Stability of non lin syst

Exercises session 2: Lyapunov functions

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Ex. 1: Given the continuous time system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2^2 \\ \dot{x}_2 = -x_2 \end{cases} \tag{1}$$

Show that

1. the $(\bar{x}_1,\ \bar{x}_2) = (0,\ 0)$ is an equilibrium point
2. and study its stability through the linearized system.

Given the function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, $\forall x_1,\ x_2 \neq 0$

3. Check that it is in fact a suitable Lyapunov function
4. and study the stability of the system with the given function

Ex. 2: Given the following parametric system

$$\begin{cases} \dot{x}_1 = x_1\ (k^2 - x_1^2 - x_2^2) + x_2\ (k^2 + x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1\ (k^2 + x_1^2 + x_2^2) + x_2\ (k^2 - x_1^2 - x_2^2) \end{cases} \tag{2}$$

and the function $V(x) = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$,

1. Check that $V(x)$ is a Lyapunov function,
2. study the stability of the origin of the system for $k = 0$
3. and for $k \neq 0$

Ex. 3: Given the continuous time system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - \alpha x_1 - (x_1 + x_2)^2\ x_2 \end{cases} \tag{3}$$

with $\alpha > 0$ and the function $V(x) = \alpha x_1^2 + x_2^2$

1. Check that $V(x)$ is a Lyapunov function,
2. study the stability of the origin of the system
3. and characterize the trajectories of the state around the origin (linearized system's eigenvalues)

Ex. 4: Given the following discrete time system

$$\begin{cases} x_1(k+1) = x_2(k) \cdot \cos(x_1(k)) \\ x_2(k+1) = x_1(k) \cdot \cos(x_2(k)) \end{cases} \tag{4}$$

1. Study the stability of the origin of the linearized system,
2. Use the following Lyapunov function to study the stability of the system, $V(x) = x_1^2(k)+x_2^2(k)$.

Ex. 5: Given the following differential equation

$$\dot{x} = -x^2 + 3x - 2 \tag{5}$$

and the steady state control input $\bar{u} = 2$

1. find the equilibrium of the system for $\bar{u} = 2$.
2. Study the plane $\dot{x}-x$ to determine the stability of the equilibrium and the region of attraction.
3. Verify the results with the Lyapunov function $V(x - \hat{x}) = \dfrac{1}{2}(x - \hat{x})^2$

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1) $\left.\frac{\partial x}{\partial t}\right|_{\bar{x}} = 0 \Rightarrow \dot{x} = 0$ [check for equilibrium]

$$\begin{cases} 0 = -x_1 + x_2^2 \longrightarrow 0 = -\bar{x}_1 + 0 \Rightarrow \bar{x}_1 = 0 \\ 0 = -x_2 \longrightarrow \bar{x}_2 = 0 \end{cases}$$

(0, 0) is an equilibrium of our system

2) stability → linearize the system: (around equilibrium) $(\bar{x}_1, \bar{x}_2) = (0, 0)$

(automomus syst, so not $\partial/\partial u$)

$$\begin{cases} \delta\dot{x}_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_{\bar{x}} \delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{\bar{x}} \delta x_2 = -1\,\delta x_1 + \cancel{2\bar{x}_2\,\delta x_2} \\ \delta\dot{x}_2 = \left.\frac{\partial f_2}{\partial x_1}\right|_{\bar{x}} \delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{\bar{x}} \delta x_2 = 0\,\delta x_1 - 1\,\delta x_2 \end{cases}$$

↓

$$\begin{cases} \delta\dot{x}_1 = -\delta x_1 \\ \delta\dot{x}_2 = -\delta x_2 \end{cases}$$

we just need to study linearized system stability

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

diag matrix, so eig(A):

$s_1 = -1$

$s_2 = -1$

$Re(s) < 0 \;\forall s$

Locally Asymp. stable

↑ because around equilibrium

- $s_i < 0 \;\forall i$ Locally asymp. stable
- IF $\exists i : s_i > 0$ unstable (globally)
- IF $\exists i : s_i = 0$ → anything can be said!

