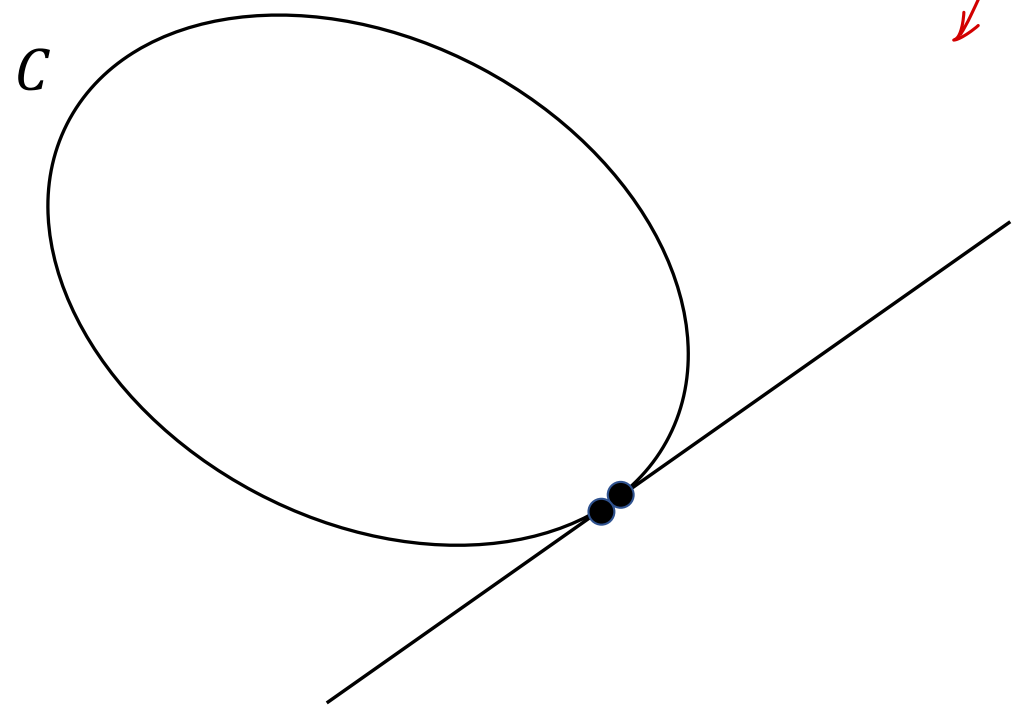
Line – conic intersection: two real distinct solutions

.

↳ cross a conic by a line… you always find two intersection!



Line – conic intersection: two real coincident solutions → **tangency**



condition of tangency
line - conic
come as 2 solution

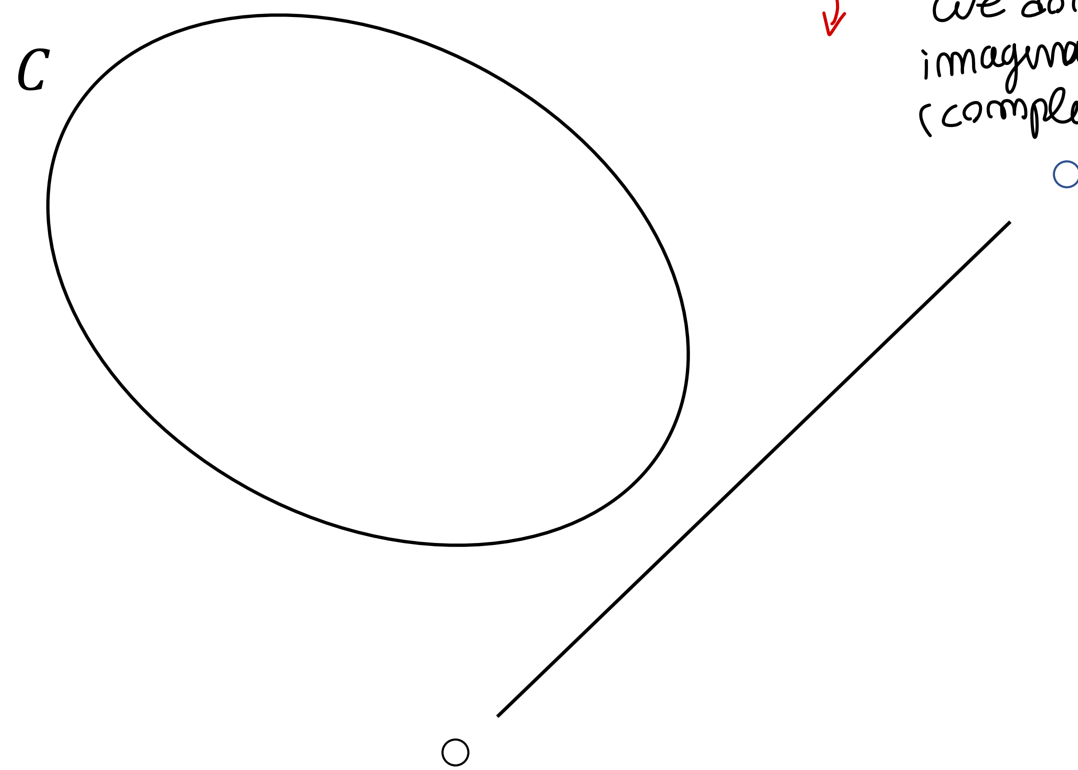
double solution

Line – conic intersection: two complex-conjugate solutions

.

NO REAL SOLUTION

we don-t see the imaginary solution (complex conj)



CHARACTERIZE and distinguish

INTERSECTION between the **LINE AT THE ∞** and a CONIC

we can identify conics type considering algebraic resolution by projective geometry, CONSIDERING

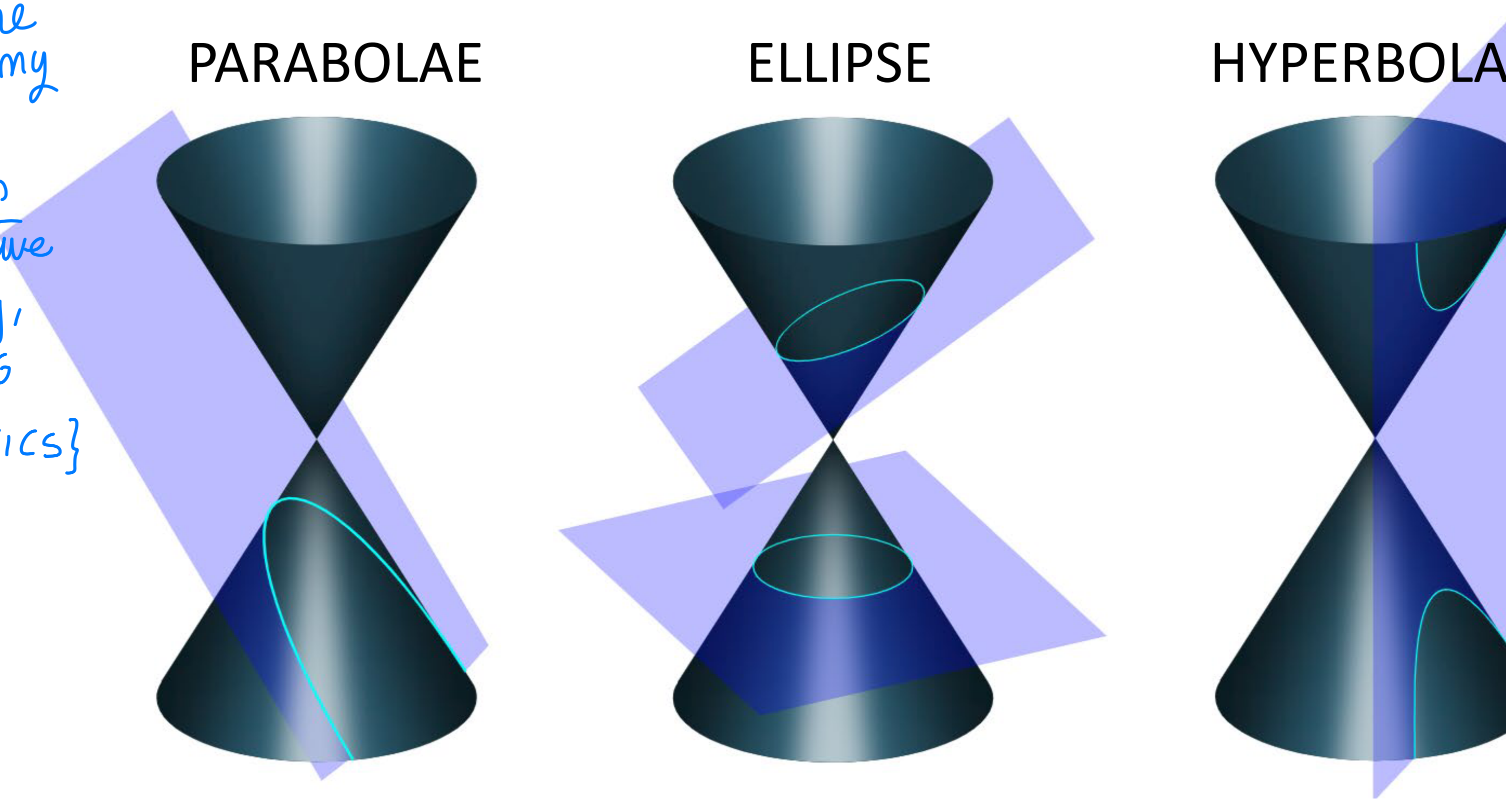
$\ell_\infty \cap \{\text{CONICS}\}$

$\parallel$

$[0 \;\; 0 \;\; 1]'$

$w=\lambda$

PARABOLAE | ELLIPSE | HYPERBOLAE



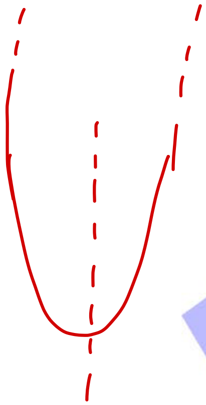
two coincident solutions: point at the ∞ along axis

two complex-conjugate solutions: no real solution

two real distinct solutions: asymptotes

give you a method to distinguish type of conics

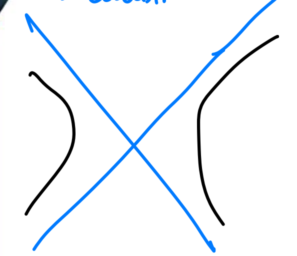
INTERSECTION between the **LINE AT THE ∞** and a CONIC



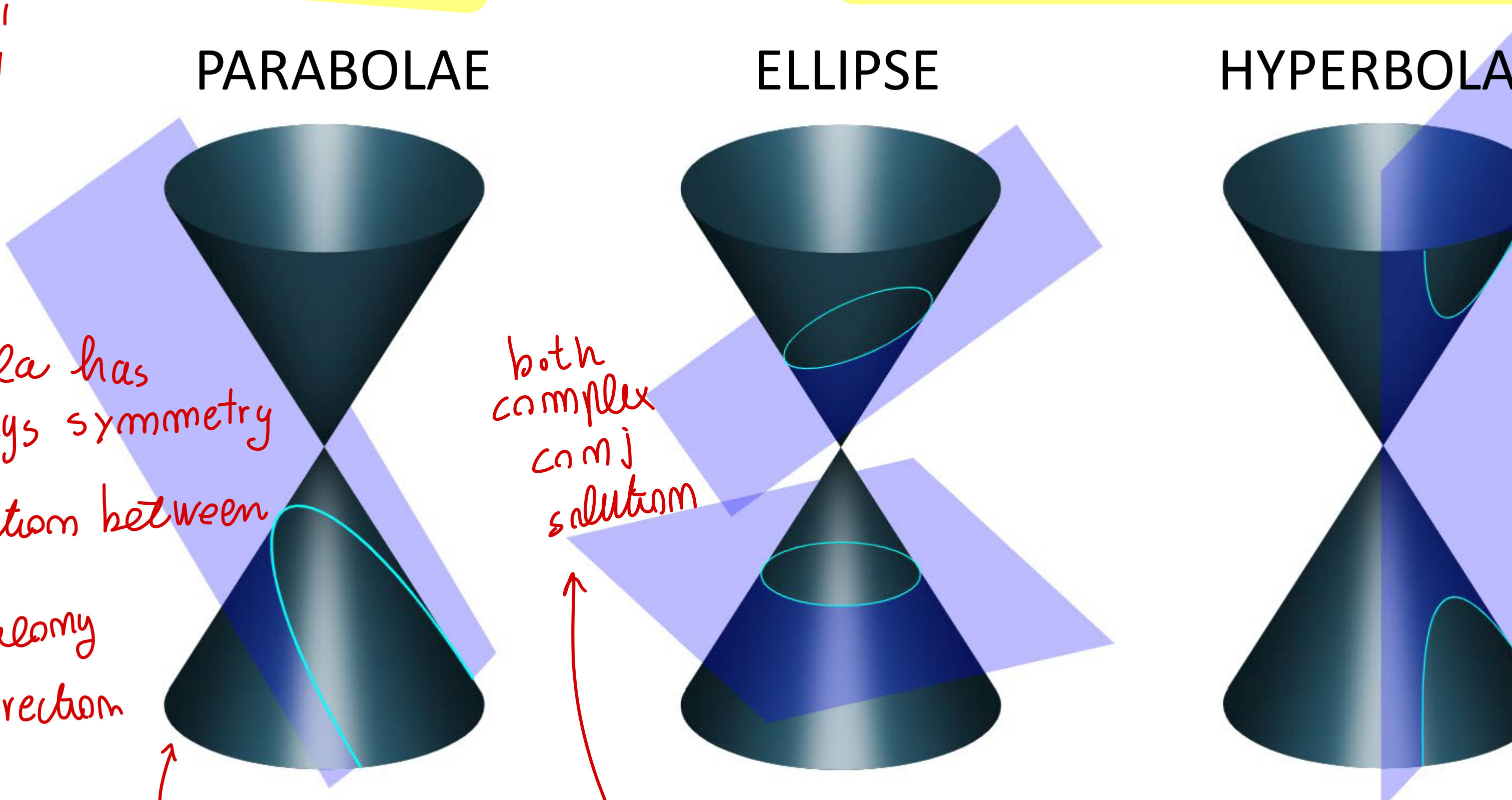
different sol.

2nd solution

1st solut.



PARABOLAE | ELLIPSE | HYPERBOLAE



parabola has always symmetry axis...

the solution between reach ∞ moving along same direction

both complex conj solution

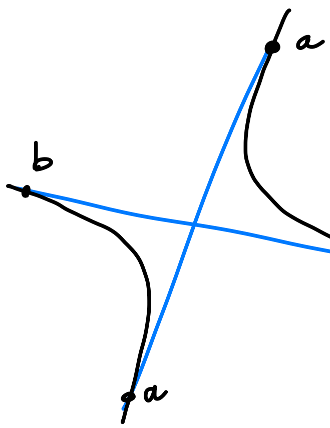
direction of the two asymtotes

two coincident solutions: point at the ∞ along axis

two complex-conjugate solutions: no real solution

because is a CLOSED CURVE

two real distinct solutions: asymptotes



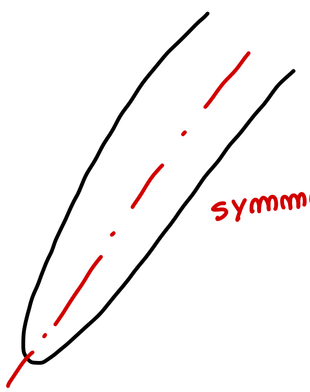
HYPERBOLAE

same line...
intersecting with point @ ∞,
you get the two asymptot
as solution of intersection,
two different points!
↓

$$\left.\begin{array}{l} [0 \;\; 0 \;\; 1]\, x = 0 \\ x^T C x = 0 \end{array}\right\}$$ give you two points at ∞ as direction of the two asymp. lines, both $\mathbb{R}$e solutions

• PARABOLAE

intersecting with
$\ell_\infty = [0 \;\; 0 \;\; 1]'$ (w=0 equation)



symmetry, the two branch proceed along same direction → they will met in the same direction
↓
double solution coincident

← this is usefull for example to look @ gravitational motion, When BASKETBALL game on camera the camera is bad placed, ↵

so parabula image of course is NOT a parabula in projective

gravitationally,

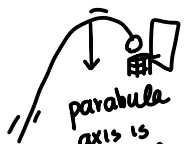
shape as conics, NOT exact parabula...

IF we find the image of the line at ∞ you can determine 3D image studying ℓ_∞ intersection with conic...

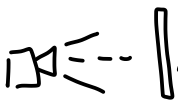
↓

analyze gravitational motion

(for example usefull in BASKETBALL)



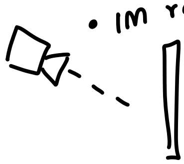
parabula axis is vertical



scene

you see parabula only if correct horiz. camera

• in reality!



you don't see a parabula but a CONIC...

to fit this conic, you consider the line at the infinity image along vertical plane, and determine 3D traj

studing $\ell_\infty \cap$ conic

you expect double coincident intersection

Good for basketball

(NOT table tennis where also friction acts)

The circular points

↓ special points very usefull in 2D reconstruction!

- useful in 2D reconstruction

↳ D

2 geometry of reconstruction problem...

↓

shape of planar object observed by uncomfortable camera position circular points

$\Downarrow$

A noteworthy example:
intersecting a circumference and the line at the ∞

|| SPECIAL CASE ||

$$\begin{cases} x^2 - 2\cancel{X_o w}\,x + \cancel{X_o^2 w^2} + y^2 - 2\cancel{Y_o w}\,y + \cancel{Y_o^2 w^2} - r^2\cancel{w^2} = 0 \\ w = 0 \end{cases}$$

considering w = 0

{ with w = 0 }

put toghether equations

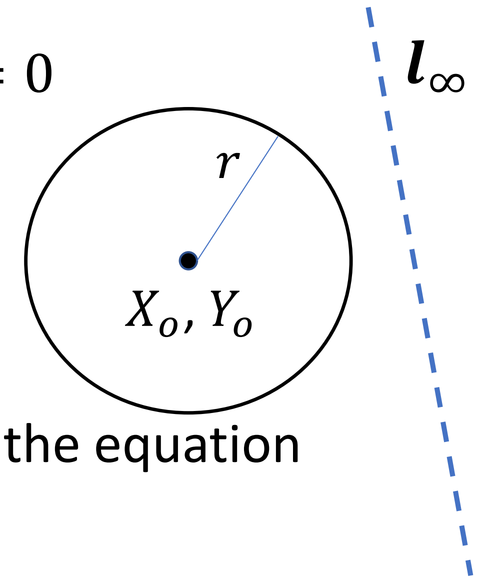
$$\rightarrow$$

$$\begin{cases} x^2 + y^2 = 0 \\ w = 0 \end{cases}$$

the only solution to this is x = 0, y = 0 as Resolution

x = y = w = 0 algebraically (0 0 0)

geometrically NO SOLUTION!



The circumference parameters (center and radius) disappear from the equation

$$\rightarrow$$

the two intersection points are the **same for all** circumferences:

$$\boldsymbol{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \text{ and } \boldsymbol{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$ (i is the imaginary unit number)

These points deserve a special name: the **CIRCULAR POINTS**

A noteworthy example: intersecting a circumference and the line at the ∞

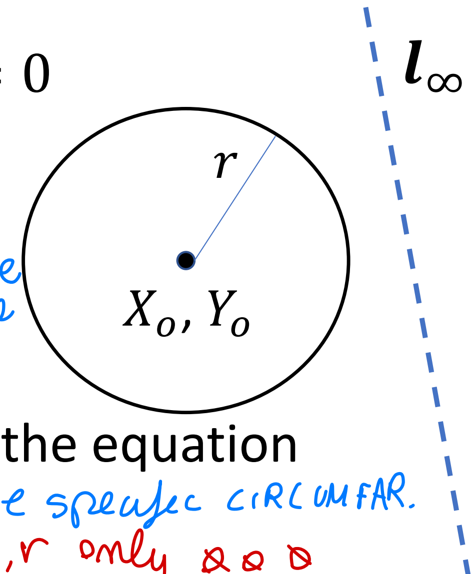
$$\begin{cases} x^2 - 2X_o w + X_o^2 w^2 + y^2 - 2Y_o w + Y_o^2 w^2 - r^2 w^2 = 0 \\ w = 0 \end{cases}$$

$\rightarrow$

$$\begin{cases} x^2 + y^2 = 0 \\ w = 0 \end{cases}$$

$w = x = y = \emptyset$ only solution NO geometrical meaning ! there is NO intersection with ℓ_∞ in any Real point

here all x_o, y_o, r disapper, No dependance on the specific circumference parameter!



The circumference parameters (center and radius) disappear from the equation

same intersection with ℓ_∞ $\forall$ circumference!

$\rightarrow$

DON'T DEPEND on the specific CIRCUMFAR. $\forall$ circumference x_o, y_o, r only $\emptyset\,\emptyset\,\emptyset$ algebrically

the two intersection points are the **same for all** circumferences:

special points! always the same!

INTERSECTION of ANY circumference with ℓ_∞

$$I = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$ (i is the imaginary unit number)

These points deserve a special name: the **CIRCULAR POINTS**

The circular points

→ very usefull in 2D reconstruction

you can derive another usefull point...

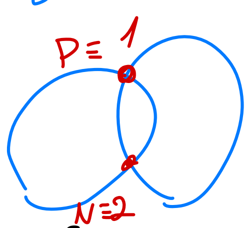
↓ intersection between two circumference

⇓

I expect to have 4 intersection

⇓

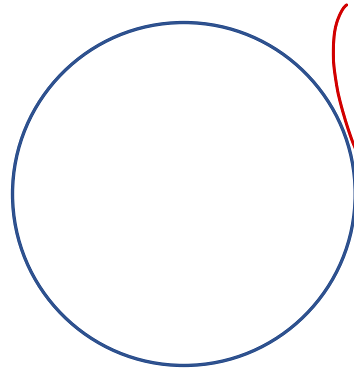
deg 2 × deg 2 = 4 deg of intersection between 2 deg 2 curves...



. I ≡ 3

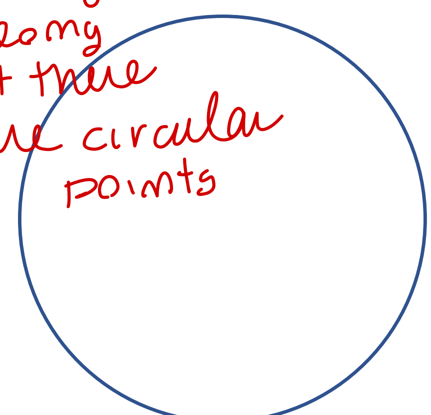
. J ≡ 4

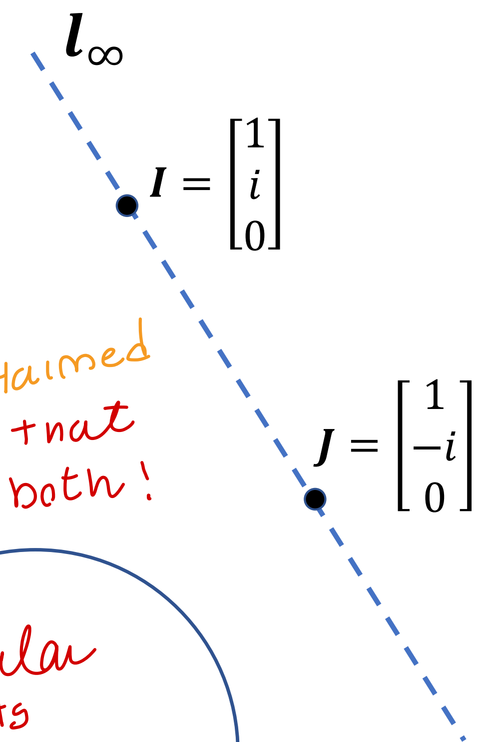
the other two intersecting points are the CIRCULAR POINTS!



being I, J the intersection between any circumference with l_∞... always contained

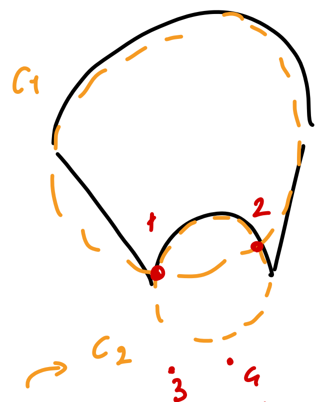
something that belongs to both! belong it there are circular points

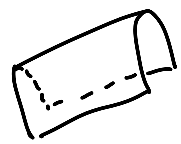




All circumferences cross the line l_∞ at the same two points I and J

in 2023 Homework → Gallery of a palace





to find stuff related to CIRCULAR POINTS
↓
you use ℓ_∞ intersection

using C_1, C_2 circumferences

intersections of the image of the circular points I, J → from image of circular point and intersection you can find image of the line at infinity

the intersection is the
↓

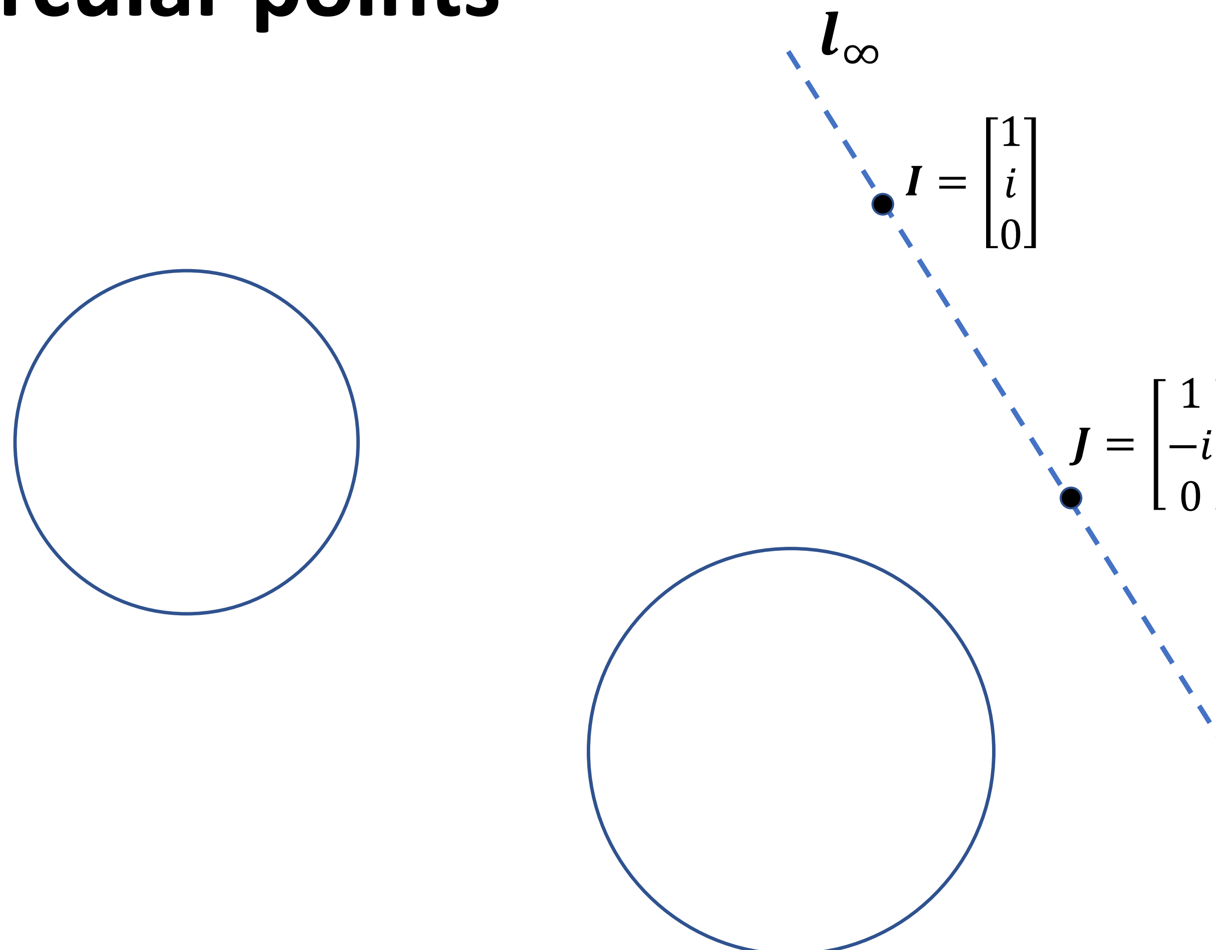
CIRCULAR POINTS are very important when we study images where there are circles..

The circular points

Remember their coordinates

$$\boldsymbol{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$



All the circumferences contain the two circular points $\boldsymbol{I}$ and $\boldsymbol{J}$

another crucial aspect →

POLARITY

← the extension of symmetry for perspective plane

when study image of symmetric objects

Polarity (conjugacy)
is the projective extension of symmetry

when you have elements that enjoy symmetry, you have POLARITY

→ polarity study what happens to the images of symmetric object (no more symmetric in perspective plane!)

useful in the analysis of images of scenes containing symmetric objects (circles, spheres, right cones, right cylinders, planar mirrors) before defining POLARITY .... we need to introduce

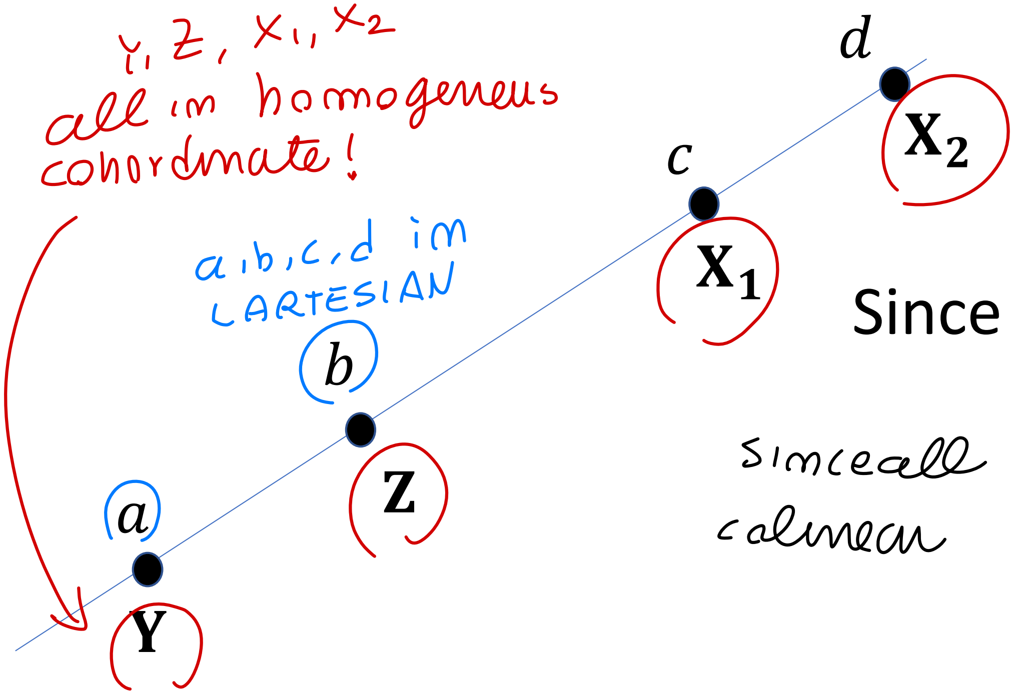
Cross ratio of a 4-tuple of colinear points

$$CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$ and linear combinations

there is a property which holds:

Y, Z, X1, X2 all in homogeneus coordinate!

a, b, c, d in CARTESIAN



Since $\mathbf{X_1}$ and $\mathbf{X_2}$ are colinear with $\mathbf{Y}, \mathbf{Z}$

since all colinear

$$\mathbf{X_1} = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

and

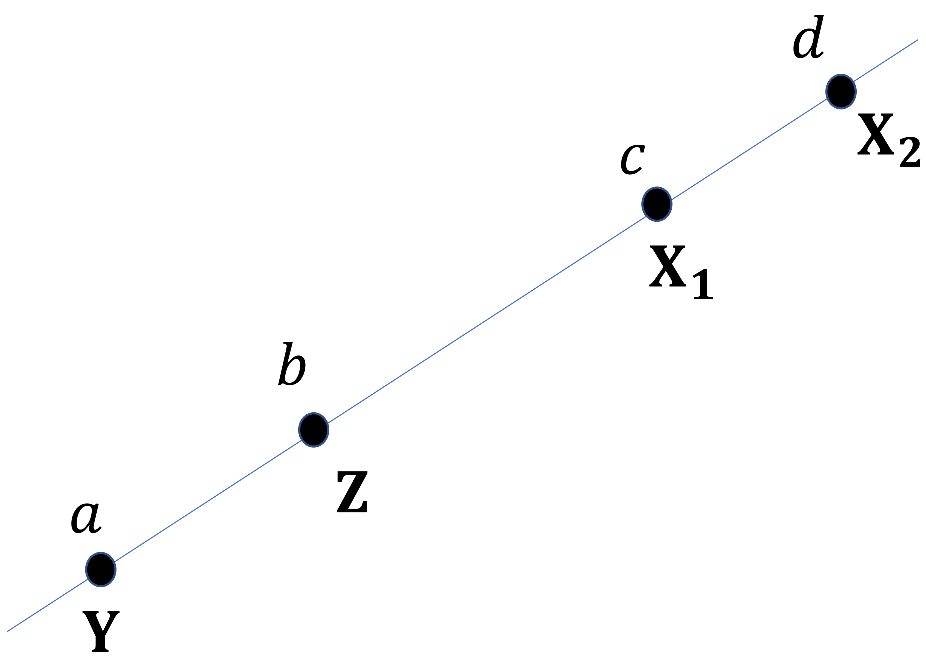
$$\mathbf{X_2} = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

it is possible to relate it $\Rightarrow$

Cross ratio and linear combination coefficients: A result

cross ratio can be expressed:

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = \frac{c-a}{c-b}\Big/\frac{d-a}{d-b} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$



easy to express the coefficients of the linear combination

Proof: since the abscissae are proportional to, e.g., the X cartesian coordinates, we can replace the abscissae by the X coordinates (let us the same names …)

Let $\mathbf{Y} = \begin{bmatrix} \boldsymbol{y} \\ * \\ \boldsymbol{v} \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} \boldsymbol{z} \\ * \\ \boldsymbol{w} \end{bmatrix}$: then $\boldsymbol{X}_1 = \begin{bmatrix} \boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z} \\ * \\ \boldsymbol{\alpha}_1 \boldsymbol{v} + \boldsymbol{\beta}_1 \boldsymbol{w} \end{bmatrix}$ and $\boldsymbol{X}_2 = \begin{bmatrix} \boldsymbol{\alpha}_2 \boldsymbol{y} + \boldsymbol{\beta}_2 \boldsymbol{z} \\ * \\ \boldsymbol{\alpha}_2 \boldsymbol{v} + \boldsymbol{\beta}_2 \boldsymbol{w} \end{bmatrix}$

The difference between the X coordinates of $\boldsymbol{X}_1$ and $\mathbf{Y}$ is

it can be proven…

$$c - a = \frac{(\boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z})\boldsymbol{v} - (\boldsymbol{\alpha}_1 \boldsymbol{v} + \boldsymbol{\beta}_1 \boldsymbol{w})\boldsymbol{y}}{(\boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z})\boldsymbol{v}} = \frac{\boldsymbol{\beta}_1(\boldsymbol{z}\boldsymbol{v} - \boldsymbol{y}\boldsymbol{w})}{(\boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z})\boldsymbol{v}}$$

The difference between the X coordinates of $\boldsymbol{X}_1$ and $\mathbf{Z}$ is

$$c - b = \frac{(\boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z})\boldsymbol{w} - (\boldsymbol{\alpha}_1 \boldsymbol{v} + \boldsymbol{\beta}_1 \boldsymbol{w})\boldsymbol{z}}{(\boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z})\boldsymbol{w}} = \frac{-\boldsymbol{\alpha}_1(\boldsymbol{z}\boldsymbol{v} - \boldsymbol{y}\boldsymbol{w})}{(\boldsymbol{\alpha}_1 \boldsymbol{y} + \boldsymbol{\beta}_1 \boldsymbol{z})\boldsymbol{w}}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Y}$ is

$$c-a=\frac{(\boldsymbol{\alpha_1 y}+\boldsymbol{\beta_1 z})\boldsymbol{v}-(\boldsymbol{\alpha_1 v}+\boldsymbol{\beta_1 w})\boldsymbol{y}}{(\boldsymbol{\alpha_1 y}+\boldsymbol{\beta_1 z})\boldsymbol{v}}=\frac{\boldsymbol{\beta_1}(\boldsymbol{zv}-\boldsymbol{yw})}{(\boldsymbol{\alpha_1 y}+\boldsymbol{\beta_1 z})\boldsymbol{v}}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Z}$ is

$$c-b=\frac{(\boldsymbol{\alpha_1 y}+\boldsymbol{\beta_1 z})\boldsymbol{w}-(\boldsymbol{\alpha_1 v}+\boldsymbol{\beta_1 w})\boldsymbol{z}}{(\boldsymbol{\alpha_1 y}+\boldsymbol{\beta_1 z})\boldsymbol{w}}=\frac{-\boldsymbol{\alpha_1}(\boldsymbol{zv}-\boldsymbol{yw})}{(\boldsymbol{\alpha_1 y}+\boldsymbol{\beta_1 z})\boldsymbol{w}}$$

The ratio between these differences is $\frac{c-a}{c-b}=-\frac{\boldsymbol{\beta_1}}{\boldsymbol{\alpha_1}}\frac{w}{v}$ and, similarly,

$$\frac{d-a}{d-b}=-\frac{\boldsymbol{\beta_2}}{\boldsymbol{\alpha_2}}\frac{w}{v}$$

Thus the cross ratio of the four colinear points $\boldsymbol{Y},\boldsymbol{Z},\mathbf{X_1},\mathbf{X_2}$ is

$$\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X_1},\mathbf{X_2}}=\frac{\boldsymbol{c}-\boldsymbol{a}}{\boldsymbol{c}-\boldsymbol{b}}\bigg/\frac{\boldsymbol{d}-\boldsymbol{a}}{\boldsymbol{d}-\boldsymbol{b}}=\frac{\boldsymbol{\beta_1}}{\boldsymbol{\alpha_1}}\bigg/\frac{\boldsymbol{\beta_2}}{\boldsymbol{\alpha_2}}$$

Q.E.D.

CROSS RATIO SPECIAL VALUE: -1 (*harmonic* 4-tuple)

When $\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X_1},\mathbf{X_2}} = \mathbf{-1}$, Y and Z are said to be **conjugate** wrt X_1 and X_2

(HARMONIC 4-TOUPLE)

OR viceversa

X_1, X_2 are CONJUGATE wrt Y and Z

Given the segment $\boldsymbol{Y}, \boldsymbol{Z}$, an instance of two conjugate points wrt $\boldsymbol{Y}, \boldsymbol{Z}$ is pair ($\mathbf{X_0}$ = midpoint of $\boldsymbol{Y}, \boldsymbol{Z}$, $\mathbf{X}_\infty$ = point at the infinity along $\boldsymbol{Y}, \boldsymbol{Z}$)

Other instances: since the cross ratio of 4 colinear points is invariant under image projection, other instances of conjugate points are images $\mathbf{y}, \mathbf{z}$ of segment endpoints, image $\mathbf{x}_{\boldsymbol{0}}$ of midpoint and vanishing point $\mathbf{x}_\infty$ (image of point at the infinity)

← representative situation of this case... is when $\begin{cases} X_1 \equiv \text{MIDPOINT } Z, Y \\ X_2 \equiv \text{point @ } \infty \end{cases}$

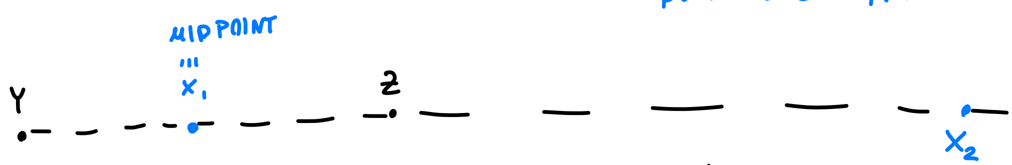
given Y, Z segment

two points s.t

$CR_{Y,Z,X_1,X_2} = -1$

is when the third point is the mid point of Y, Z

while the fourth point is the point at ∞



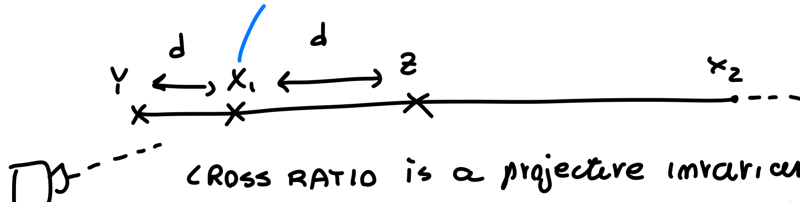
the CROSS RATIO in this situation give you -1

$$CR = \frac{\dfrac{d_{X_1 Y}}{d_{X_1 Z}} = \dfrac{+\alpha}{-\alpha} = -1}{\dfrac{d_{X_2 Y}}{d_{X_2 Z}} = \dfrac{\infty}{\infty} \simeq 1} \simeq -1$$

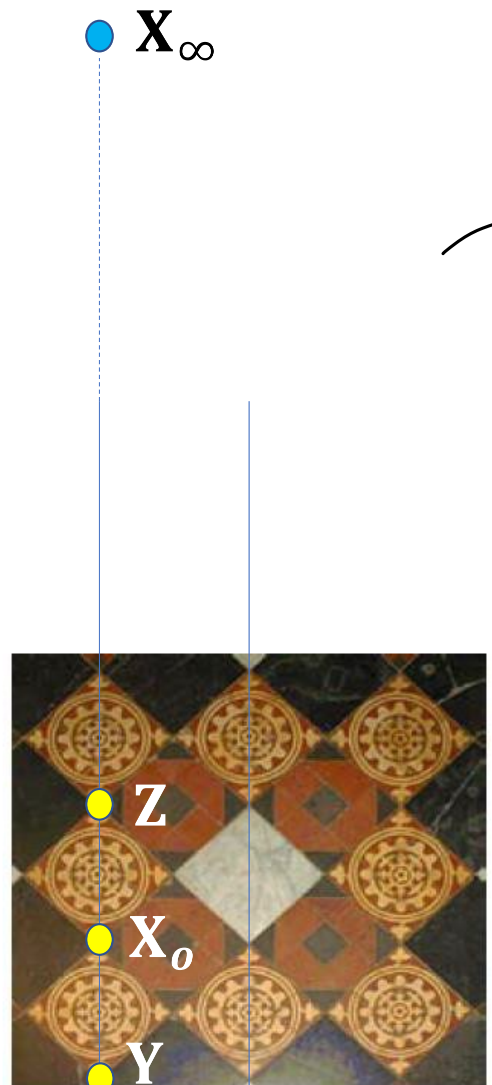
harmonic touple!

another situation occurs when you take a photo from different point

NO more midpoint of the image

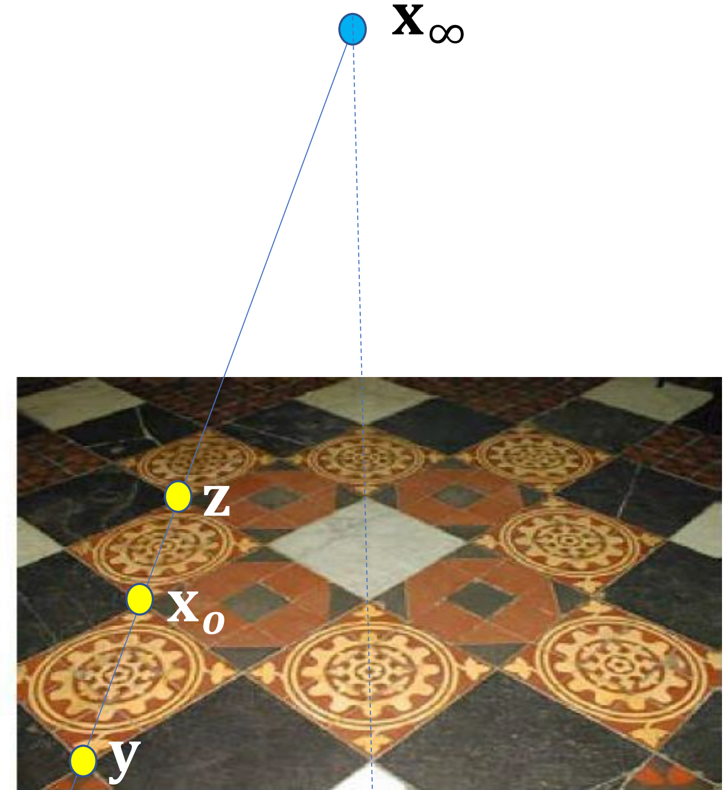


CROSS RATIO is a projective invariant! so remain the same in PROJECTIVE PLANE



a

it holds also
in PROJECTIVE
GEOMETRY



c

Harmonic 4-tuples and conjugate points

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = -1}$$

$\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are conjugate wrt $(\boldsymbol{Y}, \boldsymbol{Z})$

$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are conjugate wrt $(\boldsymbol{X_1},\boldsymbol{X_2})$

best-known case:

- two points Y and Z are the endpoints of a segment

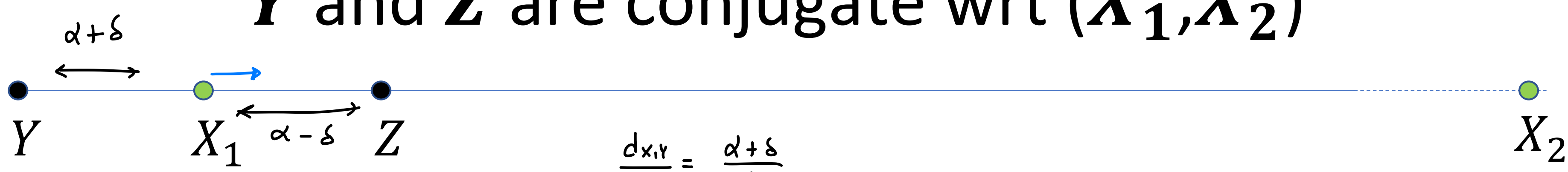
$-$ X_1 and X_2 are the midpoint of (Y, Z) and the point at the ∞

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = -1}$$

$\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are conjugate wrt $(\boldsymbol{Y}, \boldsymbol{Z})$

$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are conjugate wrt $(\boldsymbol{X_1}, \boldsymbol{X_2})$



$$CR = \frac{\dfrac{d_{X_1Y}}{d_{X_1Z}} = \dfrac{\alpha+\delta}{-(\alpha-\delta)}}{\dfrac{d_{X_2Y}}{d_{X_2Z}} = \dfrac{\infty+\delta}{\infty+\delta}} \simeq -1$$

Other case: to preserve the value of the cross ratio = -1:
the «far» point X_2 approaches a little bit
and the other point X_1 also moves to get a bit closer to X_2

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = -1$$

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$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are conjugate wrt $(\boldsymbol{X_1},\boldsymbol{X_2})$

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Harmonic 4-tuples and conjugate points

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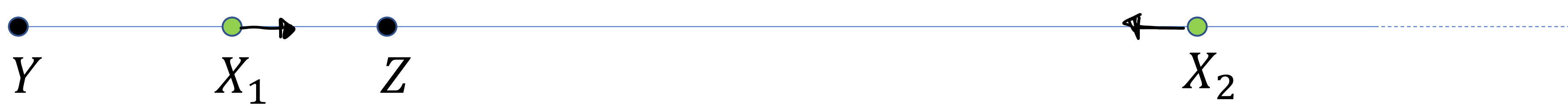


Other case: to preserve the value of the cross ratio = -1:
the «far» point X_2 approaches a little bit
and the other point X_1 also moves to get a bit closer to X_2

Harmonic 4-tuples and conjugate points

$$\frac{\frac{\alpha+\delta}{-(\alpha-\delta)}}{\frac{\ell_2-\delta}{\ell_Y-\delta}} = \frac{\frac{\alpha+\delta}{\delta-\alpha}}{\frac{\ell+2\alpha-\delta}{\ell-\delta}}$$

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = -1$$

$\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are conjugate wrt $(\boldsymbol{Y}, \boldsymbol{Z})$

$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are conjugate wrt $(\boldsymbol{X_1}, \boldsymbol{X_2})$

when inclined camera... image plane NOT parallel to scene has this image

all this 4-touples has CR=-1.. this situation may be obtained by PERPECTIVE images in Real world, with CAMERA not centered

Other case: to preserve the value of the cross ratio = -1:
the «far» point X_2 approaches a little bit
and the other point X_1 also moves to get a bit closer to X_2

different way to express the same property !

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = -1}$$

$\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are conjugate wrt $(\boldsymbol{Y}, \boldsymbol{Z})$

$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are conjugate wrt $(\boldsymbol{X_1}, \boldsymbol{X_2})$

If we revert X_1 and X_2 , the cross ratio inverts its value: but the inverse of -1 is itself!

→

In a harmonic tuple we can revert the roles of X_1 and X_2

→

X_1 and X_2 are **conjugate** points wrt Y and Z, namely both X_1 is **conjugate** to X_2 wrt Y and Z, and X_2 is **conjugate** to X_1 wrt Y and Z

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = -1$$

$\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ are conjugate wrt $(\boldsymbol{Y}, \boldsymbol{Z})$

$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are conjugate wrt $(\boldsymbol{X_1}, \boldsymbol{X_2})$

NOTE:

if X_1 and X_2 are **conjugate** points wrt Y and Z

since the cross ratio = -1 has a negative value, the simple ratios

$\boldsymbol{\frac{c-a}{c-b}}$ and $\boldsymbol{\frac{d-a}{d-b}}$ have opposite signs: i.e. one is positive and the other is negative

→

one among X_1 and X_2 is **internal** to the segment YZ, while the other is **external**

The polar line of a point wrt a conic

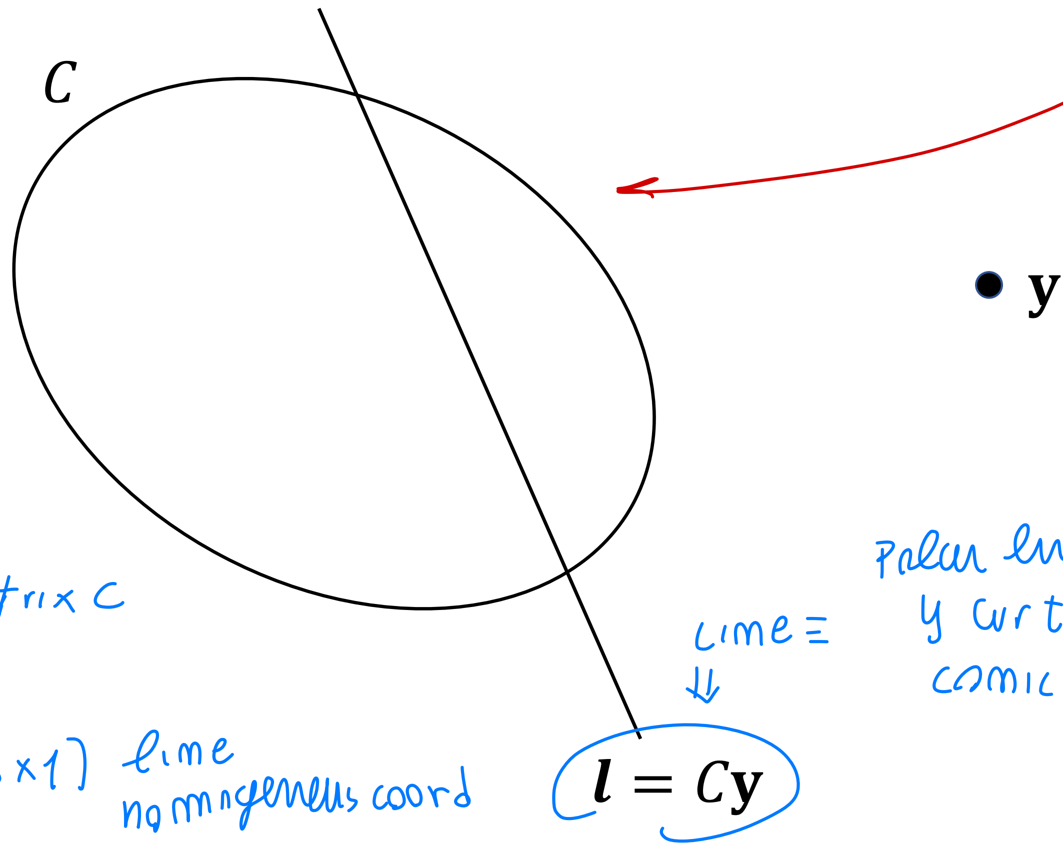
all elements conics C , point Y, we
want to define the
POLAR LINE of Y wrt conic C

Polar line of a point wrt a conic

as homogeneous coordinate line

Given a point $\mathbf{y}$ and a conic C in the plane, the line $\boxed{\boldsymbol{l} = C\mathbf{y}}$ is called the *polar line* of point $\mathbf{y}$ with respect to the conic C.

⇒ polar line of any generic point y, NOT at ∞, any point has POLAR LINE wrt a CONIC



simply as conic matrix C multiplied by point C

$[Cy] = [3 \times 1]$ line homogeneous coord

line = ⇓

Polar line of y wrt conic C

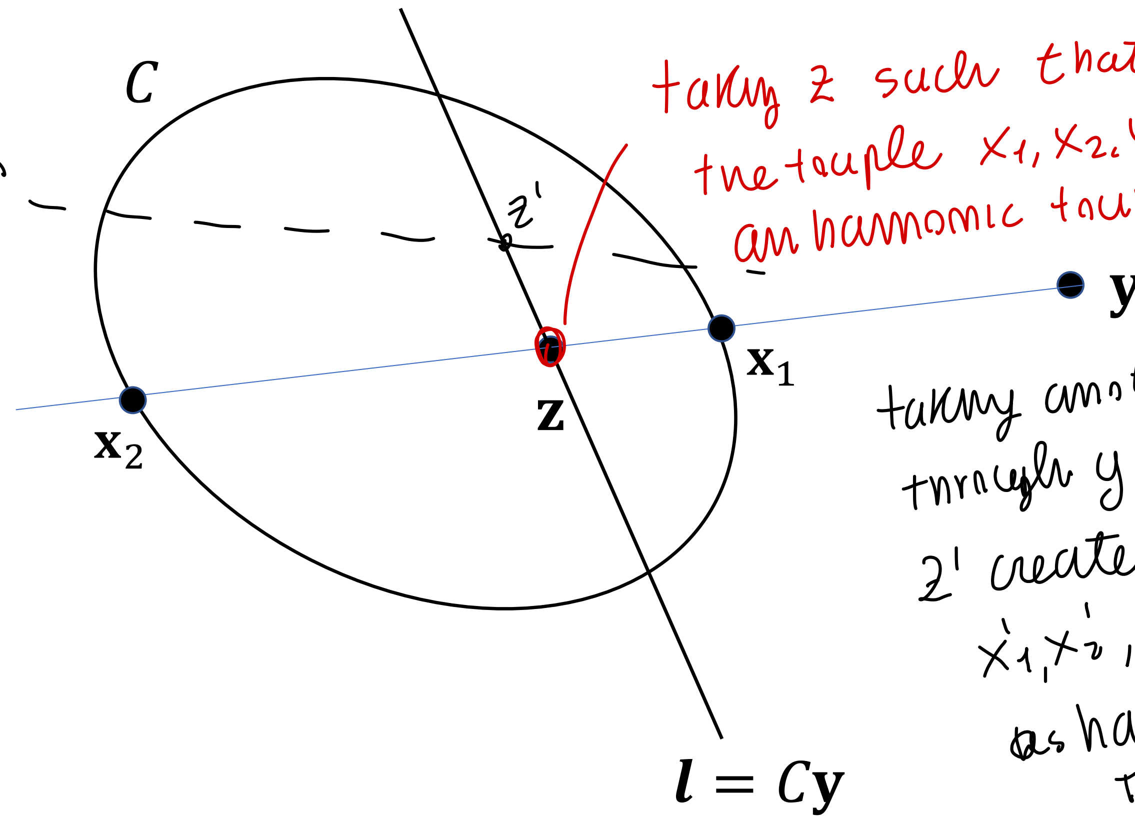
→ NICE PROPERTY of the POLAR LINE...

