DISCRETE TIME OPTIMAL CONTROL

LQ(G), (E)KF

→ extension for discrete time system the result obtained concerning OPT. CONTR. / LQ ecc.. + issues...

(NOT H_2 / H_∞ control → all considerations about shaping funct, min $\{\|\cdot\|_2\}$ ... are valid) → with different formulas

But same idea

Advanced and Multivariable Control

Optimal Control of Discrete Time Systems

Riccardo Scattolini

Optimal Control (general form problem statement)

time invariant (f NOT funct of k)

System $x(k+1) = f(x(k), u(k)),\quad x(k_0) = x_0 \qquad x \in R^n,\ u \in R^m$

initial state ($k_0 = 0$)

assume x(k) meas!

Minimize with respect to $u(k_0),\ u(k_0+1),\ \ldots,\ u(\bar{k}-1)$ the cost function

optimal control sequence

terminal state funct

$$J(x(k_0), u(\cdot), k_0) = \sum_{i=k_0}^{\bar{k}-1} l(x(i), u(i)) + m(x(\bar{k})) \quad , \quad \bar{k} > k_0$$

as done on cont. time

subject to the system's dynamics and state and input constraints:

for stabilization prob...

$$x(k) \in X, \quad u(k) \in U$$

ask states inside a given space, also for inputs!

$X \subseteq R^n,\ U \subseteq R^m$ are compact sets containing the origin

initially neglect x, u constrain! ⇒ solve unconstrained (than use MPC) to include constr.

DYNAMIC programming

Let $u_{[a,b]}$ be the control sequence $u(k)$, $k \in [a,b]$, and define

optimize cost function, by minimize

$$J^0(x(k),k) = \min_{u[k,\bar{k}-1]} J(x(k),u(\cdot),k)$$

value of cost function finding new var! optimal J^0 depends on x,k!

from this defin.

Then

@ $\bar{k}$ depends only on $m(x(\bar{k}))$

$$J^0(x(\bar{k}),\bar{k}) = \min_{u(\bar{k})} m(x(\bar{k})) = m(x(\bar{k}))$$

which represent terminal condition of J^0

and one step back (integrate backwards!)

$$J^0(x(\bar{k}-1),\bar{k}-1) = \min_{u(\bar{k}-1)} \{\underbrace{l(x(\bar{k}-1),u(\bar{k}-1))}_{\text{cost funct } l \text{ @ } \bar{k}-1} + \overbrace{J^0(x(\bar{k}),\bar{k})}^{=m(x(\bar{k}))}\}$$

ITERATION! dependance @ $\bar{k}$-1 on value at $\bar{k}$

or in general (generaliz @ generic k)

Bellman's principle of optimality

$$J^0(x(k),k) = \min_{u(k)} \{l(x(k),u(k)) + \overbrace{J^0(f(x(k),u(k)),k+1)}^{\text{optimal value @ } k+1}\}$$

Bellman's principle: IF you have an opt. sequence $k_0 \ldots \bar{k}$, reformulating optimiz problem, becomes iterative

Hamilton Jacobi Bellman's eq

HJB equation

For a fixed ***k***

$$\begin{cases} J^0(x,k) = \min_u \{ l(x,u) + J^0(f(x,u),k+1) \} \\ J^0(x,\bar{k}) = m(x) \end{cases}$$

condition, final value @ $\bar{k}$ is $m(x)$

≠ cont. time, where you have differential expression $\partial J / \partial t$...

Solution... compute min respect u

Step 1 (solve min)

compute the value u^o minimizing

optimal control var. sequence!

$$\{ l(x,u) + J^0(f(x,u),k+1) \}$$

Assuming that there exists an unique minimum, this corresponds to compute a function with arguments x and $J^0(f(x,u),k+1)$, that is

$$u^o = \kappa\left(x, J^0\right)$$

NOT yet usable, dependance on J^o get final sol.

(find J^o from HJB)

Step 2

Compute the function $J^o(x,k)$ satisfying the HJB equation

(moove backward iteratively!)

In discrete time we have difference eq. NO integration needed, BUT NOT trivial →

$$J^o(x,k) = l\left(x, \kappa\left(x, J^o(x,k)\right)\right) + J^0\left(f\left(x, \kappa\left(x, J^o(x,k)\right)\right), k+1\right)$$

with boundary condition

(approx of discrete time HJB → AI techn.)

$$J^0(x,\bar{k}) = m(x)$$

(@ end obtain final optimal cost funct)

same idea, solve min. find control law and substitute on original with final condition

FOCUS on **LQ control**

in general difficult to find HJB eq!

System linear:

$$x(k+1) = Ax(k) + Bu(k)$$

standard formulation
assuming x measurable

Cost function

summ of quadratic terms along the prediction horizon

$$J(x(k_0), u(\cdot), k_0) = \sum_{i=k_0}^{\bar{k}-1} \underbrace{[x'(i)Qx(i) + u'(i)Ru(i)]}_{\ell(x,u)} + \underbrace{x'(\bar{k})Sx(\bar{k})}_{m(x)}, \quad \bar{k} > k_0,$$

assume Q, R, S symm. matrix

$Q \geq 0$, $R > 0$, $S \geq 0$
(s.p.d) (p.d)

because you wanna weight all control vector elements and they go to ∞

problem for solving it!

Tentative solution

$$J^0(x,k) = x'P(k)x \quad , P(\bar{k}) = S$$

opt. cost as quadratic function with nucleus $P(k)$ → satisfy terminal condition automatically + solve the problem substituting to J^0 ⟹

LQ control - solution

finite horizon LQ optimal control problem

time varying gain of CNTRL LAW

(CONTROL LAW) $u(k) = -K(k)x(k)$: static state feedback control law

$$K(k) = \underbrace{(R + B'P'(k+1)B)}_{\text{Invertible, summ of p.d + sp.d !}}^{-1} B'P'(k+1)A$$

control gain

found from Riccati eq

$$P(k) = Q + A'P(k+1)A - A'P(k+1)B\left(R + B'P(k+1)B\right)^{-1} B'P(k+1)A$$

difference Riccati equation

(matrix difference eq.

$$P(\bar{k}) = S$$

P easy to compute, after initialize, iterate backward the RE to solve, until converge to $\bar{P}$!

- Slightly easier to solve than in the continuous time (difference Riccati, instead of differential Riccati)

- Same limitations of the continuous time case (finite time horizon, time varying …)

(we need practical usefull solutions!) → suitable algorithm ⟹

control var from k_0 to $\bar{k}$... what for $k > \bar{k}$?! Issue!

control law t. vary → critical analysis

time invariant, ok!

