ES 1 Given the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(t)$$

1) which output can be controlled to reference
↓ which one can be enlarged with integrator to have 0 tracking error?

conditions:

a) NO invariant zeros in 0

b) # inputs ≥ # outputs $(m \geq p)$

↳ $1 \geq 2$? NO NOT satisfied

we need to select which output control

• select y_1

$y_1(t) = C_1 x(t) = [0 \quad 1]\, x(t)$

(selecting only one output, $p = m$ OK) b ok

a) ? check of ZEROS in the origin → we need syst. matrix of y_1 as output

$$P_1(s) = \begin{bmatrix} sI - A & B_1 \\ C_1 & D_1 \end{bmatrix}$$

ZEROS of new system → $P_1(0)$? derivative action @ origin.. loose rank for $s = 0$?

$$P_1(0) = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 2 & 1 & 1 \\ \hline 0 & 1 & 0 \end{array}\right]$$

I, III row are linearly dependent

rank $(P_1(0)) = 2 <$ normal rank of $P_1 = 3$

→ There are derivative actions in the origin → NO enlargment on the first output!

• select y_2

$y_2(t) = C_2 x(t) = [1 \quad 1]\, x(t)$ ↓ checking again the conditions (b: $p = m$ ok) ✓

a:

$$P_2(0) = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

rank $(P_2(0)) = 3 =$ normal rank of P_2

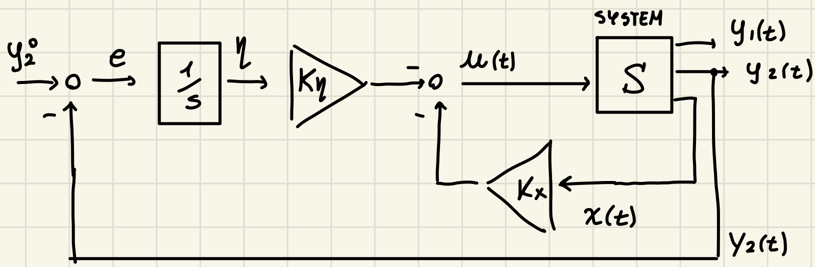
so.. NO derivative actions in the origin (a) ✓

↓

We can enlarge on output 2

2) assuming measurable states, design a state feedback control law with ZERO tracking error

↳ we can enlarge II output with integrator!



extended system scheme, to make computation

We can use for example pole placement state feedback control law!

⇓

we have to compute integrator dynamic $\eta = \frac{1}{s} e \rightarrow \dot{\eta}(t) = e(t)$ ($s\eta(s)$)

↓

error, as

$e(t) = y_2^0(t) - y_2(t)$

but we can express $y_2(t) = C_2 x(t)$

$\dot{\eta}(t) = y_2^0(t) - C_2 x(t)$

ENLARGED SYSTEM:

reference Y_2^0 acts only on integrator dynamic

to apply POLE PLACEMENT

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\eta}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \\ \eta \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\tilde{B}} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y_2^0(t)$$

enlarged Matrix

conditions: $(\tilde{A}, \tilde{B})$ reachable (on enlarged syst)

↓

By theory we proof that $(\tilde{A}, \tilde{B})$ is Reachable ⇔ (A, B) is Reachable

$$M_R = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \text{rank}(M_R) = 2 = \text{syst ORD}$$

✓ Reachable!

so we can compute gain K on enlarged syst, OR using in MATLAB place function

STATE SPACE MATRIX

poles position according to original syst dynamics!

$$K = \text{place}(\tilde{A}, \tilde{B}, [-1, -1.5, -2])$$

→ obtaining $K = [\underbrace{1.5 \quad 3.5}_{K_x \text{ Gains}} \quad \underbrace{-3}_{K_\eta \text{ Gain}}]$ → using this values on my CONTROL scheme gains K_x, K_η

on EXAM..
to compute pole placement
you should be able to compute by hand explicitly (SHIT)!

c) What if the state is unmeasurable?

↓

- We need a state estimation (OPTION 1): design a static observer

IDEA: since $y = Cx$ then $\boxed{\hat{x} = C^{-1} y}$

conditions:

C must be: (invertible)
- square
- NON singular

- (OPTION 2) we need to design a Dynamic observer

↳ K.F OR the other observer

(ENLARGED SYST OBSERVER)

conditions:

(A, C) is observable → $\mathcal{M}_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ -2 & 0 \end{bmatrix}$ rank($\mathcal{M}_o$) = 2 = syst ord OK ✓

↓

We can design an OBSERVER!

OBSERVER DYNAMIC

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \underbrace{\hat{y}}_{C\hat{x}})$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$

L computed in analogy of pole placement! With same "place" Matlab function

estimator for our state feedback control law ← $L^T = \text{place}(A^T, C^T, [-4 \quad -5])$ observer poles placed such that faster than the system

ES 2 discrete LQ control design

$$\begin{cases} x(k+1) = -x(k) + u(k) + v_x(k) & v_x(k) \sim WN(0,1) \\ y(k) = x(k) + v_y(k) & v_y(k) \sim WN(0,2) \end{cases}$$

R.E: $P(k) = Q + A^T P(k+1) A - A^T P(k+1) B (R - B^T P(k+1) B)^{-1} B^T P(k+1) A$

given during EXAM !!