3) we need to check 3 properties of $V(x)$ to ensure good lyap funct

$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

1. positive definite ✓ $V(x) > 0$
2. continuously differentiable ✓ $V(x) \in C^1$
3. radially unbounded ✓ ($V(x) \to +\infty$ if $\|x\| \to +\infty$)

yes, suitable lyap function

4) Check stability through $V(x)$:

by checking when $\dot{V}(x) \le 0$

(continuous time systems)

if < 0 GLOBAL asymp $\forall x$

↳ if ≤ 0 check for which one $\dot{V}(x) = 0$

$$\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} = \frac{1}{\cancel{2}}\Big(\cancel{2}\underbrace{x_1\dot{x}_1}_{\frac{dV}{dx_1}\frac{dx_1}{dt}} + \cancel{2}\underbrace{x_2\dot{x}_2}_{\frac{dV}{dx_2}\frac{dx_2}{dt}}\Big) \underset{(\dot{x}\text{ given})}{=} x_1(-x_1 + x_2^2) + x_2(-x_2) =$$

$$= \underbrace{-x_1^2 - x_2^2}_{<0\ \forall (x_1,x_2)\neq(0,0)} + x_1x_2^2 < 0\ ?$$

his sign depends... overall

(2 approaches):

① Approx

$$\dot{V}(x) = -x_1^2 - x_2^2 + \cancel{x_1x_2^2} \simeq -x_1^2 - x_2^2 + \cancel{O(x^3)} < 0\ !$$

$x_1x_2^2$: 3RD ORD, goes to 0 faster ($\simeq 0$)

neglect infinitesimal of III ORD

⇓ LOCALLY asymp stable

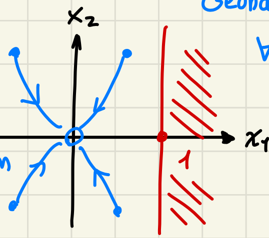
"Locally" valid because we get info near $(0,0)$

②

rewrite $\dot{V}(x) = -x_1^2 + \underbrace{x_2^2(-1 + x_1)}_{<0\ ?\ \Updownarrow\ x_1 < 1}$ → define a range of x for which $\dot{V}(x) < 0$ so asymp stable

II ORD syst:
we can plot our analysis

{found by Lyapunov} → Region of attraction



Globally asymp stable $\forall x_1 < 1$

Lyapunov is SUFFICENT : how to study Globally behav...

study trajectory of the systhem (NOT always explicitly computable)

↓

solving diff. equations $\dot{x}_2 = -x_2 \rightarrow x_2(t) = x_2(0)\, e^{-t}$

substitute it in $\dot{x}_1 = -x_1 + \underbrace{(x_2(0) e^{-t})^2}_{v} = -x_1 + v(t)$

v : Forcing input for I STATE

Global traj of state over time

$$x_1(t) = x_1(0)\, e^{-t} + (e^{-t} - e^{-2t})\, x_2^2(0)$$

$$\left.\begin{array}{l} \lim_{t \to +\infty} x_2(t) = 0 \\ \\ \lim_{t \to +\infty} x_1(t) = 0 \end{array}\right\}$$

goes to equilibrium !

$\forall\, x_1(0), x_2(0)$ the syst goes to origin for $t \to +\infty$

$\Downarrow$

{ GLOBALLY Asymp. stable }

[Lyap. funct is JUST sufficent,
we JUST know a limited region of attraction
BUT it can be Globbally anyway

Ex. 2: Given the following parametric system on parameter K

$$\begin{cases} \dot{x}_1 = x_1\,(k^2 - x_1^2 - x_2^2) + x_2\,(k^2 + x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1\,(k^2 + x_1^2 + x_2^2) + x_2\,(k^2 - x_1^2 - x_2^2) \end{cases}$$

and the function $\boxed{V(x) = \frac{1}{2}(x_1^2 + x_2^2) \geq 0,}$

1. Check that $V(x)$ is a Lyapunov function,
2. study the stability of the origin of the system for $k = 0$
3. and for $k \neq 0$

1) same as EX.1 $\Rightarrow$
- 1) pos. def ✓
- 2) continuously differentiable ✓ → OK
- 3) Radially unb. ✓

2) $V(x) = \frac{1}{2}(x_1^2+x_2^2) \rightarrow \dot{V}(x) < 0$? for which values...

$$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2(K^2 - x_1^2 - x_2^2) + \cancel{x_1x_2(K^2+x_1^2+x_2^2)} +$$

$(\dot{x}_1, \dot{x}_2)$ of s.s represent.

