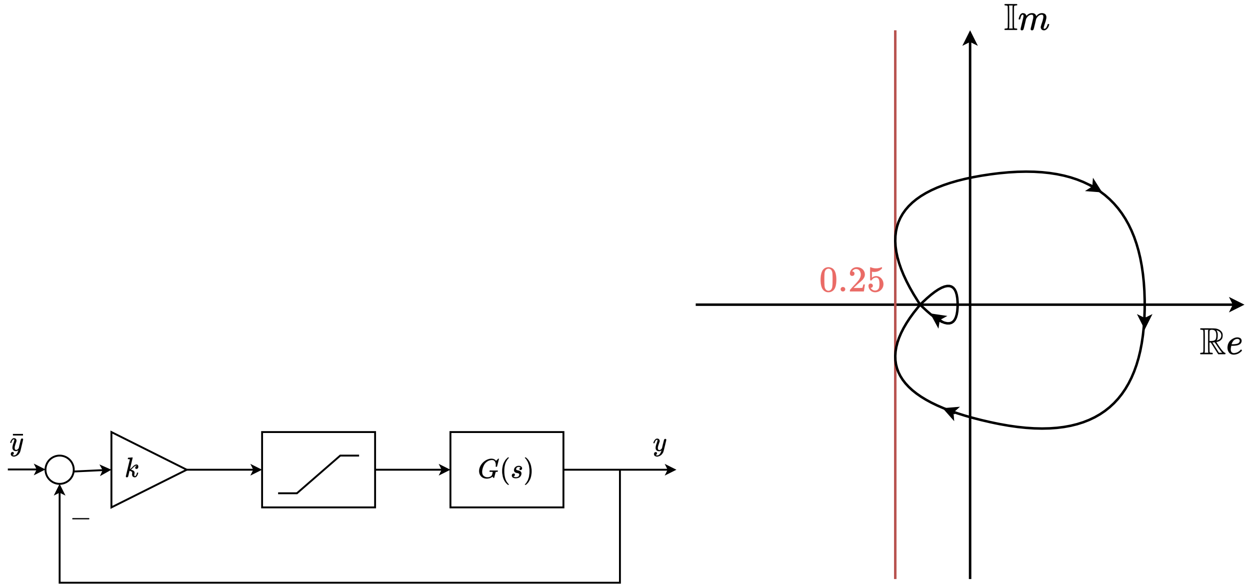
Exercises session 3: Circle Theorem, Uncertainty, MIMO Poles

Ex. 1: Consider the system depicted in Figure 1(a) where $G(s)$ is an asymptotically stable SISO system having the Nyquist diagram depicted in Figure 1(b)

1. Compute the maximum gain K that guarantees the closed-loop stability using the circle criterion.



(a) Block scheme

(b) Nyquist diagram

Figure 1

Ex. 2: Consider the nominal closed-loop system depicted in Figure 2, where

$$\bar{G}(s) \ = \ \frac{1}{1+sT}, \quad T \ > \ 0(A.S.), \tag{1}$$

while the real system is

$$G(s) \ = \ \frac{1}{(1+sT)(1+\alpha s)}, \quad \alpha \ > \ 0, \tag{2}$$

1. Model the uncertainty as both additive and multiplicative
2. Show how to design a controller which is robust to there uncertainties using the small gain theorem.

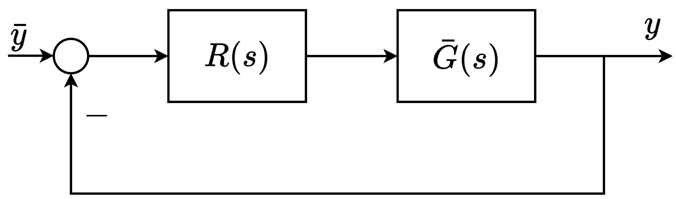


Figure 2: Ex. 2 Block diagram

Ex. 3: Given the MIMO transfer function

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} & \frac{10}{s+2} \\ \\ \frac{2}{s+1} & \frac{-0.5}{s+0.25} & \frac{10}{s+1} \end{bmatrix} \tag{3}$$

Compute

1. the poles of G(s)
2. and the zeros of G(s).