1) Design LQ control for $Q = 1$, $R = 3.75$

condition for LQ control: (✓)

$$\begin{cases} 1)\ (A,B) \text{ Reachable} \rightarrow M_R = [B] = 1\ ✓ \\ 2)\ (A, C_q) \text{ Observable} \rightarrow M_O = [C_q] = 1\ ✓ \end{cases}$$

(easy in a I ORD syst)

C_q such that $C_q^T C_q = Q = 1 \rightarrow C_q = 1$

⇓

to find LQ control law (LQ∞ horizon ∞)

↳ solve steady state R.E! when $P(k) = P(k+1)$

↓

$\bar{P} = Q + A^T \bar{P} A - A^T \bar{P} B (R - B^T \bar{P} B)^{-1} B^T \bar{P} A =$ — K_{LQ}, LQ Gain (*)

$u(k) = -K_{LQ}\, x(k)$

.. using our values $A = -1$, $B = 1$, $Q = 1$, $R = 3.75$

$$\cancel{\bar{P}} = 1 + \cancel{\bar{P}} - \frac{\bar{P}^2}{3.75 - \bar{P}} \Rightarrow \frac{-\bar{P}^2 + 3.75 - \bar{P}}{\cancel{3.75 - \bar{P}}} = 0 \rightsquigarrow \bar{P}_{1,2} = \begin{cases} 2.5 = \bar{P}_1 \\ \cancel{-1.5 = \bar{P}_2} \end{cases}$$

↑

notice [P] should by symm + pos. def!

←

$\bar{P} = 2.5$

from that we can compute K_{LQ}, this is on riccati equation (*)

↓

$K_{LQ} = -0.4$

3) Design a K.F $\rightarrow$ for K.F we need $\tilde{R}, \tilde{Q}$ ...

K.F design assuming knowing disturbance,

$$\begin{cases} V_x \sim WN(0, 1) - \tilde{Q} \\ V_y \sim WN(0, 2) - \tilde{R} \end{cases}$$

given from exercise test

{ conditions on K.F }

1) (A, C) Observable $M_0 = [C] = 1$ ✓

2) (A, B_q) Reachable $M_R = [B_q] = 1$ ✓ → K.F can be applyed!

with $B_q^T B_q = \tilde{Q}$

as done previously... the dynamic of K.F is:

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L_{KF}(y(k) - \overbrace{\hat{y}(k)}^{= C\hat{x}(k)})$$

→ L_{KF} computed using R.E. with analogy between

LQ	⟷	K.F
Q	|	$\tilde{Q}$
R	|	$\tilde{R}$
A	|	A^T
B	|	C^T
K	|	L^T

steady state R.E for K.F

$$\tilde{P} = \tilde{Q} + A\tilde{P}A^T - A\tilde{P}C^T(\tilde{R} - C\tilde{P}C^T)^{-1}C\tilde{P}A^T$$

substitute our values..

we obtain:

$$-\tilde{P}^2 + 2 + \tilde{P} = 0 \quad \begin{cases} \tilde{P}_1 = 2 \\ \cancel{\tilde{P}_2} = -1 < 0 \end{cases}$$

NO, we want pos. def

⇓

$L_{KF} = -0.5$

KF Gain from Riccati eq

4) Poles of LQG loop $\rightarrow$ closed loop poles after K.F + pole placement syst

In view of the separation principle, the poles of the closed loop system are the poles of LQ + + the poles of K.F ⇝ easy in our SISO IORD system

$(A - BK_{LQ})$: Pole of LQ $\Rightarrow$ $p_{LQ} = -1 + 1(0.4) = -0.6$ asymp stable $|p|<1$

$(A - L_{KF} C)$: Pole of KF $\Rightarrow$ $p_{KF} = -1 - (-0.5) = -0.5$

$\downarrow$

poles of closed loop system $\{-0.5, -0.6\}$

Jan 2022

1) In Lyapunov stability analysis, for a given eq. point, convergence and stability are equivalent:

~~o~~ NO

2) In MPC, presence of constraints requires the use of slack variables in order to deal with:

~~o~~ for states and output constr.

3) concerning the $\underbrace{\text{small gain theorem}}$:

- sufficient condition (NOT 1)
- possibility to study I/O stability of NC systems (NOT 2)

~~o~~ (3) allows one to study I/O stability of feedback connection systems required to be I/O stable

↓ not more powerfull than others criterion

4) concerning the EKF

- its dynamic is linear NO → used for non LIN syst!
- its gain is computed with linearization around the nominal state traj
- its gain is time inv. NO → computed ∀t using Riccati eq.
- ~~o~~ its gain is computed with the lin. around the most recent state estim.

Feb 2022

1) consider $G(s) = \left[\frac{s-1}{s+1} \;,\; \frac{s-2}{s+2} \right]$

⇓ select the WRONG answer!

↓ for sure you have poles in $-1, -2$ (multiplicity Boh!?)

- system has a pole in $s=-1$ TRUE
- ~~system has an invariant zero in $s=1$~~
- syst. does NOT have inv. zeros TRUE
- possible to design reg. with integral action on the output error
  (I can design a reg with any technique!)

2) Given a SISO or MIMO syst. with one or more (inv.) zeros with positive real part, select CORRECT case

-
- ~~the zeros limit the performance achievable by closed loop syst~~
-
- Any corresponding closed-loop syst cannot have any rob.. NO

3) consider

$$\dot{x}(t) = Ax + Bu + Md$$
$$y(t) = Cx(t) + d$$

With n states, m input, p outp., v disturb. constant but unkn.

-
- ~~It is possible to design an estimator of the disturbance d provided the $p \geq v$ and the pair (A, C) is obs~~

4) concerning the Loop Transfer Recovery procedure, WRONG?

