Advanced and Multivariable Control

Optimal Control of Discrete Time Systems

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Optimal Control

System $x(k+1)=f(x(k),u(k)), \quad x(k_0)=x_0$ $\quad x \in R^n,\, u \in R^m$

Minimize with respect to $u(k_0),\, u(k_0+1),\, ...,\, u(\bar{k}-1)$ the cost function

$$J(x(k_0),u(\cdot),k_0)=\sum_{i=k_0}^{\bar{k}-1} l(x(i),u(i))+m(x(\bar{k})) \quad , \quad \bar{k}>k_0$$

subject to the system's dynamics and state and input constraints:

$$x(k)\in X, \quad u(k)\in U$$

$X \subseteq R^n$, $U \subseteq R^m$ are compact sets containing the origin

Let $u_{[a,b]}$ be the control sequence $u(k)$, $k \in [a, b]$, and define

$$J^0(x(k), k) = \min_{u[k,\bar{k}-1]} J(x(k), u(\cdot), k)$$

Then

$$J^0(x(\bar{k}), \bar{k}) = \min_{u(\bar{k})} m(x(\bar{k})) = m(x(\bar{k}))$$

and

$$J^0(x(\bar{k}-1), \bar{k}-1) = \min_{u(\bar{k}-1)} \{l(x(\bar{k}-1), u(\bar{k}-1)) + J^0(x(\bar{k}), \bar{k})\}$$

or in general

Bellman's principle

$$J^0(x(k), k) = \min_{u(k)} \{l(x(k), u(k)) + J^0(f(x(k), u(k)), k+1)\}$$

For a fixed ***k***

$$\begin{cases} J^0(x,k) = \min_u \{l(x,u) + J^0(f(x,u),k+1)\} \\ J^0(x,\bar{k}) = m(x) \end{cases}$$

HJB equation

Step 1 compute the value u^o minimizing

$$\{l(x,u) + J^0(f(x,u),k+1)\}$$

Assuming that there exists an unique minimum, this corresponds to compute a function with arguments x and $J^0(f(x,u),k+1)$, that is

$$u^o = \kappa\left(x, J^0\right)$$

Step 2 Compute the function $J^o(x,k)$ satisfying the HJB equation

$$J^o(x,k) = l\left(x, \kappa\left(x, J^o(x,k)\right)\right) + J^0\left(f\left(x, \kappa\left(x, J^o(x,k)\right)\right), k+1\right)$$

with boundary condition

$$J^0(x,\bar{k}) = m(x)$$

LQ control

System

$$x(k+1) = Ax(k) + Bu(k)$$

Cost function

$$J(x(k_0), u(\cdot), k_0) = \sum_{i=k_0}^{\bar{k}-1} [x'(i)Qx(i) + u'(i)Ru(i)] + x'(\bar{k})Sx(\bar{k}) \quad , \quad \bar{k} > k_0, \quad Q \geq 0, \quad R > 0, \quad S \geq 0$$

Tentative solution

$$J^0(x,k) = x'P(k)x \quad , P(\bar{k}) = S$$

LQ control - solution

$$u(k) = -K(k)x(k)$$

$$K(k) = \left(R + B'P'(k+1)B\right)^{-1} B'P'(k+1)A$$

$$P(k) = Q + A'P(k+1)A - A'P(k+1)B\left(R + B'P(k+1)B\right)^{-1} B'P(k+1)A$$

difference Riccati equation

$$P(\bar{k}) = S$$

- Slightly easier to solve than in the continuous time (difference Riccati, instead of differential Riccati)
- Same limitations of the continuous time case (finite time horizon, time varying …)

Infinite Horizon LQ

$$J = \sum_{k=0}^{\infty} x'(k)Qx(k) + u'(k)Ru(k), \quad Q \geq 0, \quad R > 0$$

If
a) the pair (A, B) is reachable;
b) the pair (A, C_q) is observable, with C_q such that $Q = C_q' C_q$
then

A) the optimal control law is given by

$$u(k) = -\bar{K}x(k)$$

$$\bar{K} = \left(R + B'\bar{P}B\right)^{-1} B'\bar{P}A$$

where $\bar{P}$ is the unique solution > 0 of the stationary Riccati equation

$$\bar{P} = A'\bar{P}A + Q - A'\bar{P}B\left(R + B'\bar{P}B\right)^{-1} B'\bar{P}A$$