Ex. 4: Given the following continuos time system

$$\begin{cases} \dot{x} = A\ x + B\ u \\ y = C\ x + D\ u \end{cases} \tag{4}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

1. Compute the poles and zeros of the system
2. Check if the system is fully reachable and observable.
3. and the transfer function $G(s)$.
4. Evaluate the poles of $G(s)$.

Ex. 5: (Additional) Given the system

$$\begin{cases} \dot{x}_1 = -x_1^3 \ + \ x_2 \\ \dot{x}_2 = x_2^2 \ + \ u \end{cases} \tag{6}$$

Find the back stepping control law that stabilizes the origin given the formula

$$u = -\frac{dV_1(x_1)}{dx_1}g(x_1) - k(x_2 - \phi_1(x_1)) + \frac{d\phi_1(x_1)}{dx_1}(f(x_1) + g(x_1)x_2) \tag{7}$$

Hint: Use the extended formulation.

- CIRCLE Theorem
- MIMO poles / zeros

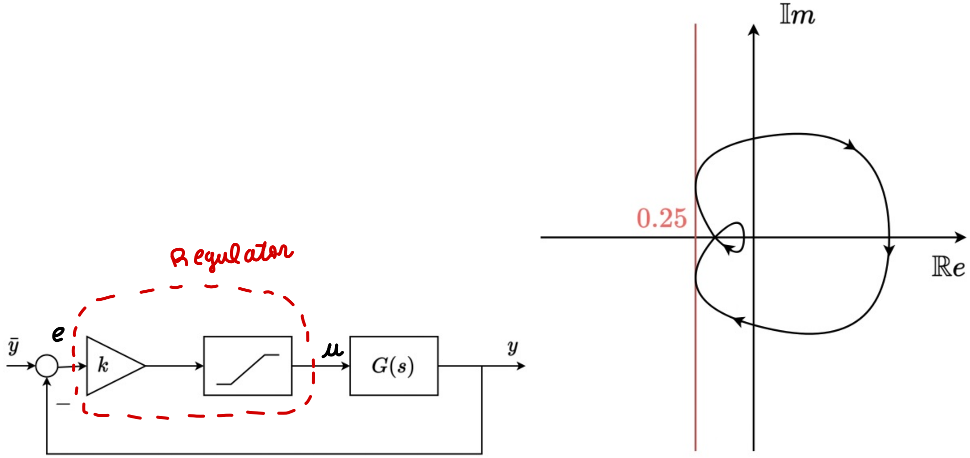
↓

CIRCLE criterion extension of Nyquist to access the system stability (NON LINEAR)

Ex. 1: Consider the system depicted in Figure 1(a) where $G(s)$ is an asymptotically stable SISO system having the Nyquist diagram depicted in Figure 1(b)

1. Compute the maximum gain K that guarantees the closed-loop stability using the circle criterion.

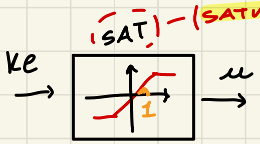
$G(s)$ asymp. stable



(a) Block scheme

(b) Nyquist diagram

Figure 1



(SAT) – (SATURATION) limit the actuator capability into a range ($u_{MIN} \div u_{MAX}$)

① K such that closed loop syst is asymp. stable using CIRCLE CRITERION...