- NOT for discrete time syst (V)
- can be used ONLY for square syst (V)
- IF applicable, guarantees that loop TF with state feedback plus

January 2022

Exercise 1 (3 marks)

In the Lyapunov stability analysis, for a given equilibrium point, convergence and stability are equivalent (select the correct answer).

- ○ Yes, provided that in the definition of Lyapunov stability the value of δ is sufficiently large
- ○ Yes
- ○ Yes, but only for asymptotic stability
- ○ No
- ○ no answer

Exercise 2 (3 marks)

In Model Predictive Control, the presence of constraints requires the use of slack variables in order to deal with (select the correct answer)

- ○ only for state constraints
- ○ the requirement of integral action in the controller
- ○ only for input constraints
- ○ for state and output constraints
- ○ no answer

Exercise 3 (3 marks)

Concerning the small gain theorem, select the correct answer

- ○ it provides necessary and sufficient conditions for the I/O stability of the feedback interconnection of linear and nonlinear systems.
- ○ it allows one to study the I/O stability of the feedback connection of systems required to be linear.
- ⊗ it allows one to study the I/O stability of the feedback connection of systems required to be I/O stable.
- ○ In the case of feedback connection of asymptotically stable linear systems, it is more powerful than the Nyquist criterion, in the sense that its applicability is wider.
- ○ no answer

Exercise 4 (3 marks)

Concerning the Extended Kalman filter, select the correct answer

- ○ its dynamics is linear
- ○ its gain is computed with the linearization around the nominal state trajectory
- ○ its gain is time invariant
- ⊗ its gain is computed with the linearization around the most recent state estimate
- ○ no answer

February 2022

Exercise 1 (3 marks)

Consider the system described by the following transfer function matrix and select the **WRONG** statement

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix}$$

- The system has a pole in *s=-1*
- The system has an invariant zero in *s=1*
- The system does not have invariant zeros
- For this system it is possible to design a regulator with integral action on the output error
- No answer

Exercise 2 (3 marks)

Given a SISO or MIMO system with one or more (invariant) zeros with positive real part, select the CORRECT statement

- In the design of a feedback regulator, the unstable zeros can be canceled by regulator poles
- The zeros limit the performance achievable by a closed-loop system
- It is impossible to stabilize the system
- Any corresponding closed-loop system cannot have any robustness property with respect to additive or multiplicative perturbations
- No answer

Exercise 3 (3 marks)

Consider the following system

$$\dot{x}(t) = Ax(t) + Bu(t) + Md$$
$$y(t) = Cx(t) + d$$

with *n* states, *m* inputs, *p* outputs, *v* disturbances assumed to be constant but unknown. For this system select the CORRECT statement.

- It is not possible to design an estimator of the disturbance *d*.
- It is possible to design an estimator of the disturbance *d* provided that $m \geq p$.
- It is possible to design an estimator of the disturbance *d* provided that $m \geq p$ and the pair *(A,C)* is observable.
- It is possible to design an estimator of the disturbance *d* provided that $p \geq v$ and the pair *(A,C)* is observable.
- no answer

Exercise 4 (3 marks)

Concerning the Loop Transfer Recovery procedure, select the WRONG statement

- ○ Cannot be used for discrete time systems. ✓ CORRECT
- ○ Can be used only for square systems. ✓ CORRECT
- ○ When applicable, guarantees that the loop transfer function with state feedback plus observer is equal to the one with state feedback only at any frequency.
- ○ Can be used only for systems with invariant zeros with negative real part.
- ○ No answer

June 2022

Exercise 1 (3 marks)

Consider the discrete-time system

$$x(k+1) = -x(k)cos^2(x(k))$$

Using a quadratic Lyapunov function, select the correct answer among the following ones

- ○ The origin is an unstable equilibrium
- ○ The origin is a locally stable equilibrium as we compute $\Delta V(x) < 0$ locally! (strictly < 0) $\Longrightarrow$
- X The origin is a locally asymptotically stable equilibrium
- ○ The origin is a globally asymptotically stable equilibrium
- ○ No answer ↑ No global result!

Exercise 2 (3 marks)

Concerning the use of the Kalman Predictor or Filter for the continuous time system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_x(t) \\ y(t) = Cx(t) + v_y(t) \end{cases}$$

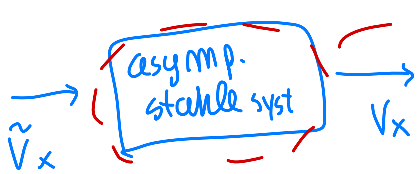
Select the correct answer

uncorrelated noises assumption only to simplify computation

- ○ It can be applied only for asymptotically stable systems (NOT required)
- ○ It can be used only if v_x and v_y are uncorrelated white gaussian noises (NOT true!)
- X It can be used also when v_x is a stationary stochastic process (with suitable modifications)
- ○ The only condition required to guarantee that the covariance of the state estimation error tends to a limiting value is that the pair (A,C) is observable
- ○ No answer