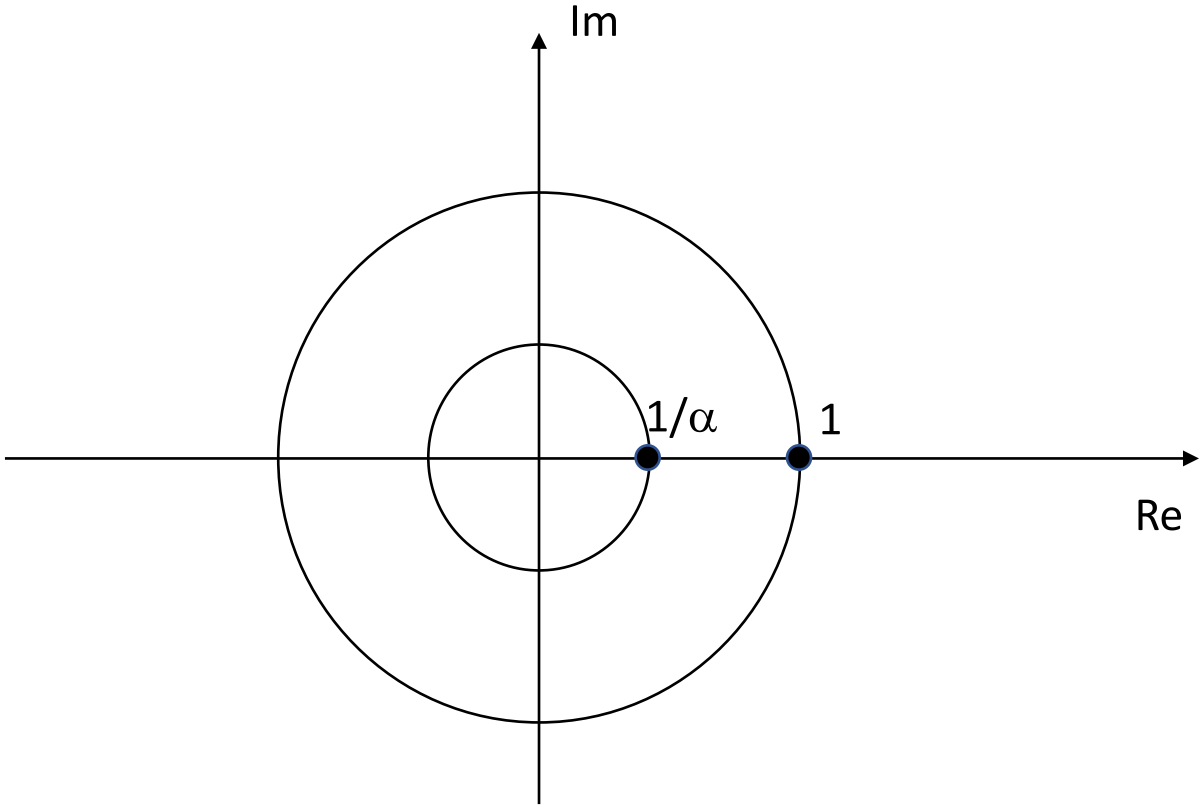
B) the closed-loop system

$$x(k+1) = \left(A - B\bar{K}\right)x(k)$$

is asymptotically stable.

LQ control with a prescribed degree of stability

Goal: we want to obtain a stabilizing regulator such that the closed-loop eigenvalues have an absolute value smaller than $1/\alpha$



Modified cost function

$$\hat{J} = \sum_{k=0}^{\infty} \{ [x'(k)Qx(k) + u'(k)Ru(k)]\alpha^{2k} \}$$

Defining $\hat{x}(k) = \alpha^k x(k)$, $\hat{u}(k) = \alpha^k u(k)$ $\longrightarrow$ $\hat{J} = \sum_{k=0}^{\infty} [\hat{x}'(k)Q\hat{x}(k) + \hat{u}'(k)R\hat{u}(k)]$

$$\alpha^{k+1}[x(k+1) = Ax(k) + Bu(k)]$$

$$\hat{A} = \alpha A, \hat{B} = \alpha B$$

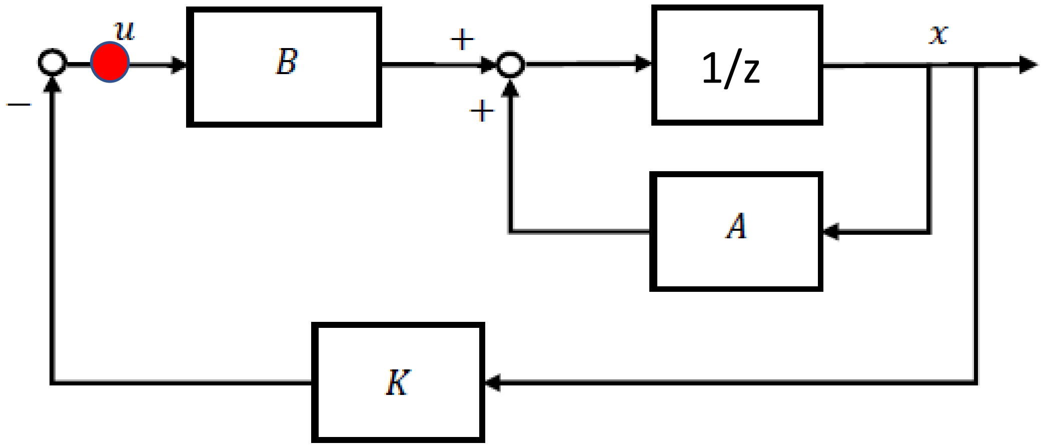
$$\hat{x}(k+1) = \alpha A\hat{x}(k) + \alpha B\hat{u}(k) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k)$$