$u = \underbrace{f(e)}$

sector Bounded → to apply CIRCLE criterion

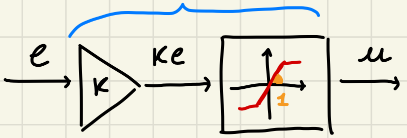
function is sector Bounded IF exist two values K_1, K_2 such that

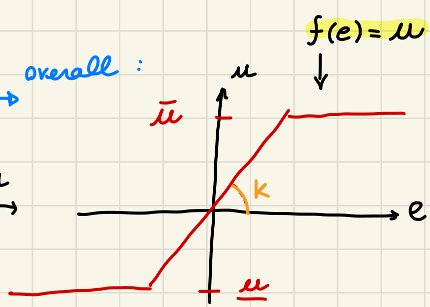
$\exists K_1, K_2 \in \mathbb{R} \mid K_1 e \leq f(e) \leq K_2 e$

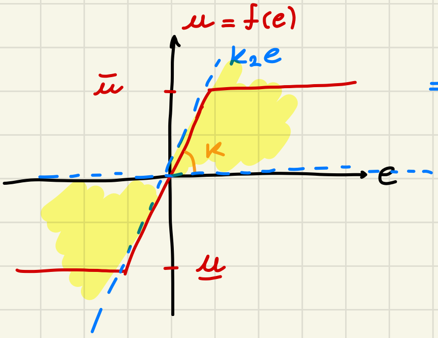
- openloop system asymp stable !

- REGULATOR as cascade of PROP + SAT

overall :







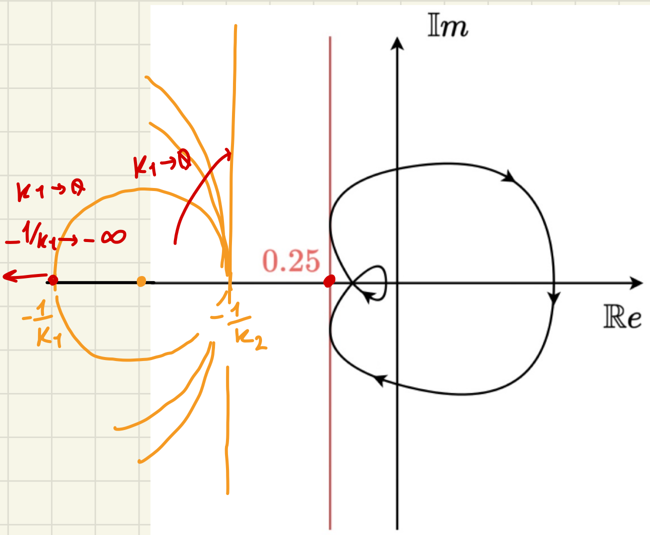
$\begin{cases} K_1 = 0 \\ K_2 = K \end{cases}$ ✓

f(e) is SECTOR bounded! it lays between $[k_1 e, k_2 e]$

circle criterion: if Nyquist diag do NOT encircle the circle with diameter $\left[-1/k_1, -1/k_2\right]$

$\Downarrow$

closed loop system is asymp. stable!



$\begin{cases} K_1 = 0 \rightsquigarrow -1/K_1 \text{ goes to } -\infty \\ K_2 = K \end{cases}$

circle degenerate, it becomes a line!

line on $-1/K_2$

so respect Nyquist diag. this must NOT encounter the line, so at least

$$\boxed{-\frac{1}{K_2} < -0.25}$$

$\Leftarrow$

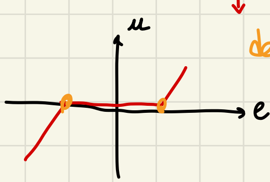
$K_2 < 4$ ($= K$)

$\Downarrow$

$K < 4$

closed loop asymp stable for $\boxed{K < 4}$

respect: $K_1 = 0 \rightsquigarrow$ only when SATURATION/ DEAD ZONE are present on the $f(e) = u$



deadzone = min error value such that the actuator work

Ex. 2: Consider the nominal closed-loop system depicted in Figure 2, where

$$\bar{G}(s) = \frac{1}{1+sT}, \quad T > 0 (A.S.), \rightsquigarrow \text{R(s)}\cdot\bar{G}\text{(s) overall close loop A.S} \tag{1}$$

while the real system is

$$\boxed{G(s) = \frac{1}{(1+sT)(1+\alpha s)}, \quad \alpha > 0,} \tag{2}$$

difference between real / nominal system → the new pole can loose stability of system closed loop in reality

