# POLE PLACEMENT in T.F DOMAIN

## For SISO syst.

even if already I know Nyquist ecc.. This procedure is really simple, simpler than apply Bode Th

# Advanced and Multivariable Control

only for SISO !!

## *Pole placement control for SISO systems described with transfer functions*

directly on TF domain, for SISO systems

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sometimes better than work on Bode Plot

## Problem statement

design R(s) for SISO by pole placement from TF representation

Given a SISO system described in transcribed function form

PLANT T.F description

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

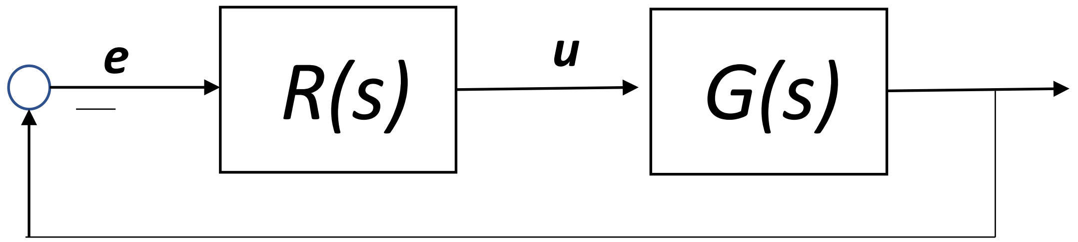
We want to find a regulator described by the transfer function

computing R(s) such that close loop system has poles where we want

prescribed

$$\boxed{R(s) = \frac{F(s)}{\Gamma(s)}}$$

design R(s)




Such that the closed-loop system has prescribed poles ! ⇒ (remaining on T.F domain)

## Remarks

- The problem is solvable if and only if the polynomials $\boxed{A(s)}$ and $\boxed{B(s)}$ do not have common factors, i.e. the system is in minimal form (recall that ***G(s)*** only describes the reachable and observable part of the system)

mooving to s.s

- The solution to this poblem can be trivially obtained by using a state space representation of the system, for instance with the reachability canonical form, and then by using the theory previously introduced

- Here we want to present a simpler solution for SISO systems, which is computationally non intensive and highly flexible previously we solve PROBLEM as :

$$\begin{cases} u = -K\hat{x} & \text{order } Q \\ \hat{x} = Ax \ldots & \text{order } n \end{cases}$$

$\downarrow$

overall $R(s)$ is of order $n$ IF full order OBS

BUT using a reduced order regulator $\leftarrow$

$n - p \rightarrow n-1$ $(p=1 : \text{SISO})$

$\Downarrow$

So combining OBS $(n-1)$ + state feedback contr law $\Big\}$ R(S) ORDER $n-1$

---

$$G(s) = \frac{B(s)}{A(s)} \quad \text{order } (n)$$

solve problem by $R(s) = \dfrac{F(s)}{\Gamma(s)}$ order $(n-1)$ $\Big\{$ reduced OBS, static feedback CL

$\Downarrow$

$P(s) = FB + A\Gamma \rightarrow$ ORDER $(n + n - 1) = 2n-1$

closed loop Syst. char. polyn

(directly from T.F representation) to assign the closed loop eig values...

(you can also consider Reg of order bigger than n-1)

## Main results

min Reg order is n-1

pole placement solved by state feedback control law + observer, so of order n overall

$u = -Kx$ + ORDER n OBS

The problem can be solved with a regulator of minimal order ***n-1*** (***n*** is the order of the system)

PROCESS T.F

$$G(s) = \frac{B(s)}{A(s)}$$

REGULATOR

$$R(s) = \frac{F(s)}{\Gamma(s)} = \frac{f_{n-1}s^{n-1}+f_{n-2}s^{n-2}+\ldots+f_1 s+f_0}{\gamma_{n-1}s^{n-1}+\gamma_{n-2}s^{n-2}+\ldots+\gamma_1 s+\gamma_0}$$

IF using R(s) of reduced order, also OBS is of order n-1

same problem solved by more tech.

POLE placement solution ⇕ equivalent

The order ***n-1*** corresponds to using a ***static feedback control law + a reduced order observer***. It is obviously possible to use a regulator of higher order

$u = -K\hat{x}$ → $\hat{x}$ given by an OBS of order n-1 ⇒ so overall (n-1) R(s) ORDER

The characteristic polynomial of the closed-loop system is

find $f_i, \gamma_i$ such that closed loop syst has desired poles...

polynomial equation

$$A(s)\Gamma(s)+B(s)F(s) = P(s)$$

⇐ ***Diophantine equation***

OF order 2n-1 ⇒ plant of order (n) * (n-1) reg. order

The polynomial ***P(s)*** is of order ***2n-1*** its, roots are the ***desired closed-loop poles***, i.e. the ***free design parameters***

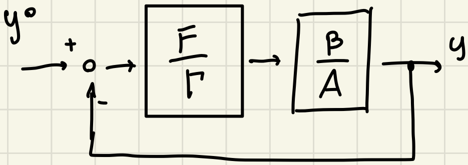
We should be able to find F(s), Γ(s) (with fixed order) such that closed loop poles on desired positions

$$P(s) = s^{2n-1}+p_{2n-2}s^{2n-2}+\ldots+p_1 s+p_0$$

We have (2n-1) closed loop poles

SPECIFICATION: specify the 2n-1 desired roots!

specify P(s) polynomial with desired roots!




$$y = \frac{FB}{A\Gamma + BF} y^\circ$$

Solution by computing left = right side →

to SOLVE the problem..

The solution to the Diophantine equation is obtained by making equal the coefficients of the powers of **s** at the left and right hand side. This corresponds to th solution of a set of linear equations

we know $A(s)$, $B(s)$, we select $P(s)$ polynomial (by our eig request), we know $F(s)$, $\Gamma(s)$ order $(n-1)$ with given coeff

Example

$$\begin{cases} A(s) = s^2 + a_1 s + a_0 \\ B(s) = b_1 s + b_0 \end{cases} \longrightarrow n=2 \longrightarrow \begin{cases} F(s) = f_1 s + f_0 \\ \Gamma(s) = \gamma_1 s + \gamma_0 \end{cases}$$

(II ORD syst.)

ORDER $n-1=1$

we can solve with $F, \Gamma$ of order $n-1=1$

$$P(s) = s^3 + p_2 s^2 + p_1 s + p_0$$

$2n-1=3$ order of $P(s)$

IF I'm able to choose $F, \Gamma$ Coefficents $f_i, \gamma_i$ such that $\forall$ power of $s$ we have requested eig val force the solution one by one @any power of s

$$A(s)\Gamma(s) + B(s)F(s) = P(s)$$

fixed $p_i$ to position the poles where we want

find $f_i, \gamma_i$ to have equivalent right/left hand side of eq. → same coefficents

$$\gamma_1 s^3 + (\gamma_0 + a_1\gamma_1 + b_1 f_1)s^2 + (a_1\gamma_0 + a_0\gamma_1 + b_1 f_0 + b_0 f_1)s + (a_0\gamma_0 + b_0 f_0) = P(s)$$

MATRIX Form equation...

by comparing equal same orders...

(Solution of pole placement) ||| solution of a system of linear equation → find $R(s)$ to fix close loop eig val.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix}$$

4 unknowns to find

linear system solution to find $\gamma, f$ of $R(s)$ coefficents

SIMPLE solution, loosing freedom, interpretation! BUT powerfull

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix}$$

depending on our pole design request

that MATRIX of coeff has a particular form.

writing the vector of unkn from $\gamma$ in descending power $\gamma_m, \gamma_{m-1}, \dots, \gamma_0, f_m, f_{m-1}, \dots, f_0$

- I ROW) has coeff of a
- II ROW) scale 0 .. a until you complete n columns
- ... III ROW) b coefficents 0 .. $b_1$ $b_0$
- IV ROW) shift !