Polar line and cross ratio (= -1)

VERY IMPORTANT! ⇑

Theorem. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points where the line through $\mathbf{y}$ and $\mathbf{z}$ crosses C: then cross ratio $\boldsymbol{CR_{Y,Z,X_1,X_2} = -1}$ ($\mathbf{y}$ and $\mathbf{z}$ are said to be **conjugate** wrt $\mathbf{x}_1$ and $\mathbf{x}_2$)

considering any line $\ell_{x_1x_2}$, than it has 2 intersection geometrically with conic...



taking z such that the touple x_1, x_2, y, z is an harmonic touple...

taking another line through y s.t ⇒ z' create x_1', x_2', y, z' as harmonic touple...

(this can be PROVEN!)

Exercise: prove it!

(or see next page)

Theorem. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points where the line through $\mathbf{y}$ and $\mathbf{z}$ crosses C: $\mathbf{y}$ and $\mathbf{z}$ are **conjugate** with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$, i.e. their cross ratio = -1.

Proof sketch. $\mathbf{x}_1$ and $\mathbf{x}_2$ are the two solutions (for $\mathbf{x}$) to the intersection problem:
$\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$ (since $\mathbf{x}$ is on line $(\mathbf{y}, \mathbf{z})$) AND $\mathbf{x}^T C\,\mathbf{x} = \mathbf{0}$ (since $\mathbf{x}$ is on conic C). Namely, $(\alpha\mathbf{y} + \beta\mathbf{z})^T C(\alpha\mathbf{y} + \beta\mathbf{z}) = 0$, with $\mathbf{z}$ on line $\boldsymbol{l} = C\mathbf{y}$, i.e. $\mathbf{z}^T C\,\mathbf{y} = 0 = \mathbf{y}^T C\mathbf{z}$ (remember C is symmetric). The remaining terms are therefore $\alpha^2\mathbf{y}^T C\mathbf{y} + \beta^2\mathbf{z}^T C\mathbf{z} = \mathbf{0}$. Thus there are two opposite solutions for the ratio β/α:

$$\beta/\alpha = \pm\sqrt{-\mathbf{y}^T C\mathbf{y}/\mathbf{z}^T C\mathbf{z}} \qquad \text{i.e. } \frac{\beta_2}{\alpha_2} = -\frac{\beta_1}{\alpha_1}$$

this can be proven using linear combination relation

Remember we are solving the intersection problem:
→ these two solutions correspond to the points $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\text{Hence } \boldsymbol{CR}_{Y,Z,\mathbf{X}_1,\mathbf{X}_2} = \frac{\boldsymbol{\beta}_1}{\boldsymbol{\alpha}_1} \Big/ \frac{\boldsymbol{\beta}_2}{\boldsymbol{\alpha}_2} = \mathbf{-1}$$

The polar line $\boldsymbol{l} = C\mathbf{y}$ is the locus of points $\mathbf{z}$ conjugate of $\mathbf{y}$ wrt conic C (more precisely, conjugate wrt to the intersection points of C with any line through $\mathbf{y}$)

a consequence of the previous theorem

Polar line and cross ratio (=-1)

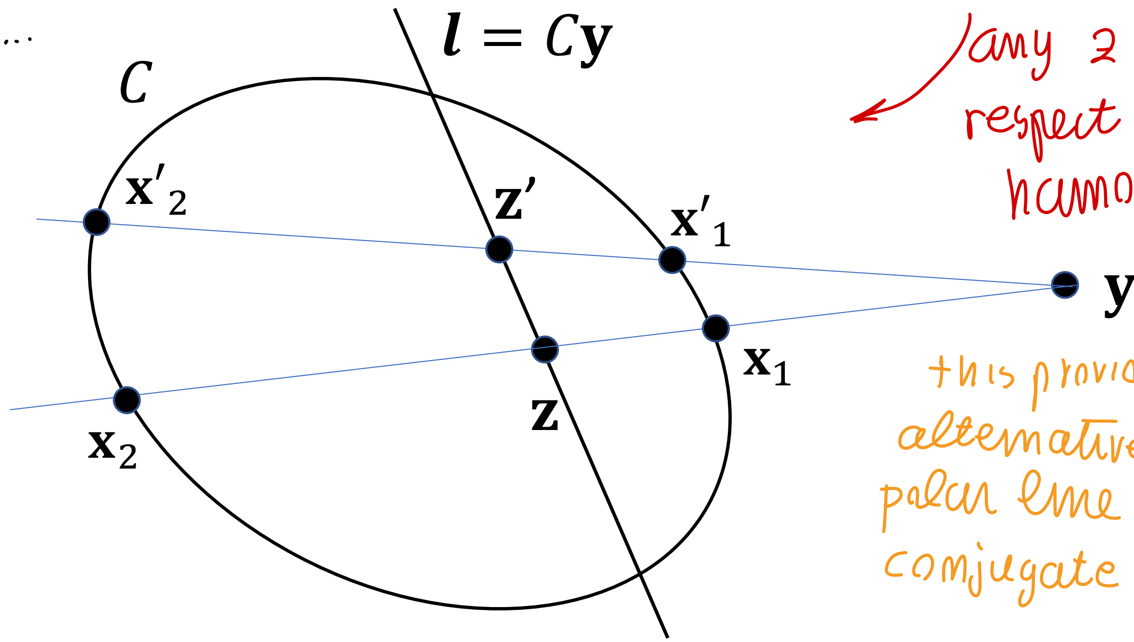
to derive another definition of POLAR LINE...

Theorem. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points where a line through $\mathbf{y}$ and $\mathbf{z}$ crosses C: cross ratio $\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X}_1,\mathbf{X}_2} = \mathbf{-1}$ ($\mathbf{y}$ and $\mathbf{z}$ are said to be **conjugate** wrt $\mathbf{x}_1$ and $\mathbf{x}_2$)

all $z^i \in l$ CO-LINEAR ...

that z^i are the polar line

any $z \in l = Cy$ respect that harmonic



this provide you an alternative definition of polar line in terms of conjugate points

CONSEQUENCE of Theorem...

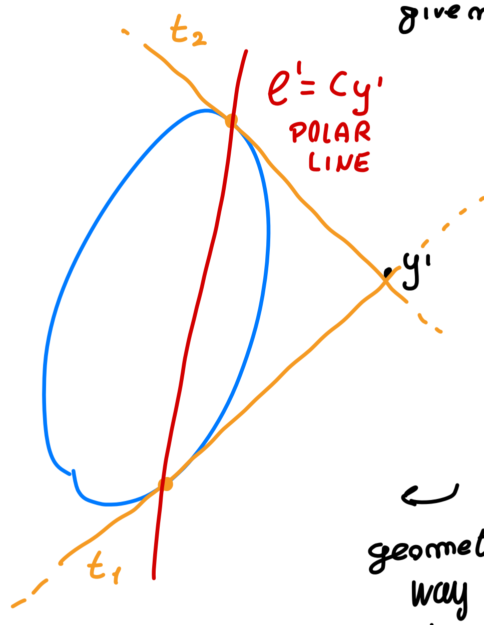
Therefore, the polar line $\boldsymbol{l}$ of $\mathbf{y}$ wrt C is the set of points $\mathbf{z}$, that are conjugate to $\mathbf{y}$ wrt the intersection points $\mathbf{x}_1$ and $\mathbf{x}_2$ between C and any line through $\mathbf{y}$.

In graphical P.O.V

you can draw $Cy = \ell$ POLAR line

given C, y

$\Rightarrow$



turns out that polar line goes through 2 special points ie tangency point

geometric way to automatically draw the POLAR LINE from C, y without computations But Geometrically

Polar line and tangency points

another
important
special case... HOW TO GRAPHICALLY
represent this POLAR LINE

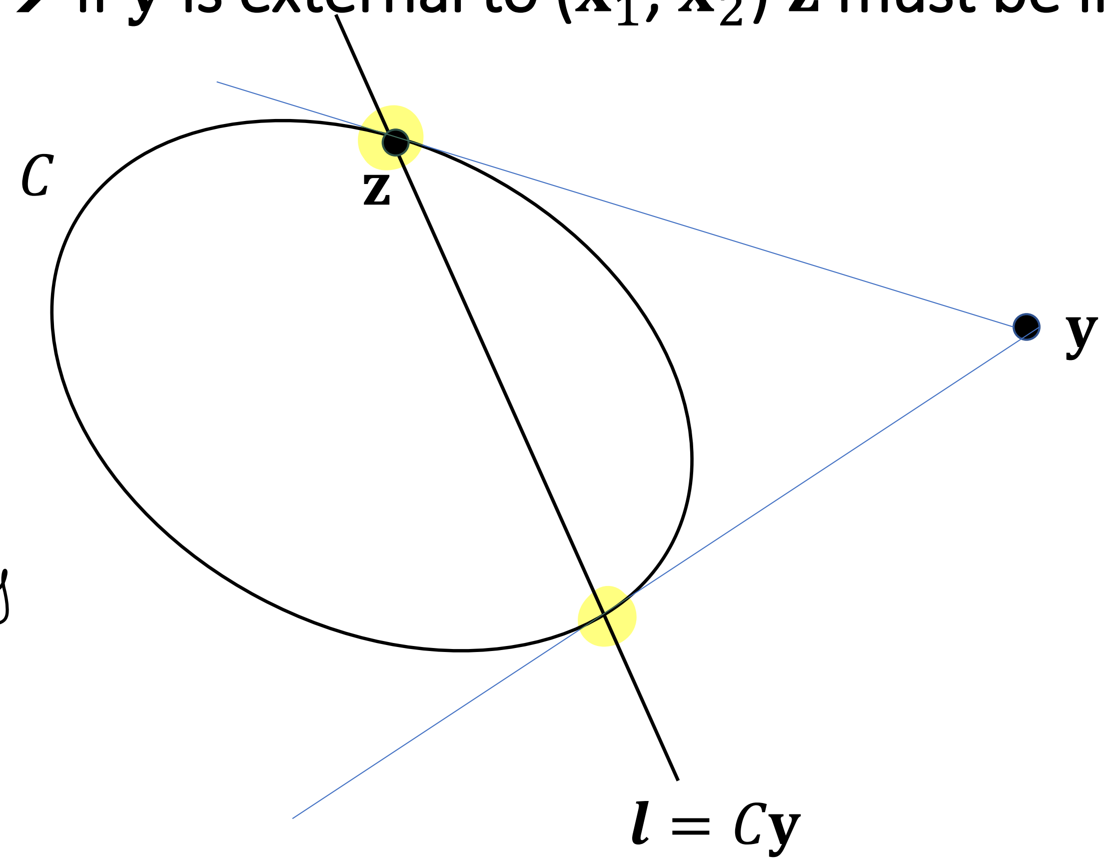
↑

there is a PROPERTY relating POLAR LINE
with TANGENCY POINTS

⇒

Property:
The polar line $\boldsymbol{l} = C\mathbf{y}$ goes through the tangency points from $\mathbf{y}$ to C
Geometric/Qualitative Proof: $\mathbf{y}$ and $\mathbf{z}$ conjugate wrt $\mathbf{x}_1$ and $\mathbf{x}_2$ → negative cross ratio → if $\mathbf{y}$ is external to $(\mathbf{x}_1, \mathbf{x}_2)$ $\mathbf{z}$ must be internal

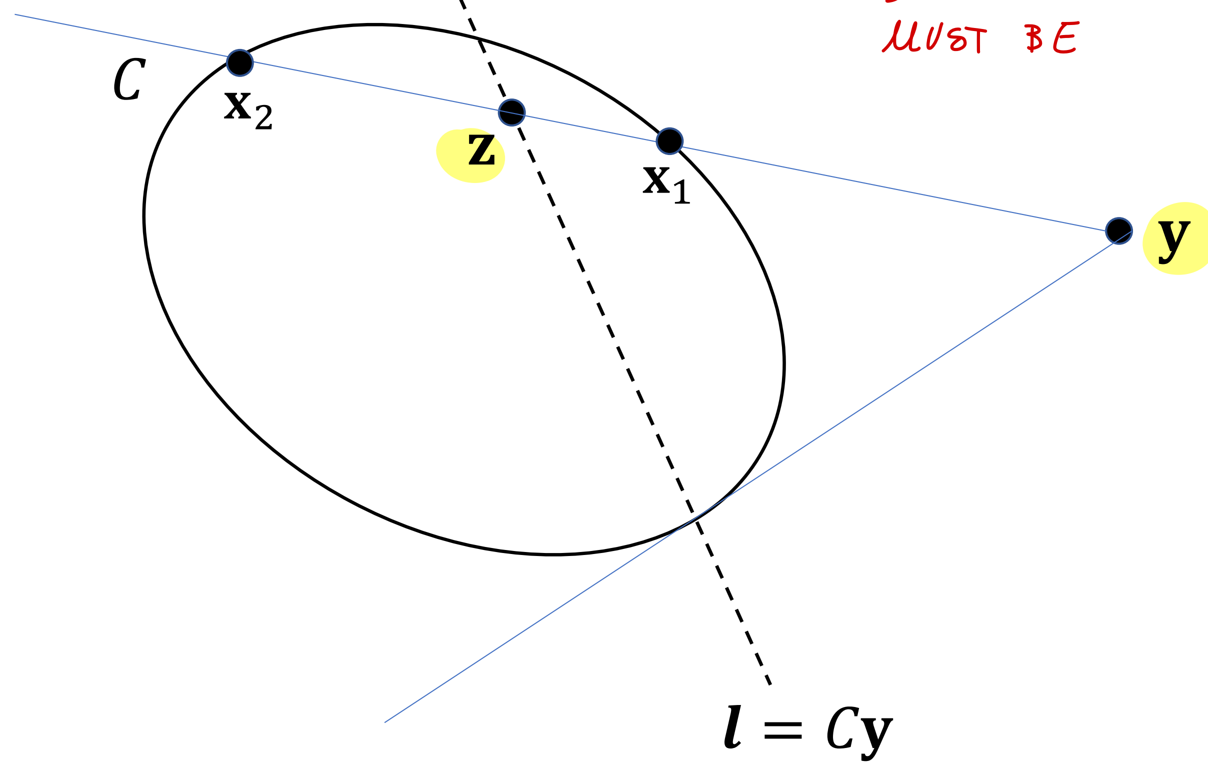
taking lines through y tangent to conics, then the tangent points are the intersection of $l = cy$ with conics



Property:
The polar line $\boldsymbol{l} = C\mathbf{y}$ goes through the tangency points from $\mathbf{y}$ to C
Geometric/Qualitative Proof: $\mathbf{y}$ and $\mathbf{z}$ conjugate wrt $\mathbf{x}_1$ and $\mathbf{x}_2$ → negative cross ratio → if y is external to $(\mathbf{x}_1, \mathbf{x}_2)$ z must be internal

↳ that PROPERTY is noticable considering that,

s.t CR = -1 MUST BE



Property:
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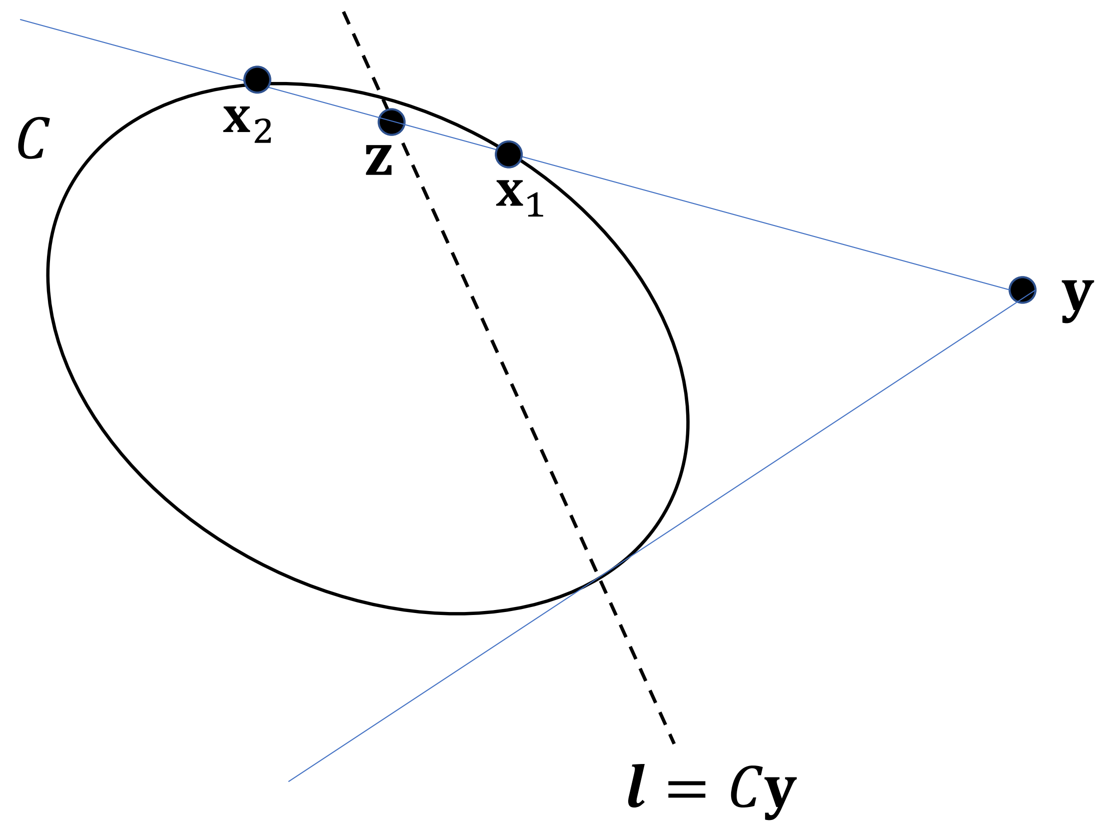
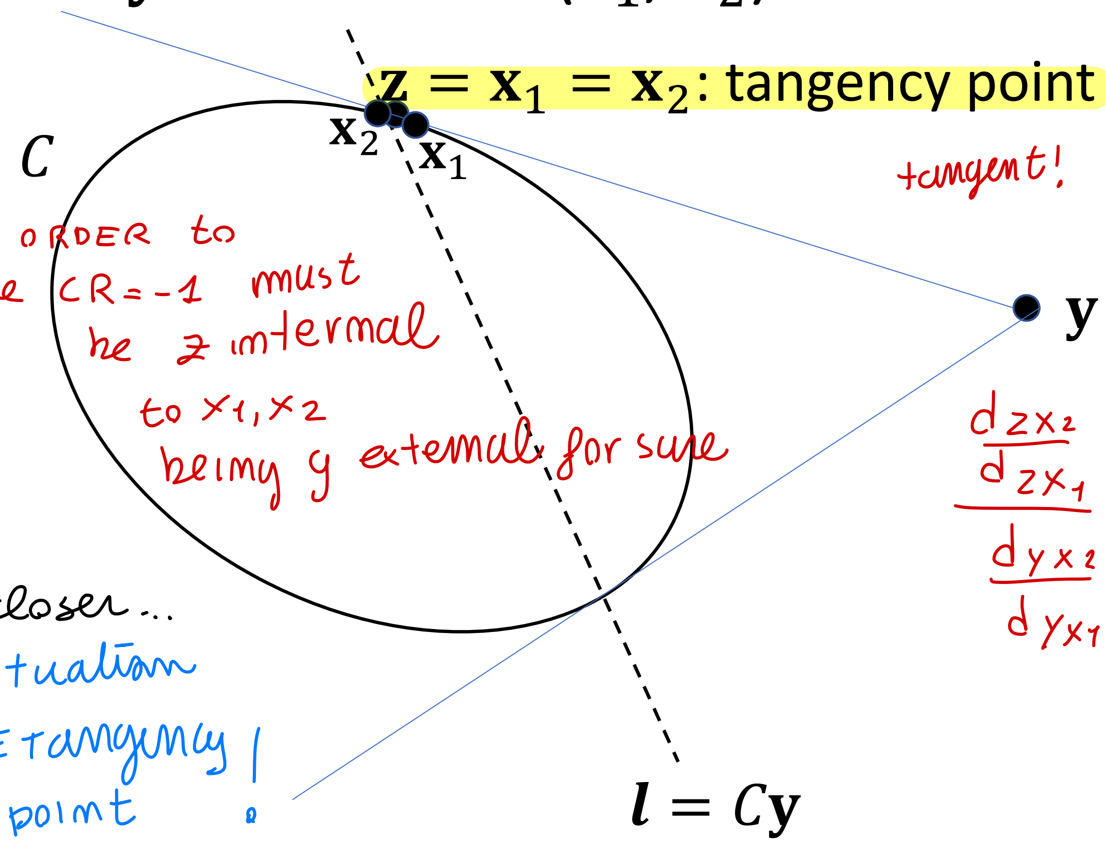


Property:

important locus of point!

The polar line $\boldsymbol{l} = C\mathbf{y}$ goes through the tangency points from $\mathbf{y}$ to C

Geometric/Qualitative Proof: $\mathbf{y}$ and $\mathbf{z}$ conjugate wrt $\mathbf{x}_1$ and $\mathbf{x}_2$ → negative cross ratio → if $\mathbf{y}$ is external to $(\mathbf{x}_1, \mathbf{x}_2)$ $\mathbf{z}$ must be internal



when we approach to the tangent, has to be $z \in (x_1, x_2)$ and y external...

intersection points become closer...

until tangent situation $x_1 \equiv x_2 \equiv z \equiv$ tangency point!

IN ORDER to be $CR = -1$ must be z internal to x_1, x_2 being y external for sure

and $x_1 \equiv x_2$ tends to coincide → tangent!

$\{ z \in (x_1, x_2)$ segment $\}$

$$\frac{\frac{d_{zx_2}}{d_{zx_1}}}{\frac{d_{yx_2}}{d_{yx_1}}} = CR = -1$$

$\ell = Cy$ POLAR LINE

is the LOCUS of the POINTS s.t conjugate to y wrt intersection between conics and any line through y

$\begin{cases} y \text{ is any line in space} \\ C \text{ is any conic} \end{cases}$

Property: ⇐ it can be proven Algebrically!

The polar line $\boldsymbol{l} = C\mathbf{y}$ goes through the tangency points from $\mathbf{y}$ to C

Algebraic Proof:

$$\mathbf{z} \in \boldsymbol{l} = C\mathbf{y} \rightarrow \boldsymbol{l}^T\mathbf{z} = \mathbf{y}^T C\mathbf{z} = \mathbf{0}$$

$C = C^T$ symmmetric

let's consider two lines..

↓ first a line s.t

$l = Cy$ and $z \in l$

⇓

being $\underbrace{z \in l}$

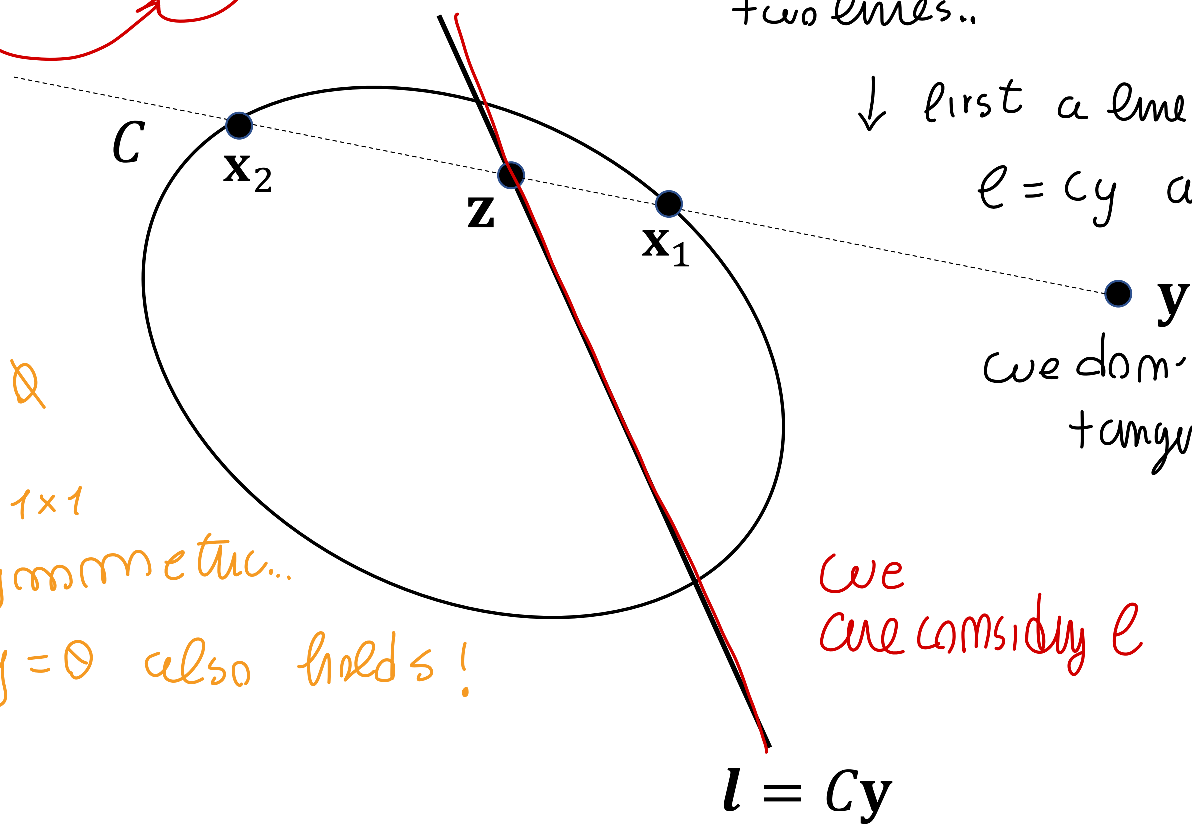
↓

$\underbrace{y^T C}\, z = 0$

this is 1×1

so is symmmetric..

$z^T C y = 0$ also holds!



we don't know y tangency

we

we consider l

Algebraic Proof: Intersect polar line $\boldsymbol{l}$ with line joining $\mathbf{y}$ and $\mathbf{z}$

$x^T C x = 0$

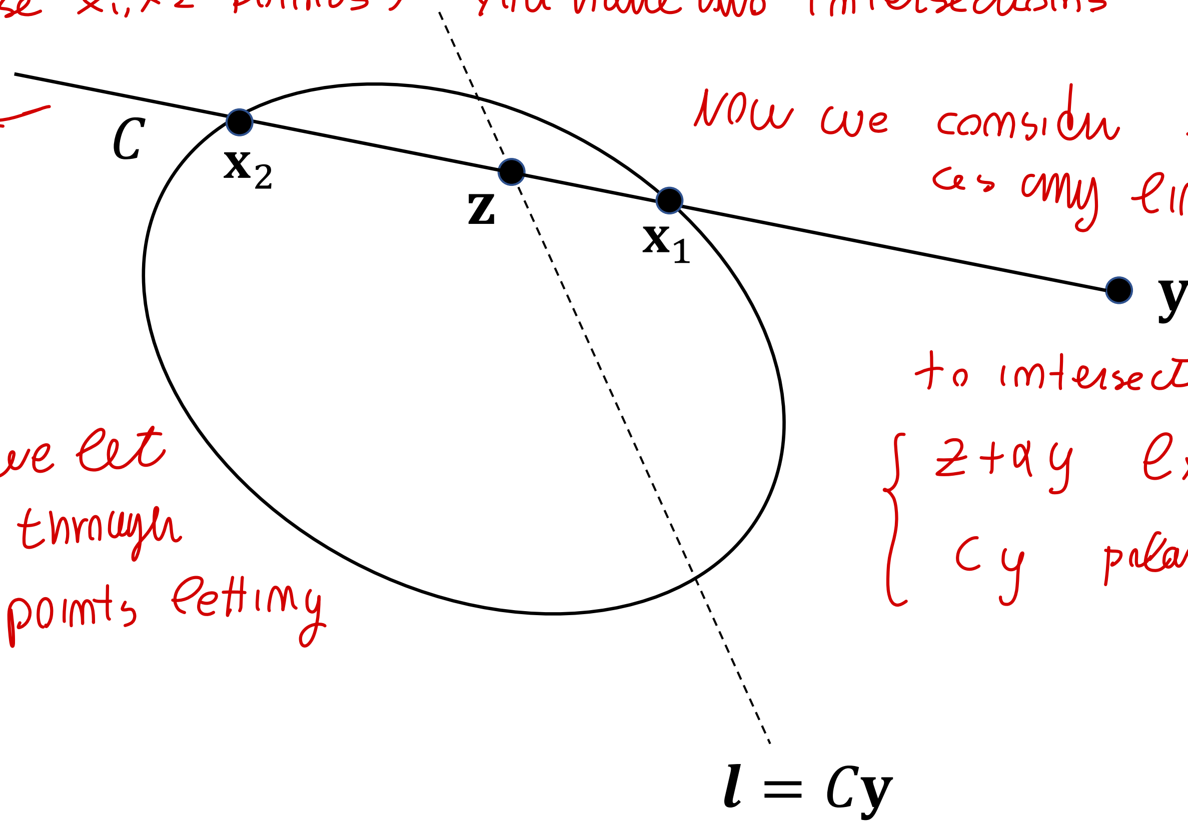
$\mathbf{z} \in \boldsymbol{l} = C\mathbf{y} \rightarrow \boldsymbol{l}^T\mathbf{z} = \mathbf{y}^T C\mathbf{z} = \mathbf{0}$ and $\mathbf{z} + \alpha\mathbf{y} \in \boldsymbol{C} \rightarrow (\mathbf{z} + \alpha\mathbf{y})^T \boldsymbol{C}(\mathbf{z} + \alpha\boldsymbol{y}) = \mathbf{0}$

belongs to conics

$\rightarrow \underbrace{\mathbf{z}^T \boldsymbol{C}\mathbf{z} + \alpha^2\, \mathbf{y}^T \boldsymbol{C}\mathbf{y} = \mathbf{0}}_{\text{to impose } x_1, x_2 \text{ points}}$ two solutions $\alpha = \pm\sqrt{-\mathbf{z}^T \boldsymbol{C}\mathbf{z} / \mathbf{y}^T \boldsymbol{C}\mathbf{y}}$

you have two intersections

line as set of points co-linear to y, z, then as $z + \alpha y \in C$ letting $\alpha \in \mathbb{R}e$ we let span the line through y and z $\Rightarrow$ all points letting α vary!



Now we consider $\ell_{x_1 x_2}$ as any line through y

to intersect
$$\begin{cases} z + \alpha y & \ell_{x_1, x_2} \\ C y & \text{polar line} = y^T C z = 0 \end{cases}$$

$$\rightarrow \quad z^T C z + \underbrace{z^T \cancel{C} \alpha y} + \underbrace{\alpha y^T \cancel{C} z} + \alpha^2 y^T C y = 0$$

$y^T C z = 0 = z^T C y$

$$\Downarrow$$

$$z^T C z + \alpha^2 y^T C y = 0$$

with two solutions

this two α give α_1, α_2

s.t $z + \alpha_i y$ give you the points x_1, x_2 intersecting the conic

part of $\ell_{x_1 x_2}$

note

$z + \alpha y = 0$

as $\alpha \in \mathbb{R}$ vary, you describe line from line ℓ_{yz} → linear combination of two points z, y give you third point co-linear, BUT hard to draw explicitly because we are working in HOMOGENEUS coordinates.

it is NOT clear the representation of the point in CARTESIAN ←

→ but for sure $z + \alpha y$ is a point on ℓ_{zy}

Algebraic Proof: Intersect polar line $\boldsymbol{l}$ with line joining $\mathbf{y}$ and $\mathbf{z}$

$\mathbf{z} \in \boldsymbol{l} = C\mathbf{y} \rightarrow \boldsymbol{l}^T\mathbf{z} = \mathbf{y}^T \boldsymbol{C}\mathbf{z} = \mathbf{0}$ and $\mathbf{z} + \alpha\mathbf{y} \in \boldsymbol{C} \rightarrow (\mathbf{z} + \alpha\mathbf{y})^T \boldsymbol{C}(\mathbf{z} + \alpha\boldsymbol{y}) = \mathbf{0}$

$\rightarrow \mathbf{z}^T \boldsymbol{C}\mathbf{z} + \alpha^2\, \mathbf{y}^T \boldsymbol{C}\mathbf{y} = \mathbf{0}$ two solutions $\alpha = \pm\sqrt{-\mathbf{z}^T \boldsymbol{C}\mathbf{z} / \mathbf{y}^T \boldsymbol{C}\mathbf{y}}$

tangency: one double solution when $-\mathbf{z}^T \boldsymbol{C}\mathbf{z} = \mathbf{0}$ i.e. $\mathbf{z} \in \boldsymbol{C} \rightarrow$ tangency point from y

$\alpha = \pm\sqrt{\dfrac{-z^T C z}{y^T C y}}$

one double solutions IFF

$\alpha_1 = -\alpha_2 = 0$

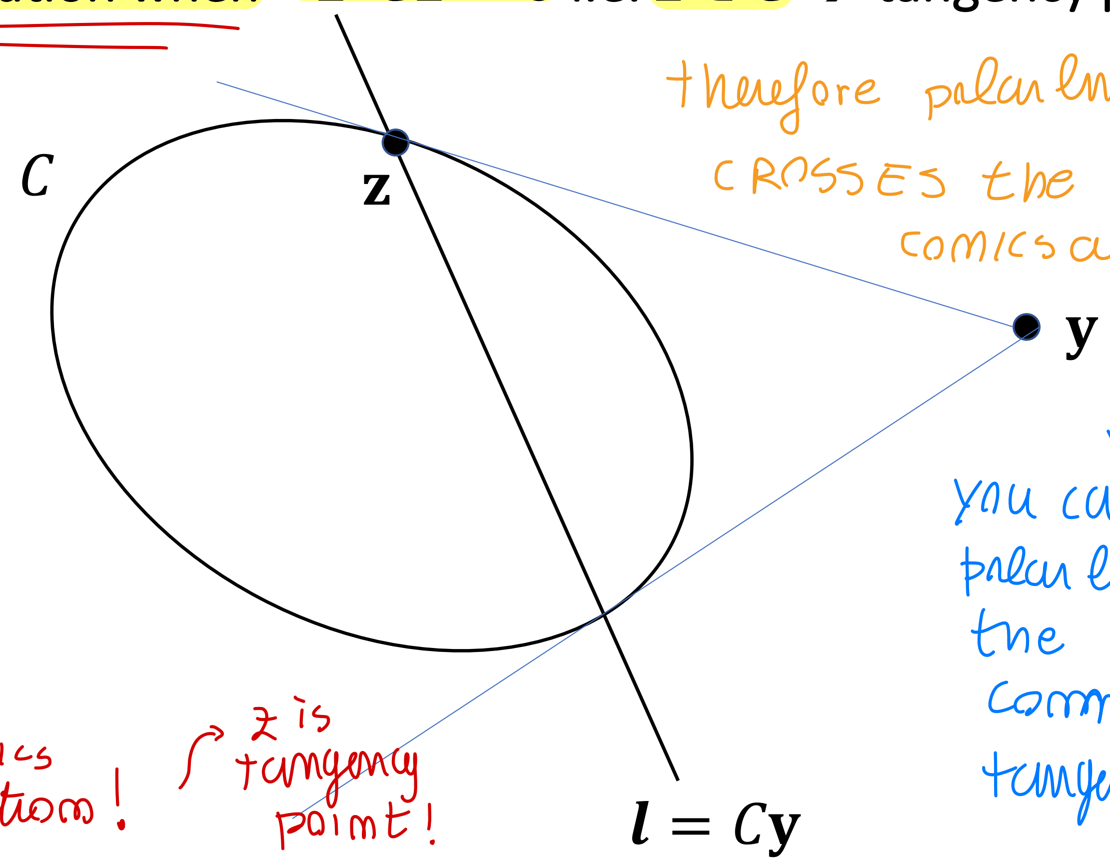
opposite values!

$\Updownarrow$

$z^T C z = 0$

$\Updownarrow$

$z \in C$ as conics equation!



z is tangency point!

therefore polar line CROSSES the conics at the tangency point

↓

you can draw polar lines from the point connecting two tangent points

The polar line of a point **ON** the conic C

02/10

as we defined the conics

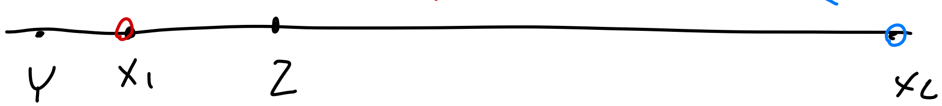
$x^T C x = 0$ with C symmm 3×3 MATRIX

and we see the CROSS-RATIO of 4-touple with value -1, is an HARMONIC 4-touple, as specific case when one point is @ ∞ and others 3 as $x_1\ x_2\ y \dots$ BUT NOT always the 4^{TH} point has to be at the infinity to be $CR = -1$!

we can have fourth point to approach the segment

this point has to move to preserve $CR = -1$

no more at the infinity



ratio tends to become $CR < -1 \dots$ to maintain $CR = -1$

also point x_1 has to move against x_2

The polar line of a point **ON** the conic C

what if $z \in C$

point on the conic... $\Rightarrow$

The polar line $\boldsymbol{l}_{\mathbf{z}} = C\mathbf{z}$ of a point the conic C is the tangent line to C through $\mathbf{z}$

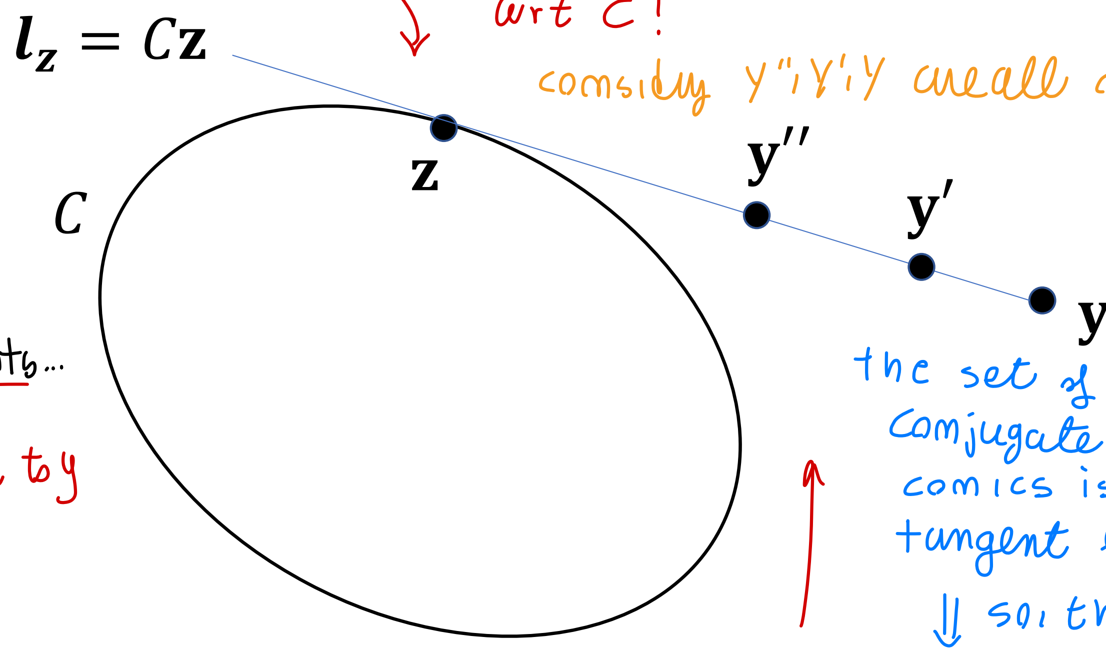
considering a line through y tangent to C in z... according to theorem, z, y are conjugate wrt intersection points...

now z is conjugate to y

how to find l_z polar line of z wrt C!

consider y'', y', y are all conjugate to z!

all as intersecty...



the set of the point conjugate of z wrt the conics is the whole tangent line!

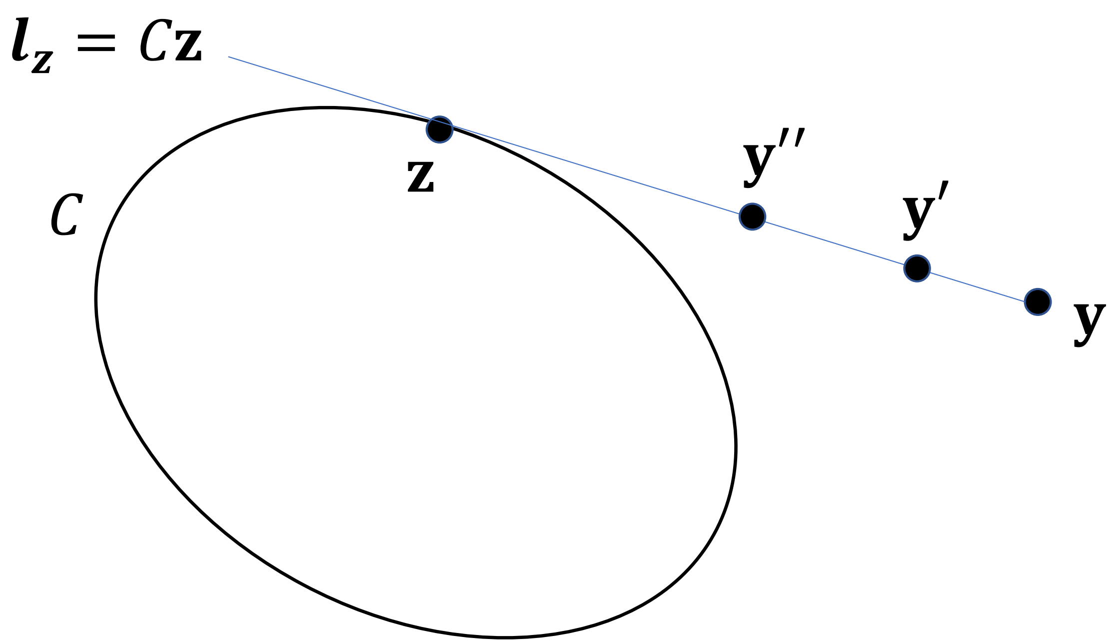
⇓ so, the polar of z wrt C is tangent line to C across z!

In fact, $\mathbf{z}$ on C is conjugate to any point $\mathbf{y}$ on the tangent to C through $\mathbf{z}$

The polar line $\boldsymbol{l_z} = C\mathbf{z}$ of a point the conic C is the tangent line to C through $\mathbf{z}$

being polar line
as locus of points
conjugate...
set of all
points conjugate to z!

important
Result!



In fact, $\mathbf{z}$ on C is conjugate to any point $\mathbf{y}$ on the tangent to C through $\mathbf{z}$

Examples: the polar line with respect to a circumference

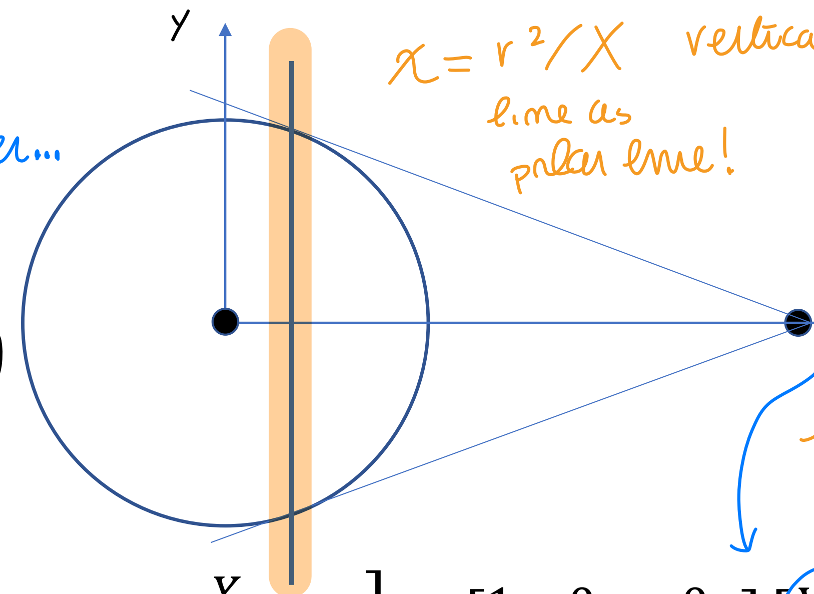
the polar of a point $\mathbf{y}$ wrt a circumference is a line perpendicular to the segment (*center*, $\mathbf{y}$)

Simple case: center on the origin, and point $\mathbf{y}$ on the X-axis

chosing an easy ref frame fixed in the circumference center...

$x = r^2/X$ vertical line as polar line!

cartesian, being $w = 1$



$$\mathbf{y} = \begin{bmatrix} \boldsymbol{X} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

homogeneus with $w=1$ is s.t. $\begin{cases} x = x \\ 0 = y \end{cases}$

by definition... (origin centered) $\begin{cases} X_o = Y_o = 0 \\ r \end{cases}$

polar line: $\boldsymbol{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ -r^2 \end{bmatrix}$

this is a line

$Xx + 0y - r^2 w = 0$ ($w=1$, CARTESIAN)

the polar line (cartesian) equation X X $-r^2 = 0$ → X $= r^2/X$: a vertical line

the polar of a point $\mathbf{y}$ wrt a circumference is a line perpendicular to the segment (*center*, $\mathbf{y}$)

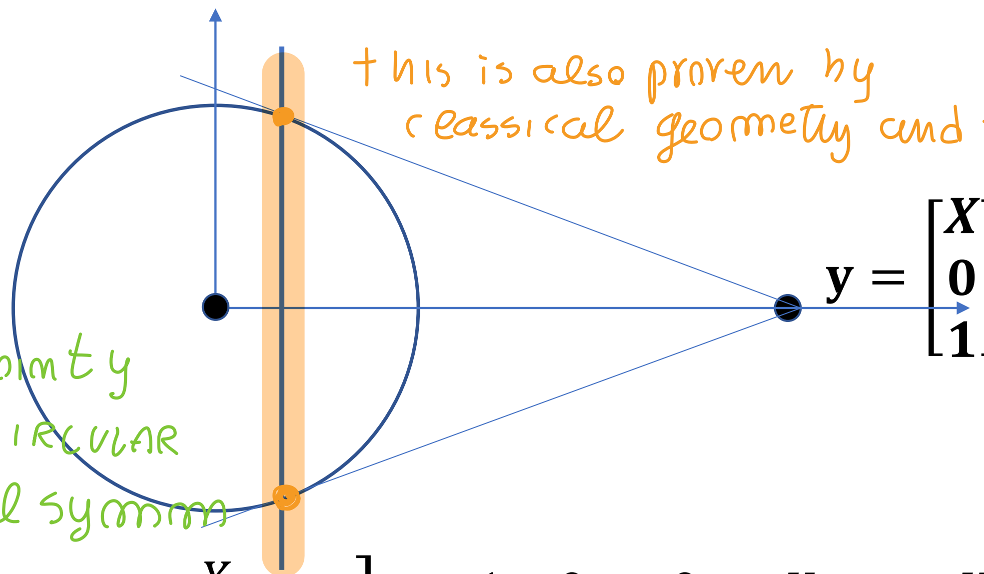
Simple case: center on the origin, and point $\mathbf{y}$ on the X-axis

(when $r=0$ or $X=0$)

....

this is also proven by classical geometry and trigonometry

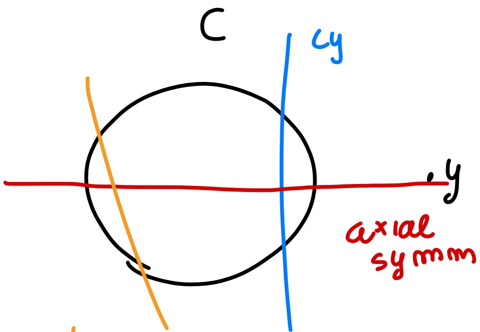
CIRCLE in 2D this is circular symm! in 3D spherical symm... adding the point y on the space you lose CIRCULAR SYMM, you preserve axial symm



$$\mathbf{y} = \begin{bmatrix} \boldsymbol{X} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

polar line: $$\boldsymbol{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ -r^2 \end{bmatrix}$$

the polar line (cartesian) equation X X $-r^2 = 0$ → X = r^2/X : a vertical line starting from 2 elements enjoying certain symmm, every construction will enjoy same symmm



any element from this C, y
↓
C·y construction which will enjoy same symmm of the two elements system

this could NOT be polar line! NO symmm wrt axis of symmm!

↑ this still enjoy axial symmm wrt that axis!

MUST be VERTICAL LINE Cy to maintain symmm

↳ polar must be symmm wrt symmm axis
↓
so ORTOGONAL to it!

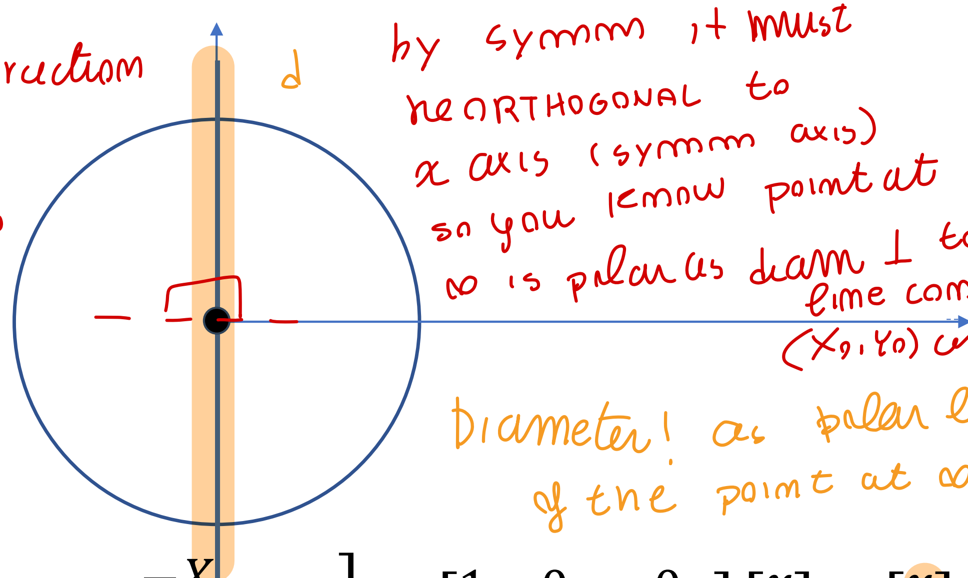
the polar of a point ***at the*** ∞ wrt a circumference

Particular case: point $\mathbf{y}$ at the ∞

It allow to solve many Reconstruction problems in man made environments full of circles

(W = 0) point at ∞ along X axis

$$\mathbf{y} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$



d

by symm it must be ORTHOGONAL to x axis (symm axis) so you know point at ∞ is polar as diam ⊥ to line connects (X_0, Y_0) with P_∞ point at ∞

Diameter! as polar line of the point at ∞ wrt circumference

explicitally

polar line: $\boldsymbol{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$

the polar line (cartesian) equation $X = 0$ is the **diameter** ⊥ direction of the point $\mathbf{y}$

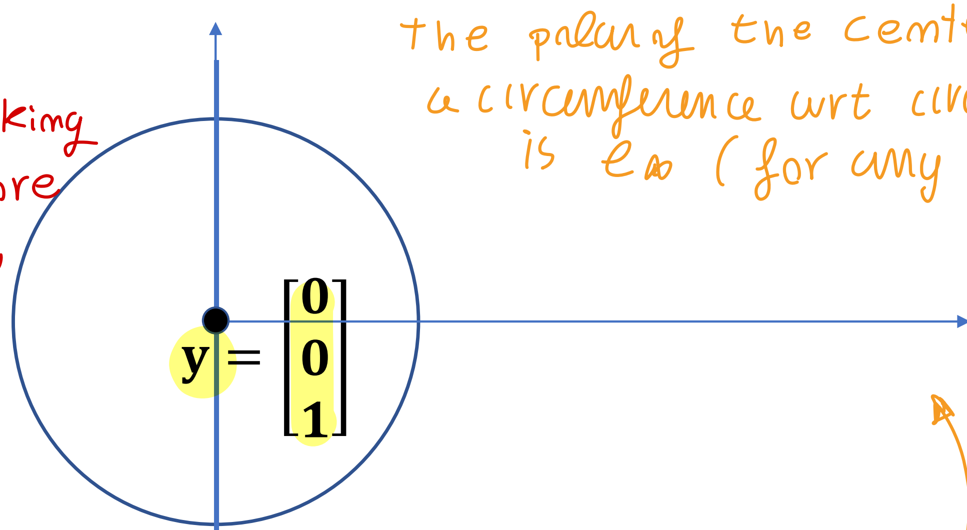
the polar line of the center of a circumference

if you want to compute polar line taking internal points! No more tangency geometrical reconstruction...

Particular case: point $\mathbf{y}$ is the origin

the polar of the center of a circumference wrt circumference is ℓ_∞ (for any circumference)

$\boldsymbol{l}_\infty$



polar line: $\boldsymbol{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r^2 \end{bmatrix}$

the polar line equation is $-r^2 w = 0$ i.e. $w = 0$ i.e. the line at the infinity $\boldsymbol{l}_\infty$

← the fact that for $y = (x_0\ y_0)$ $\quad \ell = C y = \ell_\infty$ is expected by definition!

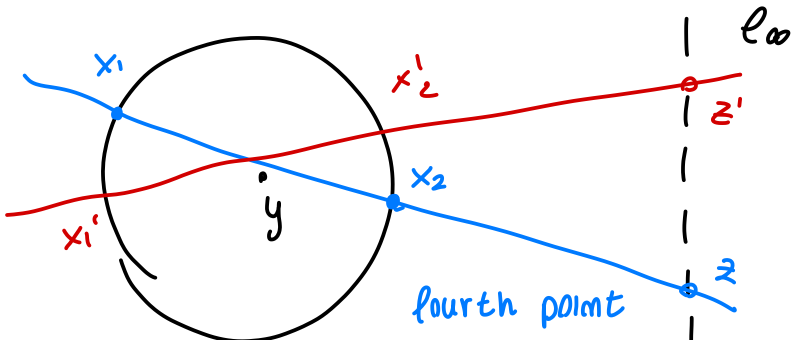
↓

in fact from the $\underbrace{\text{Algebraic proof}}$ of POLARITY...

↓

as Cy OR from $\underbrace{\text{THEOREM}}$

polar line of the point $y = (x_0\ y_0)$ is set of point which are conjugate to y wrt circumference C...



to hold all points z conjugate to Y wrt to CIRCUMFERENCE

↓

conjugate means that ratio is -1 (CROSS RATIO) CR

fourth point s.t $CR = -1$ as proven previously, when y is mid point of $x_1 x_2$... than fourth point is point at ∞ to be $CR = -1$!

⇐

then valid $\forall\ \ell_{x_1 x_2 \dots}$ the locus ℓ of points z s.t conjugate to y center is ℓ_∞!

Degenerate conics

we consider up to now non degenerate conics as the one with C NON SINGULAR

Degenerate conics

Conics: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}$ if it satisfies a homogeneous *quadratic* equation, namely $\mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{0}$, where $\boldsymbol{C}$ is a 3x3 symmetric matrix.

ellepses / parabola / hyperbole

det(C) ≠ 0!

→ C is NON SING!

Nondegenerate conics: matrix $\boldsymbol{C}$ is nonsingular, i.e. $|\boldsymbol{C}| \neq 0, \text{rank}(\boldsymbol{C}) = 3$

maximum rank matrix!