1. Model the uncertainty as both additive and multiplicative
2. Show how to design a controller which is robust to there uncertainties using the small gain theorem.

nominal system $\bar{G}(s)$:= used to design R(s), regulator ↓ simplified version of the system

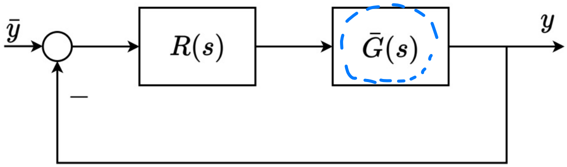


Figure 2: Ex. 2 Block diagram

α: to check when the system remain stable → uncertainty Robustness...

⇓

① model uncertainty by multiplicative / additive form

① MULTIPLICATIVE

$$G(s) = \bar{G}(s)\big(1 + \underbrace{\Delta G_m(s)}_{\text{model uncertainty}}\big)$$

knowing $G, \bar{G}$ → we can compute ΔG_m:

$$\frac{1}{\cancel{(1+sT)}(1+s\alpha)} = \frac{1}{\cancel{1+sT}}\big(1+\Delta G_m(s)\big)$$

T > 0, pole asymp. stable → can be cancel out !

$$\Delta G_m(s) = \frac{1}{1+s\alpha} - 1 = -\frac{\alpha s}{1+s\alpha}$$ multiplicative uncertainty

② ADDITIVE simply summed uncertainty

$$G(s) = \bar{G}(s) + \Delta G_a(s)$$

as before, we know $G, \bar{G}$ parametric, so we can find...

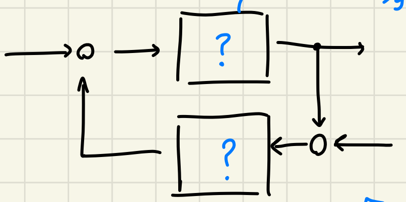
$$\frac{1}{(1+sT)(1+s\alpha)} = \frac{1}{(1+sT)} + \Delta G_a(s)$$

$$\Delta G_a(s) = \frac{\cancel{1} - \cancel{1} - \alpha s}{(1+sT)(1+s\alpha)} = -\frac{\alpha s}{(1+sT)(1+s\alpha)}$$ additive uncertainty

having both ΔG → we can check stability / Robustness using small gain theorem !

② SMALL GAIN THEOREM

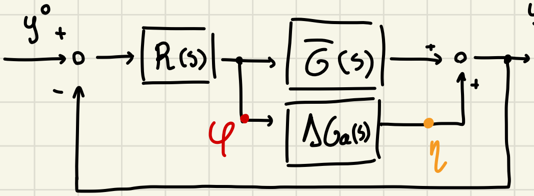
from a system



T.F coming from system to apply small Gain Th !

⇑ for an equivalent representation calling

① additive uncertainty (ΔG_a)



η output of ΔG_a
φ input of ΔG_a } $\eta = \Delta G_a(s)\varphi$

equally we can express η as... $\varphi = -R(s)y =$ (considering $y^o = 0$ reference in 0)

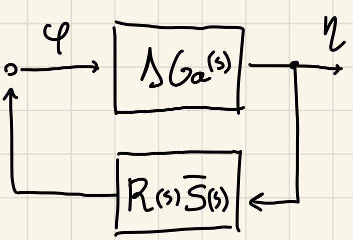
$= -R(s)\left(\varphi\bar{G}(s) + \eta\right)$

⇓

$$\left[\varphi = -\frac{R(s)\eta}{1+R(s)\bar{G}(s)}\right] = -R(s)\boxed{\frac{1}{1+R(s)\bar{G}(s)}}\eta$$

nominal sensitivity function $\bar{S}(s)$

so block scheme equivalently



sufficient condition of SMALL GAIN TH. for asymp stability of closed loop system

↳ (IF the ∞ norm of $R(s)\bar{S}(s)\Delta G_a(s)$ @ $j\omega \le 1$)

IF $\|R(j\omega)\bar{S}(j\omega)\Delta G_a(j\omega)\| \le 1$

⇒ closed loop syst asymp. stable even for uncertainty ΔG_a.