↳ STRUCTURE come from coeff of A, B (G(s))
NOT needed computation!

PROBLEM equivalent to solve set of equations...

In general, the set of linear equations to be solved is → standard structure of system... to solve

for a generic system of order n...

strictly proper syst..

n coeff for γ

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{n-1} & 1 & 0 & \dots & 0 & b_{n-1} & 0 & \dots & 0 \\
a_{n-2} & a_{n-1} & 1 & \dots & 0 & b_{n-2} & b_{n-1} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_0 & a_1 & a_2 & \dots & 1 & b_0 & b_1 & \dots & b_{n-1} \\
0 & a_0 & a_1 & \dots & a_{n-1} & 0 & b_0 & \dots & b_{n-2} \\
0 & 0 & a_0 & \dots & a_{n-2} & 0 & 0 & \dots & b_{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & a_0 & 0 & 0 & \dots & b_0
\end{bmatrix}
\begin{bmatrix}
\gamma_{n-1} \\ \gamma_{n-2} \\ \gamma_{n-3} \\ \vdots \\ \gamma_0 \\ f_{n-1} \\ f_{n-2} \\ \vdots \\ f_0
\end{bmatrix}
=
\begin{bmatrix}
1 \\ p_{2n-2} \\ p_{2n-3} \\ \vdots \\ p_{n-1} \\ p_{n-2} \\ p_{n-3} \\ \vdots \\ p_0
\end{bmatrix}
$$

*2n rows*

SET of lin eq. (2n x 2n matrix)

→ shift and repeat .. until n col. → shift for b,

*n columns* *n columns*

n coeff for f

UNKNOWN → R(s) coefficients

to have solution!!

- The matrix of coefficients (Sylvester matrix) is nonsingular iff ***A(s)*** and ***B(s)*** do not have common factors. However, it should also be well conditioned (no poles and zeros of the system too near each other) ↳ G(s) with NOT same roots! really different roots!
- The equations above can be solved one by one starting from the first one (Nowdays solved by sw)

I → $\gamma_{n-1}$
II → $\gamma_{n-2}$
---
} iterative procedure to solve set of eq

***a very simple design method***

 to design R(s) with INTEGRAL ACTION...

## Regulator with integral action

ADDITIONAL problem when design a regulator!
⇒ IF for example I wanna include INTEGRAL ACTION for STATIC requirem.

In order to include an integral action into the regulator, one can «enlarge» the the system tranfer function by considering the  system with transfer function

consider an equivalent process with integrator

↳ as automatic procedure, how to include INTEGRAL action?

$$\tilde{G}(s) = \boxed{\frac{B(s)}{sA(s)}} \text{ of order } n+1$$

ENLARGED syst.

For this new system, compute the regulator R(s) (now of order n) with the theory described to assign ***2n+1*** poles

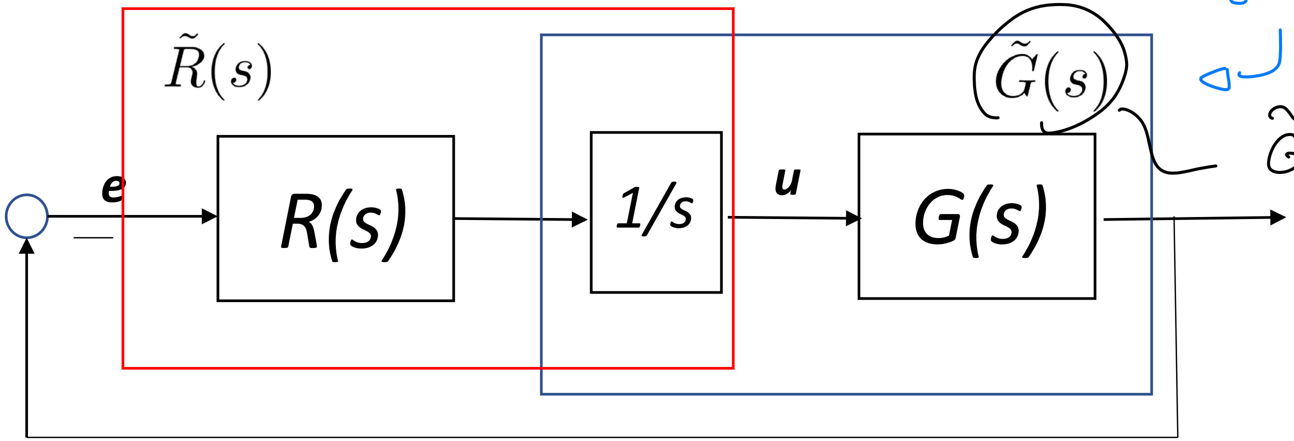
here plant of order n+1 → so R(s) of n

Finally, implement the regulator with transfer function

$$\boxed{\tilde{R}(s) = \frac{1}{s}R(s)}$$

Real Regulator is R(s) × 1/s
computed for $\tilde{G}(s)$

from $G(s) = \frac{B(s)}{A(s)}$
↓
extend to $\tilde{G}(s)$
for new syst.
⇓
@ end you implement
R(s) including the added integrator 1/s




$\tilde{G}$ of order n+1
↓
R(s) of order n at least,
so P(s) of 2n+1

{Easy procedure}

standard $\rightarrow A$ of order $n$

$\Gamma$ of order $n$

$$G(s) = \frac{B}{sA} \rightarrow \text{order } \tilde{n} = n+1$$

$$F, \Gamma \Rightarrow \tilde{n} - 1 = n$$

$$\tilde{p} = (2\tilde{n} - 1) = 2n + 2 - 1 = 2n + 1$$ {order of the new $P(s)$}

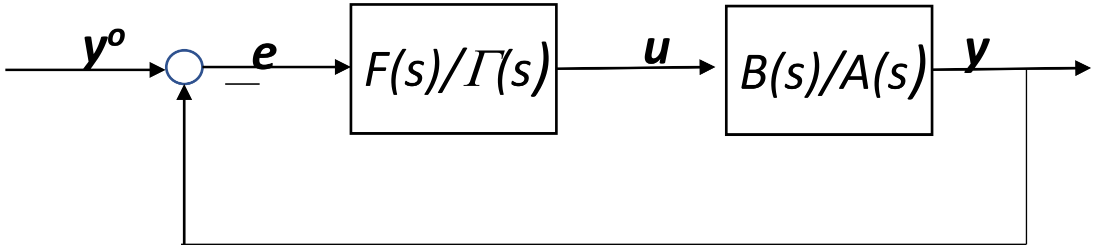
## Zeros of the closed-loop system

another critical point for this procedure is ...
once computed $F(s)/\Gamma(s)$ guarantee $P(s)$ polynomial closed loop...