because of spectral factorization theorem

dynamically can shape filter behaviour



JUNE 2022

1

$$x(k+1) = -x(k)\cos^2(x(k))$$

looking steady state $\bar{x} = -\bar{x}\cos^2(\bar{x})$

With a quadratic Lyap. function

$$V(x) = x^2 > 0$$

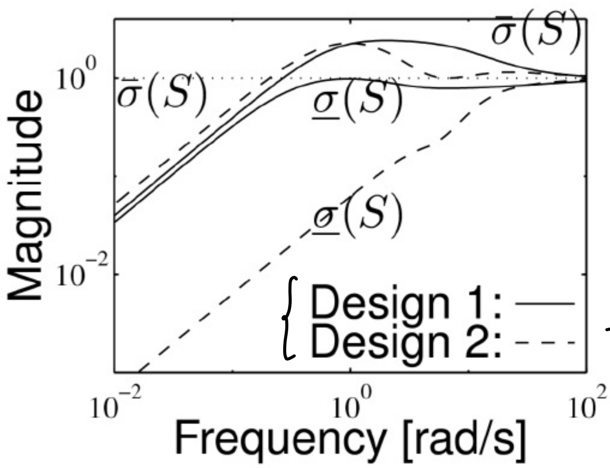
with $\Delta V(x) = \left(-x^2\cos^2(x)\right)^2 - x^2 =$

$$= \underbrace{(\cos^4(x) - 1)}_{<0 \text{ near the origin}} x^2 < 0 \text{ locally !}$$

we cannot say unstable !

Exercise 3 (3 marks)

Consider two control designs of a closed-loop system with the following sensitivity functions and select the answer that is surely true



from here we can observe that.. in MIMO system

(you can conclude looking on Bode for MIMO? NO!

- Design 1 guarantees a faster closed-loop system
- Both designs guarantee closed-loop stability
- Design 1 guarantees slightly more attenuation of high frequency measurement noise
- ~~Design 1 guarantees slightly more attenuation of low frequency process disturbances (d_y)~~
- No answer

looking to attenuation, you look to complementary sensitivity, NOT good

↳ looking to the largest singular value... this is the right answer

FOR SURE

Exercise 4 (3 marks)

In Model Predictive Control of linear systems, it is possible to guarantee robust zero steady state error regulation for constant reference signals y^o also in the case of small model uncertainties or neglected disturbances (assuming that the resulting closed-loop system is asymptotically stable) provided that

- In the cost function to be minimized it is weighted the difference between the predicted state and its asymptotic value corresponding to y^o, in nominal conditions, and the difference between the future input and its asymptotic value corresponding to y^o in nominal conditions (necessary and sufficient condition)
- The prediction horizon is chosen sufficiently long
- A model in $\delta x(k)=x(k)-x(k-1)$ and $\delta u(k)=u(k)-u(k-1)$ is used
- The prediction horizon is selected longer than the control horizon
- No answer

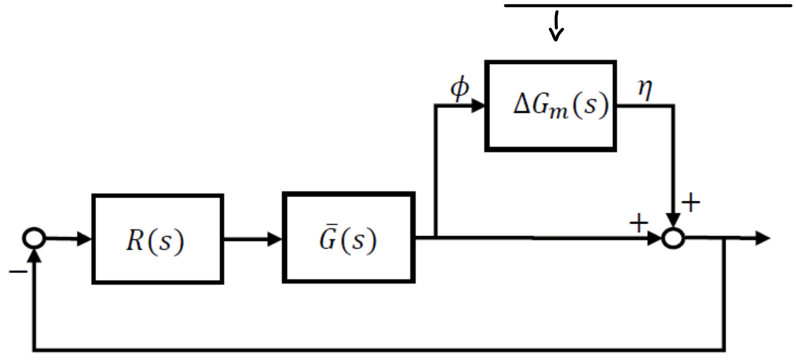
given a pair $(\bar{u}, \bar{x}) \rightarrow \bar{y} = y^o$

$$J = \sum_{k=0}^{N-1} (y^o - y(k+1))^2 + \rho (u(k+1) - \bar{u})^2 + g (y(k+N) - y^o)^2$$

↳ $(x(k+1) - \bar{x})^2 \rho$

Exercise 5 (3 marks)

Consider the following control system with multiplicative uncertainty



where

$$\bar{G}(s) = \frac{m}{1+as},\ a > 0,\ R(s) = \frac{k(1+as)}{s},\ \Delta G_m(s) = g$$

Select the sufficient condition required to guarantee the stability of the overall system

- X $\|\frac{mgk}{jw+mk}\|_\infty<1$
- ○ $\|\frac{gk(1+jaw)}{jw+mk}\|_\infty<1$
- ○ $|g| < 1$
- ○ $\|\frac{mk}{jw}\|_\infty<1$
- ○ No answer

dealing with multiplicative uncertainty, sufficient condition for asymp. stability is that

$$|T(j\omega)\cdot\Delta G_m(j\omega)| < 1 \ \forall \omega$$

with $T(s) = \frac{L(s)}{1+L(s)}$ $\quad L(s) = R(s)\cdot\bar{G}(s) = \frac{m}{1+as}\cdot\frac{k(1+as)}{s} = \frac{mk}{s}$

$$\left\|\frac{mkg}{j\omega+mk}\right\|_\infty < 1$$

from condition based on stability

July 2022

Exercise 1 (3 marks)

Concerning H_{inf} control, select the **wrong** statement among the following ones:

this concern H_2 control

- X LQG is a particular form of H_{inf} control,depending on the choice of the design parameters
- ○ The structure of H_{inf} and H_2 regulators is made by a state feedback control law and a state observer
- ○ The use of shaping functions is required to impose the form of the sensitivity functions TRUE
- ○ the shaping functions at the process inputs or at the process outputs must be asymptotically stable obvious TRUE looking the shaping function!
- ○ No answer

Exercise 2 (3 marks)

Consider the following discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u(k)}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

you can compute T.F
$u \to y_1$ (derivative term)
$u \to y_2$

NOT bring y_1 to equilibrium..

which output/s can be asymptotically track a given constant reference signal?