Standard LQ problem for the system in the variables $\hat{x}, \hat{u}$

One has to solve the LQ problem with $\hat{A}, \hat{B}$ and then use the obtained gain $\hat{K}$ in the control law

$$u(k) = -\hat{K}x(k)$$

Robustness of *LQ_{inf}* control with respect to uncertainties *at the plant input*



*It is **stabilizing**, it is **optimal** (with respect to the selected design parameters), but is it also **robust** with respect to uncertainties at the plant input?*

With simple manipulations of the Riccati equation, it is possible to obtain the following relationship

$$G_c'(-z)QG_c(z)+R=\Gamma'(-z)(R+B'PB)\Gamma(z)$$

$$G_c(z)=(zI-A)^{-1}B$$ transfer function from ***u*** to ***x***

$$\Gamma(z)=I+K(zI-A)^{-1}B$$ Inverse of the *sensitivity function* (*return difference*)

$$G_c'(-z)QG_c(z) + R = \Gamma'(-z)(R + B'PB)\Gamma(z)$$

Single input systems, m=1

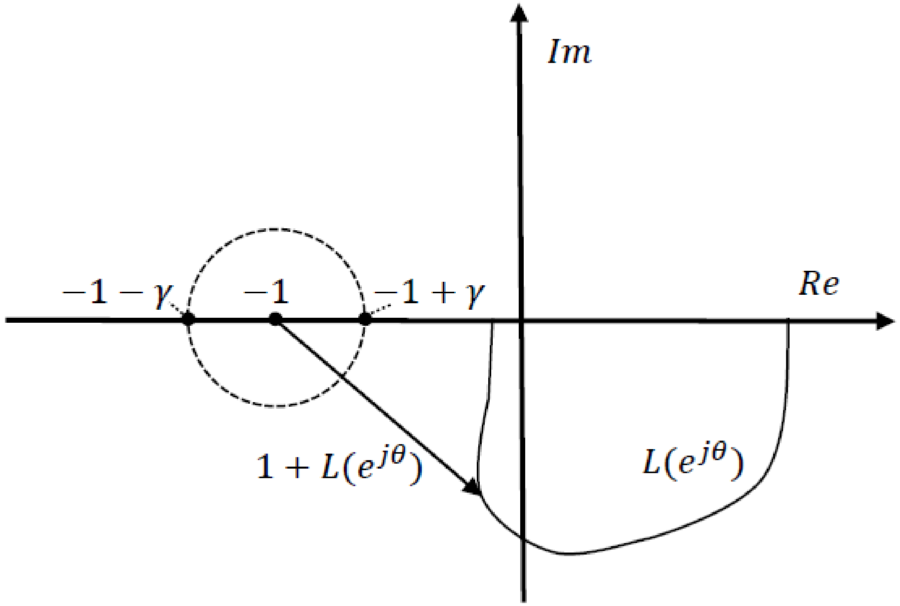
$$\left|1 + \underbrace{K(zI - A)^{-1}B}_{L(z)}\right|^2 \geq \frac{R}{R + B'PB} = \bar{\gamma} < 1$$

the polar diagram of $L\left(e^{j\theta}\right)$, $\theta \in [0, \pi]$, does not intersect the circle with center in -1 and radius $\gamma = \bar{\gamma}^{1/2}$,

$$\boxed{\text{gain margin} \in \left(\tfrac{1}{1+\gamma}, \tfrac{1}{1-\gamma}\right)}$$

Smaller than the one in continuous time LQ

No phase margin



Consider the system described by $G(z) = \dfrac{1}{z-2}$

- Design a LQ controller with $Q = R = 1$
- Analyse its robustness properties with respect to gain variations

Solution

The system can be given the state space form

$$x(k+1) = 2\,x(k) + u(k) \quad , \quad A = 2,\ B = 1$$

Riccati equation $\quad A'PA + Q - A'PB\underbrace{(R + B'PB)^{-1}B'PA}_{K} - P = 0$