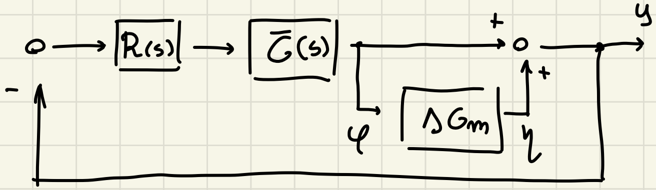
$R(j\omega)$, $\bar{S}(j\omega)$

$\underbrace{\phantom{R(j\omega),\ \bar{S}(j\omega)}}_{\text{design parameters}}$

$$\| R(j\omega) \cdot \bar{S}(j\omega)\, \Delta G_a(j\omega) \|_{\infty} \leq 1$$

true $\forall \omega$ (due to ∞ norm) $\Rightarrow$ sufficient condition!

2 multiplicative (ΔG_m)



$$\eta = \Delta G_m(s)\, \varphi$$

on that case

$$\varphi = -R(s)\,\bar{G}(s)\, y =$$

$$= -R(s)\,\bar{G}(s)\,(\varphi + \eta)$$

$$\Downarrow$$

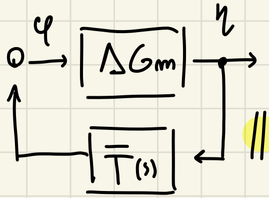
$$\varphi = -\boxed{\frac{R(s)\,\bar{G}(s)}{1 + R(s)\,\bar{G}(s)}}\,\eta$$

nominal complementary sensitivity T.F $\bar{T}(s)$

(neglecting y°)

to apply Small Gain Th:

from this loop...



$$\| \bar{T}(j\omega)\, \Delta G_m(j\omega) \|_{\infty} < 1$$

↑ usually computed by sw help!

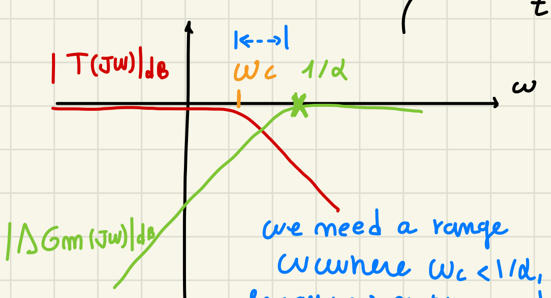
$\Downarrow$

here $\|\cdot\|_\infty$ can be computed by hand:

on dB: $|A \cdot B| < 1 \Rightarrow |A|_{dB} + |B|_{dB} < 0\,dB$ (dB)

$$\forall \omega: \ |\bar{T}(j\omega)|_{dB} + |\Delta G_m(j\omega)|_{dB} < 0\,dB$$

from theory $T(j\omega)$ has the form...



so it is possible to find α such that closed loop system is robust

{ if $\omega_c < 1/\alpha$ the system is asym stable }

we need a range ω where $\omega_c < 1/\alpha$, they must NOT overlap for any ω or the condition $< 0\,dB$ is NOT respected!

Ex. 3: Given the MIMO transfer function

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} & \frac{10}{s+2} \\ \frac{2}{s+1} & \frac{-0.5}{s+0.25} & \frac{10}{s+1} \end{bmatrix}$$

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$

- 3 minors of order 2
- 6 minors of order 1

Compute

1. the poles of G(s)
2. and the zeros of G(s).

