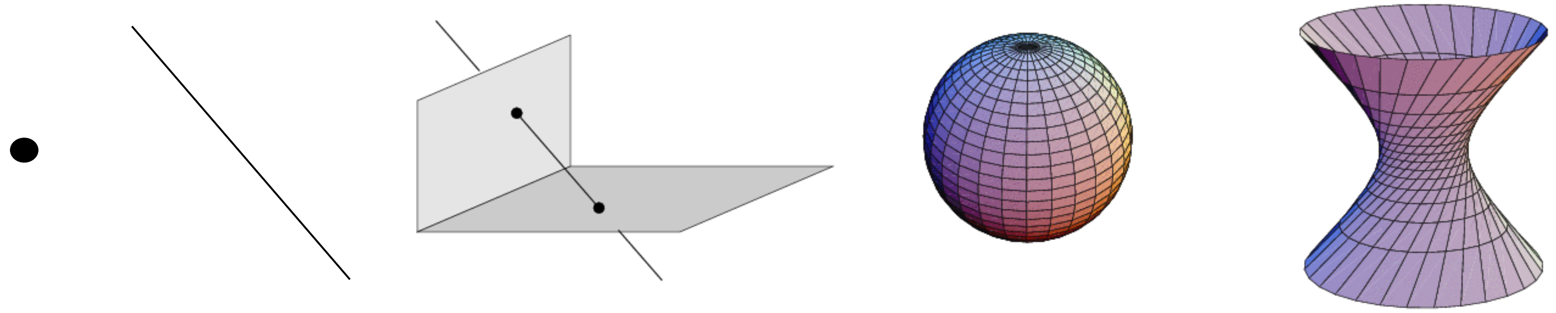
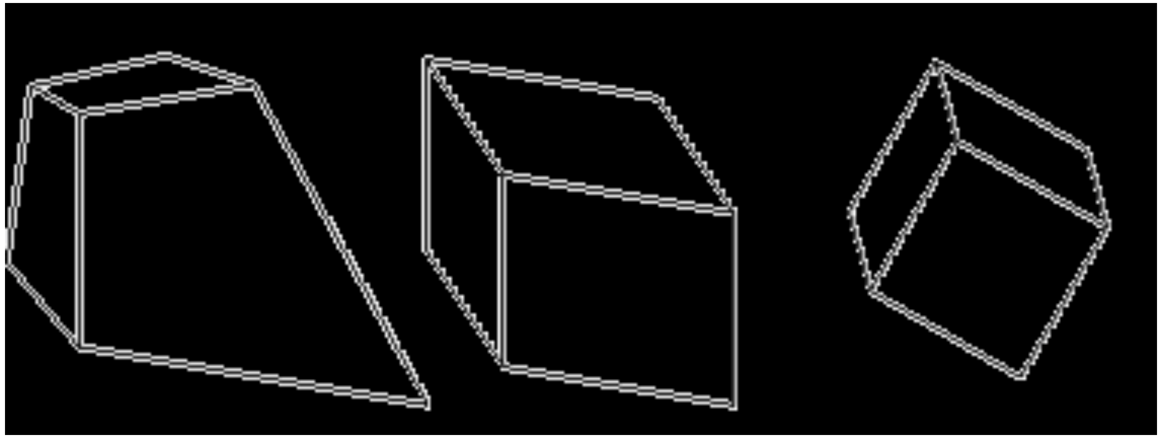
Space (3D) Projective Geometry