Solution ~~P~~ =

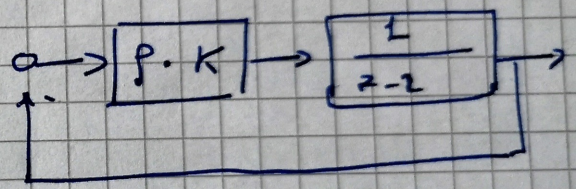
K

Solution

$P = 4.2361, \quad K = 1.618, \quad A - BK = 0.382$

closed-loop eigenvalue

<u>Robustness analysis</u>



characteristic equation

$$z - 2 + \rho \cdot 1.618 = 0$$

$$z = 2 - 1.618\rho \longrightarrow -1 < 2 - 1.618\rho < 1$$

$$\Downarrow$$

$$\rho \in [0.618, 1.8541]$$

Theoretical robustness bound

$$\rho \in \left[\frac{1}{1+\gamma}, \frac{1}{1-\gamma} \right]$$

$$\gamma = \sqrt{\bar{\gamma} = \frac{R}{R+B'PB}} = 0.4370$$

$$\rho \in [0.6959, 1.7762]$$ smaller than the real one

THEORETICAL

REAL

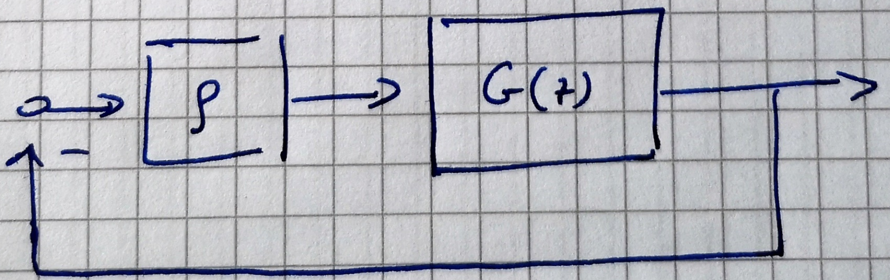
$$\rho = [0.618, 1.8541]$$

Why for discrete time systems we cannot have a gain margin $\longrightarrow \infty$?

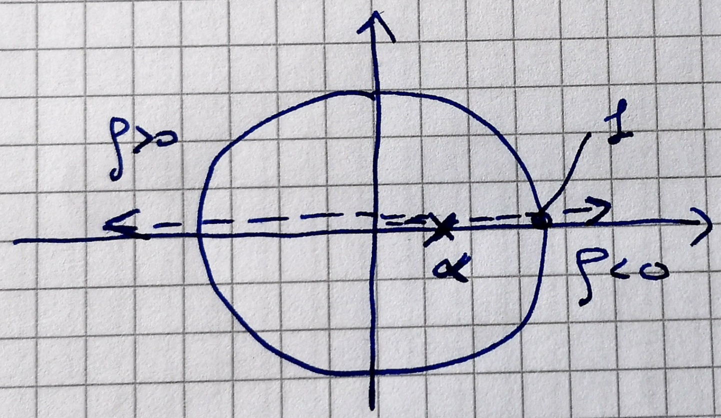
Consider a SISO system with transfer function

$$G(z) = \frac{B(z)}{A(z)} \quad \begin{array}{l} \leftarrow \text{order } n-1 \\ \leftarrow \text{order } n \end{array}$$

and



The root locus will have at least (if the relative degree is equal to 1) a branch going outside the stability region for $\rho \to \infty$



$$G(z) = \frac{1}{z-\alpha}$$

The stability region is bounded!!

Example

Consider again the system

$$x(k+1) = 2x(k) + u(k)$$

and design a regulator with LQ ($Q=1$, $R=1$) such that the closed-loop eigenvalue has modulus smaller than 0.25

Solution

$$\frac{1}{\alpha} = 0.25 \longrightarrow \alpha = 4$$

Define $\tilde{A} = 4a = 8$, $\tilde{B} = 4B = 4$

Solve the Riccati equation with $\tilde{A}$, $\tilde{B}$

$$P = 4.9501, \quad \tilde{K} = 1.9751$$

$$A - B\tilde{K} = 0.0249$$

Kalman predictor and filter

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v_x(k) \\ y(k) &= Cx(k) + v_y(k) \end{aligned}$$