Degenerate conics: matrix $\boldsymbol{C}$ is singular, i.e. $|\boldsymbol{C}| = 0$, $\text{rank}(\boldsymbol{C}) < 3$

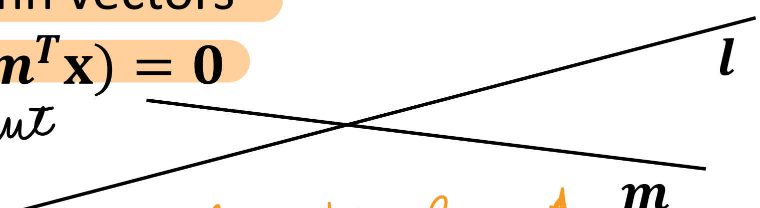
you lose rank when degenerate! singular C

↓ rank(C) = 2 OR rank(C) = 1

two cases: in 3×3 MATRIX with rank=2, you can write it as combination

- rank $(C) = 2$, $C = \boldsymbol{lm}^T + \boldsymbol{ml}^T$ with $\boldsymbol{l}$ and $\boldsymbol{m}$ are column vectors

$\rightarrow$ $\mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{x}^T\left(\boldsymbol{lm}^T + \boldsymbol{ml}^T\right)\mathbf{x} = 2\mathbf{x}^T \boldsymbol{lm}^T \mathbf{x} = \mathbf{2}(\mathbf{x}^T\boldsymbol{l})(\boldsymbol{m}^T\mathbf{x}) = \mathbf{0}$



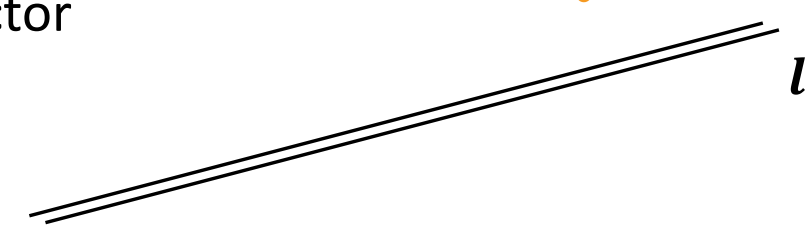
$\mathbf{x}^T\boldsymbol{l} = \mathbf{0}$ OR $\boldsymbol{m}^T\mathbf{x} = \mathbf{0}$ → with l, m independent columns...

set of point x on l   set of point x on m   x ∈ l ∨ m → UNION set of two line

- rank $(C) = 1$, $C = \boldsymbol{ll}^T$ with $\boldsymbol{l}$ column vector

$\rightarrow$ $\mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{x}^T \boldsymbol{ll}^T \mathbf{x} = (\mathbf{x}^T\boldsymbol{l})(\boldsymbol{l}^T\mathbf{x}) = \mathbf{0}$

$\mathbf{x}^T\boldsymbol{l} = \mathbf{0}$ counted 2 times



Degenerate conics

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NON degenerate, CURVE

Nondegenerate conics: matrix $\boldsymbol{C}$ is nonsingular, i.e. $|\boldsymbol{C}| \neq 0$, $\text{rank}(\boldsymbol{C}) = 3$

Degenerate conics: matrix $\boldsymbol{C}$ is singular, i.e. $|\boldsymbol{C}| = 0$, $\text{rank}(\boldsymbol{C}) < 3$

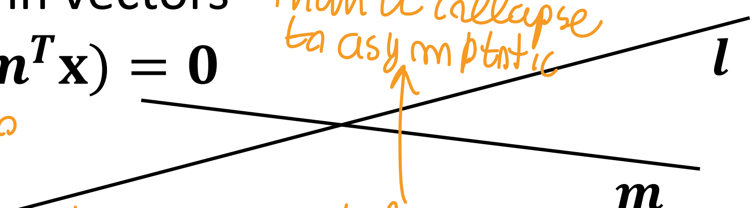
rank(C) = 3 ↓ rank (C') = 2 change C to C' than it collapse to asymptotic

two cases:

- rank $(C) = 2$, $C = \boldsymbol{l}\boldsymbol{m}^T + \boldsymbol{m}\boldsymbol{l}^T$ with $\boldsymbol{l}$ and $\boldsymbol{m}$ are column vectors

$\rightarrow\ \mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{x}^T(\boldsymbol{l}\boldsymbol{m}^T + \boldsymbol{m}\boldsymbol{l}^T)\mathbf{x} = \mathbf{2}\mathbf{x}^T \boldsymbol{l}\boldsymbol{m}^T\mathbf{x} = \mathbf{2}(\mathbf{x}^T\boldsymbol{l})(\boldsymbol{m}^T\mathbf{x}) = \mathbf{0}$

$\mathbf{x}^T\boldsymbol{l} = \mathbf{0}$ OR $\boldsymbol{m}^T\mathbf{x} = \mathbf{0}$ → this is similar to the two asymptotic lines of an hyperbolae! → if you modify C hyperbole s.t

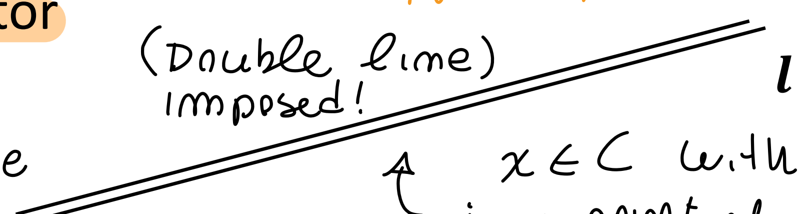


- rank $(C) = 1$, $C = \boldsymbol{l}\boldsymbol{l}^T$ with $\boldsymbol{l}$ column vector

$\rightarrow \mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{x}^T \boldsymbol{l}\boldsymbol{l}^T\mathbf{x} = \underbrace{(\mathbf{x}^T\boldsymbol{l})(\boldsymbol{l}^T\mathbf{x})} = \mathbf{0}$

$\mathbf{x}^T\boldsymbol{l} = \mathbf{0}$ counted 2 times ⇒ same line

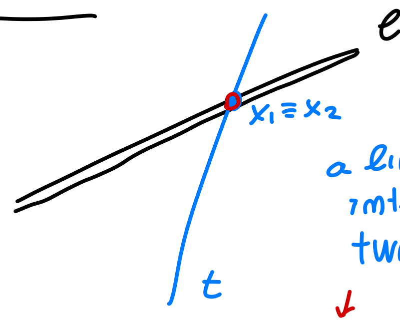
(Double line) imposed!



x ∈ C with C of rank=1 is a point of l doubled

this conic C with $\text{rank}(C) = 1$
is a <u>DOUBLE LINE</u>

(NOT JUST one line)



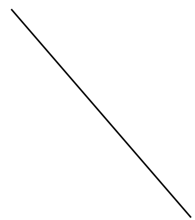
double line conic!

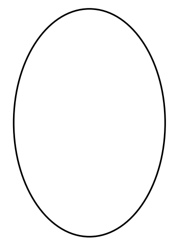
a line t intersect C two times

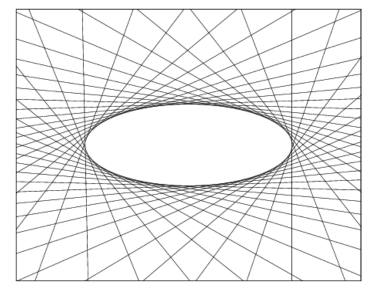
everytime you cross a conic with a line you MUST have 2 solutions

Planar Projective Geometry

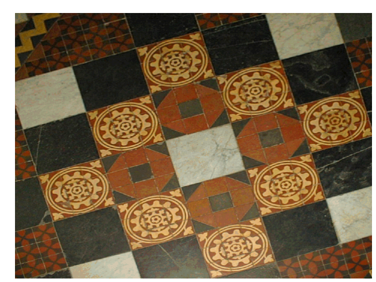
- **Elements**
- Points
- Lines
- Conics
- **Dual conics**







- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



DUAL CONICS

↓

Overlaps conics and Duality concept!

let's model DUALITY...

Conics: a conic is a set of points $\mathbf{x}$ that satisfy equation

$$\mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{0}$$

where $\boldsymbol{C}$ is a 3x3 symmetric matrix.

its dual: (point $\rightarrow$ lines) DUALITY

Dual conics: a *dual conic* is a set of *lines* $\boldsymbol{l}$ that satisfy equation

QUADRATIC set of lines!

$$\boldsymbol{l}^T \boldsymbol{C}^* \boldsymbol{l} = \mathbf{0}$$

where $\boldsymbol{C}^*$ is a 3x3 symmetric matrix.

difficult to draws! it covers all the space... $\downarrow$ SET OF LINES...

$\downarrow$ a set of lines which satisfy quadratic equation (set of points $\rightarrow$ set of lines (DUALITY))

↳ to understand what happens

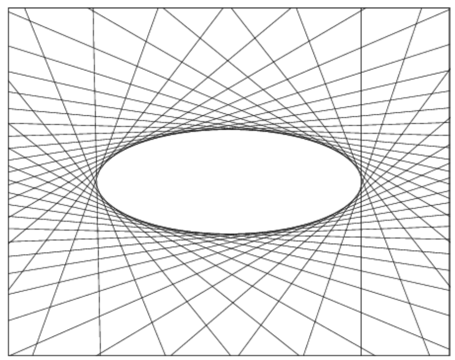
Nondegenerate dual conics: a nondegenerate dual conic is a dual conic whose matrix $\boldsymbol{C}^*$ is NONSINGULAR, i.e., . $|\boldsymbol{C}^*| \neq 0$, $\mathrm{rank}(\boldsymbol{C}^*) = 3$

⇓

you take a conic C, than all possible eimes tangent at different points...

↓

dual conic!



geometrically, it can be proven that...

valid for NON DEGENERATE dual conics C^* and conic C

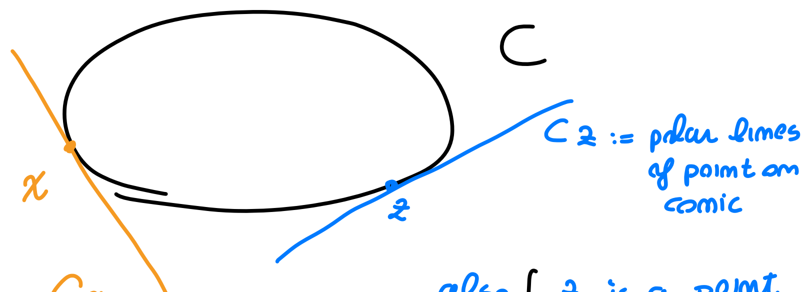
A nondegenerate dual conic $\boldsymbol{C}^*$ is the set of lines TANGENT to a nondegenerate conic $\boldsymbol{C}$

What is the relation between matrixes $\boldsymbol{C}^*$ and $\boldsymbol{C}$?

$$\boxed{\boldsymbol{C}^* = \boldsymbol{C}^{-1}}$$

taking lines ℓ tangent to conic C, than

we can describe it as POLARS of points on the conics



also $\begin{cases} z \text{ is a point } z \in C \\ x \in C \end{cases}$

$x^T C x = 0$ (because $x \in C$)

We consider set of lines ℓ POLAR to points, tangent...

$\ell^T C^* \ell = 0$

(WRONG PROOF)

C IS NOT singular, to consider all LINES tangent to the conics $x^T \ell = 0$ to be line ℓ polar

$x^T C x = 0$ to solve this...

because NON DEGENERATE C^{-1} exists!

$\ell = C x \longrightarrow x = C^{-1}\ell \rightarrow x^T = \ell^T C^{-1}$

$(C^{-1})^T = C^{-1}$

$(C^{-1}\ell)^T C (C^{-1}\ell) =$

$= \ell^T \underbrace{(C^{-1})^T C C^{-1}}_{I} \ell = \ell \underbrace{C^{-1}}_{\Downarrow} \ell$

We constraint ℓ as set of line tangent to C

$x = \ell C^* \ell$ is set of points tangent to C !

we get a quadratic equation in $\ell^T C^{-1} \ell$ → then C^{-1} represent dual conic

as dual conic

quadratic equation in ℓ !

$$C^* = C^{-1}$$

as can be proven...

NON DEG Dual conic, as set of lines tangent to the conic s.t $C = (C^*)^{-1}$

so it can be drawn!

degenerate dual conics

we define it using DUALITY...

Degenerate dual conics

very ABSTRACT elements...

Dual Conics: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}^*$ if it satisfies a homogeneous *quadratic* equation, namely $\boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \mathbf{0}$, where $\boldsymbol{C}^*$ is a 3x3 symmetric matrix.

Nondegenerate conics: matrix $\boldsymbol{C}^*$ is nonsingular, i.e. $|\ \boldsymbol{C}^*| \neq 0$, rank $\boldsymbol{C}^* = 3$

Degenerate conics: matrix $\boldsymbol{C}^*$ is singular, i.e. $|\ \boldsymbol{C}^*| = 0$, rank $\boldsymbol{C}^* < 3$

(same procedure as before...)

two cases:

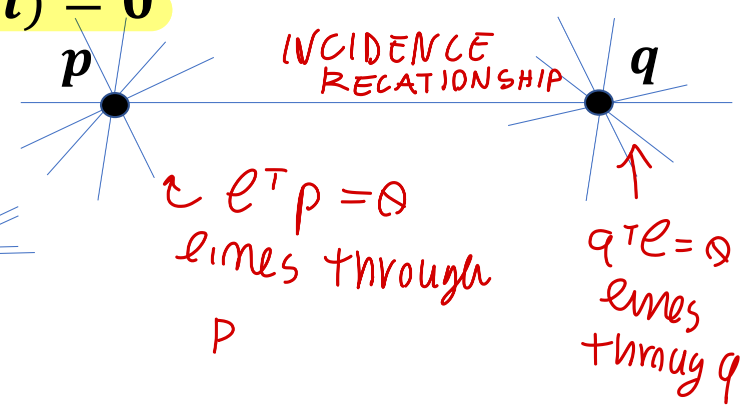
- rank $\boldsymbol{C}^* = 2$, $\boldsymbol{C}^* = \boldsymbol{p}\boldsymbol{q}^T + \boldsymbol{q}\boldsymbol{p}^T$ with $\boldsymbol{p}$ and $\boldsymbol{q}$ are column vectors

p, q are constant coming from them, l is variable

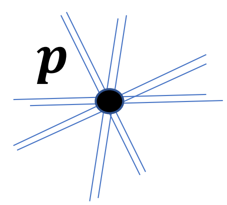
$\rightarrow \boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \boldsymbol{l}^T(\boldsymbol{p}\boldsymbol{q}^T + \boldsymbol{q}\boldsymbol{p}^T)\boldsymbol{l} = \mathbf{2}\boldsymbol{l}^T\boldsymbol{p}\boldsymbol{q}^T\boldsymbol{l} = \mathbf{2}(\boldsymbol{l}^T\boldsymbol{p})(\boldsymbol{q}^T\boldsymbol{l}) = \mathbf{0}$

$\boldsymbol{l}^T\boldsymbol{p} = \mathbf{0}$ OR $\boldsymbol{q}^T\boldsymbol{l} = \mathbf{0}$ p, q fixed

linear equations, as set of lines



- rank $\boldsymbol{C}^* = 1$, $\boldsymbol{C}^* = \boldsymbol{p}\boldsymbol{p}^T$ with $\boldsymbol{p}$ column vector



$\rightarrow \boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \boldsymbol{l}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{l} = (\boldsymbol{l}^T\boldsymbol{p})(\boldsymbol{p}^T\boldsymbol{l}) = \mathbf{0}$

$\boldsymbol{l}^T\boldsymbol{p} = \mathbf{0}$ counted 2 times

Degenerate dual conics

Dual Conics: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}^*$ if it satisfies a homogeneous *quadratic* equation, namely $\boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \mathbf{0}$, where $\boldsymbol{C}^*$ is a 3x3 symmetric matrix.

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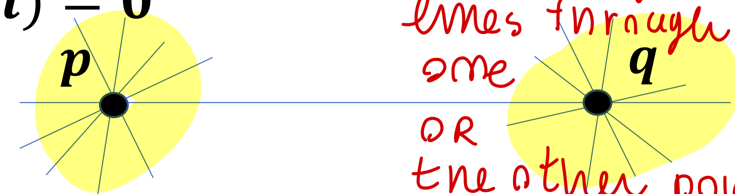
dual of conic, as the other was set of points on two lines,
↕ here is set of lines through one
OR
the other point
(DUAL!)

two cases:

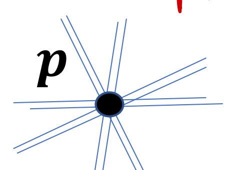
- rank $\boldsymbol{C}^* = 2$, $\boldsymbol{C}^* = \boldsymbol{p}\boldsymbol{q}^T + \boldsymbol{q}\boldsymbol{p}^T$ with $\boldsymbol{p}$ and $\boldsymbol{q}$ are column vectors

$\rightarrow \boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \boldsymbol{l}^T(\boldsymbol{p}\boldsymbol{q}^T + \boldsymbol{q}\boldsymbol{p}^T)\boldsymbol{l} = \mathbf{2}\boldsymbol{l}^T\boldsymbol{p}\boldsymbol{q}^T\boldsymbol{l} = \mathbf{2}(\boldsymbol{l}^T\boldsymbol{p})(\boldsymbol{q}^T\boldsymbol{l}) = \mathbf{0}$

$\boldsymbol{l}^T\boldsymbol{p} = \mathbf{0}$ OR $\boldsymbol{q}^T\boldsymbol{l} = \mathbf{0}$ → union set of lines concurrent at p and q



- rank $\boldsymbol{C}^* = 1$, $\boldsymbol{C}^* = \boldsymbol{p}\boldsymbol{p}^T$ with $\boldsymbol{p}$ column vector



$\rightarrow \boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \boldsymbol{l}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{l} = (\boldsymbol{l}^T\boldsymbol{p})(\boldsymbol{p}^T\boldsymbol{l}) = \mathbf{0}$

$\boldsymbol{l}^T\boldsymbol{p} = \mathbf{0}$ counted 2 times → lines going through same point doubled

A special dual conic

← When rank $(C^*) = 2$, you could write $C^* = pq^T + qp^T$

If we use instead of p, q the two $\underbrace{\text{CIRCULAR POINTS}}$

↓

a construction from special points give you NEW special points! ← you get a special dual-conics

(CIRCULAR POINTS = intersection between ANY CIRCUMFERENCE and ℓ_∞) ← strange points with $\mathbb{C}$ coordinates (same $\forall$ C) every circumference contains this points at ∞

usefull degenerate Dual conic

The (degenerate) conic-dual to the circular points

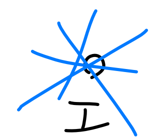
what is the degenerate dual conic $\boldsymbol{C}^{*} = \boldsymbol{p}\boldsymbol{q}^{T} + \boldsymbol{q}\boldsymbol{p}^{T}$ when points $\boldsymbol{p}$ and $\boldsymbol{q}$ are the circular points?

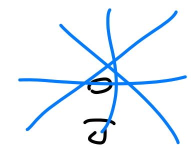
$$I = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad (I \text{ point at } \infty)$$

$$J = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

degenerate dual conics With I, J

$$\boldsymbol{C}^{*}_{\infty} = \boldsymbol{I}\boldsymbol{J}^{T} + \boldsymbol{J}\boldsymbol{I}^{T} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$





{special diagonal matrix}

as union of set of lines passing throug I, J

$\boldsymbol{C}^{*}_{\infty}$ is called the conic dual to the circular points: it will be useful in the 2D reconstruction of planar scenes (or, **image rectification**)

(dual conic, degenerate dual of rank 2)

usefull because it is equivalent to circular points (usefull in 2D reconstruction also), this is an alternative way to carry same INFORMATION of CIRCULARPOINTS in this unique entity

Angle between two lines via the conic $\boldsymbol{C}_{\infty}^{*}$ dual to the circular points

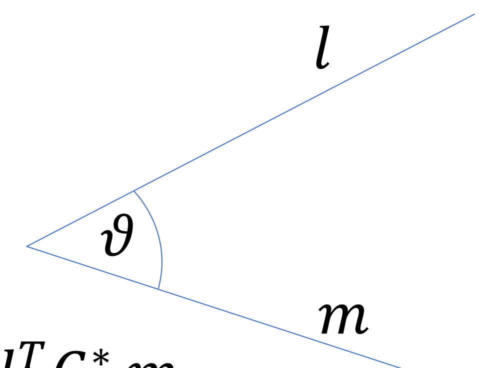
Angle between two lines l and m

↓ to compute angle...
using $\cos(\vartheta)$

The angle ϑ between $l = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $m = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is the angle between their normals $[a_1 \quad b_1]$ and $[a_2 \quad b_2]$:

angle ϑ
depends on lines
ORIENTATION... accounted
by a, b (NOT w), c_i takes into accont
distance from origin of line!

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1{}^2 + b_1{}^2)(a_2{}^2 + b_2{}^2)}}$$



But, e.g., $a_1 a_2 + b_1 b_2 = [a_1 \quad b_1 \quad c_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = l^T C_\infty^* m$

$$\rightarrow \cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

[ai, bi] is the direction NORMAL to the line
while ci accounts for distance between line / ORIGIN!

Angle between two lines l and m

even in homogeneous coordinate a_i, b_i accounts for the DIRECTION of line l_i

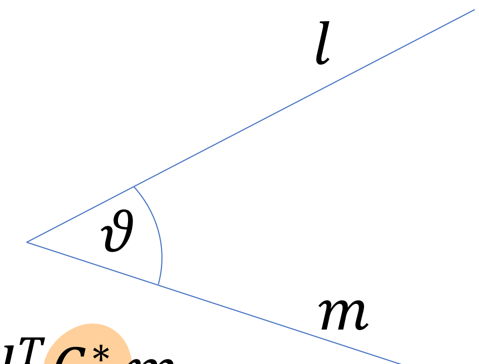
The angle ϑ between $l = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $m = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is the angle between their normals $\underbrace{[a_1 \quad b_1]}$ and $\underbrace{[a_2 \quad b_2]}$:

orientation fully determined by those

$\cos\vartheta = \dfrac{l \cdot m}{|l||m|}$ (cos of angle in between)

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}}$$

$\frac{b_2 - b_1}{a_2 - a_1}$ represent tangent of inclination of NORMAL...



But, e.g., $a_1 a_2 + b_1 b_2 = [a_1 \quad b_1 \quad c_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = l^T C_\infty^* m$

using also the third element c_i

! It is the conic dual to CIRCULAR points!

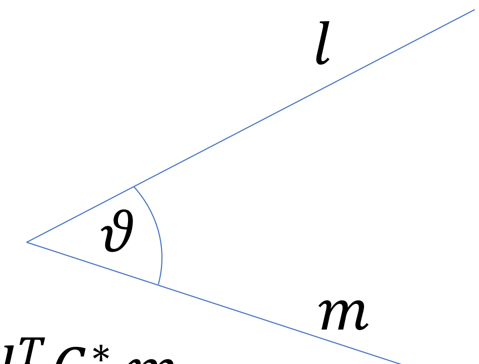
$$\begin{cases} {a_1}^2 + {b_1}^2 = l^T C_\infty^* l \\ {a_2}^2 + {b_2}^2 = m^T C_\infty^* m \end{cases}$$

$$\rightarrow \cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

Angle between two lines l and m

The angle ϑ between $l = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $m = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is the angle between their normals $[a_1 \quad b_1]$ and $[a_2 \quad b_2]$:

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}}$$



this formula is almost the same when we move from real line and the image plane...

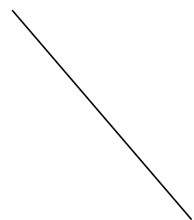
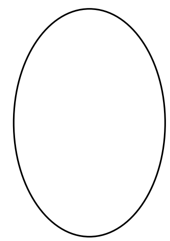
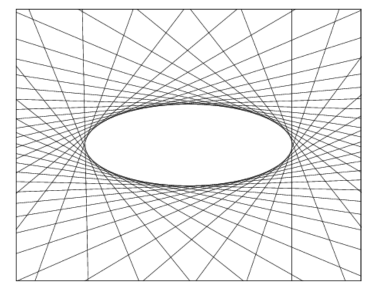
But, e.g., $a_1 a_2 + b_1 b_2 = [a_1 \quad b_1 \quad c_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = l^T C_\infty^* m$

this is very usefull when considering the RELATIONSHIP between the Real lines and their images!

$$\rightarrow \cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

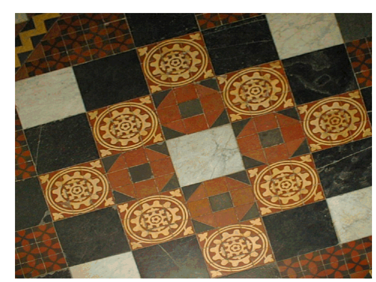
Planar Projective Geometry

- **Elements**
- Points
- Lines
- Conics
- Dual conics

between all this elements!

- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities


08/10

we wanna consider the transformation between objects in planar geometry framework. Attention to

Projective mappings

it is important ← to study projection between planar scene and the image, to do the opposite RECONSTRUCTION (image data → planar scene) ← RECONSTRUCTION is a crucial part..

image projection between planar scene and image is example of it!

(TRANSFORMATION)

Def. *A **projective mapping** between a projective plane* $\mathbb{P}^2$ *and an other projective plane* $\mathbb{P}'^2$ *is an **invertible** mapping which preserves colinearity:*

ALWAYS INVERSE EXIST

(the same or a different one)

$$h: \mathbb{P}^2 \rightarrow \mathbb{P}'^2, \mathbf{x}' = h(\mathbf{x}), \ \mathbf{x}_1, \ \mathbf{x}_2, \mathbf{x}_3 \text{ are colinear}$$

$$\leftrightarrow$$

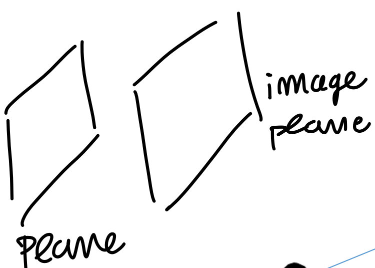
↓ preserves colinearity

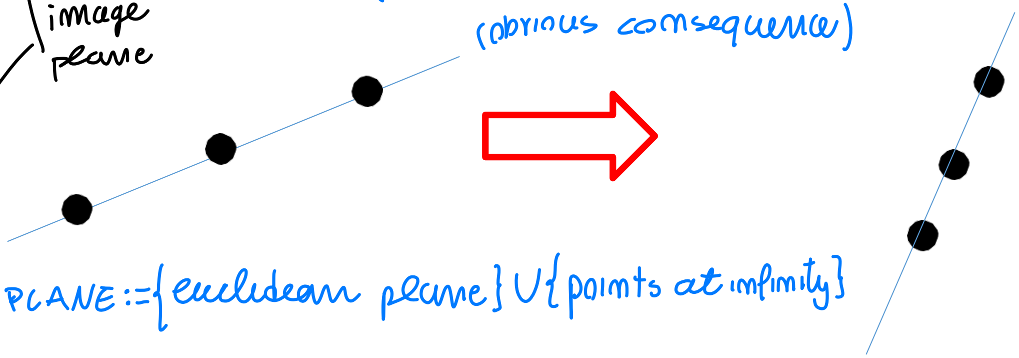
$$\mathbf{x}'_1 = h(\mathbf{x}_1), \ \mathbf{x}'_2 = h(\mathbf{x}_2), \mathbf{x}'_3 = h(\mathbf{x}_3) \text{ are colinear}$$

Whenever we have 3 CO-LINEAR points, the mapping of those points transform it in 3 new points still colinear

in our case, after we go from a PLANAR scene to its IMAGE

Then Also h^{-1} is a PROJECTIVE MAPPING (obvious consequence)





PROJECTIVE PLANE := {euclidean plane} ∪ {points at infinity}

Alternative names:

- *Projectivity*
- *Homography*

IF $h(\cdot)$ is a PROJECTIVE MAPPING, then also $h^{-1}(\cdot)$ is PROJECTIVE (as direct consequence of the definition)

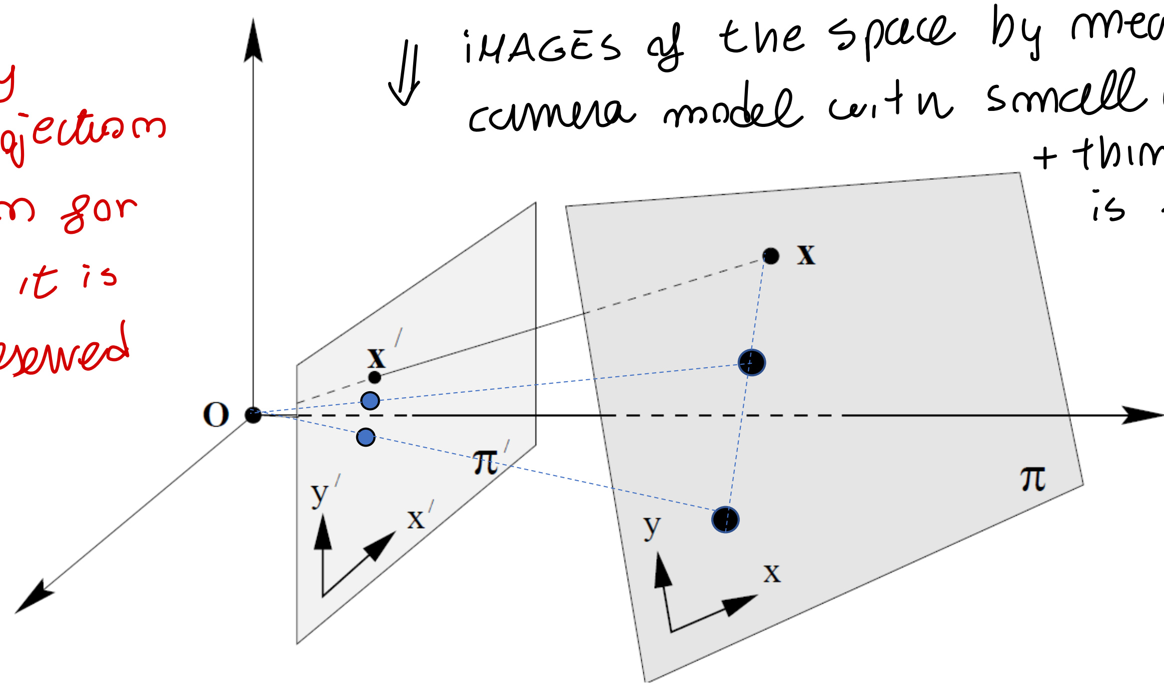
also called "HOMOGRAPHY" / "PROJECTIVITY"

Examples of projective mappings

Mappings between two planes induced by central projection are projective, since they preserve colinearity

the image device (camera) works by apply central projection of the scene... then for the definition itself it is s.t colinearity is preserved

⇓ IMAGES of the space by means of camera model with small angles + thin lenses is PROJECTIVE



Examples of projective mappings

Mappings between two planes induced by central projection are projective, since they preserve colinearity

because there is an unique plane containing the CENTER of projection and the three points.

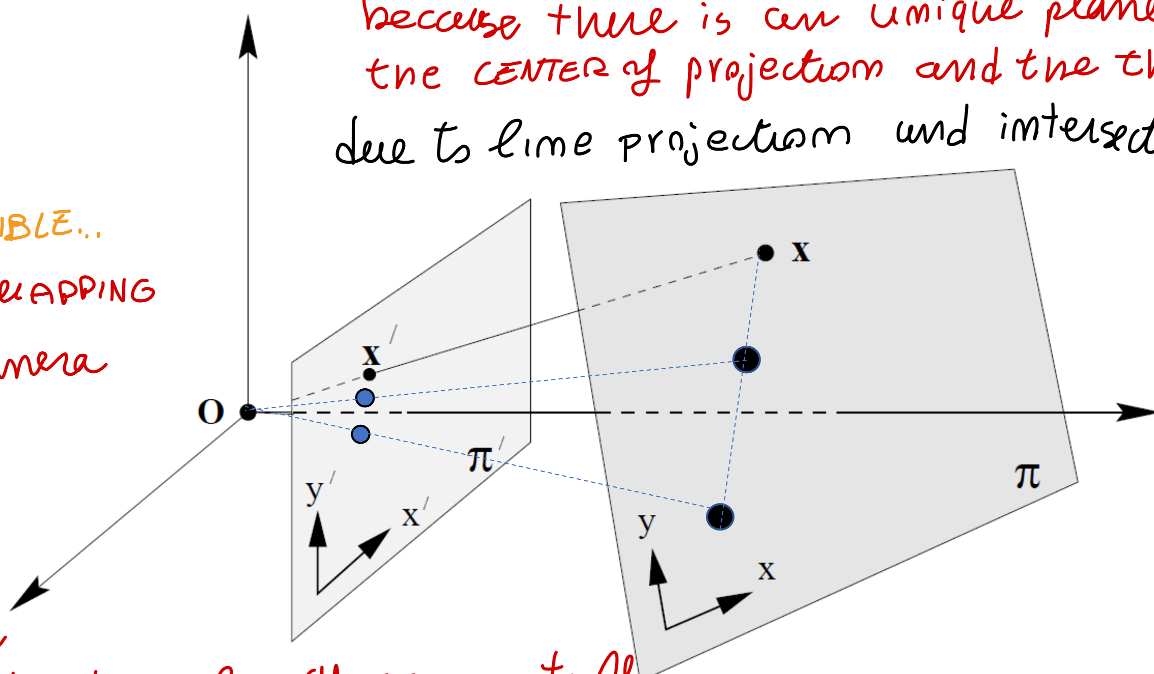
due to line projection and intersection reasons...

also images is COLINEAR

And it is INVERTIBLE...

We lose PROJECTIVE MAPPING when DEGENERATE camera plane goes to ∞

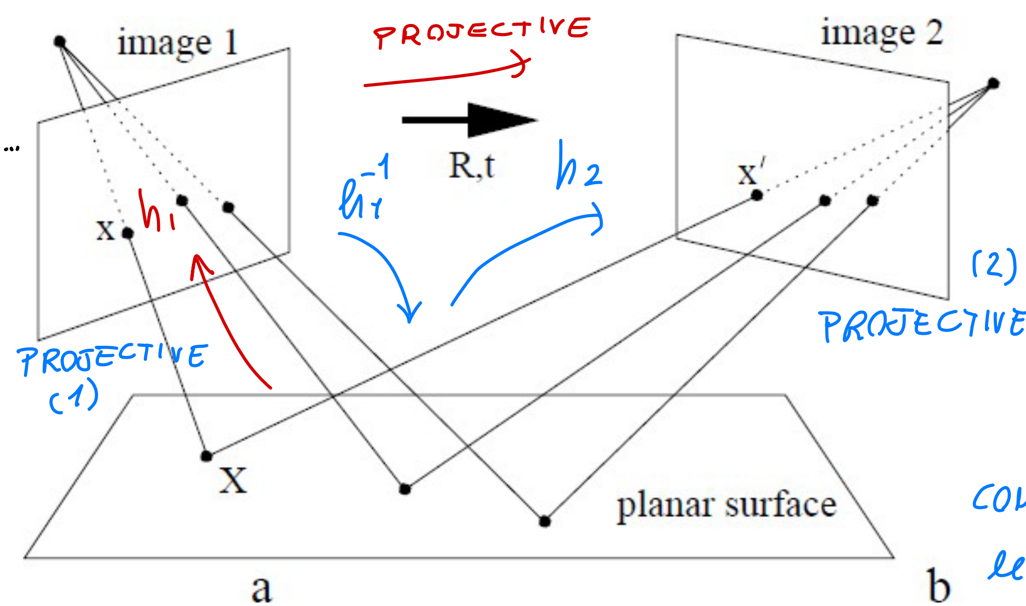
but a phisical camera nominal... the image is projectively related to the planar scene itself



Examples of projective mappings

Mapping between two images of a planar scene is a homography a composition of the 1° image → scene homography and the scene → 2° image homography ⇨ it is a **homography**

planar scene taken by two cameras... the 2 images are related by projective mapping



as composition of this two mappings as $h_1^{-1} * h_2$

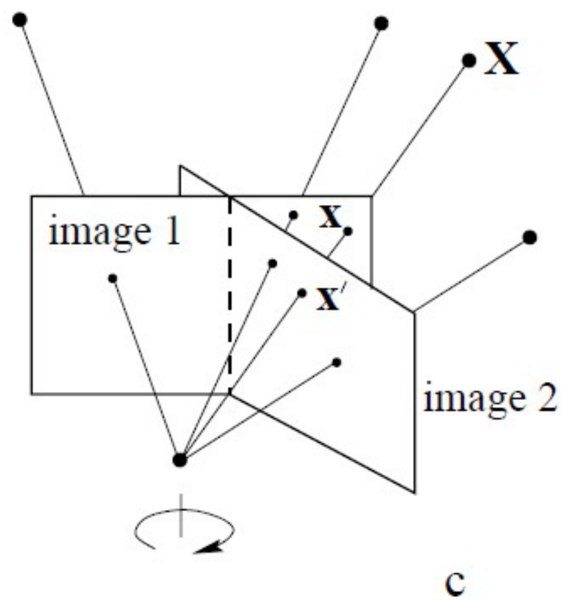
Combination as COMPOSITION of 2 PROJ mapping are still a PROJ - MAPPING

the two separate image are related by projective mapping

Examples of projective mappings

Two images of a 3D scene, taken by a camera rotating around its center are related by a **homography**, since the 2° image can be regarded as a central projection of the 1° image

important example... two images of 3D scene of points on different planes (3D set of POINTS)



take 2 images of set of points such that these two images have same lense center... FOR example with camera on same position moving view with same center! with for example eye moving a BIT rotating to observe scene

Examples of projective mappings

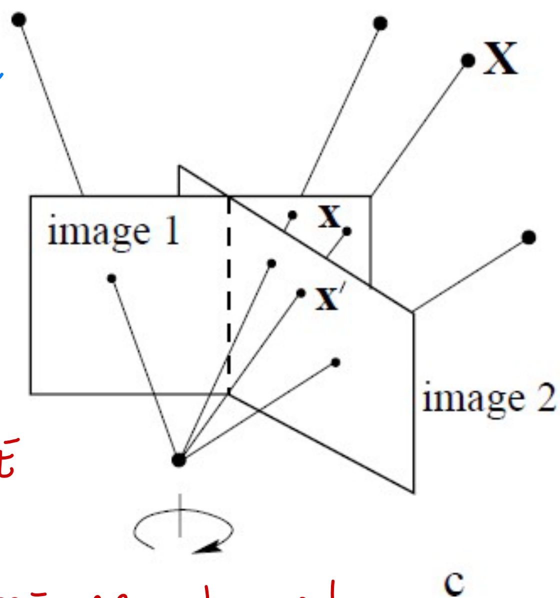
Two images of a 3D scene, taken by a camera rotating around its center are related by a **homography**, since the 2° image can be regarded as a central projection of the 1° image

still PROJ MAPPING, because relation between two images are like

1) "image" (PLANAR)
2) "image of the image"

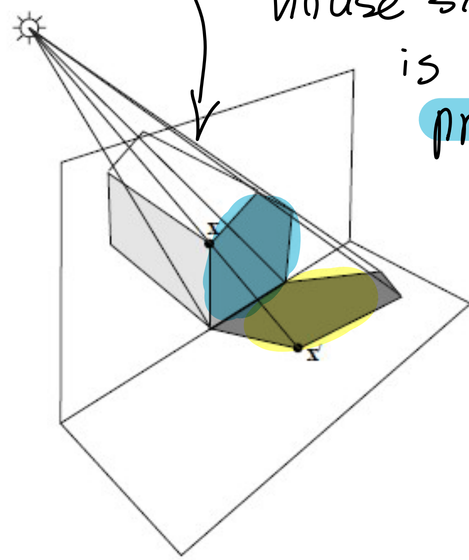
relationship in between is PROJECTIVE

I remove the same lense, CENTER of PROJ maintained



Examples of projective mappings

The shadow cast by a **planar** silhouette onto a **ground plane** is a projective transformation of the planar silhouette, since they are related by a central projection



house silhouette shadow
is a PROJ TRANSFORMATION,
projectively related to
the silhoutte

(both planar,
PROJECIVE MAPPING)

Fundamental Theorem of Projective Geometry

Theorem: *A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}'^2$ is projective if and only if there exists an invertible 3x3 matrix H such that for any point in $\mathbb{P}^2$ represented by the vector $\mathbf{x}$, is $h(\mathbf{x}) = H\,\mathbf{x}$*

Whenever we have a PROJECTIVE TRANSFORM. we can express $h(\cdot)$ as linear by H

$h(x) = Hx$ linear in HOMOGENEUS COORDINATES

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

this is very important because it simplify a lot how we can compute it

i.e. projective mappings are LINEAR in the homogeneous coordinates

(they are not linear in cartesian coordinates)

PROJ MAPPING

maybe NON LIN in cartesian, as seen from CAMERA device description (before)

$X = x\frac{f}{z}$ NOT linear in cartesian!

in HOMOGENEUS coordinates the mappings become linear Hx

this represents the HOMOGRAPHY

then ensure that conics and transformed points of conics $x \in C$ is given by $Hx = x'$

NOT function of x, constant MATRIX

← this can be PROVEN, one direction of theorem is easy to prove... IFF (the sufficent / necessary condition)

- supposing that $h(\cdot)$ is PROJECTIVE, so preserve colinearity...

IF I apply it to x_1, x_2, x_3 co-linear points

$x_3 = \alpha x_1 + \beta x_2$ than mapping as Hx does it preserve co-linearity?

Hx_1, Hx_2, Hx still co-linear? yes!

$$Hx_3 = H\alpha x_1 + H\beta x_2 = \alpha \underbrace{(Hx_1)}_{x_1'} + \beta \underbrace{(Hx_2)}_{x_2'} = \alpha x_1' + \beta x_2'$$

x_3' is a linear combination of x_1', x_2'... co-linearity is PRESERVED!

$h(\cdot) = Hx$ PRESERVES CO-LINEARITY (PROJ)

SO we PROVE that

IF $h(x) = H \cdot x \implies h(\cdot)$ is PROJECTIVE

$\Leftarrow$

VICEVERSA
is hard to PROOF

It is IMPORTANT to reconstruct the required DOF to univocally identify H

important because, Fitting H is required during RECONSTRUCTION, to know the amount of required equations

Homography: 8 degrees of freedom

H is 3×3 → apparently 9 DOF, BUT being x homogeneous

From the theorem $\lambda H x \equiv H x$

$$h(\mathbf{x}) = \mathbf{x}' = H\,\mathbf{x}$$

Therefore, if we multiply the matrix H by any nonzero scalar λ, the relation is satisfied by the same points

9 DOF but one freedom!

$(9-1) = 8$ DOF

$$\mathbf{x}' = \lambda H\,\mathbf{x}$$

Thus any nonzero multiple of the matrix H represents the same projective mapping as H.

Hence H is a homogeneous matrix: in spite of its 9 entries,

H has only 8 degrees of freedom, namely the ratios between its elements.

Homography estimation

H has only 8 degrees of freedom, namely the ratios between its elements.
E.g.

how many pairs of points we need to reconstruct H...
to map the two planes,
knowing x_1, x_1' ← you have 2 informations

$$H = \begin{bmatrix} A_{11} & A_{12} & t_1 \\ A_{21} & A_{22} & t_2 \\ v_1 & v_2 & 1 \end{bmatrix}$$

each pair of corresponding points carry 2 equations
⇒ you need 4 pairs!

KNOWN

KNOWN

$x_1' = H x_1$ MAPPING

4 POINTS
$x_i \rightarrow x_i'$
$i = 1, .., 4$

Therefore, it can be estimated by just FOUR point correspondences,

since each point correspondence $\mathbf{x}' = H\,\mathbf{x}$ yelds **two** independent equations

$$X_1' = H X_1 \rightarrow \begin{bmatrix} x \\ y \\ w \end{bmatrix}' = H \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

than W is NOT a real DOF
2 equality constraint

In OUR Framework we assume that we have the PHOTOGRAPH (image)!

IF I already know 4 points coordinate in your scene (to RECTIFY the image)

Homography estimation

H has only 8 degrees of freedom, namely the ratios between its elements. E.g.

$$H = \begin{bmatrix} A_{11} & A_{12} & t_1 \\ A_{21} & A_{22} & t_2 \\ v_1 & v_2 & 1 \end{bmatrix}$$

Therefore, it can be estimated by just FOUR point correspondences,

since each point correspondence $\mathbf{x}' = H\,\mathbf{x}$ yelds **two** independent equations from Imge I can know 4 points in Real world, of the plane... → I use the information from plane x_i and I can extract x'_i

than I write:

$$\begin{cases} x_1' = H x_1 \\ x_2' = H x_2 \\ \vdots \end{cases}$$

4 equations each with 2 equations

↳ I solve it in the elements of H, fit and solve

↓

so I know H, and from image I can compute $x = H^{-1} x'$ as Rectified image

sometimes som x_i real plane points are KNOWN!

...

... computing equation using the homogeneus coordinate w

$$w' X' = x'$$

X': CARTESIAN

it is USELESS INFORMATION!

we already know $X' = x'/w'$

In equation we know x', x ...

And if we have 4 points we can find H

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & e \end{bmatrix} = H$$

unknown constants, to obtain from the equations

Transformation of points, lines, conics, dual conics

How points/... transforms by PROJECTIVE MAPPING

└ how other elements are transformed? →

$$\underbrace{x' = Hx}$$

TRANSFORMATION RULE

= relationship between point to transform and transformed point

Transformation rules for the plane elements

A homography transforms **each point $\mathbf{x}$** into a point $\mathbf{x}'$ such that:

$$\mathbf{x} \rightarrow H\mathbf{x} = \mathbf{x}'$$

from this, we derive the laws for other elements

EASY to PROOF → (*)

A homography transforms **each line $\boldsymbol{l}$** into a line $\boldsymbol{l}'$ such that:

$$\boldsymbol{l} \rightarrow H^{-T}\boldsymbol{l} = \boldsymbol{l}'$$

apply same on other objects...

(**)

A homography transforms **each conic C** into a conic C' such that:

$$C \rightarrow H^{-T} C H^{-1} = C'$$

(exactly same proof, using C^* procedure with set of lines transformation)

A homography transforms **each dual conic C^*** into a dual conic $C^{*\prime}$

$$C^* \rightarrow H C^* H^T = C^{*\prime}$$

(*) **LINE**: transformed by

$$\ell \longrightarrow \boxed{H} \longrightarrow \ell' = H^{-T}\ell$$

a PROJECTIVE MAPPING TO a line give you still a line (as expected because of CO-LINEARITY PRESERVATION!) all transformed points MUST remain co-linear after PROJECTIVE MAP

Knowing that $x' = Hx$ transformed point

a line ℓ is s.t $\ell^T x = 0$ (equation of line points)

$x = H^{-1}x'$ (because invertible mapping)

since ℓ is considered by set of points x: $\ell^T x = 0$ on the line

I want to transform the CONSTRAINT as a constrain in the TRANSFORMED projective plane!

so I want to express in the set of points in $\mathbb{P}'^2$

$$\ell^T x = \ell^T(H^{-1}x') = \underbrace{\ell^T H^{-1} x' = 0}$$

CONSTRAINT acting on elements on second plane...

What object is $H^{-1}x'$.... ?

Lines are characterized as set of points x solutions of linear homogeneous eq. $\ell^T x$ while conics to quadratic

|| $\ell^T H^{-1} x' = 0$ is LINEAR equation in x also HOMOGENEOUS, being "$= 0$" ||

this is a linear homog. equation in x' points on transformed plane

then, it is a LINE!

$$\ell'^T x' = \ell^T H^{-1} x' = 0$$

$$\ell' = (H^{-1})^T \ell = H^{-T}\ell$$

(**) <u>CONICS</u> even if increasing DOF of curve

knowing $x' = Hx$ $\qquad x \to \boxed{H} \to x'$

considering $x : \; x^T C x = 0 \quad (x \in \text{conic})$

then, as before.. $x = H^{-1} x'$

$$(H^{-1}x')^T C (H^{-1}x') = x'^T \boxed{H^{-T} C H^{-1}} x' = (x')^T \boxed{C'} x' = 0$$

this is still a conic, in PROJECTIVE PLANE ← still quadratic equation in x', HOMOGENEUS...

⇓

{ Projective transformation of a conic is another conic }

NOT valid in a "general" transformation, it holds because PROJECTIVE

Vanishing points ⟹

frequent situation when considering scenes with man made objects (buildings, etc...) architecture etc (MANY PARALLEL LINES) there exist a Theorem involving it..

defined as the IMAGE of a POINT at the infinity

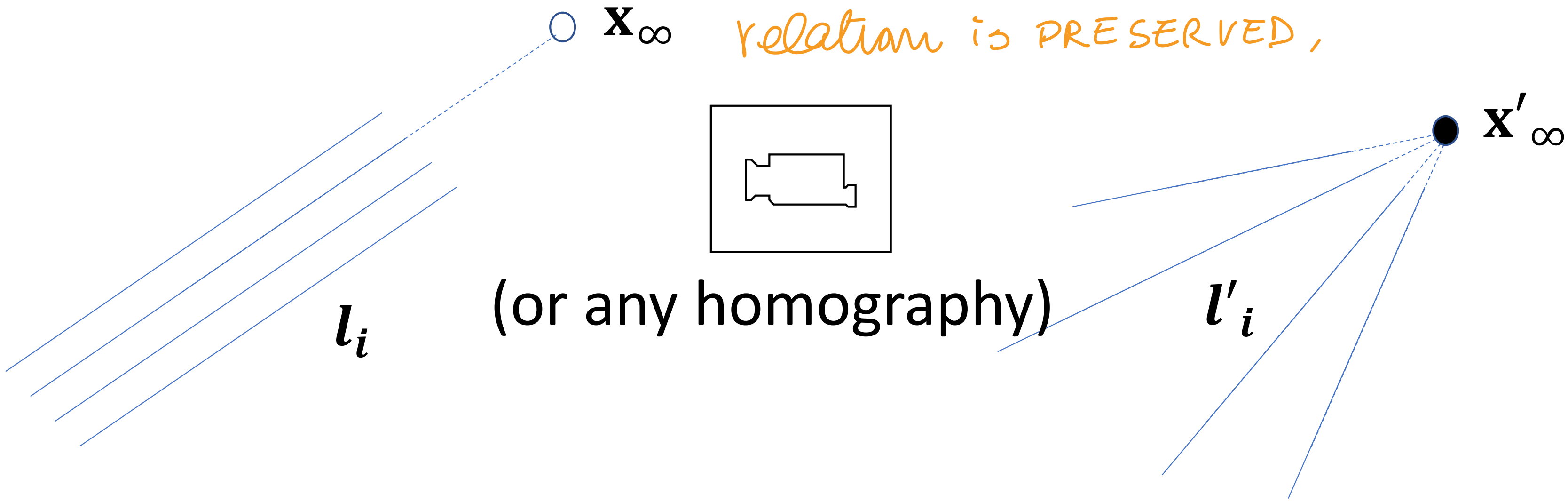
vanishing point

≃ "PUNTO DI FUGA"

Theorem: *the image of a set of parallel lines* $\boldsymbol{l_i}$ *is a set of lines* $\boldsymbol{l'_i}$ ***concurrent*** *at a common point* $\boldsymbol{x'}$ *called the vanishing point of the direction of lines* $\boldsymbol{l_i}$

in the image, those parallel lines has common intersection point! they intersect @ ∞, since all this lines have one point in common x_∞ → CONCURRENT at x_∞

TAKING an image of this lines, incidence relation is PRESERVED,



vanishing point

Theorem: *the image of a set of parallel lines $\boldsymbol{l_i}$ is a set of lines $\boldsymbol{l'_i}$ **concurrent** at a common point $\boldsymbol{x'}$ called the vanishing point of the direction of lines $\boldsymbol{l_i}$*

preserved.. IF $l^T x_\infty = 0$ than $l^T \underbrace{H^{-1} H}_{I} x_\infty = (H^{-T} l)^T x'_\infty = 0$ so stiel $(l')^T x'_\infty = 0$

(transforming the line, each the points x transformed into x' still are points of l')

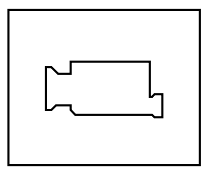
all points $x \rightarrow x'$ transform into New set of lines

so from a set of lines l they transforms into concurrents lines l'... and they concur into $x'_\infty = H x_\infty$

transforming a point x_∞ at ∞ you obtain x'_∞ no more at infinity, you obtain a transformed point NOT ∞..

$\mathbf{x}_\infty \xrightarrow{H} \mathbf{x}'_\infty$

this point is "VANISHING POINT"



(or any homography)

$\boldsymbol{l_i}$ $\qquad$ $\boldsymbol{l'_i}$

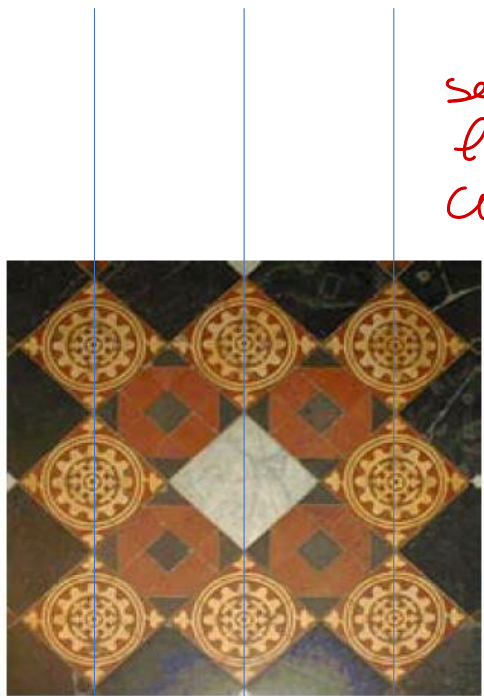
EXAMPLE

Vanishing point: image of a point at the ∞, (where images of parallel lines concur)

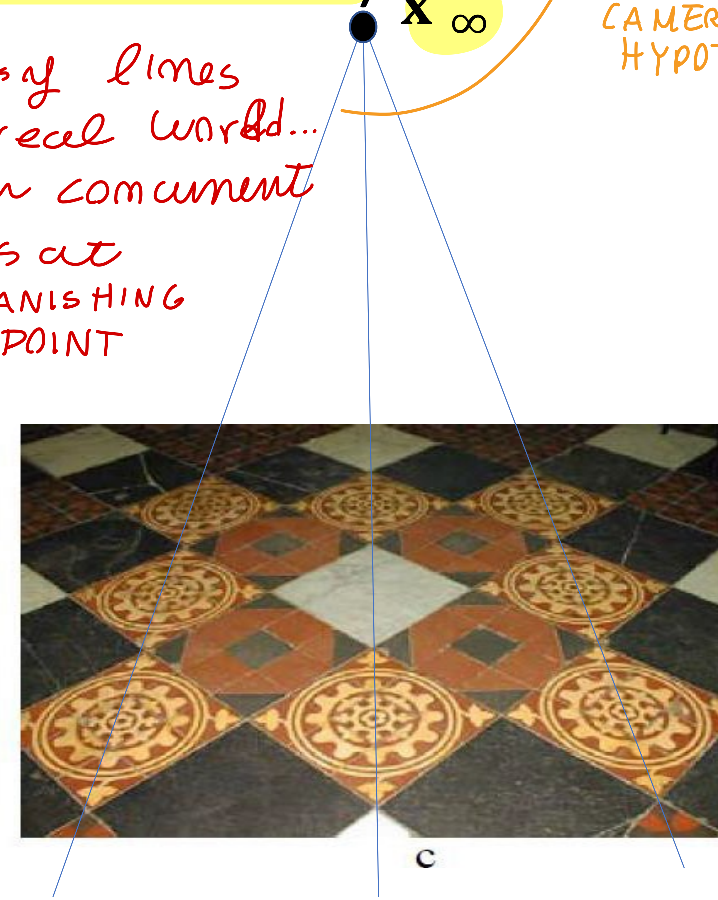
when thin lense and small angles, CAMERA HYPOT.

set of parallel lines in real world must concur

images of lines // in real world... then concurrent lines at VANISHING POINT



a



c

The vanishing line

new concept, regarded as
the IMAGE of the LINE AT the INFINITY

vanishing line $\boldsymbol{l}'_\infty$ *or horizon* = image of the set of the points at the ∞ = the image of $\boldsymbol{l}_\infty$ (must be a line: why?)

