June 2011

$\boxed{\text{Ex 1}}$

a)
$$\begin{cases} \bar{x}_1 = 0.5\bar{x}_1 + 3\bar{x}_1^2 \\ \bar{x}_2 = 0.5\bar{x}_2 + \bar{x}_1\bar{x}_2 \end{cases} \qquad \begin{cases} 0 = (-0.5 + 3\bar{x}_1)\bar{x}_1 \\ 0 = (-0.5 + \bar{x}_1)\bar{x}_2 \end{cases}$$

$$\begin{cases} \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{cases} \quad , \quad \begin{cases} \bar{x}_1 = 1/6 \\ \bar{x}_2 = 0 \end{cases}$$

b) Linearised model

$$\begin{cases} \delta x_1(k+1) = 0.5\,\delta x_1(k) + 6\bar{x}_1\delta x_1(k) \\ \delta x_2(k+1) = 0.5\,\delta x_2(k) + \bar{x}_2\,\delta x_1(k) + \bar{x}_1\,\delta x_2(k) \end{cases}$$

equilibrium $\bar{x}_1 = \bar{x}_2 = 0$

$$\begin{cases} \delta x_1(k+1) = 0.5\,\delta x_1(k) \\ \delta x_2(k+1) = 0.5\,\delta x_2(k) \end{cases} \rightarrow A = \begin{vmatrix} 0.5 & 0 \\ 0 & 0.5 \end{vmatrix}$$ asymptotic stability

equilibrium $\bar{x}_1 = 1/6$, $\bar{x}_2 = 0$

$$\begin{cases} \delta x_1(k+1) = 1.5\,\delta x_1(k) \\ \delta x_2(k+1) = 0.5\,\delta x_2(k) + \frac{1}{6}\delta x_2(k) \end{cases} \rightarrow A = \begin{vmatrix} 1.5 & 0 \\ 0 & 0.5+1/6 \end{vmatrix}$$

unstable equilibrium

c)

$$\Delta V(x) = \alpha x_1^2(k+1) + \beta x_2^2(k+1) - \alpha x_1^2(k) - \beta x_2^2(k), \quad \alpha > 0,\ \beta > 0$$

(equilibrium $\bar{x}_1 = \bar{x}_2 = 0$)

$$\Delta V(x) = \alpha\left(0.25\, x_1^2 + 3x_1^2 + 9x_1^4\right) + \beta\left(0.25\, x_2^2 + x_1 x_2^2 - x_1^2 x_2 \ldots\right.$$

$$- \alpha x_1^2 - \beta x_2^2$$

$$= \underbrace{(0.25\alpha - \alpha)\, x_1^2 + (0.25\beta - \beta)\, x_2^2}_{<0} + \text{higher order terms}$$

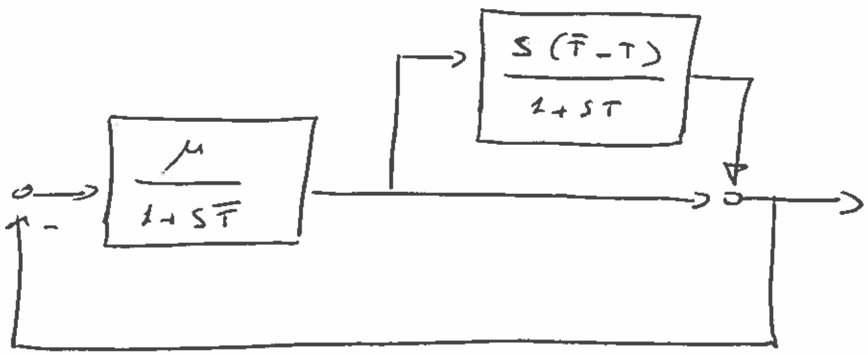
locally $\Delta V(x) < 0$

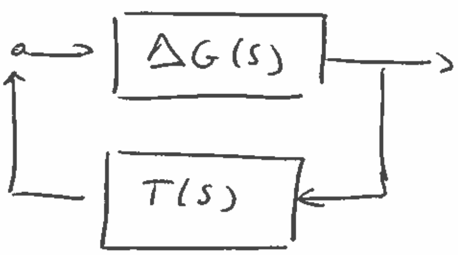
$\boxed{\text{Ex 2}}$

$$G(s) = \bar{G}(s)\left(1 + \Delta G(s)\right)$$

$$\frac{\mu}{1 + sT} = \frac{\mu}{1 + s\bar{T}}\left(1 + \Delta G(s)\right)$$

$$\Delta G(s) = \frac{1 + s\bar{T}}{\mu}\left(\frac{\mu}{1 + sT} - \frac{\mu}{1 + s\bar{T}}\right) = \frac{s(\bar{T} - T)}{1 + sT}$$