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \text{ white gaussian noise with } E[v] = 0$$

for simplicity

$$E\left[v(k_1)v'(k_2)\right] = V\delta(k_1 - k_2) \qquad V = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{R} \end{bmatrix}, \quad \tilde{Q} \in R^{n,n}, \quad \tilde{R} \in R^{p,p}, \quad \tilde{Q} \geq 0, \quad \tilde{R} > 0$$

$$x(0) \text{ gaussian with } E[x(0)] = \bar{x}_0,\ E\left[(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)'\right] = \tilde{P}_0 \geq 0$$

$$E\left[x(0)v'(k)\right] = 0, \quad \forall k \geq 0$$

Kalman predictor

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L(k)\left[y(k) - C\hat{x}(k|k-1)\right]$$

Let $\hat{e}(k|k-1) = x(k) - \hat{x}(k|k-1)$ be the estimation error

$$\begin{aligned} \hat{e}(k+1|k) &= [A - L(k)C]\,\hat{e}(k|k-1) + v_x(k) - L(k)v_y(k) \\ &= [A - L(k)C]\,\hat{e}(k|k-1) + \begin{bmatrix} I & -L(k) \end{bmatrix} v(k) \\ &= A_c(k)\hat{e}(k|k-1) + B_c(k)v(k) \end{aligned}$$

$$A_c(k) = [A - L(k)C]\,, \quad B_c(k) = \begin{bmatrix} I & -L(k) \end{bmatrix}$$

$$E\left[\hat{e}(k+1|k)\right] = A_c(k)E\left[\hat{e}(k|k-1)\right]$$

$$E\left[\hat{e}(k+1|k)\right] = A_c(k)E\left[\hat{e}(k|k-1)\right]$$

If $\hat{x}(0|-1) = \bar{x}_0 \longrightarrow E\left[\hat{e}(0|-1)\right] = 0$

$$E\left[\hat{e}(k+1|k)\right] = 0, \quad \forall k$$

Covariance of the estimation error

$$\tilde{P}(k|k-1) = E\left[\hat{e}(k|k-1)\hat{e}'(k|k-1)\right] \quad , \quad \tilde{P}(0|-1) = \tilde{P}_0$$

Choice of the gain ***L(k)***

$$\min_{L(k)} \gamma' \tilde{P}(k+1|k)\gamma \qquad \gamma \text{ generic vector}$$

The gain $L(k)$ minimizing $\gamma' \tilde{P}(k+1|k)\gamma$ is

$$L(k) = A\tilde{P}(k|k-1)C' \left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1}$$

where $\tilde{P}(k|k-1)$ is the solution to the Riccati equation

$$\begin{aligned}
& \tilde{P}(k+1|k) \\
= \; & A\tilde{P}(k|k-1)A' + \tilde{Q} \\
- \; & A\tilde{P}(k|k-1)C' \left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1} C\tilde{P}(k|k-1)A'
\end{aligned}$$

with initial condition

$$\tilde{P}(0|-1) = \tilde{P}_0$$

Duality

LQ

$$u(k) = -K(k)x(k)$$

$$K(k) = \left(R + B'P'(k+1)B\right)^{-1} B'P'(k+1)A$$

$$P(k) = Q + A'P(k+1)A - A'P(k+1)B\left(R + B'P(k+1)B\right)^{-1} B'P(k+1)A$$

$$P(\bar{k}) = S$$

KP

$$L(k) = A\tilde{P}(k|k-1)C'\left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1}$$

$$\begin{aligned}\tilde{P}(k+1|k) &= A\tilde{P}(k|k-1)A' + \tilde{Q} \\ &- A\tilde{P}(k|k-1)C'\left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1} C\tilde{P}(k|k-1)A'\end{aligned}$$

$$\tilde{P}(0|-1) = \tilde{P}_0$$

$$\begin{array}{cc} LQ & KP \\ k & -k \\ A & A' \\ B & C' \\ Q & \tilde{Q} \\ R & \tilde{R} \\ P & \tilde{P} \end{array}$$

$$K(k) \longleftrightarrow L'(k)$$

Stability of the stationary Kalman Predictor

If

1. the pair (A, B_q), with B_q such that $\tilde{Q} = B_q B_q'$, is reachable;
2. the pair (A, C) is observable;

then

A) the optimal predictor is

$$\begin{aligned} \hat{x}(k+1|k) &= A\hat{x}(k|k-1) + Bu(k) + \bar{L}[y(k) - C\hat{x}(k|k-1)] \\ &= (A - \bar{L}C)\hat{x}(k|k-1) + Bu(k) + \bar{L}y(k) \end{aligned}$$

with

$$\bar{L} = A\overline{\tilde{P}}C' \left[C\overline{\tilde{P}}C' + \tilde{R}\right]^{-1}$$

and $\overline{\tilde{P}}$ is the unique positive definite solution of the stationary Riccati equation

$$\overline{\tilde{P}} = A\overline{\tilde{P}}A' + \tilde{Q} - A\overline{\tilde{P}}C' \left[C\overline{\tilde{P}}C' + \tilde{R}\right]^{-1} C\overline{\tilde{P}}A'$$

B) the estimator is asymptotically stable, that is all the eigenvalues of $(A - \bar{L}C)$ have modulus less than 1.