3D Space Projective Geometry

- **Elements**
- Points
- Planes
- Quadrics
- Dual quadrics

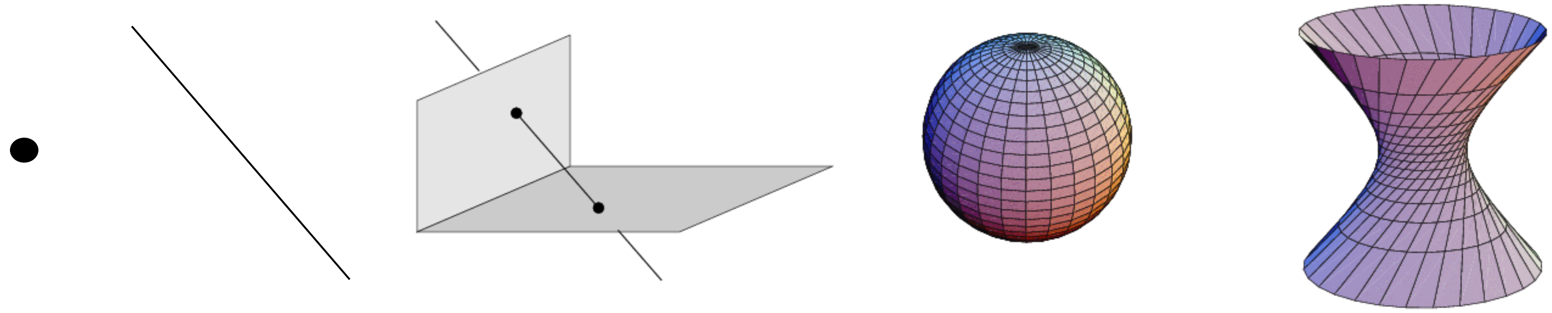


- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities

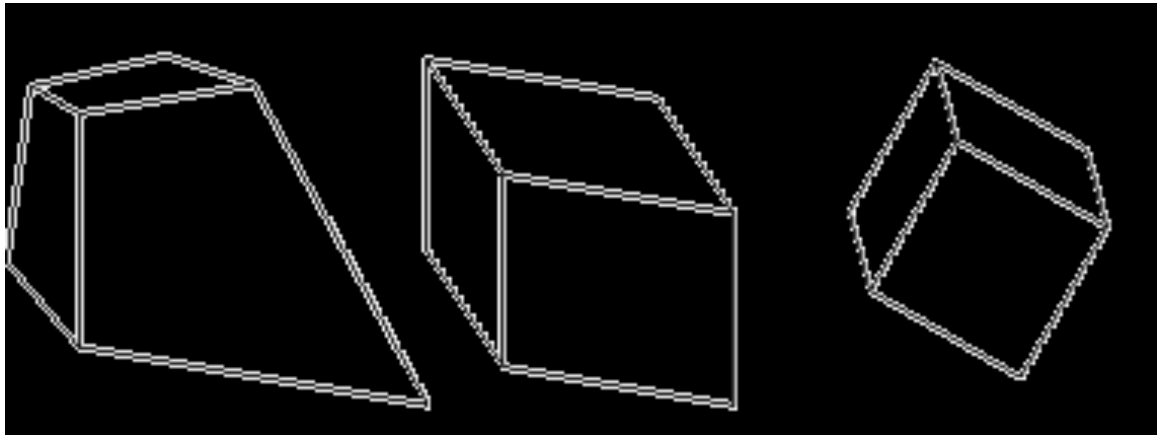


3D Space Projective Geometry

- **Elements**
- **Points**
- Planes
- Quadrics
- Dual quadrics

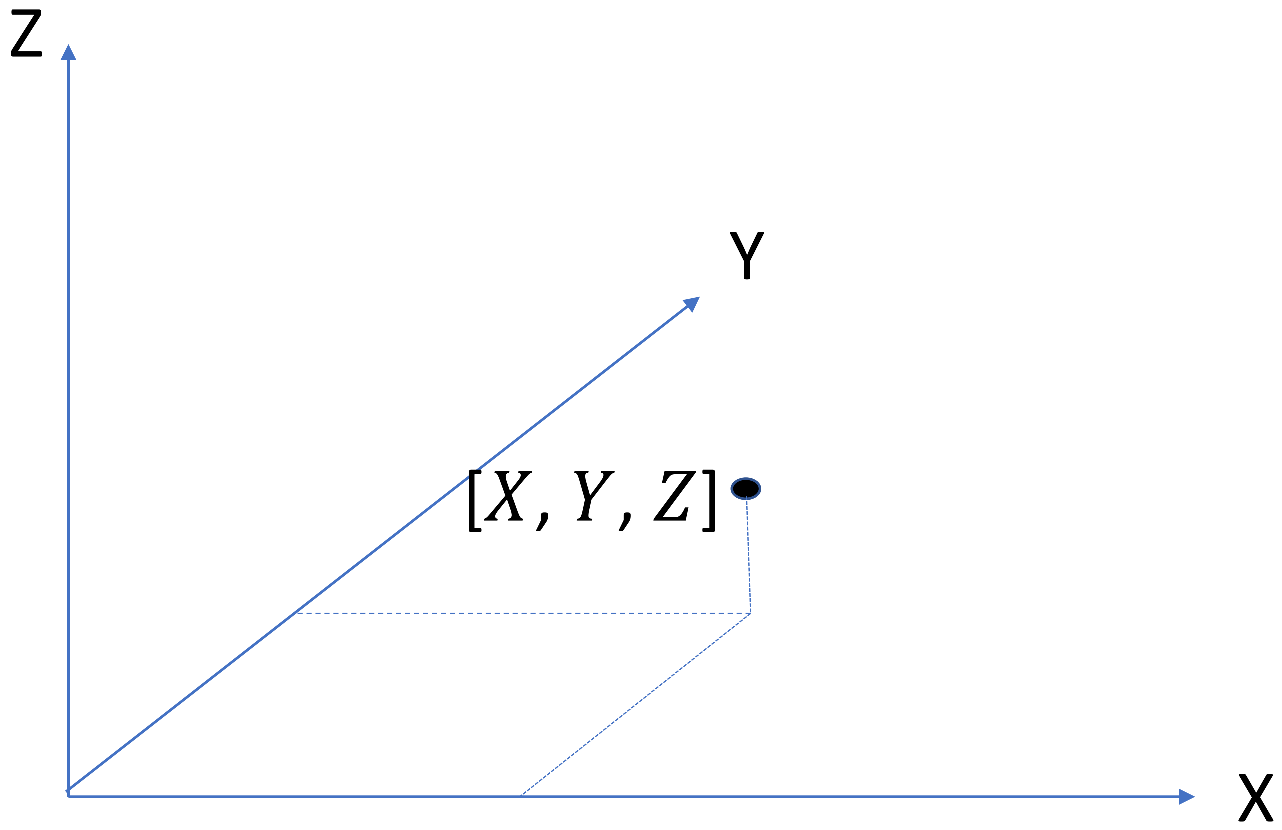


- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities

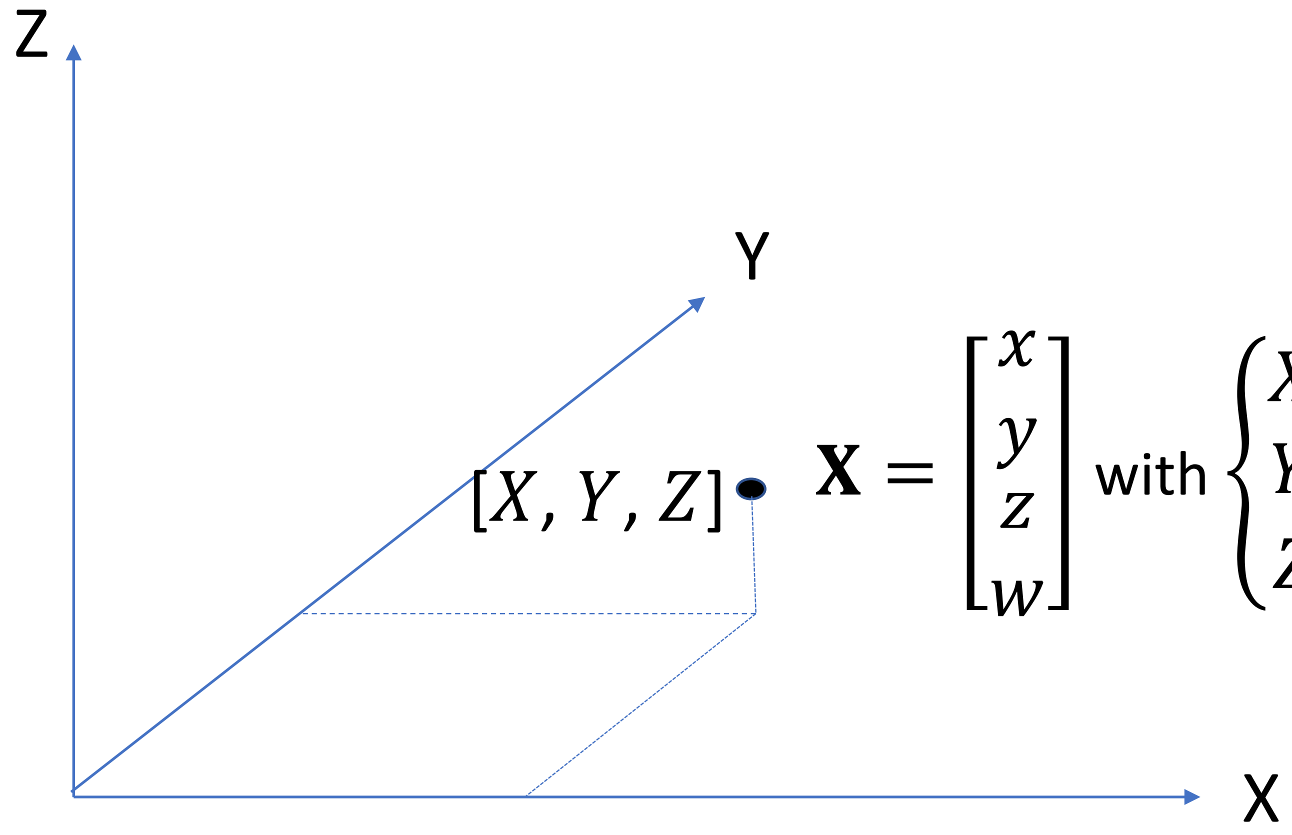


Points in the projective space

Euclidean space (3D) cartesian coordinates

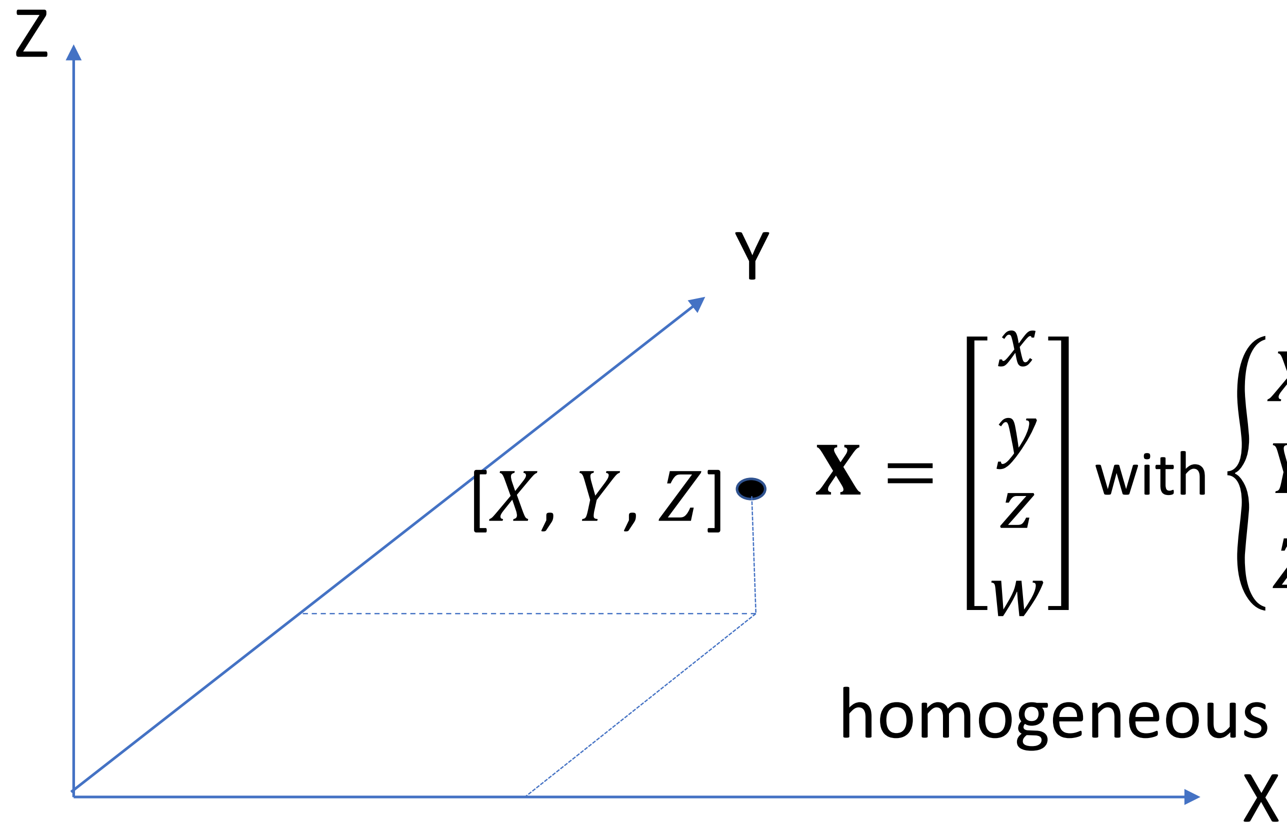


Projective space (3D)
4 homogeneous coordinates



$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

Projective space (3D)
4 homogeneous coordinates