Infinite Horizon LQ

cost function from $k_0 = 0$ → properly tune Q, R.. Q>R..

$$J = \sum_{k=0}^{\infty} x'(k)Qx(k) + u'(k)Ru(k), \quad Q \geq 0, \quad R > 0$$

properly weighted on ∞ horizon

NOT including termal cost, @ ∞ final step we have m = 0 ⇓ final state 0! state must go to 0 @ ∞ or J → ∞! NO

conditions to verify @ PRIORI!

If

→ NOT strictly necessary, need stabilizable at least!

a) the pair (A, B) is reachable;

b) the pair (A, C_q) is observable, with C_q such that $Q = C_q' C_q$

↳ easy to satisfy for small Q ...

then

A) the optimal control law is given by

optim. c.l computed solving stationary R.E, take $\bar{P}$, and define $\bar{K}$

$$u(k) = -\bar{K}x(k)$$

$$\bar{K} = \left(R + B'\bar{P}B\right)^{-1} B'\bar{P}A$$

NOT GIVEN @ EXAM

and similar solution to cont. time

where $\bar{P}$ is the unique solution > 0 of the stationary Riccati equation

GIVEN DURING EXAM

$$\bar{P} = A'\bar{P}A + Q - A'\bar{P}B\left(R + B'\bar{P}B\right)^{-1} B'\bar{P}A$$

ALGEBRAIC equation

In a practical POV same conditions as cont. time! (a) + (b)

and.. valid also

B) the closed-loop system

stabilizing state feedback C. Law

$$x(k+1) = \left(A - B\bar{K}\right) x(k)$$

this theory example solved for I ORD syst to simplify, even IF strenght is for MIMO syst!

is asymptotically stable.

here on discrete time... you neglect it

$\Downarrow$

BUT in some cases could be usefull to consider additional term!

to modify system responce $\rightarrow$ You modify by properly define Q, R...

related to mooving the stability region so that eig of closed loop system are on the left of a region

$\downarrow$

stabilizing (guarantee by LQ)

$\downarrow$

+ additional request on eig on closed loop $\Longrightarrow$ ...

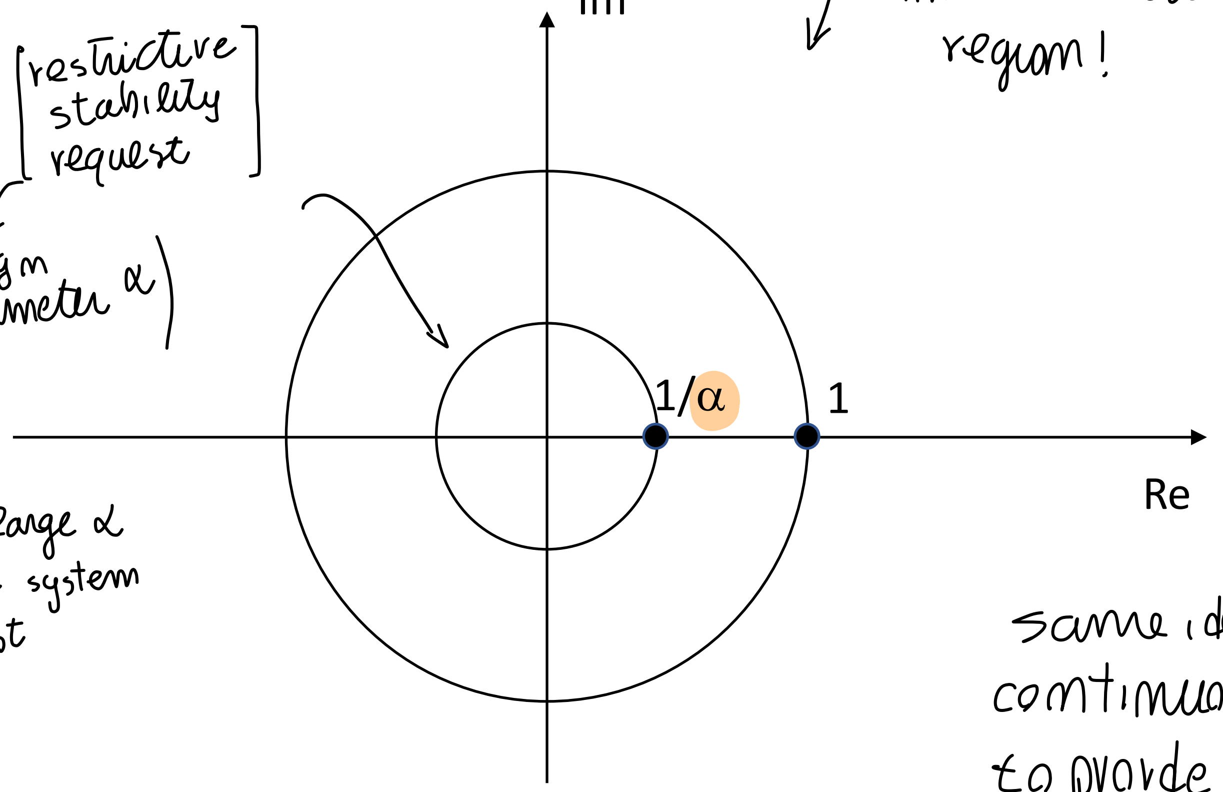
LQ control with a prescribed degree of stability

Goal: we want to obtain a stabilizing regulator such that the closed-loop eigenvalues have an absolute value smaller than $1/\alpha$

inside a certain region!

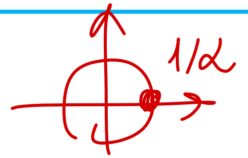
[restrictive stability request]

(design parameter α)



IF too large α require system too fast

same idea of continuous time to provide this result =>



Modified cost function

$$\hat{J} = \sum_{k=0}^{\infty} \{[x'(k)Qx(k) + u'(k)Ru(k)]\alpha^{2k}\}$$

multiply by α^{2k} ($\alpha > 1$) to modify the result as desired

new state   new control var.

