OPTIMAL CONTROL (HJB equation)

Advanced and Multivariable Control

Optimal Control

different control approach...
CONTROL problem
$\updownarrow$
OPTIMIZATION problem $\Rightarrow$

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Optimal control

Basic idea: the control problem is transformed into an ***optimization*** one, where the goal is to compute the control variable by ***minimising a suitable performance index*** (or cost function) under ***constraints*** on the input, state, and output variables → (on continuous time)

Extremely flexible approach, which allows to consider nonlinear systems and to formulate different objectives and constraints → ex: cases hard to approach on freq domain

Widely used in many fields, such as all the ***engineering*** problems, and in particular in ***aerospace, mechanical, chemical*** fields, but also in ***economics***, ***finance***, ***biological*** systems,…

→ 2 ways!

Different approaches to its solution, based on sufficient conditions formuated by means of ***dynamic programming***, or necessary conditions (Pontryagin's ***Maximum Principle***) (2 APPROACHES)

It may be very difficult to find a solution, many ***numerical methods*** are available. Close connections with ***Reinforcement Learning*** (comes from dynamic programming)

It is the precursor ***of Model Predictive Control***, the most popular method for advanced process control (last Chapter of our course)

↓ after initial description we specialize for particular simple case...

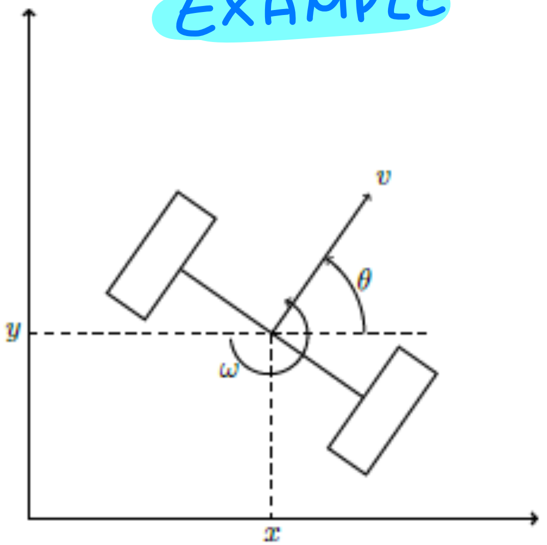
We'll give a hint of the dynamic programming approach, and then we'll specialize to the simplest case of application to ***linear systems*** with simple, ***quadratic cost functions***

Find v, ω on $[0,T]$ over the best choices

EXAMPLE

Mobile robot - path tracking problem

model of the system to control



"main problem elements"

difficult ↳ by using standard technique

- **Model - unicycle**

$$\begin{aligned} \dot{x} &= \cos\theta\, v \\ \dot{y} &= \sin\theta\, v \\ \dot{\theta} &= \omega. \end{aligned}$$

control variables

v, ω

(track the reference)

- reference trajectory: $x^o(t),\ y^o(t),\ t \in [0,T]$

(suitable chosen)

weights

Optimization problem

tracking from $[0,T]$

- *performance index, or cost function*

$$\min_{v,\omega} \int_0^T ((x(\tau) - x^o(\tau))^2 + (y(\tau) - y^o(\tau))^2) + r_1 v^2(\tau) + r_2 \omega^2(\tau))d\tau +$$

$$+q((x(T) - x^o(T))^2 + (y(T) - y^o(T)^2))$$

other terms by tuning param. weighting the difference squared @ T → respect final value of system

↑ sometimes you want to use a limitation weighting v^2, ω^2, limit the usage of v^2, ω^2 weighted

(x, y, θ possible!)