The roots of the polynomial ***F(s)*** are not design parameters, and the algorithm can set them to values which can seriously deteriorate the closed-loop performance. Remember that «unstable zeros» of ***nonminimum phase*** systems, produce the so-called ***inverse response***

automatic procedure to find $F(s)$, $\Gamma(s)$ → [but can fix zeros on BAD positions]

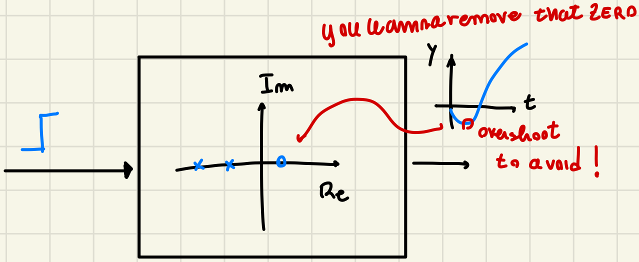
zeros are the ones of $F(s)$, $B(s)$
output of algorithm
IF zero on $Re(s)>0$
↳ step response
very Bad!! undershoot
performance!
⇓
you can obtain bad perform
because of some zeros of
$F(s)$ ⇝ how to
remove that?
↳




once computed
$R(s)$ by algorithm
↓
the T.F $\frac{Y(s)}{Y^o(s)}$

$$Y(s)=\frac{B(s)F(s)}{A(s)\Gamma(s)+B(S)F(s)}Y^o(s)=\frac{B(s)F(s)}{P(s)}Y^o(s)$$

has the zeros of the plant $B(s)$ (NOT modificable)
+ zeros of $F(s)$ → No way to force condition of $F(s)$ roots → so poor placement, Bad transient!


you wanna remove that ZERO
Im
Re
y
t
overshoot
to avoid !

chosen by us to place the ZEROS!

To avoid this problm, define a polynomial $\Delta(s) = s^{n-1} + \delta_{n-2}s^{n-2} + \ldots + \delta_1 s + \delta_0$ with «stable» roots, and implement the scheme

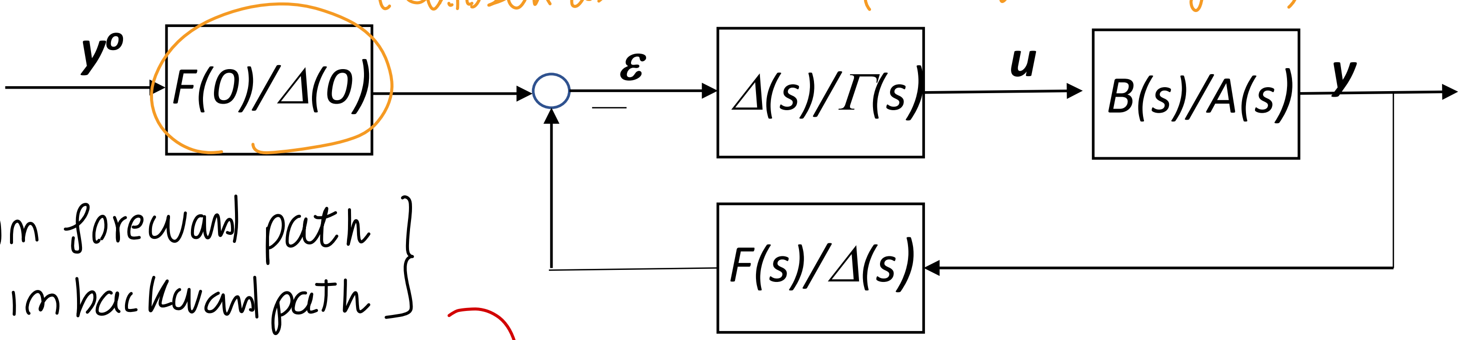
$\Delta(s)$ with same order of $F(s)$, $\Gamma(s)$ + poles with desired roots

pre-compensator to have the same static gain! (chosen as a Gain equal to feedback gain)

after $F(s)$, $\Gamma(s)$ standard theory implementation... you implement

$R'(s) = \Delta(s)/\Gamma(s)$ in foreward path

$R''(s) = F(s)/\Delta(s)$ in backward path




$F(0)/\Delta(0)$ as static gain of feedback term

both proper T.F of order $n-1$ @ num, den

Now $Y(s) = \boxed{\dfrac{B(s)\Delta(s)}{P(s)} \dfrac{F(0)}{\Delta(0)}} Y^o(s)$ ⇐ so you get that $Y(s)/Y^o(s)$

NOW you have $\Delta(s)$ roots! → you can implement $R(s)$ fixing ZEROS

- The precompensator is used to guarantee that the signal ***ε*** in the scheme at the steady state is proportional to the error ***y°-y***
- There is a cancellation of the roots of ***Δ(s)***, but this is allowed since the polynomial is chosen with asymptotically stable roots

so on steady state overall $\varepsilon \simeq y^o - y$ not influenced by the feedback block!

important! If integral action ~ input to 0, 0 error only if input of controller depend on error!

now $y/y^\circ$ T.F become

$$\frac{Y}{Y^\circ} = \frac{\frac{\Delta B}{\Gamma A}}{1 + \frac{\cancel{\Delta} B}{\Gamma A}\frac{F}{\cancel{\Delta}}} = \frac{\Delta B}{\underbrace{A\Gamma + BF}_{P(s)}}$$

[No more the roots of F(s) @ numerator]

(feedback term)

cancellation of polynomial with chosen roots ⇒ STABLE!

While the feed foreward block $\frac{F(0)}{\Delta(0)}$ is to maintain the STATIC gain

@ steady state

$\varepsilon \sim (y^\circ - y)$
only y multiplyed by $F/\Delta$ → maintain $\varepsilon$ value

- **How to cancel poles or zeros?**

another flexibility of this algorithm to cancel poles/zeros

(to cancel singularities)

Assume that

$$A(s) = (s+a)A'(s)$$

(other part of polynomial)

and we want to design a regulator that cancels the pole in $\boxed{s=-a}$

we wanna cancel that pole

we wanna cancel a plant pole (design R(s) to cancel poles/zeros)

It is sufficient to choose

$$P(s) = (s+a)P'(s)$$

to cancel that pole, you need to place that poles among the pole of the desired polynomial P(s)

using than the classical algorithm procedure

In fact, the Diophantine equation becomes

$$\overbrace{(s+a)A'(s)}^{A(s)}\Gamma(s) + B(s)F(s) = (s+a)P'(s) \longrightarrow B(s)F(s) = (s+a)[P'(s) - A'(s)\Gamma(s)]$$

B.F will be a polynom. containing the root in s=-a !

This means that ***(s+a)*** must be a factor of ***B(S)F(s)***, so that it will be automatically included into ***F(s)***

IF F(s) contain (s+a)

and a cancellation occurs between the terms ***(s+a)*** in ***A(s)*** and ***F(s)***

you have the desired cancellation

algorithm force the route s=-a on polynomial F(s)!

and because B(s) is the num. polynomial of T.F (fixed!)

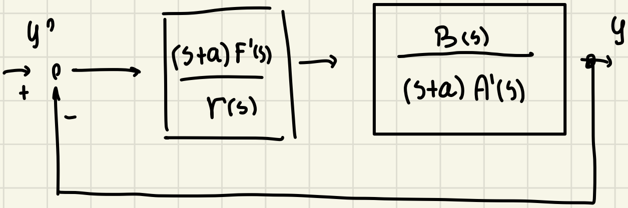
The same procedure can be followed to cancel zeros of ***B(s)***

G(s) has root in s=-a and also F(s)

↳ cancellation !!

(as done on classical design procedure)

from the system




$$P(s) = \underbrace{(s+a)\overbrace{P'(s)}^{\text{order } 2m-2}}_{\text{order } 2m-1}$$

$\downarrow$

$F(s) = (s+a)F'(s)$ automatically by solving diophantine eq.