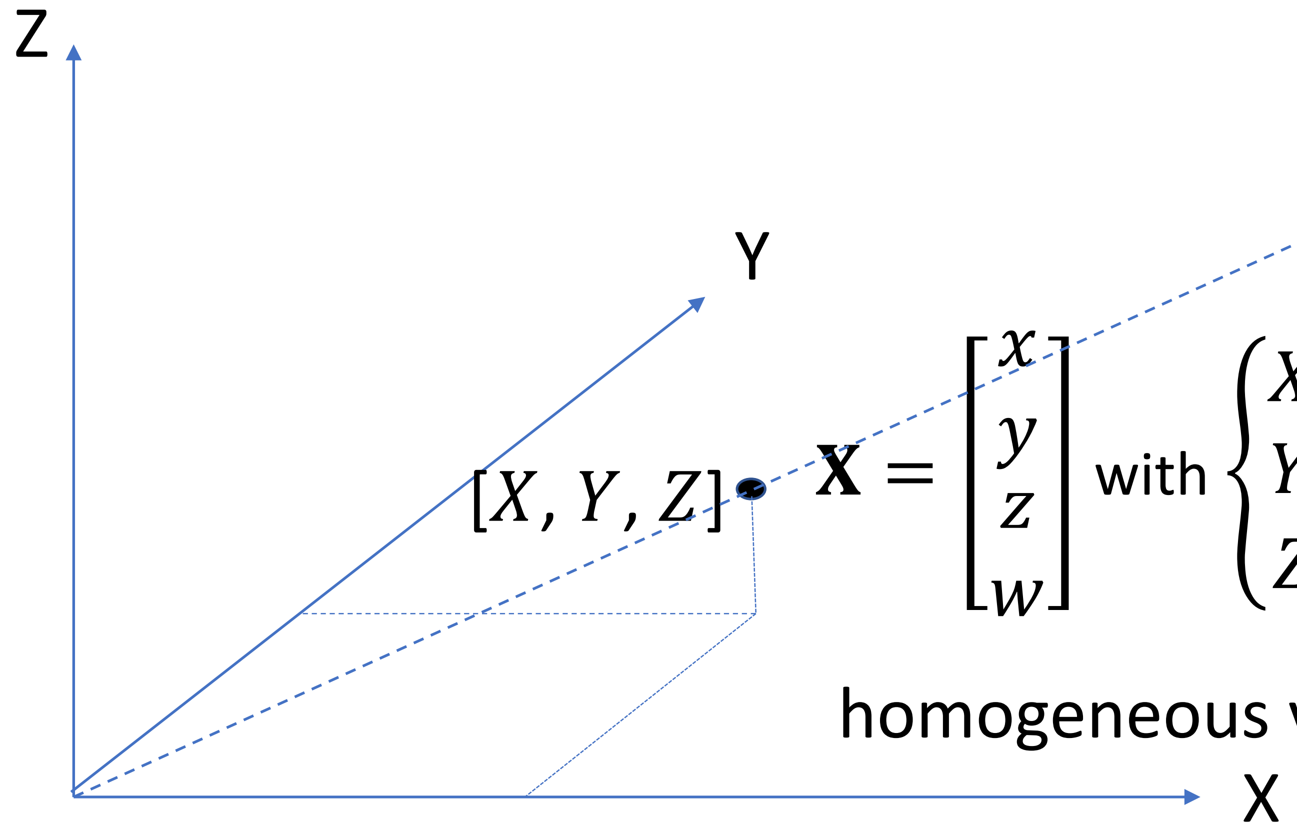


$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

homogeneous vector: $\lambda\mathbf{X} \Leftrightarrow \mathbf{X}$

Projective space $\mathbb{P}^3$: points at the ∞

$$\mathbf{X}_\infty = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$



$$[X, Y, Z] \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

homogeneous vector: $\lambda\mathbf{X} \Leftrightarrow \mathbf{X}$

$$\mathbb{P}^3 = \{\mathbf{X} \in \mathbb{R}^4 - \{[0 \quad 0 \quad 0 \quad 0]^T\}\}$$

redundancy

4 homogeneous coordinates to represent points in the 3D space (3 dof)

an infinite number of equivalent representations for a single point, namely all nonzero multiples of the vector $[X \quad Y \quad Z \quad 1]^T$

