Exercises session 7: Model Predictive Control

Ex. 1: Given the discrete time system

$$\begin{cases} x(k+1) = 5x(k) + u(k) \end{cases} \tag{1}$$

and the cost function

$$J(x,u) = \sum_{i=0}^{N-1} \left[(x(k+i))^2 + (u(k+i))^2 \right] + \mathcal{S}x(k+N) \tag{2}$$

1. With $\mathcal{S} = 1$, find the minimum prediction horizon N which ensure the closed-loop stability.
2. If $N = 1$, which is the minimum value of $\mathcal{S}$ which guarantees the closed loop stability?
3. Consider $N = 2$, $S = 1$ and $x(k) = 0.2$. Moreover, assume the following constraints

$$\begin{cases} 0 \leq x(k+i) \leq 2 \\ -1 \leq u(k+i) \leq 1 \end{cases} . \tag{3}$$

Show how to set-up a Quadratic programming optimization problem.

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   Show how to set-up a Quadratic programming optimization problem.

1) $J(x,u) = \sum_{i=0}^{N-1} \left[\underbrace{x^2(k+i)}_{\text{STATES}} + \underbrace{u^2(k+i)}_{\text{CONTROL VAR.}} \right] + \underbrace{S\,x(k+N)}_{\text{TERMINAL COST.}}$

STATES: try to decrease it as a cost

CONTROL VAR.: it is a cost, we try to limit it.

TERMINAL COST.: ultimate goal on the states..

↓

MPC formulation

$$\min_{u(k),\dots,u(k+N-1)} \left\{ J(x,u) \right\} = \sum_{i=0}^{N-1} \left(x^T(k+i)\,Q\,x(k+i) + u^T(k+i)\,R\,u(k+i) \right) + x^T(k+N)\,S\,x(k+N)$$

s.t :

$$x(k+i+1) = A\,x(k+i) + B\,u(k+i) \quad \forall i = 0,\dots,N-1$$

{ dynamic of discrete time syst happening in time.. }

dym eq.. need one more sample $x(k+N)$ respect u until $u(k+N-1)$ non dym

$$x(k+i) \in X \quad \forall i = 0,\dots,N$$

$$u(k+i) \in U \quad \forall i = 0,\dots,N-1$$

states prediction horiz.

Boundaries..

not last state

⇓

MPC steps to obtain control input...

PRINCIPLE of MPC

1) sample state $X(k)$ (IF non available → use observer)

2) Solve optimization Problem

3) Apply $u^*(K)$

4) Repeat at $K+1$

in practice, similar to the "driving" car procedure → where you observe the street (samples) apply first command and repeat... achieving control

without constraints

↓

analytical solution feasible.. you JUST need to solve R.E. of LQ finite horizon

⇒ solve MPC analytically!

time varying gain found by R.E.

LQ FINITE HORIZON =
$$\begin{cases} u(k+i) = -K(i)\,x(k+i) \\ K(i) = (R + B^T P(i+1) B)^{-1} B^T P(i+1) A \\ P(i) = A^T P(i+1) A + Q - A^T P(i+1) B (R + B^T P(i+1) B)^{-1} B^T P(i+1) A \\ P(N) = S \end{cases}$$

unconstrained MPC as an LQ finite horiz problem!

→ to obtain an optimal control sequence $\forall i$ and Repeat the process...

- min pred horiz N ensuring closed-loop stability → N?

- We start with $N=1$ → $u(K) = -K(0)\,x(K)$ optimal value we aim for

from formulas... $K(0) = (R + B^T P(1) B)^{-1} B^T P(1) A =$

$(Q = 1,\ R = 1)$ $\quad A = 5 \quad B = 1$ from syst dyn eq..

⇓

$$K(0) = (1 + 1^T P(1) 1)^{-1} 1^T P(1) 5 = (2)^{-1} 5 = 5/2$$

we need to compute $P(1)$..

↓

$P(1) = P(N) = S = 1$

↳ $u(K) = 5/2\ x(K)$

$x(k+1) = Ax(k) + Bu(k) = (A - BK(0))x(k) = (5 - 5/2)x = 2.5\,x(k)$

UNSTABLE!!

NOT OK $N=1$!

⇓

- we can try $N=2$ repeating the procedure...

$K(0) = (R + B^T P(1) B)^{-1} B^T P(1) A$

$$\begin{cases} P(1) = A^T P(2) A + Q - A^T P(2) B (R + B^T P(2) B)^{-1} B P(2) A \\ P(2) = P(N) = S = 1 \end{cases}$$

substituting...