- ○ None of them
- ○ The first one y_1 → you can bring y_2 to proper value BUT not y_1!
- X The second one y_2
- ○ Both of them → for sure we can remove both of them!
- ○ No answer

$$\begin{cases} Y_1(z) = \dfrac{(-z+1)}{\quad} u(k) \\ Y_2(z) = \dfrac{1}{z^2 - z + 0.25} u(k) \end{cases}$$

derivative action you cannot use u!

SOLUTION

$\boxed{P}$

Exercise 3 (3 marks)

Consider a nonlinear discrete time system $x(k+1)=f(x(k),u(k))$, an equilibrium point, and the corresponding linearized system $\delta x(k+1)=A\delta x(k)+B\delta u(k)$. Given the Lyapunov equation

$$A'PA-P=-Q$$

with $Q>0$, it results $P>0$. Then

on discrete time lyap $\Downarrow$

- ○ Nothing can be said on the stability of the equilibrium since it should be checked that for all the possible $Q>0$ one has $P>0$ WRONG interpretation of lyap result!
- X The equilibrium is locally asymptotically stable
- ○ The equilibrium is globally stable impossible to say from linearized syst.
- ○ The equilibrium is globally asymptotically stable
- ○ No answer

Exercise 4 (3 marks)

The poles of the following system are the system

computation..
forsline at
least pole in
-1, -3, -2 , look
to minors of order 2
to check multiplicity

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+3} \\ \dfrac{s-1}{(s+2)(s+3)} & \dfrac{1}{s+3} \\ \dfrac{s+2}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix}$$

- ○ Poles: s=-1 (double) s=-2, s=-3
- ○ Poles: s=-1 (triple) s=-2, s=-3
- ○ Poles: s=-1 (double) s=-2, s=-3 (double)
- ☒ Poles: s=-1 (triple) s=-2, s=-3 (double)
- ○ No answer

{ after computation }

September 2022

Exercise 1 (3 marks) **SELECT THE CORRECT ANSWER**

Consider the continuous-time linear system:

$$\dot{x}(t) = Ax(t)$$

and assume that $A+A'$ is definite negative. Then, with the Lyapunov theory it is possible to conclude

- ○ Nothing about the stability of the system.
- ○ The system is stable
- ☒ The system is asymptotically stable
- ○ The system is unstable
- ○ No answer

$A'P + PA = -Q$

$P = I$

$A' + A = -Q$

asymp stable

Exercise 2 (3 marks) **SELECT THE CORRECT ANSWER**

The zeros of continuous or discrete time systems are important because:

- ○ They influence the stability of the system NO
- ○ They influence the static and dynamic performances which can be achieved with the design of a closed-loop system. WRONG
- ○ They influence only the static performances which can be achieved with the design of a closed-loop system. NO, we have seen how unstable zeros influence performance
- ✗ They influence the possibility to design LQ or LQG control laws.
- ○ No answer

Exercise 3 (3 marks) **SELECT THE CORRECT ANSWER**

The design of a state-feedback pole placement control law for a reachable multi-input system

- ○ Can be always completed only by using all the inputs → NO, you can also use only one input!
- ○ Can be always completed by using only one input → NOT always
- ✗ Can be completed using one or more inputs, it depends on the specific problem
- ○ Can be completed only if the system does not have zeros ↑ in general true assumption
- ○ No answer

Exercise 4 (3 marks) **SELECT THE CORRECT ANSWER**

Imposing the terminal constraint $x(k+N)=0$ in the formulation of MPC to guarantee stability

- ○ Can be made only for unconstrained problems NOT TRUE!
- ○ Can be made, but it is then impossible to use the Receding Horizon approach and all the sequence of future computed control moves $u(k),\ldots,u(k+N-1)$ must be used
- ✗ Can be made, provided that the state $x(k)$ at the current time instant k must be in the set of states where a solution exists
- ○ Can only be made for linear systems
- ○ No answer

Exercise 5 (3 marks) **SELECT THE CORRECT ANSWER**

Concerning the Infinite Horizon LQ control for discrete time systems:

- ○ It guarantees at the same time gain and phase robustness margins
- ○ It guarantees either gain or phase robustness margins
- ✗ It guarantees gain margins smaller than in continuous time — LQ control in descrete time has limited property!
- ○ It guarantees gain margins smaller, or equal, or larger than in continuous time
- ○ No answer

Es 1. discrete time Lyapunov

$$\begin{cases} x_1(k+1) = x_2(k)\cos(x_1(k)) \\ x_2(k+1) = x_1(k)\cos(x_2(k)) \end{cases} \rightsquigarrow$$

using a quadratic Lyap as usual for discrete syst.

$$V(x) = x_1^2(k) + x_2^2(k)$$

$$\downarrow$$

$$\Delta V(x) = x_1^2(k+1) + x_2^2(k+1) - x_1^2 - x_2^2 =$$

$$= x_2^2\cos^2(x_1) + x_1^2\cos^2(x_2) - x_1^2 - x_2^2 =$$

$$= \underbrace{x_1^2}_{>0}\underbrace{[\cos^2(x_2)-1]}_{<0 \text{ near origin}} + \underbrace{x_2^2}_{>0}\underbrace{[\cos^2(x_1)-1]}_{<0 \text{ near origin}} \leq 0$$

near the origin!

(semi def neg)!

FROM HERE STUDY TRAJ

$\Delta V(x) \leq 0$ for a small perturb $= 0$

in the neighbourghr of the origin:

$\Downarrow$

~~locally asymp. stable equilibrium~~!