$$+\cancel{x_2(-x_1)(K^2+x_1^2+x_2^2)} + x_2^2(K^2 - x_1^2 - x_2^2)$$

$$\dot{V}(x) = (x_1^2+x_2^2)(K^2-x_1^2-x_2^2)$$

[for which values $\dot{V} < 0$?]

for $\boxed{K=0}$

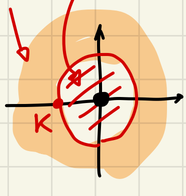
$$\dot{V}(x) = \underbrace{(x_1^2+x_2^2)}_{>0 \text{ always}}\underbrace{(-x_1^2-x_2^2)}_{<0} < 0 \quad \forall x \neq (0,0)$$

↳ ORIGIN GLOBALLY asymp. stable

3) for $\boxed{K \neq 0}$

$$\dot{V}(x) = \underbrace{(x_1^2+x_2^2)}_{>0}\underbrace{(K^2-x_1^2-x_2^2)}_{<0\,?} \longrightarrow \begin{matrix} K^2 - x_1^2 - x_2^2 < 0 \\ x_1^2 + x_2^2 > K^2 \end{matrix}$$

Circumference of radius K centered around (0,0)



the ORIGIN lays inside UNSTABLE REGION!

$\Rightarrow \forall K \neq 0$ OUR SYSTEM IS UNSTABLE!

Ex. 3: Given the continuous time system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 \end{cases}$$

with $\alpha > 0$ and the function $V(x) = \alpha x_1^2 + x_2^2$ [parametric Lyap function]

1. Check that $V(x)$ is a Lyapunov function,
2. study the stability of the origin of the system
3. and characterize the trajectories of the state around the origin (linearized system's eigenvalues)

1) Lyap funct? → ① $\alpha > 0 \to$ pos. def ✓

② $V(x) \in C^1$ ✓ → suitable Lyap funct

③ Radially unb. ✓

2) $\bar{x} = (0,0)$ is an equilibrium

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} = 2\alpha x_1 \dot{x}_1 + 2 x_2 \dot{x}_2 =$$

$$= 2\alpha x_1 x_2 + 2 x_2 \left(-x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2\right) =$$

$$= \cancel{2\alpha x_1 x_2} - 2x_2^2 - \cancel{2\alpha x_1 x_2} - 2x_2^2 (x_1 + x_2)^2$$

↓

$$\dot{V}(x) = \underbrace{-2x_2^2}_{\leq 0} \underbrace{\left(1 + (x_1+x_2)^2\right)}_{>0 \text{ always}} \leq 0 \quad \forall x \text{ but here}$$

$\dot{V}(x) = 0 \quad \forall (x_1, 0)$

↑ independently from the value of x_1

{ NO INFO about origin stability! } ← $\dot{V}(x) = 0 \;\; \forall x_1 \neq 0$

⇓

We need to use KRASOWSKI-LESECLE th for that cases when $\dot{V}(x) = 0$

"energetically" $\dot{V}(x)$ is the energy change → $\dot{V}(x) < 0 =$ "dissipation of energy by the syst"

When $\dot{V}(x) = 0$ system fixed on a certain energy state

↳ check if traj where $\dot{V} = 0$?

if $\dot{V}=0$ only on equilibrium point → asymp stable..
IF for some $x \neq eq$ $\dot{V}=0$ (No conclusions!)

$$\dot{V}(x)=0 \longrightarrow x_2=0$$

from our model when $x_2=0$
$$\begin{cases} \dot{x}_1=0 \\ \dot{x}_2=-0-\alpha x_1-0=-\underbrace{\alpha}_{>0 \text{ always}} x_1 \end{cases} \rightarrow \dot{x}_2=0 \text{ only if } x_1=0$$

only traj such that $\dot{V}=0$
is the origin! → origin asym stable
for KRASOWSKI LESELLE

3) characterize
traj of state around origin (can be done by lin. syst)

using the eig values of linearized syst I know how it behaves

Linearize syst:
(around $\bar{x}=(0,0)$)
$$\begin{cases} \delta\dot{x}_1=\frac{\partial f_1}{\partial x_1}\Big|_{\bar{x}}\delta x_1+\frac{\partial f_1}{\partial x_2}\Big|_{\bar{x}}\delta x_2 \\ \delta\dot{x}_2=\frac{\partial f_2}{\partial x_1}\Big|_{\bar{x}}\delta x_1+\frac{\partial f_2}{\partial x_2}\Big|_{\bar{x}}\delta x_2 \end{cases} \rightarrow \begin{cases} \delta\dot{x}_1=\delta x_2 \\ \delta\dot{x}_2=-\alpha\,\delta x_1+(-\delta x_2) \end{cases}$$

find eig values (to characterize traj)
$$A=\begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix} \rightarrow$$
compute eig values as solution of
$$\det(sI-A)=0$$
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity of A order