Defining $\hat{x}(k) = \alpha^k x(k)$, $\hat{u}(k) = \alpha^k u(k)$ $\longrightarrow$ $\hat{J} = \sum_{k=0}^{\infty} [\hat{x}'(k)Q\hat{x}(k) + \hat{u}'(k)R\hat{u}(k)]$

new cost function in $\hat{x}, \hat{u}$

$(|eig| < 1/\alpha)$

$$\alpha^{k+1}[x(k+1) = Ax(k) + Bu(k)]$$

all expressed on new var!

differently from cont. time you modify both A,B

In practice you have just to compute $\hat{A} = \alpha A$, $\hat{B} = \alpha B$ and solve in the same way LQ control problem with standard software solution

$$\hat{A} = \alpha A, \hat{B} = \alpha B \implies$$

$$\hat{x}(k+1) = \alpha A\hat{x}(k) + \alpha B\hat{u}(k) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k) \longrightarrow$$

rewrite syst matrix according to new syst def. (standard system)

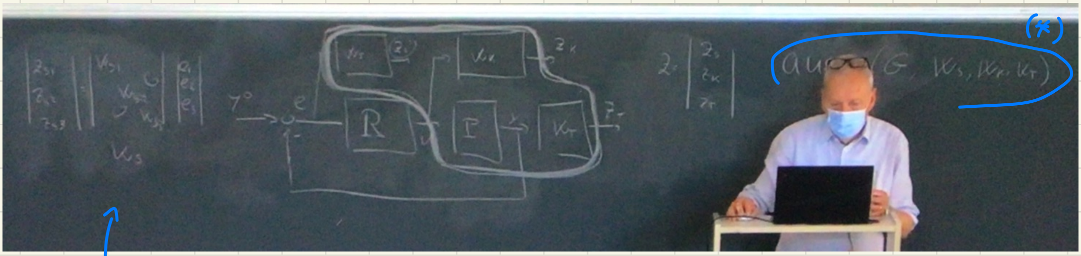
Standard LQ problem for the system in the variables $\hat{x}, \hat{u}$

One has to solve the LQ problem with $\hat{A}$, $\hat{B}$ and then use the obtained gain $\hat{K}$ in the control law

$$u(k) = -\hat{K}x(k)$$

stabilizing the syst with this control law!

[RECAP] note on LEZ12



(when considering shaping function W_s)

As we see on simulation LEZ 12 (slide 60)

WWS, WWK, WWT.. ⇒ defined as diag matrix, same for all shaping function

define enlarged block

augW(·) put all togheter (*)
↳ overall syst represent.

on
H_2 - H_∞ dyn. due to observer of same order of syst.ord

P+W+K+... enlarged overall syst.

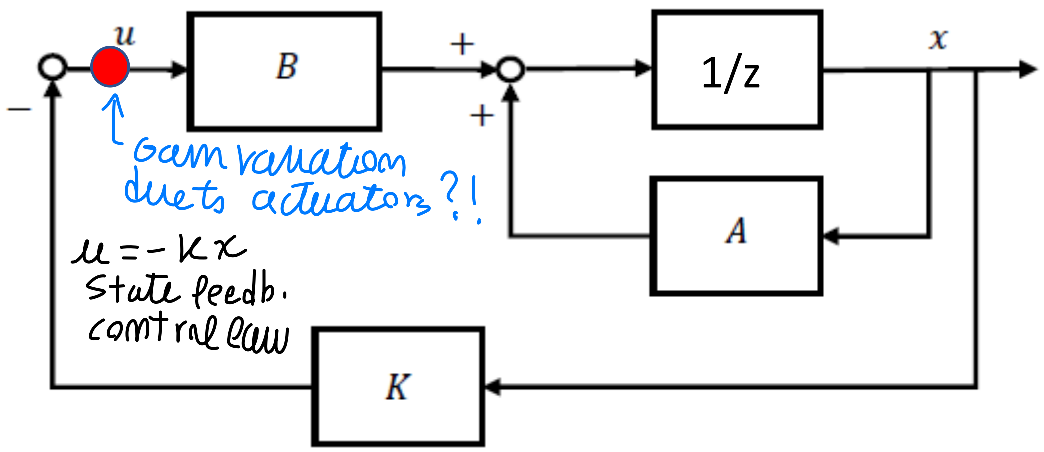
overall high ORD REG!

sometimes too large → we try to reduce the order without performance deterioration

through model order reduction → hig reduction significant!

Robustness of LQ_{inf} control with respect to uncertainties *at the plant input*

consideration respect ROBUSTNESS
Guarantee of design → properties
respect gain/ phase margin!



Gain variation due to actuators?!

$u=-Kx$
State feedb.
control law

*It is **stabilizing**, it is **optimal** (with respect to the selected design parameters), but is it also **robust** with respect to uncertainties at the plant input?*

you have gain margin?

With simple manipulations of the Riccati equation, it is possible to obtain the following relationship

from (RE)

control variable weight matrix

$$G_c'(-z)QG_c(z)+R=\Gamma'(-z)(R+B'PB)\Gamma(z)$$

G, Γ defined as...

with

$$G_c(z)=(zI-A)^{-1}B$$ transfer function from ***u*** to ***x*** → open loop T.F

$$\Gamma(z)=I+\underbrace{K(zI-A)^{-1}B}_{\text{LOOP T.F : } L(z)}$$ Inverse of the *sensitivity function* (*return difference*)

$\simeq(1+L(z))$

$1/(1+L)$ =

function

In cont. time, including phase variation ($\pm 60°$) } ROBUSTNESS
gain variation $(+0.5, \infty)$ } guarantee !

What about Discrete time system ?

neglect, ≥0 for sure

$$\underbrace{G_c'(-z)QG_c(z)}_{\text{neglect}} + R = \Gamma'(-z)(\overbrace{R + B'PB}^{\text{scalar}})\Gamma(z)$$

[we can divide for this term!]

respect SISO

Single input systems, m=1

with just one input:

taking the norm...

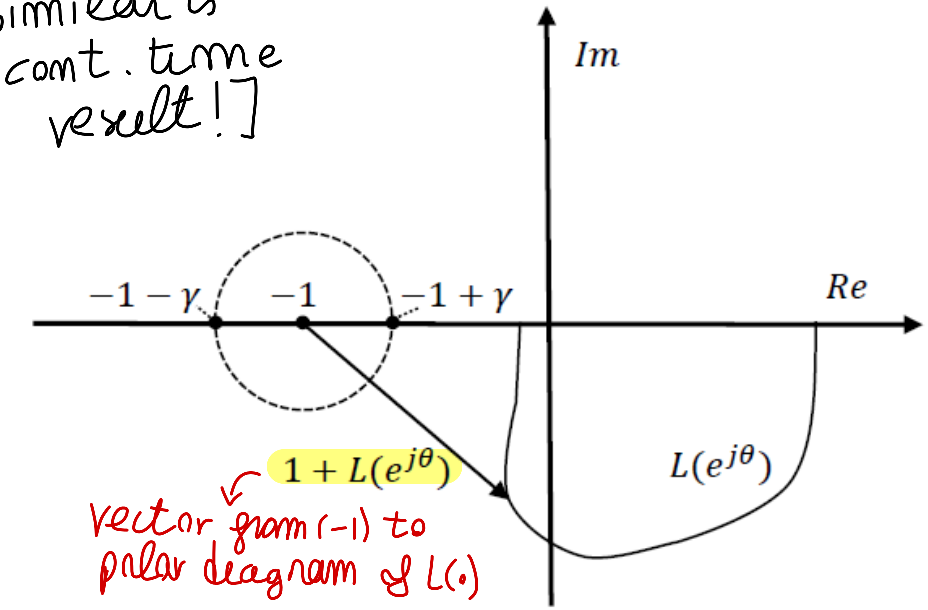
$$\Big|1 + \underbrace{K(zI - A)^{-1}B}_{L(z)}\Big|^2 \geq \frac{R}{R + B'PB} = \bar{\gamma} < 1$$

(by definition)

$L(z) \rightarrow$ $1 + L(z)$ is the vector from (-1) to the polar diag

the polar diagram of $L(e^{j\theta})$, $\theta \in [0, \pi]$, does not intersect the circle with center in -1 and radius $\gamma = \bar{\gamma}^{1/2}$,

[similar to cont. time result!]