- *constraints*

subject to the system's dynamics and

$$|v(t)| \le \bar{v},\ |\omega(t)| \le \bar{\omega},\ t \in [0,T]$$

max value of control var! LIMITATION

- is r_1, r_2 big $\rightarrow$ I have a slow system with high limitation

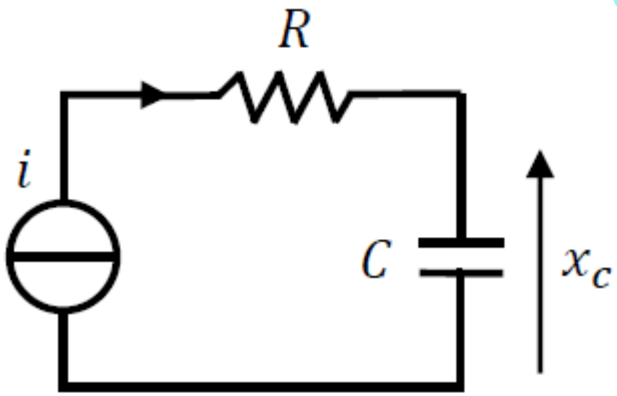
  r_1, r_2 small $\rightarrow$ lot control var, $x, y \rightarrow x^\circ, y^\circ$ very fast follow

- IF q high $\rightarrow$ I am more interested on the final value @ $t = T$

problem formulated

for non lin syst, quadratic cost funct, linear constrain

EXAMPLE 2



Model

$$C\dot{x}_c(t) = i(t)$$

Goals

(1) minimize the energy stored in the capacitor at time ***t=T*** given a known initial condition

(2) minimize the power dissipated in the resistance

weight of square system @ t=T, requirement (1)

"S" specify relative importance of (1) Goal

weight

terminal cost

requirement (2) to minimize power dissipation

$$\min_i J = s x_c^2(T) + \int_{t_0}^{T} R i^2(\tau) d\tau, \quad s \geq 0$$

performance index, or cost function

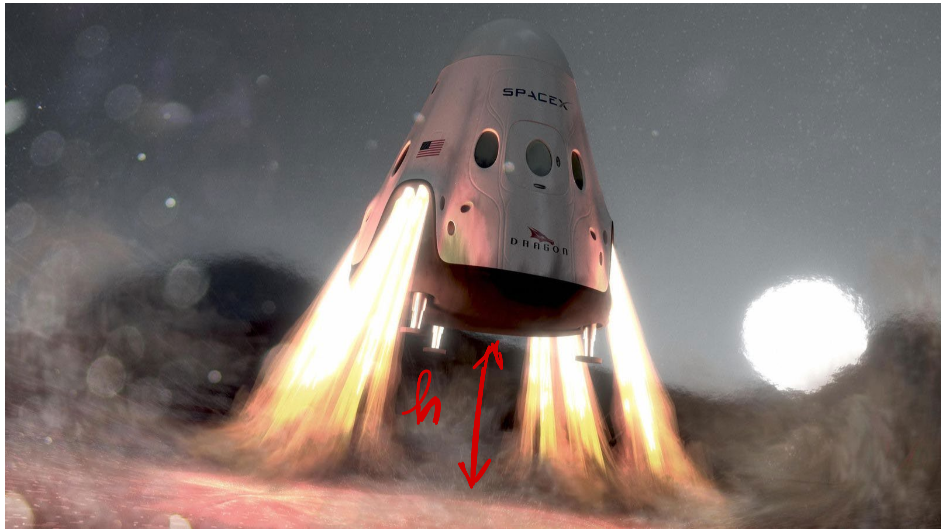
↳ related to system evolution

⇓

subject to the system's dynamics
and to possible constraints on ***i*** and ***x***

constraints

limitations



Spacecraft landing

EXAMPLE 3

Model

(Gravit. force) total mass

(control var) Jet thrusters

mass change when using thruster

$$M\ddot{h} = -gM + u \qquad \dot{M} = -ku$$

$$M(0) = M_0,\ h(0) = h_0,\ \dot{h}(0) = \dot{h}_0, \text{ and } k > 0.$$

Optimal control problem: optimally manage the thrusters ***u*** in order to minimize the final time ***T*** under constraints

$\min_{u} T$ — performance index, or cost function

min time for landing T

→ min time problem with control var constraints

$$M(t) \geq m \qquad h(t) \geq 0$$

$$h(T) = \dot{h}(T) = 0$$

constraints

Generic stabilization problem

design parameter (T, Q, R, S) → to be chosen well to represent well the requirements

design parameters

spd $Q \geq 0$ weights the deviation of the state from zero

p.d $R > 0$ weights the input

spd $S \geq 0$ weights the deviation of the final state from zero

you have ↳ a quadratic cost function..

you penalyze usage of control!