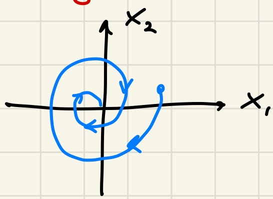
$$\det\begin{pmatrix} s & -1 \\ \alpha & s+1 \end{pmatrix}=s^2+s+\alpha=0$$
↳ (2 eig values of syst)

$$s_{1,2}=\frac{-1\pm\sqrt{1-4\alpha}}{2}$$
↓
wrt varying α value.

a. $1-4\alpha<0$ (Im eig val)
→ $\alpha>1/4$

$$s_{1,2}=-\frac{1}{2}\pm i\,\frac{1}{2}\sqrt{4\alpha-1} \rightarrow \begin{Bmatrix} \text{complex} \\ \text{conj stable} \\ \text{poles} \end{Bmatrix}$$
Re(s)<0 STABLE

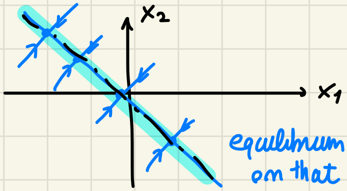
stable focus



b. $\alpha = 0$ $\quad s_1 = -1$, $s_2 = 0$

stable eig val

null eig value

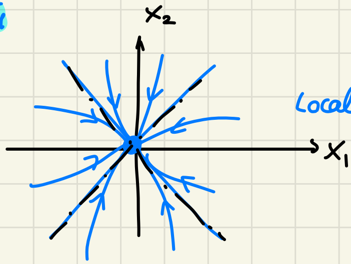


equilibrium lays on that line reading with that traj

c. $0 < \alpha < 1/4$

$s_{1,2} < 0$

Re eig val



Locally unique equilibrium

Ex. 4: Given the following discrete time system

$$\begin{cases} x_1(k+1) = x_2(k) \cdot \cos(x_1(k)) \\ x_2(k+1) = x_1(k) \cdot \cos(x_2(k)) \end{cases} \tag{4}$$

1. Study the stability of the origin of the linearized system,
2. Use the following Lyapunov function to study the stability of the system, $V(x) = x_1^2(k) + x_2^2(k)$.

1)

$$\begin{cases} \delta x_1(k+1) = \left.\frac{\partial f_1}{\partial x_1}\right|_{\bar{x}} \delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{\bar{x}} \delta x_2 \\ \delta x_2(k+1) = \left.\frac{\partial f_2}{\partial x_1}\right|_{\bar{x}} \delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{\bar{x}} \delta x_2 \end{cases} \rightarrow \begin{cases} \delta x_1(k+1) = \cancel{-\sin(\bar{x}_1)\bar{x}_2}\, \delta x_1(k) + \overset{1}{\cancel{\cos(\bar{x}_1)}}\, \delta x_2(k) \\ \delta x_2(k+1) = \overset{1}{\cancel{\cos(\bar{x}_2)}}\, \delta x_1(k) + \cancel{(-\sin(\bar{x}_2)\bar{x}_1)}\, \delta x_2(k) \end{cases}$$

$$\begin{cases} \delta x_1(k+1) = \delta x_2 \\ \delta x_2(k+1) = \delta x_1 \end{cases} \longrightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

anti-diag matrix

make conclusion from lin. syst

$$\downarrow$$

$$\text{eig}(A) \Rightarrow \det(zI - A) = z^2 - 1 = 0 \quad \underbrace{z_{1,2} = \pm 1}$$

lin. simply stable ← [on the unitary circle!]

↓

NOTHING can be inferred about syst stability

Discrete time { asymp stable are eig inside $|z| < 1$! }

NO INFO about stability

2)

$V(x) = x_1^2(k) + x_2^2(k)$ → suitable Lyapunov

{ 1. pos def ✓
2. cont. differentiable ✓ → OK!
3. radially unb ✓ }

[property 3 guarantee that the conclusions are global] usefull when we have only one eq.