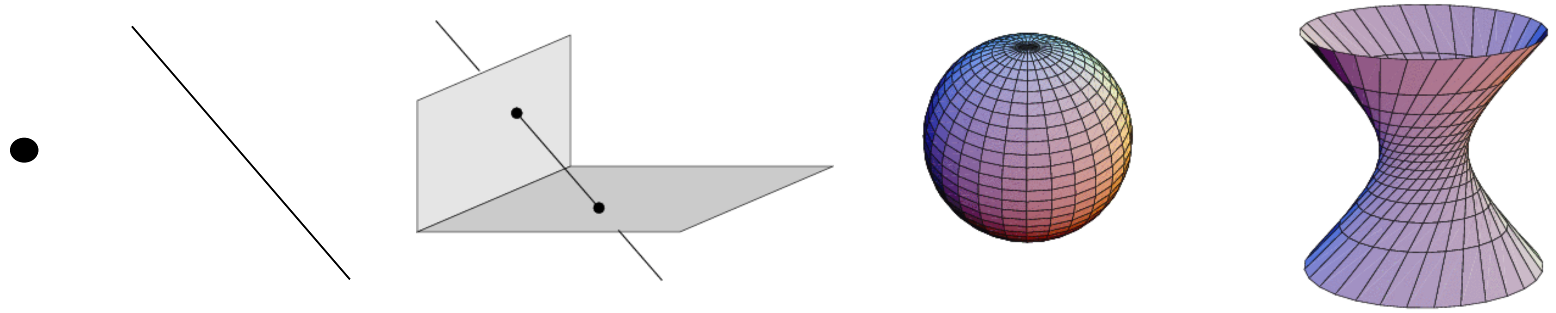
the null vector $[0 \quad 0 \quad 0 \quad 0]^T$ **does not** represent any point

→ Projective space $\mathbb{P}^3 = \{[x \quad y \quad z \quad w]^T \in \mathbb{R}^4\} - \{[0 \quad 0 \quad 0 \quad 0]^T\}$

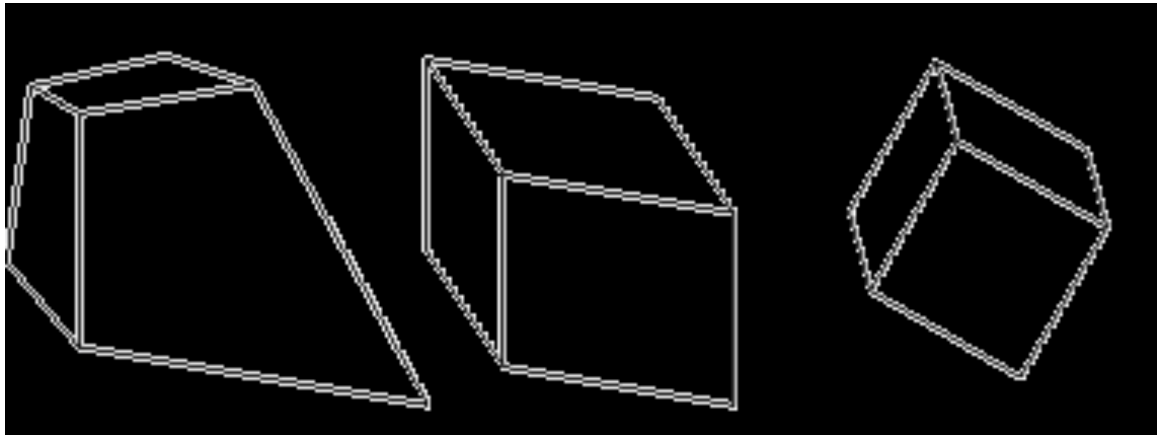
→ its three degrees of freedom are the three independent ratios between the four coordinates $x : y : z : w$

3D Space Projective Geometry

- **Elements**
  - Points
  - **Planes**
  - Quadrics
  - Dual quadrics

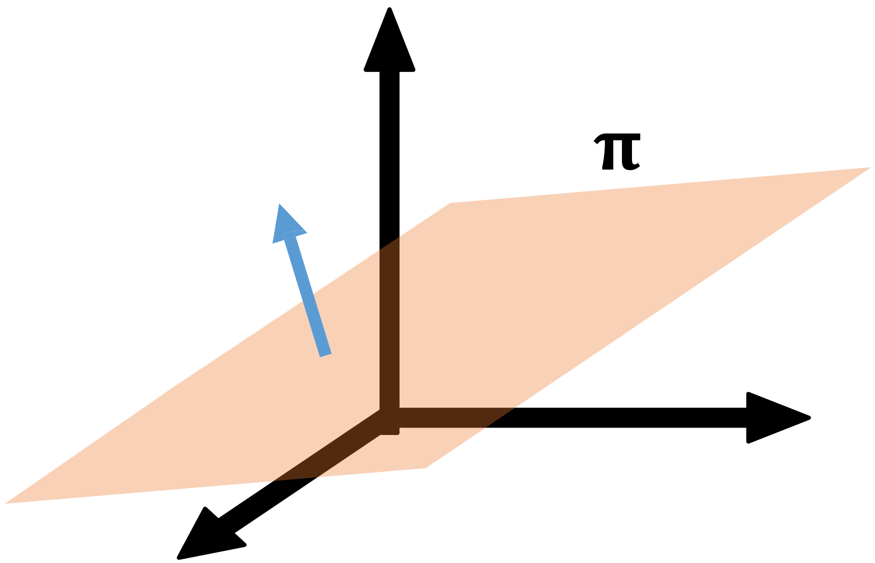


- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities



Planes in the projective space

Planes in 3D Projective Geometry



$\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with direction $\begin{pmatrix} a & b & c \end{pmatrix}$ normal to the plane,

and $\frac{-d}{\sqrt{a^2+b^2+c^2}}$ = the distance between the origin and the plane

$\boldsymbol{\pi}$ is a homogeneous vector: $\lambda\boldsymbol{\pi} \Leftrightarrow \boldsymbol{\pi}$

redundancy

4 homogeneous parameters to represent planes in the 3D space (3 dof)

an infinite number of equivalent representations for a single planes, namely all nonzero multiples of the homogeneous vector $[a \quad b \quad c \quad d]^T$