T chosen

$$\min_u J = \int_{t_0}^{T} (\underbrace{x'(\tau)Qx(\tau)}_{\text{quadratic term in state}} + \underbrace{u'(\tau)Ru(\tau)}_{\text{quadratic } u(t)})d\tau + \underbrace{x'(T)Sx(T)}_{\text{quadratic state}}$$

← cost function for stab. problem

Example: second order system

I restrict using diag matrix

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \cancel{S = 0}$$

we don't weight final value of state

the integrand is → $q_1 x_1^2 + q_2 x_2^2 + r_1 u_1^2 + r_2 u_2^2$

q_1, q_2, r_1, r_2 free design choices

FOR DIFFERENT q, r choices..

don't interested on the $u(t)$ value, quick result

(q_1,q_2) >> (r_1,r_2) → I am mainly interested to quickly bring the state to zero (leads to a fast closed-loop system)

using much $u(t)$

limit control!

(r_1,r_2) >> (q_1,q_2) → I don't want to use too much the control variables (typically, I'll obtain a slow closed-loop system)

q_1 >> q_2 → I want that the first state x_1 goes much faster to zero than the second state x_2

$x_1 \to 0$ fast $\quad x_2 \to 0$ slower dynamic

and so on…

Formal problem statement (OPTIMAL CONTROL) (Reformulation of problem)

instead of classical specification we use technical demand on the system!

Given the syst. described by diff eq.

$$\dot{x}(t) = f(x(t), u(t)), \quad x \in R^n, \quad u \in R^m$$

If x to estimate we use an observer

f continuously differentiable with respect to its arguments, x measurable

Knowledge about the state (time invariant system)

Goal: compute an "optimal control" $u^o(t)$, $t \in [t_0, T]$ minimizing

cost function

[OPTIMIZATION problem] ⇒

$$J(x(t_0), u(\cdot), t_0) = \boxed{\int_{t_0}^{T} l(x(\tau), u(\tau))d\tau + m(x(T))}$$

(General formulation)

l, m continuously differentiable

respect its argument

⇒ subject to the system's dynamics and state and input constraints:

$$x(t) \in X, \quad u(t) \in U$$

constraints

$\boxed{X \subseteq R^n, U \subseteq R^m}$ are compact sets containing the origin

Denote by $u_{[a,b]}$ the control functions $u(\cdot)$ in the interval $[a,b]$ and define

optimal value of J @ t

$$J^0(x(t),t) = \min_{u[t,T]} J(x(t),u(\cdot),t) = \boxed{\int_t^T l(x(\tau),u(\tau))d\tau + m(x(T))} \qquad t \in [t_0,T]$$

Note that J^0 and J depend on $x(t)$, i.e. on the current value of the state, while they do not depend on the state evolution up to time t

@ t generic instant → depend only on x(t), NOT on past values

To proceed, we need the Bellman's principle of optimality

{Fundamental Result}

proof of the solution ⟹

(NOT ASK on EXAM :))