Consider the system

$$\begin{cases} x(h+1) = +\,x(h) + b\,u(h) + v_x(h) \\ y(h) = x(h) + v_y(h) \end{cases}$$

where $v_x \sim WN(0, q)$, $v_y \sim WN(0, r)$

and all the other conditions required to apply the theory of the Kalman predictor are satisfied.

Note that, although $y(h) = x(h) +$ disturbance, the use of a KP (or a filter) is recommended to filter out the effect of the noises.

$$P = APA' + \tilde{Q} - \underbrace{APC'(CPC' + \tilde{R})^{-1}}_{L} CPA'$$

$$\hat{x}(k+1|k) = (A - LC)\hat{x}(k|k-1) + Bu(k) + Ly(k)$$

In our case $A = C = 1$, $\tilde{R} = 1$

For different values of q, one has

$q = 0.5 \longrightarrow P = 1 \longrightarrow L = 1/2 \longrightarrow \underline{A - LC = 0.5}$

$q = 1 \longrightarrow P = 1.618 \longrightarrow L = 0.618 \longrightarrow \underline{A - LC = 0.382}$

$q = 2 \longrightarrow P = 2.7321 \longrightarrow L = 0.7321 \longrightarrow \underline{A - LC = 0.2679}$

The predictor becomes faster and faster as q increases

Kalman Filter

$$\tilde{x}(k+1|k+1) = A\tilde{x}(k|k) + Bu(k) + L(k+1)\left[y(k+1) - C\left(A\tilde{x}(k|k) + Bu(k)\right)\right]$$

where

$$\begin{aligned} L(k) &= \tilde{P}(k|k-1)C'\left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1} \\ \tilde{P}(k|k-1) &= A\hat{P}(k-1|k-1)A' + \tilde{Q} \\ \hat{P}(k|k) &= \tilde{P}(k|k-1) - L(k)C\tilde{P}(k|k-1) \\ \hat{P}(0|0) &= \tilde{P}_0 \end{aligned}$$

$\hat{P}(k|k)$ is the covariance of the estimation error $\hat{e}(k|k) = x(k) - \hat{x}(k|k)$.

Convergence and stability results similar to the ones of KP can be established

Extended Kalman Filter (Predictor) - *EKF*

A very important method, **widely used many fields** of engineering and science to estimate **the state** of nonlinear systems and **the parameters** of grey box models

All the details on its development are reported in the textbook

System

$$\begin{aligned} x(k+1) &= f\left(x(k), u(k)\right) + v_x(k) \\ y(k) &= g\left(x(k)\right) + v_y(k) \end{aligned}$$

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \text{ white gaussian noise with } E[v] = 0$$

for simplicity

$$E\left[v(k_1)v'(k_2)\right] = V\delta(k_1 - k_2) \qquad V = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{R} \end{bmatrix}, \quad \tilde{Q} \in R^{n,n}, \quad \tilde{R} \in R^{p,p}, \quad \tilde{Q} \geq 0, \quad \tilde{R} > 0$$

$$x(0) \text{ gaussian with } E[x(0)] = \bar{x}_0,\ E\left[\left(x(0) - \bar{x}_0\right)\left(x(0) - \bar{x}_0\right)'\right] = \tilde{P}_0 \geq 0$$

$$E\left[x(0)v'(k)\right] = 0, \quad \forall k \geq 0$$

Structure of EKF

$$\hat{x}(k+1|k) = \underbrace{f\left(\hat{x}(k|k-1), u(k)\right)}_{\textit{mimics the dynamics of the system}} + \hat{L}(k)\underbrace{\left[y(k) - g(\hat{x}(k|k-1))\right]}_{\textit{output estimation error}}$$

gain of the filter

$$\hat{L}(k) = \hat{A}(k)\tilde{P}(k|k-1)\hat{C}(k)'\left[\hat{C}(k)\tilde{P}(k|k-1)\hat{C}(k)' + \tilde{R}\right]^{-1}$$

where

$$\hat{A}(k) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\hat{x}(k|k-1),u(k)}, \quad \hat{C}(k) = \left.\frac{\partial g(x)}{\partial x}\right|_{\hat{x}(k|k-1)}$$

and

$$\begin{aligned} \tilde{P}(k+1|k) &= \hat{A}(k)\tilde{P}(k|k-1)\hat{A}(k)' + \tilde{Q} + \\ - \quad & \hat{A}(k)\tilde{P}(k|k-1)\hat{C}(k)'\left[\hat{C}(k)\tilde{P}(k|k-1)\hat{C}(k)' + \tilde{R}\right]^{-1}\hat{C}(k)\tilde{P}(k|k-1)\hat{A}(k)' \end{aligned}$$

$$\tilde{P}(0|-1) = \tilde{P}_0$$

Comments on *EKF*

Very useful to estimate unknown parameters (see the next slide)

Continuous time formulations for continuous time systems available

In case of a continuous time system, the system can be discretized first (Euler, for example), and then the filter is applied to the discretized model. The approximation errors due to discretization can be significant.