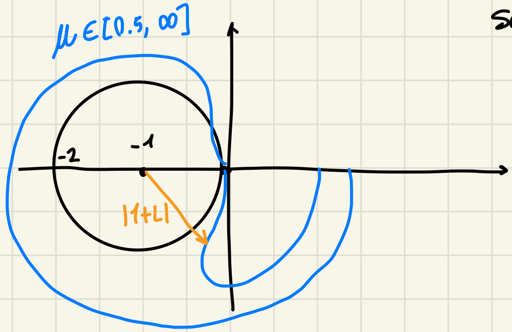
$$\boxed{\text{gain margin} \in \left(\frac{1}{1+\gamma}, \frac{1}{1-\gamma}\right)}$$

(smaller range than cont. time!) → less ROBUSTNESS, limitations

- Smaller than the one in continuous time LQ

- No phase margin → not defined on discrete time systems!

In cont. time



same consideration on discrete time
↓
but now for different value of μ range to guarantee stability

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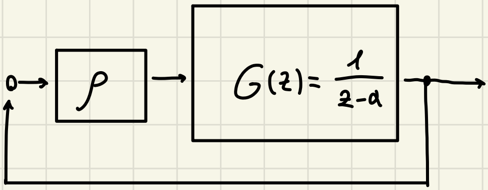
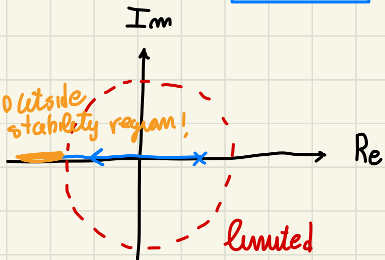
Given a Discrete time system



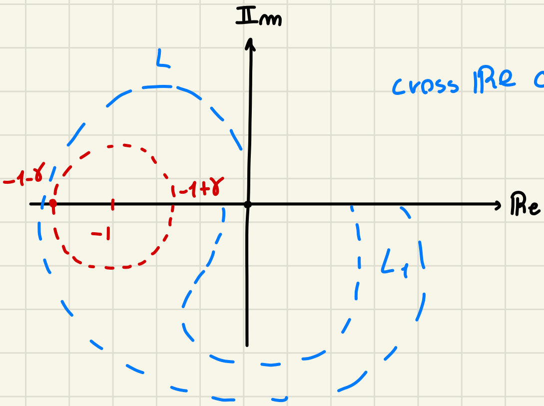
Root locus



Gain cannot go to $(+\infty)$!

due to limited space stability region respect cont. time case!

so we can have 2 possibilities..



cross Re axis on left of $-1-\gamma$
or on right of $-1+\gamma$

$\Downarrow$

than to compute
Gain margin! := defined as
the max value of a gain in
the loop such that you
arrive inside forbidden region!

[so gain margin in
between $\left\{ \frac{1}{1+\gamma}, \frac{1}{1-\gamma} \right\}$] → loss stability...

here we have lower, upper
limit $\Downarrow$

LQ guarantee gain margin BUT with smaller
guarantee!

(EXAM): look gain margin and define it!

Example

Consider the system described by $\boxed{G(z) = \frac{1}{z-2}}$

- 1) Design a LQ controller with $Q = R = 1$
- 2) Analyse its robustness properties with respect to gain variations

Solution

1 The system can be given the state space form

unstable! eig on 2!

compute solution

$$x(k+1) = 2x(k) + u(k) \quad , \quad A = 2, \; B = 1 \quad \leftarrow \text{S.S form}$$

Riccati equation $\quad A'PA + Q - \underbrace{A'PB(R+B'PB)^{-1}B'PA}_{K} - P = 0$ steady state riccati eq...

Solution ~~P~~ = consider only pos def sol. ⟶

easy solve..

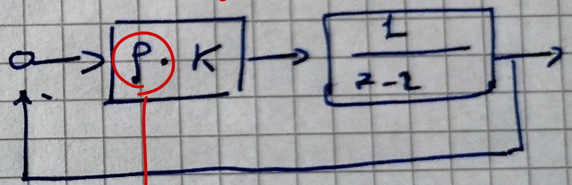
Solution

K

$P = 4.2361$, $K = 1.618$, $A - BK = 0.382$ stabilized!

closed-loop eigenvalue (asymp stable)

static feedback



to analyze gain margin, I use ρ!

2 Robustness analysis

characteristic equation

$$z - 2 + \rho \cdot 1.618 = 0$$

$$z = 2 - 1.618\rho \longrightarrow -1 < 2 - 1.618\rho < 1$$

$$\Downarrow$$

$$\rho \in [0.618,\ 1.8541]$$

real system gain margin

"Theoretical" robustness bound respect found result on discrete time system!

$$\rho \in \left[\frac{1}{1+\gamma}, \frac{1}{1-\gamma} \right]$$

$$\gamma = \sqrt{\bar{\gamma} = \frac{R}{R+B'PB}} = 0.4370$$

$$\rho \in [0.6959, 1.7762]$$

|| smaller than the real one ||

THEORETICAL

REAL

$$\rho = [0.618, 1.8541]$$

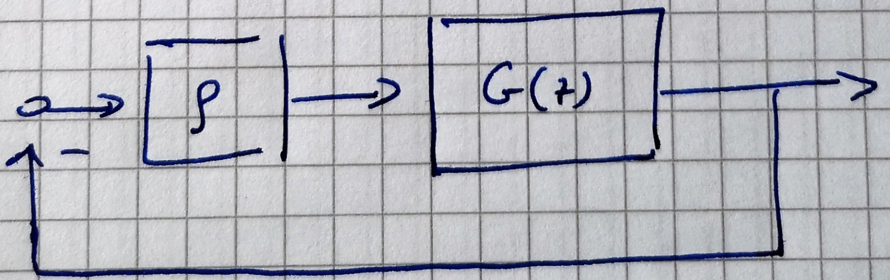
Wider gain margin! Better result real!

- Why for discrete time systems we cannot have a gain margin $\longrightarrow \infty$? (like in cont. time) $\longrightarrow$ NOT a weakness of LQ control!