Diophantine eq: $(s+a)A'(s)\Gamma(s) + (s+a)F'(s)\,B(s) =$

$$= (s+a)P'(s)$$

(EXTENSION)

*What about discrete time systems?*

Nothing changes in the algorithm save for the fact that

algebraic problem, all remain the same

1. ***s --> z***
2. ***1/s --> z/(z-1) or 1/(z-1)*** integrator definition
3. ***s=0 --> z=1*** static performance on $z=1$

## Exercise

Solution on 3 ways: ( don't matter for the algorithm ! work also for unstable systems )

unstable system

Given the system $\boxed{G(s)=\frac{1}{s-1}}$ design a regulator with integral action such that the closed-loop poles are in $s=-1$

Use 3 different approaches: a) with state space formulation, → as discussed on LEZ 6 from s.s

↓ I ORD system (state feedback)

b) use pole placement based on the transfer function → LEZ 7

approach c) a PI regulator tuned with the root locus

↑ approach NOT systematic

### Solution a - state space (a)

find ss form easily!

$$G(s)=\frac{1}{s-1} \longrightarrow \boxed{\begin{cases}\dot{x}=x+u\\ y=x\end{cases}}$$

⇓ Enlarged state with integrator

like for MIMO system design by putting integrator

integrator on each channel then stabilize

original system + integrator

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix}=\underbrace{\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}}_{\bar{A}}\begin{bmatrix}x\\ v\end{bmatrix}+\underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{\bar{B}}u$$

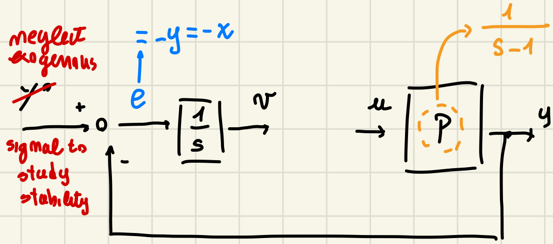
eig PLANT

INTEGRATOR $s=0$

(System + Integrator)

neglect $y^o \simeq 0$

$$\dot{v}=-y=-x$$



neglect exogenous
$\cancel{y^*}$
signal to study stability
+
-
$e = -y = -x$
$\frac{1}{s}$
$v$
$u$
$P$
$\frac{1}{s-1}$
$y$

then → design state feedback controller which guarantee poles in $s=-1$ (don't need OBSERVER)

Easy to verify that $(\bar{A}, \bar{B})$ is reachable and the invariant zeros (if any) of the original system are not at the origin (obvious, $G(s)$ does not have zeros) ← (necessary to solve the problem)

For the enlarged system we compute a pole placement control law

STATE feedback control law...

$$\mu = -\bar{K} \begin{vmatrix} x \\ v \end{vmatrix}$$

(such that eig of $sI-\bar{A}-\bar{B}\bar{K}$ are in (-1))

(note that the overall state is measurable)

⇓

We can use the Ackermann's formula or, simply, set

(SINGLE INPUT system)

↓ (to avoid formula by memory...)

2 rows for $u$

$$\bar{K} = |k_x \ \ k_v|$$

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 1-k_x & -k_v \\ -1 & 0 \end{bmatrix} \rightarrow \det(sI-\bar{A}+\bar{B}\bar{K}) = s^2+(k_x-1)s-k_v$$

characteristic polinomial

requirements

We set $\det(sI - (\bar{A} - \bar{B}\bar{K})) = s^2 + 2s + 1 = \boxed{(s+1)^2}$

(closed-loop poles in $s=-1$)

we have $k_x - 1 = 2 \rightarrow k_x = 3$

$-k_v = 1 \rightarrow k_v = -1$

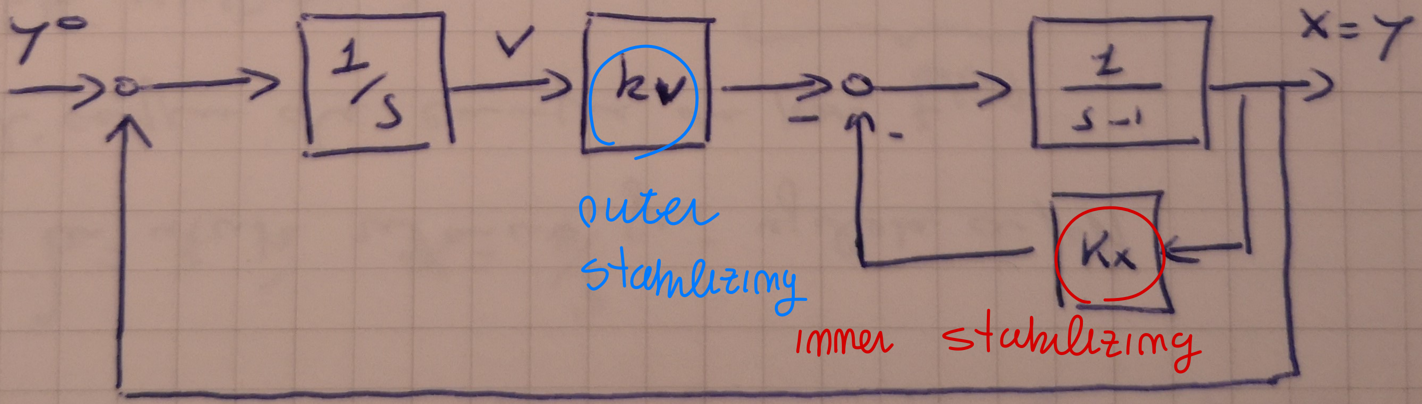
Result  I SOLUTION

$\Downarrow$

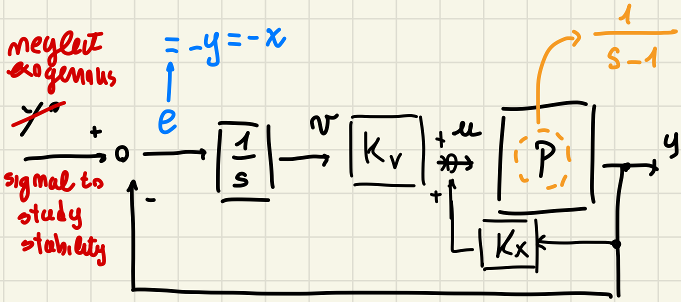
ORDER 1 Regulator

Overall scheme

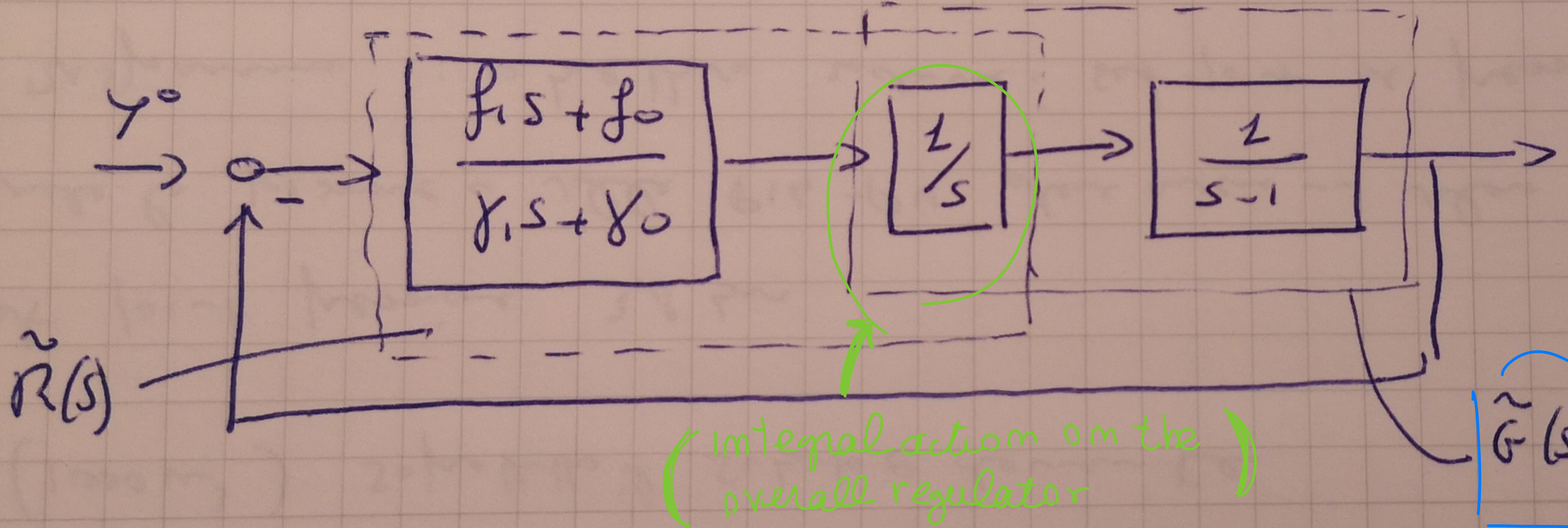
(2 feedback loop)