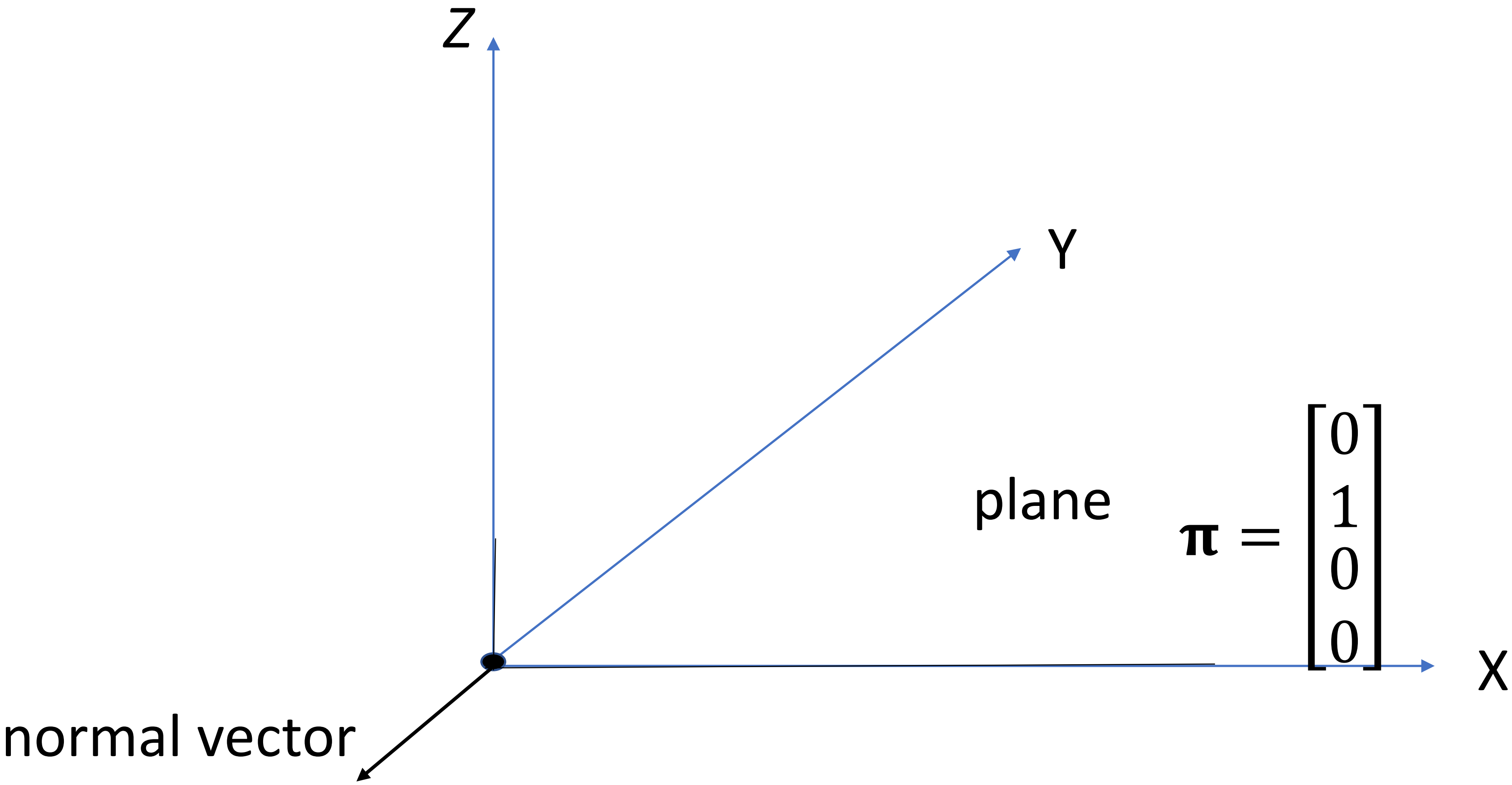
the null vector $[0 \quad 0 \quad 0 \quad 0]^T$ **does not** represent any plane

→ its three degrees of freedom are the three independent ratios between the four parameters a $: b : c : d$

remark

If the fourth parameter d is null, $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, then the plane goes through point $[0,0,0]$

Example: the X-Z plane



$$\boldsymbol{\pi} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

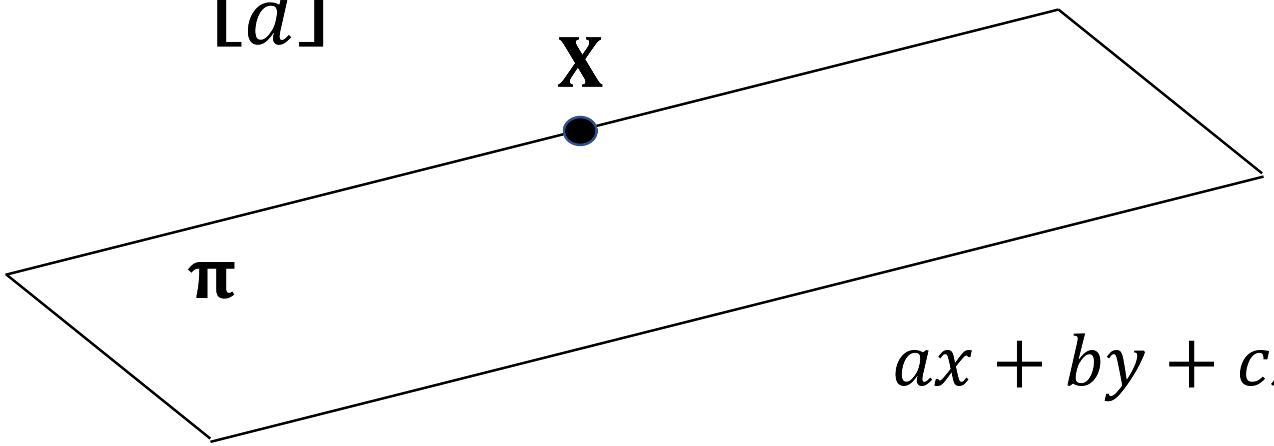
The incidence relation:
a point is on a plane (or a plane goes through a point)

Incidence relation

the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is on the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

or

the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ goes through the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$



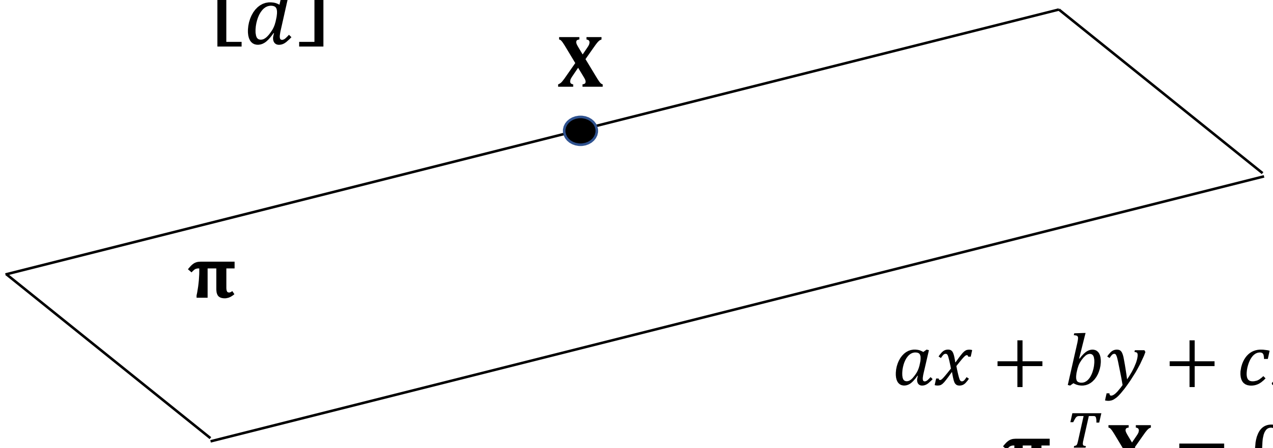
$$ax + by + cz + dw = 0$$

Incidence relation

the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is on the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

or

the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ goes through the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$



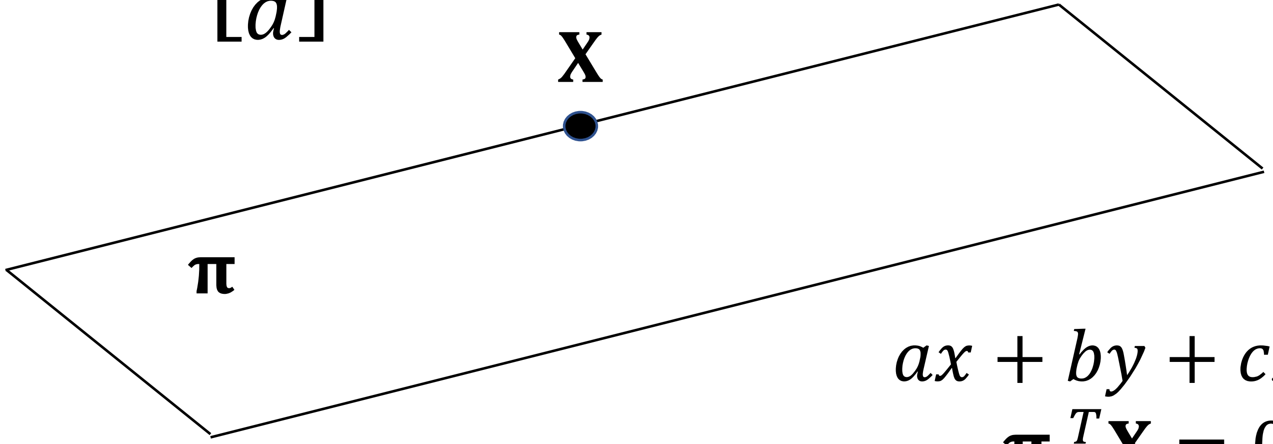
$$ax + by + cz + dw = 0$$
$$\boldsymbol{\pi}^T\mathbf{X} = 0 = \mathbf{X}^T\boldsymbol{\pi}$$