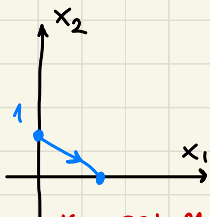
GLOBAL? → not with KRAKOSWI-LESELLE → you should loop the trajectory

taking $\begin{cases} x_1(0) = 0 \\ x_2(0) = 1 \end{cases}$ (mooving outside the origin) := when $\cos^2 - 1 = 0$

from syst. equation

$$\begin{cases} x_1(1) = 1 \\ x_2(1) = 0 \end{cases} \rightarrow \begin{cases} x_1(2) = 0 \\ x_2(0) = 1 \end{cases}$$

back on the initial condition



NOT Globally asymp!

only local stability

- using the linearization approach

$$\begin{cases} \delta x_1(k+1) = \cancel{-\bar{x}_2\, 2\sin(\bar{x}_1)\,\delta x_1(k)} + \cos(\bar{x}_1)\,\delta x_2(k) \\ \delta x_2(k+1) = \cos(\bar{x}_2)\,\delta x_1(k) \cancel{-\bar{x}_1\sin(\bar{x}_2)\,\delta x_2(k)} \end{cases}$$

on the origin

$$\begin{cases} \delta x_1(k+1) = \delta x_2(k) \\ \delta x_2(k+1) = \delta x_1(k) \end{cases} \rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \text{eig}(A) := \det(zI - A) =$$

$$= \begin{vmatrix} z & -1 \\ -1 & z \end{vmatrix} = z^2 - 1 = 0$$

$\{ z = \pm 1 \}$ → Nothing can be concluded! eig on the border of stability region!

→ linearization don't give you any answer!
while Lyap give local result

ES 2, Theory QUESTION

- on SISO, you have just one Bode
  while on MIMO, system gain is inside a certain range causing difficulties

ES3

study a 1ORD syst $\dot{x} = \overbrace{[1 \quad 2]}^{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + d$ — 2 input

$A = 0$

1) assume $d = 0$ → compute LQ controller with

$\begin{cases} Q = 4 \\ R = 5 I_2 \end{cases}$ D.R.E : $\dot{P}_t + Q - P_t B R^{-1} B^T P_t^T + P_t A + A^T P_t^T = 0$

∞ horizon LQ Controller ⇒ $\dot{P}_t = 0$, solve for $\bar{P}$ constant matrix!

$Q - \bar{P} B R^{-1} B^T \bar{P} = 0$ ($\bar{P}$ $N \times N$ sym matrix → $N=1$ → $\bar{P}$ scalar)

↳ $4 - \bar{P}^2 = 0$ ⇒ $\bar{P} = \begin{cases} +2 & \text{take pos. def value} \\ \cancel{-2} & \text{not neg value} \end{cases}$

with $\bar{P} = +2$ we can compute K

$$K = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2 = \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$$

so I can compute the closed loop eig value by using

$u = -Kx$ as control law

⇓ SYSTEM

$\dot{x} = (\cancel{A}^{0} - BK)x =$

$= -[1 \quad 2] \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix} x = -2x$, ← stabilized system!

(with #input > #state easy..)

2) how to estimate d assuming it is constant

(full / reduced order observer)

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + d \\ \dot{d} = 0 \\ y = x \end{cases} \longrightarrow \begin{bmatrix} \dot{x} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = x$$

$\Downarrow$ using a Full order observer!

IF reduced order OBS instead
n states, p outputs

you should make a transformation such that ? BHO.

$$\left(\tilde{x} = Tx, \ \tilde{x} = \begin{bmatrix} y \\ \tilde{x}_{red} \end{bmatrix} \right\} n-p$$

$\downarrow$

$$\begin{cases} \dot{d} = 0 \\ \underbrace{\dot{y} - u_1 - 2u_2}_{\text{new output transformation}} = d \end{cases}$$

$\dot{y}$ can be hard (not accurate) to compute
you mimic

$$\dot{\hat{d}} = L\big[\underbrace{\dot{y} - [1 \quad 2]u}_{\eta} - \hat{d}\big]$$

$\downarrow$

$$\dot{\hat{d}} - L\dot{y} = -L\hat{d} - L[1 \quad 2]u$$

Now we should compute L to obtain a stable reduced order observer!

then redefine
$\hat{d} - Ly \simeq \xi$ to get a better represent.

$$\dot{\xi} = -L\hat{d} - L[1 \ 2]u \pm L^2 y = \dots$$

$$\dot{\xi} = -L\hat{d} - L[1\ \ 2]u \pm L^2 y = -L\underbrace{\xi}_{L(\hat{d}-Ly)} - L^2 y - L[1\ \ 2]u$$

project completed!

$\dot{\xi} = ...$ observer of order 1 (reduced)

known terms here, depending on y, u to compute ξ → $\xi = d - Ly$

allow to estimate d

on that exercise we want to find an observer BUT we found a reduced order observer JUST to cry more

Es. 4

Consider a system with step responce coefficients:

$s_1 = 0.25$

$s_2 = 0.75$

$s_3 = s_4 = 1$

→ 1) derive step impulse responce

$g_i = s_i - s_{i-1}$ ⟹ $g_1 = 0.25$

$g_2 = 0.5$

$g_3 = 0.25$

$g_4 = 0$

$\dots\ g_i = 0 \quad i \geq 4$

2) Show how to build a prediction of $y(k+1)$, $y(k+2)$

usefull for MPC

for STABLE SYST

here 0

remember that
$$\begin{cases} y(k+1) = g_1 u(k) + g_2 u(k-1) + g_3 u(k-2) + \cancel{\dots} \\ y(k+2) = g_1 u(k+1) + g_2 u(k) + g_3 u(k-1) + \cancel{\dots} \end{cases}$$

