EXERCISES OF ADVANCED AND MULTIVARIABLE CONTROL

Lyapunov stability

LY1 10/6/2008

Consider the system

$$\dot{x}(t) = ax(t)(1 - bx(t)), \quad a > 0, b > 0$$

A) Compute the equilibria;
B) Analyze the stability of the equilibria by means of the linearized model
C) Check the previous result with a graphical analysis of the plane $x - \dot{x}$;
D) Check the previous result with the Lyapunov function

$$V(x - \bar{x}) = \frac{1}{2}(x - \bar{x})^2.$$

Solution

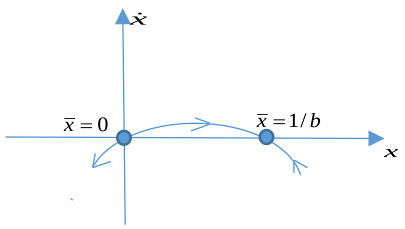
$A)\ \bar{x} = 0\ ,\ \bar{x} = 1/b$

$B)\ \delta\dot{x} = \{a(1 - b\bar{x}) - ab\bar{x}\}\delta x$

$\bar{x} = 0 \ \rightarrow\ \delta\dot{x} = a\delta x \quad \text{unstable}$

$\bar{x} = 1/b \ \rightarrow\ \delta\dot{x} = -a\delta x \quad \text{asymptotically stable}$

C)



$D)\ \bar{x} = 0 \ \rightarrow\ V(x) = 0.5x^2\ ,\ \dot{V}(x) = ax^2 - abx^3 > 0 \text{ in a neighbor of the origin}$

$\bar{x} = 1/b \ \rightarrow\ V(x) = 0.5(x - \bar{x})^2 \ \rightarrow\ x = (1/b) + \delta x \rightarrow\ \delta\dot{x} = -a\delta x - ab\delta x^2$

$V(\delta x) = 0.5\delta x^2 \ \rightarrow\ \dot{V}(x) = -a\delta x^2 - ab\delta x^3 < 0 \text{ in a neighbor of the origin}$

LY2 25/9/2006

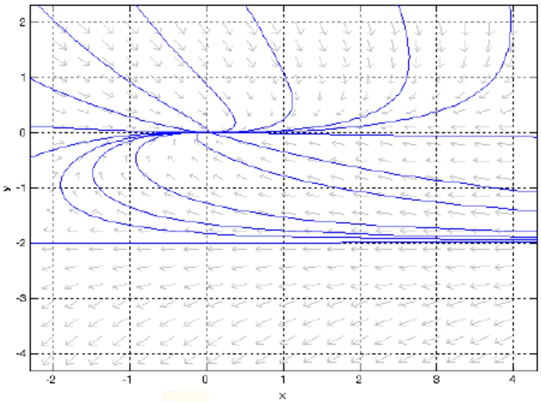
Given the system

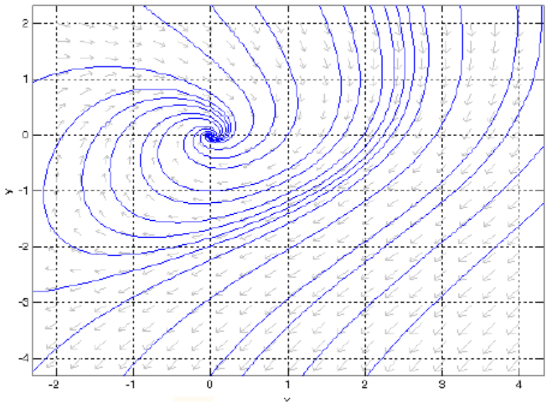
$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + 2x_2(t) \\ \dot{x}_2(t) &= -2x_1(t) - x_2(t) - x_2^2(t) \end{aligned}$$

A) Analyze the stability of the origin with the Lyapunov function

$$V(x) = x_1^2(t) + x_2^2(t)$$

B) Can the origin be a globally asymptotically stable equilibrium?
C) Which one of these figures represents the phase plane of the system?