Consider a SISO system with transfer function

$$G(z) = \frac{B(z)}{A(z)} \quad \begin{matrix} \leftarrow (\text{order } n-1) \\ \leftarrow (\text{order } n) \end{matrix} \quad \left(\begin{matrix} n \text{ poles and} \\ n-1 \text{ zeros} \end{matrix} \right)$$

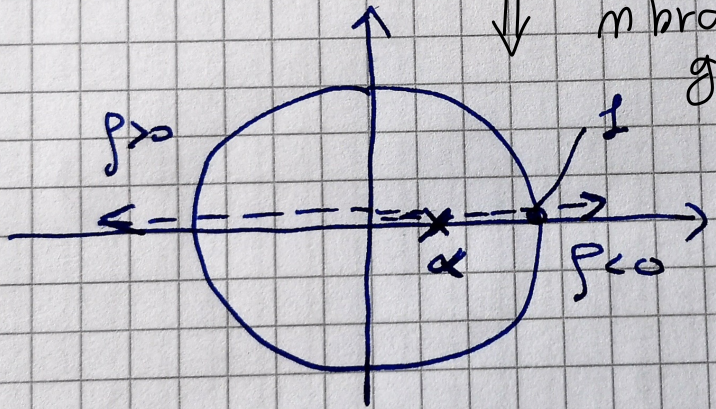
and



The root locus will have at least (if the relative degree is equal to 1) a branch going outside the stability region for $\rho \to \infty$

$\Downarrow$ closed loop syst has n branches from the poles going to ZEROS!

@ least one branch goes to ∞ → one cross stability region
↓
on discrete time syst (NO) ∞ gain
or one branch go outside stability

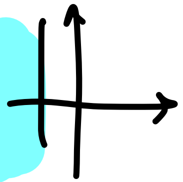


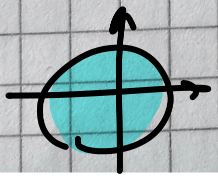
$$G(z) = \frac{1}{z-\alpha}$$

EXAMPLE

The stability region is bounded!!

LIMITATION! in cont. time sometimes you can have ∞ gain margin because ∞ stability region





While discrete time has limited stability region

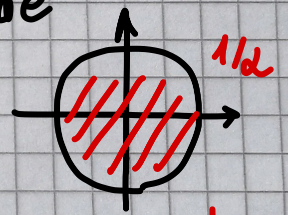
Example

Consider again the system

$$x(k+1) = 2x(k) + u(k)$$

and design a regulator with LQ $(Q=1, R=1)$ such that the closed-loop eigenvalue has modulus smaller than 0.25

We ask that the poles are inside



Solution

$$\frac{1}{\alpha} = 0.25 \longrightarrow \alpha = 4$$

Define $\tilde{A} = 4a = 8$, $\tilde{B} = 4B = 4$

redefine $\tilde{A} = \alpha A$, $\tilde{B} = \alpha B$

Solve the Riccati equation with $\tilde{A}$, $\tilde{B}$

$$P = 4.9501, \quad \tilde{K} = 1.9751$$

Fast system $\rightarrow$ $\cancel{A-B}$ $A - B\tilde{K} = 0.0249 \ll 0.25$ ok extension of ideas on discrete time

Kalman predictor and filter

(Important syst control technique widely used!)

[v_y: noise meas error
v_x: noise acting on syst]

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + v_x(k) \\ y(k) &= Cx(k) + v_y(k) \end{cases}$$

$x(k)$ unknown

noises acting on the system

- $v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ white gaussian noise with $E[v] = 0$

covariance matrix

v_x cov matrix

(**for simplicity**, assume NOT correlated noises)

$$E\left[v(k_1)v'(k_2)\right] = V\delta(k_1 - k_2) \qquad V = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{R} \end{bmatrix}, \quad \tilde{Q} \in R^{n,n}, \quad \tilde{R} \in R^{p,p}, \quad \tilde{Q} \geq 0, \quad \tilde{R} > 0$$

v_y cov

$x(0)$ gaussian with $E[x(0)] = \bar{x}_0$, $E\left[(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)'\right] = \tilde{P}_0 \geq 0$

$E\left[x(0)v'(k)\right] = 0, \quad \forall k \geq 0$

No correlation between initial state and noise vector

$$x(0) = (\bar{x}_0, \tilde{P}_0)$$

$$\hat{x}(0) \stackrel{?}{=} \bar{x}_0$$

"OBSERVERS which estimate a state @ k given meas up to k-1) = PREDICTOR
≠ FILTER which gave info given meas until k

Kalman predictor

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + \underbrace{L(k)}_{\text{Gain}}\underbrace{[y(k) - C\hat{x}(k|k-1)]}_{\text{innovation error}}$$

Let $\hat{e}(k|k-1) = x(k) - \hat{x}(k|k-1)$ be the state estimation error

$$\Downarrow$$

(by substituting...)

$$\begin{aligned}
\hat{e}(k+1|k) &= [A - L(k)C]\,\hat{e}(k|k-1) + v_x(k) - L(k)v_y(k) \\
&= [A - L(k)C]\,\hat{e}(k|k-1) + \begin{bmatrix} I & -L(k) \end{bmatrix} v(k) \\
&= A_c(k)\hat{e}(k|k-1) + B_c(k)v(k)
\end{aligned}$$

$$A_c(k) = [A - L(k)C]\,, \quad B_c(k) = \begin{bmatrix} I & -L(k) \end{bmatrix}$$

$$\downarrow$$

$$E\left[\hat{e}(k+1|k)\right] = A_c(k)E\left[\hat{e}(k|k-1)\right]$$

evolution of
expected value of estim error

$$E\left[\hat{e}(k+1|k)\right] = A_c(k)E\left[\hat{e}(k|k-1)\right]$$

setting properly $\hat{x}(0|-1) = \bar{x}_0 = \mathbb{E}[x(0)]$

If $\hat{x}(0|-1) = \bar{x}_0$ $\longrightarrow$ $E\left[\hat{e}(0|-1)\right] = 0$

starts from 0
so remain in 0
without input

$$E\left[\hat{e}(k+1|k)\right] = 0, \quad \forall k$$

Covariance of the estimation error   knowledge of system matrix...

and given initial

$$\tilde{P}(k|k-1) = E\left[\hat{e}(k|k-1)\hat{e}'(k|k-1)\right] \quad , \quad \tilde{P}(0|-1) = \tilde{P}_0$$

condition

Choice of the gain ***L(k)***   minimize a quadratic form of $\tilde{P}$

$$\min_{L(k)} \gamma' \tilde{P}(k+1|k)\gamma \qquad \gamma \text{ generic vector}$$

...