Mixed continuous – discrete formulations are widely used. The dynamics of the system is in continuous time, the gain update is in discrete time to reduce the computational effort and to consider the availability of discrete measurements (wide literature available, see also next slides)

Very few convergence results available (basically, if you start near the real value of the state, convergence is guaranteed

Estimation of unknown parameters with *EKF*

Consider the system

$$
\begin{aligned}
x(k+1) &= f\left(x(k), u(k), \alpha\right) + v_x(k) \\
y(k) &= g\left(x(k), \alpha\right) + v_y(k)
\end{aligned}
$$

$\boxed{\alpha \text{ unknown constant parameter}}$

Add a fictitious dynamics

$$\alpha(k+1) = \alpha(k)$$

Enlarge the state $\tilde{x} = \begin{bmatrix} x \\ \alpha \end{bmatrix}$

Apply EKF to

$$
\begin{aligned}
\tilde{x}(k+1) &= \tilde{f}\left(\tilde{x}(k), u(k)\right) + \tilde{v}_x(k) \\
y(k) &= \tilde{g}\left(\tilde{x}(k)\right) + v_y(k)
\end{aligned}
$$

Example EKF

$$x(k+1) = ax(k) + u(k)$$
$$y(k) = x(k) + v(k)$$

a unknown parameter to be estimated

$$x(k+1) = a(k)x(k) + u(k)$$
$$a(k+1) = a(k)$$
$$y(k) = x(k) + v(k)$$

$$\longrightarrow \quad A(k) = \begin{bmatrix} \hat{a}(k) & \hat{x}(k) \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Simulations

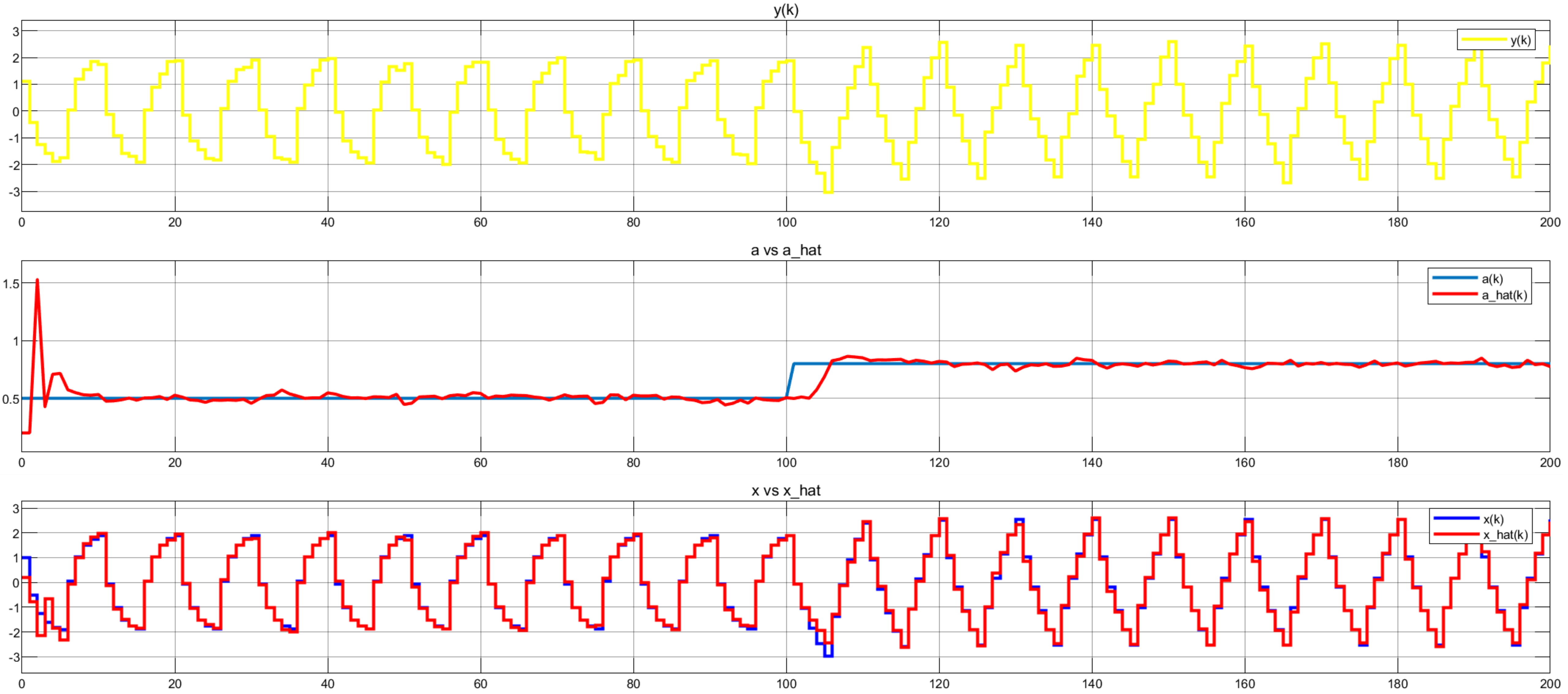
$k < 100 \rightarrow a = 0.5$
$k \geq 100 \rightarrow a = 0.8$