Solution

A) $\dot{V}(x) = -2x_1^2 - 2x_2^2(1+x_2) < 0$ in a neighbor of the origin

B) there is another equilibrium $\bar{x}_1 = -10, \bar{x}_2 = -5$ therefore the origin cannot be asymptotically stable

C) the linearized model (at the origin) is

$$\delta\dot{x}_1 = -\delta x_1 + 2\delta x_2$$
$$\delta\dot{x}_2 = -2\delta x_1 - \delta x_2$$
$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \ \det(sI - A) = s^2 + 2s + 5 \ , \ s = -1 \pm j2 \rightarrow \text{stable focus, figure on the right}$$

LY3 3/9/2009

Given the system

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) - x_1^3(t) \\ \dot{x}_2(t) &= x_1(t) - x_2^3(t) \end{aligned}$$

A) Check if it is possible to study the stability of the origin by looking at the linearized system.
B) Analyze the stability of the origin with by selecting a suitable Lyapunov function.

Solution

A) Linearized model

$$\delta\dot{x}_1 = -\delta x_2$$
$$\delta\dot{x}_{12} = \delta x_1$$

Matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ eigenvalues $s = \pm j$

Nothing can be concluded since the real part of the eigenvalues is null.

B) $V(x) = 0.5(x_1^2 + x_2^2) \rightarrow \dot{V}(x) = -(x_1^4 + x_2^4) < 0$ the origin is asymptotically stable.

LY4 14/7/2009

Consider the system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \cos(x_2(t)) \\ \dot{x}_2(t) &= x_1(t) \end{aligned}$$

A) Compute the equilibria, and study their stability and characteristics with the corresponding linearized system.
B) Consider the asymptotically stable equilibria and check the previous result with the Lyapunov function $V(x-\bar{x}) = 0.5(x_1 - \bar{x}_1)^2 - \cos(x_2 - \bar{x}_2) + 1$

Solution

A) $\bar{x}_1 = 0$, $\cos(\bar{x}_2) = 0 \rightarrow \bar{x}_2 = (0.5+i)\pi$, i integer

Linearized model

$$\delta\dot{x}_1 = -\delta x_1 - (\sin \bar{x}_2)\delta x_2$$
$$\delta\dot{x}_2 = \delta x_1$$

i even :

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \text{ , } \det(sI-A) = s^2 + s + 1 \text{ , complex conjugate eigenvalues} \rightarrow \text{stable focus}$$

i odd :

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \text{ , } \det(sI-A) = s^2 + s - 1 \text{ , eigenvalues } s_1 = -1.62, s_2 = 0.62 \rightarrow saddlepo\text{int}$$

B)

Let $\Delta x_1 = x_1$, $\Delta x_2 = x_2 - (0.5+i)\pi$ i even. The system can be written as

$$\Delta\dot{x}_1 = -\Delta x_1 - \sin(\Delta x_2)$$
$$\Delta\dot{x}_2 = \Delta x_1$$

The function *V* becomes $V(\Delta x) = 0.5\Delta x_1^2 - \cos(\Delta x_2) + 1 > 0$. Moreover $\dot{V}(\Delta x) = -\Delta x_1^2 \leq 0$
However $\dot{V}(\Delta x) = -\Delta x_1^2 = 0$ for $\Delta x_1 = 0$ and the system becomes

$$0 = \sin(\Delta x_2)$$
$$\Delta\dot{x}_2 = 0$$

So that $\Delta x_2 = \Delta\bar{x}_2 = i\pi$, *i* even. Therefore, for any equilibrium there is a neighbor where $\dot{V}(\Delta x) \neq 0$, save for the equilibrium itself. Therefore, in view of the Krasowski LaSalle Thm. these equilibria are asymptotically stable. Since there are many equilibria, they cannot be globally stable.