↓

study syst stability

in discrete time know we study $\Delta V(x)$ instead of $\dot{V}(x)$

$$\Delta V(x) = V(x(k+1)) - V(x(k)) =$$

$$= \underbrace{x_2^2(k)\cos^2(x_1(k)) + x_1^2(k)\cos^2(x_2(k))}_{V(x(k+1))} - \underbrace{x_1^2(k) - x_2^2(k)}_{V(x(k))} =$$

$$= \underbrace{-x_1^2(k)}_{<0}\underbrace{(1 - \cos^2(x_2(k)))}_{\sin^2(x_2(k))} - \underbrace{x_2^2(k)}_{<0}\underbrace{(1 - \cos^2(x_1(k)))}_{\sin^2(x_1(k))} = -x_1^2(k)\sin^2(x_2(k)) - x_2^2(k)\sin^2(x_1(k))$$

$\Delta V(x) \leq 0$ always!

$\Delta V(x) = 0$ both for $\begin{cases} \bullet\ x_1 = 0, \forall x_2 \\ \bullet\ x_2 = 0, \forall x_1 \end{cases}$ Nothing can be concluded by Lyapunov

here with multiple conditions to check $\Leftarrow$ $\left\{ \begin{array}{l} \text{KRASOWSKI-LESELLE} \\ \text{hard to apply} \end{array} \right\}$

(condition 1) $x_1 = 0$

$$\begin{cases} x_1(k+1) = \bar{x}_2 \cos(0) \\ x_2(k+1) = 0 \cos(\bar{x}_2) \end{cases}$$

evolution of syst over time

@ $k=0$ with $x_1(0)=0, x_2(0)=\bar{x}_2$

$$\begin{cases} x_1(1) = \bar{x}_2 \\ x_2(1) = 0 \end{cases}$$

over time... from initial condition the two value swap (OSCILLATION!)

@ $k=1$

$$\begin{cases} x_1(2) = 0 \cos(\bar{x}_2) = 0 \\ x_2(2) = \bar{x}_2 \cos(0) = \bar{x}_2 \ \ldots \end{cases}$$

undamped oscillation

$(0, \bar{x}_2) \to (\bar{x}_2, 0) \to (0, \bar{x}_2) \ldots.$

(condition 2) $x_2 = 0$

with $x_1(0) = \bar{x}_1$, $x_2(0) = 0$

@ $k=0$

$$\begin{cases} x_1(1) = x_2(0) \cos(x_1(0)) = 0 \\ x_2(1) = \underbrace{\bar{x}_1}_{x_1(0)} \cos(\bar{x}_2(0)) = \bar{x}_1 \ \ldots \end{cases}$$

also here

$(0, \bar{x}_1) \to (\bar{x}_1, 0) \ldots$

undamped oscillation

developing trajectory on this example

analyzing traj $\rightarrow$ (simply stable system) because undamped oscillation

Ex. 5 Given the following differential equation

I ORD system

$$\dot{x} = -x^2 + 3x - u(t) \quad (5)$$

(to get graphical study)

and the steady state control input $\bar{u} = 2$

1. find the equilibrium of the system for $\bar{u} = 2$.
2. Study the plane $\dot{x} - x$ to determine the stability of the equilibrium and the region of attraction.
3. Verify the results with the Lyapunov function $\boxed{V(x - \hat{x}) = \frac{1}{2}(x - \hat{x})^2}$

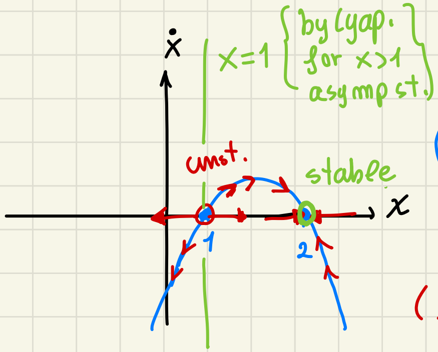
1) we know $\bar{u} = 2$

→ set $\dot{x} = 0$, $\quad 0 = -\bar{x}^2 + 3\bar{x} - \bar{u}$

$$\bar{x}_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} \quad \begin{matrix} \nearrow \bar{x}_1 = 2 \\ \searrow \bar{x}_2 = 1 \end{matrix}$$

candidate equilibrium for $\bar{u} = 2$ control input!