Input: square wave ±1 with period 5 , $x(0) = 1$

Output noise: white, zero mean, variance 0.01

$$P(0) = 100I \ , \qquad \begin{bmatrix} \hat{x}(0) \\ \hat{a}(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$

Covariance matrices *Q=0.1 I , R = 1*


y(k)
y(k)
3
2
1
0
-1
-2
-3
0
20
40
60
80
100
120
140
160
180
200
a vs a_hat
a(k)
a_hat(k)
1.5
1
0.5
x vs x_hat
x(k)
x_hat(k)

Continuous Extended Kalman Filter with *discrete measurements*

System

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) + v_x(t) \\
y(k) &= g(x(k)) + v_y(k) \\
x(k) &= x(t_k)
\end{aligned}
\qquad v_x = wgn(0, \tilde{Q}) \quad v_y = wgn(0, \tilde{R})
$$

EKF – initialization

$$
\begin{aligned}
\hat{x}(0|0) &= E[x(t_0)] \\
P(0|0) &= E[(x(t_0) - \hat{x}(t_0))(x(t_0) - \hat{x}(t_0))']
\end{aligned}
$$

EKF – predictor step

continuous dynamics

$$\begin{aligned}\dot{\hat{x}}(t) &= f(\hat{x}(t), u(t)) \\ \dot{P}(t) &= A(t)P(t) + P(t)A'(t) + \tilde{Q}\end{aligned}$$

$$\begin{aligned}\hat{x}(t_{k-1}) &= \hat{x}(k-1|k-1) \\ P(t_{k-1}) &= P(k-1|k-1)\end{aligned}$$

$$\begin{aligned}\hat{x}(k|k-1) &= \hat{x}(t_k) \\ P(k|k-1) &= P(t_k)\end{aligned}$$

$$A(t) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\hat{x}(t),u(t)}$$

EKF – update step

$$\begin{aligned}L(k) &= P(k|k-1)C'(k)(C(k)P(k|k-1)C'(k) + \tilde{R})^{-1} \\ \hat{x}(k|k) &= \hat{x}(k|k-1) + L(k)(y(k) - g(\hat{x}(k|k-1))) \\ P(k|k) &= (I - L(k)C(k))P(k|k-1)\end{aligned}$$

$$C(t) = \left.\frac{\partial g(x)}{\partial x}\right|_{\hat{x}(k|k-1)}$$

discrete update

LQG control

System

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v_x(k) \\ y(k) &= Cx(k) + v_y(k) \end{aligned}$$

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}, \; x(0) \text{ satisfy the assumption adopted to derive the KF}$$

Cost function

$$J = \lim_{\bar{k}\to\infty} \frac{1}{\bar{k}} E\left[\sum_{k=0}^{\bar{k}-1} x'(k)Qx(k) + u'(k)Ru(k)\right]$$

LQG control - solution

1. determine with the stationary Kalman predictor (or filter) the Kalman gain $\bar{L}$ and the estimate $\hat{x}(k|k-1)$ ($\tilde{x}(k|k)$);

2. compute the optimal LQ control law, that is the gain $\bar{K}$;

3. apply the control law

$$u(k) = -\bar{K}\hat{x}(k\,|k-1\,) \text{ in the predictor case}$$

or

$$u(k) = -\bar{K}\tilde{x}(k\,|k\,) \text{ in the filter case.}$$

The closed-loop eigenvalues are those of $(A - B\bar{K})$ and of $(A - \bar{L}C)$ (if the predictor is used)

end