LY4 6/9/2010

Given the discrete time system with $\gamma \neq -1$

$$
\begin{aligned}
x_1(k+1) &= x_2(k)(\alpha x_1(k)+\gamma) \\
x_2(k+1) &= x_1(k)(\alpha x_2(k)-1)
\end{aligned}
$$

A) compute the equilibria;
B) Compute the values of α,γ such that the origin is an asymptotically stable equilibrium;
C) Set γ=0.5 and check the stability of the linearized model with the Lyapunov equation and

$$P=\begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

D) With the same P matrix check the stability of the equilibrium of the nonlinear system with the Lyapunov function *V(x)=x'Px*

Solution

A) The equilibrium conditions are

$$\bar{x}_1 = \alpha\bar{x}_1\bar{x}_2 + \gamma\bar{x}_2$$

$$\bar{x}_2 = \alpha\bar{x}_1\bar{x}_2 - \bar{x}_1$$

and the unique solution is $\bar{x}_1 = \bar{x}_2 = 0$

B) The linearized model at the origin is

$$\begin{aligned}\delta x_1(k+1) &= \gamma\delta x_2(k) \\ \delta x_2(k+1) &= -\delta x_1(k)\end{aligned} \rightarrow A=\begin{bmatrix} 0 & \gamma \\ -1 & 0 \end{bmatrix} \rightarrow \det(zI\_A) = z^2+\gamma \rightarrow |\gamma|<1$$

C) Lyapunov equation

$$A'PA-P=-\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}<0$$

$$\Delta V(x) = \begin{bmatrix} x_2(\alpha x_1+\gamma & x_1(\alpha x_2-1 \end{bmatrix} P \begin{bmatrix} x_2(\alpha x_1+\gamma) \\ x_1(\alpha x_2-1) \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \end{bmatrix} P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

D)

$$= 2\alpha^2 x_1^2 x_2^2 + 4\alpha\gamma x_1 x_2^2 + 2\gamma^2 x_2^2 + 1.5\alpha^2 x_1^2 x_2^2 - 3\alpha x_1^2 x_2 + 1.5x_1^2 - 2x_1^2 - 1.5x_2^2$$

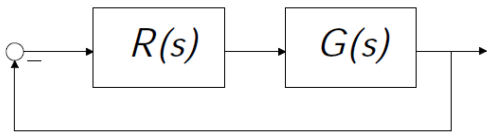
E) Considering only the second order terms

$$\Delta V(x) = -0.5x_1^2 - x_2^2 < 0$$

Poles and zeros

PZ1 (10/6/2008)

Given the system described by the block scheme



where

$$G(s) = \left| \begin{matrix} \frac{s-1.5}{s+1} & 0 & \frac{1}{s+1} \\ \frac{s-1}{s+1} & \frac{s-2}{s-1.5} & \frac{2}{s+1} \end{matrix} \right|, \quad R(s) = \left| \begin{matrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \\ \frac{s+1}{s-2} & \frac{1}{s+1} \end{matrix} \right|$$

A) Compute the poles and zeros of G(s) and R(s),

B) Consider if it is possible, without additional computations, to analyze the stability of the feedback system by simply looking at the sensitivity function.

Solution

A) Poles of *G(s):* s=-1, s=-1, s=1.5
   Zeros of *G(s)*: *s=2*
   Poles of *R(s): s=0, s=0, s=-1, s=2*
   Zeros of *R(s): none*

B) No, due to the presence of a pair zero/pole in *s=2*. More computations are required.

PZ2 (4/9/2007)

Given the system described by

$$G(s) = \begin{bmatrix} \frac{1}{s+3} & \frac{s+1}{(s+3)(s+2)} \\ \frac{s+3}{s+1} & \frac{s}{s+1} \\ \frac{2}{s+3} & \frac{1}{s+1} \end{bmatrix}$$

A) Compute poles and zeros;

B) Compute the static gain (s=0) corresponding to the inputs

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

and provide a lower and upper estimation of the minimum and maximum singular values.