⇓

$$P(1) = 5 \cdot 1 \cdot 5 + 1 - \frac{5 \cdot 1 \cdot 1}{1+1} 1 \cdot 1 \cdot 5 = 25 + 1 - \frac{25}{2} = 13.5$$

$$K(0) = (1 + 1 \cdot 13{,}5 \cdot 1)^{-1} 1 \cdot 13{,}5 \cdot 1 = 4{,}65$$

⇓

$$A - B \cdot K(0) = 5 - 4{,}65 = 0{,}35$$

Asymp. stable!

OK $N=2$

② min S such closed loop stability

↓

given $N=1$ ⇒ find $K(0)$ stabilizing

$K(0) = (R + B^T P(1) B)^{-1} B^T P(1) A =$ (*)...

looking @ closed loop syst.. asymp stable $\iff |\det(zI - (A - BK(0)))| < 1$

↓

the ROOTS z_i of $\det(zI \dots) = 0$ are such that $|z_i| < 1 \;\; \forall i = 1, \dots$

SISO, I ORD system

↓

$|A - BK(0)| < 1$ to be A.S

Knowing $P(1) = P(N) = S$ (as given by text $N=1$)

$$\Rightarrow P(1) = S \longrightarrow K(0) \overset{(*)}{=} (1 + 1 \cdot S \cdot 1)^{-1} 1 \cdot S \cdot 5 = \frac{5S}{1+S}$$

$$\Rightarrow \begin{cases} 5-\dfrac{5s}{s+1} < 1 \longrightarrow \cancel{5s}+5-\cancel{5s} < s+1 \longrightarrow s>4 \\ -1 < 5-\dfrac{5s}{s+1} \longrightarrow -s-1 < \cancel{5s}+5-\cancel{5s} \longrightarrow s>-6 \end{cases}$$

$s>0$

$$\boxed{s>4}$$

3 $N=2,\ S=1$

$x(k)=0.2$

assume $\begin{cases} 0 \le x(k+i) \le 2 \\ -1 \le u(k+i) \le 1 \end{cases}$

What is a Q.P

$$\min_{\varphi} \left\{ J(\varphi)=\frac{1}{2}\varphi^T H \varphi + f^T\varphi \right\}$$

s.t $\quad A_{eq}\,\varphi = b_{eq} \qquad l_b \le \varphi \le u_b$

$\quad A_{in}\,\varphi \le b_{in}$

typical problems easy to solve by solver!

$$\begin{cases} x(k+1)=Ax(k)+Bu(k) \\ x(k+2)=Ax(k+1)+Bu(k+1)=A^2x(k)+ABu(k)+Bu(k+1) \\ \vdots \end{cases}$$

We wanna collect the states in a vector

$$\begin{bmatrix} x(k+1) \\ x(k+2) \end{bmatrix} = \begin{bmatrix} A \\ A^2 \end{bmatrix} \underbrace{x(k)}_{\text{measured}} + \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix}$$

{ System dynamic }

$$X = A_c\, x(k) + B_c \cdot U$$

analyze the COST FUNCTION of Q.P

$$J(x,u)=\sum_{i=0}^{N-1} \left(x^T(k+i)\,Q\,x(k+i) + u^T(k+i)\,R\,u(k+i) \right) + x^T(k+N)\,S\,x(k+N) =$$

don't participate on J

$$= \cancel{x^T(k)\,Q\,x(k)} + \underbrace{\begin{bmatrix} x^T(k+1) & \dots & x^T(k+N) \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q_S \end{bmatrix}}_{Q_c} \underbrace{\begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N) \end{bmatrix}}_{=X} + \dots \longrightarrow$$

Writing the same conclusions.. for the control var.

$$\dots + \underbrace{\left[U^T(k) \;\dots\; U^T(k+N-1) \right]}_{U^T} \underbrace{\begin{bmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{bmatrix}}_{R_c} \underbrace{\begin{bmatrix} U(k) \\ \vdots \\ U(k+N-1) \end{bmatrix}}_{U}$$

We develop the same cost function J

↓

Quadratic J can be expressed... $J(X,U) = X^T Q_c X + U^T R_c U$

↓

defining $\varphi = \begin{bmatrix} X \\ U \end{bmatrix}$ we can write

$$J(\varphi) = \varphi^T \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix} \varphi = \frac{1}{2} \varphi^T H \varphi$$

$$H = 2 \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix}$$

Q.P cost function

rewriting our problem

Respect the constraints: from the syst dynamic..