① POLES 1st step) compute all the minors of the function:
(of any order)

minor of a Matrix:

{ • minor of order r is the determinant of a SUBMATRIX r×r }

in our case

$$M_{12} = \det[C_1 \ C_2]$$

$$M_{23} = \det[C_2 \ C_3]$$

$$M_{13} = \det[C_1 \ C_3]$$

2nd step) find characteristic polinomial $\varphi(s) :=$ least common denominator of all minors

3rd step) find the roots of $\varphi(s)$ ⇝ you found the system poles!

NOTICE) from a.s.s representation: easier you take A and compute eig values from here

① $$M_{12} = \det\begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{2}{s+1} & -\frac{0.5}{s+0.25} \end{bmatrix} = -\frac{0.5}{(s+1)(s+0.25)} + \frac{2}{(s+1)^2} = \dots = \frac{1.5s}{(s+1)^2(s+0.25)}$$

$$M_{13} = \det\begin{bmatrix} \frac{1}{s+1} & \frac{10}{s+2} \\ \frac{2}{s+1} & \frac{10}{s+1} \end{bmatrix} = \frac{10}{(s+1)^2} - \frac{20}{(s+1)(s+2)} = \frac{(s+2)10-20(s+1)}{(s+1)^2(s+2)} = \frac{10s}{(s+1)^2(s+2)}$$

$$M_{23} = \det\begin{bmatrix} -\frac{1}{s+1} & \frac{10}{s+2} \\ -\frac{0.5}{s+0.25} & \frac{10}{s+1} \end{bmatrix} = -\frac{10}{(s+1)^2} + \frac{5}{(s+2)(s+0.25)} = \frac{-10(s+2)(s+0.25)+5(s+1)^2}{(s+2)(s+1)^2(s+0.25)} =$$

$$= \boxed{\frac{-5s(s+2.5)}{(s+1)^2(s+2)(s+0.25)}}$$

We also have minor of order 1 !

least common den of all minor of all matrix $\rightsquigarrow$ $(s+1)^2(s+2)(s+0.25) = \varphi(s)$

$\downarrow$

the roots are all the poles..

Poles) $\begin{cases} s_{1,2} = -1 \\ s_3 = -2 \\ s_4 = -0.25 \end{cases}$

4 poles $\rightarrow$ they all appear on $G(s)$, BUT we must apply the procedure to find multiplicity

$\Downarrow$

MIMO T.F poles must be found by this procedure

2 ZEROS from T.F $\rightsquigarrow$ standard procedure to compute it

1st step) from MIMO T.F : compute NORMAL RANK r_n of the matrix $G(s)$

- r_n Rank of $G(s)$ $\forall s$ except for a finite number of s (if there is ans such that G lose rank we don't look at it)

$r_n = 2$ on all case (2 rows lin indip) $\rightsquigarrow$ 2 rows lin indip

2nd step) compute all minors of order r_n written such that they have $\varphi(s)$ as denominator fixing the numerator

3rd step) compute the polynomial $z(s) :=$ maximum common divisor of all the minors r_n

4th step) roots of $z(s)$ are the zeros of the system

1 $r_n = 2$, already minors of order r_n computed

2 $$M_{12} = \frac{1.5s \quad (s+2)}{(1+s)^2(s+0.25)\,(s+2)}$$ ← such that $\varphi(s)$ is the denominator

$$M_{13} = -\frac{10s \quad (s+0.25)}{(s+1)^2(s+2) \quad (s+0.25)}$$

$$M_{23} = \frac{-5s(s+2.5)}{(s+1)^2(s+2)(s+0.25)}$$

3 max common divisors = common factor along numerator!

$Z(s) = s$ → 4 $\underbrace{s = 0}$ is the only ZERO of our system!