$\boldsymbol{l}'_\infty$

In a planar scene, the set of all the points at infinity is a line in the image plane, as w=0 (linear in w, line)

$\boldsymbol{l}'_\infty$ can be found from two vanishing points:

transforming the line, we still have a line, in IMAJE PROJECTION it is HORIZON LINE

Also all the other vanishing points are on the line $\boldsymbol{l}'_\infty$

vanishing line $\boldsymbol{l}'_\infty$ *or horizon* = image of the set of the points at the ∞ = the image of $\boldsymbol{l}_\infty$ (must be a line: why?)

$x'_{\infty 2}$

$x'_{\infty 1}$

Knowing that some lines are images of parallel lines... you simply construct the two vanishing points

$\boldsymbol{l}'_\infty \equiv$ set of the images of points at ∞

being $l_\infty = U x_\infty$
$l'_\infty = U x'_\infty$
vanishing line

$\boldsymbol{l}'_\infty$ can be found from two vanishing points:

Also all the other vanishing points are on the line $\boldsymbol{l}'_\infty$

on planar scene, all vanishing points are co-linear (on same line)

Vanishing points on different directions are in the same line horizon tends to be CIRCUMFERENCE, because of earth shape!

IF planet were planar

↓

in PLANAR SCENE, we call the vanishing line ~ set of images of line @ infinity

"

VANISHING LINE ~ HORIZON

← that has important consequences on RECONSTRUCTION and other image extraction...

for example find x_∞ of a direction in an image, without KNOWING the direction, you connect line with 2 vanishing points!

vanishing line ℓ'_∞ is $\bigcup x'_\infty$ associated to all directions

Polarity is presesved under projective mapping

easy to
prove in
a simple way:

give point y and conic C, and
polar line $l = Cy$

i.e.

then tally image of it
preserve the
property!

$$\boldsymbol{l} = C\mathbf{y} \Rightarrow \boldsymbol{l}' = C'\mathbf{y}'$$

Proof. From transformation rules for conics, lines and points:

$$C'\mathbf{y}' = H^{-T}CH^{-1}\mathbf{y} = H^{-T}CH^{-1}H\mathbf{y} = H^{-T}C\mathbf{y} = H^{-T}\boldsymbol{l} = \boldsymbol{l}'$$

Polarity is preserved under projective mappings

FOR EXAMPLE

this is preserved any HOMOGRAPHY
also taking image

↓ POLAR RELATIONSHIP

tangents from the point at the ∞

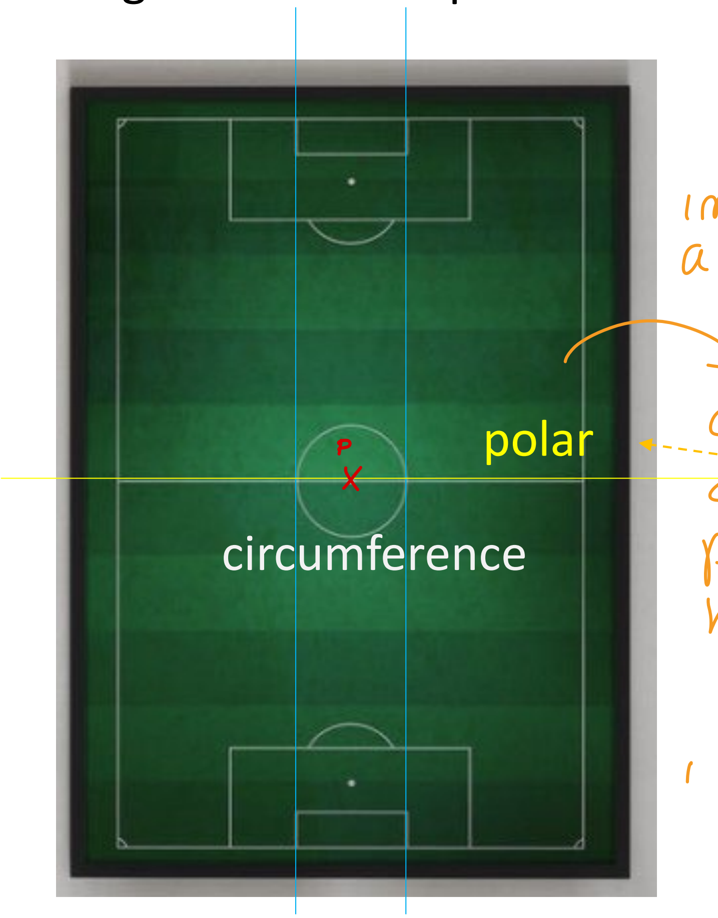
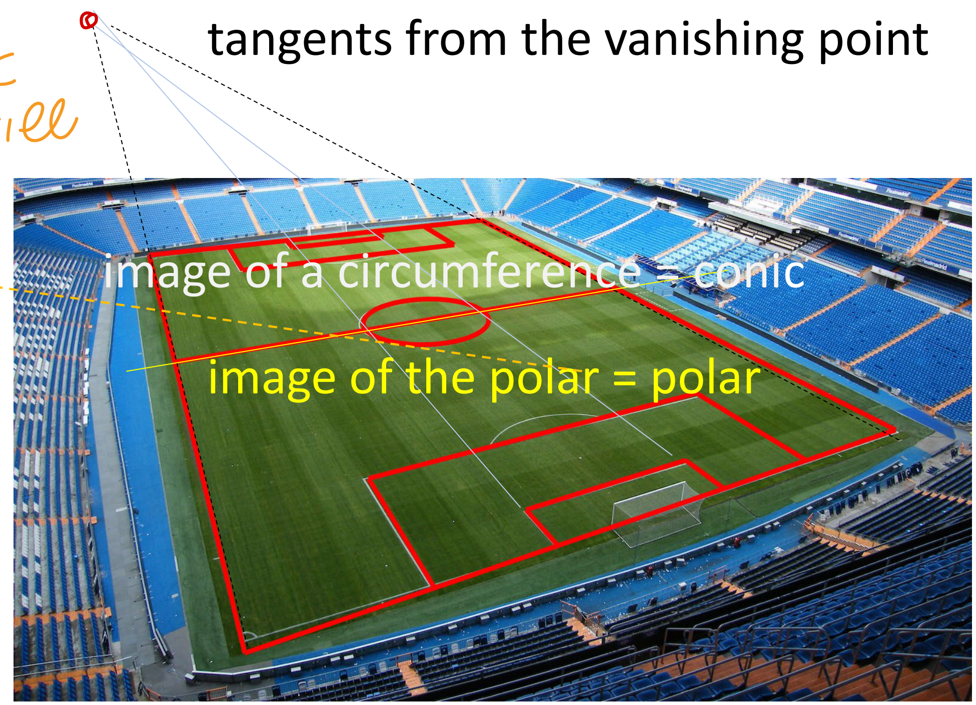


image of a conic C is C' still a conic as proven before!
⇓

x'_∞

tangents from the vanishing point



Polarity is preserved under projective mappings

tangents from the point at the ∞

Also by an image, we preserve polarity!

so you can use projective to map cell..

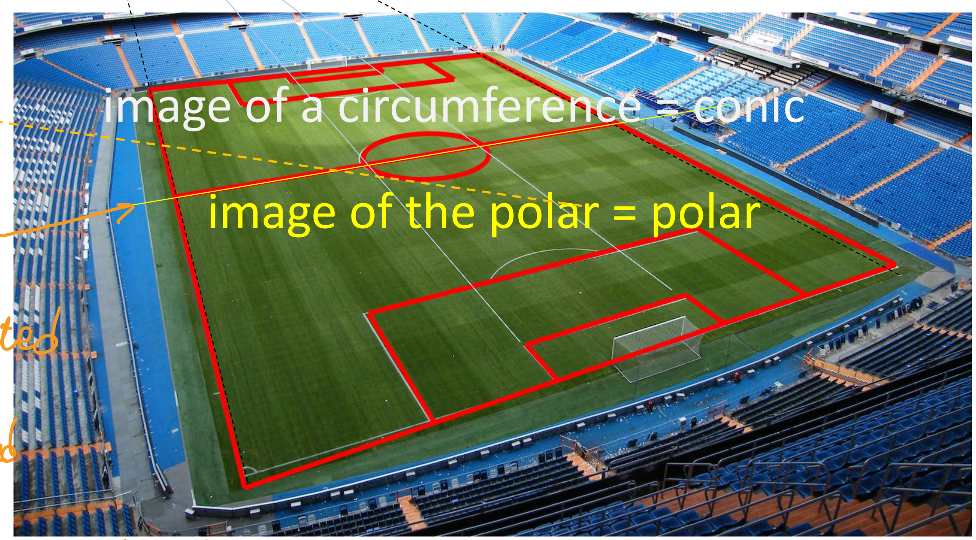
H^{-1}

the POLAR line of the circumference center is diameter

$\Uparrow$

x'_∞

tangents from the vanishing point



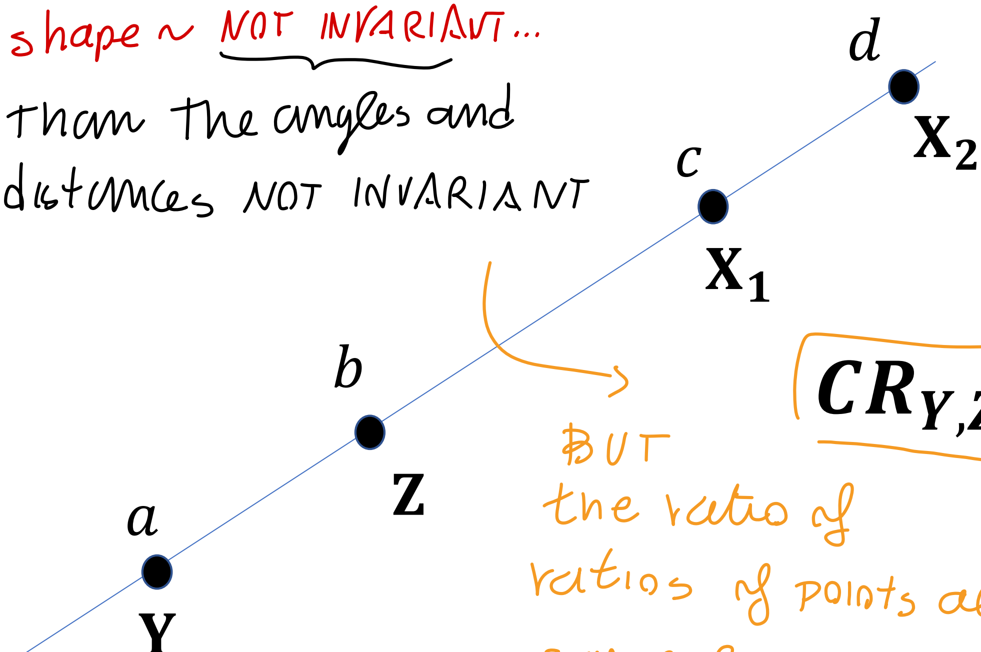
this is projected and maintained as polar in the image plane

What is INVARIANT under PROJECTIVE transformation?
⇓

Cross ratio: a projective invariant

In the image of something there are invariant properties of the scene, like the "SHAPE"? (comics) NO!
"PARALLEL" → CONCURRENT etc.

shape ~ $\underbrace{\text{NOT INVARIANT}}$...
than the angles and distances NOT INVARIANT



BUT
the ratio of ratios of points along same line is INVARIANT

$$\boxed{CR_{Y,Z,X_1,X_2}} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

Cross ratio invariance under projective mappings

$$CR_{Y,Z,X_1,X_2} = CR_{Y',Z',X'_1,X'_2}$$

Proof:

this can be PROVEN!

- applying any homography H, the coefficients of the linear combinations remain the same

$$\mathbf{X}_1 = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

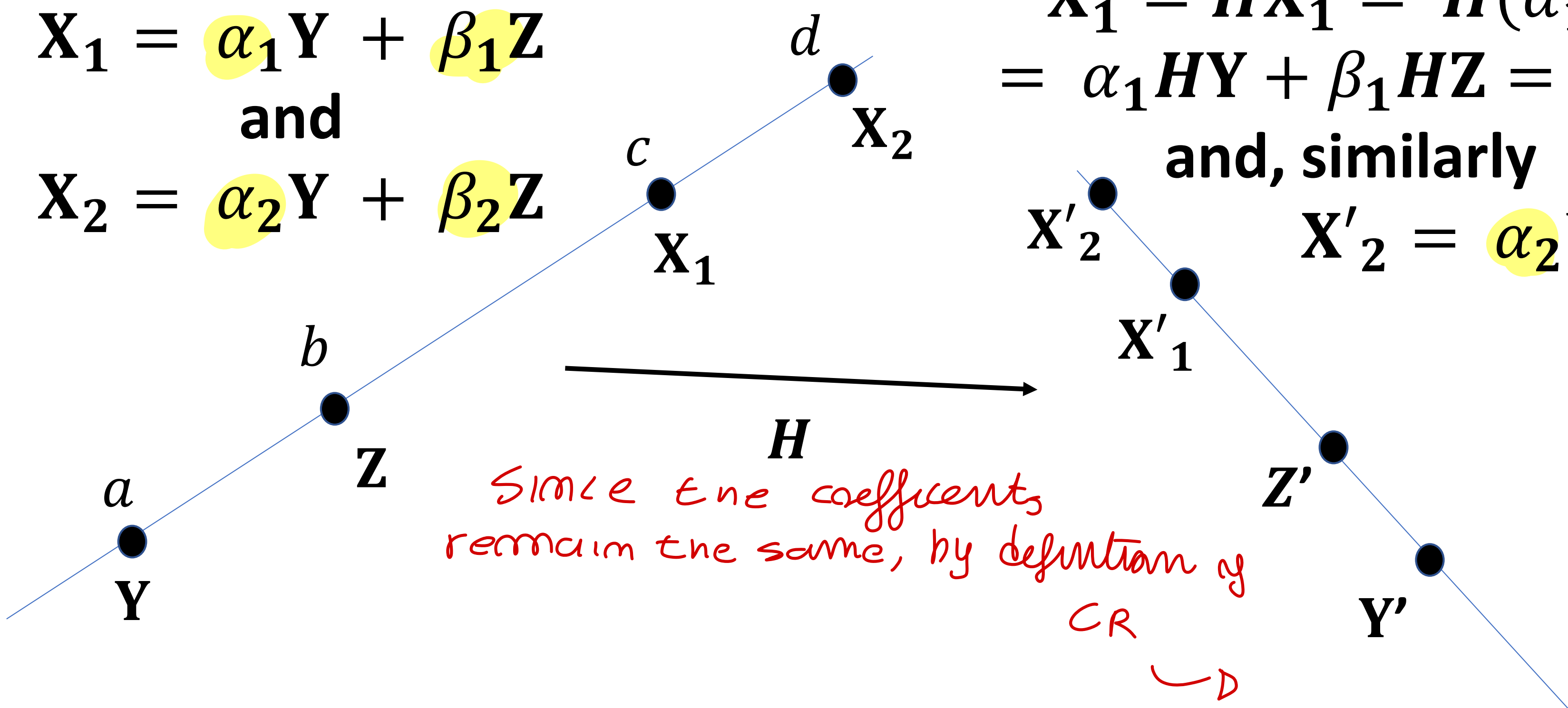
and

$$\mathbf{X}_2 = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

$$\mathbf{X}_1' = \boldsymbol{H}\mathbf{X}_1 = \boldsymbol{H}(\alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z})$$
$$= \alpha_1 \boldsymbol{H}\mathbf{Y} + \beta_1 \boldsymbol{H}\mathbf{Z} = \alpha_1 \mathbf{Y}' + \beta_1 \mathbf{Z}'$$

and, similarly

$$\mathbf{X}'_2 = \alpha_2 \mathbf{Y}' + \beta_2 \mathbf{Z}'$$



Since the coefficients remain the same, by definition of CR

↳ D

and, from result

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = \frac{c-a}{c-b}\Big/\frac{d-a}{d-b} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$

$$\mathbf{X_1} = \alpha_1\mathbf{Y} + \beta_1\mathbf{Z}$$

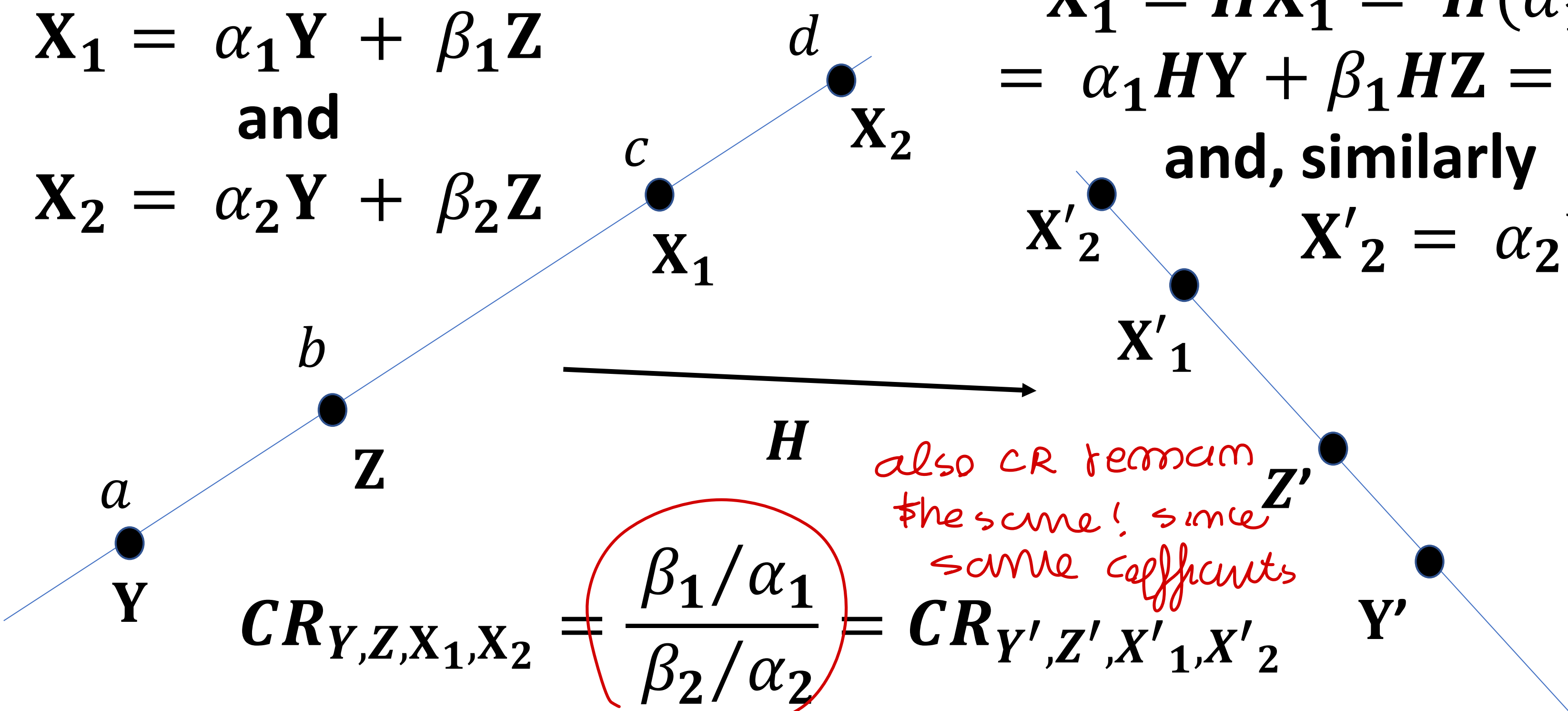
and

$$\mathbf{X_2} = \alpha_2\mathbf{Y} + \beta_2\mathbf{Z}$$

$$\mathbf{X}'_1 = \boldsymbol{H}\mathbf{X_1} = \boldsymbol{H}(\alpha_1\mathbf{Y} + \beta_1\mathbf{Z})$$
$$= \alpha_1\boldsymbol{H}\mathbf{Y} + \beta_1\boldsymbol{H}\mathbf{Z} = \alpha_1\mathbf{Y}' + \beta_1\mathbf{Z}'$$

and, similarly

$$\mathbf{X'_2} = \alpha_2\mathbf{Y}' + \beta_2\mathbf{Z}'$$



also CR remain the same! since same coefficients

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2} = \boldsymbol{CR_{Y',Z',X'_1,X'_2}}$$

consequence of CROSS RATIO invariance...

you can complete cutted images

SOCCER FIELD

$\Rightarrow$ the AREA close to the goal $\rightarrow$ if you take an image and it is cutted/occlusion

you have a', b', c' points

you can compute where d' should be by using $CR = CR'$ equaty

$CR(a,b,c,d) = CR'(a',b',c',d')$

you can find just d' from the others!

[only one unknown for reconstruction!]

and it is a LINEAR EQUATION! so easy to compute (NOT quadratic / fractional)

so CROSS RATIO invariance allow to recover missing part of the scene

Hierarchy of projective transformations

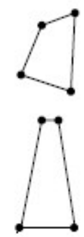
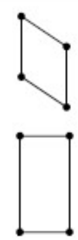
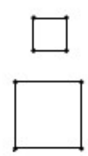
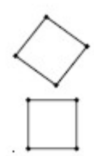
↳ to understand PROJ TRANSF. we have to see particular classes of it!

↓

the more complex is the general one we introduced,

BUT simpler one exists ⟹

Hierarchy of projective transformations

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$. +
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3). + +
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area + + +

SIMPLER

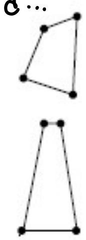
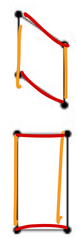
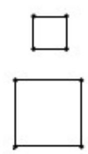
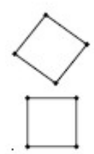
rigid motion = (ROTOTRANSL) TRANSFORMATION

↳ apply rigid motion, no changes in shape, particular prop (colinearity preserved!)

Hierarchy of projective transformations

general one! here ONLY CO-LINEARITY is preserved...

degree of ORDER of CONTACT preserved...

Group	Matrix	Distortion	Invariant properties
" Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$. +
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3). + +
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area + + +

don't preserve shape and angles, only the ZERO... angle is preserved -> PRESERVE PARALLELISM!

parallelism preserved

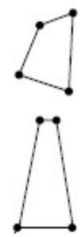
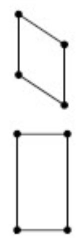
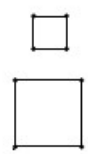
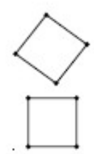
with an additional complex. = + include euclidean transform

DISTANCES NOT PERSERVED... angles preserved -> shape preserved NOT dimension

ROTOTRANSLATING + HOMOGENEUS scaling

friendly PROJ TRANSF = ROTOTRANSL (all euclidean transf are SIMILARITY, But NOT viceversa)

Hierarchy of projective transformations

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections contact of ORDER 3 (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$. +
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3). + +
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area + + +

IN DETAILS ... when H satisfy certain constrain ~ it belongs to a certain restricted class!

↓

We look at the MATRIX represents the transformation H to recognize what kind of TRANSF we deal with

Isometries (or Euclidean mappings)

↓ ROTOTRANSLATION

⇓ RIGIDLY MOVE the point, more invariants

$$H_I = \begin{bmatrix} R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

homogeneus

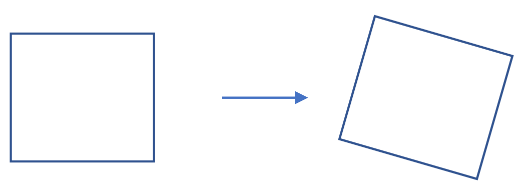
$R_\perp$ is an orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^T$

$\det {R_\perp}^{-1} = 1$ planar rigid displacement (-1 for reflection)

3 dofs: translation $\boldsymbol{t}$ + rotation angle ϑ

Invariants: lenghts, distances, areas → shape and size → relative positions

preserved!



"ISO" ~ the same metrics

Isometries (or Euclidean mappings)

ROTATION ANGLE

RIGID MOTION!

2×2 submatrix as ORTHOGONAL Rotation matrix

$$H_I = \begin{bmatrix} R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

TRANSLATION vector

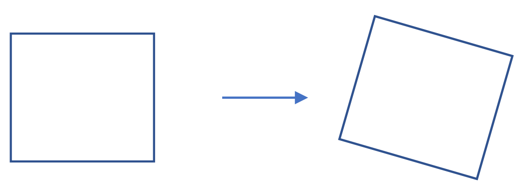
$R_\perp$ is an orthogonal matrix: $R_\perp^{-1} = R_\perp^{T}$ {usefull property!}

det $R_\perp^{-1} = 1$ planar rigid displacement (-1 for reflection)

Rigid motion on a plane has 3DOF ← **3 dofs**: translation $\boldsymbol{t}$ + rotation angle ϑ

Invariants: lenghts, distances, areas → shape and size → relative positions

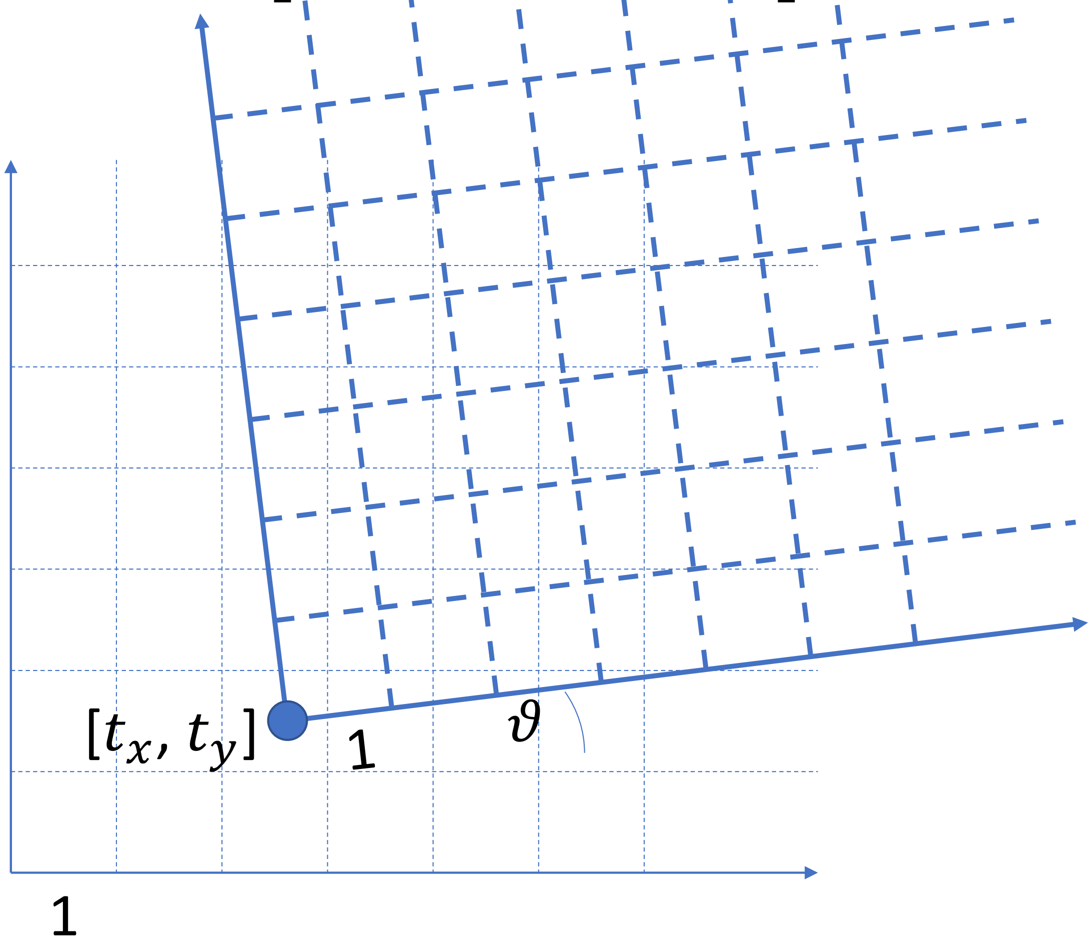
by moving (2 DOF) and rotate (1 DOF) } you read it in the MATRIX



same invariants!

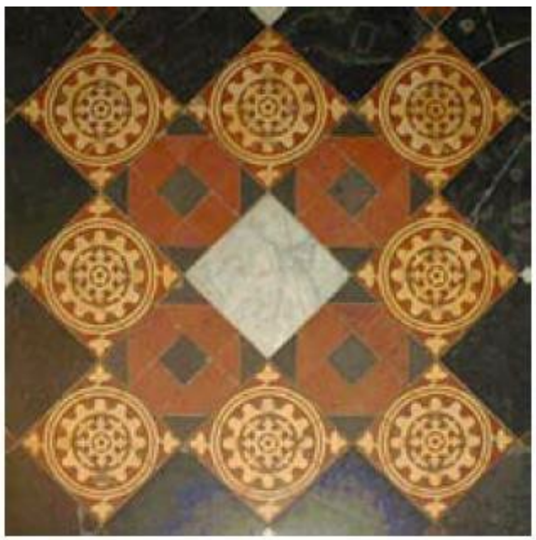
EXAMPLE :

Isometry $H_I = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$: 3 degrees of freedom



moving the frame by that transformation H_I,

we get this new reference

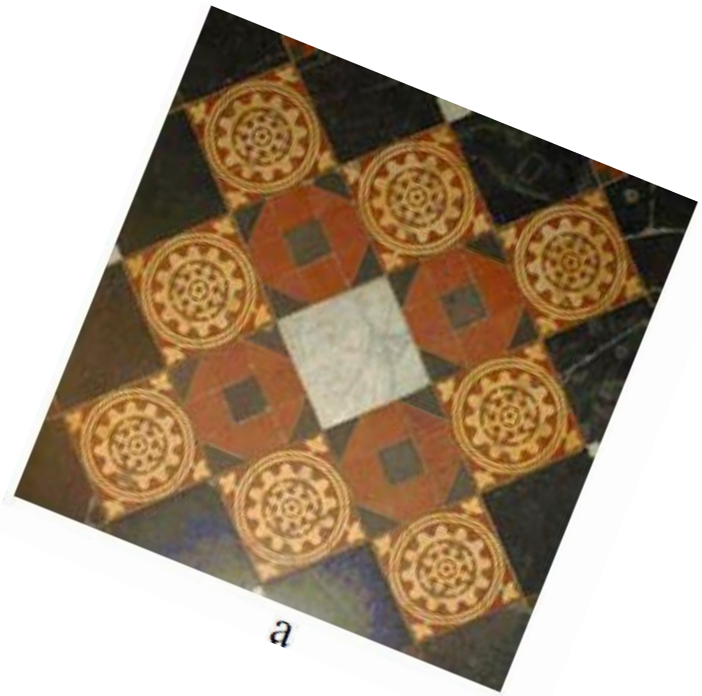


a

applyed to scene...
same
size and
shape

isometry

preserve lenght/
destance/shape/
size!



a

Similarities

multiply by common scaling factor "s"

ROTATION orthogonal matrix * s

one additional DOF, the scale (s) wrt isometry,

$$H_S = \begin{bmatrix} s\,R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

4 dof

$\{\vartheta, s, t_x, t_y\}$

{ s=2 := DOUBLE size
s=3 := TRIPLE
s<1/2 := HALF SIZE }

{ ROTOTRANSLATION + SCALING }

$R_\perp$ is an orthogonal matrix: $R_\perp{}^{-1} = R_\perp{}^T$

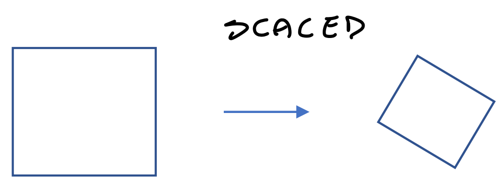
4 dofs: rigid displacement + *scale*

Invariants: ratio of lenghts, angles → shape (not size)

{ invariant angles }

the circular points ***I*** and ***J*** are INVARIANTS!

algebrically invariants are the CIRCULAR POINTS!



also RATIO of shapes remains the same! "SIMILAR"

Similarities

$$H_S = \begin{bmatrix} s\,R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\ \sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$R_\perp$ is an orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^{T}$

4 dofs: rigid displacement + *scale*

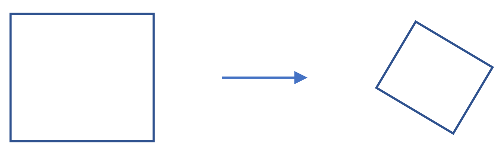
therefore also angles

Invariants: ratio of lenghts, angles → shape (not size)

scaling factor s

the circular points ***I*** and ***J***

in compact version...
we can show
CIRCULAR POINTS
≡
intersections between
any CIRCUMFERENCE and ℓ_∞

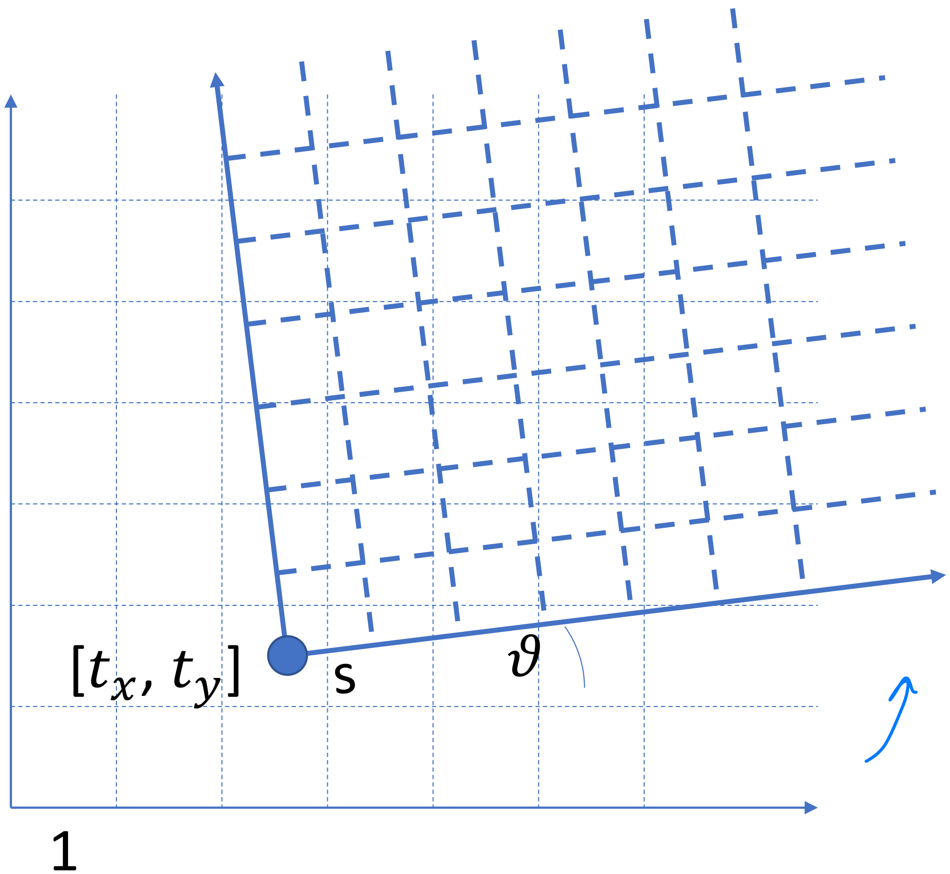


which are invariant under similarities

you find I, J points ~ CIRCULAR POINTS with complex coordinates

$$\text{Similarity } H_S = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix} : 4 \text{ degrees of freedom}$$

how SIMILLIRATIES works...



ROTOTRANSLATION
+
scaling

15/10

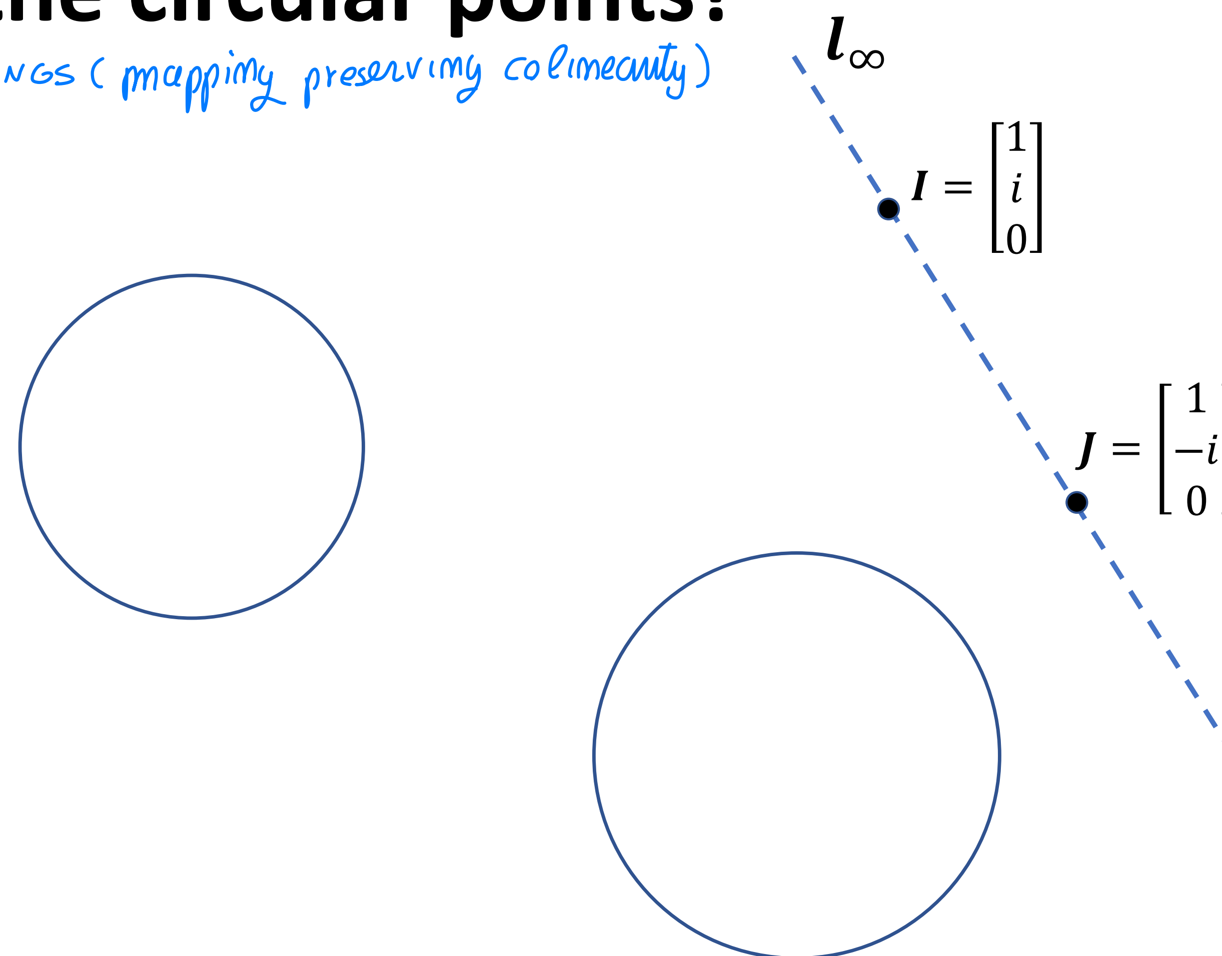
remember **the circular points?**

↳ deal with subclasses of PROJECTIVE MAPPINGS (mapping preserving colinearity)

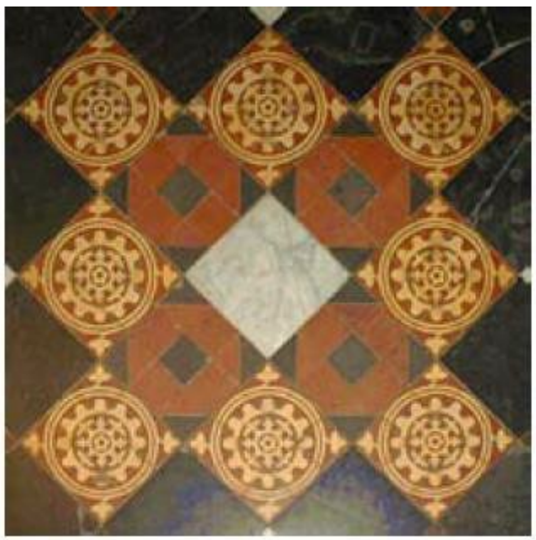
Remember their coordinates

$$\boldsymbol{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

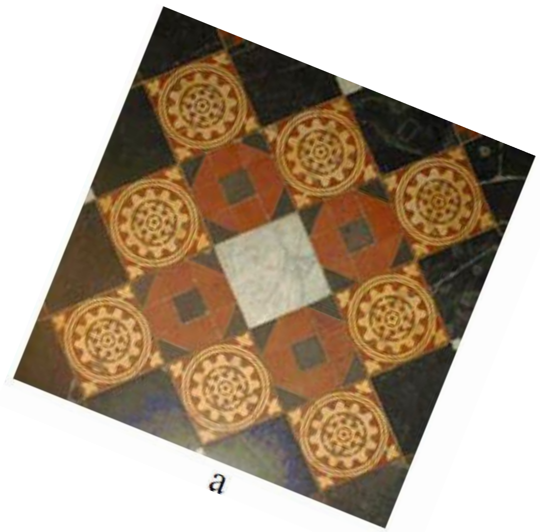


All the circumferences contain the two circular points $\boldsymbol{I}$ and $\boldsymbol{J}$



a

similarity



AFFINE TRANSFORMATION

Affinities (or affine mappings)

↝ MORE General transformation

4 components (free)

$$H_A = \begin{bmatrix} A & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} & \begin{matrix} t_x \\ t_y \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

translation components

A is any 2x2 rank-2 matrix

(No orthogonality constraint, BUT still H_A must be invertible, A must be rank(A)=2 to preserve inversion!)

2 dof more than SIMILARITIES and ISOMETRY ← **6 dofs**: $A + \boldsymbol{t}$

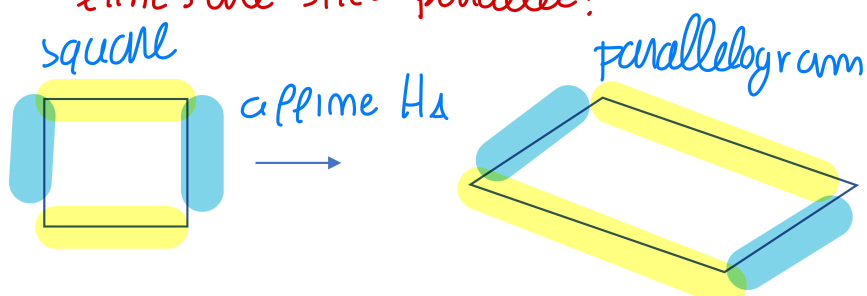
Invariants: parallelism, ratio of parallel lenghts, ratio of areas

the line at the infinity $\boldsymbol{l}_\infty = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]^T$

RATIO between lenghts NO MORE INVARIANT, → deform the shape

so neither angle and shape!

from parallel lines, transformed lines are still parallel!



shape changes

Affinities (or affine mappings)

$$H_A = \begin{bmatrix} A & & \boldsymbol{t} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

A is any 2x2 rank-2 matrix

6 dofs: A + $\boldsymbol{t}$

Invariants: parallelism, ratio of parallel lenghts, ratio of areas

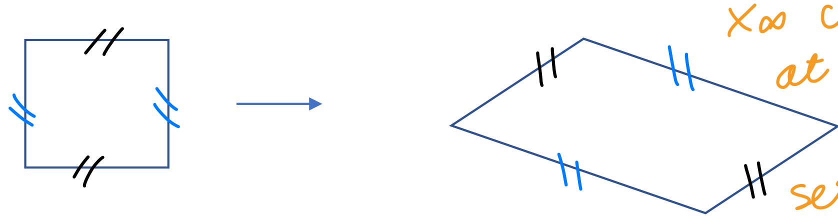
the line at the infinity $\boldsymbol{l}_\infty = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]^T$

since parallelism is kept... starting from vanishing points, those are vanishing points of parallel lines direction...

↓

transform lines into new lines still parallel... than x_∞ will be x'_∞ still at intersection.. than x_∞ as sets create ℓ_∞, preserved ℓ_∞

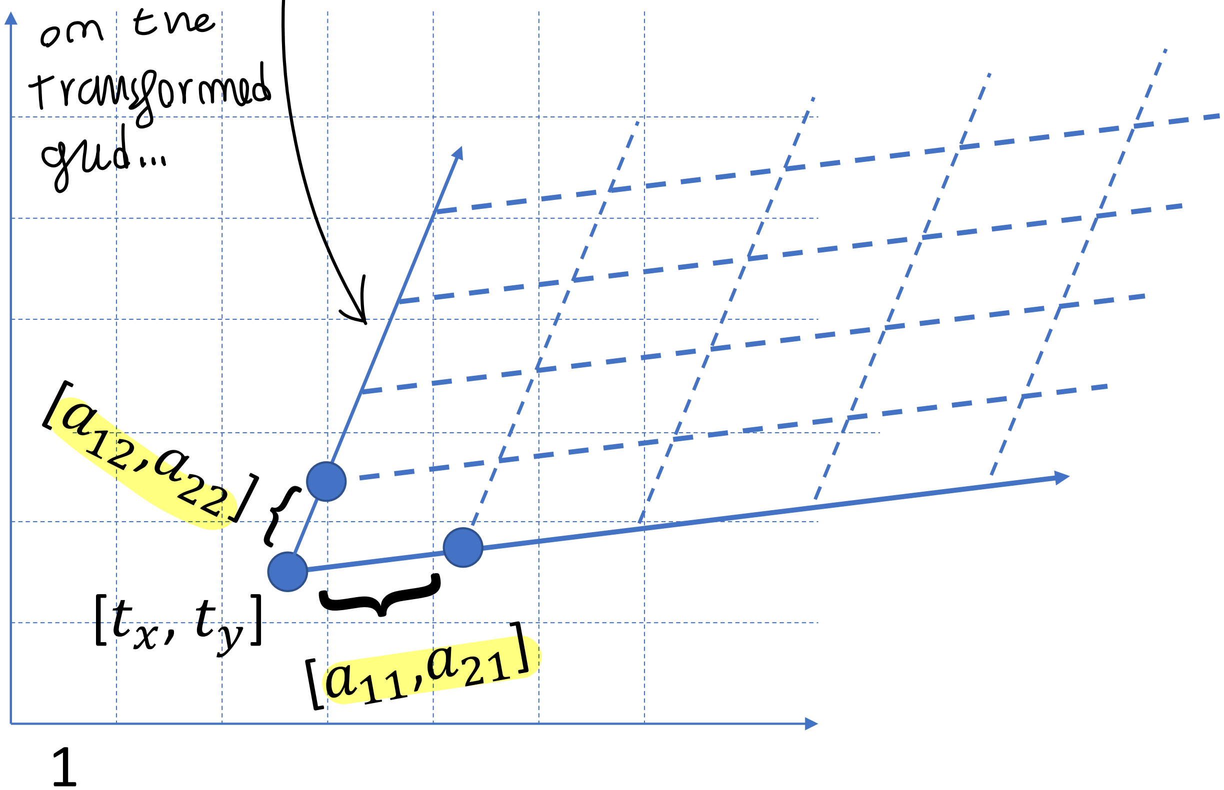
ortogonality NOT preserved... (different angles)



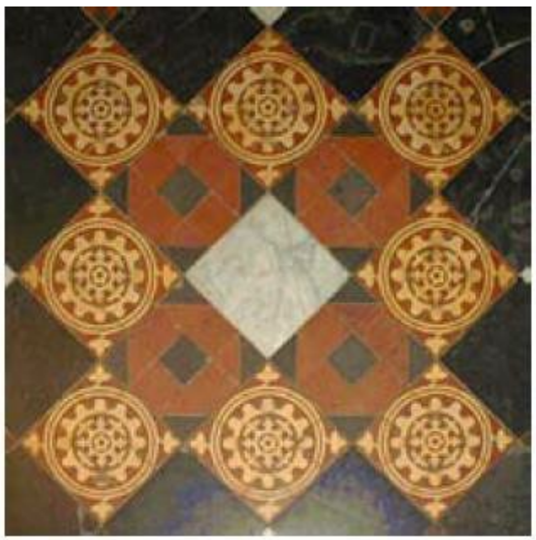
Affinity $H_A = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$: 6 degrees of freedom

you obtain a new grid no more with ORTHOGONAL lines...

you can read H_A on the transformed grid...



an image planar, after affinity gets distorted, no more perpendicular but parallelism preserved



a

affinity

b

MOST GENERAL → INVERTIBLE mapping preserving CO-LINEARITY (non singular)

Projectivities (or projective mappings, or homographies)

homogeneous matrix (independent from a common scaling factor) ←

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$$

still homogeneous matrix...

λH get the same final transformation result!

A is any 2x2 rank-2 matrix

(NON singular, because H is INVERTIBLE by definition)

YOU JUST have 8 independent numbers

8 dofs: $A + \boldsymbol{v} + \boldsymbol{t}$

$\lambda x' = \lambda H x$ ... therefore since homogeneous coord... you can scale by $1/\lambda$

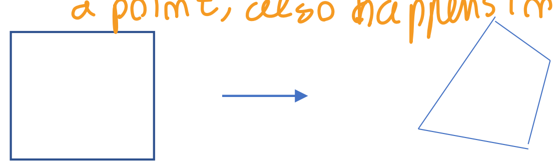
(POOR, being general) by definition

Invariants: colinearity, incidence, order of contact (crossing, tangency, inflections), the cross ratio

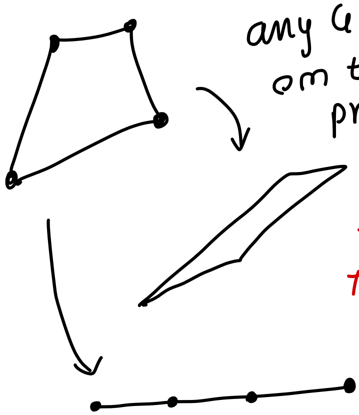
if line is going through a point, also happens in transformed

crossing 1ST ORD
tangency 2nd
inflections 3rd

preserved contact order



Projectivities (or projective mappings, or homographies)



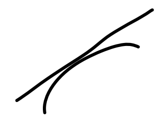
any 4 points on the plane preserving invariants!

having 8 dof, the 4 points (x,y) are freely moved!
↓
4 equations

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$$

A is any 2x2 rank-2 matrix

if you have in the plane



after H mapping

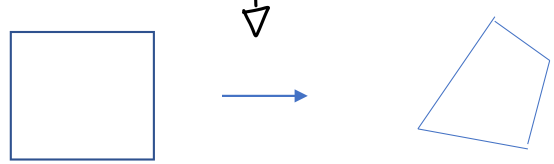
maintained

8 dofs: $A + \boldsymbol{v} + \boldsymbol{t}$

Invariants: colinearity, incidence, order of contact (crossing, tangency, inflections), the cross ratio

overall, this is preserved

having 8 dof, applying H to a square... you find ANY set of 4 points quadrangle



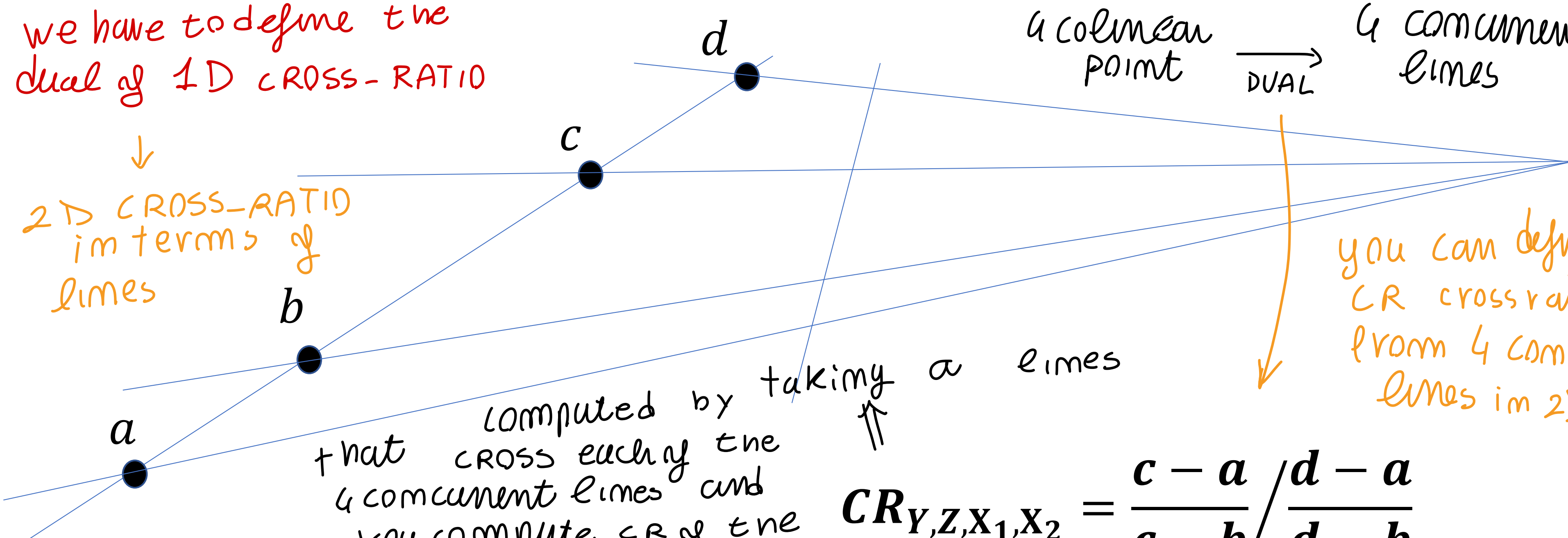
2D cross ratio of a 4-tuple of coplanar, concurrent lines: the dual of the 1D cross ratio of 4 colinear points. Take any crossing line … compute the 1D cross ratio of intersection points

we have to define the dual of 1D CROSS-RATIO

↓

2D CROSS-RATIO in terms of lines

4 colinear point $\xrightarrow{\text{DUAL}}$ 4 concurrent lines

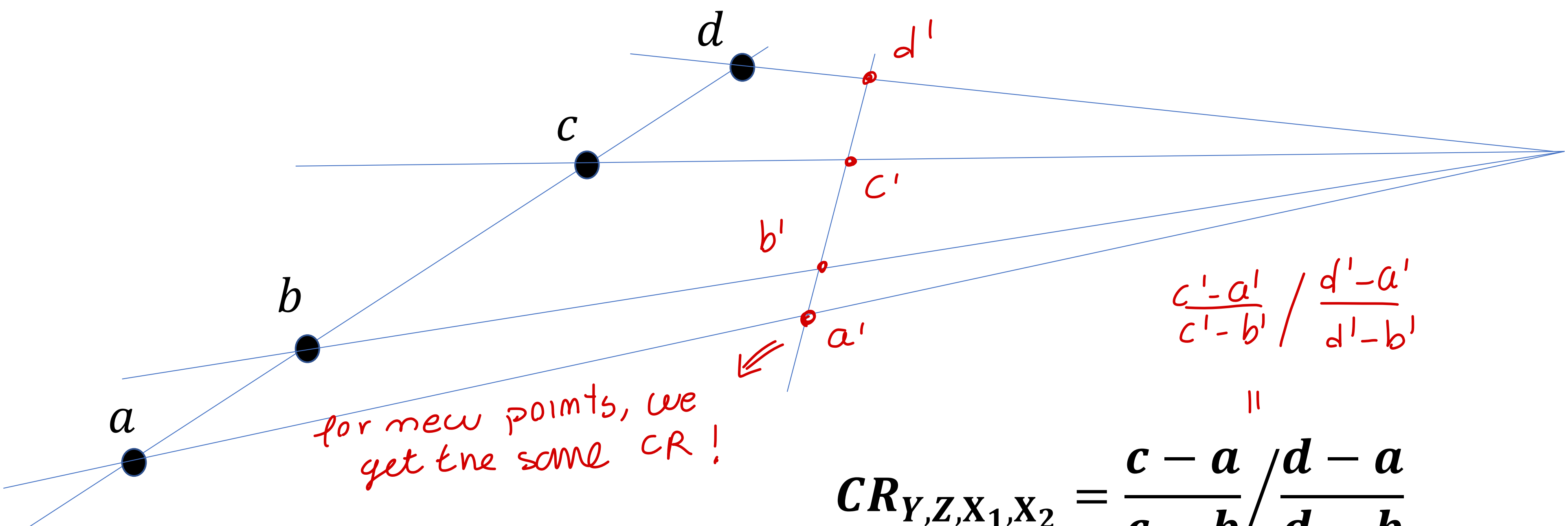


you can define CR cross ratio from 4 concurrent lines in 2D

computed by taking a lines that cross each of the 4 concurrent lines and you compute CR of the points that encounter those lines

$$CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

2D cross ratio of a 4-tuple of coplanar, concurrent lines: the dual of the 1D cross ratio of 4 colinear points. Take any crossing line … compute the 1D cross ratio of intersection points



$$\frac{c'-a'}{c'-b'} \Big/ \frac{d'-a'}{d'-b'}$$

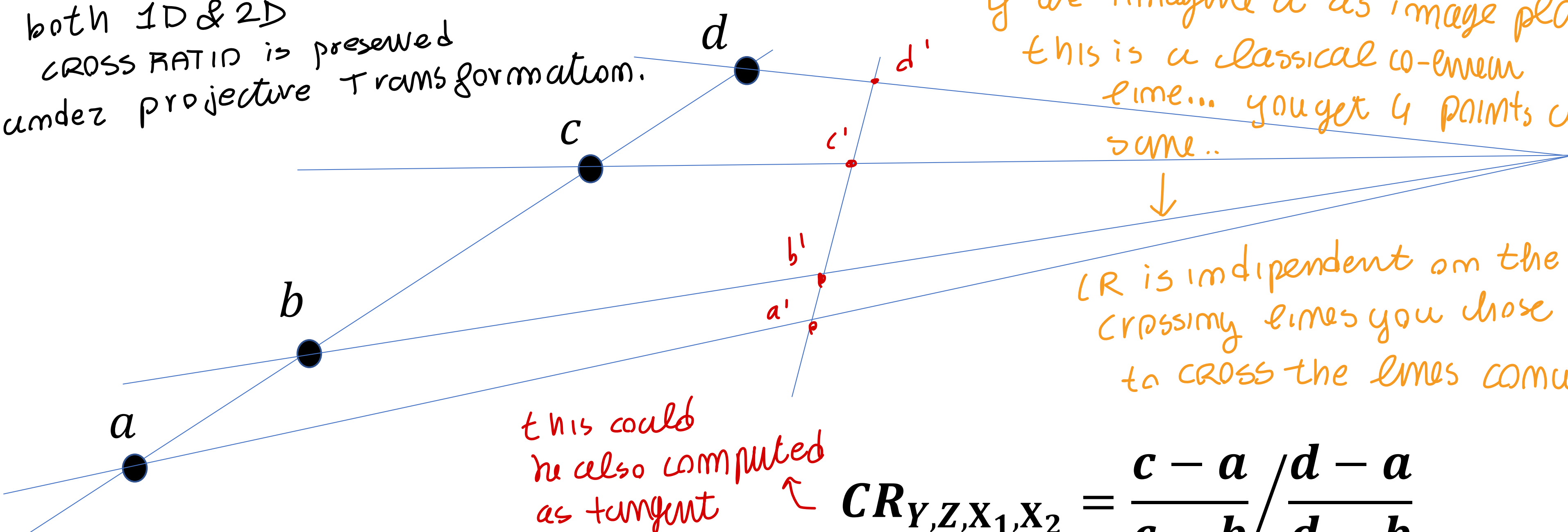
$$\parallel$$

$$\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X}_1,\mathbf{X}_2} = \frac{\boldsymbol{c}-\boldsymbol{a}}{\boldsymbol{c}-\boldsymbol{b}} \Big/ \frac{\boldsymbol{d}-\boldsymbol{a}}{\boldsymbol{d}-\boldsymbol{b}}$$

2D cross ratio of a 4-tuple of coplanar, concurrent lines: the dual of the 1D cross ratio of 4 colinear points. Take any crossing line … compute the 1D cross ratio of intersection points

both 1D & 2D CROSS RATIO is preserved under projective Transformation.

if we imagine it as image plane... this is a classical co-linear line... you get 4 points with same..