Using φ as control var → we rearrange such that

$$X = A_c x(k) + B_c U \;\longrightarrow\; X - B_c U = A_c \overbrace{x(k)}^{\text{known, measured}}$$

equality

$$\begin{bmatrix} I & -B \end{bmatrix} \underbrace{\begin{bmatrix} X \\ U \end{bmatrix}}_{=\varphi} = A_c x_k \;\longleftrightarrow\; A_{eq} \varphi = b_{eq}$$

bounds

$$\begin{cases} \underline{0} \le X \le \underline{2} \\ -\underline{1} \le U \le \underline{1} \end{cases} \;\longrightarrow\; \underbrace{\begin{bmatrix} \underline{0} \\ -\underline{1} \end{bmatrix}}_{l_b} \le \underbrace{\begin{bmatrix} X \\ U \end{bmatrix}}_{\varphi} \le \underbrace{\begin{bmatrix} \underline{2} \\ \underline{1} \end{bmatrix}}_{u_b}$$

→ implementing in matlab ...

{ optimal solution } ↳ $\varphi = \begin{bmatrix} 0.069 \\ 0.1724 \\ -0.931 \\ -0.1724 \end{bmatrix}$

When writing MPC → min probleem..

$\min_{u(k) \dots u(k+N-1)}$

↑ only control input optimiz var.

decreasing computational error → representing all dynamic as function of $u(k)$ (NOT always easy)

$\min_{\substack{u(k) \dots u(k+N-1) \\ x(k) \dots x(k+N)}}$

↑ while to represent all dynamic using also the states is more natural, including optimiz.var $\underline{x}(k)$.. so computationally harder!

Ex. 2: Given the step $u(k) = step^*(k)$, consider the system whose response to that step is

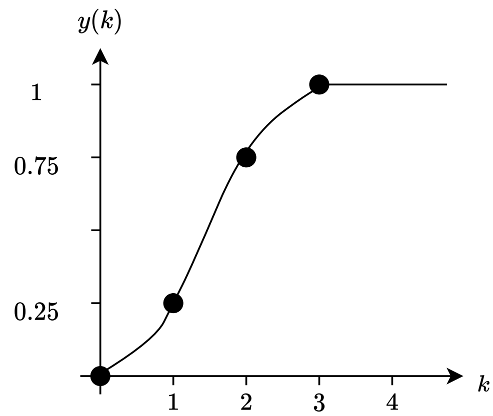

Figure 1

1. Define the step response model and the impulse response model of the system.
2. Formulate the closed-loop impulse-response predictive model with horizon N=2.
3. Formulate the closed-loop step-response predictive model with horizon N=2.

Ex. 2: Given following dynamical system

$$x(k+1) = 0.5x(k) + 0.5u(k) \tag{4}$$

1. Define the predictive model with horizon N=3 using the impulse response.
2. Show how to modify the approach to have a closed-loop control scheme.

$$P(i) = Q + A^T P(i+1) A - A^T P(i+1) B \left(R + B^T P(i+1) B\right)^{-1} B^T P(i+1) A$$

R.E
desncte
time

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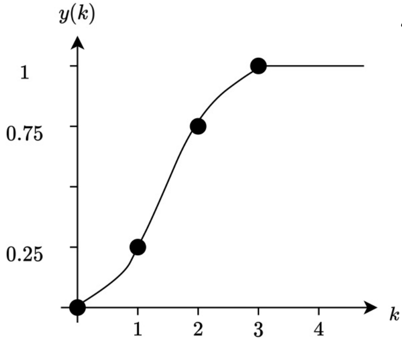

step responce $y(k)$

Figure 1

1. Define the step response model and the impulse response model of the system.
2. Formulate the closed-loop impulse-response predictive model with horizon N=2.
3. Formulate the closed-loop step-response predictive model with horizon N=2.

1) to define it we assume the system to be asymp. stable
(so it will not diverge if we apply a step)

from graph → $y(1) = 0.25 = s_1$ ⇒ step, impulse responce?
$y(2) = 0.75 = s_2$
$y(3) = 1 = s_3$
$y(4) = 1 = s_4$

impulse responce
$y(1) = g_1$ (different notation)
$y(2) = g_2$
$y(3) = g_3$
...
$y(n) = g_n$: for $n \to +\infty$ $g_n \to 0$ impulse resp if asymp. stable syst

Relation between IMPULSE-STEP responce model

$$s_i = \sum_{J=1}^{i} g_J \Rightarrow s_i - s_{i-1} = g_i$$

↑ from this relation, given the step responce $y(k)$ we can find:

$g_1 = s_1 - s_0 = 0.25$
$g_2 = s_2 - s_1 = 0.5$
$g_3 = s_3 - s_2 = 0.25$
$g_4 = s_4 - s_3 = 0$ ← already for $n=4$ $g_n = 0$

from impulse responce $\rightsquigarrow$ impulse responce model ?

$$y(k) = \sum_{J=1}^{\infty} g_J \cdot u(k-J)$$

computational problem ! $\rightarrow$ we can instead use the summ until m, such that for $J \geq m$ $g_J = 0$

Denote with M the value such that $g_J = 0 \quad \forall J \geq M$

then... $y(k) = \sum_{J=1}^{M} g_J \, u(k-J)$ (Just for notation..) general formulation of I.R model

to get prediction from this model...

FUTURE OUTPUTS $\quad y(k+i) = \sum_{J=1}^{M} g_J \, u(k-J+i)$ prediction @ time i out of impulse model

while.. STEP RESPONCE MODEL

$$g_i = s_i - s_{i-1}$$

$\curvearrowright$ using known relationship

$$y(k) = \sum_{J=1}^{+\infty} g_J \, u(k-J) = \sum_{J=1}^{+\infty} (s_J - s_{J-1}) \, u(k-J)$$

... developing this summ...

$$y(k) = \underbrace{(s_1 - s_0)\, u(k-1)}_{J=1} + \underbrace{(s_2 - s_1)\, u(k-2)}_{J=2} + \underbrace{(s_3 - s_2)\, u(k-3)}_{J=3} + \dots =$$

collecting s_1, s_2..

$$y(k) = s_1 \big(\underbrace{u(k-1) - u(k-2)}_{\delta u(k-1)}\big) + s_2 \big(\underbrace{u(k-2) - u(k-3)}_{\delta u(k-2)}\big) + \dots =$$

increment in the $u(k)$ CONTROL INPUT value

$$y(k) = \sum_{J=1}^{+\infty} s_J \, \delta u(k-J)$$

stepresponce model better formulation !

FUTURE OUTPUTS $\downarrow$ prediction formulation

$$y(k+i) = \sum_{J=1}^{+\infty} s_J \, \delta u(k-J+i)$$

$+\infty$! problem computation, but $s_i \cancel{\to} 0$ don't go to 0 we have to treat it properly to write that prediction formula in a better usable way!

2) from our model

$$y(k+i) = \sum_{J=1}^{3} g_J \, u(k+i-J) \qquad (M=3 \text{ in our case})$$

$i=1$: $y(k+1) = g_1 u(k+\cancel{1-1}) + g_2 u(k+1-2) + g_3 u(k+1-3) =$

$= g_1 u(k) + g_2 u(k-1) + g_3 u(k-2)$

prediction on $k+1$ depends on current k and past $k-1, k-2$ values

$i=2$: $y(k+2) = g_1 u(k+2-1) + g_2 u(k+2-2) + g_3 u(k+2-3) =$

$= g_1 \underset{\uparrow}{u(k+1)} + g_2 u(k) + g_3 u(k-1)$

future control action!

Representing in VECTOR FORM

$$\begin{bmatrix} y(k+1) \\ y(k+2) \end{bmatrix} = \underbrace{\begin{bmatrix} g_1 & 0 \\ g_1 & g_2 \end{bmatrix}}_{B} \overbrace{\begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix}}^{U} + \underbrace{\begin{bmatrix} g_2 & g_3 \\ g_3 & 0 \end{bmatrix}}_{B^{OLD}} \overbrace{\begin{bmatrix} u(k-1) \\ u(k-2) \end{bmatrix}}^{U^{OLD}}$$

$y(k)$?

↑ decision variables, future
FUTURE CONTROL INPUTS

↑ KNOWN, belongs to the past
PAST CONTROL INPUTS

$$Y(k) = B\,U(k) + B^{OLD}\,U^{OLD}(k)$$

output prediction based on past/future control inputs!

We miss $y(k)$ influencing the system ?! → because we compute the open loop model!