Solution

A) Poles of *G(s):* s=-3, s=-3, s=-2, s=-1, s=-1
   Zeros of *G(s)*: none

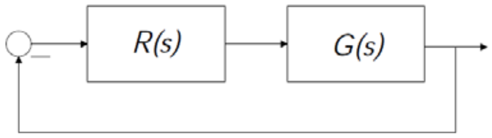
B) $G(0) = \begin{bmatrix} 1/3 & 1/6 \\ 3 & 0 \\ 2/3 & 1 \end{bmatrix}$

$\|y_1\|_2 = 3.09$, $\|y_2\|_2 = 1.01$, $\|y_3\|_2 = 2.45$

$\overline{\sigma}(G(0)) \geq 3.09$, $\underline{\sigma}(G(0)) \leq 1.01$

PZ3 (14/7/2009)

Consider the system



where

$$G(S) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s+2)(s-1)} \\ \frac{b}{s-1} & \frac{2}{s-1} \end{bmatrix}, \quad R(S) = \begin{bmatrix} \frac{c}{(s-2)(s+1)} & \frac{-1}{s+1} \\ \frac{2}{s-2} & \frac{1}{s+1} \end{bmatrix}$$

Compute the values of the parameters *b* and *c* such that it is possible to study the stability of the feedback system simply by looking at the sensitivity function.

Solution

Poles of *G(s)*: *s=1, s=1, s=-2;*

Zeros of *G(s)*: *s=(b-4)/2*

The fist condition requires to avoid cancellations inside *G(s)* between unstable poles and zeros: *(b-4)/2≠1* → *b≠6*

Poles of *R(s)*: *s=-1, s=-1, s=2*

Zeros of *R(s)*: *s=(-2-c)/2*

To avoid cancellations inside *R(s)* of unstable poles and zeros (-2-c)/2 *≠2* → *c≠-6*

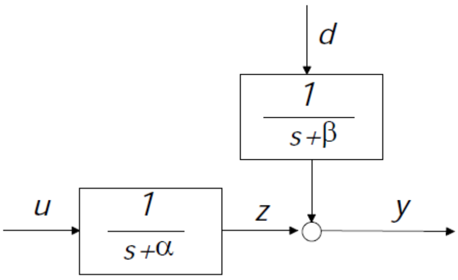
To avoid cancellations between unstable zero of *G(s)* and poles of *R(s): (b-4)/2≠2* → *b≠8*

To avoid cancellations between unstable zero of *R(s)* and poles of *G(s):* (-2-c)/2 *≠1* → *c≠-4*

Pole placement and disturbance estimation

PP1 (10/6/2008)

Consider the system



Where $\alpha>0$, $\beta>0$ and d is a constant, but unknown signal.

A) Show how to design with pole placement a regulator with integral action such that all the closed-loop poles are in s=-1.
B) Show how to estimate the disturbance from the measures of *u* and *y* and the condition to be verified on α, β.

Solution

A) Enlarge the plant with an integrator. The polynomials of the enlarged transfer function are

$$A(s) = s^2 + \alpha s \ , \ B(s) = 1$$

The desired polynomial is

$$P(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

The stabilizing part of the regulator is defined by

$$R'(s) = \frac{F(s)}{\Gamma(s)} = \frac{f_1 s + f_0}{\gamma_1 s + \gamma_0}$$

and its coefficients must be computed solving the linear system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

B)

Denote by η the output of the block with input *d*. Then, the enlarged system, introducing a fictitious dynamics for *d*, is described by

$$\begin{bmatrix} \dot{z} \\ \dot{\eta} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\beta & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ \eta \\ d \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} z \\ \eta \\ d \end{bmatrix}$$

This new system must be observable. Applying the observability test, the condition required is $\alpha \neq \beta$. Then, a standard observer, or a reduced one (note that the output already coincides with a state) can be used.

PP2 (4/9/2007)

Given the system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

A) Compute a state feedback control law such that the closed-loop eigenvalues are in *s=-1*;

B) Compute a state observer with eigenvalues in s=-1;

C) Compute (in symbolic form) the transfer function of the regulator obtained by combining the results of the previous steps A and B.