$$u = -\begin{bmatrix}K_x & K_v\end{bmatrix}\underline{x} = -\underbrace{\begin{bmatrix}3 & -1\end{bmatrix}}_{\bar{K}}\begin{bmatrix}x\\ v\end{bmatrix}$$



## b Solution 2 - pole placement with transfer functions




↓ with integrator into extended T.F

enlarge plant with integrator

$$\tilde{G}(s) = \frac{1}{s(s-1)}$$

$A(s) = s^2 - s$, $B(s) = 1$, $P(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$ request to have roots on $\boxed{s=-1}$

$P(s)$ of order $2n-1 = 3$

$\tilde{G}(s)$ of order $n = 2$, so $R(s)$ must be of order $n-1 = 1$ @ num /den so we solve diophanten equation in MATRIX FORM

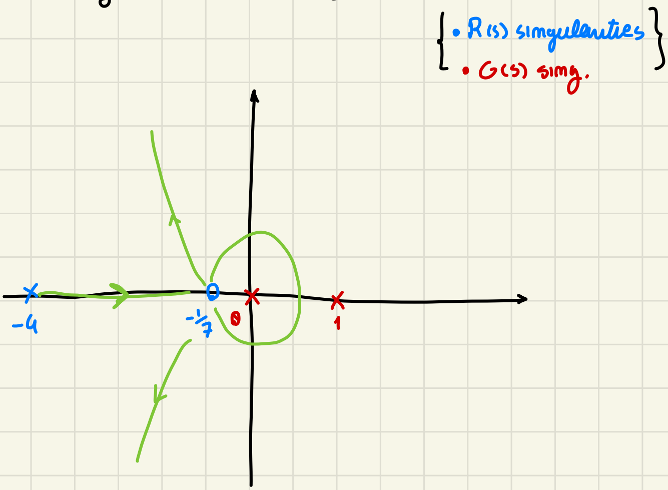
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{vmatrix} = \begin{vmatrix} 1 \\ 3 \\ 3 \\ 1 \end{vmatrix} \xrightarrow{\text{SOLVING}} \begin{cases} \gamma_1 = 1 \\ \gamma_0 = 4 \\ f_1 = 7 \\ f_0 = 1 \end{cases}$$

$$\boxed{\tilde{R}(s) = \frac{7\left(s + 1/7\right)}{s\,(s+4)}}$$

{ standard Regulator form }

↳ $\boxed{\text{ORDER } 2}$

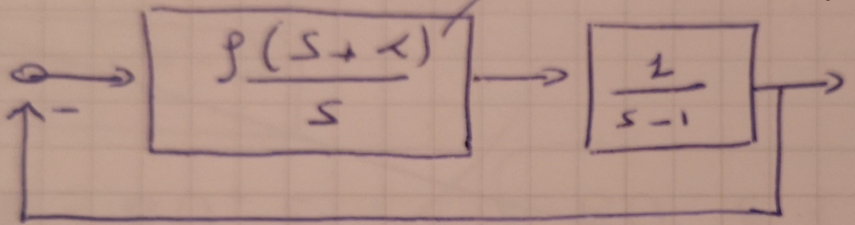
looking the ROOT locus



ORDER 1 Regulator

Solution 3 C

PI




unstable system → you cannot use Bode criterion! ⇒ difficult to find PI, we use root locus method

PI defined by $K_P, K_I$ ...

in general by a ZERO $\alpha$ + GAIN $\rho$

writing char eq. I force the coefficient equals to desired P(s) coeff!

Characteristic equation

$$s(s-1)+\rho(s+\alpha) = \underbrace{s^2+2s+1}_{P(s)}$$

NOT systematic approach!

$$\Downarrow$$

$\rho = 3$, $\alpha = 1/3$

possible approach only possible for simple systems

hard to extend

**New Example**

design a regulator for an unlucky system

$$G(s)=\frac{s-2}{(s-1)(s+3)}$$

system unstable and nonminimum phase

unstable pole $s=+1$

ZERO on $Re(s)>0$ limits bandwidth ($s=+2$)

→ NO Bode criterion!

Goal: assign poles in $-1$.

$$R(s)=\left(\frac{f_1 s+f_0}{\gamma_1 s+\gamma_0}\right) \text{ ORDER 1}, \quad P(s)=(s+1)^3$$

$2n-1=3$

$n=2 \rightsquigarrow R(s)$ of order $n-1=1$

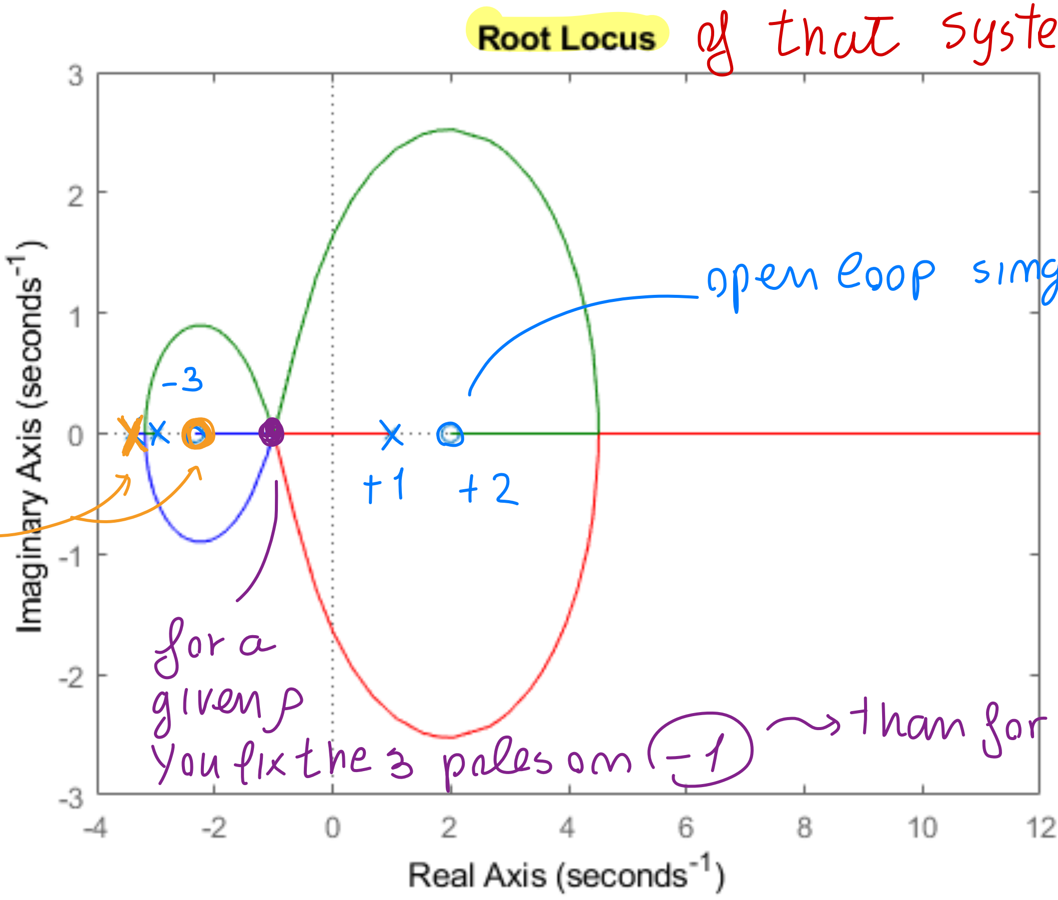
for pole placement this is a normal problem to solve, NO PROBLEMS! standard procedure