$$T(s) = \frac{L(s)}{1 + L(s)} =$$

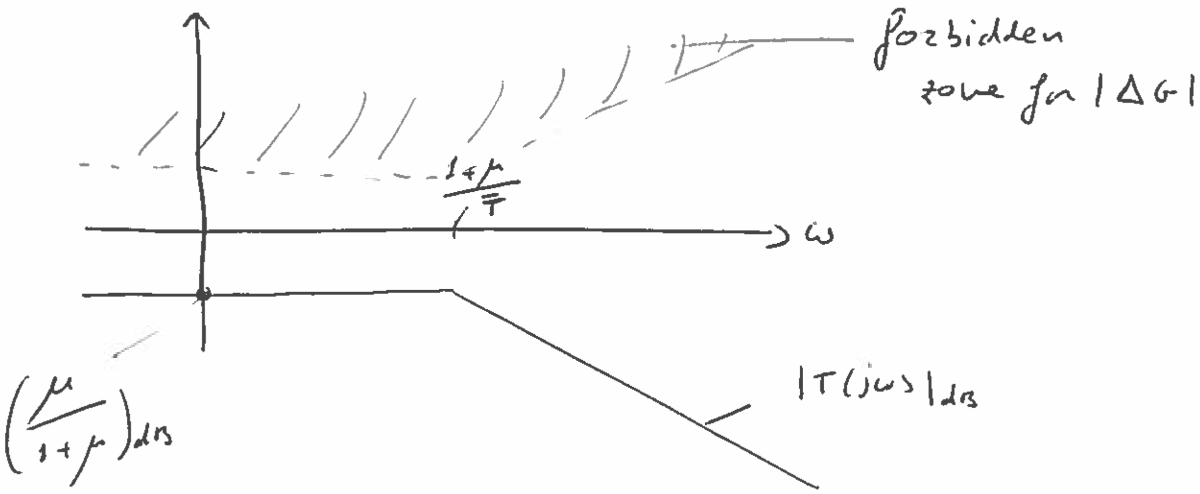
$$= \frac{\frac{\mu}{1 + s\bar{T}}}{1 + \frac{\mu}{1 + s\bar{T}}} = \frac{\mu}{1 + \mu + s\bar{T}}$$

$$T(s) = \frac{\frac{\mu}{1+\mu}}{1 + \frac{T}{1+\mu} s}$$

Condition

$$|T(j\omega)\,\Delta G(j\omega)| < 1 \qquad \forall \omega$$

$$|\Delta G(j\omega)| < \frac{1}{|T(j\omega)|}$$



$\boxed{\text{EX 3}}$

$$G_d(s) + G_u(s) \longrightarrow \begin{cases} \dot{x} = -a\,x + u & (G_u(s)) \\ \dot{z} = w & (G_d(s)) \end{cases}$$

$$\begin{cases} \begin{vmatrix} \dot{x} \\ \dot{z} \end{vmatrix} = \overbrace{\begin{vmatrix} -a & 0 \\ 0 & 0 \end{vmatrix}}^{A} \begin{vmatrix} x \\ z \end{vmatrix} + \overbrace{\begin{vmatrix} 0 \\ 1 \end{vmatrix} w}^{V_x} \\ y = \underbrace{\begin{vmatrix} 1 & 1 \end{vmatrix}}_{C} \begin{vmatrix} x \\ z \end{vmatrix} + V \quad , \quad \tilde{R} = 1 \end{cases}$$

$$E[V_x V_x'] = \begin{vmatrix} 0 \\ 1 \end{vmatrix} |0 \; 1| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = \tilde{Q}$$

Observability of (A, c):

$$M_o = \begin{vmatrix} c \\ cA \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -a & 0 \end{vmatrix} \longrightarrow a \neq 0 \quad \text{(as assumed)}$$

Reachability of $(A, \tilde{Q}^{1/2})$, $\tilde{Q} = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$

$$M_r = \begin{vmatrix} \tilde{Q}^{1/2} & A\tilde{Q}^{1/2} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad (\tilde{Q}^{1/2} = \tilde{Q})$$

the reachability condition is not verified, the matrix $\tilde{P}(t)$ of the Riccati equation of the predictor can only converge to a solution $\tilde{P} \geq 0$, <u>not</u> to $\tilde{P} > 0$