Solution

A) $K = [k_1 \quad k_2] \to \det(sI-(A-BK)) = \det\begin{bmatrix} s-k_1+2 & -1-k_2 \\ k_1 & s-2+k_2 \end{bmatrix} = s^2+(k_2-k_1)s+2k_2-4+3k_1$

To assign the eigenvalues in *s=-1* we must impose

$$k_2 - k_2 = 2$$

$$2k_2 - 4 + 3k_1 = 1 \quad \to \quad k_1 = 0.2, \quad k_2 = 2.2$$

B)

We must select the gain of the observer *L* such that the eigenvalues of *A-LC* are in *s=-1.*

Select $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, then with the same kind of computations we obtain l_1*=2,* l_2*=9*.

C)

$$R(s) = K(sI-(A-BK-LC))L = \frac{20.2s+40.6}{s^2+4s+30.2}$$

LQ control

LQ1 (10/6/2008)

Given the system

$$\dot{x}(t) = 8x(t) + u(t)$$

A) Design a LQ controller, with Q=40, R=1/20, such that the closed loop system has eigenvalues smaller than -2.

B) Describe the robustness characteristics of continuous-time and discrete-time LQ control.

Riccati equation:

$$\dot{P}(t) + Q - P(t)BR^{-1}B'P(t) + P(t)A + A'P(t) = 0$$

Solution

A) $A=8, B=1 \to \overline{A} = A+2 = 10$, solution of the Riccati equation $\overline{P} = 2, \overline{K} = R^{-1}B'\overline{P} = 40$
$A - B\overline{K} = -32$

B) See the notes

LQ2 (25/9/2006)

Given the system

$$\dot{x}(t) = x(t) + u_1(t) + 2u_2(t)$$

A) Compute the LQ control law with Q=1, R=I

B) Compute the closed-loop eigenvalue

C) Assuming that the system is now described by

$$\begin{aligned}\dot{x}(t) &= x(t) + u_1(t) + u_2(t) + v_x(t)\\ y(t) &= x(t) + v_y(t)\end{aligned}$$

where v_x and v_y are white noises with zero mean and variances equal to one, compute the Kalman filter

D) compute the control law obtained by combining the previous results and the corresponding regulator
E) write the eigenvalues of the closed-loop system.

Solution
A)

$$A = 1,\ B = \begin{bmatrix}1 & 2\end{bmatrix} \rightarrow \text{stationary Riccati equation } 2\overline{P} + 1\text{-}\overline{P}^2\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 0 \rightarrow \overline{P} = 0.69 \rightarrow \overline{K} = \begin{bmatrix}0.69\\1.38\end{bmatrix}$$

B) $A - B\overline{K} = -2.45$

C) Stationary Riccati equation for the Kalman filter

$$\tilde{P}A + A\tilde{P} + \tilde{Q} - \tilde{P}C'\tilde{R}^{-1}C\tilde{P} = 0\ ,\ \tilde{Q} = \tilde{R} = C = 1 \rightarrow \tilde{P} = 2.41 \rightarrow L = \tilde{R}^{-1}C\tilde{P} = 2.41 \rightarrow A - LC = -1.41$$

D) Control law

$$u = -\begin{bmatrix}0.69\\1.38\end{bmatrix}\hat{x}$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + Ly$$

$$u = -K(sI - (A - BK - LC)^{-1}Ly = -\begin{bmatrix}0.69\\1.38\end{bmatrix}(s + 4.86)^{-1}2.41y$$

E) in view of the separation principle, the eigenvalues are those of *A-BK*=-2.45 and *A-LC=-1.41*

LQ3 (3/9/2009)

Given the system

$$\dot{x}(t) = 5x(t) + bu(t)$$

A) Assuming *b*=1, design a LQ regulator with weighting matrices Q=20, R=0.1 and compute the corresponding closed-loop eigenvalue.
B) Determine the set of values of *b* such that the control law previously computed guarantees the asymptotic stability of the closed-loop system and compare the result with the theoretical robustness bounds guaranteed by LQ control theory.