CR is indipendent on the crossing lines you chose to cross the lines concurrent

this could be also computed as tangent double ratio...

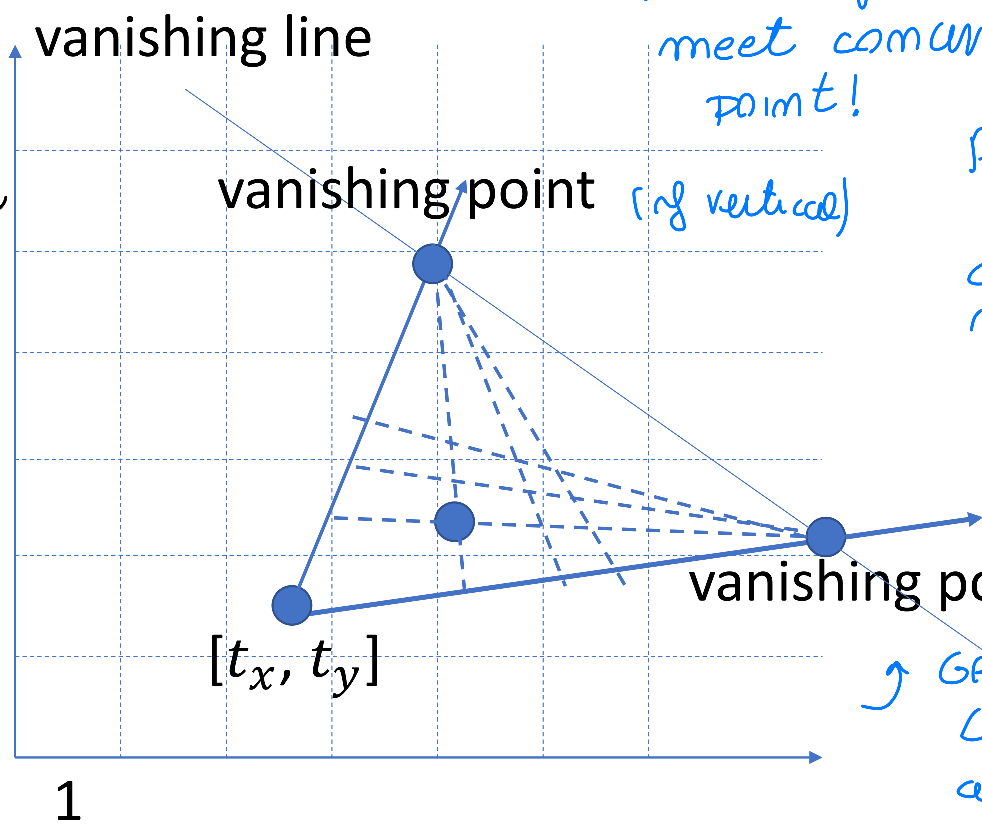
$$\boldsymbol{CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}}$$

General Projective mapping

grid lines meet @ vanishing point at vertical/horizontal direction

the aspect of general projective transform..

no more parallel lines... they meet concurrently at vanishing point!

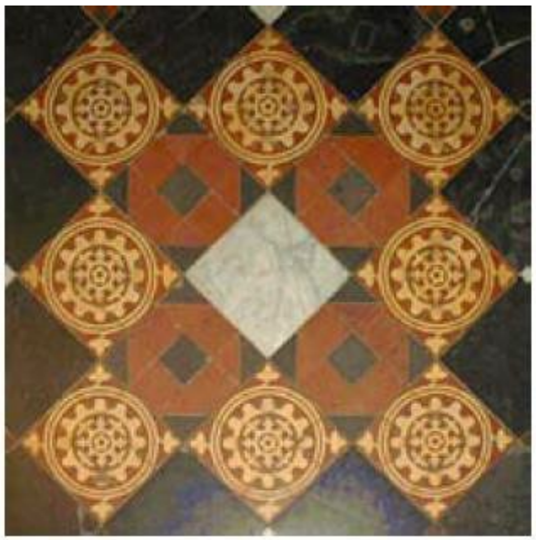


(of vertical)

parallel
↓
concurrent lines meeting @ vanishing point

(of horizontal lines)

GRID lines becomes as concurrent at X_∞ point at infinity!



a

general image

- NO parallelism!
- only concurrency is invariant
- and CR of colinear points is the same

projectivity

in original and transformed points

this is what happens in a CAMERA (perfect model!)

while in Real CAMERA, distorsion can occur when straight line is CURVED!

c

when curved ... the curvature is the DISTORSION!

in wide angle camera you can perceive distorsion

NOW we consider NO distorsion camera where straight line → remain so

2D reconstruction (image rectification)

↳ with this KNOWLEDGE
we try to solve main problem
of 2D shape reconstruction by a usefull
tool ≡ is in PLANAR WORD
↓
also known as IMAGE RECTIFICATION

you rectify the image of the planar scene
to retreive the original one
(Recover property)

RECONSTRUCTION ~ shape estimation → RECTIFICATION in such a way that we get the correct shape from the image

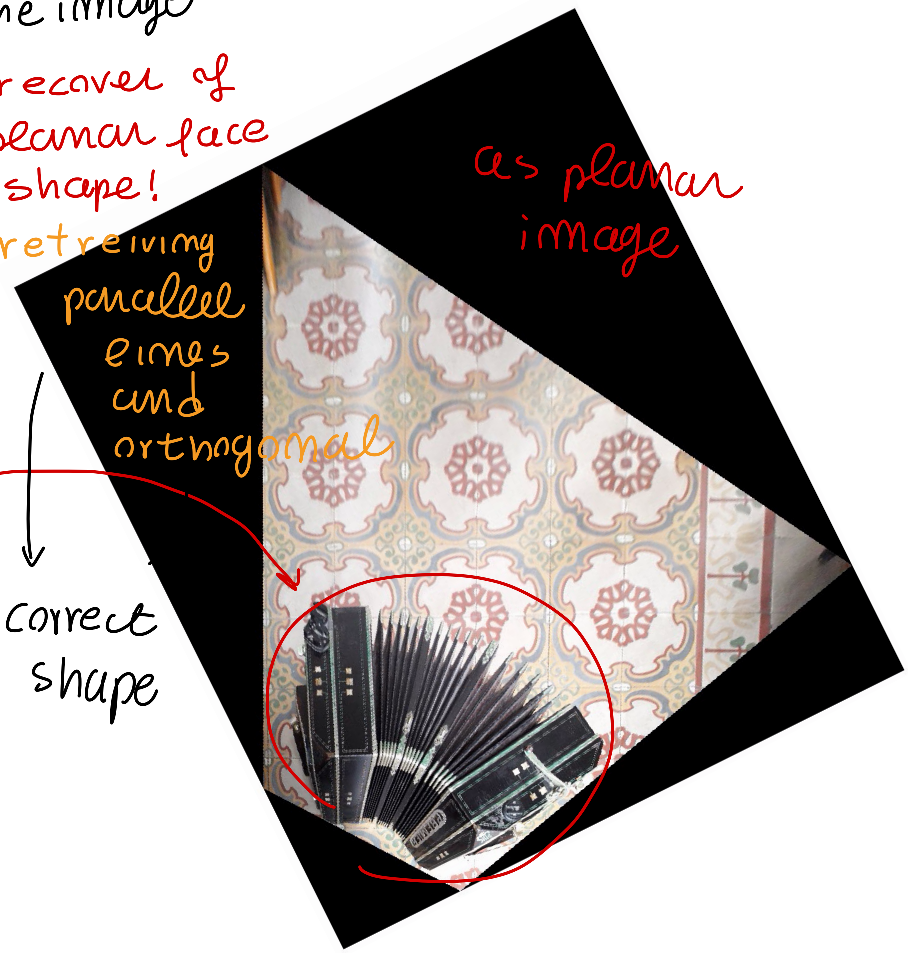
original image

rectified image

recover of planar face shape!

retreiving parallel lines and orthogonal

correct shape

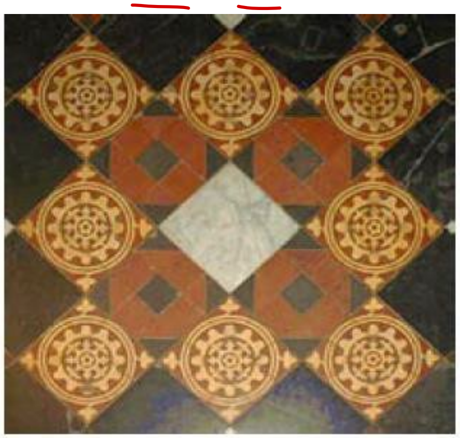


as planar image

2D reconstruction (image rectification): adopted framework

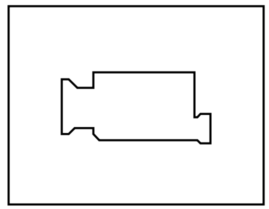
If in principle

planar scene UNKNOWN



a

projective transformation As image of planar scene



You just know its PROJ. then concurrency is INVARIANT

projectivity H

you want to reconstruct ORIGINAL scene model

you don't know camera position and parameter

image

KNOWN

c

You can extract relevant points in the image

IMAGE ANALYSIS!

- planar scene: unknown
- relative pose of scene and camera: unknown
- camera parameters (e.g. focal distance): unknown

UNKNOWNS!

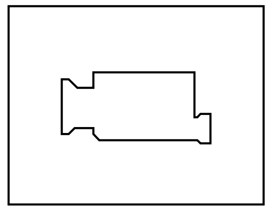
→

scene-to-image projective **mapping** $\boldsymbol{H}$ **: unknown**

2D reconstruction (image rectification): adopted framework

planar scene $x = H^{-1} x'$ $\leftarrow$ $x' = H x$ UNKNOWN! image

a



projectivity H

IF H was known, you can just map image to scene by H^{-1}

c

- planar scene: unknown
- relative pose of scene and camera: unknown
- camera parameters (e.g. focal distance): unknown

$\rightarrow$

scene-to-image projective **mapping** $\boldsymbol{H}$: **unknown**

2D reconstruction (or image rectification) problem formulation

definition

given: an image of an **unknown** planar scene

recover a ***model*** of the scene

Scene: a set of **unknown** points $\mathbf{x}_i$ on a plane (original scene)

Image: a projective transformation of the scene → $\mathbf{x}'_i = H\mathbf{x}_i$

the image points $\mathbf{x}'_i$ are **known**, but H is **unknown**

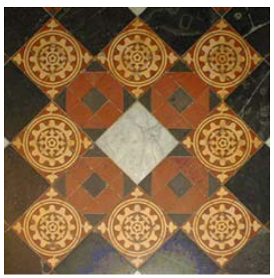
of recovering the model...

Difficulty: the projective mapping H is **unknown**

→ we can not simply invert the mapping H

2D reconstruction problem: utopistic formulation

⇓ in practice...



unknown
planar scene

unknown
projective
map $H??$

H

known image

design a
mapping
H_R
$\simeq$
H^{-1}

recover original image by reconstructing H_R, BUT this is HARD because there are MANY unknowns!

$\simeq H \cdot H^{-1} = I_3$

recovered
(reconstructed)
scene

(should be)

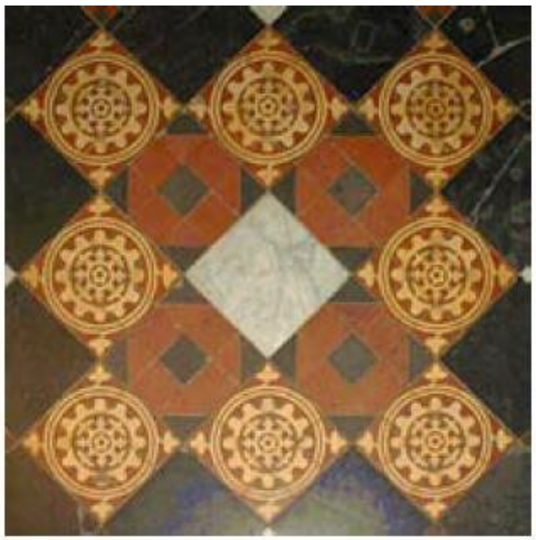
But you don't know H!

combined mapping: an identity $\mathbf{I}_3$

if only the image is given → too many unknowns (8)

to transform the PROBLEM from unsolvable to solvable you have 2 main steps...

2 STRATEGY to combine



a

?

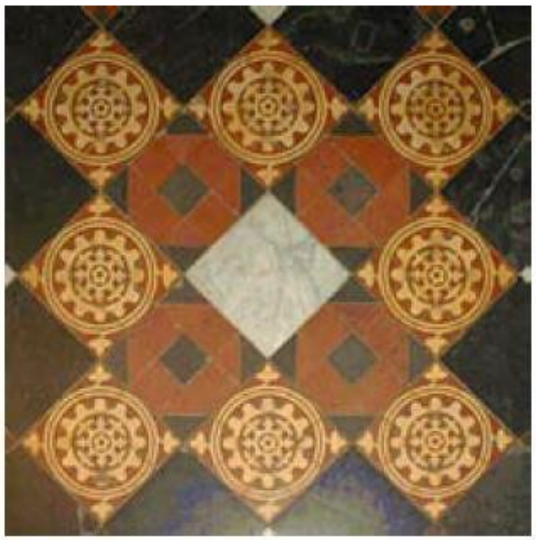
c

NOT use only the image... add additional information!

Combined strategy:

1. **reduce unknowns**
2. **add constraints** (try to add constraint)

if only the image is given → too many unknowns (8)



a

?

c

Use additional KNOWLEDGE
- some lines are parallel
- grid is rectangular....

ADD INFORMATION about scene elements

Combined strategy:

1. reduce unknowns

2. add constraints

combine this to make it solvable!

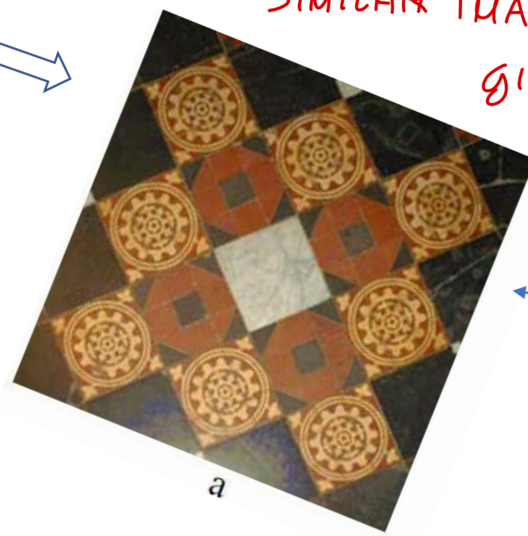
1. Reduce unknowns

shape reconstruction: 4 unkns = 8 - 4

You are happy to recover a SIMILAR IMAGE, not original 100%.
give up on 8 unkns,
just SHAPE!

c

given image



a

similarity

a

reconstructed scene:
same shape as the original,
but different size and pose

Instead of reconstruct original scene, it is enough to reconstruct the shape often! NOT Absolute position...
JUST the shape is enough, NOT exact pose!

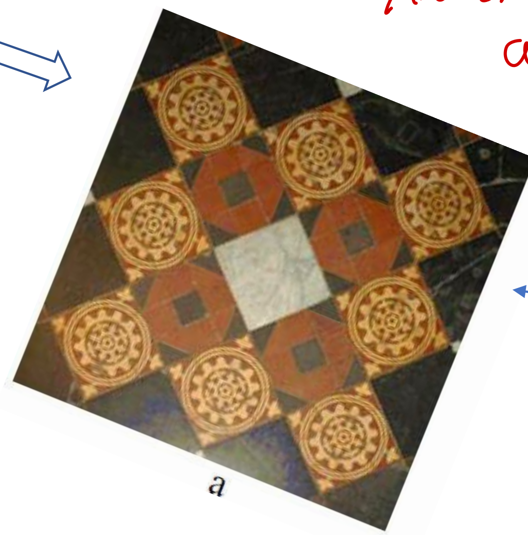
shape reconstruction: 4 unkns = 8 - 4

give up on size, position, orientation...
you skip x, y, θ position/orient.
and a scaling factor s size!

happy with $8 - 4 = 4$ unkn
(4 dof less)

c

given image



a

similarity

a

it is often enough
computer vision
achievement...

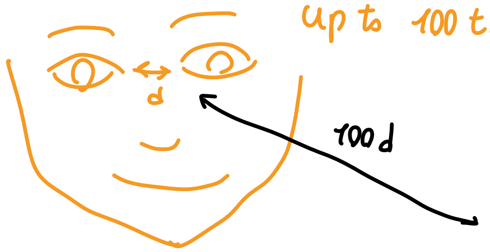
reconstructed scene:
same shape as the original,
but different size and pose

in CV we don't care
about size sometimes, in fact, since we use a CAMERA,
due to central PROJECTION MODEL...

in CV we don't care about size sometimes, in fact, since we use a CAMERA, due to central PROJECTION MODEL... → you don't distinguish between object with same shape one FAR (big) while other small CLOSE

image analysis often we don't care too much about scale (hard to discriminate) ←

against daily experience, we know how to guest size, BUT we use knowledge about object + 2 eyes → I use triangulation to reconstruct shape + size up to 100 times eyes distance



discriminate size

When further, even 2 eyes view don't count size, image order doesn't work...

I discriminate size observing invariant property like gravity affecting oscillations when walking larger than small person, due to timing of steps related to invariant gravity + common size knowledge by human experience!

we use also element relative size...

+ we guest by looking motions etc...

We use several strategies to discover size

BUT from multiple images without view point distance, size cannot be recognized automatically

Also ORIENTATION is hard,

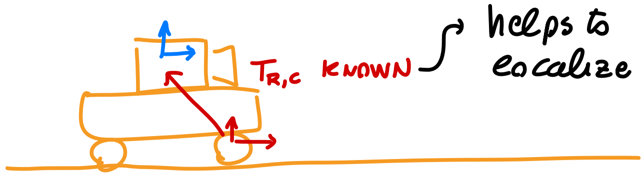
But can be important to find it!

↓

to discover also about orientation you calibrate the camera determining its pose wrt some KNOWN points of the world (like the wheels of a ROBOT)

↓

determining relative position between camera and important reference, LOCALIZATION between estimation of pose of object wrt camera

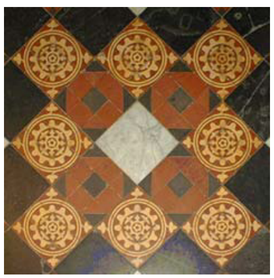


After reducing unkn from 8 to 4... Realistic PROBLEM

2D reconstruction: 1. reduce unknowns

its enough to rectify image related to original by SIMILARITY!

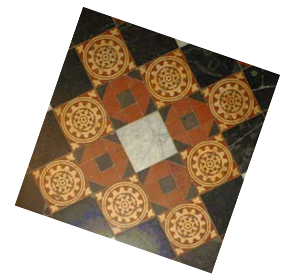
unknown projective map $H??$



unknown original scene

known image

design a mapping H_R



recovered (reconstructed) scene: same **shape** as original

we focus on ||SHAPE RECONSTRUCTION||

combined mapping: a **similarity** H_S not an identity

only 4 unkn instead of 8!

METHOD 2 to reduce UNKN

1. Reduce unknowns

(less satisfactory) affine reconstruction: often used as an intermediate step towards shape reconstruction

sometimes as first step towards shape reconstruction

① Affine reconstruction (initial step intermediate)

NOT so good

② from Affine to Shape

⋮

affine reconstruction: 2 unkns = 8 - 6

some improvement from picture

c

given image

observed image of unkn planar scene

closer to original ←

b

what we reconstruct is still NOT same shape, BUT parallelism is preserved! NOT final result but good

affinity

related by affinity

a

reconstructed scene:
parallelism preserved
different shape, pose and size

affine reconstruction: 2 unkns = 8 - 6

6 DOF on affinity,
(x, y, θ, s + 2 additional DOF)
due to shape lost

c

given image

recover model as affine model of real scene (more distant from original scene rather than similarity!) → NOT very happy about this! BUT could be necessary as 1st step

b

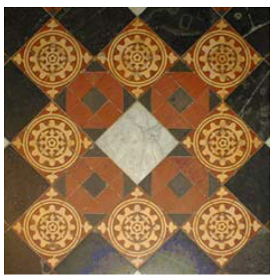
reconstructed scene:
parallelism preserved
different shape, pose and size

affinity

a

2D reconstruction: 1. reduce unknowns

unknown projective map $H??$



unknown original scene

known image

design a mapping H_R

recovered (reconstructed) scene: affine to the original

Affinity relationship

(less powerfull than before)

combined mapping: an affinity H_A

STRATEGY 2 to solve reconstruction

2. Add constraints

additional information extracted from some source to make reconstruction feasible

Add constraints (from additional information): two cases

typical additional information depends on our goal..

GOAL OF

1) - **affine** reconstruction

we need 2 equations constraint, by affine we have to go from 8 to 6, so 2 CONSTRAINTS needed

(reconstructed scene is an affine mapping of the original one)

2) - **shape** reconstruction

we need 4 constraints add info to reduce from 8 to 4!

(reconstructed scene is a similarity mapping of the original one)

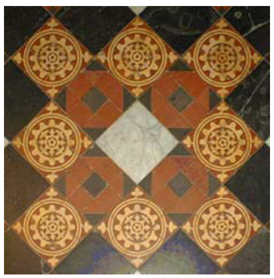
IF we first do AFFINE Reconstruction, than from that shape require 2 additional ... (2 + 2 into 2 separate steps)

(2.1) fix just 2 constraints

affine reconstruction

2D affine reconstruction problem

unknown projective map $H??$



unknown original scene

known image

design a mapping H_{AR}

recovered (reconstructed) scene: affine to the original

must be affine reconstruction..

combined mapping: an affinity H_A

A theorem on an affine invariant

Usefull for AFFINE RECONSTRUCT.

Theorem. *A projective transformation H maps the line at the infinity $\boldsymbol{l}_\infty$ onto itself (i.e., $\boldsymbol{l}_\infty$ is invariant under the projective transformation H)*

If it preserve ℓ_∞ not point by point, but as recombination

$\Updownarrow$ (IFF)

since we want combined transformation AFFINE, this Theorem provide us additional useful information to accomplish this TASK

H is ***affine***

$\Downarrow$

ℓ_∞ is invariant under any AFFINE trans

Proof: Any point at the infinity $\mathbf{x}_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is mapped onto a point $\mathbf{x}' = H\mathbf{x}_\infty$ also at the infinity if and only if the third coordinate of $\mathbf{x}'$ is = 0, i.e.,

transformation

$$\forall (x, y)\ [v_1 \quad v_2 \quad 1] \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0 \rightarrow [v_1 \quad v_2 \quad 1] = \underbrace{[0 \quad 0 \quad 1]}$$

it depends on the third row of the matrix H

with this last column you ensure that TRANSFORMATION is affine and a point @ ∞ remain still at ∞

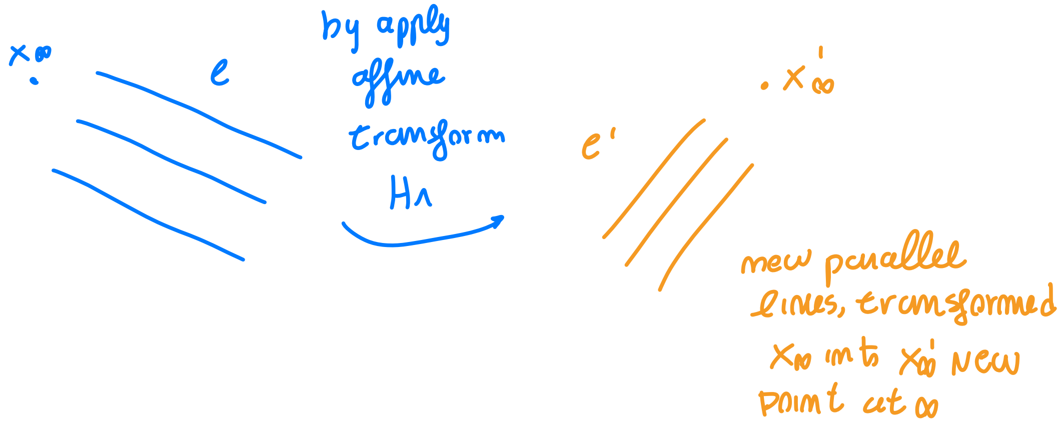
namely,

$$H = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \text{ is } \textbf{affine}$$

other then algebrically,

GEOMETRICALLY the theorem is proven by the fact that parallelism is presewed in Affinity!

the point at infinity x_∞ is assouated to a direction



How to apply the THEOREM...

Application to affine reconstruction

The given image is a general projective mapping of the original scene

→ from planar scene, MAPPING

the vanishing line $\boldsymbol{l'}_{\infty}$ (i.e. the image of the line at the infinity $\boldsymbol{l}_{\infty}$ of the scene) is in general $\neq \boldsymbol{l}_{\infty}$!! New line NOT as l_∞, NOT affine in principle !

l'_∞ remain @ ∞ if H is affine, BUT NOT occurs often! → l' in general

Use $\boldsymbol{l'}_{\infty}$ as additional information: if we apply to the image a new projective mapping H_{AR} that maps $\boldsymbol{l'}_{\infty}$ back to $\boldsymbol{l}_{\infty}$, we obtain a new, modified image

The (new) image of the line at the infinity $\boldsymbol{l}_{\infty}$ is again $\boldsymbol{l}_{\infty}$ (itself)

IF in real scene we can find l'_∞ new line at infinity

→

From the theorem, the obtained model (i.e. the new image) is an **affine** mapping of the original scene

→

The obtained model is an **affine reconstruction** of the scene whenever ℓ_∞ is mapped onto itself... the transform is affine!

Application to affine reconstruction

$\ell'_\infty \to \ell''_\infty \equiv \ell_\infty$

↳ starting from original scene → image is a general mapping (PROJECTIVE)

The given image is a general projective mapping of the original scene

this maps ℓ_∞ into a generic finite line! → if we can discover where is ℓ'_∞ image... Then just take any H_{AR} that map ℓ'_∞ to ℓ_∞

the vanishing line $\boldsymbol{l}'_\infty$ (i.e. the image of the line at the infinity $\boldsymbol{l}_\infty$ of the scene) is in general $\neq \boldsymbol{l}_\infty$!!

because of Theorem, just reprojecting ℓ'_∞ to ℓ_∞, we can use this partial transform, than combined mapping

- **Use $\boldsymbol{l}'_\infty$ as additional information**: if we apply to the image a new projective mapping H_{AR} that maps $\boldsymbol{l}'_\infty$ back to $\boldsymbol{l}_\infty$, we obtain a new, modified image

The (new) image of the line at the infinity $\boldsymbol{l}_\infty$ is again $\boldsymbol{l}_\infty$ (itself)

IF ℓ_∞ is mapped to ℓ'_∞ infinity: AFFINE

→

COMBINED transformation $H \to H_{AR}$ is AFFINE

From the theorem, the obtained model (i.e. the new image) is an **affine** mapping of the original scene

→

The obtained model is an **affine reconstruction** of the scene

• mapping 1 line = -2 DOF

l_∞ ≡ ORIGINAL ℓ_∞ of scene

H_{AR}

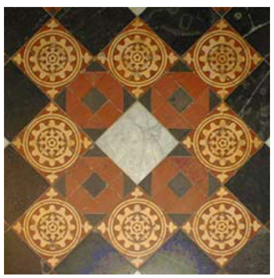
design mapping

6 DOF ! AFFINE

8 DOF

unknown projective map H??

l'_∞



unknown original scene

known: Image + l'_∞

design a mapping H_{AR}

recovered (reconstructed) scene model: affine to the original

combined mapping: affine mapping H_A

I fixed 2 DOF by ℓ_∞ mapping, eq. to AFFINE MODEL, better than the generic projective mapping ...

IN TECHNICHAL TERMS... how to find H_{AR} once you have ℓ'_∞ (image of ℓ_∞)

Affine rectification from the image $\boldsymbol{l}'_\infty$ of the vanishing line $\boldsymbol{l}_\infty$

suppose you KNOW it

- Image of points at the infinity = vanishing points $\mathbf{v_1}, \mathbf{v_2}$ → vanishing line $\boldsymbol{l}'_\infty$

a good matrix is ANY MATRIX!

$$\boldsymbol{l}'_\infty = \mathbf{v_1} \times \mathbf{v_2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

maintain invertibility + any NON SINGULAR matrix here is OK!

- Affine rectification matrix

$$H_{AR} = \begin{bmatrix} * & * & * \\ * & * & * \\ & \boldsymbol{l}'^{T}_\infty & \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ a & b & c \end{bmatrix}$$

WHY?

transformation that maps back ℓ'_∞ to ℓ_∞...

- Affine reconstructed model $M_A = H_{AR} * \text{Image}$

Affine rectification from the image $\boldsymbol{l}'_\infty$ of the vanishing line $\boldsymbol{l}_\infty$

- Image of points at the infinity = vanishing points $\mathbf{v_1}, \mathbf{v_2}$ → vanishing line $\boldsymbol{l}'_\infty$

this H_{AR} choice is good because,

$$\boldsymbol{l}'_\infty = \mathbf{v_1} \times \mathbf{v_2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

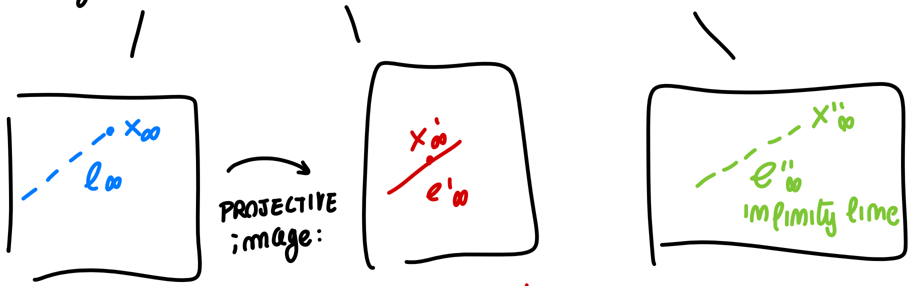
- Affine rectification matrix

$$H_{AR} = \begin{bmatrix} * & * & * \\ * & * & * \\ & \boldsymbol{l}'^{T}_\infty & \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ a & b & c \end{bmatrix}$$

WHY?

- Affine reconstructed model $M_A = H_{AR} * \text{Image}$

from original scene, image, New reconstructed model



$$\begin{cases} \ell_\infty = [0 \;\; 0 \;\; 1]^T \\ x_\infty = [x \;\; y \;\; 0]^T \end{cases}$$

$x'_\infty = [x' \; y' \; w']$ new point, no more @ infinity ↳ generic point

$\equiv$ belongs to e'_∞ (being e'_∞ as set of image of x_∞)

since incident

→ H_{AR}

$$e'^{T}_\infty x'_\infty = 0 \text{ (incidence)}$$

than taking $H_{AR} = \begin{bmatrix} * \\ e'^{T}_\infty \end{bmatrix}$ this will be such that x''_∞ as reconstruction of x'_∞ to x_∞

$H_{AR} \cdot x'_\infty$? using e'_∞ as last row..

$$x''_\infty = [* \;\; * \;\; \underset{\uparrow\; (e'^{T}_\infty)\, x'_\infty}{0}]$$

so it is a POINT at infinity by definition

than, with this choice of H_{AR} any x'_∞ projection of points on ℓ_∞ into ℓ'_∞ will be mapped into x''_∞ all [* * 0] points at infinity! so combine mapping is AFFINE overall

OUR OBJECTIVE!

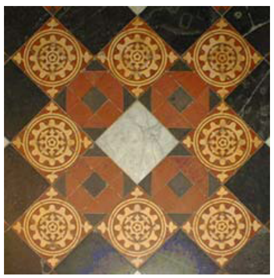
Shape $\underbrace{\text{(Euclidean)}}_{\text{in our restricted planar world}}$ reconstruction

PURPOSE

2D shape reconstruction problem

unknown projective map $H??$

UNKNOWN CAMERA



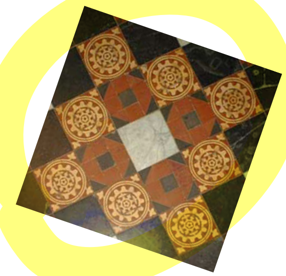
unknown original scene

known image

PROJECTIVE image

"Shape Reconstruction"

design a mapping H_{SR}



recovered (reconstructed) scene model: same shape as original

combined mapping: a **similarity** $\boldsymbol{H_S}$

↝ recover model similar to original one!

↓

to do that, we can use a technique similar to the one before

- find some invariant property under a mapping!

↳ we do the same and use a Theorem

A theorem on an invariant under similarities

Theorem. *A projective transformation H maps the **circular points** $\boldsymbol{I}$ and $\boldsymbol{J}$ onto themselves (i.e., $\boldsymbol{I}$ and $\boldsymbol{J}$ are invariant under the projective transformation H)*

$\Updownarrow$ (IFF) valid in both way

*H is **a similarity***

2 kinds of proof: 1)

Proof: Multiplying a similarity matrix H_S by circular point $\boldsymbol{I}$, a multiple of $\boldsymbol{I}$ is obtained. Analogous result is obtained for the other circular point $\boldsymbol{J}$

more simply

$\{\boldsymbol{I}, \boldsymbol{J}\}$ = circle ∩ $\boldsymbol{l}_\infty$ ------- H_S ------> other circle ∩ $\boldsymbol{l}_\infty$ = $\{\boldsymbol{I}, \boldsymbol{J}\}$

• Proof 1: by multiply H_s similarity by I or J you still obtain CIRCULAR POINT

$$H_s = \begin{bmatrix} R & \begin{matrix} t_x \\ t_y \end{matrix} \\ 0 \; 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = [1 \;\; i \;\; 0]^T$$

$$H_s \cdot I = \begin{bmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \\ 0 \end{bmatrix}$$

this is still a CIRCULAR point! it is a point at infinity and is CIRCULAR by means of i

• Proof 2: a circular point is the intersection between any circle and ℓ_∞

than, the transformed circular point after similarity dan be found by apply similarity to circumference and ℓ_∞

- $H_s \ell_\infty \rightarrow \ell'_\infty$ as ℓ @ ∞

  any similarity is also Affine transform, them ℓ_∞ is mapped onto itself

- C circumference after similarity H_s.. since similarity presere schape... $H_s C = C'$ is still a circumference!

SO, the I, J circular points are mapped into themself by any similarity

↓ use the same reasoning of AFFINE RECONSTRUCTION, but NOW considering CIRCULAR points and their image

Application to shape reconstruction

so, if I know I',J' image of the CIRCULAR POINTS... we can use it to solve shape reconstruct.

The given image is a general projective mapping of the original scene

→

(KNOWN as additional INFORMATION)

the image $(\boldsymbol{I}',\boldsymbol{J}')$ of circular points $(\boldsymbol{I},\boldsymbol{J})$ of the scene plane is in general $\neq (\boldsymbol{I},\boldsymbol{J})$.

↓

Use $(\boldsymbol{I}',\boldsymbol{J}')$ as **additional information**: if we apply to the image a new projective mapping H_{SR} that maps $(\boldsymbol{I}',\boldsymbol{J}')$ back to $(\boldsymbol{I},\boldsymbol{J})$, we obtain a new, modified image

The (new) image of the circular points $(\boldsymbol{I},\boldsymbol{J})$ is again $(\boldsymbol{I},\boldsymbol{J})$ (themselves)

→

From the theorem, the obtained model (new image) is *similar* to the original scene

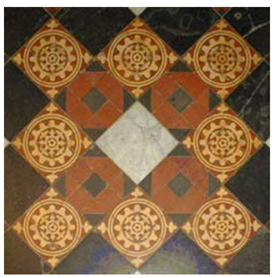
→

The obtained model is a **shape reconstruction** of the scene

2D shape reconstruction problem

you have KNOWLEDGE of I',J' circular points

unknown projective map $H??$



unknown original scene

known: image + (I', J')

design a mapping H_{SR}

maps into a RECTIFIED image with ORIGINAL SHAPE

you map I',J' into original I,J
↓
so, combined mappings will map I,J to themself
↓
SIMILARITY

reconstructed scene model: same shape as original

|| you need to estimate image of CIRCULAR POINTS ||!

combined mapping: a **similarity** H_S

Shape (Euclidean) reconstruction via Singular Value Decomposition

→ we need a Technicality to do that!

(by using some tools like MATLAB, in Algebra)

to reconstruct HSR we rely on SVD matrix decomposition!

(well done in MATLAB by using svd(Matrix))

We use SVD because our idea is to take the image of circular points I', J' and map them in I, J by H_{SR}
original

BUT we have to deal with a system of equations!

impose $\begin{cases} I' \rightarrow I \\ J' \rightarrow J \end{cases}$ $\underbrace{\text{pair of points}}$ to force map

complex entity to manage algebrically

$\downarrow$

in help to this problem, we can use the dual conics to circular point...

as degenrate dual conic based on two points I, J and cares the same information as the two points

$\downarrow$

you take $C^*_\infty = IJ^T + JI^T$ $\leftarrow$ building this conics, you get

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} = C^*_\infty$$ dual degenrate conic rank = 2

conic dual to CIRCULAR POINTS

it is useful because it is a simple element no more pair of entity
↳ more comfortable to manipulate algebrically BUT keep same information! equivalent to I, J and from this C^*_∞ we can derive I, J

↳ so we can start from C'^*_∞

Shape (Euclidean) reconstruction

Instead of using the image I' and J' of the circular points, one can use the image $C_\infty^{*\prime} = I' J'^T + J' I'^T$ of the conic dual to the circular points C_∞^*

↳ we want to find reconstruction mapping that map $C_\infty^{*\prime}$ into C_∞^*

$$C_\infty^* = IJ^T + JI^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as advantage, it is a single entity, you know how to manipulate this single matrix!

$$C_\infty^{*\prime} = I'J'^T + J'I'^T \neq C_\infty^*$$

(simpler than manipulating couple of points..)

Singular value decomposition

taking a symm matrix, $C_\infty^* = C_\infty^{*T}$ also $C_\infty^{*\prime}$ is symm (its image)

$$\text{Why} \quad \mathbf{C}_\infty^{*}{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^{\mathsf{T}} \ ?$$

Normally: SVD (Singular Value Decomposition)

It can be decomposed as product of 3 entity

IN SYMMETRIC CASE

SINGULAR VALUES

$$\mathbf{C}_\infty^{*}{}' = \mathbf{V}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\mathbf{U}^{\mathsf{T}}$$ with **U** and **V** orthogonal

But $\mathbf{C}_\infty^{*}{}'$ is symmetric → $\mathbf{C}_\infty^{*}{}'^{\mathsf{T}} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathsf{T}} = \mathbf{V}\mathbf{D}\mathbf{U}^{\mathsf{T}} = \mathbf{C}_\infty^{*}{}'$

and SVD is unique → $\mathbf{U} = \mathbf{V}$ for symmetric MATRIX

Observation : **H=U** orthogonal (3x3): not a P^2 isometry

Use of $C_\infty^{*\prime}$ in shape reconstruction

← that's usefull, since:

finding a projectivity H_{SR} that maps $(\boldsymbol{I}', \boldsymbol{J}')$ back to $(\boldsymbol{I}, \boldsymbol{J})$

reduces to finding a projectivity H_{SR} that maps $C_\infty^{*\prime}$ back to C_∞^{*}

Using the transformation rule for dual conics under projective mappings we get

$C_\infty^{*\prime} = I' J'^T + J' I'^T$

$$C_\infty^{*} = H_{SR}\, C_\infty^{*\prime}\, H_{SR}^{\ T}$$

solving this for $C_\infty^{*\prime}$ yields

↳ we want to apply transform that brings I', J' to I, J back to original circular points

$$C_\infty^{*\prime} = H_{SR}^{\ -1} C_\infty^{*}\, H_{SR}^{\ -T} = H_{SR}^{\ -1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (H_{SR}^{\ -1})^T$$

but, from SVD applied to the symmetric matrix $C_\infty^{*\prime}$

$$(SVD)\, C_\infty^{*\prime} = U_\perp \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_\perp^{\ T} \rightarrow H_{SR}^{\ -1} = U_\perp \rightarrow H_{SR} = U_\perp^{\ -1} = U_\perp^{\ T}$$

i.e., one of the possible ∞^4 solutions

Use of $C_\infty^{*\prime}$ in shape reconstruction

finding a projectivity H_{SR} that maps $(\boldsymbol{I}', \boldsymbol{J}')$ back to $(\boldsymbol{I}, \boldsymbol{J})$

reduces to finding a projectivity H_{SR} that maps $C_\infty^{*\prime}$ back to C_∞^{*}

Using the transformation rule for dual conics under projective mappings we get

as we have seen, given transformation H, then $C^* = H C^{*\prime} H^T$ transformation rule ∀ dual conic

$$C_\infty^* = H_{SR}\, C_\infty^{*\prime}\, H_{SR}{}^T$$

we WANT that H_{SR} bring $C_\infty^{*\prime}$ back to C_∞^* original

solving this for $C_\infty^{*\prime}$ yields

from that equation $\Rightarrow$
$$C_\infty^{*\prime} = H_{SR}{}^{-1} C_\infty^{*}\, H_{SR}{}^{-T} = H_{SR}{}^{-1}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}(H_{SR}{}^{-1})^T$$

we KNOW that shape

it is exactly as SVD ...

decomposing MATRIX by $U C U^T$ (diag)

but, from SVD applied to the symmetric matrix $C_\infty^{*\prime}$

$$(SVD)\, C_\infty^{*\prime} = U_\perp \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} U_\perp{}^T \rightarrow H_{SR}{}^{-1} = U_\perp \rightarrow H_{SR} = U_\perp{}^{-1} = U_\perp{}^T$$

i.e., one of the possible ∞^4 solutions

Use of $C_\infty^{*\prime}$ in shape reconstruction

finding a projectivity H_{SR} that maps $(\boldsymbol{I}', \boldsymbol{J}')$ back to $(\boldsymbol{I}, \boldsymbol{J})$
reduces to finding a projectivity H_{SR} that maps $C_\infty^{*\prime}$ back to C_∞^*
Using the transformation rule for dual conics under projective mappings we get

$$C_\infty^* = H_{SR}\, C_\infty^{*\prime}\, {H_{SR}}^T$$

solving this for $C_\infty^{*\prime}$ yields

as result of svd we get U, σ_i, V
in our case $U = V$

$$C_\infty^{*\prime} = {H_{SR}}^{-1} C_\infty^*\, {H_{SR}}^{-T} = {H_{SR}}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ({H_{SR}}^{-1})^T$$

this is equivalent to SVD of dual conic

but, from SVD applied to the symmetric matrix $C_\infty^{*\prime}$

ORTHOGONAL

$$(SVD)C_\infty^{*\prime} = U_\perp \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} {U_\perp}^T \rightarrow {H_{SR}}^{-1} = U_\perp \rightarrow H_{SR} = {U_\perp}^{-1} = {U_\perp}^T$$

i.e., one of the possible ∞^4 solutions

so I just need $C^{*'}_\infty$ image of the conics dual to circular points

$\downarrow$

then just $svd(C^{*'}_\infty) = U, \sigma_i, V$

$(C^{*'}_\infty = I'J'^T + J'I'^T)$

$\downarrow$

$U = H_{SR}^{-1}$

$\downarrow$

$U^{-1} = U^T = H_{SR}$

this is the mapping we were looking for

$\uparrow$

ORTHOGONAL

$\leftarrow$ we saw 2D reconstruction by MATRIX operation involved by SVD,

by I', J' OR directly $C^{*'}_\infty$ $\Leftarrow$ used to reconstruct planar scene from an image

$\uparrow$

this can be computed as $= I'J'^T + J'I'^T$

you can use it to find shape reconstructing homography which bring $C^{*'}_\infty$ back to C^*_∞ original H_s

RECTIFYING HOMOGRAFY

designed by SVD, apply it to $C^{*'}_\infty$ than you get

$= U C^*_\infty V^T$ this hold since this mimic transformation rule of similarity

$V = U$ in symmm

$\Leftarrow$ here you can go froming to model by $H_{SR}^{-1} = U_\perp$

we need $H_{SR} = U_\perp^T$ (orth) $\Leftarrow$ you get it from SVD

There is a numerical issue in image rectification

22/10

← as seen from some examples, this algebraic theory has UNCERTAINTY issue involved on that... ⟹

Accuracy issue in image rectification

Ideal situation

- In principle, given $C_\infty^{*\prime}$

$$\text{svd}(C_\infty^{*\prime}) = U\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

this should be like that

- But, due to noise and numerical errors, SVD gives

purpose is to find Hrect even if NOT ORTHOGONAL!

$$\text{svd}(C_\infty^{*\prime}) = U\begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & 0\end{bmatrix}U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

Numerical problem, you don't get $\begin{bmatrix}\lambda & & \\ & \lambda & \\ & & 0\end{bmatrix}$ but different SV

- In principle, given $C_\infty^{*\prime}$

$$\text{svd}(C_\infty^{*\prime}) = U\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

- But, due to noise and numerical errors, SVD output is

we can rewrite the matrix like that...

we don't need orthogonal H_{rect}, we can decompose such that $\begin{pmatrix}1 & & \\ & 1 & \\ & & 0\end{pmatrix}$ in the middle...

$$\text{svd}(C_\infty^{*\prime}) = U\begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & 0\end{bmatrix}U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

$$\text{svd}(C_\infty^{*\prime}) = U\begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & 0\end{bmatrix}U^T = U\begin{bmatrix}\sqrt{a} & 0 & 0\\0 & \sqrt{b} & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{a} & 0 & 0\\0 & \sqrt{b} & 0\\0 & 0 & 1\end{bmatrix}U^T$$

real svd decomposition

$$\rightarrow H_{rect}^{-1} = \underbrace{U\begin{bmatrix}\sqrt{a} & 0 & 0\\0 & \sqrt{b} & 0\\0 & 0 & 1\end{bmatrix}}\ (-1) \rightarrow H_{rect} = \begin{bmatrix}1/\sqrt{a} & 0 & 0\\0 & 1/\sqrt{b} & 0\\0 & 0 & 1\end{bmatrix}U^T$$

How we use svd in practice!

To sum up: image rectification from the image $(\boldsymbol{I'}, \boldsymbol{J'})$ of the circular points $(\boldsymbol{I}, \boldsymbol{J})$

1. • Image of the circular points → image of the conic dual to the circular points
(suppose it's known)

$$C^{*\prime}_{\infty} = \boldsymbol{I'}\boldsymbol{J'}^{T} + \boldsymbol{J'}\boldsymbol{I'}^{T}$$

2. • Singular value decomposition

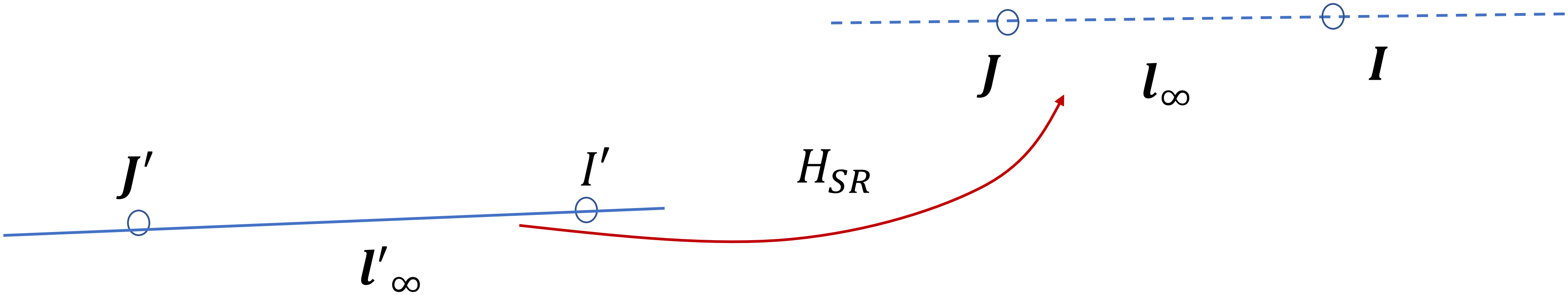
$$\text{svd}(C^{*\prime}_{\infty}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^{T} = H_{SR}^{-1} C^{*}_{\infty} H_{SR}^{-T}$$

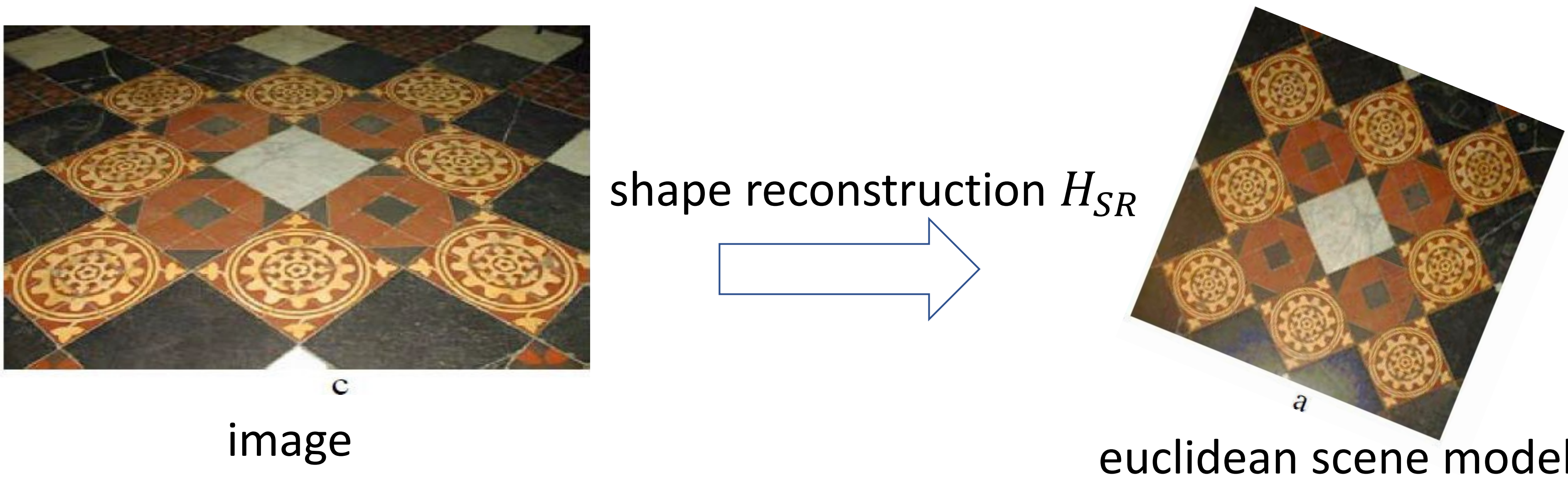
3. • Rectifying transformation (from svd output U)

$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{T}$$

this is enough to recover image up to similarity

4. • Euclidean reconstructed model $M_S = H_{SR} * \text{Image}$


J
l_∞
I
J'
I'
H_{SR}
l'_∞


c
image
shape reconstruction H_{SR}
a
euclidean scene model

How to find $(\boldsymbol{I}', \boldsymbol{J}')$ (or $C_\infty^{*\prime}$) in practical cases?

in practice, finding I', J' is NOT trivial!

- complex entity, NOT visible in the image as features (ABSTRACT entity)

How to find (I', J') (or $C_\infty^{*\prime}$) in practical cases?