The gain $L(k)$ minimizing $\gamma' \tilde{P}(k+1|k)\gamma$ is

$$L(k) = A\tilde{P}(k|k-1)C' \left[\overbrace{C\tilde{P}(k|k-1)C' + \tilde{R}}^{\text{pos. def ! OK}} \right]^{-1}$$

computed $\forall k$
↓
use K.F formula
updating $L(k)$

where $\tilde{P}(k|k-1)$ is the solution to the Riccati equation

$$\begin{aligned}
& \tilde{P}(k+1|k) \\
= \quad & A\tilde{P}(k|k-1)A' + \tilde{Q} \\
- \quad & A\tilde{P}(k|k-1)C' \left[C\tilde{P}(k|k-1)C' + \tilde{R} \right]^{-1} C\tilde{P}(k|k-1)A'
\end{aligned}$$

with initial condition

$$\tilde{P}(0|-1) = \tilde{P}_0$$

{ trivial implementation } → sometimes we preffer a const gain → it converges for $k \to \infty$? ⇒

Duality

To Remember for exam...

LQ

$$u(k) = -K(k)x(k)$$

$$K(k) = (R + B'P'(k+1)B)^{-1} B'P'(k+1)A$$

$$P(k) = Q + A'P(k+1)A - A'P(k+1)B\,(R + B'P(k+1)B)^{-1} B'P(k+1)A$$

$$P(\bar{k}) = S$$

KP

proper transposition

$$L(k) = A\tilde{P}(k|k-1)C'\left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1}$$

same Riccati equation

$$\tilde{P}(k+1|k) = A\tilde{P}(k|k-1)A' + \tilde{Q} - A\tilde{P}(k|k-1)C'\left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1} C\tilde{P}(k|k-1)A'$$

$$\tilde{P}(0|-1) = \tilde{P}_0$$

LQ	KP
k	$-k$
A	A'
B	C'
Q	$\tilde{Q}$
R	$\tilde{R}$
P	$\tilde{P}$

$$K(k) \longleftrightarrow L'(k)$$

Stability of the stationary Kalman Predictor

If

(Bq partitioning of $\tilde{Q}$)

1. the pair (A, B_q), with B_q such that $\tilde{Q} = B_q B_q'$, is reachable;
2. the pair (A, C) is observable;

then $\Downarrow$

$\Rightarrow$ A) the optimal predictor is

$$\begin{aligned} \hat{x}(k+1|k) &= A\hat{x}(k|k-1) + Bu(k) + \bar{L}[y(k) - C\hat{x}(k|k-1)] \\ &= (A - \bar{L}C)\hat{x}(k|k-1) + Bu(k) + \bar{L}y(k) \end{aligned}$$

with

$$\bar{L} = A\bar{\tilde{P}}C'\left[C\bar{\tilde{P}}C' + \tilde{R}\right]^{-1}$$

and $\bar{\tilde{P}}$ is the unique positive definite solution of the stationary Riccati equation

$$\bar{\tilde{P}} = A\bar{\tilde{P}}A' + \tilde{Q} - A\bar{\tilde{P}}C'\left[C\bar{\tilde{P}}C' + \tilde{R}\right]^{-1} C\bar{\tilde{P}}A'$$

take only the p.d solution !

$\Rightarrow$ B) the estimator is asymptotically stable, that is all the eigenvalues of $(A - \bar{L}C)$ have modulus less than 1.

Example

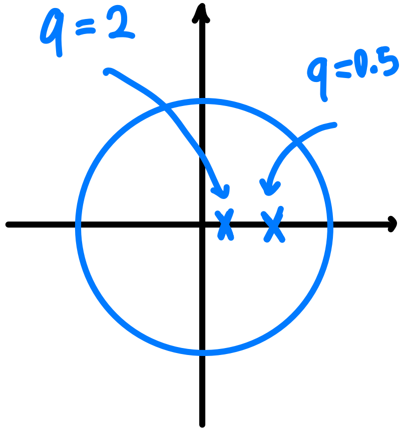
Consider the system → integrator system + error

$$\begin{cases} x(k+1) = +x(k) + b\,u(k) + v_x(k) \\ y(k) = x(k) + v_y(k) \end{cases}$$

where $v_x \sim WN(0, q)$, $v_y \sim WN(0, 1)$

and all the other conditions required to apply the theory of the Kalman predictor are satisfied.

Note that, although $y(k) = x(k) +$ disturbance, the use of a KP (or a filter) is recommended to filter out the effect of the noises.



$$P = APA' + \tilde{Q} - \underbrace{APC'(CPC' + \tilde{R})^{-1}}_{L} CPA'$$

$$\hat{x}(k+1|k) = (A - LC)\hat{x}(k|k-1) + Bu(k) + Ly(k)$$

In our case $A = C = 1$, $\tilde{R} = 1$

For different values of q, one has

$$\begin{cases} q = 0.5 \rightarrow P = 1 \rightarrow L = 1/2 \rightarrow A - LC = 0.5 \\ q = 1 \rightarrow P = 1.618 \rightarrow L = 0.618 \rightarrow A - LC = 0.382 \\ q = 2 \rightarrow P = 2.7321 \rightarrow L = 0.7321 \rightarrow A - LC = 0.2679 \end{cases}$$

The prediction becomes faster and faster as q increases

Kalman Filter

$$\tilde{x}(k+1|k+1) = A\tilde{x}(k|k) + Bu(k) + L(k+1)\,\underbrace{[y(k+1) - C\left(A\tilde{x}(k|k) + Bu(k)\right)]}_{\text{innovation error}}$$

↓ (similar to pole placement observer..)

where

↳ different formulas than predictor ⇒

$$\begin{cases} L(k) &= \tilde{P}(k|k-1)C'\left[C\tilde{P}(k|k-1)C' + \tilde{R}\right]^{-1} \\ \tilde{P}(k|k-1) &= A\hat{P}(k-1|k-1)A' + \tilde{Q} \\ \hat{P}(k|k) &= \tilde{P}(k|k-1) - L(k)C\tilde{P}(k|k-1) \\ \hat{P}(0|0) &= \tilde{P}_0 \end{cases}$$

$\hat{P}(k|k)$ is the covariance of the estimation error $\hat{e}(k|k) = x(k) - \hat{x}(k|k)$.

Convergence and stability results similar to the ones of KP can be established

Extended Kalman Filter (Predictor) - *EKF*

Extension for NON LINEAR systems! allow to enstimate!

A very important method, **widely used many fields** of engineering and science to estimate **the state** of nonlinear systems and **the parameters** of grey box models (very usefull, lot used!)