(PROBLEM)
the model does NOT depend on $y(k)$ (measured Data)

closed loop formulation ? → using a fictitious disturbance

Assume the system to be affected by a CONSTANT DISTURBANCE

$$\underbrace{y(k)}_{\text{measured}} = \underbrace{\sum_{J=1}^{M} g_J\, u(k-J)}_{\text{based on known values!}} + \underbrace{d(k)}$$

↳ we can get an estimate of it:

$$d(k) = y(k) - \sum_{J=1}^{M} g_J\, u(k-J)$$

→ basing prediction on current measure

we can use a constant disturbance

↓

$$y(k+i) = \sum_{J=1}^{M} g_J\, u(k-J+i) + d(k)$$

also @ k+i it is d(k)

⇓

CLOSED LOOP IMPULSE RESPONCE MODEL

$$y(k+i) = \sum_{J=1}^{M} g_J\, u(k+i-J) + y(k) - \sum_{J=1}^{M} g_J\, u(k-J)$$

CLOSED LOOP formulation, including also feedback $y(k)$

In our example ... with $M=3$ for $i=1,2$

$i=1$:

$$y(k+1) = g_1 u(k) + g_2 u(k-1) + g_3 u(k-2) + y(k) - g_1 u(k-1) - g_2 u(k-2) - g_3 u(k-3)$$

$i=2$

$$y(k+2) = g_1 u(k+1) + g_2 u(k) + g_3 u(k-1) + y(k) - g_1 u(k-1) - g_2 u(k-2) - g_3 u(k-3)$$

↓ VECTOR FORM

CLOSED LOOP IMPULSE RESPONCE MODEL

measured data

$$\begin{bmatrix} y(k+1) \\ y(k+2) \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ g_2 & g_1 \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y(k) + \begin{bmatrix} (g_2-g_1) & (g_3-g_2) & -g_3 \\ (g_3-g_1) & -g_2 & -g_3 \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \end{bmatrix}$$

3) NOW WE WANNA FIND "STEP RESPONCE"

While for impulse we go from open loop to closed loop,
in step responce, openloop have ∞ summ, so cannot be computed

↓

considering directly closed loop :=> Assume a CONSTANT DISTURBANCE affecting the system

$$y(k) = \sum_{J=1}^{+\infty} S_J \, \delta u(k-J) + d(k)$$

estimating the disturbance ↳

$$d(k) = y(k) - \sum_{J=1}^{+\infty} S_J \, \delta u(k-J)$$

considering future output model:

constant disturbance

$$y(k+i) = \sum_{J=1}^{+\infty} S_J u(k-J+i) + d(k) =$$

$$= \sum_{J=1}^{+\infty} S_J \, \delta u(k-J+i) + y(k) - \sum_{J=1}^{+\infty} S_J \, \delta u(k-J)$$

in our example

• $i=1$

$$y(k+1) = S_1 \delta u(k) + S_2 \delta u(k-1) + S_3 \, \delta u(k-2) + \ldots + y(k) - S_1 \delta u(k-1) - S_2 \delta u(k-2) + $$
$$- S_3 \, \delta u(k-3) - \ldots$$

↓

$$y(k+1) = S_1 \delta u(k) + (S_2 - S_1) \delta u(k-1) + (S_3 - S_2) \delta u(k-2) + \overset{0}{(S_4 - S_3)} \delta u(k-3) + \overset{0}{\cancel{\ldots}} + y(k)$$

so from our system step responce, for $J \geq 3$ $S_J = 1$ steady state value

for $J = M = 3$ $S_J = 1$ $\forall J \geq 3$!

(even if we had an ∞ summ we find a way to troncate it!)

$S_4 - S_3 = 0$
$S_5 - S_4 = 0$
...

→ 0 values, we can neglect and obtain a finite summ model

$$y(k+1) = S_1 \delta u(k) + (S_2 - S_1) \delta u(k-1) + (S_3 - S_2) \, \delta u(k-2) + y(k)$$

same reasoning for

• $i=2$

$$y(k+2) = S_1 \delta u(k+1) + S_2 \delta u(k) + (S_3 - S_1) \delta u(k-1) + (S_4 - S_2) \delta u(k-2) + y(k)$$

↳ as for Impulse responce model → VECTOR FORM: =>

CLOSED LOOP STEP RESPONCE MODEL

$$\begin{bmatrix} y(k+1) \\ y(k+2) \end{bmatrix} = \begin{bmatrix} s_1 & 0 \\ s_1 & s_2 \end{bmatrix} \underbrace{\begin{bmatrix} \delta u(k) \\ \delta u(k+1) \end{bmatrix}}_{\text{FUTURE CONTROL sequence}} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y(k) + \begin{bmatrix} (s_2-s_1) & (s_3-s_2) & 0 \\ (s_3-s_1) & (s_4-s_2) & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \delta u(k-1) \\ \delta u(k-2) \\ \delta u(k-3) \end{bmatrix}}_{\text{PAST CONTROL sequence}}$$

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