3) prediction of $y(k+1)$, $y(k+2)$ using step responces

knowing $g_i = s_i - s_{i-1}$ $\quad y(k+1) = g_1 u(k) + g_2 u(k-1) + g_3 u(k-2) =$

$$= (s_1 - \cancel{s_0})u(k) + (s_2 - s_1)u(k-1) + (s_3 - s_2)u(k-2) + \overset{0}{\dots} =$$

$$= s_1 \underbrace{(u(k) - u(k-1))}_{\delta u(k)} + s_2 (u(k-1) - u(k-2)) + s_3 u(k-2) =$$

writing it in $\delta u(k) \dots$ →

$$y(k+1) = s_1 \delta u(k) + s_2 \delta u(k-1) + s_3 u(k-2)$$

not a variation

from Imp. responce to step resp, we express in δu

this expression for an MPC feedback controller → NOT a real feedback

so you consider an additional disturbance on the system

from this approach...

$$\begin{cases} d(k) = y(k) - s_1 \delta u(k-1) - s_2 \delta u(k-2) + \\ \quad - s_3 \delta u(k-3) \ldots \\ d(k+i) = d(k) \end{cases}$$

assume constant in the future

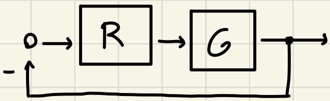
$$y(k+1) = s_1 \delta u(k) + s_2 \delta u(k-1) + s_3 \cancel{u(k-2)} + d(k)$$

which cancel out this term..

so it appear $y(k)$ and you have a feedback system!

{ passage to remember! }

ES.



$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-0.5} \\ 0 & \frac{1}{s+2} \end{bmatrix}, \quad R(s) = \begin{bmatrix} 1 & 1 \\ 0 & -\frac{s-0.5}{s+1} \end{bmatrix}$$

$\downarrow$ compute poles and invariant zeros

$G(s)$ poles, for sure in $s=-1$, $s=-2$, $s=0.5$ POLE of $G(s)$ (*)

multiplicity?

$$\det G(s) = \frac{1}{(s+1)(s+2)} = \uparrow \text{ ok, all multiplicity } 1$$

$\Downarrow$

to compute inv. zeros $= \dfrac{(s-0.5)}{(s+1)(s+2)(s-0.5)} \longrightarrow$ ZERO in $s=0.5$

for regulator $R(s)$: we have pole in $-1=s$

$$\det(R(s)) = -\frac{s-0.5}{s+1}$$ ZERO in $s=0.5$ (*) ZERO of $R(s)$

unstable!

But we are NOT sure that there is a cancellation

You should study singularities of $\leftarrow$

$R(s)G(s)$, $G(s)\cdot R(s)$ to check if there are cancellations

$\Downarrow$

$$G(s)R(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & -\frac{s-0.5}{(s+1)(s+2)} \end{bmatrix}$$

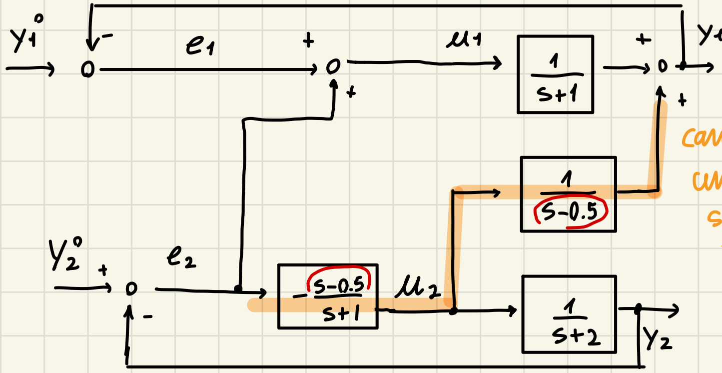
No more pole in 0.5,

|| cancellation! ||

so we can study the system, what is happening?

looking at $\Longrightarrow$

IF you see the system as:



cancellation of unstable singularities through this path!

↓

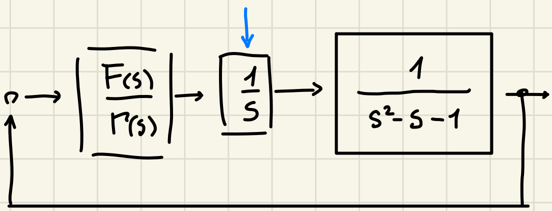
{ look the example to check cancellation }

NOT full order controller used

Es. how to add integral action to my system?

Knowing the total order of the system $n = 3$ enlarged

$\frac{1}{z-1}$ OR $\frac{z}{z-1}$ for discrete time syst



$$F(s) = f_2 s^2 + f_1 s + f_0$$

$$\Gamma(s) = \gamma_2 s^2 + \gamma_1 s + \gamma_0$$

↓

$$P(s) = s^5 + p_4 s^4 + \dots + p_1 s + p_0$$

$$A(s) = s^3 - s^2 - s$$ (desired den!)

write problem as linear syst

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}^{A(s)}_{\underbrace{}_{n}\ \underbrace{}_{num.}} \begin{bmatrix} \gamma_2 \\ \gamma_1 \\ \gamma_0 \\ f_2 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix}$$

⟶ solve to find γ, f

ES. EKF

delay $K+1$, $K-1 \to$ 2 ORDER, order given by max delay

$$\begin{cases} z(k+1) = u(k) + \alpha z^2(k-1) + v_x(k) \\ y(k) = z(k) + v_y(k) \end{cases}$$

$v_x \sim WN(0,1)$

$v_y \sim WN(0,1)$

$\downarrow$ EKF to estimate also α!