All the details on its development are reported in the textbook

(additive disturbances)

System generally non linear

$$\begin{cases} x(k+1) = f\left(x(k), u(k)\right) + \boxed{v_x(k)} \\ y(k) = g\left(x(k)\right) + \boxed{v_y(k)} \end{cases}$$

$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ white gaussian noise with $E[v] = 0$

same assumptions as (KF)

for simplicity

$$E\left[v(k_1)v'(k_2)\right] = V\delta(k_1 - k_2) \qquad V = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{R} \end{bmatrix}, \quad \tilde{Q} \in R^{n,n}, \quad \tilde{R} \in R^{p,p}, \quad \tilde{Q} \geq 0, \quad \tilde{R} > 0$$

$$x(0) \text{ gaussian with } E[x(0)] = \bar{x}_0,\ E\left[\left(x(0) - \bar{x}_0\right)\left(x(0) - \bar{x}_0\right)'\right] = \tilde{P}_0 \geq 0$$

$$E\left[x(0)v'(k)\right] = 0, \quad \forall k \geq 0$$

Structure of EKF

final idea is as general K.P — using linearized model

$$\hat{x}(k+1|k) = \underbrace{f\left(\hat{x}(k|k-1), u(k)\right)}_{\text{mimics the dynamics of the system}} + \hat{L}(k)\underbrace{\left[y(k) - g(\hat{x}(k|k-1))\right]}_{\text{output estimation error}}$$

using the N.L dynamics KNOWN

use linearization properly

gain of the filter

$$\hat{L}(k) = \hat{A}(k)\tilde{P}(k|k-1)\hat{C}(k)'\left[\hat{C}(k)\tilde{P}(k|k-1)\hat{C}(k)' + \tilde{R}\right]^{-1}$$

where

linearization matrix of the system

$$\hat{A}(k) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\hat{x}(k|k-1),u(k)}, \quad \hat{C}(k) = \left.\frac{\partial g(x)}{\partial x}\right|_{\hat{x}(k|k-1)}$$

A, C change overtime!

and

{ Riccati equation } ⇒

$$\tilde{P}(k+1|k) = \hat{A}(k)\tilde{P}(k|k-1)\hat{A}(k)' + \tilde{Q} +$$
$$- \hat{A}(k)\tilde{P}(k|k-1)\hat{C}(k)'\left[\hat{C}(k)\tilde{P}(k|k-1)\hat{C}(k)' + \tilde{R}\right]^{-1}\hat{C}(k)\tilde{P}(k|k-1)\hat{A}(k)'$$

using $\hat{A}$, $\hat{C}$ matrix linearized!

$$\tilde{P}(0|-1) = \tilde{P}_0$$

time (k) dependent, We cannot have steady state EKF!

as:

$$x(k+1) = \underbrace{A\,x(k) + B\,u(k)}_{f(x(k),\,u(k))} + v_x$$

$$\hat{x}(k+1|k) = A\,\hat{x}(k|k-1) + B\,u(k) + L(k)[\cdots]$$

(EKF) $$\hat{x}(k+1|k) = f(\hat{x}(k|k-1), u(k)) + L(k)[\cdot\cdot]$$

Comments on *EKF*

- Very useful to estimate unknown parameters (see the next slide)

- Continuous time formulations for continuous time systems available

$\Downarrow$

- In case of a continuous time system, the system can be discretized first (Euler, for example), and then the filter is applied to the discretized model. The approximation errors due to discretization can be significant.

  Mixed continuous – discrete formulations are widely used. The dynamics of the system is in continuous time, the gain update is in discrete time to reduce the computational effort and to consider the availability of discrete measurements (wide literature available, see also next slides)

- Very few convergence results available (basically, if you start near the real value of the state, convergence is guaranteed

  theoretically EKF is very limited! → NOT SO meaningful results!

W. B model → all KNOWN
B. B model → all UNKNOWN
G. B model → some param unknown
↳ use K. F for estimation of parameters

Estimation of unknown parameters with *EKF*

Consider the system

(for example parameter on physical system to estimate)

$$\begin{cases} x(k+1) = f\left(x(k), u(k), \alpha\right) + v_x(k) \\ y(k) = g\left(x(k), \alpha\right) + v_y(k) \end{cases}$$

GRAY BOX modelling →

α unknown constant parameter

Add a fictitious dynamics

$$\alpha(k+1) = \alpha(k)$$

{ important also in Black Box model }
when I wanna compute some internal param

Enlarge the state $\tilde{x} = \begin{bmatrix} x \\ \alpha \end{bmatrix}$

to estimate both state and param!

Apply EKF to

$$\begin{cases} \tilde{x}(k+1) = \tilde{f}\left(\tilde{x}(k), u(k)\right) + \tilde{v}_x(k) \\ y(k) = \tilde{g}\left(\tilde{x}(k)\right) + v_y(k) \end{cases}$$

Example EKF

$$\begin{cases} x(k+1) = ax(k) + u(k) \\ y(k) = x(k) + v(k) \end{cases}$$

$\downarrow$

a unknown parameter to be estimated

non linearity!

$$\begin{cases} x(k+1) = a(k)x(k) + u(k) \\ a(k+1) = a(k) \\ y(k) = x(k) + v(k) \end{cases} \longrightarrow \hat{A}(k) = \begin{bmatrix} \hat{a}(k) & \hat{x}(k) \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

enlarge the state!

Now II ORD system,

(with dynamic nonlinear, here linearized matrix)

Simulations

what happens for different values of a

$$\begin{cases} k < 100 \rightarrow a = 0.5 \\ k \geq 100 \rightarrow a = 0.8 \end{cases}$$

Input: square wave ±1 with period 5 , $\underbrace{x(0) = 1}_{\text{initial state}}$

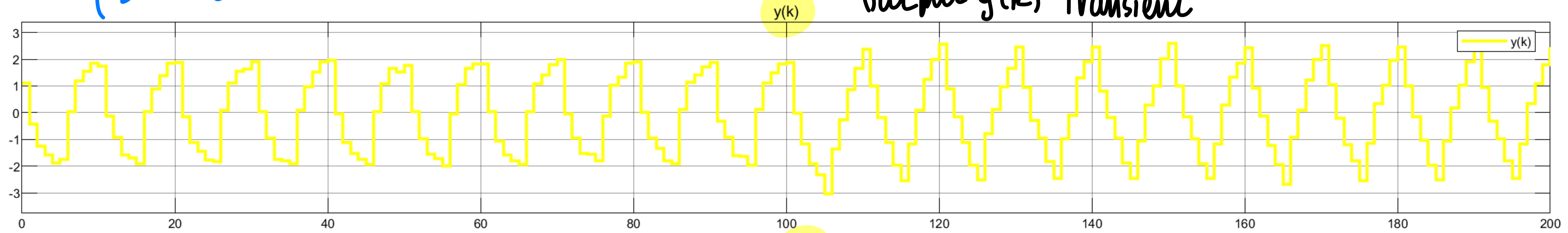
Output noise: white, zero mean, variance 0.01

$P(0) = 100I$, $\begin{bmatrix} \hat{x}(0) \\ \hat{a}(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$ (set initial values...)

