ADVANCED AND MULTIVARIABLE CONTROL

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Exercise 1

Consider the system

$$\dot{x}_1(t) = -x_1(t) + x_1^2(t)$$
$$\dot{x}_2(t) = -x_2(t) + 2x_2^2(t) + x_1^2(t)$$

A. Compute the equilibria
B. Determine the stability properties of the equilibria with the linearized systems.
C. Determine if the equilibrium at the origin is asymptotically stable using the Lyapunov theory.

Solution Exercise 1

a) $$0 = \overline{x}_1(1-\overline{x}_1) = 0 \rightarrow \overline{x}_1 = 0, \overline{x}_1 = 1$$
$$0 = -\overline{x}_2 + 2\overline{x}_2^2 + \overline{x}_1^2$$

Neglecting non real solutions, the two equilibria are $\begin{matrix} \overline{x}_1 = 0, \overline{x}_2 = 0 \\ \overline{x}_1 = 0, \overline{x}_2 = 0.5 \end{matrix}$

b) Linearized model

$$\delta\dot{x}_1(t) = -\delta x_1(t) + 2\overline{x}_1\delta x_1(t)$$
$$\delta\dot{x}_2(t) = -\delta x_2(t) + 4\overline{x}_2\delta x_2(t) + 2\overline{x}_1\delta x_1(t)$$

At the equilibrium

$$\delta\dot{x}_1(t) = -\delta x_1(t)$$
$$\delta\dot{x}_2(t) = -\delta x_2(t)$$

Two eigenvalues in -1: equilibrium asymptotically stable

At point *(0,0.5)*

$$\delta\dot{x}_1(t) = -\delta x_1(t)$$
$$\delta\dot{x}_2(t) = \delta x_2(t)$$

One eigenvalue in 1: equilibrium unstable

c) $$V(x) = 0.5\left(x_1^2 + x_2^2\right) \rightarrow \dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 + x_1^3 - x_2^2 + 2x_2^3 + x_1^2 x_2$$

d) In a neighbor of the origin the second order terms (both negative) dominate so that $\dot{V}(x) < 0$ locally.

Exercise 2

Consider the system

$$\dot{x}_1(t) = bu(t)$$
$$\dot{x}_2(t) = x_1(t) + u(t)$$

and assume that you want to design an infinite horizon LQ control with *Q=diag(q₁,q₂), R=1.*

A. Compute the conditions guaranteeing that the solution of the infinite horizon LQ control is stabilizing.
B. With *Q=I*, *b=1*, the solution of the steady-state Riccati equation is *P=I*. Check the stability of the closed-loop system a) computing the closed-loop eigenvalues, b) by using a suitable Lyapunov function (the one used for the stability analysis of LQ control).
C. Assume now to implement the feedback control law $u(t) = -\rho K x(t)$ (*K* is again the solution of the LQ problem), specify the set of values of ρ guaranteed by LQ control so that the closed-loop system remains asymptotically stable.

Steady-state Riccati equation: $A'P + PA + Q - PBR^{-1}B'P = 0$

Solution Exercise 2

A.

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} b \\ 1 \end{bmatrix} \rightarrow \text{reachability matrix } M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} b & 0 \\ 1 & b \end{bmatrix} \rightarrow \text{ condition } b \neq 0$$

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \rightarrow Q^{1/2} = \begin{bmatrix} \sqrt{q_1} & 0 \\ 0 & \sqrt{q_2} \end{bmatrix} \rightarrow \text{observability matrix } M_o = \begin{bmatrix} Q^{1/2} \\ Q^{1/2}A \end{bmatrix} = \begin{bmatrix} \sqrt{q_1} & 0 \\ 0 & \sqrt{q_2} \\ 0 & 0 \\ \sqrt{q_2} & 0 \end{bmatrix} \rightarrow q_2 > 0$$

B.

$$K = R^{-1}B'P = \begin{bmatrix} 1 & 1 \end{bmatrix} \rightarrow A - BK = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \rightarrow \text{eigenvalues } -1, -1$$

$$J^o = x'Px = x'x > 0 \rightarrow \frac{\partial J^o}{\partial t} = x'(A - BK)'x + x'(A - BK)x = -x' \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x < 0$$

C.

In view of the robustness properties of LQ, one has $\rho \in (0.5, \infty)$ guaranteed.

In this specific case, one has $\det(sI - (A - \rho BK)) = s^2 + 2\rho s + \rho$ and the condition to have all roots with real part smaller than 0 is simply *ρ>0,* so that the real robustness property is greater than the theoretical one.

Exercise 3

Consider the following system, where *d* is an unknown constant,

$$x_1(k+1) = x_2(k) + d$$
$$x_2(k+1) = x_1(k) + x_2(k) + u(k)$$
$$y(k) = x_1(k)$$

A. Design a reduced order observer for this system in order to estimate the state x_2 and the disturbance d.
B. B. Based on the estimated state, compute a state feedback control law which places all closed-loop the eigenvalues in z=0.5.

Solution Exercise 3

A.

By defining the fictitious dynamics *d(k+1)=d(k)* to *d*, the system equations can be written as

$$x_2(k+1) = x_2(k) + y(k) + u(k)$$
$$d(k+1) = d(k)$$
$$y(k+1) = x_2(k) + d(k)$$

Or

$$\begin{bmatrix} x_2(k+1) \\ d(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(k) \\ u(k) \end{bmatrix}$$

output transformation

$$y(k+1) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_2(k) \\ d(k) \end{bmatrix}$$

The observer is

$$\hat{x}_2(k+1) = \hat{x}_2(k) + y(k) + u(k) + l_1\left(y(k+1) - \hat{x}_2(k) - \hat{d}(k)\right)$$
$$\hat{d}(k+1) = \hat{d}(k) + l_2\left(y(k+1) - \hat{x}_2(k) - \hat{d}(k)\right)$$

The gain $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ must be chosen to assign the eigenvalues of $A - LC,\ A_{obs} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_{obs} = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Note however that, in this case, the pair (A_{obs},C_{obs}) is not observable, so that one cannot freely impose the observer eigenvalues.

B.

As for the control law, set

$$u(k) == -K\hat{x}(k), \quad \hat{x}(k) = \begin{bmatrix} x_1(k) \\ \hat{x}_2(k) \end{bmatrix} \text{ , eig}(A\text{-}BK) \text{ in } 0.5 \rightarrow K = \begin{bmatrix} 1.25 & 0 \end{bmatrix}$$

Exercise 4

With reference to the system used in *H2, Hinf* control

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t)$$
$$z(t) = Cx(t) + D_{11} w(t) + D_{12} u(t)$$
$$v(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t)$$

A. Explain what is the meaning of the variables *z, w, v*;
B. Define the goals of *H2, Hinf* control
C. Specify what are the shaping functions and how they can be chosen in a SISO control problem
D. Describe the main steps to be performed to use model reduction techniques

Solution Exercise 4

A.

See the notes, pag. 57

B.

See the notes, pag. 56

C.

See the notes pag. 54-56 or pag. 155-160

D.

See the notes pag. 164-165