2) cartesian plane $\dot{x} - x$ (only for I ORD syst)



$\dot{x} = -x^2 + 3x - \bar{u}$ parabola intercepting in $\bar{x} = 2, 1$

(from equilibrium point)

if $x \neq 1, 2$... moving out of eq

$(\bar{x} = 1)$: moving in $x < 1 \rightarrow \dot{x} < 0 \Rightarrow x \downarrow$

moove away

We go away from $\bar{x} = 1$ ← moving in $x > 1 \rightarrow \dot{x} > 0 \Rightarrow x \uparrow$

UNSTABLE EQ.

$(\bar{x} = 2)$ moving in $x < 2 \rightarrow \dot{x} > 0 \Rightarrow x \uparrow$

in $x > 2 \rightarrow \dot{x} < 0 \Rightarrow x \downarrow$ come back

(stable equilibrium)

it can be verifyed with Lyapunov

3) $V(x-\hat{x}) = \frac{1}{2}(x-\hat{x})^2$ (Lyapunov)

change of var to be sure $V=0$ @ equilibrium

$\bar{x} = x_2 \rightarrow \hat{x} = \bar{x}_2 = 2$ ($\bar{x}=2$ constant)

define new var $\Delta x = x - \bar{x} \Rightarrow \Delta\dot{x} = \dot{x}$

$$V(\Delta x) = \frac{1}{2}\Delta x^2$$

pos def ✓
C^1 ✓
rad. unb ✓
OK

$$\dot{V}(\Delta x) = \frac{\partial V}{\partial \Delta x}\frac{\partial \Delta x}{\partial t} = \Delta x \cdot \Delta\dot{x} = \Delta x\, \dot{x} =$$

└ diff eq.

$$= \Delta x(-x^2 + 3x - 2) =$$

└ $\bar{u}=2$, analyze for that control input

$x = \Delta x + \bar{x}$

$$\dot{V}(\Delta x) = \Delta x\left(-(\Delta x+\bar{x})^2 + 3(\Delta x + \bar{x}) - 2\right) =$$

$$= \Delta x(-\Delta x^2 \cancel{-\bar{x}^2} - 2\Delta x\bar{x} + 3\Delta x \cancel{+3\bar{x}-2})$$

$-\bar{x}^2 + 3\bar{x} - 2 = 0$ as we find during equilibrium research

$$\dot{V}(\Delta x) = \Delta x(-\Delta x^2 - 2\bar{x}\Delta x + 3\Delta x) =$$

$$= \Delta x^2 \underbrace{(-\Delta x - 2\bar{x} + 3)}$$ check for which values $\dot{V}(\Delta x) < 0$

>0 always

$$-\Delta x - 2\bar{x} + 3 < 0 \Leftrightarrow \Delta x > 3 - 2\bar{x}$$

for our two equilibrium

- $\bar{x} = 1$

$$\dot{V}(\Delta x) = \Delta x^2(-\Delta x - 2 + 3) = \Delta x^2(-\Delta x + 1)$$

$$\dot{V}(\Delta x) \simeq \Delta x^2 \cancel{-\Delta x^3}$$

>0 neglected

with approx Δx^3 higher order! here near 0

UNSTABLE! (not "locally" → overall)

- $\bar{x} = 2$

↑ in accordance to previously found results

>0 always

$$\dot{V}(\Delta x) = \Delta x^2(-\Delta x - 4 + 3) = \Delta x^2\underbrace{(-\Delta x - 1)}_{?}$$ → ① approx. near $\Delta x = 0$

$\dot{V}(\Delta x) \simeq -\Delta x^2 < 0$ locally asymp. st.

(neglect $\Delta x^3 \simeq 0$) (approx) holds only near $\Delta x = 0$

② analytically

$(\Delta x = x - \bar{x})$ $\quad \bar{x} = 2$

$-\Delta x - 1 < 0 \Leftrightarrow \Delta x > -1 \rightarrow x - \bar{x} > -1 \Rightarrow x > -1 + 2 = 1$ region of attraction

0.1 Additional exercises

Ex. 6: Consider the following differential equation

$$\dot{x} = -x^3 \tag{6}$$

1. Study the equilibrium points of the system
2. and analyse its stability with both the linearized system
3. and using the Lyapunov function $V(x) = x^2$.