In 2D rectification we use (I', J') , or equivalently $C_\infty^{*\prime}$,

as additional information

im PRACTICE, you know something like

How can we find (I', J') (or $C_\infty^{*\prime}$) **?**

from information on the observed scene we derive constraints on $C_\infty^{*\prime}$

possible knowledge:

, in the real scene, you see on image, know 2 lines are ⊥ or parallel (usefull to extract knowledge)

a. known angles between lines (couple of lines)

b. known shape of objects, e.g., circumferences

some ellipses in scene is circumference

c. combinations of a. and b.

c. observation of rigid planar motion → give unknown planar motion, you can extract equivalent info to $C_\infty^{*\prime}$

a. known angles between lines

you know original angle, but
you see the image on projective plane →

REMEMBER: Angle between two lines l and m

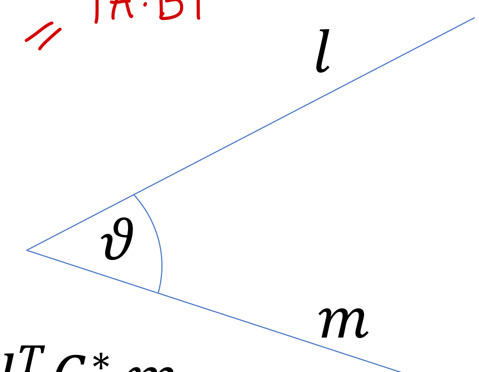
The angle ϑ between $l = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $m = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is the angle between their normals

$\underbrace{[a_1 \quad b_1] \text{ and } [a_2 \quad b_2]}_{}$:

direction of lines! ai bi express direction of normals

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}} = \frac{A \cdot B}{|A \cdot B|}$$

$A = [a_1 \ b_1]$
$B = [a_2 \ b_2]$



we want to use homogeneus coordinate

But, e.g., $a_1 a_2 + b_1 b_2 = [a_1 \quad b_1 \quad c_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \underline{l^T C_\infty^* m}$

using the whole homogeneus vector line

by algebraic computation...

$$\rightarrow \cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

Express the angle ϑ between two lines **in the scene** in terms of elements **in the image**: $l', m', C_\infty^{*\prime}$

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}} = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

in view of transformation rules

$$\text{e.g., } l^T C_\infty^* m = l'^T H H^{-1} C_\infty^{*\prime} H^{-T} H^T m' = l'^T C_\infty^{*\prime} m'$$

line transform → conic transform

all the H are simplyfyed, so it is preserved!

$$\begin{cases} l = H^T l' \\ m = H^T m' \end{cases}$$

inverse mapping

Express the angle ϑ between two lines **in the scene** in terms of elements **in the image**: $l', m', C_\infty^{*\prime}$

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}} = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

in view of transformation rules

$$\text{e.g., } l^T C_\infty^* m = l'^T H H^{-1} C_\infty^{*\prime} H^{-T} H^T m' = l'^T C_\infty^{*\prime} m'$$

$\rightarrow$

so it is preserved!
(also for ($e^T C_\infty^* e$))

$$\cos\vartheta = \frac{l'^T C_\infty^{*\prime} m'}{\sqrt{(l'^T C_\infty^{*\prime} l')(m'^T C_\infty^{*\prime} m')}}$$

cos(θ) of a known angle on original scene, as function of image lines and $C_\infty^{*\prime}$

Express the angle ϑ between two lines **in the scene** in terms of elements **in the image**: l', m', $C_\infty^{*\prime}$

known

$$\underbrace{\cos\vartheta}_{} = \frac{l'^T C_\infty^{*\prime} m'}{\sqrt{(l'^T C_\infty^{*\prime} l')(m'^T C_\infty^{*\prime} m')}}$$

we constraint $C_\infty^{*\prime}$, this correspond to one constraint on $C_\infty^{*\prime}$ unknown!

Here, l' and m' are extracted from the image (see later), whereas $\boldsymbol{C_\infty^{*\prime}}$ is the **unknown** matrix we want to estimate

Known angle ϑ between two **scene lines** → nonlinear eqn on $C_\infty^{*\prime}$

simple constraint

if the scene lines are perpendicular, $\cos \underset{\downarrow}{\overset{(\pi/2)}{\vartheta}} = 0$ → $\boxed{l'^T C_\infty^{*\prime} m' = 0}$ **linear**

it become linear constraint!

Knowledge of l'_∞ → constraints on $C^{*\prime}_\infty$

$$C^{*\prime}_\infty l'_\infty = (I'J'^T + J'I'^T)l'_\infty = I'(\underbrace{J'^T l'_\infty}_{0}) + \mathrm{J}'(\underbrace{I'^T l'_\infty}_{0}) = \mathbf{0}$$

$$C^{*\prime}_\infty l'_\infty = 0$$

2 constraints

line at the infinity l'_∞ = RNS($C^{*\prime}_\infty$)

How many constraints are needed to find $C^{*\prime}_{\infty}$?

3x3 but symm

$\begin{bmatrix} a & d & e \\ * & b & f \\ * & * & c \end{bmatrix}$

in principle 6 dof

(up to scaling factor) −1   −1

Matrix $C^{*\prime}_{\infty}$ is symmetric, homogeneous, and **singular**

↑ by definition it has rank = 2

→ 4 degrees of freedom (e.g. $\boldsymbol{I}'$, $\boldsymbol{J}'$)

→ 4 **independent** constraints are needed

(in principle)

however

singularity is in general a nonlinear constraint

→use 5 linear constraints (orthogonal lines)

or

if l'_{∞} can be derived from parallelism (or two pairs of orthogonal lines with one line in common), use

$l'_{\infty} = RNS(C^{*\prime}_{\infty})$: 2 linear constraints which implies singularity

+ 2 linear constraints from pairs of orthogonal lines

Are the constraints associated to the 5 indicated pairs of orthogonal lines independent? NO: l'_∞ can be derived

you look
for orthogonal
lines in the
original
scene...

NOT all
independent!

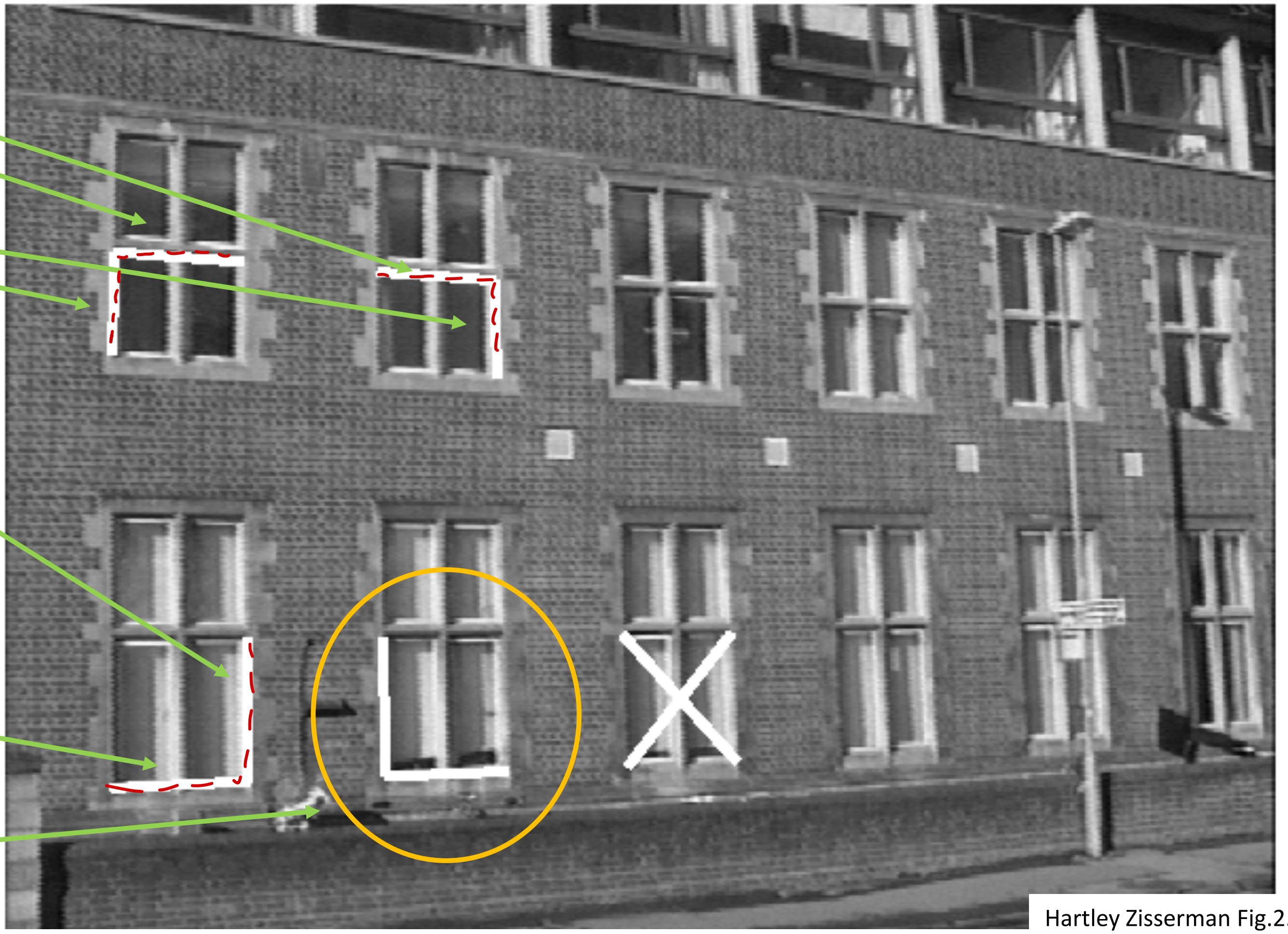
colinear

→ parallel

concurrent at same
vertical vanish. point

→ must concur at same
horizontal vanish. point

→ derivable from
previous contraints



Hartley Zisserman Fig.2.17

some constraints are dependent to others... chose lines carefully

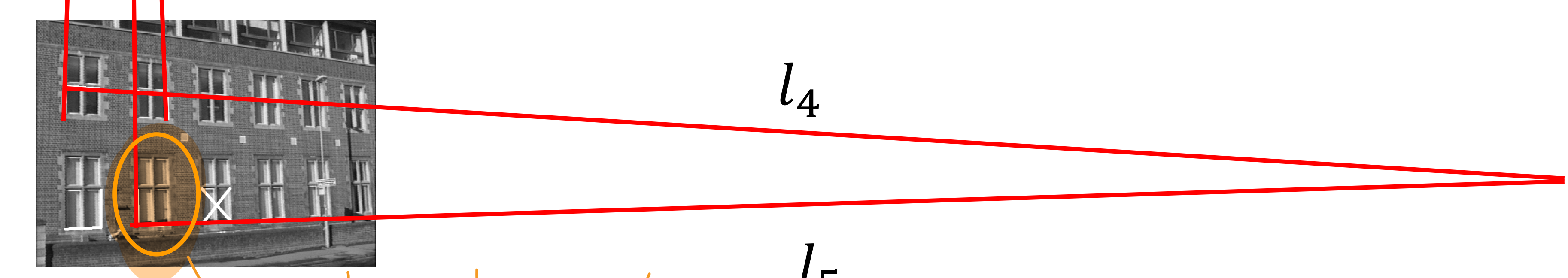
Left-low window lines are constrained by the others:

- vertical line must concur where other vertical lines do
- horizontal line must be colinear to l_5

l_1 l_2 l_3

- → four pairs are sufficient

sometimes 4 are sufficient!



l_4

l_5

this depends on others

a. rectification from pairs of orthogonal lines

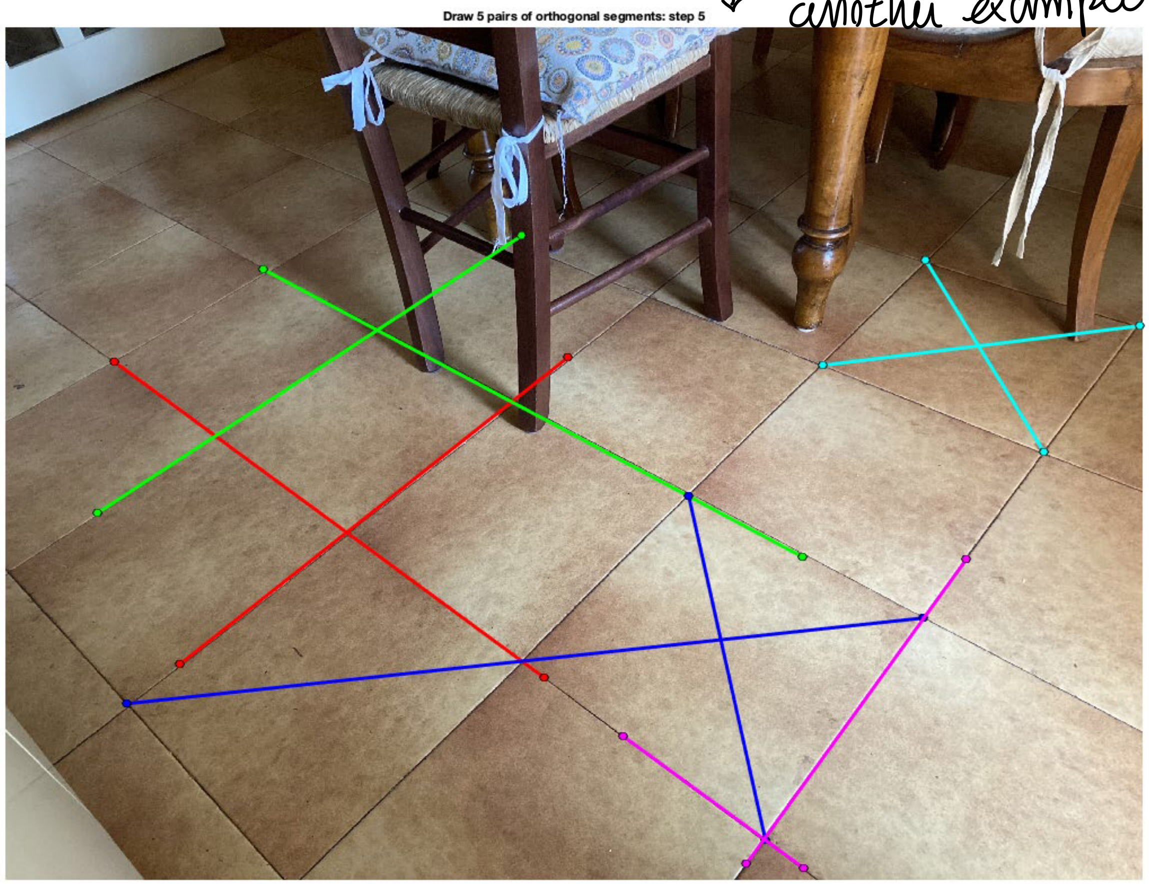
rectifyed image using 4 OR 5 constraints in the form of couples of perpendicular line

$l'^T C^{*'}_{\infty} m' = 0$

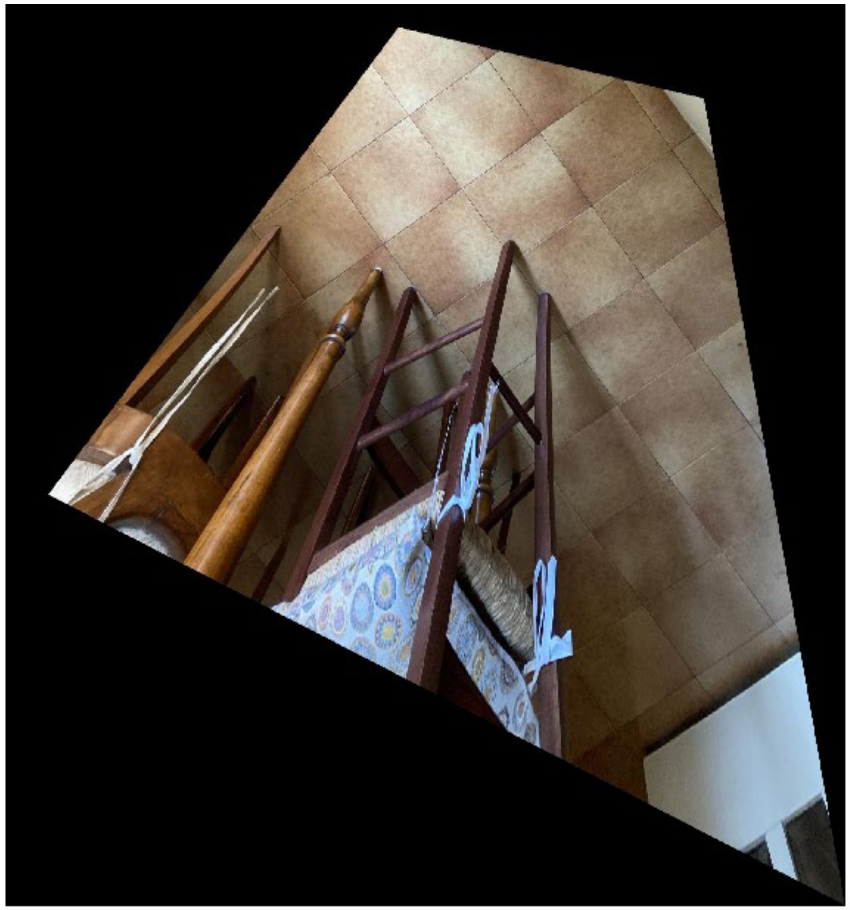
Hartley Zisserman Fig.2.17

Are they all independent?

another example



rectified image



credits: Luca Magri

the use of 4 or 5 pairs... ? (of constraints)

in principle $\underbrace{\det(C^{*\prime}_{\infty})} = 0$

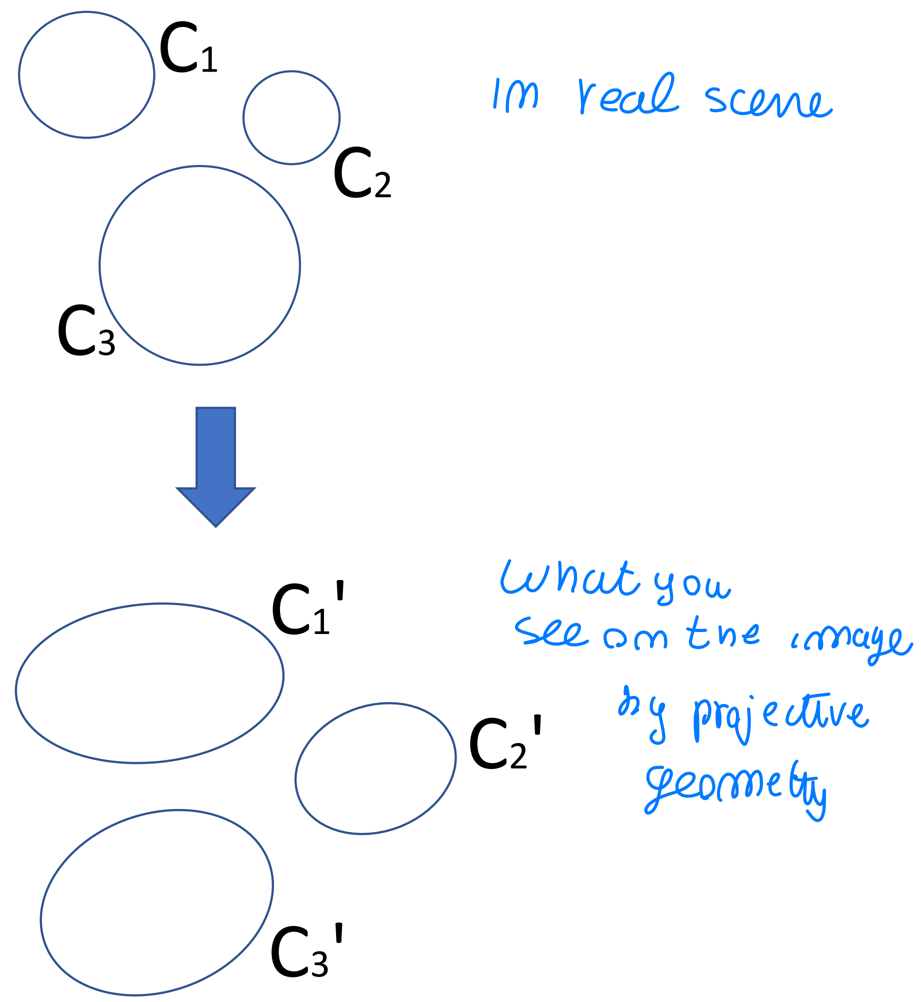
singularity of $C^{*\prime}_{\infty}$ → you need complex composition, singularity cause det = 0, you have complex function of elements... you use 5 constraints

when lot of ⊥ lines, I preffer all linear constraints rather than 4 linear + 1 non linear ← 5 constraints

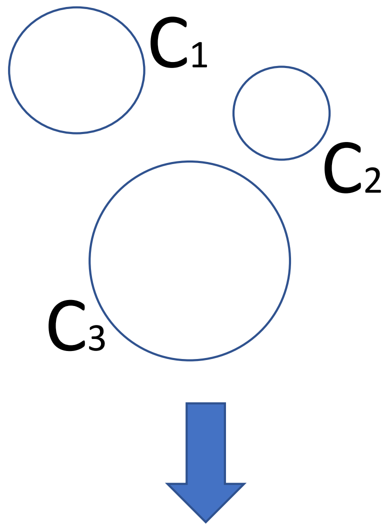
better to use more constraints than how much we need

b. Example: image of circumferences

Example:
image of
circumferences



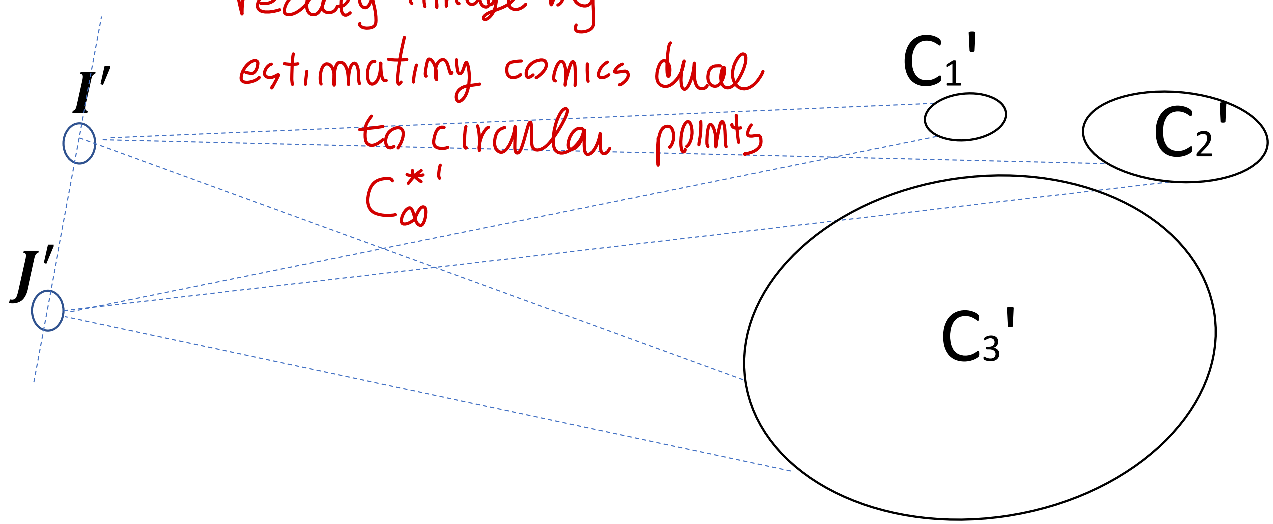
Example: image of circumferences



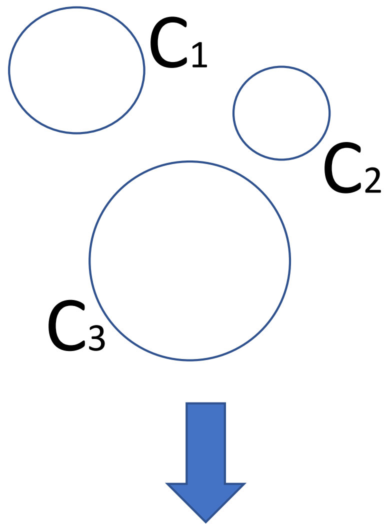
3 circumferences on same plane of table

intersect the images of circumferences
→ image of circular points

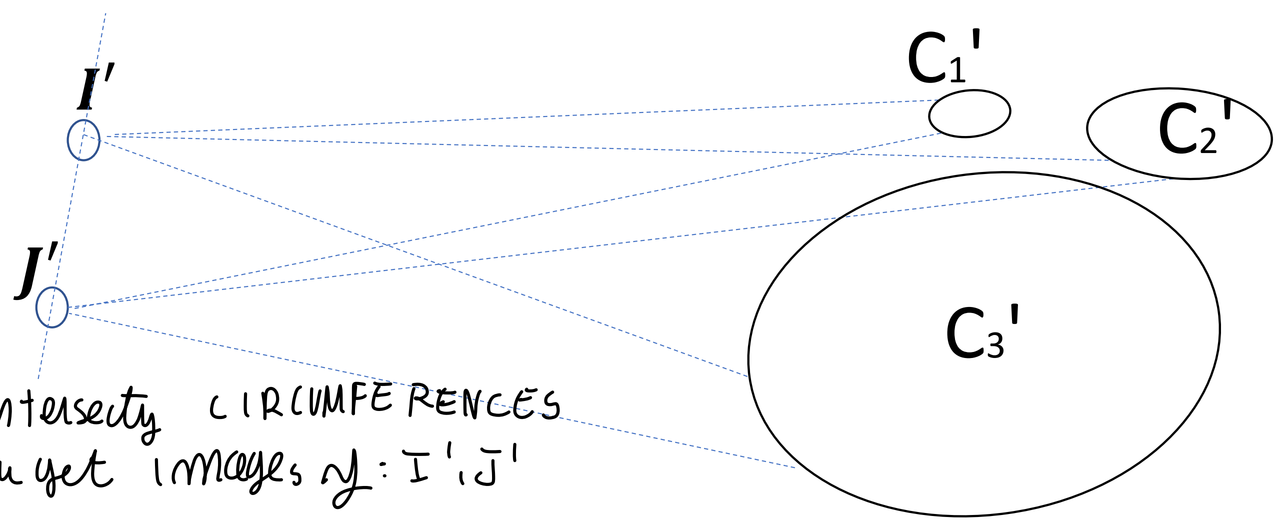
rectify image by estimating conics dual to circular points $C_\infty^{*\prime}$



Example: image of circumferences



intersect the images of circumferences
→ image of circular points



any circumference C
contain I, J info, and
any C' also contains I', J' ⇒ intersecting circumferences
you get images of: I', J'

by intersecting 2 circumferences images

↓

just by 2 you get 4 intersection points
(2 deg 2 problems → gives you $\underline{4 \text{ dof}}$)

↑
4 points
solutions

IF you use
3 circumference,
you can produce solutions
with different pairs! → using 3 circumferences, you get I', J'

Image rectification from the image of circular points: $\{I', J'\} = C_1' \cap C_2' \cap C_3'$

- Image of the circular points → image of the conic dual to the circular points

$$C_\infty^{*\prime} = \boldsymbol{I'J'^T} + \boldsymbol{J'I'^T}$$

usual procedure as before!

- Singular value decomposition

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^* H_{SR}^{-T}$$

- Rectifying transformation (from svd output U)

$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} * \text{Image}$

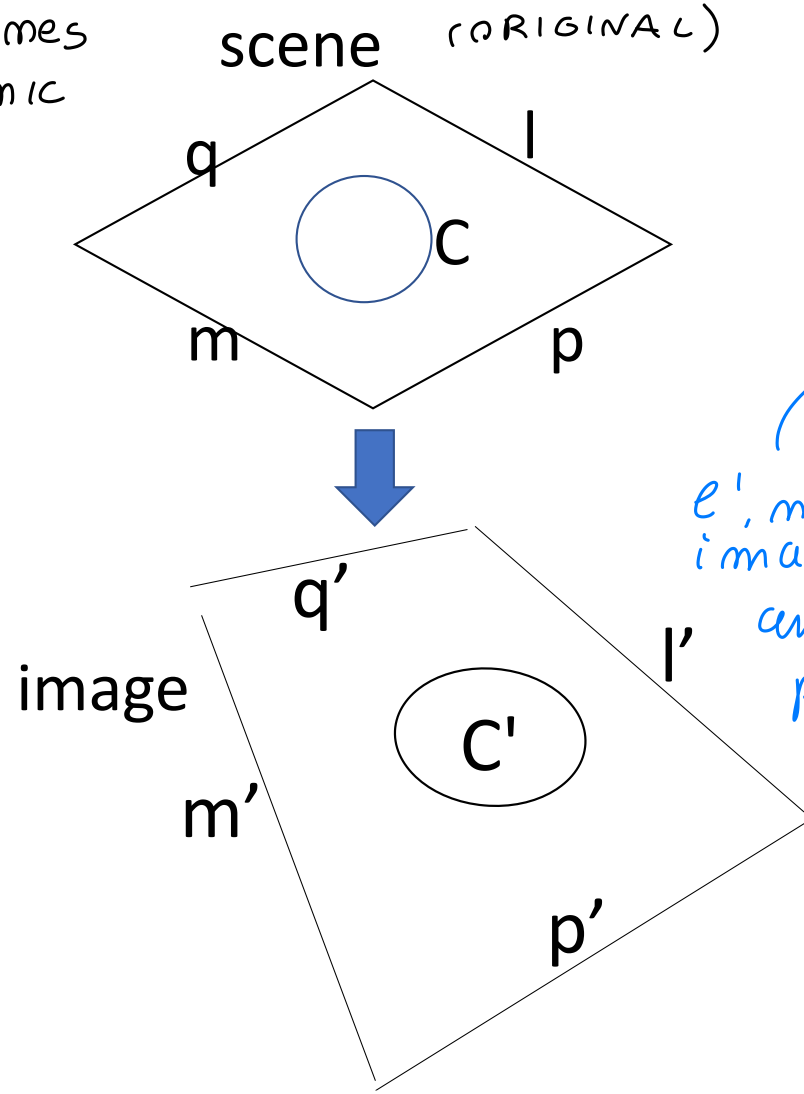
c. Example: circumference + parallelogram

by combining circumference and lines →

(a & b)

Example

you extract some lines and conic



l', m' are images of l // m and also p', q' as p // q

$$\mathbf{v}_2 = \mathrm{l}' \times m'$$

first find the two
VANISHING POINTS
↓ than ℓ'_∞
is the line joining
vanishing points

$$\mathbf{v}_1 = p' \times q'$$

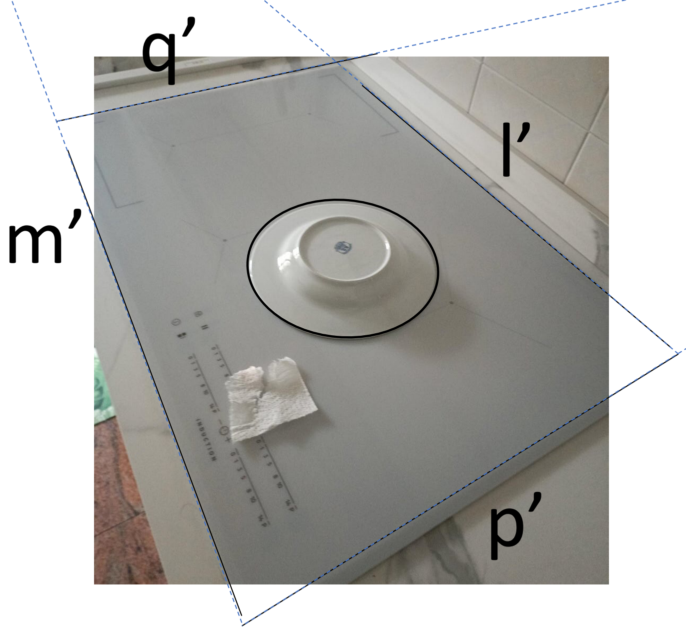


image of the line at the infinity

$\mathbf{v}_2 = \mathrm{l}' \times m'$

by concurrent line to v_1, v_2

$\mathbf{v}_1 = p' \times q'$

$l'_\infty = \mathbf{v}_1 \times \mathbf{v}_2$

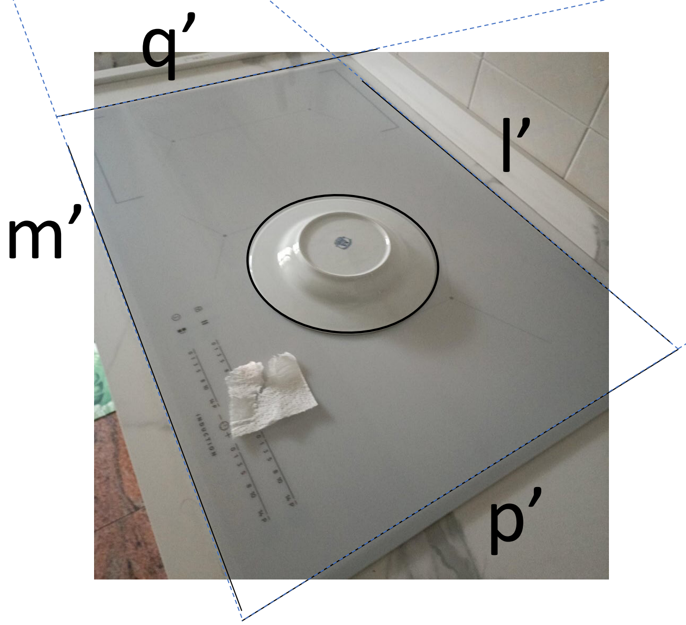
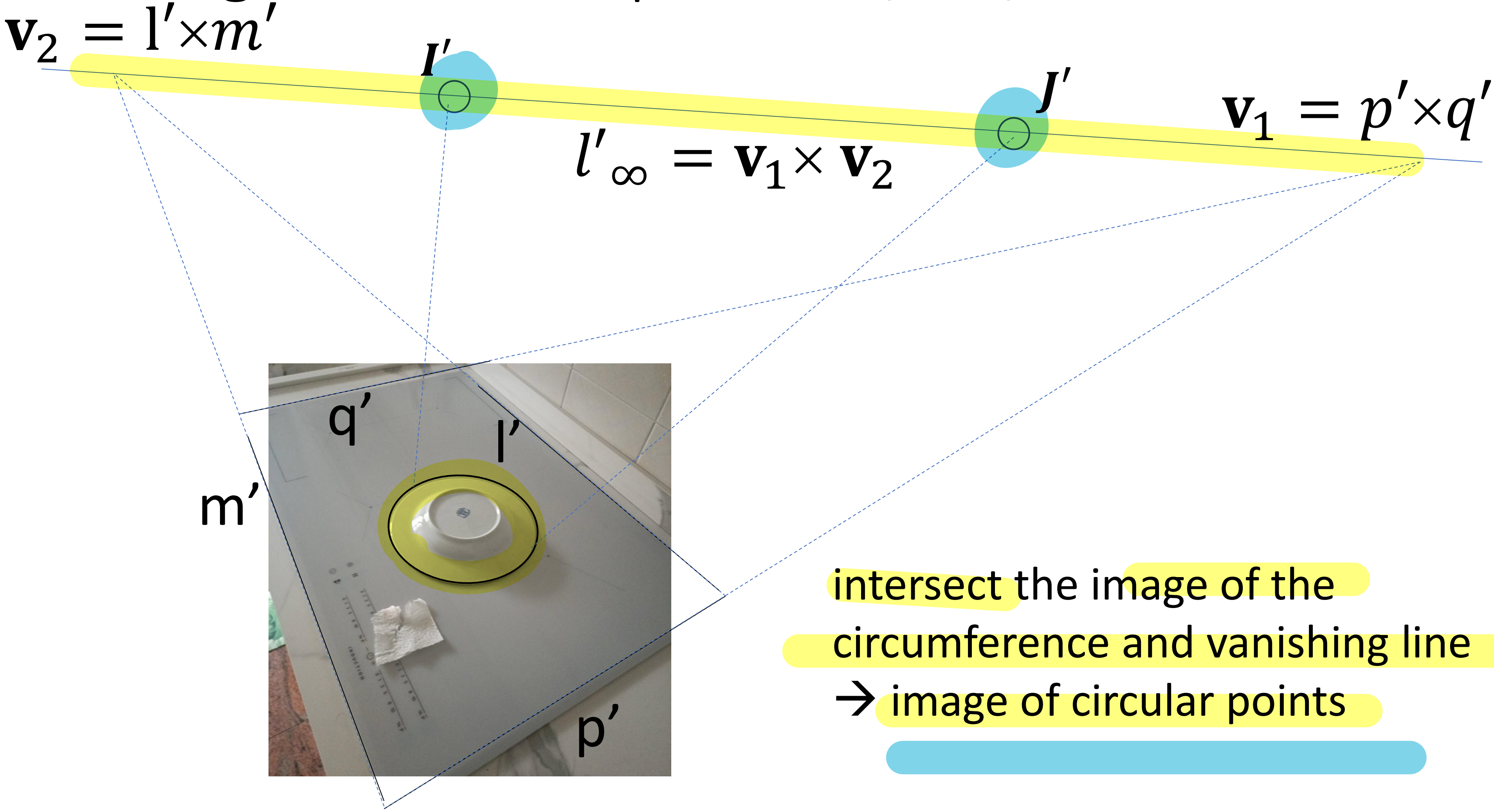


Image of circular points: {I’, J’} = $l’_{\infty} \cap C’$



intersect the image of the
circumference and vanishing line
→ image of circular points

Image rectification from the image of circular points: $\{I', J'\} = l'_\infty \cap C'$

than, usual procedure

...

- Image of the circular points → image of the conic dual to the circular points

$$C_\infty^{*\prime} = \boldsymbol{I'J'^T} + \boldsymbol{J'I'^T}$$

- Singular value decomposition

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^* H_{SR}^{-T}$$

- Rectifying transformation (from svd output U)

$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} * \text{Image}$

d. Example: rectification from planar displacement

↓

given fixed camera, you
see plane moving, and
the plane is moving
with displacement planar
(NOT PITCH / ROLL OR Z motion)

If you can observe both images... you can reconstruct planar scene shape!

Image of a rigid planar motion

Just rotate, and move, on the same plane!

(Fixed camera)

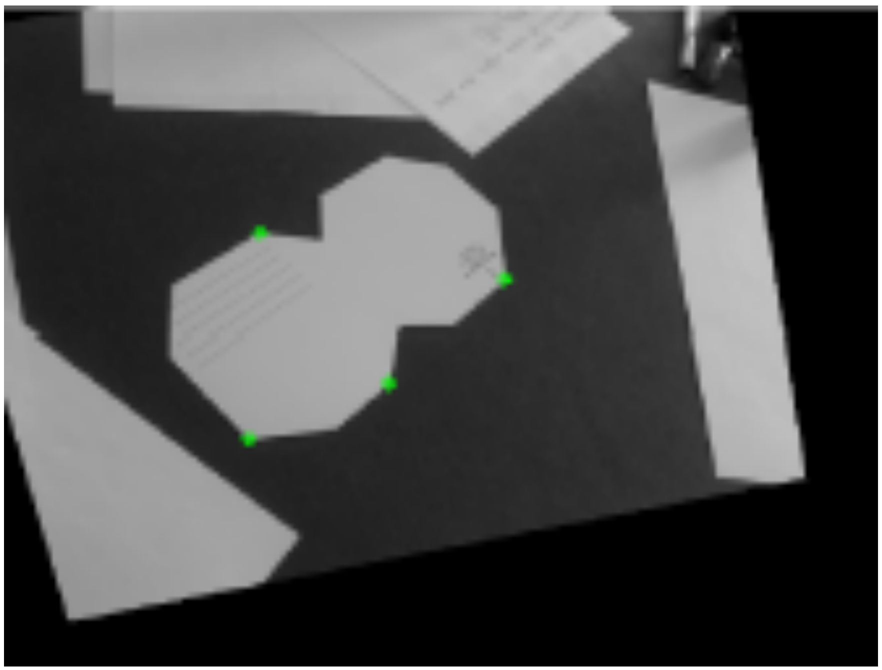


image before displacement

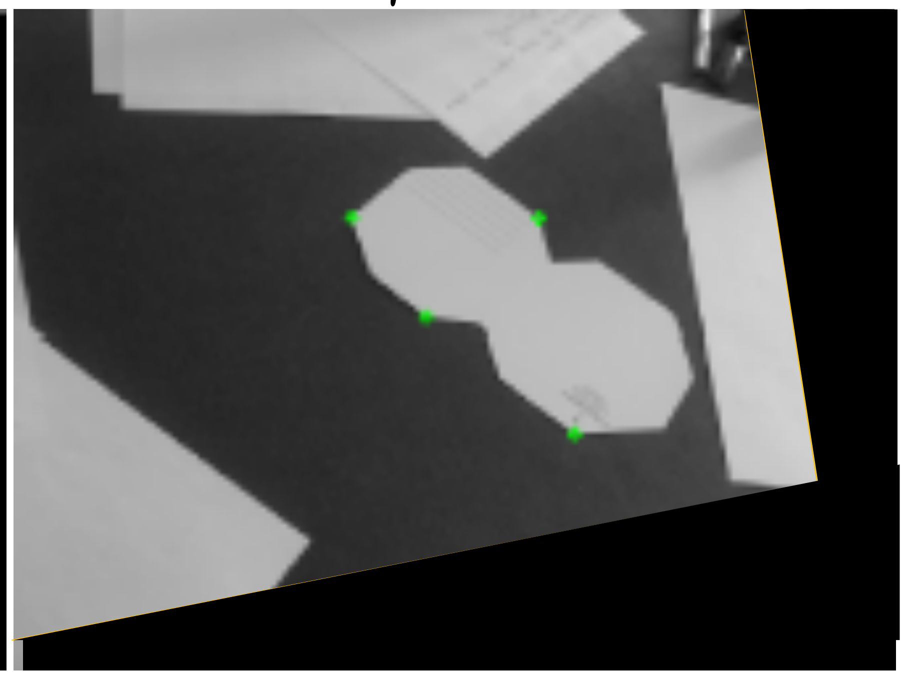
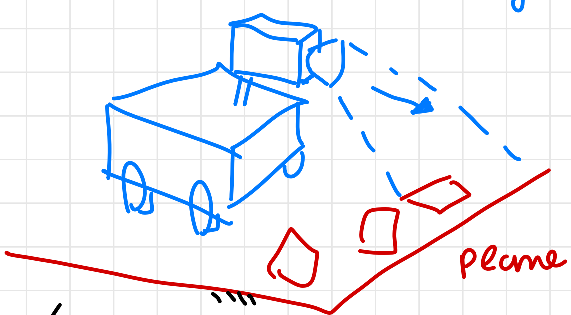


image after displacement

- **A use case example:**

Mobile Robot mounting camera navigating on planar floor

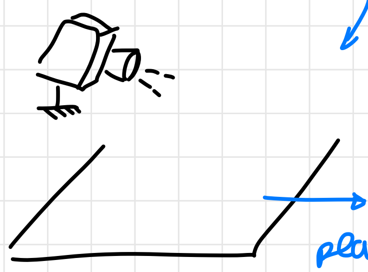
↲ this is the same as the discussed problem (relative motion)



relative displacement is as before!

same issue of RIGID PLANAR MOTION reconstruction

↕ same concept



this is very usefull for **VISUAL ODOMETRY**

≡

in which you use visual sensors to do odometry

($\sim$ odos ≡ path ← measuring the path)

↑

apply reconstruction with  → you can reconstruct floor shape

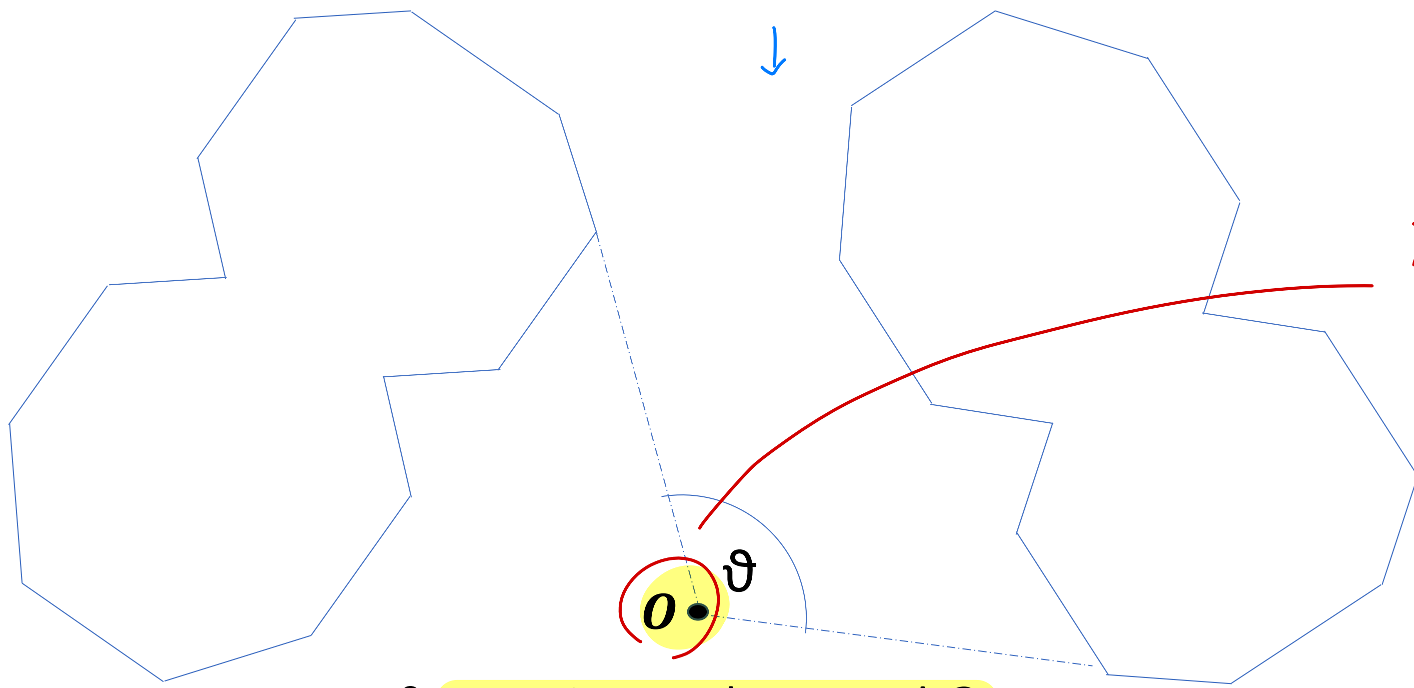
and so measuring apparent motion ↲

once floor relative position reconstructed ⇒ measure motion

Property of rigid planar motion:
any **roto-translation** is a **pure rotation**

(NOT valid in space!)

you can represent any rigid motion as pure Rotation, ignoring translation!

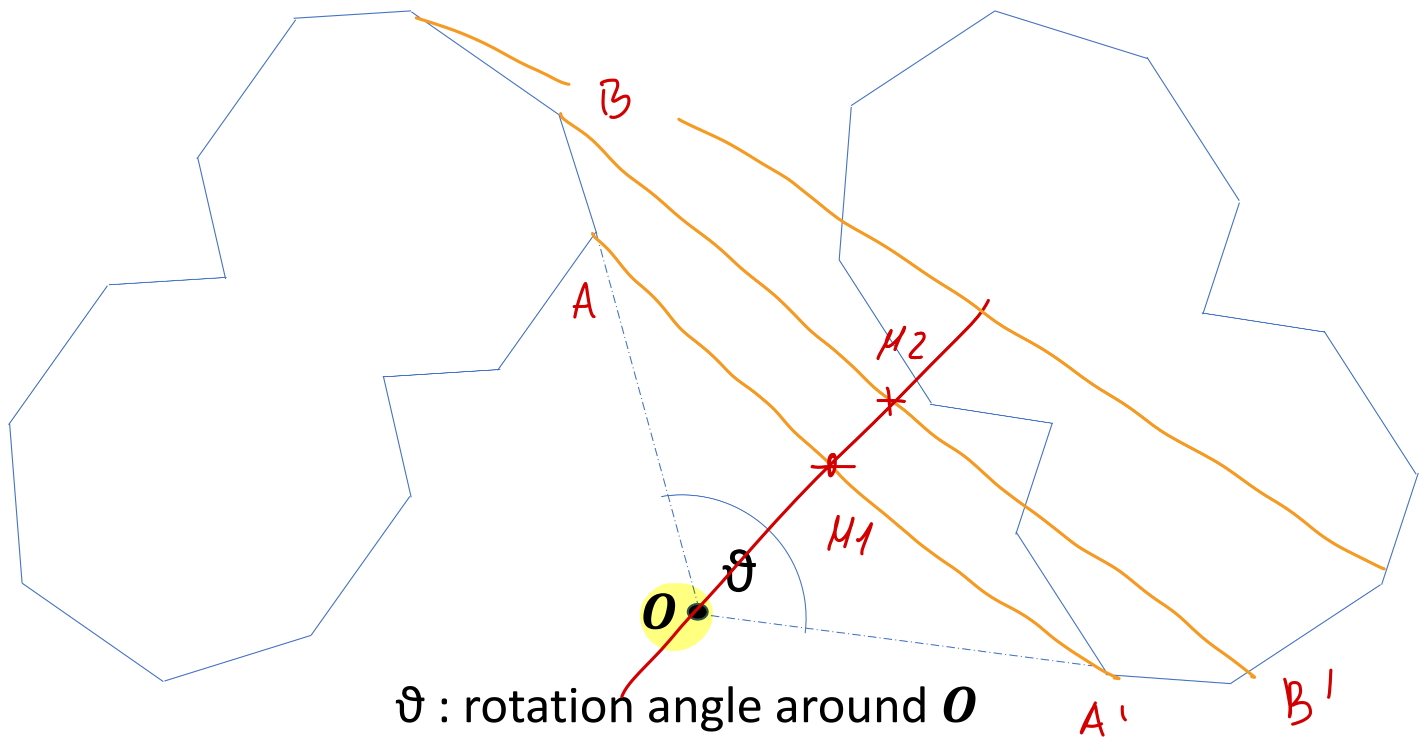


ϑ : rotation angle around ***O***

but how to find O center of Rotation ?

If you can freely chose center of rotation ⇒ any ROTOTRANSLATION ≡ simple ROTATION around O

Property of rigid planar motion: **any roto-translation** is a **pure rotation**



ϑ : rotation angle around $\boldsymbol{O}$

to derive O, you compute point vector before/after moving
by intersecting median line of some segments just doing for two segments is enough

PLANAR ROTATION ≡ ISOMETRY!

rigid planar motion

(All ISOMETRY are SIMILARITY!)

simplest class of projective transformation

- Planar rigid motion: isometry → is a similarity → **invariant** circ. pts $\boldsymbol{I}, \boldsymbol{J}$
- any (planar) rigid displacement is a pure rotation: 3 dofs X_o, Y_o, ϑ

→ Center of rotation $\boldsymbol{O}$ is also invariant

(under ROTATION)

→3 **invariants**: circular points $\boldsymbol{I}, \boldsymbol{J}$ and center of rotation $\boldsymbol{O}$

BUT this is wrt CAMERA... observe in camera view, we have to analyze images before/after moving...

these 3 points, namely, $\boldsymbol{I}, \boldsymbol{J}$ and $\boldsymbol{O}$ remain fixed during motion

because STATIC camera, floor moves, if position despite of motion are fixed, than also I', J', O' also are fixed!

→

also their images $\boldsymbol{I}', \boldsymbol{J}'$ and $\boldsymbol{O}'$ remain fixed during motion image points $\boldsymbol{I}'$, $\boldsymbol{J}'$ and $\boldsymbol{O}'$ remain fixed

corresponding features → estimate homography H

$\boldsymbol{I}'$, $\boldsymbol{J}'$ and $\boldsymbol{O}'$ are invariant under mapping H

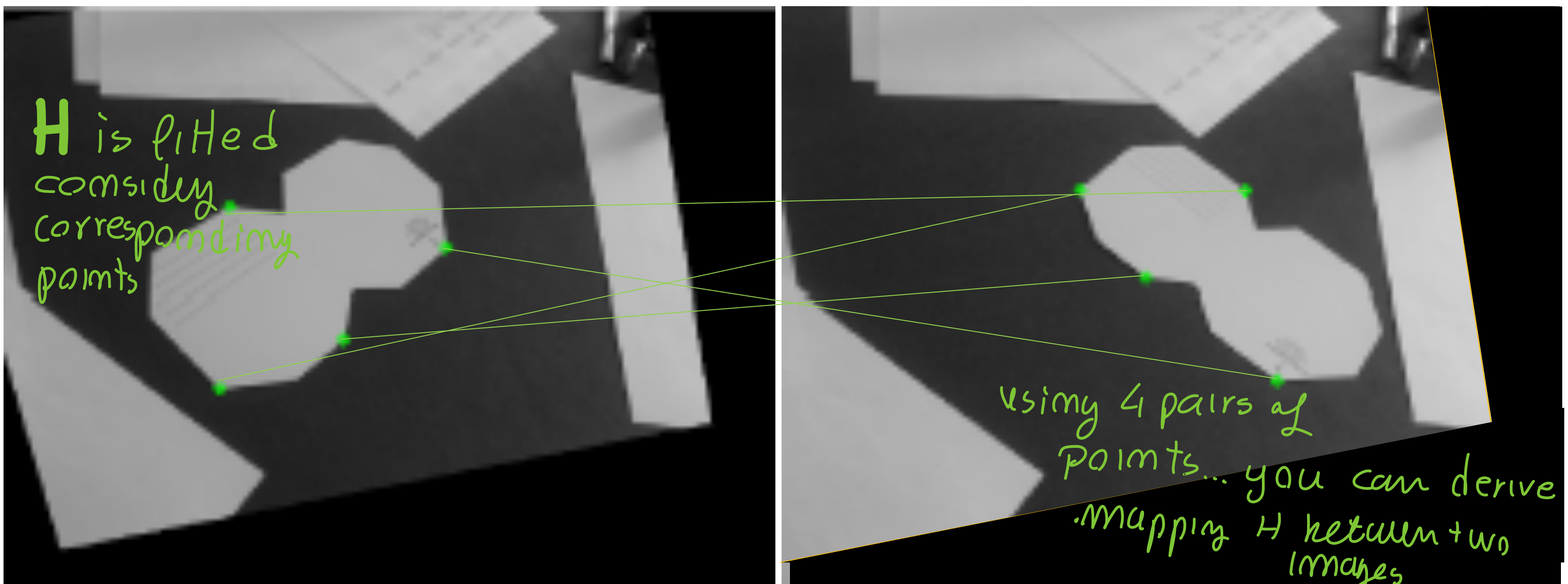


image before displacement → image after displacement

we get planar motion HOMOGRAPHY, two images related by H

KNOWING 4 points A, B, C, D

$$\begin{cases} A' = HA \\ B' = HB \\ C' = HC \\ D' = HD \end{cases}$$

at least with 4 pairs of corresponding points

↓

usual fitting of transformation

features extraction, points, edges etc can be automated...

← that H can be fitted using the invariants and taking 4 pairs of points

↓

we also know how to compute invariants!

↓

than once I', J', O' computed associate to I, J, O

↓

given H how can compute invariant?

↓

simply from H

just found fitting img points (feature) before and after

img before — H → img after

↓

we can find invariance of given homography H

...

$Hx = x$ looking for invariant

$$\boxed{Hx = \lambda x}$$

└ homogeneus entity, up to factors

this is eig problem $\begin{cases} x\text{: eig vector} \\ \lambda\text{: eig value} \end{cases}$ of the matrix H

(simple to solve with MATLAB)

└ give you as output both x, λ

In 3×3 matrix H

(IN PLANAR DOMAIN YES!) → we have x_1, x_2, x_3 $\lambda_1, \lambda_2, \lambda_3$ 3 eig!