Incidence relation

the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is on the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

or

the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ goes through the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$



$$ax + by + cz + dw = 0$$

$$\boldsymbol{\pi}^T\mathbf{X} = 0 = \mathbf{X}^T\boldsymbol{\pi}$$

Dividing by $\boldsymbol{w}$ we find the cartesian coordinates again

The plane at the infinity:
the locus of the points at the infinity

The «locus» of the points at the infinity

As there are infinite points at the infinity (one for each direction), what is the aspect of the set of these points at the infinity?

Simply: $$w = 0$$

This set is a plane: $[\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c} \quad \boldsymbol{d}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\\\boldsymbol{w}\end{bmatrix} = 0,$ actually $[\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\\\boldsymbol{w}\end{bmatrix} = 0$

namely, **the plane at the infinity** $\boldsymbol{\pi}_\infty = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

NOTE: this plane has undefined normal direction

The duality principle between points and planes

2. Since dot product is commutative → incidence relation is commutative

$$\boldsymbol{\pi}^T\mathbf{X} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix} = 0 = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = 0 = \mathbf{X}^T\boldsymbol{\pi} = \mathbf{0}$$

point $\mathbf{X}$ is on plane $\boldsymbol{\pi}$

$\Leftrightarrow$

point $\boldsymbol{\pi}$ is on **plane** $\mathbf{X}$

point $\mathbf{X}$ is on plane $\boldsymbol{\pi}$ (i.e. plane $\boldsymbol{\pi}$ goes through point $\mathbf{X}$)

$\leftarrow\rightarrow$

point $\boldsymbol{\pi}$ is on **line** $\mathbf{X}$ (i.e. line $\mathbf{X}$ goes through point $\boldsymbol{\pi}$)

Principle of **duality** between points and planes
in 3D Projective Geometry

For any true sentence containing the words

- point
- plane
- is on
- goes through

there is a DUAL sentence -also true- obtained by substituting, in the previous one, each occurence of

- point	by	- plane
- plane	by	- point
- is on	by	- goes through
- goes through	by	- is on

31/10/2023

CONTINUE of IACV-B, 05/11

The point on three planes

intersection between three planes

equation of plane through three points, due to duality

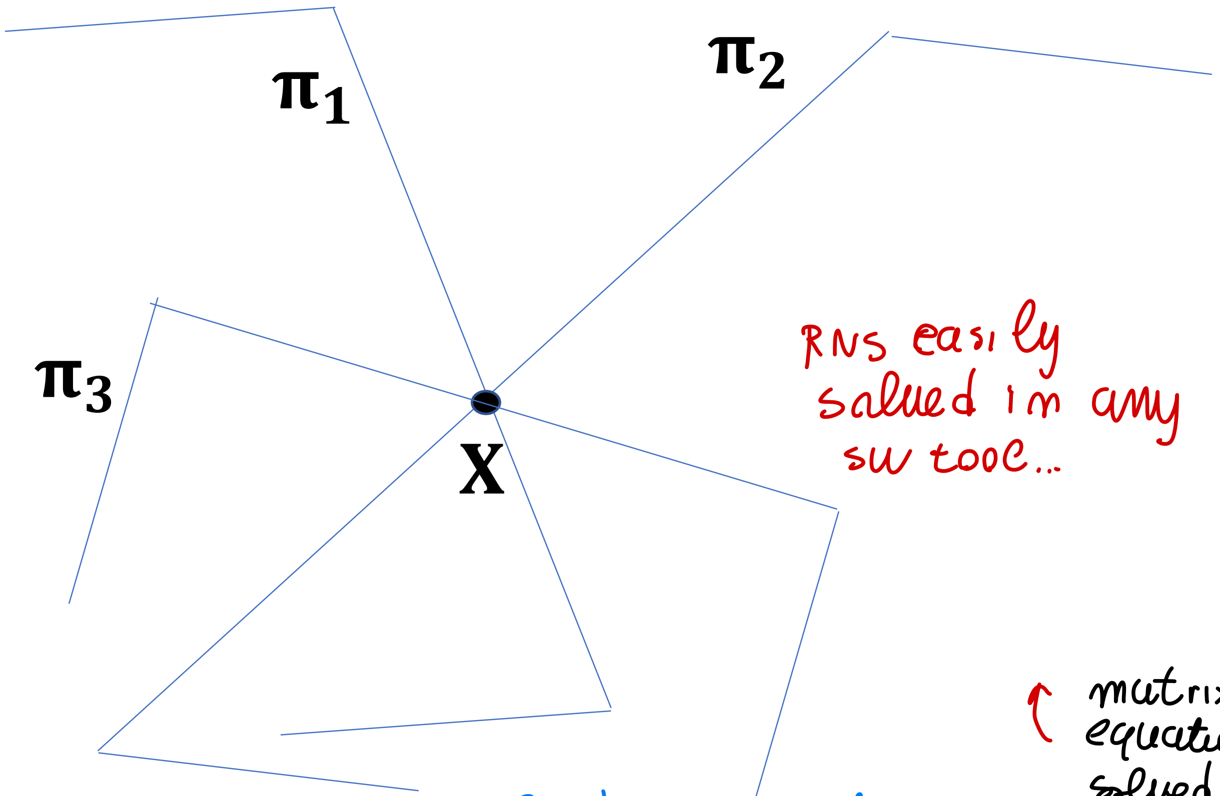
just study one is enough

the point on three planes

X which satisfy 3 constraints

$$\begin{cases} \boldsymbol{\pi}_1{}^T \mathbf{X} = 0 \\ \boldsymbol{\pi}_2{}^T \mathbf{X} = 0 \\ \boldsymbol{\pi}_3{}^T \mathbf{X} = 0 \end{cases}$$

$X \in \pi_1$
$X \in \pi_2$
$X \in \pi_3$



RNS easily solved in any sw tool...