Bellman's principle of optimality

dynamic programming approach

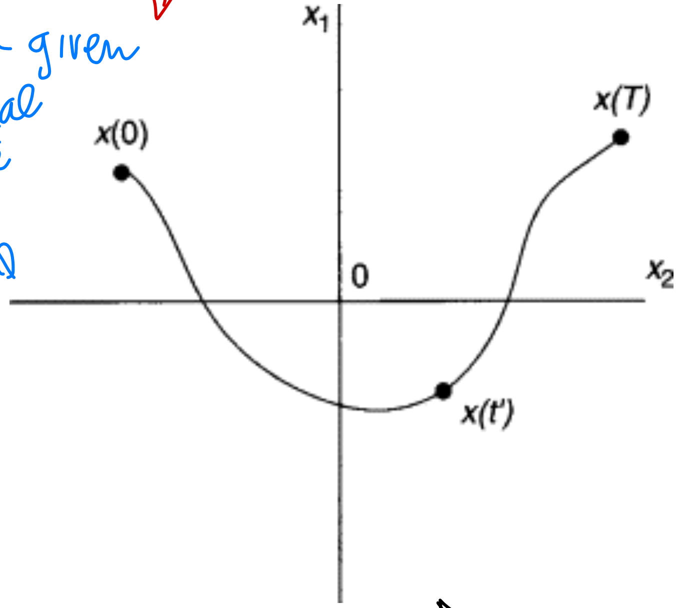
From any point of an optimal trajectory, the remaining trajectory is optimal for the corresponding problem over the remaining number of stages, or time interval, initiated at that point

starting from $x(0)$ you wanna reach @T: $x(T)$

solving optim. of problem you obtain a certain optimal traj → take any point → optimize ∀ point the traj

for any point we have an optimal traj... different

from a given initial state $x(0)$ @ $t=0$



Richard Bellman, the father of Dynamic Programming

If some trajectory in the phase space connects the initial ***x(0)*** and terminal ***x(T)*** points and is optimal in the sense of some cost functional, then the sub-trajectory, connecting any intermediate point ***x(t')*** of the same trajectory with the same terminal point ***x(T)***, should also be optimal.

Lot used in many optimization application

Alexander S. Poznyak, in Advanced Mathematical Tools for Automatic Control Engineers: Deterministic Techniques, Volume 1, 2008

First we ignore $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ constrain → than we will include that by model predictive control

From

from a generic time instant "t"

not function of $u(\cdot)$ anymore...

$$J^0(x(t),t) = \min_{u[t,T]} J(x(t),u(\cdot),t) = \int_t^T l(x(\tau),u(\tau))d\tau + m(x(T))$$

optimal value will not depend on $u(\cdot)$ for which we solve the problem

$$J^0(x(t),t)$$

$$= \min_{u[t,t_1]} \left\{ \min_{u[t_1,T]} \left[\int_t^{t_1} l(x(\tau),u(\tau))d\tau + \int_{t_1}^T l(x(\tau),u(\tau))d\tau + m(x(T)) \right] \right\}$$

don't depend on what happens from $t_1 \div T$

$$\min_{u[t_1,T]} \left[\underbrace{\int_t^{t_1} l(x(\tau),u(\tau))d\tau}_{\text{does not depend on } u[t_1,T]} + \int_{t_1}^T l(x(\tau),u(\tau))d\tau + m(x(T)) \right]$$

does not depend on ***u[t₁,T]***

So we can remove it from min{·}

$$J^0(x(t),t)$$

$$= \min_{u[t,t_1]} \left\{ \int_t^{t_1} l(x(\tau),u(\tau))d\tau + \min_{u[t_1,T]} \left[\int_{t_1}^T l(x(\tau),u(\tau))d\tau + m(x(T)) \right] \right\}$$

$$J^0(x(t),t)$$

$$= \min_{u[t,t_1]} \left\{ \int_t^{t_1} l(x(\tau),u(\tau))d\tau + \underbrace{\min_{u[t_1,T]} \left[\int_{t_1}^{T} l(x(\tau),u(\tau))d\tau + m(x(T)) \right]} \right\}$$

optimal solution is the one computed from (t, t_1)

optimal solution

recall Bellman!