$$G(s)=\frac{s-2}{(s^2+2s-3)}, \quad \underbrace{P(s)=s^3+3s^2+3s+1}_{\text{all poles fixed in } (s=-1) \text{ on closed loop}}$$

shift shift

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -3 & 2 & -2 & 1 \\ 0 & -3 & 0 & -2 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \rightarrow \boxed{R(s)=\frac{-2.45-5.6}{s+3.4}}$$

automatic solution of the problem




of that system G(s)

open loop singularities

singularities introduced by the R(s)

for a given ρ you fix the 3 poles on −1 → than for larger gain → instability

because non minimum phase system

strange Root locus

for large ρ it become unstable closed loop

(cancellation problem)

↓

MATLAB

"motore.m"

+ SIMULINK

file projects

Example

V disturbance torque

related to $\omega_1$

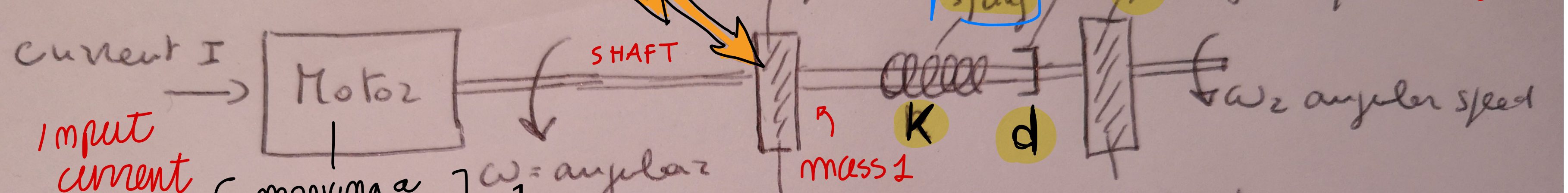
$\varphi_1$ angular position

flexibility

damper

spring

$\varphi_2$ angular position of second mass




moving a flexible manipulator

$\omega_1$ = angular speed of shaft

$J_1$ = inertia mass 1

$J_2$ = inertia mass 2

⇓

Model

mech

$\boxed{\text{system}}$

$$\begin{cases} \dot\varphi_1 - \dot\varphi_2 = \omega_1 - \omega_2 \quad \text{(angles / speed relation)} \\ J_1 \dot\omega_1 = d(\omega_2 - \omega_1) + k(\varphi_2 - \varphi_1) + k_I \cdot I + V \\ J_2 \dot\omega_2 = d(\omega_1 - \omega_2) + k(\varphi_2 - \varphi_1) \end{cases}$$

disturbance torque

control variable

damper of system

(III ORD system)

3 STATE variable

↳ $x \in \mathbb{R}^3$

$$\begin{cases} x_1 = \varphi_1 - \varphi_2 \\ x_2 = \omega_1 \\ x_3 = \omega_2 \end{cases}, \quad u = \begin{vmatrix} I \\ V \end{vmatrix}, \quad y = \omega_2$$

real control variable

[disturbance] torque

2 inputs

output

We wanna design a control to control it

## Numerical values

$J_1 = \frac{10}{9}$, $J_2 = 10$, $k_e = 1$, $d = 0.1$, $k_I = 1$

$$\omega_2 = \frac{0.01s + 0.1}{s^3 + 0.1s^2 + 0.8s} I$$

relationship $\omega_2 / I$

III ORD syst

$\Downarrow$

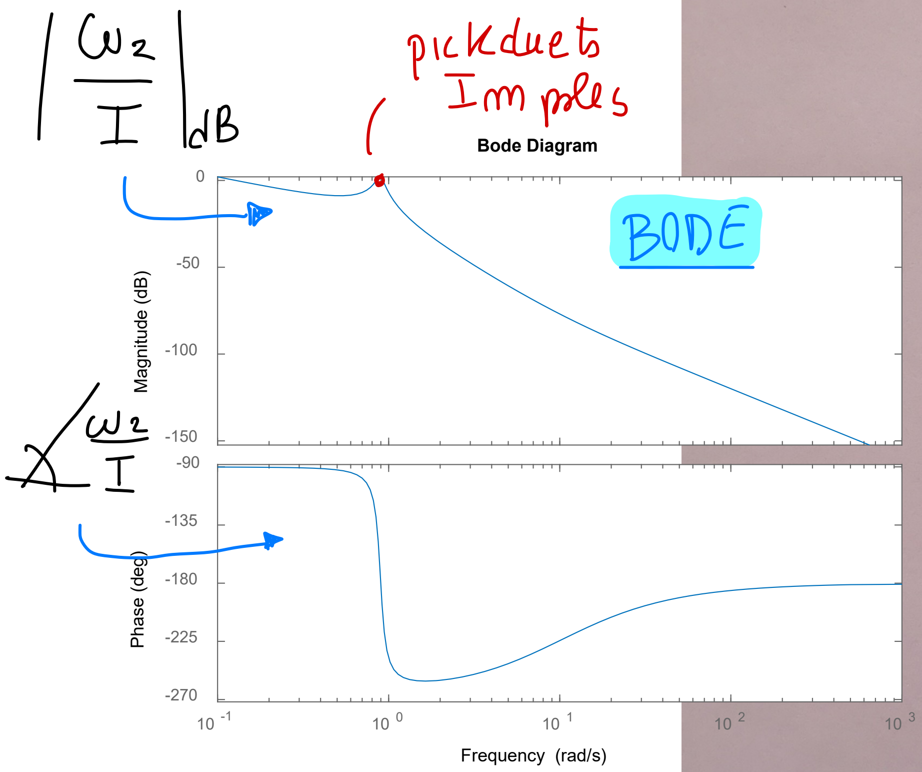
poles $\begin{cases} s = 0 \\ s = -0.05 \mp j\,0.89 \end{cases}$ (poorly damped)

$\omega_p = 0.894$

$\xi_p = 0.0559$

complex conj poles poorly damp. near Im axis

zero $s = -10$ (high frequency)



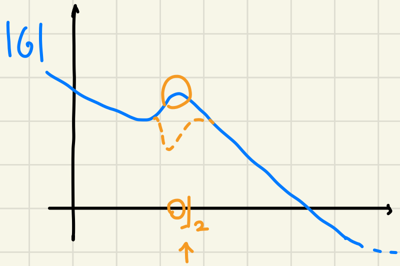

(Classical mech system)

(you need integral action on Regulator to remove disturbance..)