{ zero which don't appear on any minor of I ORD of $G(s)$ }

Ex. 4: Given the following continuos time system

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x + D\,u \end{cases}$$ s.s form

poles / zeros from s.s
+ OBS/ REACH check
and comparison between
T.F / S.S computation

where

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

1. Compute the poles and zeros of the system
2. Check if the system is fully reachable and observable.
3. and the transfer function $G(s)$.
4. Evaluate the poles of $G(s)$.

1) poles from s.s can be easily computed from matrix A:

$$\varphi(s) = \det(sI - A) = \det\begin{bmatrix} s & 0 \\ 1 & s+1 \end{bmatrix} = s(s+1) \rightarrow \text{poles} \begin{cases} s_1 = 0 \\ s_2 = -1 \end{cases}$$

instead to compute the zeros of MIMO system from ss (long procedure)
↓ we can use system MATRIX:

$$P(s) = \left[\begin{array}{c|c} sI - A & -B \\ \hline C & D \end{array}\right] \Rightarrow$$

ZEROS: are values of s such that
$\text{rank}(P(s)) < r_m$ (P(s) lose rank!)

↓

$$P(s) = \left[\begin{array}{cc|cc} s & 0 & 1 & 1 \\ 1 & s+1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

r_m: rank for any s except for some value of s
⇓
here all rows independent
$r_m = 4$ (except for a finite number of values of s)

$$\left[\begin{array}{cccc} 1 & s+1 & 1 & 1 \\ 1 & s+1 & 1 & 1 \\ - & - & - & \end{array}\right]$$

these are indep. ∀s
while on our case dep.

⇓ computing det(P(s))

det of a 4×4 matrix:
computed by decomposition

ZEROS $\det(P(s)) = 0$
↳

$$P(s) = \left[\begin{array}{cc|cc} s & 0 & 1 & 1 \\ 1 & s+1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \det(P(s)) = s(-1)^{1+1}\det\begin{bmatrix} s+1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} +$$

$$+ \cancel{0(-1)^3 \det[\,//\,]} + 1(-1)^4 \det\begin{bmatrix} 1 & s+1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} +$$

$$+ \cancel{1(-1)^5 \det\begin{bmatrix} 1 & s+1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}^{\det = 0} =$$

dipendent columns

{ with same procedure }

$\Downarrow$

$$= s\left[\cancel{1(-1)^{3+1}\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}^{\det=0}_{\text{dependent}} + 1(-1)^{3+2}\det\begin{bmatrix} s+1 & 1 \\ 1 & 1 \end{bmatrix}\right] +$$

$$+ \left[1(-1)^4 \det\begin{bmatrix} s+1 & 1 \\ 1 & 1 \end{bmatrix} + \cancel{1(-1)^5 \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}\right] =$$

$$= -s\,[s + \cancel{1} - \cancel{1}] + [s + \cancel{1} - \cancel{1}] = s - s^2 = s(1-s) = \det(P(s)) = 0$$

$\Rightarrow$ ZEROS $\begin{cases} s = 0 \\ s = 1 \end{cases}$

considerations...

- pole in $s=0$ / zero in $s=0$
  ↳ cancellation?
  in MIMO we cannot say a priori if the couple cancel out! $\Rightarrow$ to check for cancellation
  ← we can use

to check IF

Is there a cancellation ?

1. Reachability matrix & Observability matrix ⟶ if cancellation happen, the Reach/Obs matrix lose rank $\Rightarrow$ no longer fully Reach/Obs !

2. Poles & Zeros of MIMO T.F

2 REACHABILITY

$n = 2$ (syst order)

1 $M_R = [B \quad AB \dots A^{n-1}B]$

$(n=2) \downarrow$ $M_R = [B \quad AB] = \left[\begin{array}{cc|cc} -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 2 \end{array}\right]$ rows elimindip $\Downarrow$ rank = 2

rank$(M_R) = 2$
↳ fully Reachable !