↓ in MATRIX form

$$\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \\ \boldsymbol{\pi}_3{}^T \end{bmatrix}_{3 \times 4} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 D space ≡ vector & all its multiples

(∞^1)

1D set of solution IN HOMOGENEUS X coordinate

matrix equation easy solved by RNS

$$\mathbf{X} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \\ \boldsymbol{\pi}_3{}^T \end{bmatrix}_{3 \times 4} \right)$$

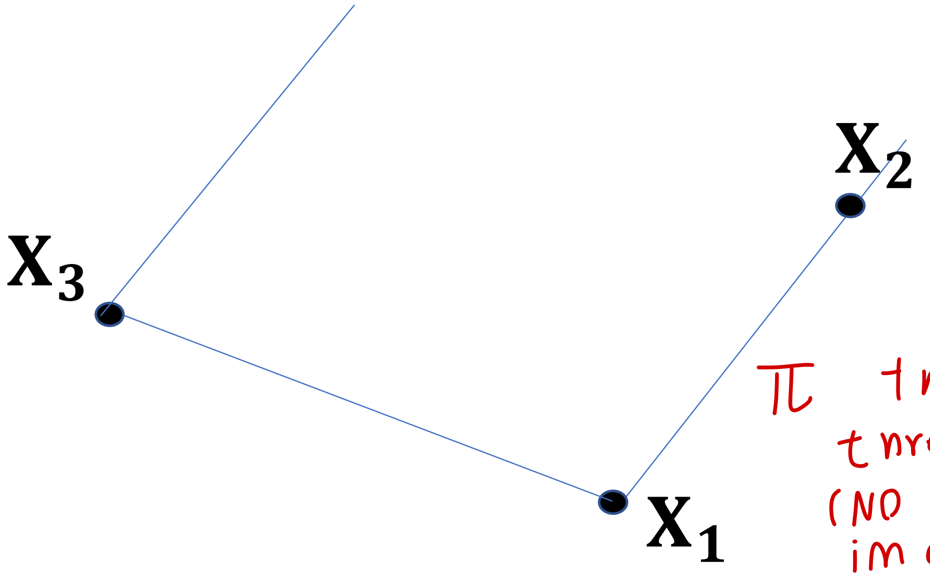
you find many vector up to a scaling factor due to homogeneity

a solution vector + all its mutiples

the plane through three points

the plane through three points:
dual of the point through three planes

the plane through three points



Π through this three points (NO colinear, in general position)

↓

only ONE PLANE through this

same equation wrt $X_i \in \Pi$, $i = 1,2,3$

$$\begin{cases} \mathbf{X_1}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_2}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_3}^T \boldsymbol{\pi} = 0 \end{cases}$$

⇓ construct matrix

$$\underbrace{\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \\ \mathbf{X_3}^T \end{bmatrix}}_{3 \times 4} \boldsymbol{\pi} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

same math of the other problem! →

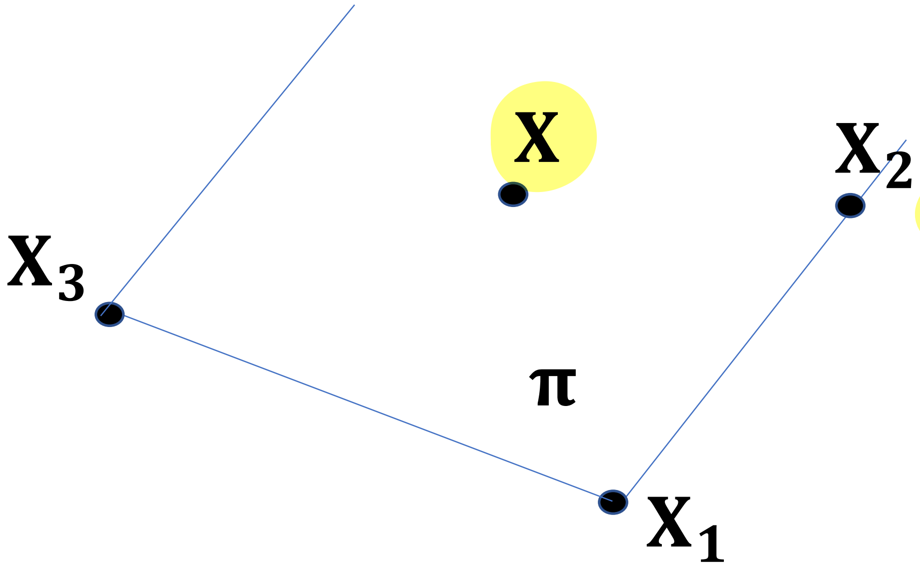
$$\boldsymbol{\pi} = \mathbf{RNS}\left(\underbrace{\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \\ \mathbf{X_3}^T \end{bmatrix}}_{3 \times 4}\right)$$

a solution vector + all its mutiples

the plane as its span

space covered by letting an element vary according to some rule

usefull to develop further knowledge



$\mathbf{X}$ is a linear combination $\overbrace{\alpha \mathbf{X_1} + \beta \mathbf{X_2} + \gamma \mathbf{X_3}}^{\text{span of } x_1, x_2, x_3}$

→ $\mathbf{X}$ is coplanar to $\mathbf{X_1}, \mathbf{X_2},$ and $\mathbf{X_3}$

span of this points

i.e. $\underbrace{\mathbf{X} = [\mathbf{X_1} \quad \mathbf{X_2} \quad \mathbf{X_3}] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}}_{\text{we cover the span of this points}} = \mathrm{M}\mathbf{x}$ where

(space spanned)

$\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ can be regarded as homogeneous coordinates within the 2D geometry of plane $\boldsymbol{\pi}$

letting $\alpha, \beta, \gamma \in \mathbb{R}$ any (NOT $\alpha=\beta=\gamma=0$!)

X is on the plane going through x_1, x_2, x_3... plane is a span of three points, we cover all π by any x_1, x_2, x_3 combination

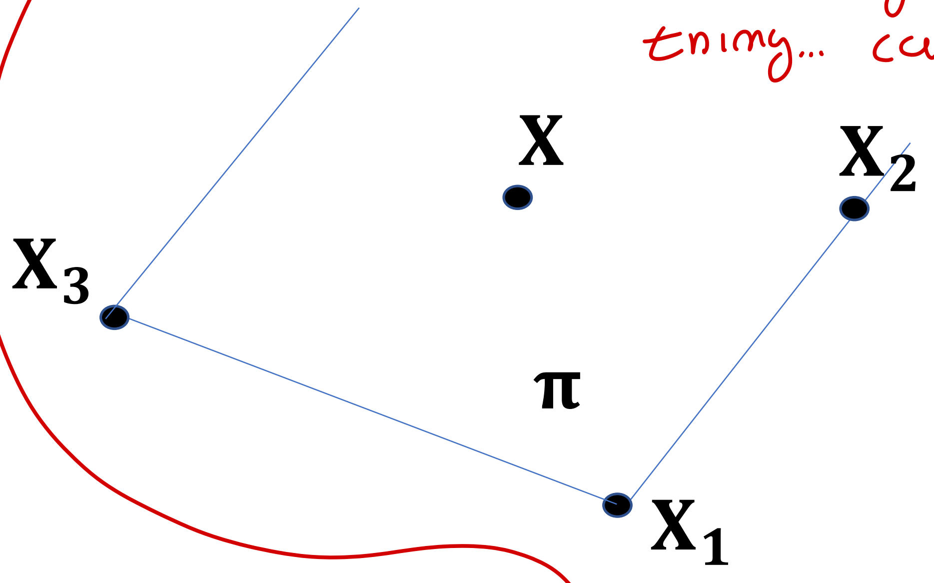
$$\mathbf{X} = \mathrm{M}\mathbf{x}$$

we can represent by constraint $X = Mx$ $M[4\times3]$ matrix * $x \in \mathbb{R}^3$

the plane as its span

represent a plane NOT as 3 comp. vector, but with Mx, constraining

math way to describe same thing... can be usefull later on



$\mathbf{X}$ is a linear combination $\propto \mathbf{X_1} + \beta \mathbf{X_2} + \gamma \mathbf{X_3}$

→ $\mathbf{X}$ is coplanar to $\mathbf{X_1}, \mathbf{X_2}$, and $\mathbf{X_3}$

i.e. $\mathbf{X} = [\mathbf{X_1} \quad \mathbf{X_2} \quad \mathbf{X_3}] \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix} = \mathrm{M}\mathbf{x}$ where

if $2x$, we find same point! homogeneous coord

$\mathbf{x} = \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix}$ can be regarded as homogeneous

↓ 3 homogeneous coord

coordinates within the 2D geometry of plane $\boldsymbol{\pi}$

as $[\alpha \quad \beta \quad \gamma]$ vary, we move on π plane! ⇒ we get span of π... the point $X \in \pi$ as $X \in \mathbb{R}^4$ point on space... x reduction on π

$$\boxed{\mathbf{X} = \mathrm{M}\mathbf{x}}$$

NO minimal parametrization → ≡ a way to represent entity by smallest # of parameters as # DOFs
of lines in 3D space ←

LINES ?