↳ PROPER Regulator ? ⟹ hard to define...

the design of R(s) is HARD

~> with **Bode CRITERION**




↑ with classical approach, you introduce ZEROS to modify behav } Notch Filter design

{ NOT so good, you could have mistakes in the model, so maybe no good cancellation! }

↳ design a controller by NOTCH FILTER or Pole placement

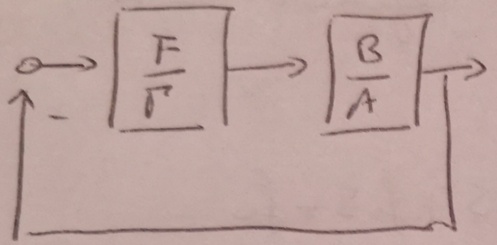
NOTCH FILTER: design to cancel

Pole placement: moove the pole with higher damping, NO cancellation

Order of the system $n=3$, order of the regulator $= 2$

apply pole placement theory

$R(s)$ of order $n-1=2$




$$\begin{cases} F = f_2 s^2 + f_1 s + f_0 \\ \Gamma = \gamma_2 s^2 + \gamma_1 s + \gamma_0 \end{cases}$$

Desired closed loop polynomial $P(s)$ of order $2n-1 = 5$

↓ different possible poles selection:

(NOTCH FILTER design) Project 1

↓

(cancel the poles!)

second couple placed where there are the open loop poles

$$P(s) = \underbrace{\left(s^2 + 2\xi_n\omega_n s + \omega_n^2\right)}_{\omega_n = 0.5,\ \xi = 0.7} \boxed{\left(s^2 + 2\xi_p\omega_p s + \omega_p^2\right)} \underbrace{(s+10)}$$

far from origin! (cancelling the system ZERO)

$\xi_p = 0.056$

$\omega_p = 0.89$

complex conj poles with $\xi = 0.7$, well dumped

[remove singularities on closed loop BUT not on all dynamics!]

(pole placement for cancellation)

← cancellation of poorly demped poles of system

↑ these are also open-loop poles. Cancellation!

⇒ STANDARD ALGORITHM

**Project 1**: good response to the reference signal, poor response to disturbance variations. Why?

↓ from this Regulator form:

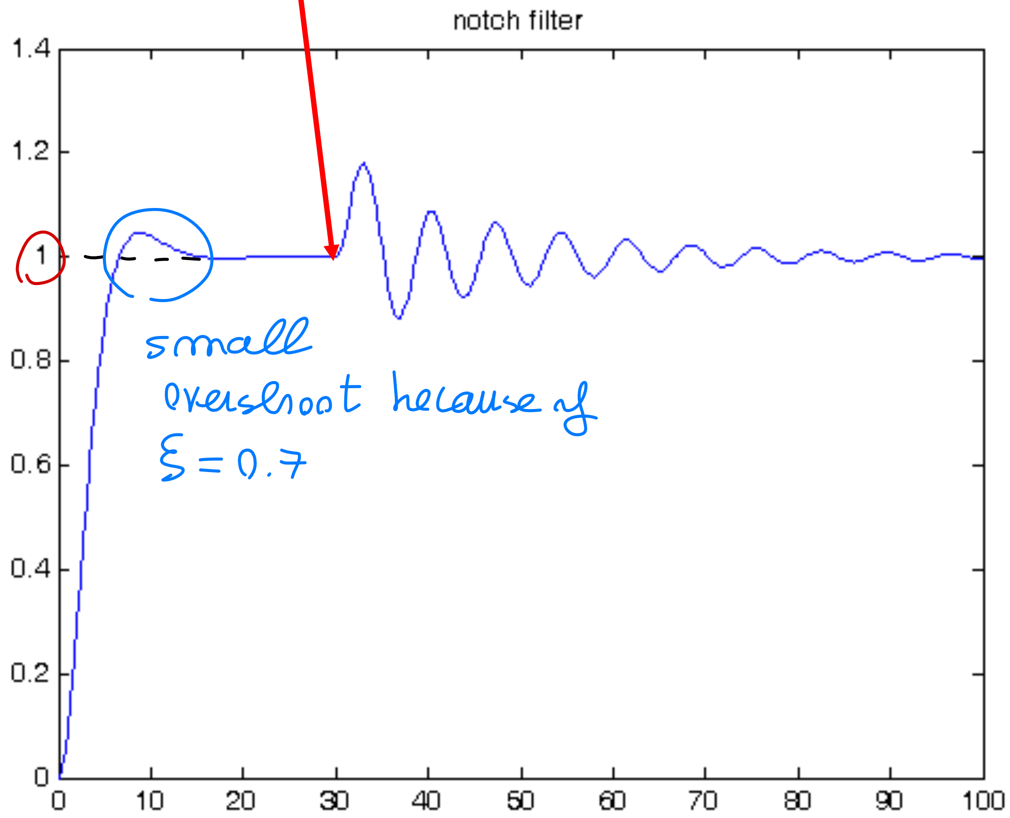
disturbance

When d(t) act → bad responce (you see cancelled pole effect!)

↳ in fact

SIMULATION

SET POIN




small overshoot because of $\xi = 0.7$

to track a reference equal to 1
⇓
small overshoot for the complex conj high dumped poles

Project 1   from theory:

$A(s) = \underbrace{A'(s)}_{\text{poles with}}\, s$   1 Include the pole to cancel on the $A'(s)$ polynomial

$\begin{cases} \omega_p = 0.89 \\ \xi_p = 0.056 \end{cases}$

solving diophantic equation

If $\boxed{P(s)} = A'(s)\, P'(s) \rightarrow A(s)\Gamma(s) + B(s)F(s) = P(s)$

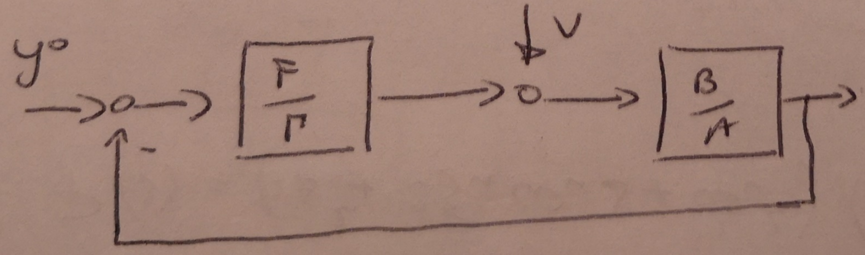
(so in the $R(s)$ dynamic!)

$\underbrace{A'(s)\cdot(s+10)}_{A(s)}\, \Gamma(s) + B(s)\, F(s) = A'(s)\, P'(s)$

$A'(s)$ in the roots of $F(s)$

$\Downarrow$

$F(s)$ contains the term $A'(s)$, $F(s) = \bar{f}\, A'(s)$



If $P(s) = A'(s)\,P'(s) \rightarrow A(s)\Gamma(s) + B(s)F(s) = P(s)$

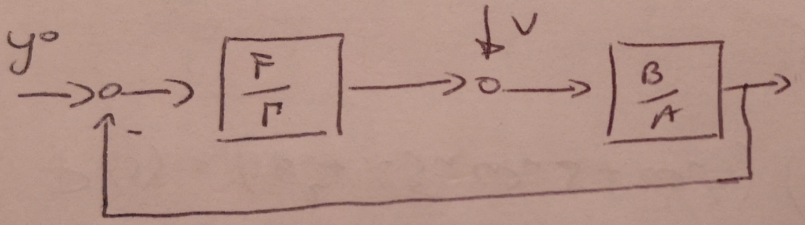
$$\underbrace{A'(s)\cdot(s+10)}_{A(s)}\,\Gamma(s) + B(s)F(s) = A'(s)\,P'(s)$$

$\Downarrow$

$F(s)$ contains the term $A'(s)$, $F(s) = \bar f A'(s)$

To cancel some dynamic you use KNOTCH FILTER
↓
(cancelling singularities)

emphasise that by automatic procedure, the R(s) design is NOT so easy




poles cancelled by the $Y/Y^\circ$ but remain on $Y/V$!