$(H - \lambda I)x = 0$ eig PROBLEM → $\det(H - \lambda I) = 0$ eig problem qubic eq

invariants under a projective mapping H

Homogeneous coordinates:

invariants He = λe ←→ **eigenvectors**

the solutions of qubic equation has @ least $\lambda_1 \in \mathbb{R}$
and λ_2, λ_3 complex conj
(OR 3 real)

it can be shown that:

- eigenvectors $\boldsymbol{I'}, \boldsymbol{J'}$ correspond to complex eigenvalues (complex λ correspond to complex x)
- the phase of their eigenvalues is the rotation angle
- eigenvector $\boldsymbol{O'}$ correspond to the real eigenvalue (real λ correspond to real x)

invariants under a projective mapping H

Homogeneous coordinates:

invariants He = λe ←→ **eigenvectors**

$e^{-i\theta} = \cos\theta - i\sin\theta$ $e^{+i\theta} = \cos\theta + i\sin\theta$ are the solutions

corresponding to images of circular points!

it can be shown that:

- eigenvectors $\boldsymbol{I}', \boldsymbol{J}'$ correspond to complex eigenvalues
- the phase of their eigenvalues is the rotation angle
- eigenvector $\boldsymbol{O}'$ correspond to the real eigenvalue

to find I', J' needed to reconstruct the floor, when floor has some feature KNOWN

↓

reconstruct shape by moving and taking 2 images

before / after motion

then you solve eig(H) ← You fit H before / after motion

↓

3 $\lambda_1, \lambda_2, \lambda_3$ $\lambda_1 \in \mathbb{R} \rightarrow x_1 = O'$ centering rotation image (usefull in visual odometry)

$\lambda_2, \lambda_3 \in \mathbb{C} \rightarrow x_2, x_3 = I', J'$

- images of circular points

↳

this is for SHAPE RECONSTRUCTION up to a scaling factor (further info for size estimate)

Rectification from planar motion

found by fitting two motion images

- find eigenvector-eigenvalues of H:
- eigenvalues are proportional to $\lambda' = 1$, $\lambda'' = e^{i\theta}$, $\lambda''' = e^{-i\theta}$
- eigenvector e' associated to $\lambda' = 1$ is the image of the C.O.R. $\boldsymbol{O}$
- angle θ is the rotation angle
- eigenvectors e'' and e''' associated to λ'' and λ''' are the images $\boldsymbol{I}', \boldsymbol{J}'$ of the circular points $\boldsymbol{I}, \boldsymbol{J}$
- thus $C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^T + \boldsymbol{J}'\boldsymbol{I}'^T$
- apply singular value decomposition svd($C_\infty^{*\prime}$) $= UC_\infty^{*}U^T$
- we obtain the rectification matrix $H_{SR} = U^T$

Remember: angle between scene lines expressed in terms of image lines

- From transformation rules for lines and dual conics:

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)\ (m^T C_\infty^* m)}} = \frac{l\prime^T C_\infty^{*\prime} m\prime}{\sqrt{(l\prime^T C_\infty^{*\prime} l\prime)\,(\,m'^T C'^*_\infty m\prime)}}$$

known angle → equation on $C_\infty^{*\prime}$

e.g., for perpendicular lines $l'^T C_\infty^{*\prime} m' = 0$ **linear**

Example 1.
rectification from two planar rectangles

Image rectification from two coplanar rectangles

to do this information, RECTIFICATION

KNOWING it is a RECTANGLE, it carry information that angles $\simeq \pi/2$, PERPENDICULAR sides

co-planar

or … rectification from vanishing points

Direct method

1 Direct method

1. Find $C_\infty^{*\prime}$ (related to I', J')
2. Compute H_{rect} by svd

- 4 dof for $C_\infty^{*\prime}$:

  9 elements minus

  - 3x3 symmetric → 3 constraints;

  - homogeneous matrix → 1 constraint;

  - singular matrix → 1 constraint

direct method to reconstruct the upper face

(common plane...)

FIRST, you find:

vanishing points

x_{v_1} x_{v_2}

When open instrument we are interested in α

You know they are images of parallel lines

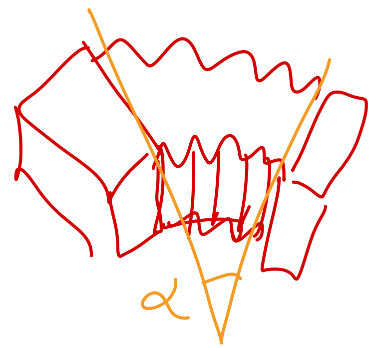


image of the line at the infinity

line joining vanishing points is l'_∞

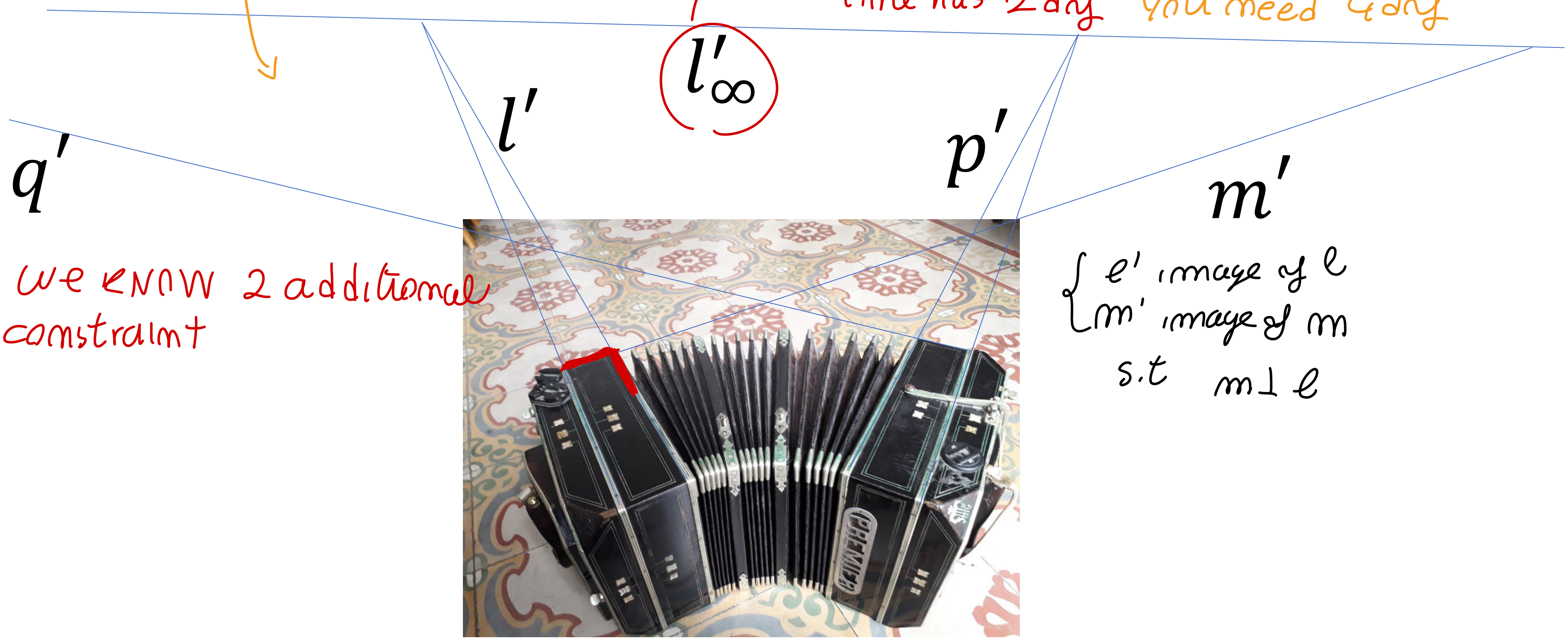
$$l'_\infty$$

you discover new equations

known angles (pairs of perpendicular lines)

l'_∞ tells you 2 equations as fixing $l'_\infty = H l_\infty$

line has 2 dof   you need 4 dof



we know 2 additional constraint

$\begin{cases} l' \text{ image of } l \\ m' \text{ image of } m \end{cases}$

s.t $m \perp l$

not always lines choice is the best, we must carefully chose constraints!

IF MAKE IMPLICIT singularity constraints, you can use just 4 constraint

known angles (pairs of perpendicular lines)

x_∞ point at infinity along m' which is image of m ! e

same for q'

$q \perp p$

IN image NOT ORTHOGONAL (perspective alterate)

5^{TH} constraint will identify a new point on l'_∞ which you already impose! NOT KNOWN equation we select indipendent constraints!

4 linear constraints on $C_\infty^{*\prime}$

using also Angle KNOWLEDGE

$$l'^T C_\infty^{*\prime} m' = 0$$ (1 constr) ($l \perp m$)

because $C_\infty^{*\prime}$ is written as $I'J'^T + J'I'^T$

$$p'^T C_\infty^{*\prime} q' = 0$$ (1 constr) ($p \perp q$)

4 constr

this is by definition

but
$J'^T l'_\infty = 0$
$I'^T l'_\infty = 0$
because $I', J' \in l'_\infty$ so concurrent

$$0 = C_\infty^{*\prime} l'_\infty = (I'J'^T + J'I'^T) l'_\infty =$$

$$= I'(\underbrace{J'^T l'_\infty}_{0}) + J'(\underbrace{I'^T l'_\infty}_{0}) = \mathbf{0}$$ (2 constr)

KNOWLEDGE of l'_∞ give us additional 2 constraints

RIGHT NULL SPACE

line at the infinity l'_∞ = RNS($C_\infty^{*\prime}$)

Image rectification

- From the above constraints → find $C_\infty^{*\prime} = \boxed{\boldsymbol{I'J'^T + J'I'^T}}$

this kind of expression is valid for any rank=2 when degenerate conics of rank 2 we write as vector product summ (3×3)

this imply singularity! CONSTRAINT it

- Singular value decomposition

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^{*} H_{SR}^{-T}$$

- Rectifying transformation (from svd output U)

$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} * \text{Image}$

reconstruction of the upper face

You don-t preserve
shape but you know
It is properly reconstructed,
thanks to fact that
floor has $\square$
square

29/10) whenever $C^{*'}_{\infty}$ OR I', J' is available,

↓

you apply SVD to $C^{*'}_{\infty}$

then use formula considering also numerical errors

(we will NOT obtain diag matrix with $\sigma_1 = \sigma_2 = 1$, $\sigma_3 = 0$)

↓

in general we obtain different singular values $\sigma_1 = a$ $\sigma_2 = b$, so we compensate it

UP to now we apply DIRECT method, to search $C^{*'}_{\infty}$ directly, then SVD

↳ ALTERNATIVE path follows a two step method ⟹

29/10

different method, which don't need svd, consisting in →

Stratified method

1) Affine reconstruction (less satisfactory)
- then...
2) upgrade Affine to Metric reconstruction (without need of svd)

Stratified method

PROJECTIVE of original scene

obtained by proj. transformation

⇑

original information

1. First step: affine reconstruction - from projective to affine

then

lower distance from proj to original scene

(SIMILARITY)

2. Second step: shape reconstruction - from affine to metric
(= image rectification or metric or euclidean reconstruction)

euclidean metric, closer to the original scene as SIMILARITY,

- sometimes reduces numerical errors

this preserve parallel lines.. but NOT same angles!

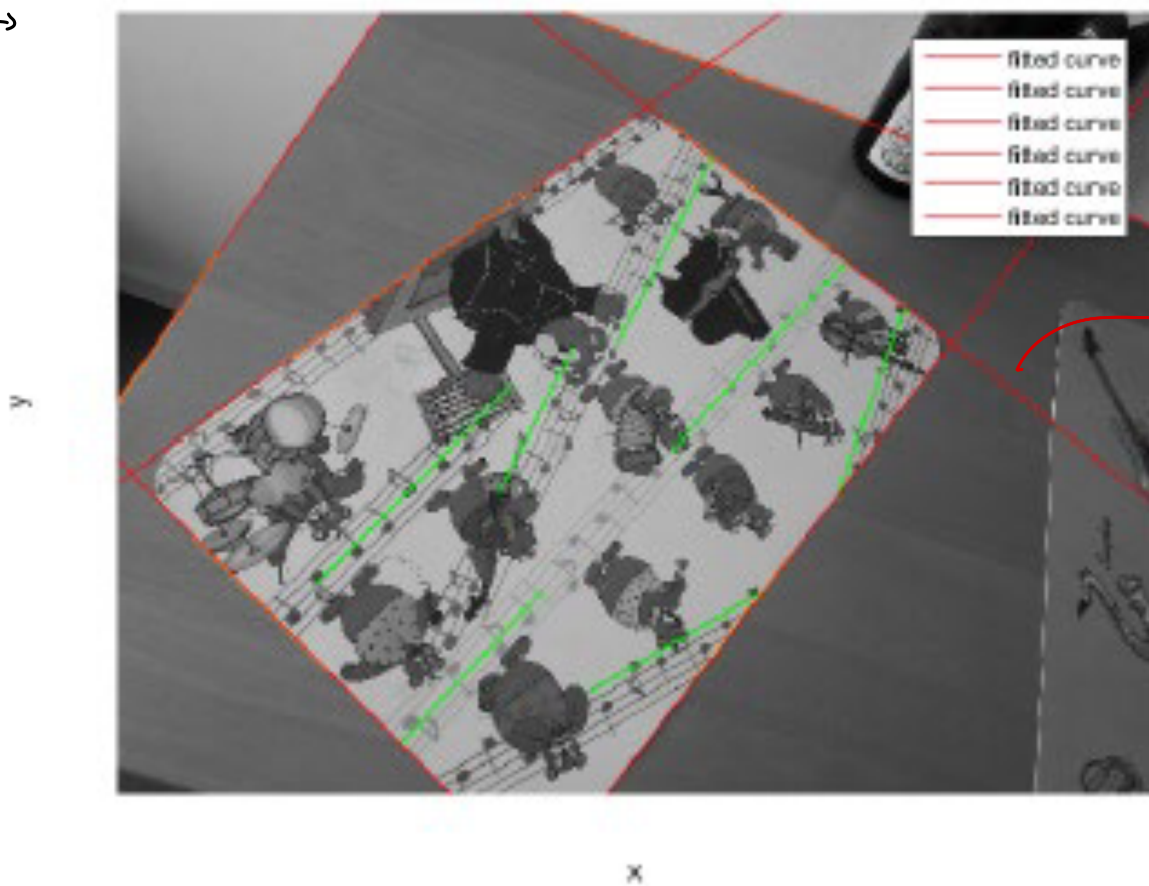
← preserving shape

Stratified method

1. $\underbrace{\text{affine reconstruction}}_{\text{to upgrade projective image ...}}$ - from projective to affine

we know how to do this from a given image of unknown planar scene look at images of lines which are parallel ↳ in original scene

Images of pairs of parallel lines



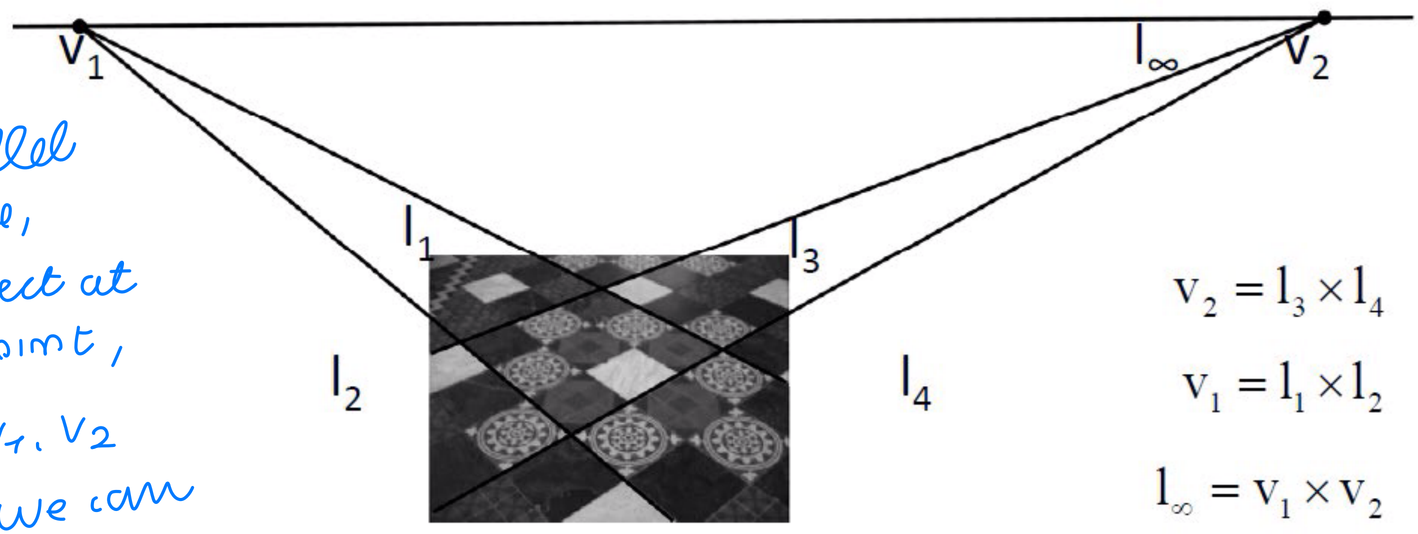
parallel lines

Image of **two** pairs of parallel lines

Affine rectification

lines on tiles border are parallel

since parallel in real scene, they intersect at vanishing point, we find V_1, V_2 and then we can obtain l'_∞ as projection of l_∞, then create H that map l'_∞ to l_∞ for affine reconstruction



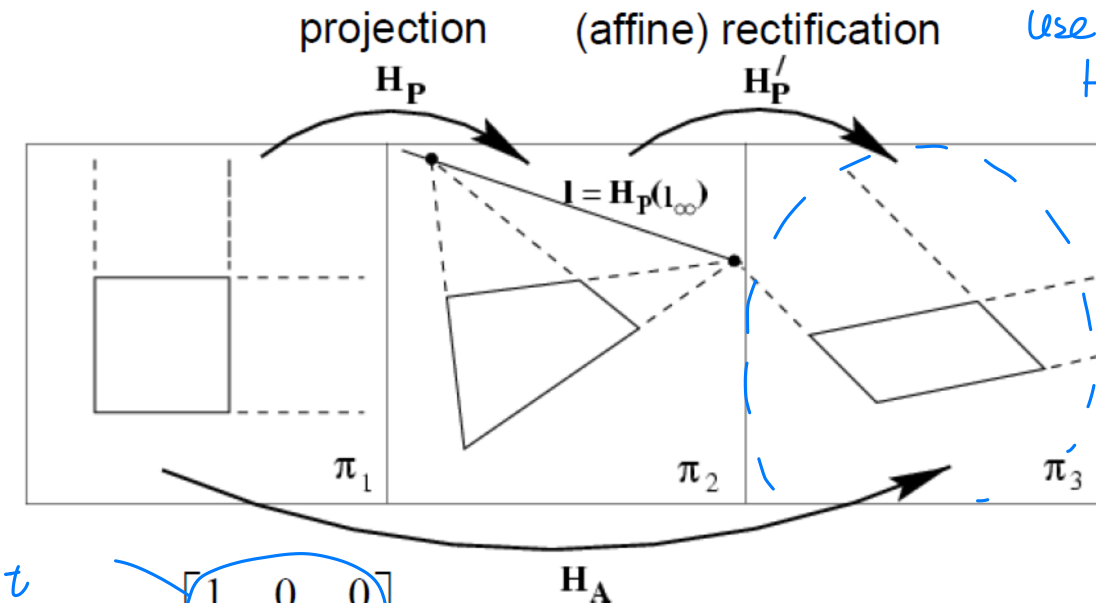
$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$

Affine properties from images

↓ to use ℓ_∞ information is easily used to reconstruct H_P



affine Reconstruction of the scene (BUT without shape map)

free to be chosen, s.t rank = 2

$$\mathbf{H'}_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \mathbf{H}_A \qquad \mathbf{l'}_\infty = [l_1 \quad l_2 \quad l_3]^{\mathrm{T}}, l_3 \neq 0$$

in fact, any point x on l'_∞ is mapped to a point at the ∞

Affine rectification

- Apply the above mapping $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{vmatrix}$ to image lines and points

- while following transformation rules for lines and points respectively

In this phase, no need to use information about conic dual to circular points

IF we start from Affine Rectification, it is easy to go to Metric Rectification respect a one step

Affine rectification

(with l_∞ information, using parallel lines information)

direct rectification versus stratified rectification

second step?

Metric properties from images

$$
\begin{aligned}
\mathbf{C}_\infty^{*}{}' &= (\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)\mathbf{C}_\infty^{*}(\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)^\top \\
&= (\mathbf{H}_P\mathbf{H}_A)\mathbf{H}_S\mathbf{C}_\infty^{*}\mathbf{H}_S^\top(\mathbf{H}_P\mathbf{H}_A)^\top \\
&= (\mathbf{H}_P\mathbf{H}_A)\mathbf{C}_\infty^{*}(\mathbf{H}_P\mathbf{H}_A)^\top \\
&= \begin{bmatrix} \mathbf{K}\mathbf{K}^\top & \mathbf{K}^\top \mathrm{v} \\ \mathrm{v}^\top \mathbf{K} & \mathrm{v}^\top \mathrm{v} \end{bmatrix}
\end{aligned}
$$

Rectifying transformation from SVD

$$
\mathbf{C}_\infty^{*}{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^\top \qquad \mathbf{H} = \mathbf{U}^\top
$$

after affine rectification

Second step

notice what happen after Affine

- Relationship between

original conic dual to the circular points $C_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and

affine(ly rectified) image of the conic dual to the circular points $C_\infty^{*\prime}$: **affine**

when you have done Affine rectification...

affine mapping HA

ORIGINAL

$$C_\infty^{*\prime} = H_A C_\infty^* H_A^T = \underbrace{\begin{bmatrix} G & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix}}_{\text{general form of Affine mapping}} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} G^T & \boldsymbol{0} \\ \boldsymbol{t}^T & 1 \end{bmatrix} = \begin{bmatrix} GG^T & 0 \\ \boldsymbol{0}^T & 0 \end{bmatrix}$$

ORIGINAL one

this simplify NEXT STEP, NO SVD is needed

after affine rectification

No more svd! we need to find affine reconstructing transformation from intermediate to metric reconstruction...

- Relationship between

original conic dual to the circular points $C_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and

affine(ly rectified) image of the conic dual to the circular points $C_\infty^{*\prime}$: **affine**

$$C_\infty^{*\prime} = H_A C_\infty^* H_A^T = \begin{bmatrix} G & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} G^T & \boldsymbol{0} \\ \boldsymbol{t}^T & 1 \end{bmatrix} = \begin{bmatrix} GG^T & 0 \\ \boldsymbol{0}^T & 0 \end{bmatrix}$$

In direct method

$$\begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$ found is 3×3 orthogonal with NON ZERO elements on last row in general

While the Reconstructing transform is affinity $\longrightarrow$

after affine rectification

we need to apply transformation to image, such that, affine transform.

- **Important Property** after an affine rectification, $C_\infty^*{}'$ can be written as

$$C_\infty^*{}' = \begin{bmatrix} GG^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

where K is a 2×2 invertible matrix (cfr HZ example 2.26)

- this is the image of the canonical C_∞^* through an affine transformation since similarities do not change C_∞^*, and projective transformation has been removed by the affine rectification

Stratified method

1. First step: affine reconstruction - from projective to affine

then

2. Shape reconstruction - from affine to metric

used for Metric RECONSTRUCTION

Remember: costraints from known angles

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{l^T C_\infty^* l \;\; m^T C_\infty^* m}} \overset{\text{PRESERVED}}{=} \frac{l'^T C_\infty^{*\prime} m'}{\sqrt{l'^T C_\infty^{*\prime} l' \;\; m'^T C'^*_\infty m'}}$$

using known angles & cos(θ) formula

known angle → equation on $C_\infty^{*\prime}$

⇒ and for perpendicular lines $\boxed{l'^T C_\infty^{*\prime} m' = 0}$ **linear**

image rectification from orthogonal lines

applying to dual conic $C_\infty^{*\prime}$ with known shape...

- If $\boldsymbol{l}'$ and $\boldsymbol{m}'$ are images two orthogonal lines $\boldsymbol{l}$ and $\boldsymbol{m}$ (in the 3D world), then

$$\boldsymbol{l}'^{\top} C_\infty^{*\prime} \boldsymbol{m}' = 0 \qquad (*)$$

- Let us use this information to compute $C_\infty^{*\prime}$, thus compute H
- Remember that, after an affine rectification

$$C_\infty^{*\prime} = \begin{bmatrix} GG^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

zeros simplify

- and that $S = GG^\top$ is **a symmetric homogeneous matrix** $\left((GG^\top)^\top = GG^\top\right)$, thus there are only 2 unknowns to identify $C_\infty^{*\prime}$
- Each pair of orthogonal lines yield a single equation

by apply the equation (*) on perpendicular lines... reduce to

$$\boldsymbol{l}'(1{:}2)^\top S\, \boldsymbol{m}'(1{:}2) = 0$$

second line is all zero, only one meaningfull equation remain

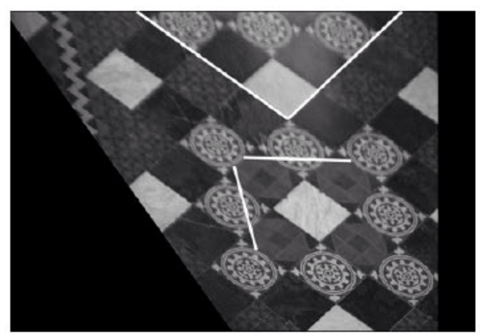
Metric from affine

l' and ***m'*** are lines in the affinely rectified image; they are image of orthogonal lines

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{K}\mathbf{K}^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

$$(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2)(k_{11}^2 + k_{12}^2, k_{11}k_{12}, k_{22}^2)^\mathsf{T} = 0$$

from two pairs
→ estimate GG^T
(homogeneous)



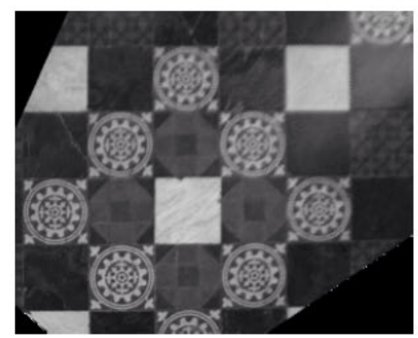


image rectification from orthogonal lines

- If $\boldsymbol{l}'$ and $\boldsymbol{m}'$ are images two orthogonal lines $\boldsymbol{l}$ and $\boldsymbol{m}$ (in the 3D world), then

$$\boldsymbol{l}'^{\top} C_{\infty}^{*}{}' \boldsymbol{m}' = 0$$

- Let us use this information to compute $C_{\infty}^{*}{}'$, thus compute H
- Remember that, after an affine rectification

$$C_{\infty}^{*}{}' = \begin{bmatrix} \boxed{GG^{\top}} & \mathbf{0} \\ \mathbf{0}^{\top} & 0 \end{bmatrix}$$

(being $C_{\infty}^{*}{}'$ symmetric)

- and that $S = GG^{\top}$ **is a symmetric homogeneous matrix** ($GG^{\top} = G^{\top}G$), thus there are only 2 unknowns to identify $C_{\infty}^{*}{}'$

there are just 2 DOFs on this...

- Each pair of orthogonal lines yields a single equation

$$\boldsymbol{l}'(1{:}2)^{\top} S\, \boldsymbol{m}'(1{:}2) = 0$$

Two pairs of orthogonal lines are enough to identify $C_{\infty}^{*}{}'$

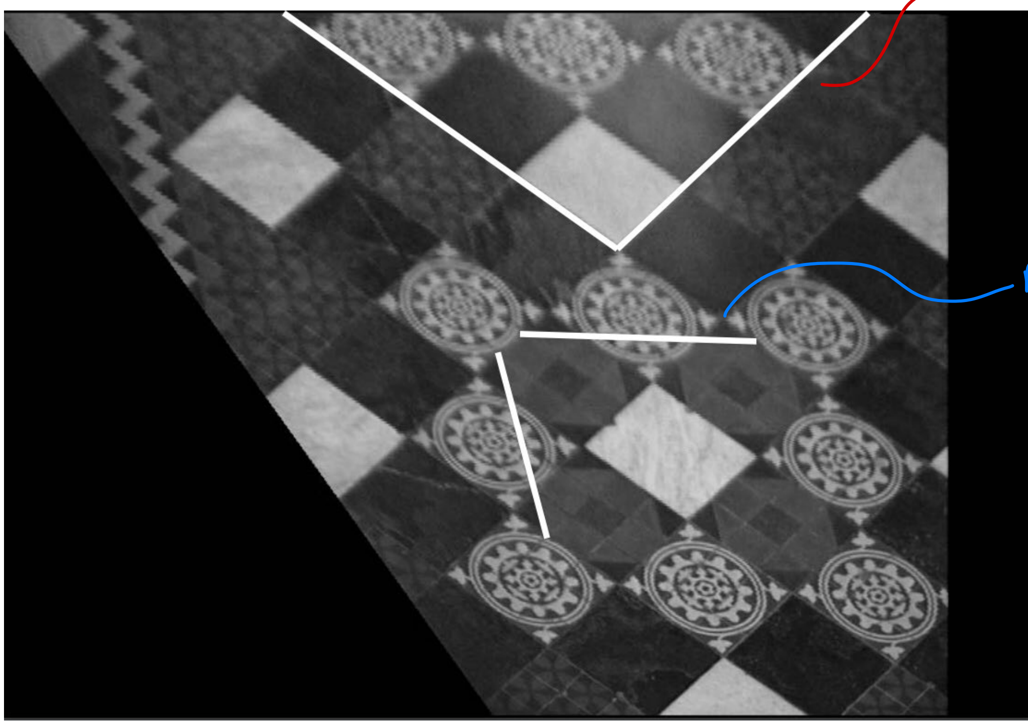
Stratified Rectification from Orthogonal Lines

- $\boldsymbol{l}'(1{:}2)^\intercal S\, \boldsymbol{m}'(1{:}2) = 0$

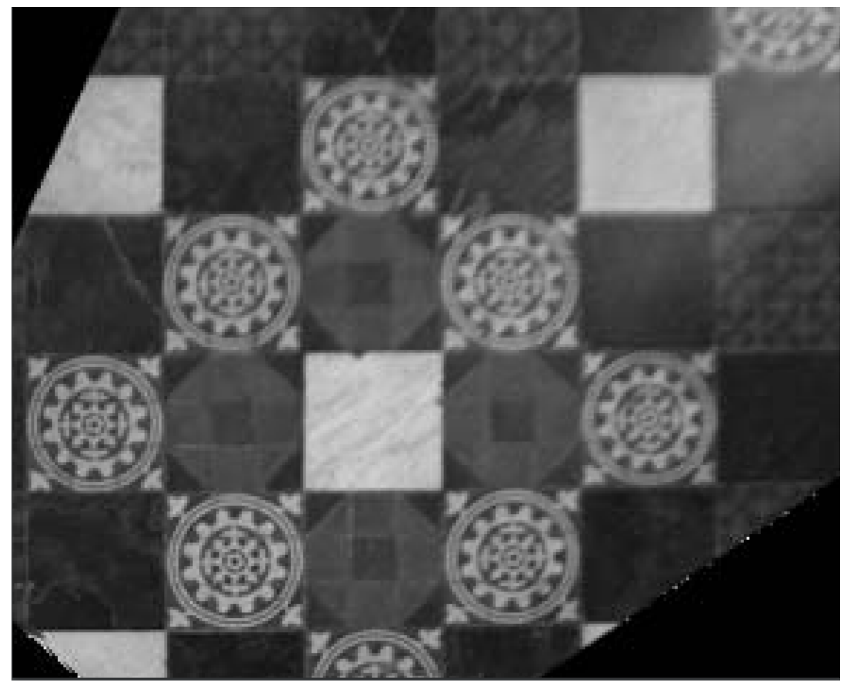
2 pairs of orthogonal lines are enough

- Two (different, i.e. not parallel) pairs of orthogonal lines are enough to estimate $S = GG^\intercal$, (thus G through Choleski factorisation)

PAIR 1 (⊥)



PAIR 2 (⊥)



Hartley Zisserman Fig.2.17

On the Affine Rectified image:

↳

$$l'^T \begin{vmatrix} GG^T & 0 \\ 0 & 0 \end{vmatrix} m' = 0$$

l'

m'

$$p'^T \begin{vmatrix} GG^T & 0 \\ 0 & 0 \end{vmatrix} q' = 0$$

Use image of pairs of orthogonal lines to estimate GG^T

we found GG^T from that

remember: just 2 pairs are needed

(one possible G)

Apply Cholesky factorisation to find G
(an upper triangular matrix G is provided)

this gives you one good G (there exist many G s.t $GG^T = S$)

Cholesky chose G upper triangular, to remove ambiguity

NOTE: **no SVD is needed**

just eimean steps

From affine reconstruction to metric reconstruction

From $C_\infty^{*\prime} = \begin{bmatrix} GG^T & 0 \\ 0 & 1 \end{bmatrix} = H_A C_\infty^* H_A^T = \begin{bmatrix} G & \boldsymbol{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G^T & \mathbf{0} \\ \boldsymbol{t}^T & 1 \end{bmatrix}$

you use GG^T to find

and $H_{rect} = H_A^{-1}$

$$\| \quad H_{rect} = \begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix}^{-1}$$

transformation from Affine to shape Reconstruction

where

t is a free (but useless) vector

use property co-linearity / parallel / perpendicular

and other knowledge to discover like vanishing points etc..

↳ in such a way that ℓ_∞ can be found

after ℓ_∞' has been found, instead of 5 constraints

1) ℓ_∞' find! (by this method of discover)

2) find two pairs of orthogonal lines.

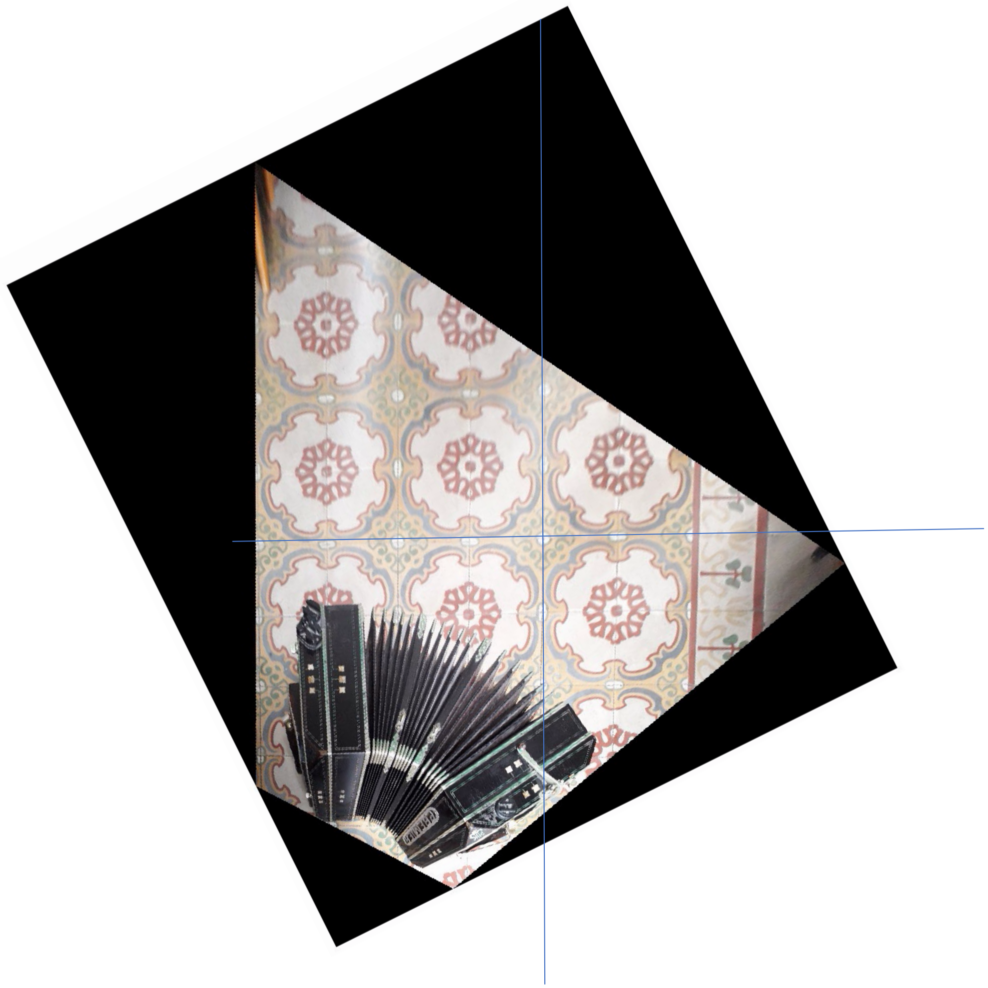
When you cannot select two parallel lines, you cannot use ℓ_∞ intermediate score...

? NOT UNDERSTOOD !

Result of metric (shape) reconstruction

same stratified approach for

reconstructed upper face



Some accuracy issues in image rectification

↓

2D shape reconstruction, is crucial in homework

NOTICE that by apply vanishing point etc.

→ UNCERTAINTY affect all lines

due to how corner / edge point etc are selected,

NOT for sure correct image point!

⇓

From UNCERTAIN points, you get

uncertain results...

Some accuracy issues in image rectification

1. Noise and numerical errors (already seen)

affecting SVD

$$\mathrm{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

$$\neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- In principle, given $C_\infty^{*\prime}$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

- But, due to noise and numerical errors, SVD output is

to solve, we enrich by pre-multiply

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = U \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

$$\rightarrow H_{rect}^{-1} = U \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow H_{rect} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

Some accuracy issues in image rectification

2. Poor estimation:

e.g., intersection of two lines, that are too close to eachother

when using close image...

you choose lines too close to each other

intersection of two lines, that are too close to eachother

Uncertainty in the intersection point

too much parallel in image, so in plane intersection is large uncertain

↓

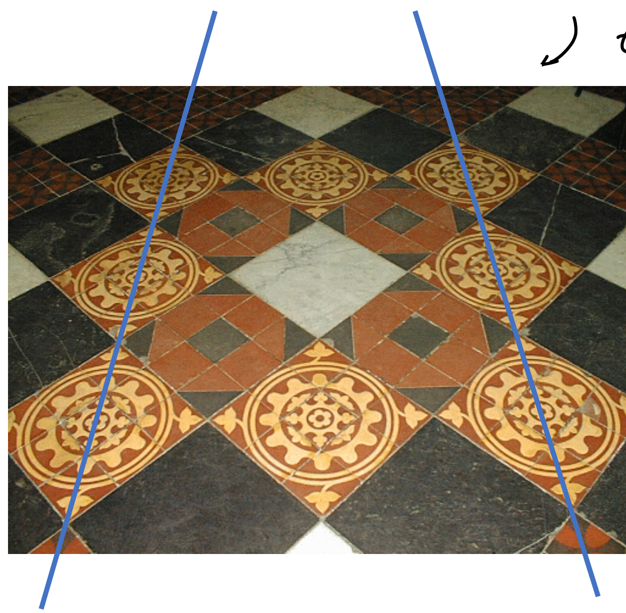
lines affected by uncertainty which lines intersected get more results!

Some accuracy issues in image rectification

2. Poor estimation:

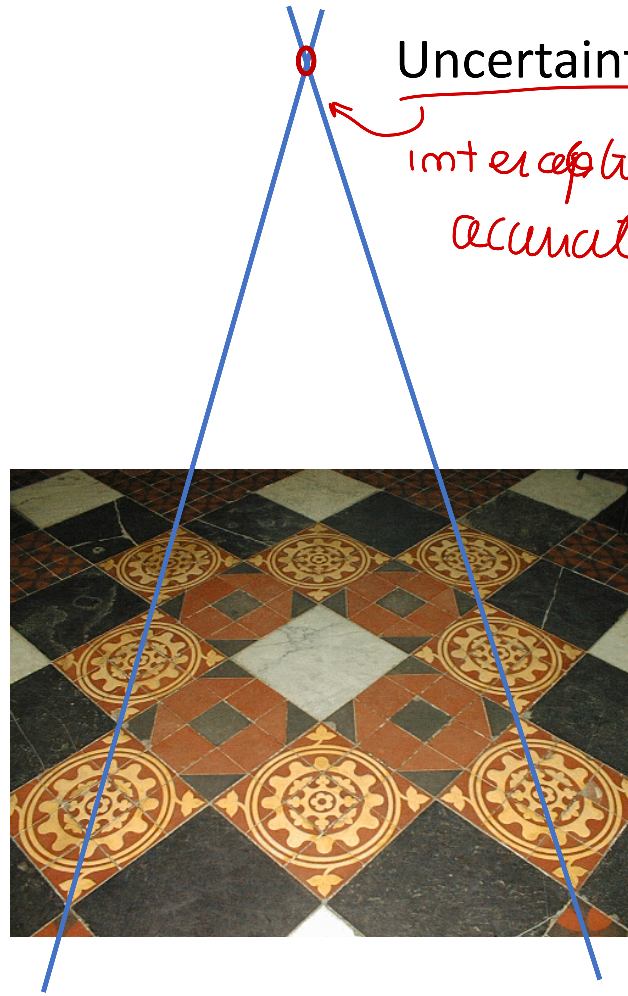
e.g., intersection of two lines, better **use lines that are far from eachother**

this avoid unaccurate intersection



… better **use lines that are far from eachother**

When computing parameter in objects, remember to chose wisely lines on image!



Uncertainty in the intersection point

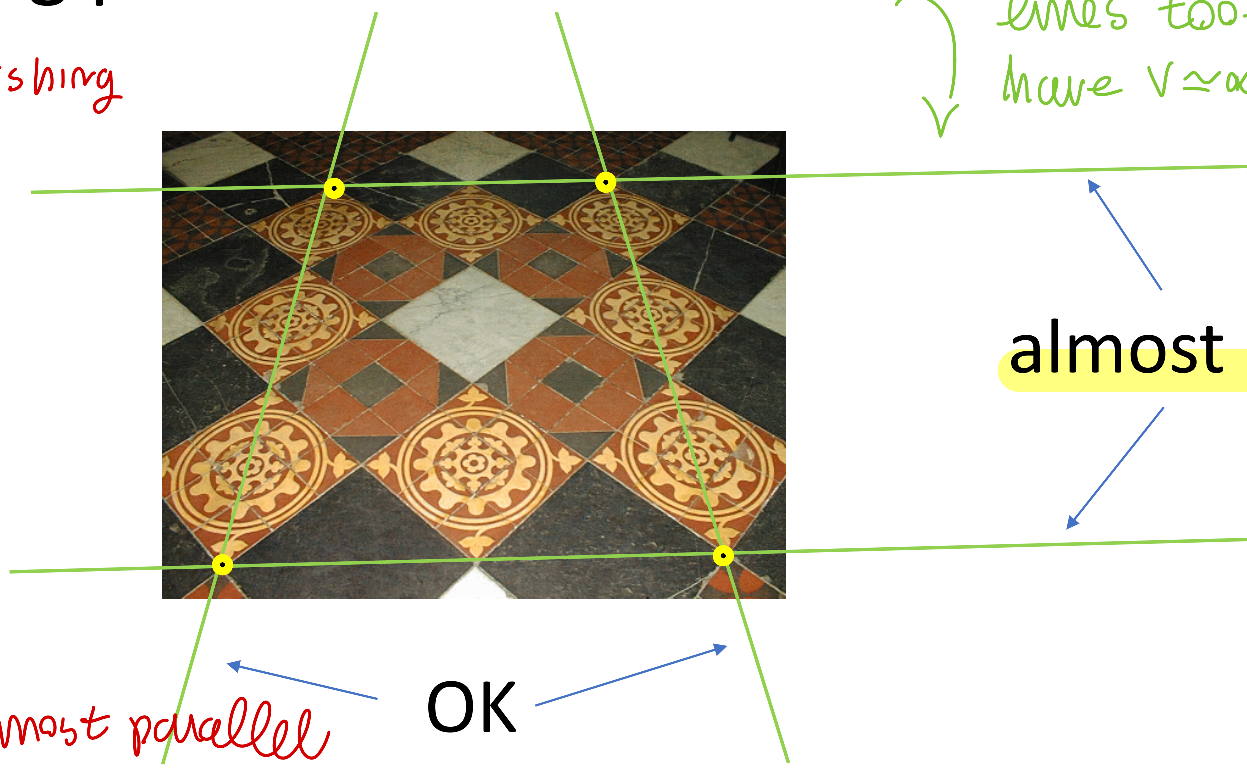
intersection more accurate!

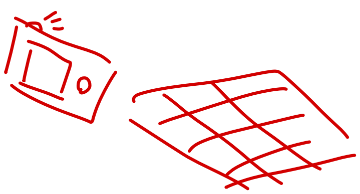
Some accuracy issues in image rectification

3. Vanishing point almost at the ∞

to find v_1, v_2 two vanishing points...
lines too far have $v \simeq \infty$

not all vanishing points are easy to find!





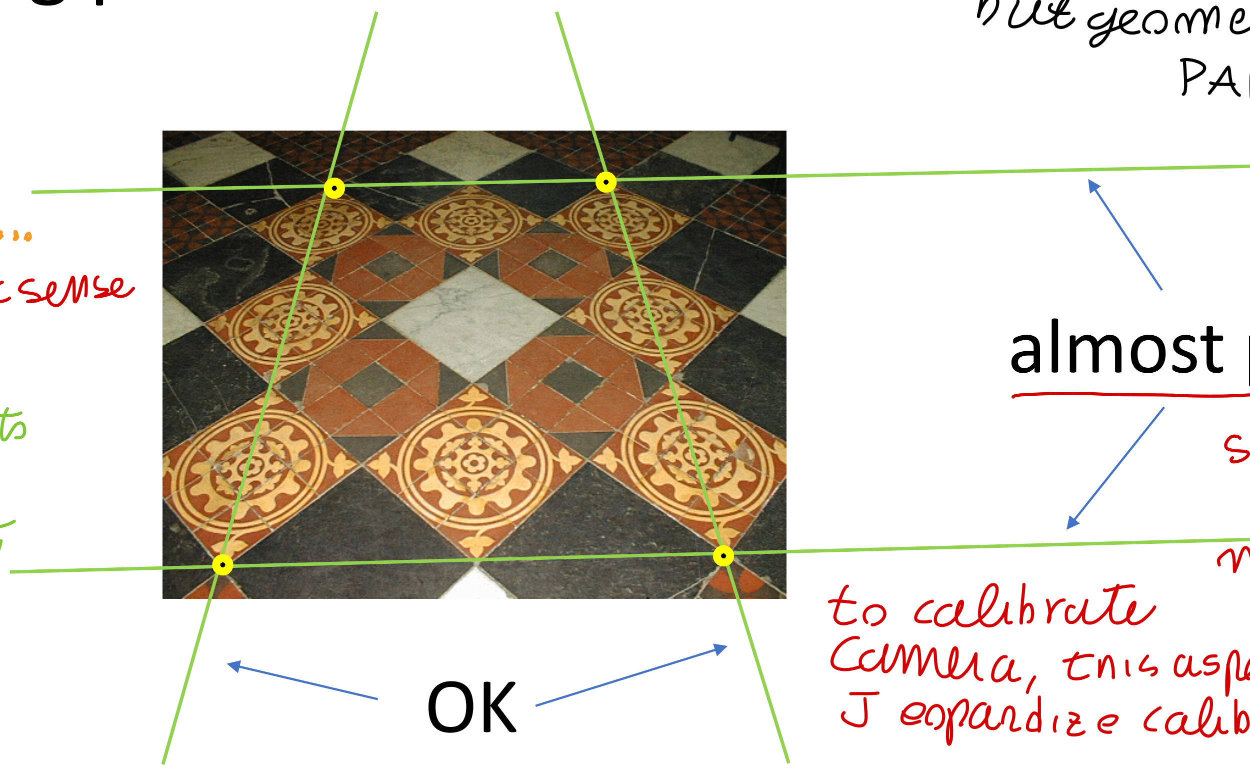
in vertical image plane...
often lines almost parallel

Some accuracy issues in image rectification

3. Vanishing point almost at the ∞

Mathematically Bad
but geometrically Good
PARADOX

this is a
Degenerate
Bad situation...
in Algebraic sense
↓
But in practice, to
Rectify, this is
what we want
to achieve!



almost parallel

so $V_i \simeq \infty$
NOT good
numerically
⇓
to calibrate
camera, this aspect
Jeopardize calibration

OK

Vanishing points towards the ∞

Big problem to apply formulas...

When the images of parallel lines are almost parallel

also in the image...

→ Estimate of vanishing points: not accurate

→ estimate of l'_∞ (image of l_∞): not accurate

→ affine rectification: not accurate

Stratified «geometric» method

lines in image almost parallel as
in original scene. Bad Algebraically
already rectified directions!

↓

this is a Good situation

you rely
on Geometric
ideas instead
of Algebra!

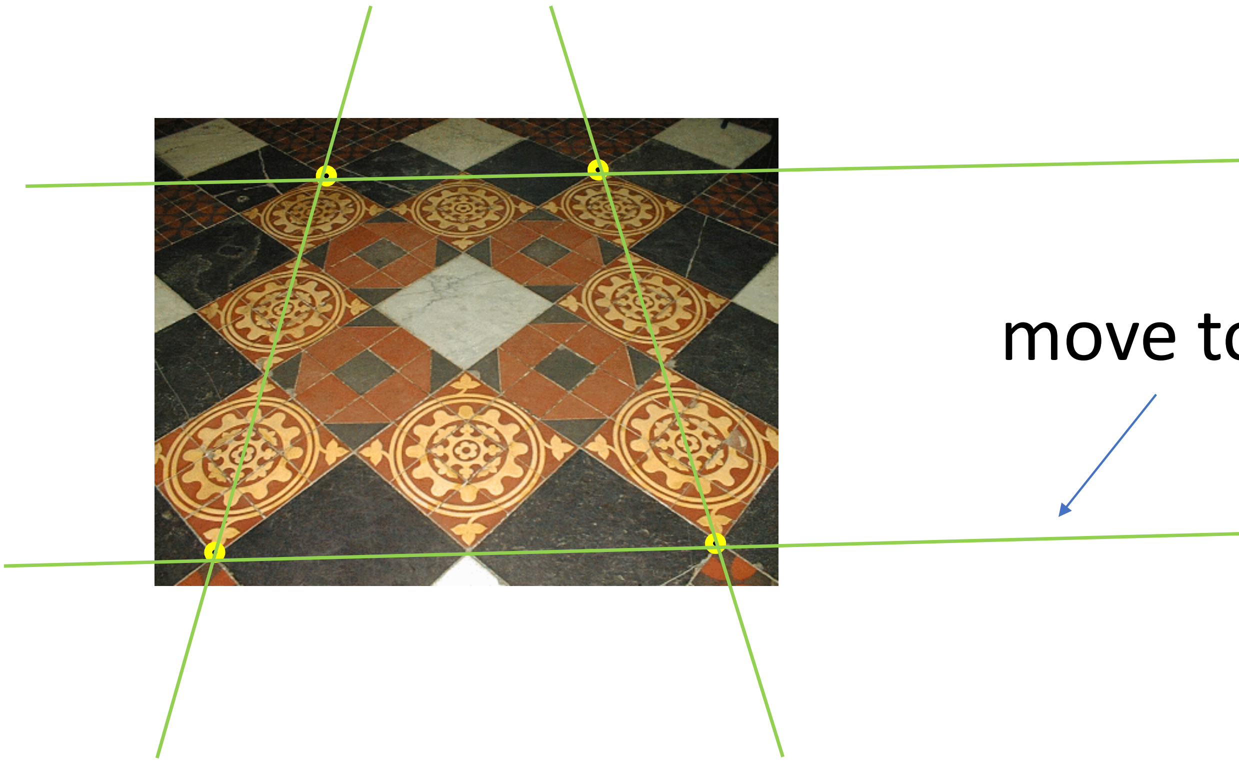
Stratified «geometric» method

first affine
then metric

1) Affine:
you need
model where parallelism
is preserved, it is
here as if Affine
already done...

you simply
apply Geometric rectification

1° step: affine rectification



move to parallel

1° step: affine rectification

we have a couple already parallel + new to rectify

you move by means of parallel transformation

rectified

move to parallel

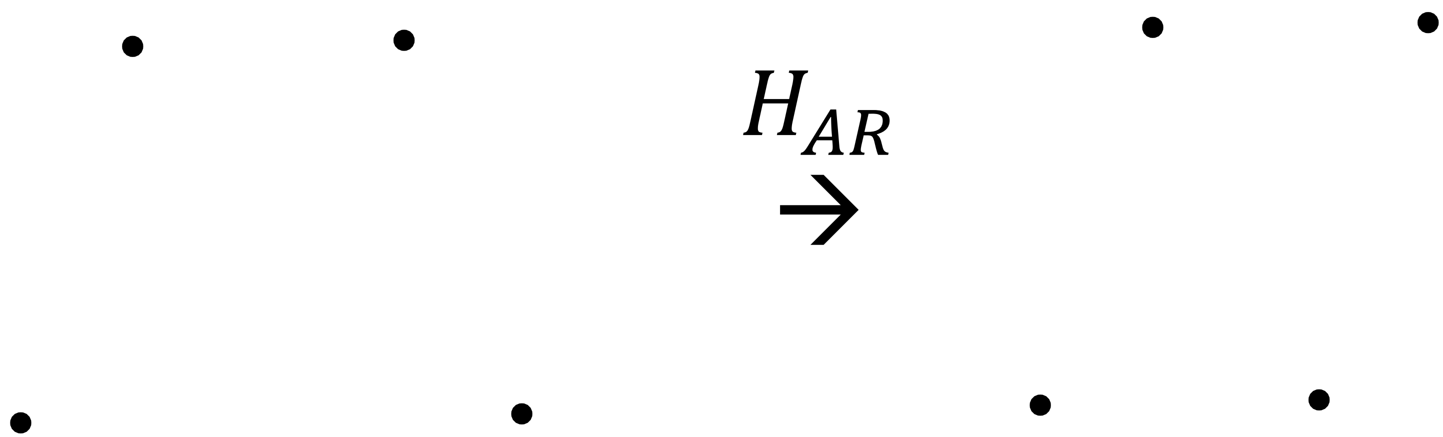
line

to let it become parallel

we achieve that by moving the 4 points in the new desired positions!

4 points → 4 points HOMOGRAPHY

in order to transform lines to become parallel



you obtain affine rectification by moving points without using algebra

4 points → 4 points HOMOGRAPHY

to do Geometric transformation

A • B •

H_{AR}
→

A' • B' •

you find the good points to use, and you move it to desired positions

move

C • D •

C' • D' •

where H_{AR} is the solution of

$$\left[\begin{aligned} \bullet\ A' &= H_{AR}\mathrm{A} \\ \bullet\ B' &= H_{AR}\mathrm{B} \\ \bullet\ C' &= H_{AR}\mathrm{C} \\ \bullet\ D' &= H_{AR}\mathrm{D} \end{aligned}\right\} \text{(homogeneous coordinates)}$$

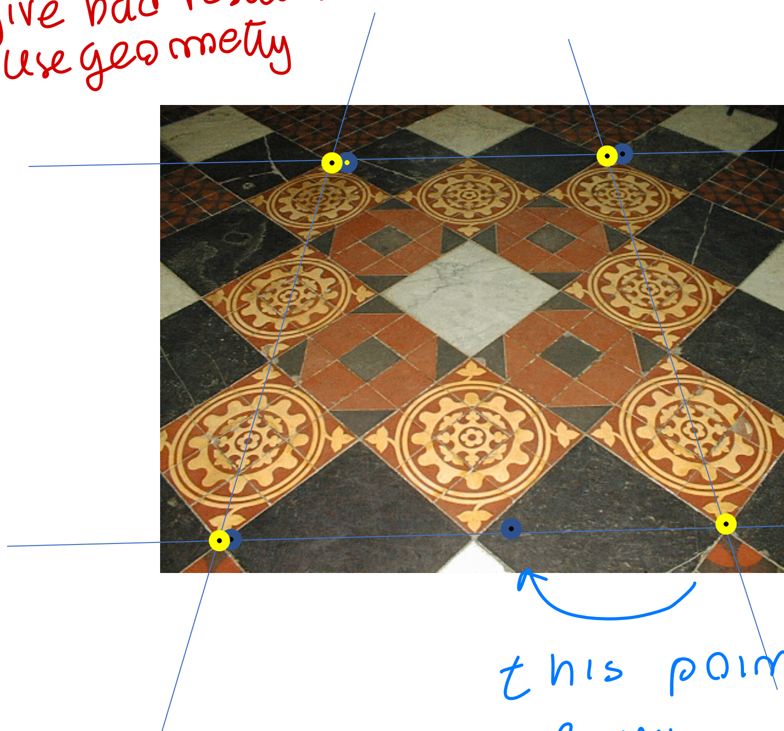
Geometrically vs Algebraic

when Algebra tends to give bad results, use geometry

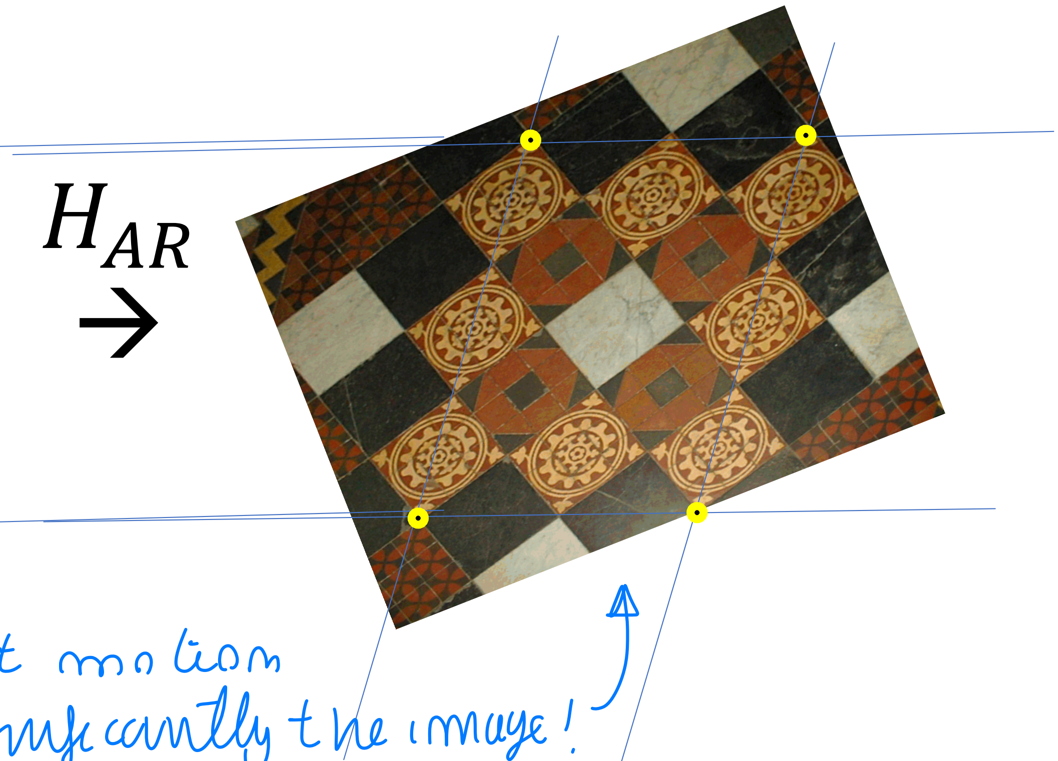
you use more points than needed, then fit transformation... tend to decrease numerical errors

affine reconstruction

you exploit basic geometry to find affine reconstruction



$$H_{AR} \rightarrow$$



this point motion change significantly the image!