Solution

A) LQ solution $\overline{P} = 2,\ \overline{K} = 20,\ A - B\overline{K} = -15$

B) The system is asymptotically stable if *5-20b<0* that is *b>0.25*. This result is consistent with the theoretical gain margin (0.5,∞).

Optimal control

OC1 (10/6/2008)

Given the dynamical system

$$\dot{x}(t) = x^2(t) + 0.5u(t)$$

Set the solution, with the Hamilton-Jacobi-Bellman equation, of the optimal control problem

$$\min J = \frac{1}{2}\int_0^T (x(t) + u^2(t))dt + x^2(T)$$

Solution

$$\frac{\partial J^o}{\partial t} = -\min_u \left\{0.5x + 0.5u^2 + \frac{\partial J^o}{\partial x}(x^2 + 0.5u)\right\}$$

$$\frac{\partial}{\partial u} = u + 0.5\frac{\partial J^o}{\partial x} \rightarrow u = -0.5\frac{\partial J^o}{\partial x}$$

HJB equation

$$\frac{\partial J^o}{\partial t} = -0.5x - 0.125\left(\frac{\partial J^o}{\partial x}\right)^2 - \frac{\partial J^o}{\partial x}x^2 + 0.25\left(\frac{\partial J^o}{\partial x}\right)^2$$

$$\downarrow$$

$$\frac{\partial J^o}{\partial t} = -0.5x + 0.125\left(\frac{\partial J^o}{\partial x}\right)^2 - \frac{\partial J^o}{\partial x}x^2$$

$$J^o(x,T) = x^2(T)$$

Extended Kalman filter

EKF1 (4/9/2007)

Given the discrete-time system

$$\begin{aligned} z(k+1) &= \alpha z(k)u(k) + z^2(k-1) + v_z(k) \\ y(k) &= z(k) + v_y(k) \end{aligned}$$

Where v_z and v_y are white noises with zero mean and null variance, while a is a unknown constant coefficient.

A) Show how to represent the system in state variables;

B) Show how to design an Extended Kalman Filter (structure of the predictor, covariances of the disturbances, matrices to be used in the computation of the predictor gain).

Solution

$$x_1(k) = z(k)$$
$$x_2(k) = z(k-1)$$
$$x_3(k) = \alpha$$
$$\downarrow$$

A)

$$x_1(k+1) = x_1(k)x_3(k)u(k) + x_2^2(k) + v_z(k)$$
$$x_2(k+1) = x_1(k)$$
$$x_3(k+1) = x_3(k)$$
$$y(k) = x_1(k) + v_y(k)$$
$$Q^* = diag(1,0,0)$$

B) $$A(k) = \begin{bmatrix} \hat{x}_3(k)u(k) & 2\hat{x}_2(k) & \hat{x}_1(k)u(k) \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$C(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Model Predictive Control

MPC1 (3/9/2009)

Given the discrete-time system

$$x(k+1) = 2x(k) + u(k)$$

and the cost function

$$J = \sum_{i=0}^{N-1} (x^2(k+i) + 2u^2(k+i)) + 0.5x(k+N)$$

A) Compute the solution of the infinite time LQ control (N=∞) and the corresponding closed-loop eigenvalue.
B) Using the Receding Horizon strategy, compute the control law corresponding to the prediction horizon N=3.
C) Describe how to define the MPC problem when there are constraints on the control variable (theoretical answer is required).

Solution

A) LQ solution

$$\overline{P} = 7.275, \overline{K} = 1.569, A\text{-}B\overline{K} = 0.431$$

B)

$$P(3) = S = 0.5, K(2) = 0.4, A\text{-}BK(2) = 1.6$$
$$P(2) = 2.6, K(1) = 1.13, A\text{-}BK(1) = 0.869$$
$$P(1) = 5.523, K(0) = 1.468, A\text{-}BK(0) = 0.5318$$