$$J^0(x(t),t) = \min_{u[t,t_1]} \left\{ \int_t^{t_1} l(x(\tau),u(\tau))d\tau + J^o(x(t_1),t_1) \right\}$$

if the optimal control has been applied in the interval $[t_1, T)$, the optimal cost of the state trajectory starting at t is obtained by minimizing the sum of the cost incurred from t to t_1 plus the optimal cost from t_1 to T

$$J^0(x(t),t) = \min_{u[t,t_1]} \left\{ \int_t^{t_1} l(x(\tau),u(\tau))d\tau + J^o(x(t_1),t_1) \right\}$$

taking $t_1 = t+dt$ → you can rewrite

mean value theorem with $\alpha \in [0,1]$

$$t_1 = t + dt$$

$$J^0(x(t),t) = \min_{u[t,t+dt]} \{ l(x(t+\alpha dt),u(t+\alpha dt))dt + J^o(x(t+dt),t+dt) \}$$

expand $\underbrace{J^o(x(t+dt),t+dt)}_{\text{taylor expansion...}}$

$$J^o(x(t+dt),t+dt) = J^o(x(t),t) + \frac{\partial J^o(x(t),t)}{\partial x}\frac{dx(t)}{dt}dt + \frac{\partial J^o(x(t),t)}{\partial t}dt + O(dt)^2$$

$$\cancel{J^0(x(t),t)}$$

$$= \min_{u[t,t+dt]} \Big\{ l(x(t+\alpha dt), u(t+\alpha dt))dt + \cancel{J^o(x(t),t)} + \frac{\partial J^o(x(t),t)}{\partial x}\frac{dx(t)}{dt}dt$$

$$+ \frac{\partial J^o(x(t),t)}{\partial t}dt + O(dt)^2 \Big\}, \quad \alpha \in [0,1]$$

divide by dt, and let $dt \to 0$

does not depend on ***u***

$$0 = \min_{u[t]} \left\{ l(x(t),u(t)) + \frac{\partial J^o(x(t),t)}{\partial x} f(x(t),u(t)) + \frac{\partial J^o(x(t),t)}{\partial t} \right\}$$

at a fixed time t, x and u must be considered as vectors, instead of functions of time properly chose in some way!

$$\boxed{\frac{\partial J^o(x,t)}{\partial t} = -\min_u \left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\}}$$

Hamilton Jacobi Bellman equation

partial diff eq. NOT possible to solve in practice

$$J(x,u(\cdot),T) = m(x)$$

*does not depend on **u***

@ T fixed cost function depends on x only

$$\Updownarrow$$

$$J = \cancel{\int_t^T //} + m(x)$$

(terminal condition)

$$\boxed{J^o(x,T) = m(x)}$$

How to use the HJB equation?

IN PRACTICE

$$\frac{\partial J^o(x,t)}{\partial t} = -\min_u \left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\}$$

$$J^o(x,T) = m(x)$$

function to define for optimization

Step 1 compute the value u^o minimizing

$$\left\{ l(x,u) + \frac{\partial J^o(x,t)}{\partial x} f(x,u) \right\} \longrightarrow u^o = \kappa\left(x, \frac{\partial J^o(x,t)}{\partial x}\right)$$

optimal INPUT variable

unknown...

{we write it analytically}

use this u^o on HJB eq.

Step 2 compute the function $J^o(x,t)$ satisfying the HJB equation

$$\Downarrow$$

$$\frac{\partial J^o(x,t)}{\partial t} = -l\left(x, \kappa\left(x, \frac{\partial J^o(x,t)}{\partial x}\right)\right) - \frac{\partial J^o(x,t)}{\partial x} f\left(x, \kappa\left(x, \frac{\partial J^o(x,t)}{\partial x}\right)\right) \quad , \quad J^o(x,T) = m(x)$$

Step 3 use $\frac{\partial J^o(x,t)}{\partial x}$ in the control law $u^o = \kappa\left(x, \frac{\partial J^o(x,t)}{\partial x}\right) \longrightarrow u^o = \kappa(x,t)$

J^o evaluated...

(optimal cost function)

Comments

typically Dynamic programming (iterative methods)

- From a computational point of view, this is a very tough probem, many different approaches have been proposed

- The computations must proceed backwards in time. You start from the final value $J^o(x,T)=m(x)$ and move on with reverse time

- The resulting control law will be of the form ***u=κ(x,t)***, i.e. a state feedback control law even though an open-loop optimization problem has been formulated

↓

on Reinforcement Learning from a givenstate , Try more times until good resolution by Records