2° step: affine → metric (Euclidean)

in Algebraic method
you use two pairs of
orthogonal lines

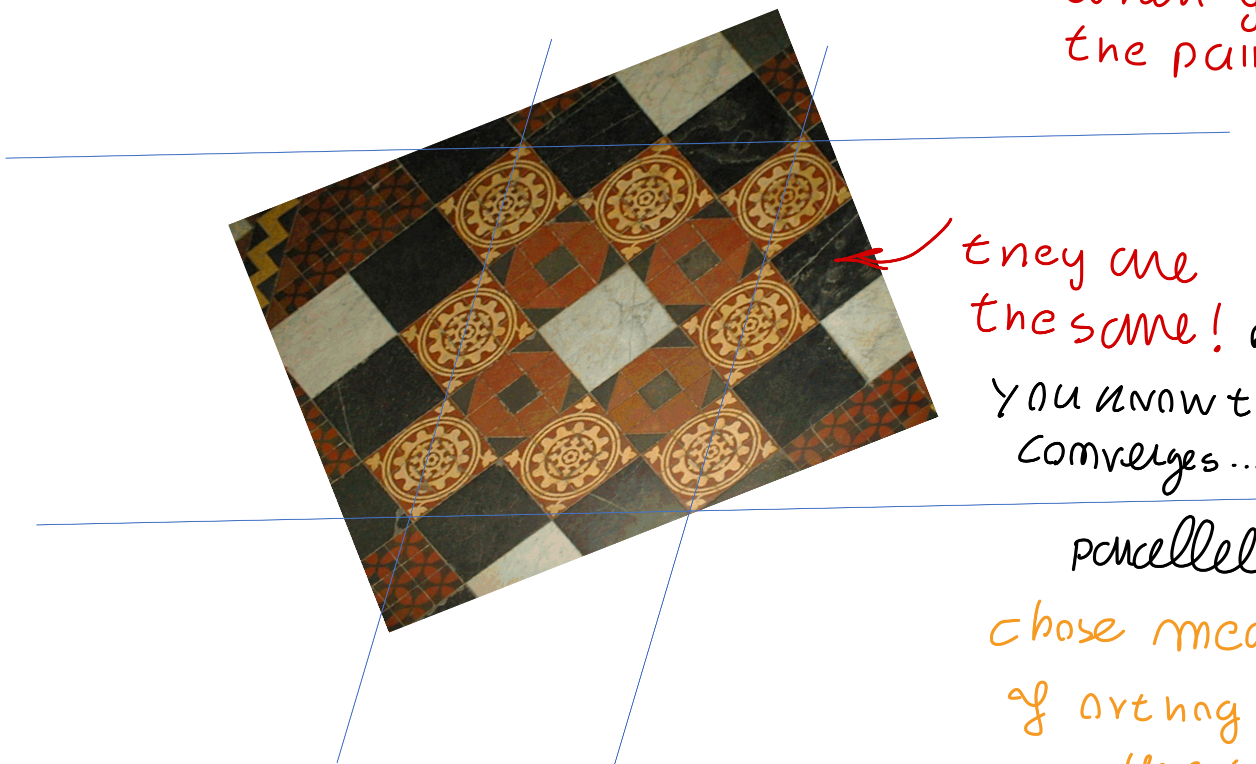
two INDEPENDENT constraints:

- two pairs of image lines, that are images of orthogonal lines

$$\begin{cases} l'^T C_\infty^{*\prime} m' = 0 \\ p'^T C_\infty^{*\prime} q' = 0 \end{cases}$$

$\begin{cases} l \perp m \\ p \perp q \end{cases}$ original lines

these two pairs lead to the same constraint
→ they are not independent

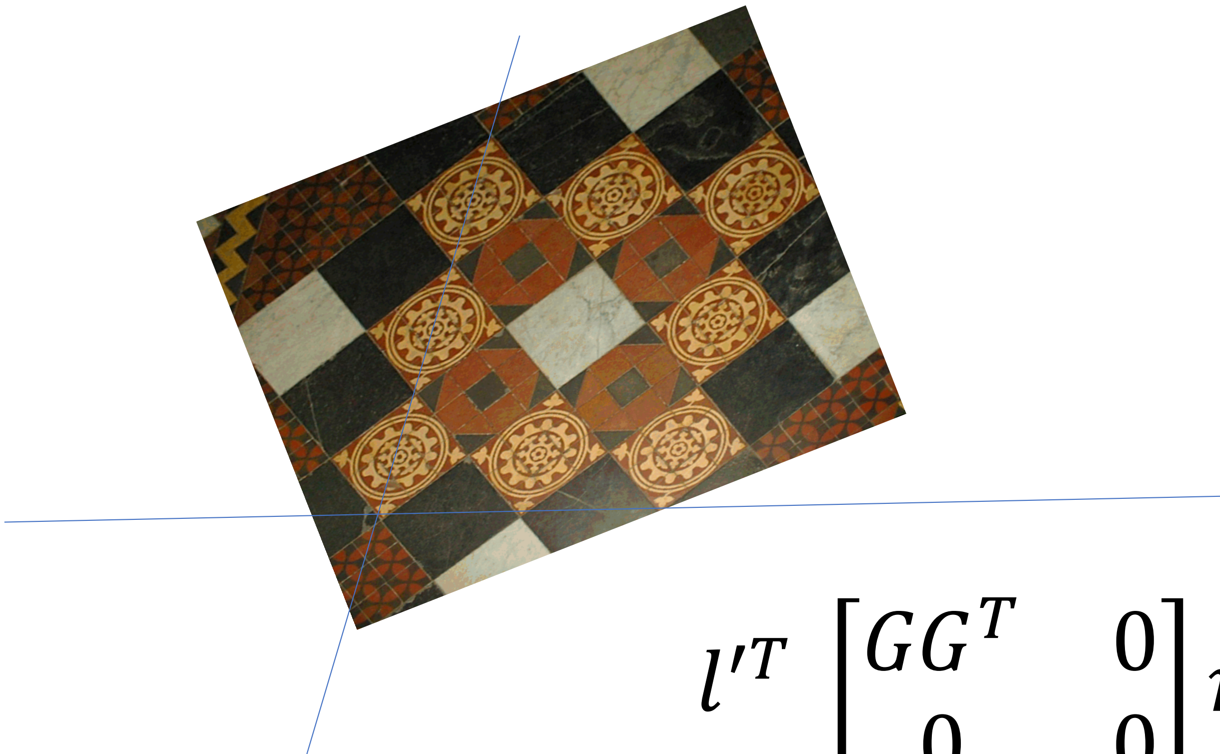


when you chose the pairs of orthogonal lines is important to chose independent

they are the same! once you know ℓ'_∞, you know this lines converges... images of parallel lines couples

chose meaningfull couple of orthogonal lines! use wisely pairs

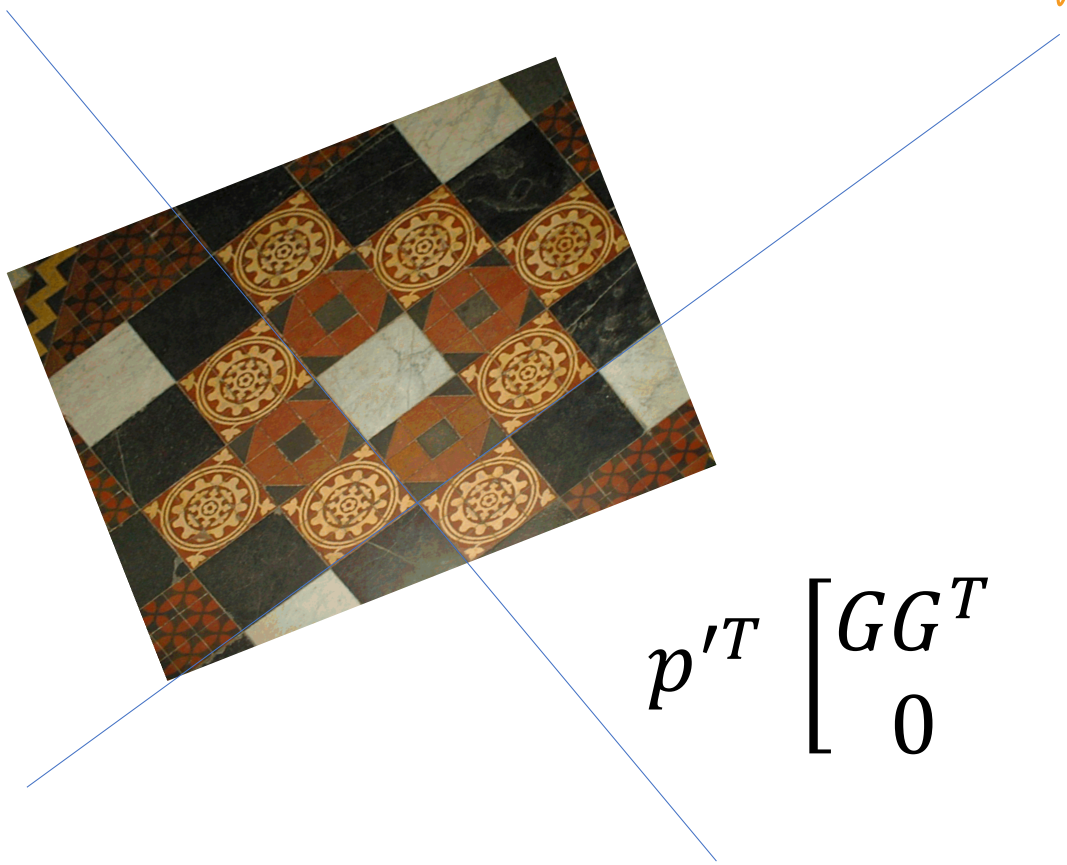
→ use only one of the two pairs as first pair



$$l'^T \begin{bmatrix} GG^T & 0 \\ 0 & 0 \end{bmatrix} m' = 0$$

→ use a new pair as a second pair

INDIPENDENT from previous



$$p'^T \begin{bmatrix} GG^T & 0 \\ 0 & 0 \end{bmatrix} q' = 0$$

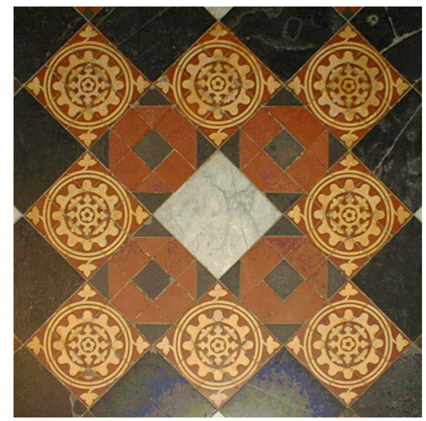
Once GG^T has been estimated, find G by Cholesly Factorisation

then apply rectifying transformation

$$H_{rect} = \begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix}^{-1}$$

Metric rectification from affine

H_{rect}



affine rectification

metric rectification

from affine to metric: **geometric** method with two pairs of orthogonal lines

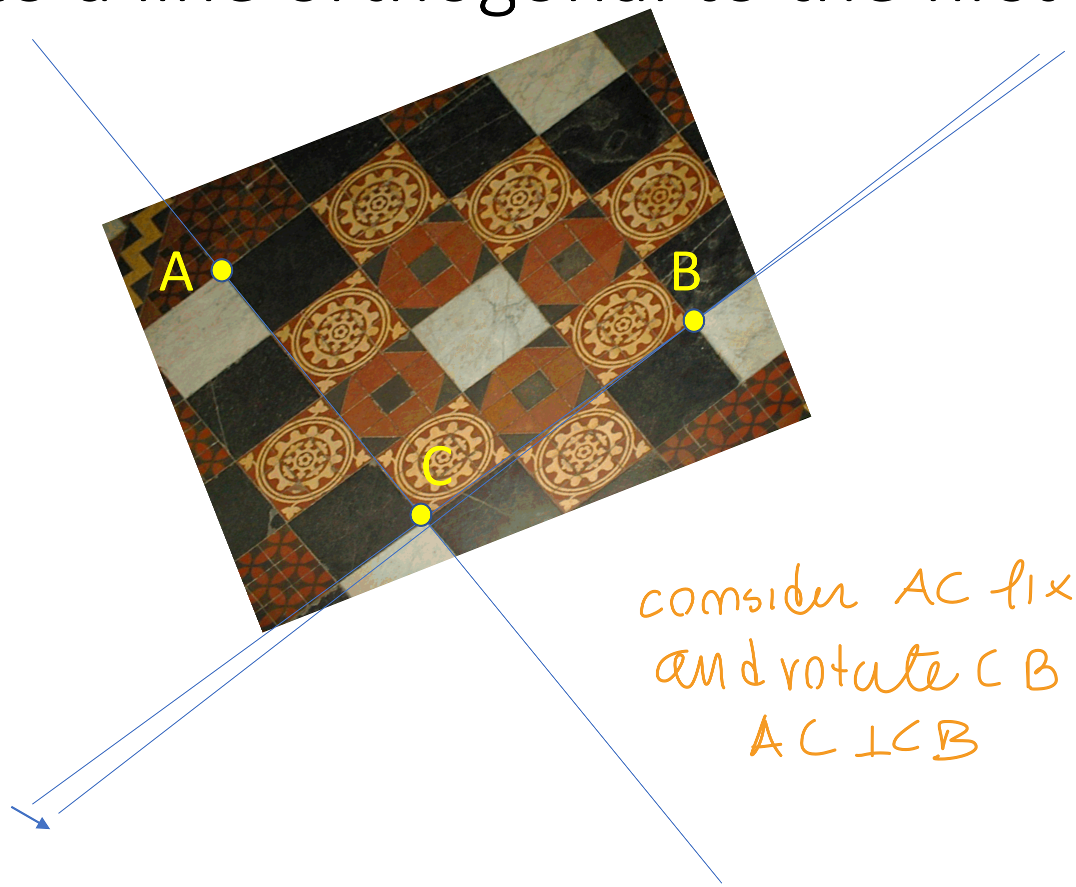
having two pairs of orthogonal lines.

first pair, this should be ⊥ in reconstruction

⇓

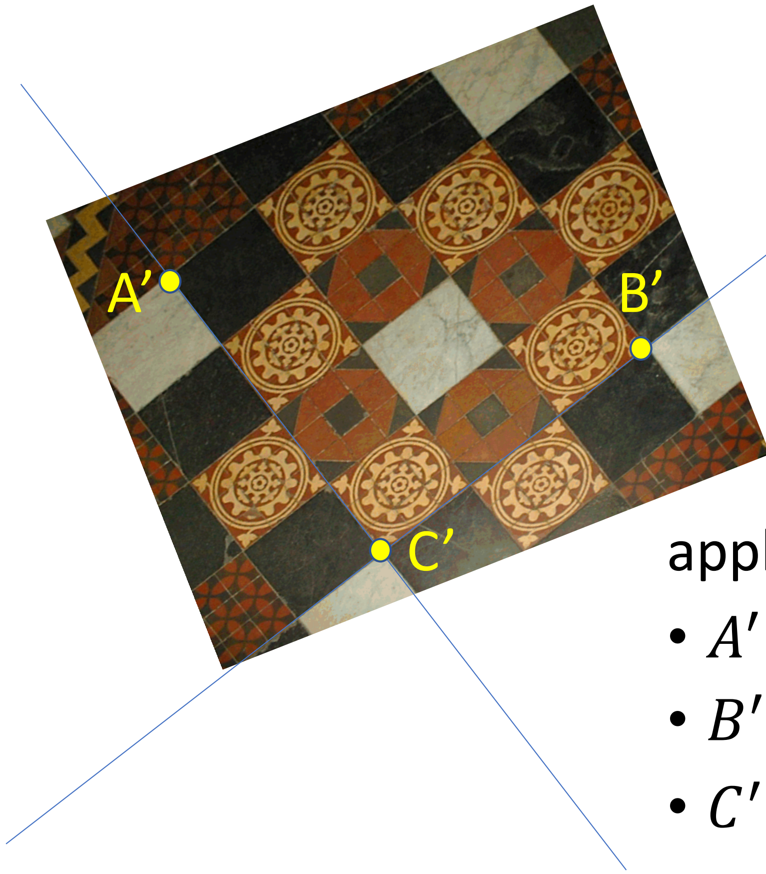
so you simply take one line invariant and rotate the other one

First pair of orthogonal lines: apply a homography (in fact, an **affinity**) that maps the second line onto a line orthogonal to the first line



consider AC fixed
and rotate CB to make
AC ⊥ CB

First pair of orthogonal lines: apply an affinity that maps the second line onto a line orthogonal to the first line



A', B' = A, B

while C → C'
move to new point

apply affinity H_A, which solves

- $A' = H_A \mathrm{A}$
- $B' = H_A \mathrm{B}$
- $C' = H_A \mathrm{C}$

Second pair of images of orthogonal lines

after first transformation

new lines that should be ⊥ in original scene

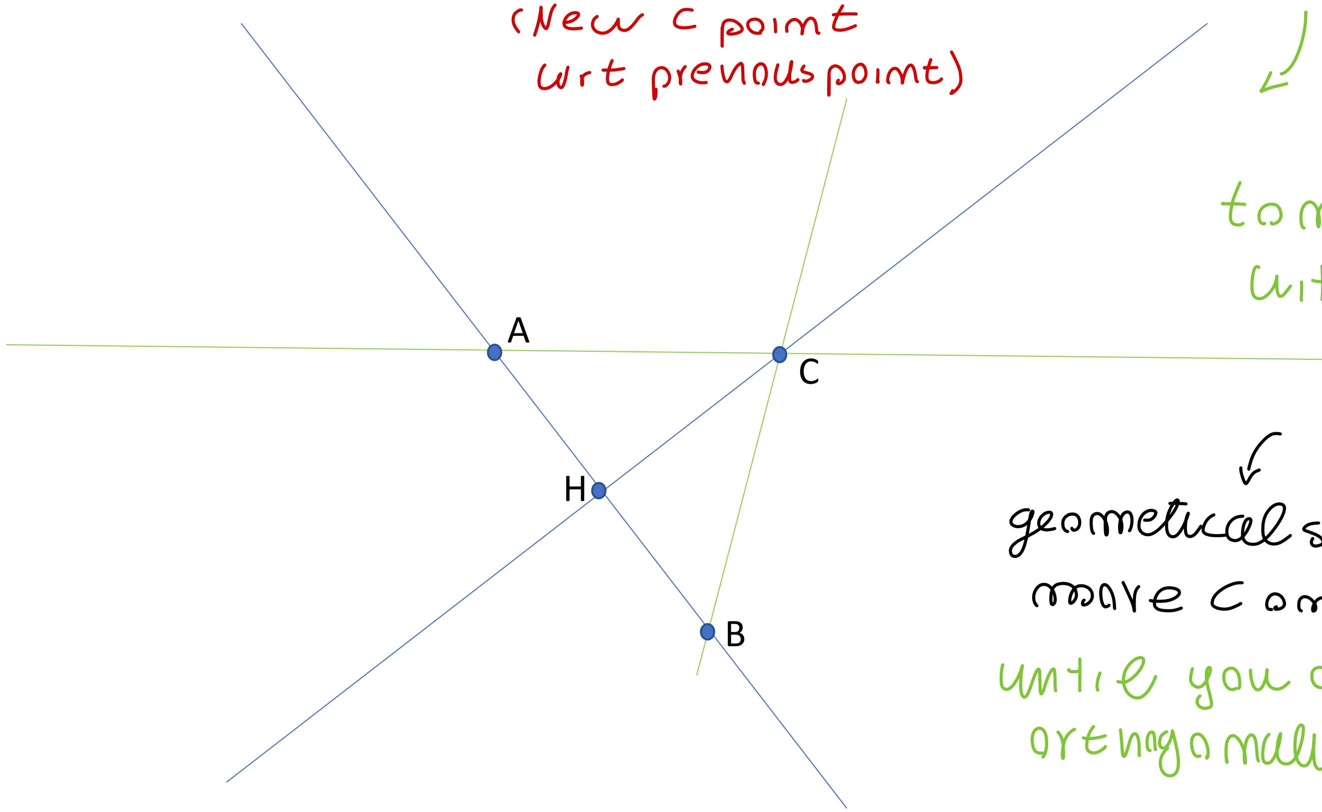
I do the same ?

NO, you must do it without changing previously adjusted lines

we will ruin previous correction if we just move that..

Second pair of image lines must be mapped onto orthogonal lines while keeping the first pair orthogonal

(New C point wrt previous point)

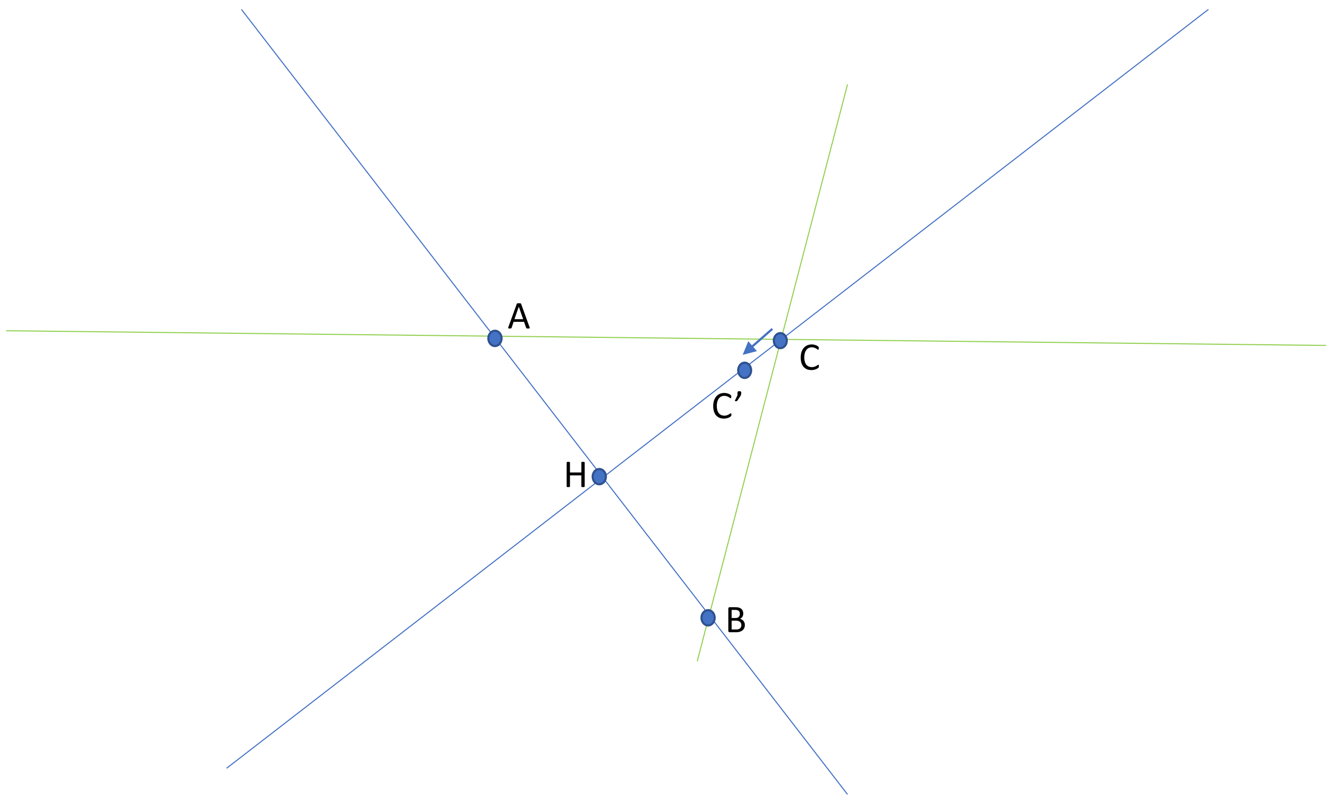


by isolating points

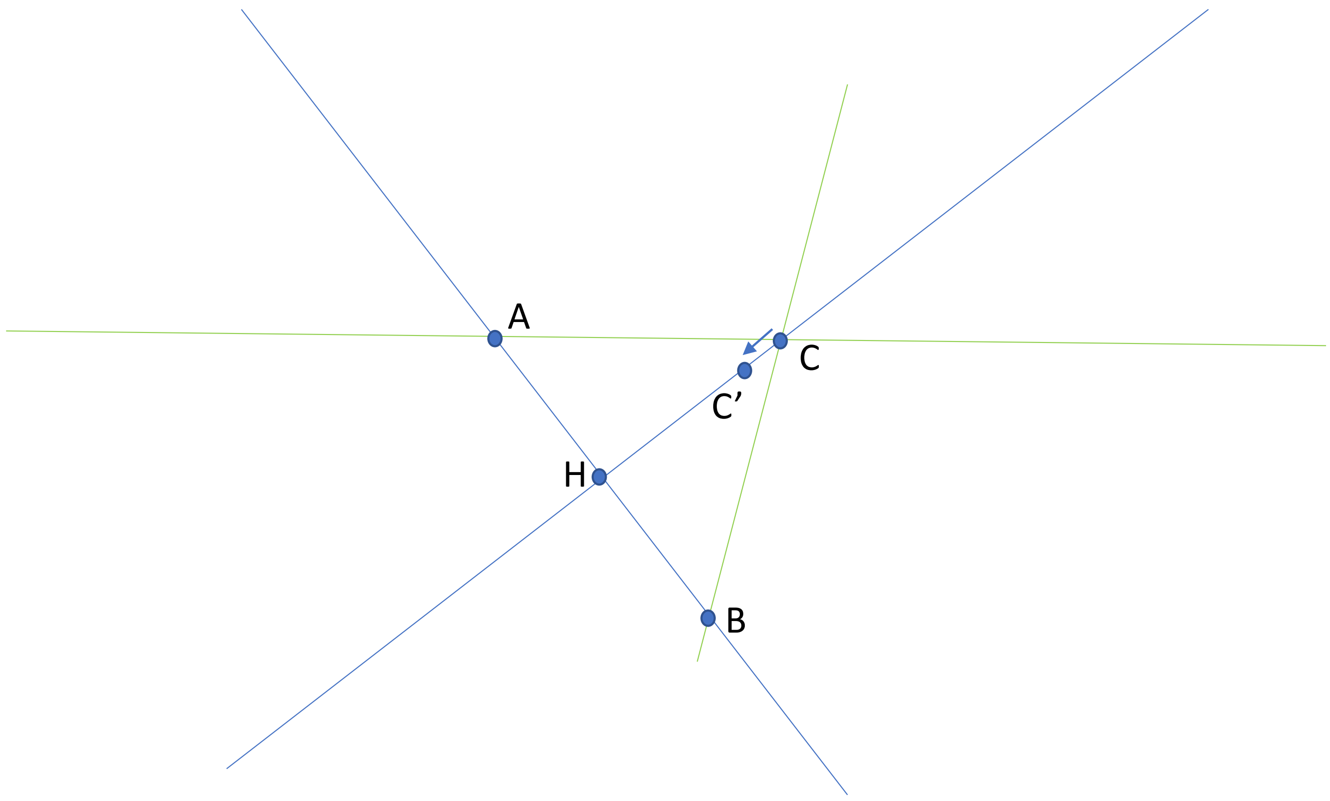
to make $AC \perp CB$ without ruining $AH \perp HC$...

geometrical solution, move C on HC line until you obtain orthogonality

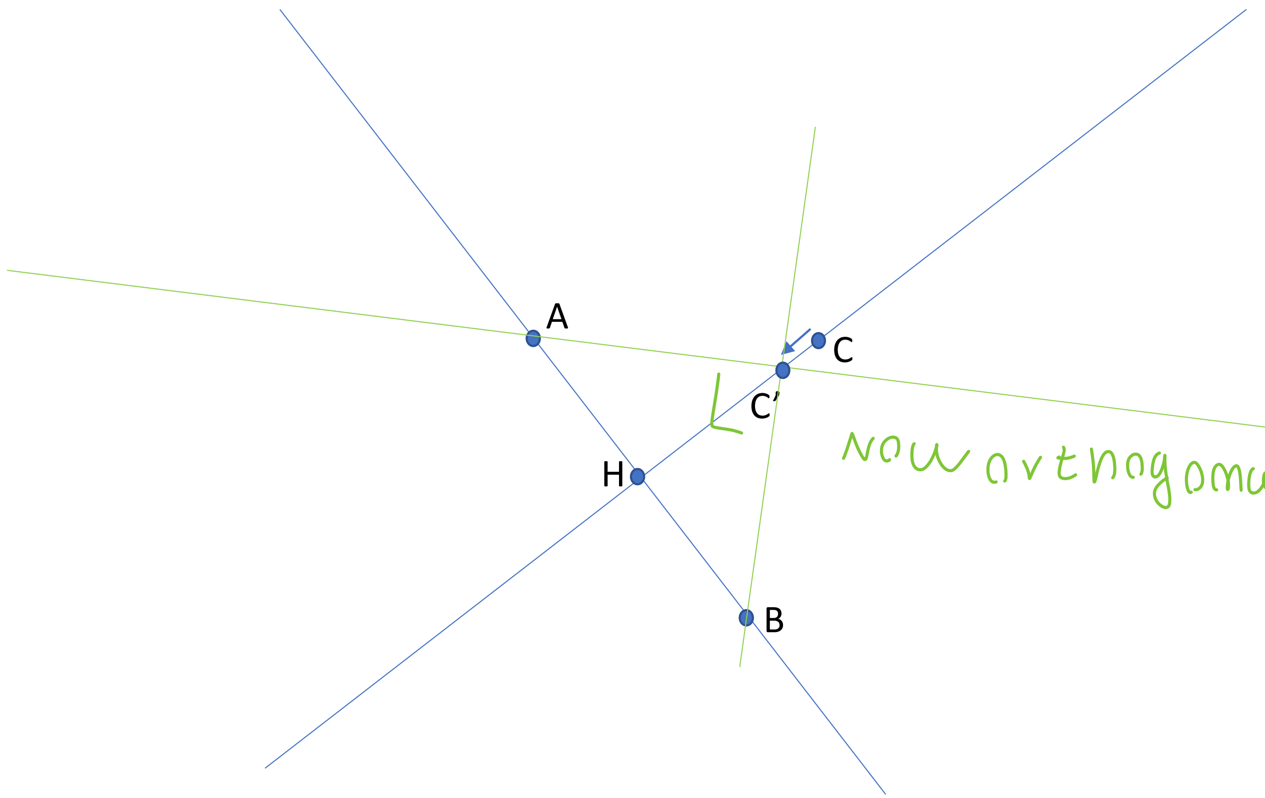
Map point C onto a point C’ still on the line HC such that line HC’ is orthogonal to line AB



Euclide’s theorem: ABC’ is a rectangular triangle iff HC’ = $\sqrt{AH\ HB}$



Euclide's theorem: ABC' is a rectangular triangle iff HC' = $\sqrt{AH\ HB}$

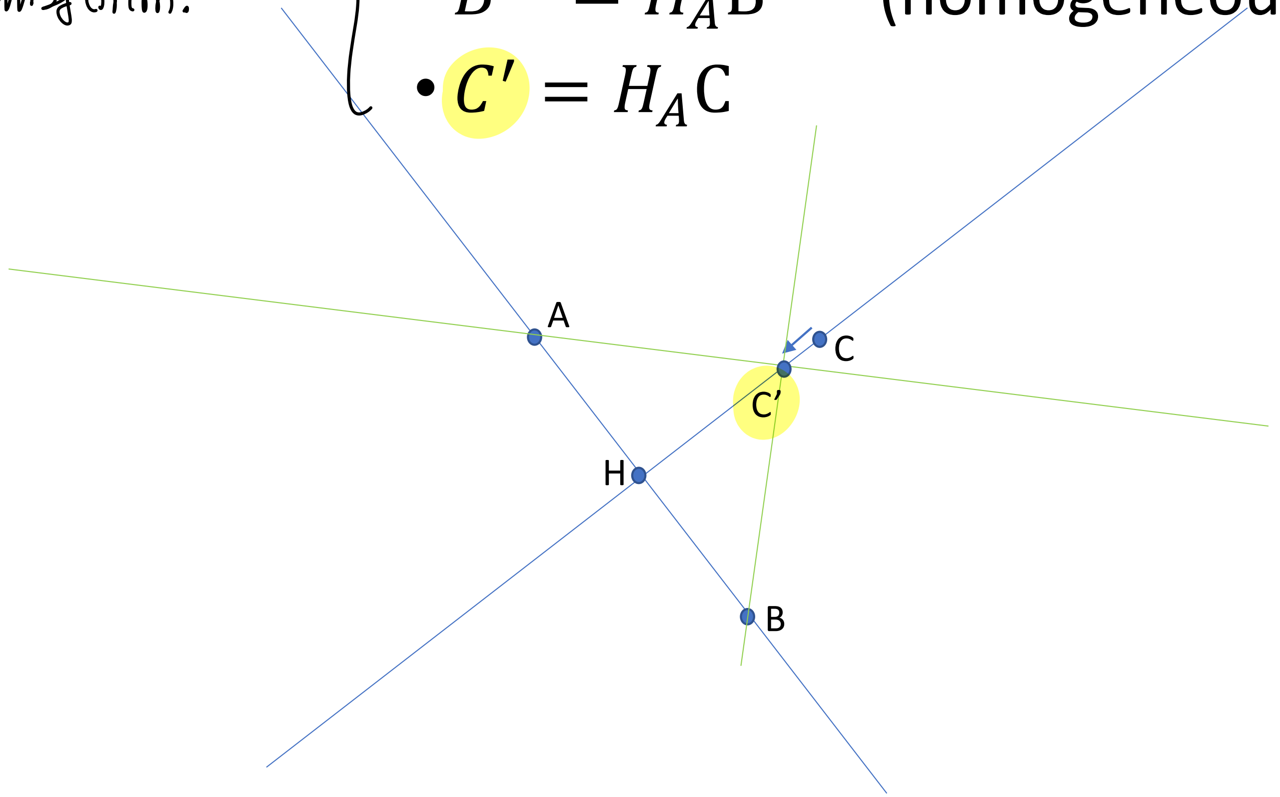


now orthogonal!

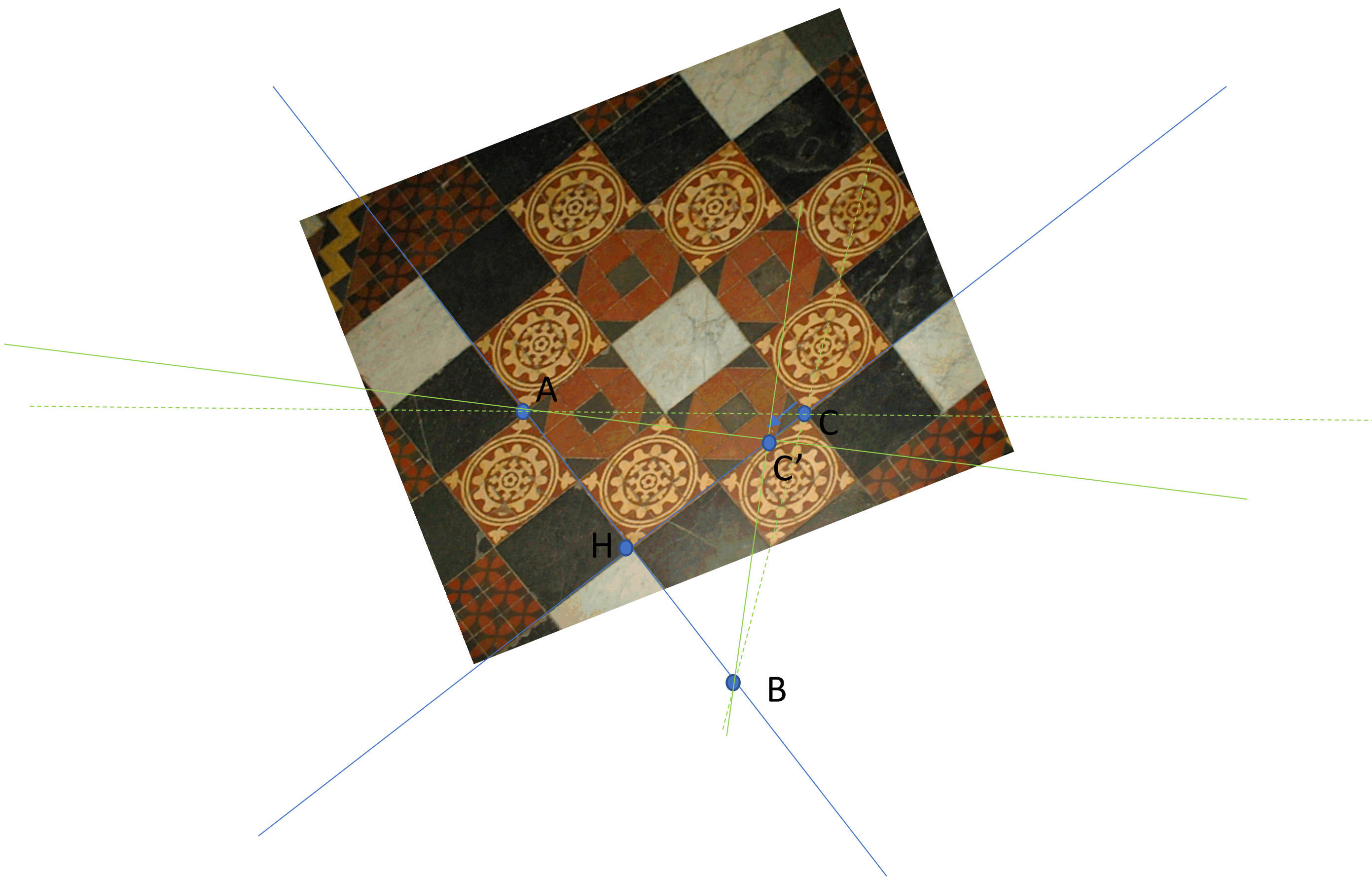
you can estimate C' by equality rectangular triangle you apply new transform.

apply affinity H_A, where H_A is the solution of

- $A \quad = H_A \mathrm{A}$
- $B \quad = H_A \mathrm{B}$ (homogeneous coordinates)
- $C' = H_A \mathrm{C}$

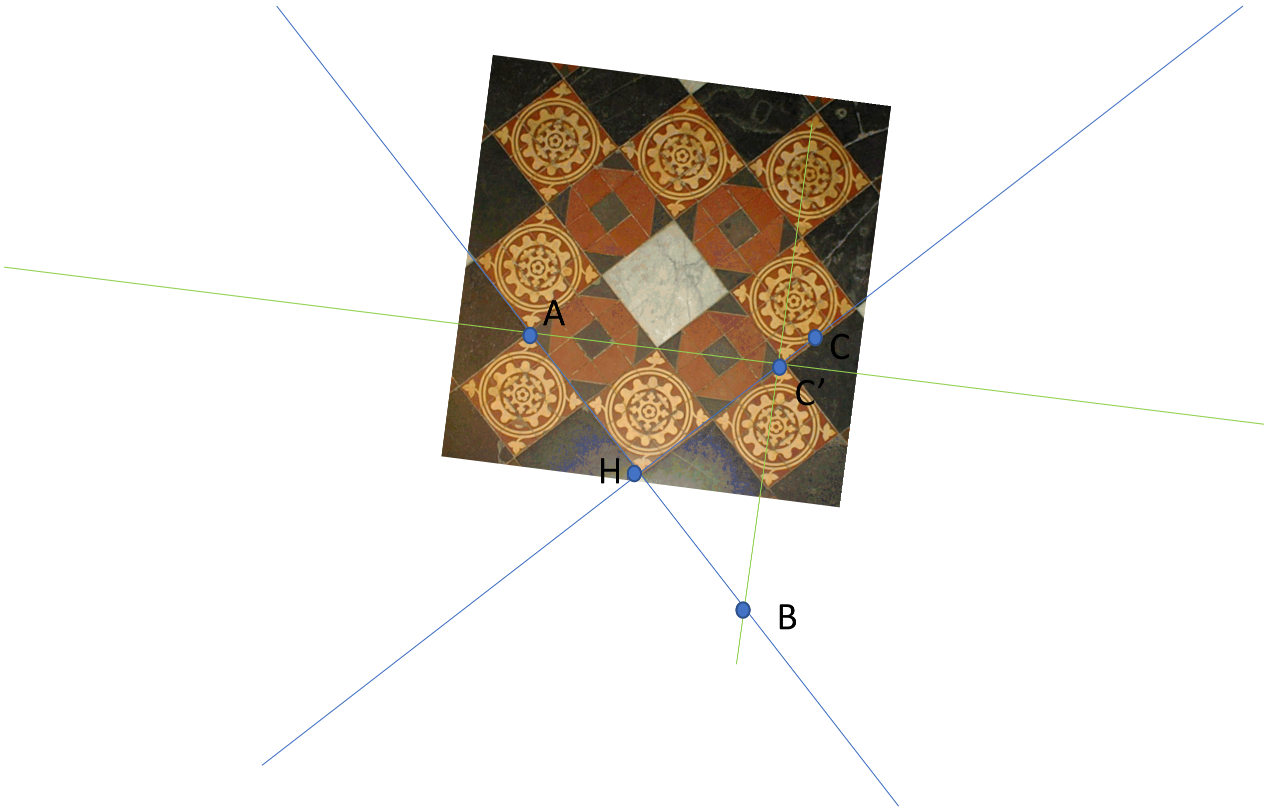


Apply affinity H_A



Apply affinity H_A

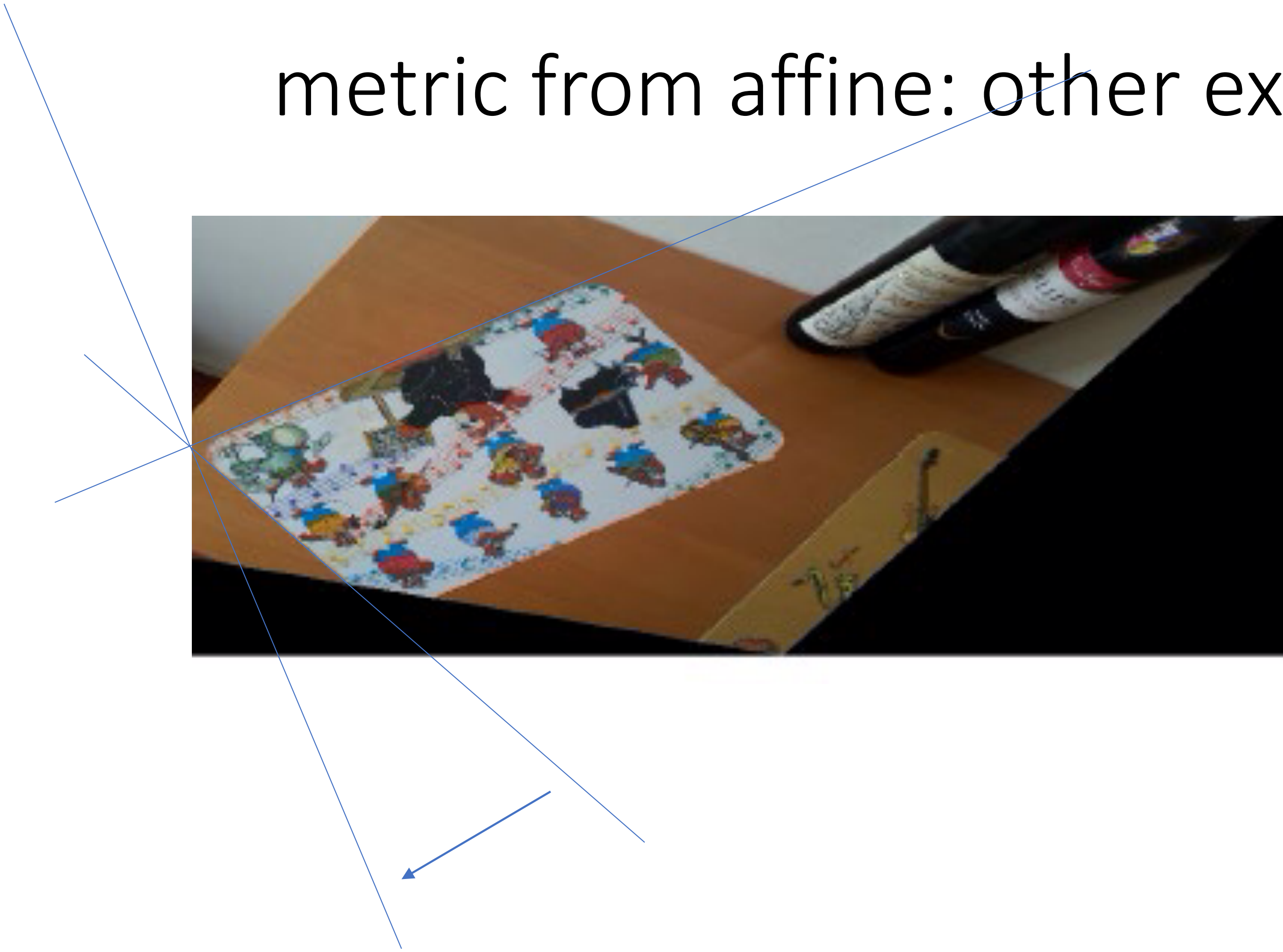
→ reconstructed shape



overall at the end you move all points on final desired pose

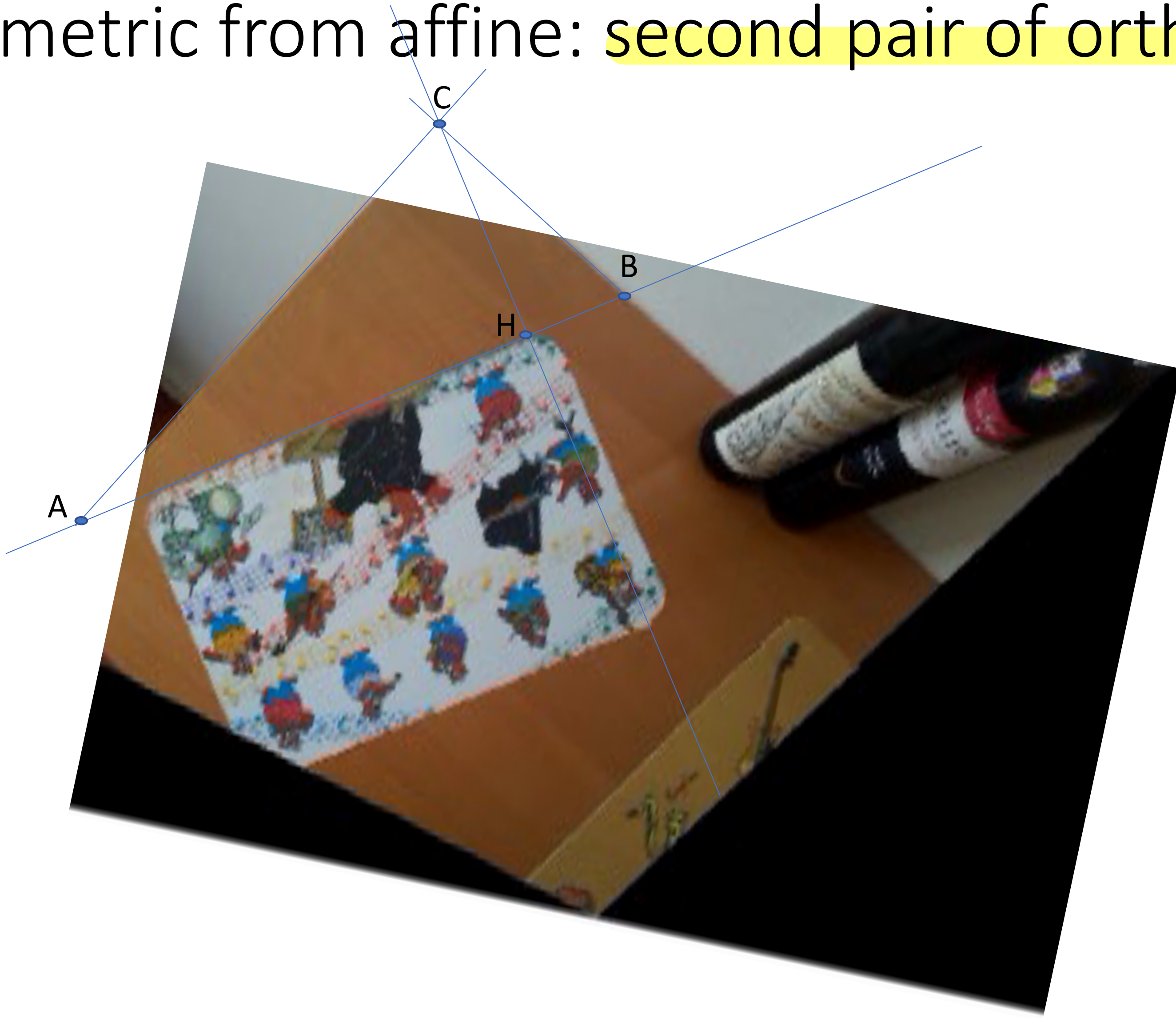
metric from affine: other example

metric from affine: other example



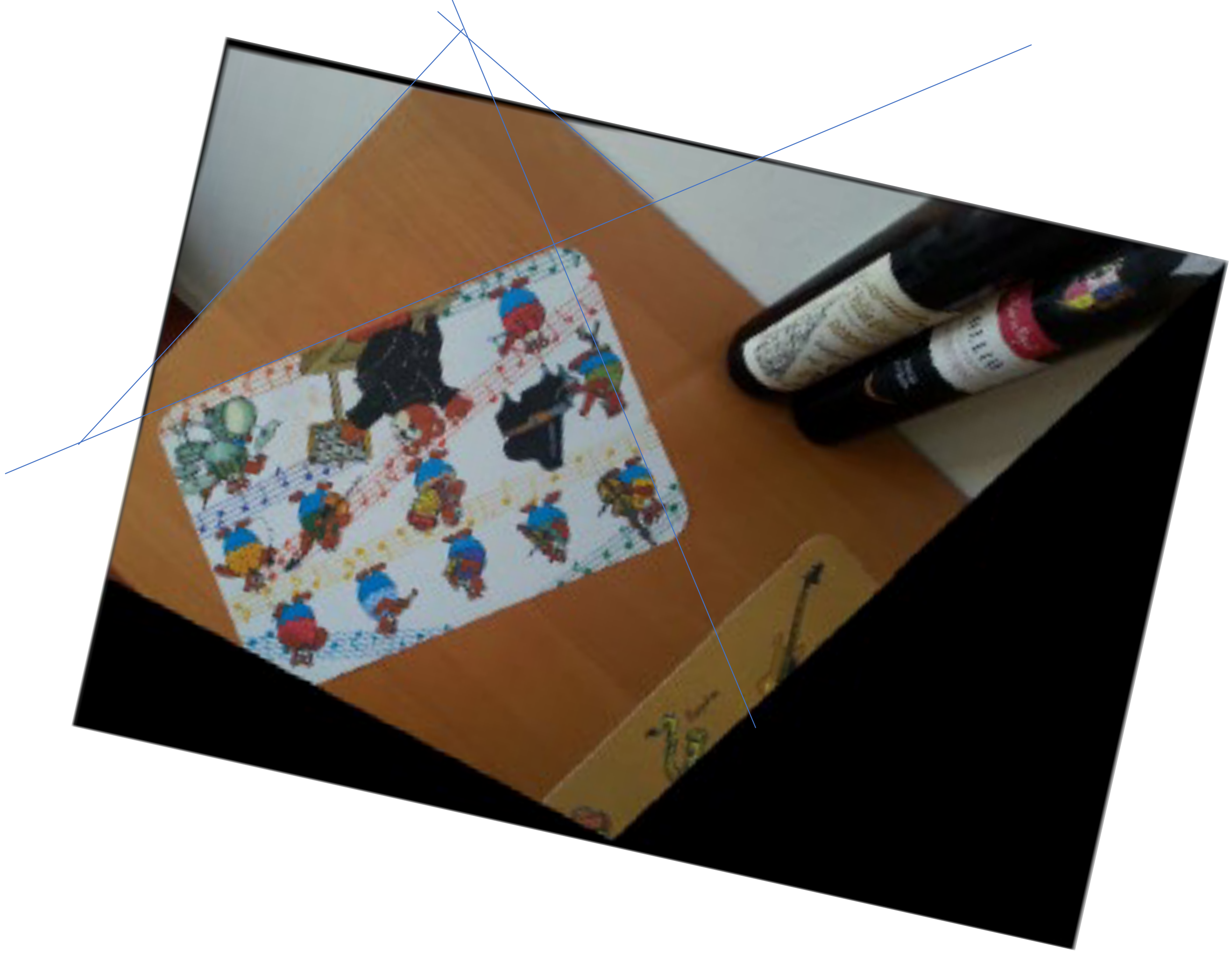
metric from affine: first pair of orthogonal lines
→ make images orthogonal too

metric from affine: second pair of orthogonal lines



second pair of orthogonal lines $HC'^2 = AH \times BH$
Euclides Theorem

rectified image (2D shape reconstructed)



← suggestion on what to do when

Algebra fails due to vanishing
point @ ∞ or too parallel lines etc..

Use Geometry as plan B
When Algebraic fails!

↓

you prefer Algebra because more constraints
can be imposed, reducing errors...

While in Geometry you chose points and move it..
IF one of those is affected by errors for point
extraction
↳ bad Result!

(better to avoid Geometry
→ When NO Degenerate!)

high responsability to chosen points

- Introduction and the Camera Optical System
- Planar (2D) Projective Geometry
- **Space (3D) Projective Geometry**
- Camera Geometry (3D → 2D Projection)

apply concept of 2D into 3D !

NOT all in depth... 2D reconstruction we use additional info + 1 image

while in 3D we need SEVERAL IMAGE (multi view Geometry)

for NOW JUST 3D geometry, usefull to understand CAMERA model Geometry