<u>Gain of the predictor with the stationary $\tilde{P} \geq 0$</u>

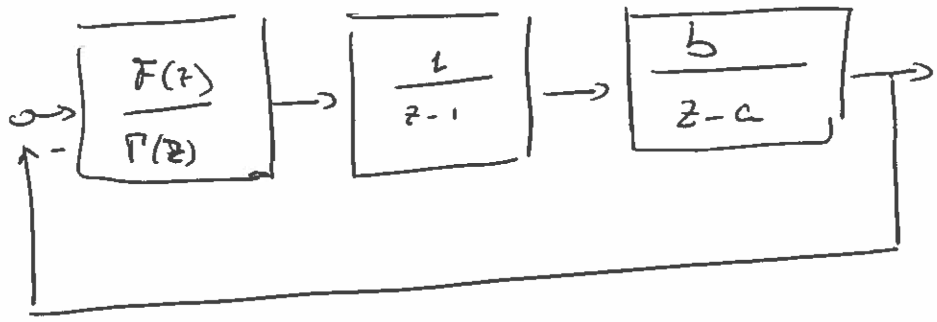
$$L = \tilde{P} c' \tilde{R}^{-1} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

$$A - Lc = \begin{vmatrix} -a & 0 \\ 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 \\ 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \end{vmatrix} = \begin{vmatrix} -a & 0 \\ 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -a & 0 \\ 0 & -1 \end{vmatrix}$$ eigenvalues in $-a$ and -1 (the predictor is asympt. stable)

Ex 4

a)



$$F(z) = f_1 z + f_0 \quad , \quad A(z) = (z-1)(z-a) = z^2 - (a+1)z + a$$

$$\Gamma(z) = \gamma_1 z + \gamma_0 \quad , \quad B(z) = b$$

$$P(z) = z^3 + p_2 z^2 + p_1 z + p_0 \quad \text{(design parameter)}$$

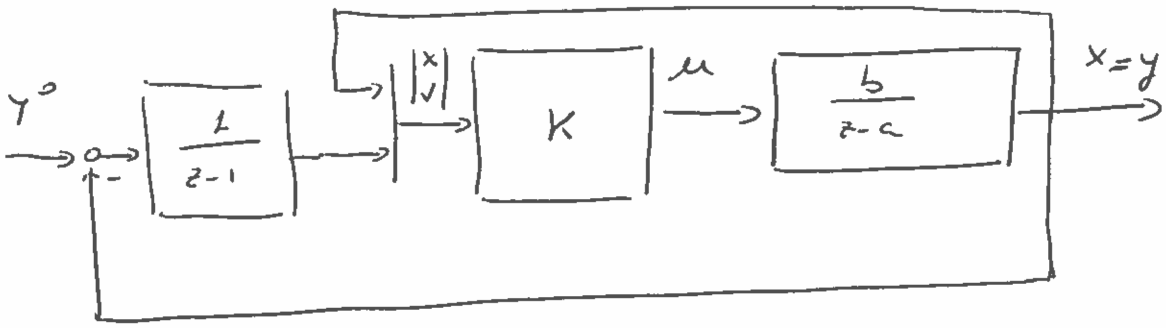
$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -(a+1) & 1 & 0 & 0 \\ a & -(a+1) & b & 0 \\ 0 & a & 0 & b \end{vmatrix} \begin{vmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{vmatrix} = \begin{vmatrix} 1 \\ p_2 \\ p_1 \\ p_0 \end{vmatrix}$$

b)

$$\begin{cases} x(h+1) = a\,x(h) + b\,u(h) \\ y(h) = x(h) \\ v(h+1) = v(h) + y^\circ - y(h) = v(h) + y^\circ - x(h) \end{cases}$$

$$\begin{cases} \begin{vmatrix} x(h+1) \\ v(h+1) \end{vmatrix} = \overbrace{\begin{vmatrix} a & 0 \\ -1 & 1 \end{vmatrix}}^{A} \begin{vmatrix} x(h) \\ v(h) \end{vmatrix} + \overbrace{\begin{vmatrix} b \\ 0 \end{vmatrix}}^{B} u(h) + \begin{vmatrix} 0 \\ 1 \end{vmatrix} y^\circ \\ y(h) = \begin{vmatrix} 1 & 0 \end{vmatrix} \begin{vmatrix} x(h) \\ v(h) \end{vmatrix} \end{cases}$$

Control scheme



K is designed such that $A-BK$ is asymptotically stable (with pole placement, LQ, H_2, H_∞, ...)

Ex 5

See the notes