$\Downarrow$

NOT in standard form!

FIRST put in normal form

then apply standard K.F

$$\begin{cases} x_1(k) = z(k) \\ x_2(k) = z(k-1) \end{cases} \longrightarrow \begin{cases} x_1(k+1) = u(k) + \alpha x_2^2(k) + v_x(k) \\ x_2(k+1) = x_1(k) \end{cases}$$

$$y(k) = x_1(k) + v_y(k)$$

add an additional state $x_3(k) = \alpha(k)$

$\Downarrow$

$$\begin{cases} x_1(k+1) = u(k) + x_3(k)\, x_2^2(k) + v_x(k) \\ x_2(k+1) = x_1(k) \\ y(k) = x_1(k) + v_y(k) \end{cases}$$

form used to apply EKF

$\Downarrow$

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \tilde{R} = 1$$

$\uparrow$ only v_x which appear on $x_1(k+1)$

$\Rightarrow$ so, to design your Kalman predictor... you can simply mimic system equation..

$\Downarrow$

$$\begin{cases} \hat{x}_1(k+1) = u(k) + \hat{x}_3(k)\,\hat{x}_2^2(k) + \ell_1(k)\left[y(k) - \hat{x}_1(k)\right] \\ \hat{x}_2(k+1) = \hat{x}_1(k) + \ell_2(k)\left[y(k) - \hat{x}_1(k)\right] \\ \hat{x}_3(k+1) = \hat{x}_3(k) + \ell_3(k)\left[y(k) - \hat{x}_1(k)\right] \\ y(k) = x_1(k) + v_y(k) \end{cases}$$

$\hookrightarrow$

you should compute $L(k) = \begin{bmatrix} l_1(k) \\ l_2(k) \\ l_3(k) \end{bmatrix}$ through a linearization approach

compute

$A(k)$ linearizing the system

$$A(k) = \left.\begin{bmatrix} 0 & 2x_2x_3 & x_2^2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|_{\hat{x}_1, \hat{x}_2, \hat{x}_3}$$

{ @ any k $A(k)$ changes, depending on $\hat{x}$ }

$C = [1 \; 0 \; 0]$ linear time invariant

R.E computed $\forall k$ and compute $L(k)$ + implement EKF → NOT steady state

in general $A(k)$ changes $\forall k$

on EKF NOT so convergence results! → but the most used approach for estimation

Im Practice

↳ in cont time EKF can easy implemented using a continuous model of syst + predictor gain discrete time

ES. LQ CONTROL

Given the system:

$$\begin{cases} \dot{x}_1 = b u \\ \dot{x}_2 = x_1 + u \end{cases} \qquad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \qquad R = 1$$

① stabilizable with LQ control?

conditions:

1) (A,B) reachable $\longrightarrow$ with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} b \\ 1 \end{bmatrix}$

$$\Downarrow$$

$$\mathcal{M}_R = \begin{array}{cc} \begin{bmatrix} b & 0 \\ 1 & b \end{bmatrix} \\ B \quad AB \end{array} \iff \text{Reachable } b \neq 0 ! \checkmark$$

(otherwise $\dot{x}_1 = 0$ so take a free value)

2) (A, C_q) observable

$\downarrow$

C_q as partition related to Q

$$C_q = \begin{bmatrix} \sqrt{q_1} & 0 \\ 0 & \sqrt{q_2} \end{bmatrix} \quad \text{OBS. ?}$$

$$\mathcal{M}_O = \begin{bmatrix} C_q \\ C_q A \end{bmatrix} = \begin{bmatrix} \sqrt{q_1} & 0 \\ 0 & \sqrt{q_2} \\ 0 & 0 \\ \sqrt{q_2} & 0 \end{bmatrix} \to \text{condition of obs} \iff q_2 \neq 0 \text{ OBS!} \checkmark$$

(otherwise, x_2 depending on x_1, you should weight x_2 otherwise $x_2 \to +\infty$)

② taking $Q = I$ and $b = 1$, $R = 1$

With Riccati eq solution $\bar{P} = 1$ $\to$ compute K and eig values of closed loop syst.

$$-\dot{P}(t) + A^T P(t) + P(t) A + Q - P(t) B \underbrace{R^{-1} B^T P(t)}_{K(t)} = 0$$

given B, R, P

$$K = 1 \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\Downarrow$$

So you have the following control law: $u(t) = -K x(t) = -\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$$= -x_1 - x_2$$

closed loop eig value:

$$A - BK = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \rightsquigarrow$$ 2 eigen values in $s=-1$

3) consider the control law: $u = -\rho K x$
check stability? (Gain margin)

Theoretically: to guarantee gain margin $\rho \in (+0.5, \infty)$ (cont. time)

While on this case: $(A - B\rho K) = \begin{bmatrix} -\rho & -\rho \\ 1-\rho & -\rho \end{bmatrix}$

$\Downarrow$

$$\det(sI - (A - B\rho K)) = \underbrace{s^2 + 2\rho s + \rho = 0}_{\text{charact. eq.}}$$

to have all same sign, CNS to have all sol on $\mathrm{Re} < 0$ $\Rightarrow$ $\rho > 0$

4) What could you do to guarantee poles of closed loop system inside stability region?

to have all $< -\alpha$ ↷ α design parameter

↓

you can use $\tilde{A} = A + \alpha I$ → not α too big or you require a very fast system

apply LQ control theory!