Ex. 7: Given the following system

$$\begin{cases} \dot{x}_1 = x_1 \; x_2^2 \; - \; x_1 \\ \dot{x}_2 = -x_1^2 \; x_2 \end{cases} \tag{7}$$

Study the stability of the origin using the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

Ex. 8: Given the following system

$$\begin{cases} \dot{x}_1 = -x_1^3 \; + \; x_2 \\ \dot{x}_2 = -x_2^2 \; + \; u \end{cases} \tag{8}$$

and the Back-stepping formula (given at the exam)

$$u = -\frac{dV_1(x_1)}{dx_1} g(x_1) - k(x_2 - \phi_1(x_1)) + \frac{d\phi_1(x_1)}{dx_1}(f(x_1) + g(x_1)x_2) \tag{9}$$

Determine a control law stabilizing the origin using the Back-stepping method.

Ex. 9: Given the following system

$$\begin{cases} \dot{x}_1 = -x_1 \; x_2^2 \; - \; x_1 \; - \; x_2^3 \; + \; x_2 \\ \dot{x}_2 = x_1^4 \; - \; x_1^2 \; + \; x_1^2 \; x_2 \; - \; x_2 \end{cases} \tag{10}$$

and the phase plane in Figure 1

1. Study the equilibrium points of the system using the given phase plane,
2. verify the stability using the linearized system

and given the matrix

$$P = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \tag{11}$$

and using Lyapunov theorem for linearized systems, assessing its stability property.

Hint: For continuous time systems use $A^T P + PA = -Q$

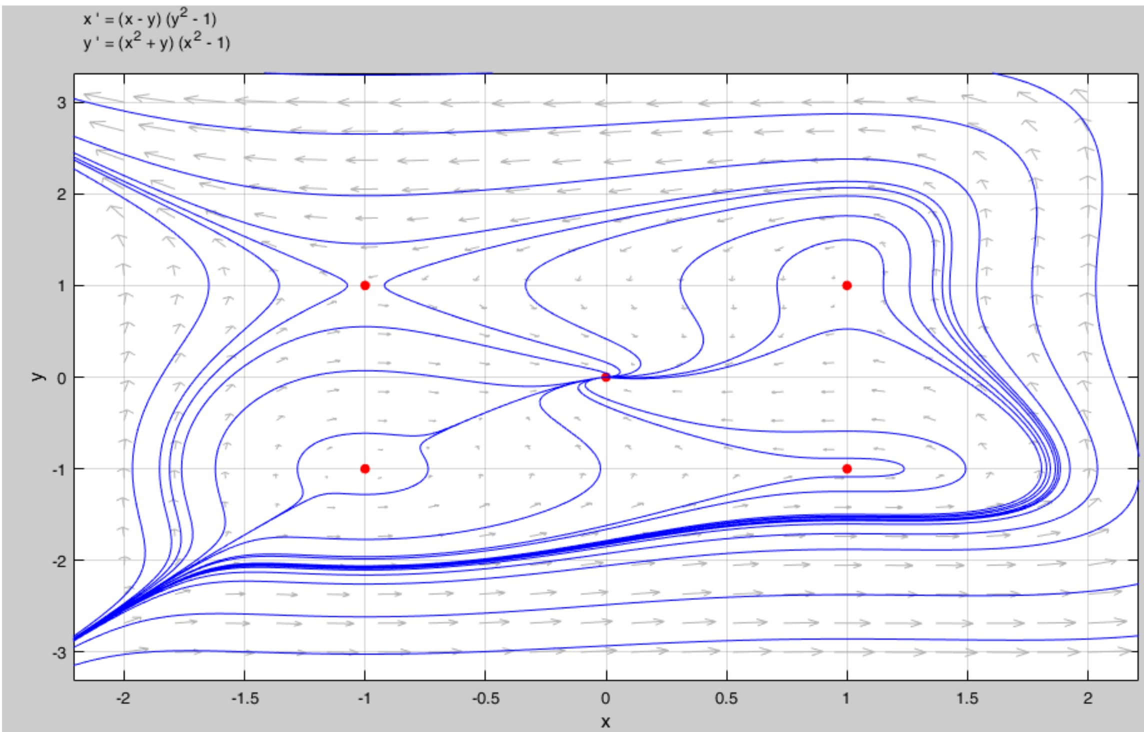


Figure 1: Phase plane of system given in exercise 9.