Covariance matrices $Q=0.1\ I$, $R = 1$

(simulation)

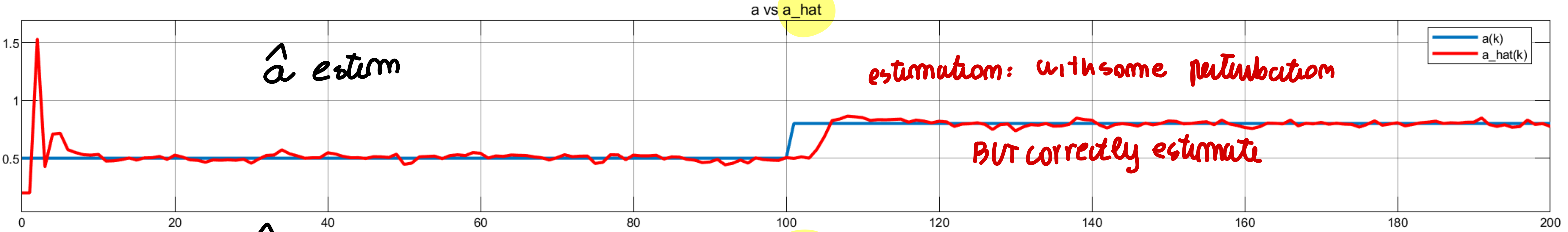
output y(k) transient



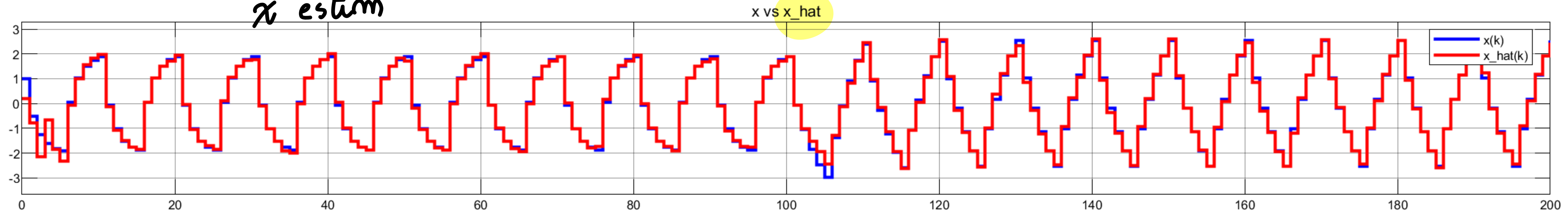
â estim

estimation: with some perturbation

BUT correctly estimate



x̂ estim



↑ estimation of state x and parameter!

DISCRETE

(NOT in exam)

Continuous Extended Kalman Filter with *discrete measurements*

System (cont. time)

if discretize the syst. we include discretization error!

$$
\begin{cases}
\dot{x}(t) = f(x(t), u(t)) + v_x(t) \\
y(k) = g(x(k)) + v_y(k) \\
x(k) = x(t_k)
\end{cases}
$$

← cont time state dynamic

BUT discrete measurement

discrete state measurements, sampling @ t_k

$$v_x = wgn(0, \tilde{Q}) \quad v_y = wgn(0, \tilde{R})$$

standard assumptions on v_x, v_y and initial values

EKF – initialization

$$
\begin{cases}
\hat{x}(0|0) = E[x(t_0)] \\
P(0|0) = E[(x(t_0) - \hat{x}(t_0))(x(t_0) - \hat{x}(t_0))']
\end{cases}
$$

(NOT in exam) smart EKF extension

EKF – predictor step

continuous dynamics

run the syst in continuous time

$$\begin{cases} \dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) \\ \dot{P}(t) = A(t)P(t) + P(t)A'(t) + \tilde{Q} \end{cases}$$

run the Riccati eq.

$$\hat{x}(t_{k-1}) = \hat{x}(k-1|k-1)$$
$$P(t_{k-1}) = P(k-1|k-1)$$

update at each sampling time t_k

$$\begin{cases} \hat{x}(k|k-1) = \hat{x}(t_k) \\ P(k|k-1) = P(t_k) \end{cases}$$

$$A(t) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\hat{x}(t),u(t)}$$

⇓ than update L(k) and estimate

running cont time model

EKF – update step

$$L(k) = P(k|k-1)C'(k)(C(k)P(k|k-1)C'(k) + \tilde{R})^{-1}$$
$$\hat{x}(k|k) = \hat{x}(k|k-1) + L(k)(y(k) - g(\hat{x}(k|k-1)))$$
$$P(k|k) = (I - L(k)C(k))P(k|k-1)$$

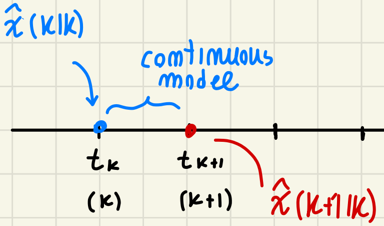
$$C(t) = \left.\frac{\partial g(x)}{\partial x}\right|_{\hat{x}(k|k-1)}$$

update with standard EKF

discrete update → powerfull!

(IDEA) given t_k, I integrate cont. time system from one sampling to another, this reduce discretiz. effect

using continuous dynamic... reducing the a priori discretization issue



predictor step

estimate new $\hat{x}(k+1|k)$

and update @ the end..

let cont. time model evolution! → [updating $\hat{x}$ given past values]

LQG control

↓

System

given linear discrete syst

(Gaussian noises

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + v_x(k) \\ y(k) = Cx(k) + v_y(k) \end{cases}$$

(known states for LQG)!

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}, \; x(0) \text{ satisfy the assumption adopted to derive the KF}$$

Cost function

↓ define the cost function as done in cont. time

(remove the problem of positive summ, so J anyway finite

$$J = \lim_{\bar{k}\to\infty} \frac{1}{\bar{k}} E\left[\sum_{k=0}^{\bar{k}-1} x'(k)Qx(k) + u'(k)Ru(k)\right]$$

stochastic x, y terms! because gaussian noise

LQG control - solution

1. determine with the stationary Kalman predictor (or filter) the Kalman gain $\bar{L}$ and the estimate $\hat{x}(k|k-1)$ $(\tilde{x}(k|k))$;
2. compute the optimal LQ control law, that is the gain $\bar{K}$;
3. apply the control law

$$u(k) = -\bar{K}\hat{x}(k|k-1) \text{ in the predictor case}$$

or

$$u(k) = -\bar{K}\tilde{x}(k|k) \text{ in the filter case.}$$

⇓ separation principle holds!

The closed-loop eigenvalues are those of $(A - B\bar{K})$ and of $(A - \bar{L}C)$ (if the predictor is used) NO problems if conditions respected

all done for H_2- H_∞ control remain valid in discrete time!