OBSERVABILITY

$M_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ \hline -1 & -1 \\ -1 & -1 \end{array}\right]$ columns dependent! $\Rightarrow$ rank=1 < 2

rank $(M_O) = 1 < 2$

NOT fully OBS. system

↳ something happen, NOT fully OBS → there is a cancellation

{ probably between pole/zero in $s=0$? }

cancellation happen

BUT we don't know where !

FROM poles/zeros from T.F of system:

2 to check cancellation, from T.F

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} s & 0 \\ 1 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{s(s+1)}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} s+1 & 0 \\ -1 & s \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{s(s+1)}\begin{bmatrix} s & s \\ s & s \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{s(s+1)}\begin{bmatrix} -2s & -2s \\ -2s & -2s \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{s(s+1)}\begin{bmatrix} s^2-s & s^2-s \\ s^2-s & -2s \end{bmatrix} = \frac{s}{s(s+1)}\begin{bmatrix} s-1 & s-1 \\ s-1 & -2 \end{bmatrix}$$

T.F of the system

$$G(s) = \frac{\cancel{s}}{\cancel{s}(s+1)} \begin{bmatrix} s-1 & s-1 \\ s-1 & -2 \end{bmatrix}$$

$\downarrow$

~~cancellation~~! But until now is a simplification...
to check if cancel ZERO/POLE evaluating

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-1}{s+1} \\ \frac{s-1}{s+1} & -\frac{2}{s+1} \end{bmatrix}$$

computing the POLES

$$M_2 = \det[G(s)] = -\frac{2(s-1)}{(s+1)^2} - \frac{(s-1)^2}{(s+1)^2} =$$

$$= \frac{-\cancel{2s}+2-s^2\cancel{+2s}-1}{(s+1)^2} = \frac{-(s^2-1)}{(s+1)^2} =$$

$$= -\frac{\cancel{(s+1)}(s-1)}{(s+1)^{\cancel{2}}} = \frac{s-1}{s+1}$$

$\varphi(s) = (s+1) \rightsquigarrow$ pole in $s=-1$

zeros

$r_n = 2 \rightarrow$ all minors of order 2 $\rightsquigarrow$ max common divisor

$Z(s) = s-1 \rightsquigarrow$ ZERO in $s=+1$

from T.F respect S.S analysis we have a cancellation on $s=0$

G(s) represent only Reach/Obs part of the system

we cannot observe the pole/zero on $s=0$

Ex. 5: (Additional) Given the system

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = x_2^2 + u \end{cases}$$

Find the back stepping control law that stabilizes the origin given the formula

$$u = -\frac{dV_1(x_1)}{dx_1} g(x_1) - k(x_2 - \phi_1(x_1)) + \frac{d\phi_1(x_1)}{dx_1} (f(x_1) + g(x_1)x_2)$$

Hint: Use the extended formulation.

0.1 Additional Informations

Procedure for poles computation

:

1. Compute ALL the minors of any order of $G(s)$,
2. Find their least common denominator $\phi(s)$,
3. Find the roots of $\phi(s)$, well done!

Remark. *o A 'minor of order r' of A is the determinant of an rxr sub matrix of A.*

o The characteristic polynomial $\phi(s)$ of a MIMO system is the least common denominator of all the minors of any order of $G(s)$.

o The poles of $G(s)$ are the roots of $G(s)$.

o If a state space representation is available, it may be easier to directly compute the eigenvalues of A.

Procedure for zeros computation

:

1. Compute the NORMAL RANK r_n of $G(s)$,
2. Compute all the minors of order r_n written to have $\phi(s)$ at the denominator,
3. Compute their maximum common divisor $z(s)$
4. Find the roots of $z(s)$, well done!

Remark. *o The 'Normal Rank' of a matrix $G(s)$ is the rank of $G(s)$ for all the values of s, except for a finite number of values.*

o The polynomial $z(s)$ is defined as the maximum common divisor of all the minors of order r_n (normal rank) of $G(s)$, written such that they have $\phi(s)$ as denominator.

o The invariant zeros are all and only the roots of $z(s)$.