$$y = \frac{FB}{P}y^\circ + \frac{B\Gamma}{P}V = \frac{\bar f' \cancel{A'(s)} B(s)}{\cancel{A'(s)}\,P'(s)}y^\circ + \frac{B\Gamma}{\boxed{A'(s)\,P'(s)}}V$$

$= P(s)$

Cancellation

no cancellation

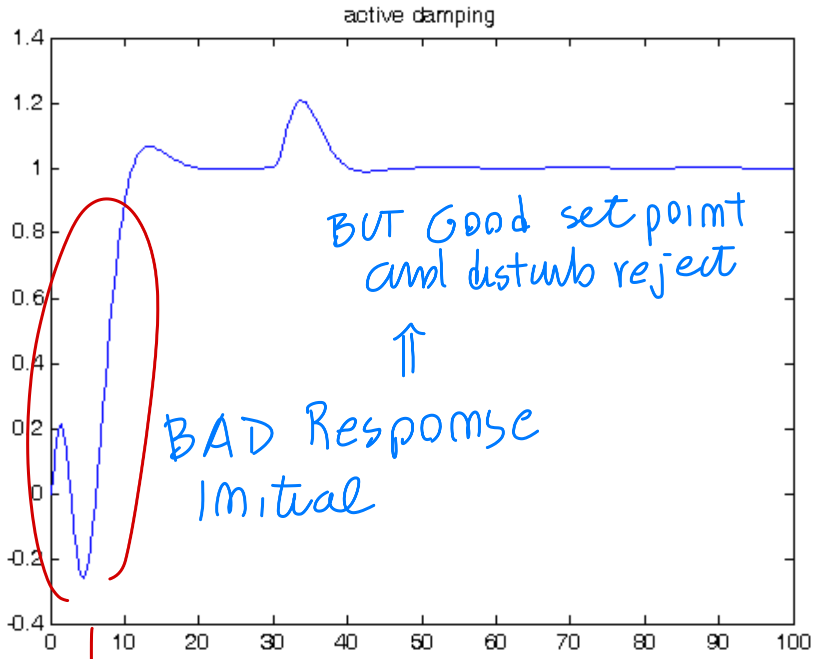
Cancellation of $A'(s)$ on F and on P
↓
so $Y/Y^\circ$ has not that poles!

BUT they appear on $Y/V$ T.F

poles undesired remain here ⇒

so disturbance has bad effect as seen!

Unacceptable «inverse response» due to the zeros of the polynomial ***F(s)***




BUT Good set point and distub reject

BAD Response Initial

PROBLEMATIC part of response! (DON'T WANT it)!

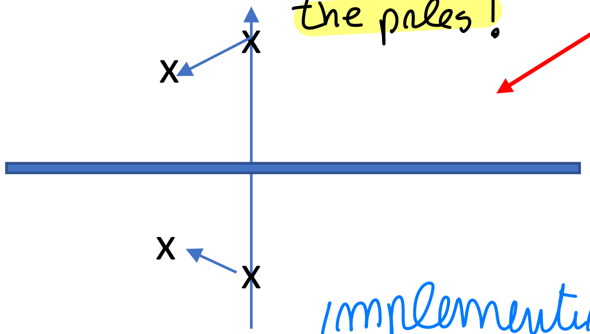
a ROBOTIC arm has issue...
→ new palynomial F(s) has roots outside unitary circle! ⟹ so we use a TRICK ⟶

other solution.. try to moove on closed loop the 2 poles, mantain same $\omega_n = 0.5$ but ask for larger dumping factor!

Project 2

$$P(s) = (s^2 + 2\xi_n \omega_n s + \omega_n^2)\underbrace{(s^2 + 2\xi_m \omega_n s + \omega_m^2)}_{\xi_m = 0.7,\ \omega_m = 0.5}(s+10)$$

better to moove the poles!



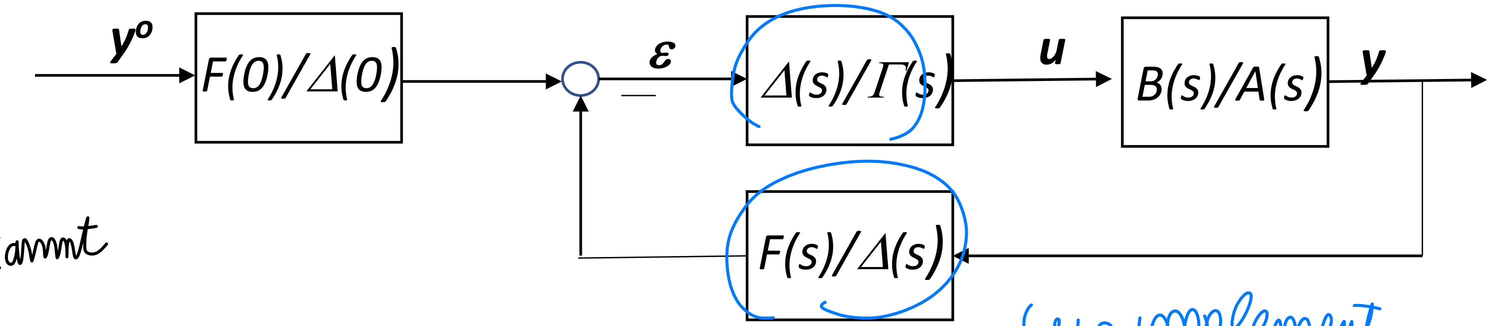
new plinomial positioning by different design

implementing differently ⟹

**Project 2 with the scheme**

We implement $\Delta(s)/\Gamma'(s)$ instead of $F(s)/\Gamma(s)$
so $y/y^o$ I have a different dynamic!

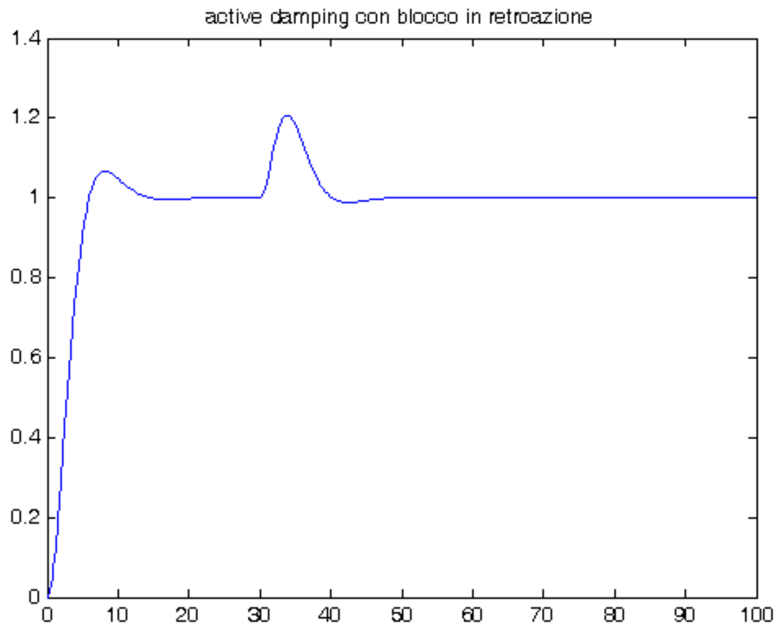
Pole placement
usefull when
with standard
technique you cannot
solve




We implement
in this way
for a
better behaviour

remove
the zeros
of $F(s)$
undesired

cancel
||
hide singularities
but not
cancel out
from closed
loop
system




$\Downarrow$

**Excellent result**

Usefull approach for
poorly damped syst
where bode techniques
are hard to use