NOT PRIMITIVE in space

POINTS / PLANES

Lines are primitive elements in the planar geometry

but they are **not** primitive elements in the space geometry

There is no minimal parameterization for lines in 3D

However

in the 3D space there are (∞) planes, and planes contain lines

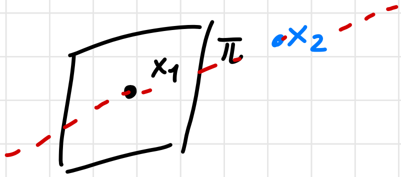
Lines are intermediate entities between points and planes

they are self-dual

you have 2 DOF to establish where is a point in line...

↓ ...

once you fix through point,
the line has (∞^2) possibility



↳ line in 3D space has ∞^4 DOF (while 2 dof in plane)

and there is no way to define it univoquely

define x_1, x_2 and orient, if line // π many represent of same line !

⇓

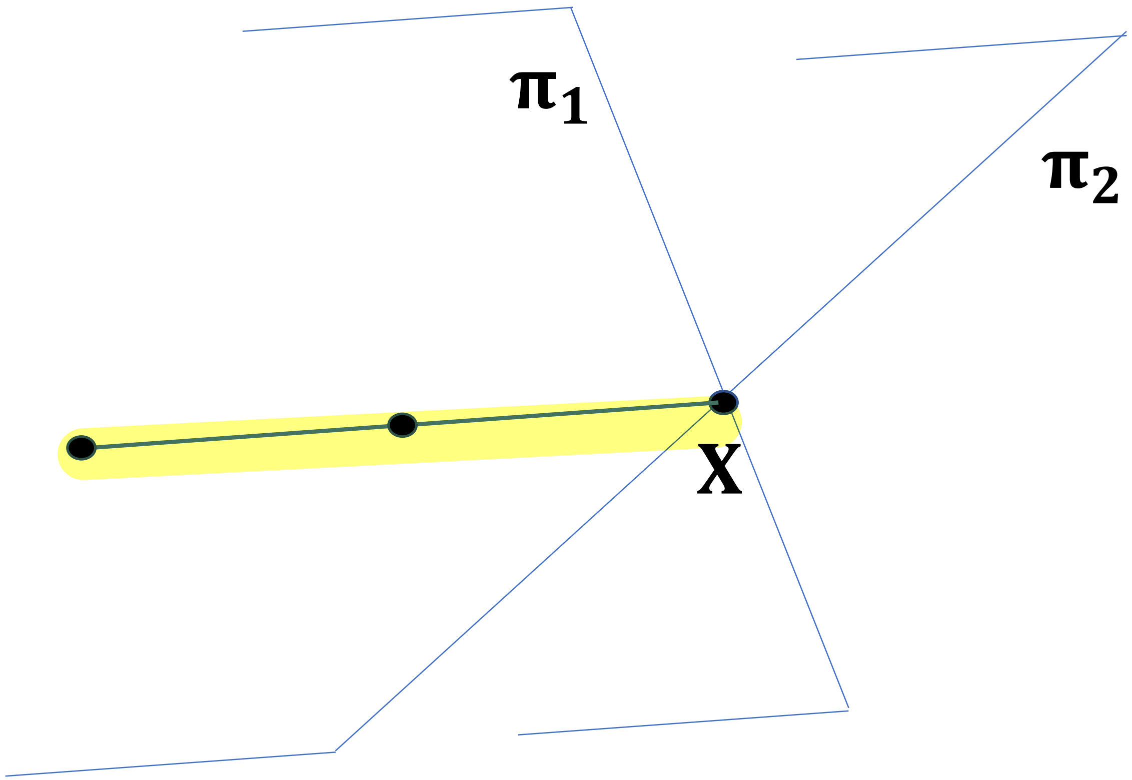
theorem ensure NO unique representation

more than 4 params to represent a line !

→ many ways BUT none is minimal → we use redundant ⇒

Line: the set of points **X** on **two** planes

possible representation of line as set of points on two planes



$$\begin{cases} \boldsymbol{\pi}_1{}^T\mathbf{X} = 0 \\ \boldsymbol{\pi}_2{}^T\mathbf{X} = 0 \end{cases}$$

constraint $X \in \pi_1, \pi_2$

in matrix form

2 eq less... you find 1D set of points, because 2D homogeneous

$$\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \end{bmatrix}_{2 \times 4} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \end{bmatrix}_{2 \times 4}\right)$$

8 parameters, more than needed ones!

(∞^2) solution

Algebrically

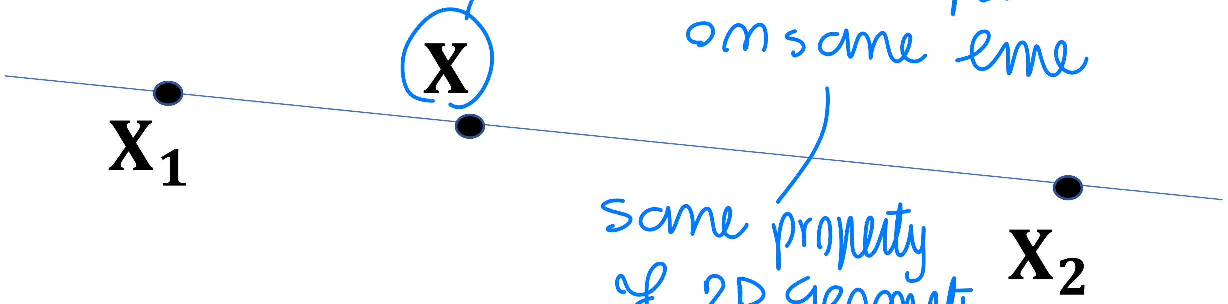
2D set of solution vectors: two points and all their linear combinations

→ due to homogeneity: 1D set of points (parameter abscissa) ← using one param to describe → euclidean using dual representation for line description $\Longrightarrow$

linear combination of two points

Property: the point $\mathbf{X}$ given by the linear combination $\mathbf{X} = \propto \mathbf{X}_1 + \beta \mathbf{X}_2$ of two points $\mathbf{X}_1$ and $\mathbf{X}_2$ is on the line $\mathbf{L}$ through $\mathbf{X}_1$ and $\mathbf{X}_2$

x as linear combination of x_1, x_2 is a point on same line



same property of 2D geometry

A **line L** can also be defined as the set of all points, that are linear combinations of two given points: $\mathbf{X} = \propto \mathbf{X}_1 + \beta \mathbf{X}_2$

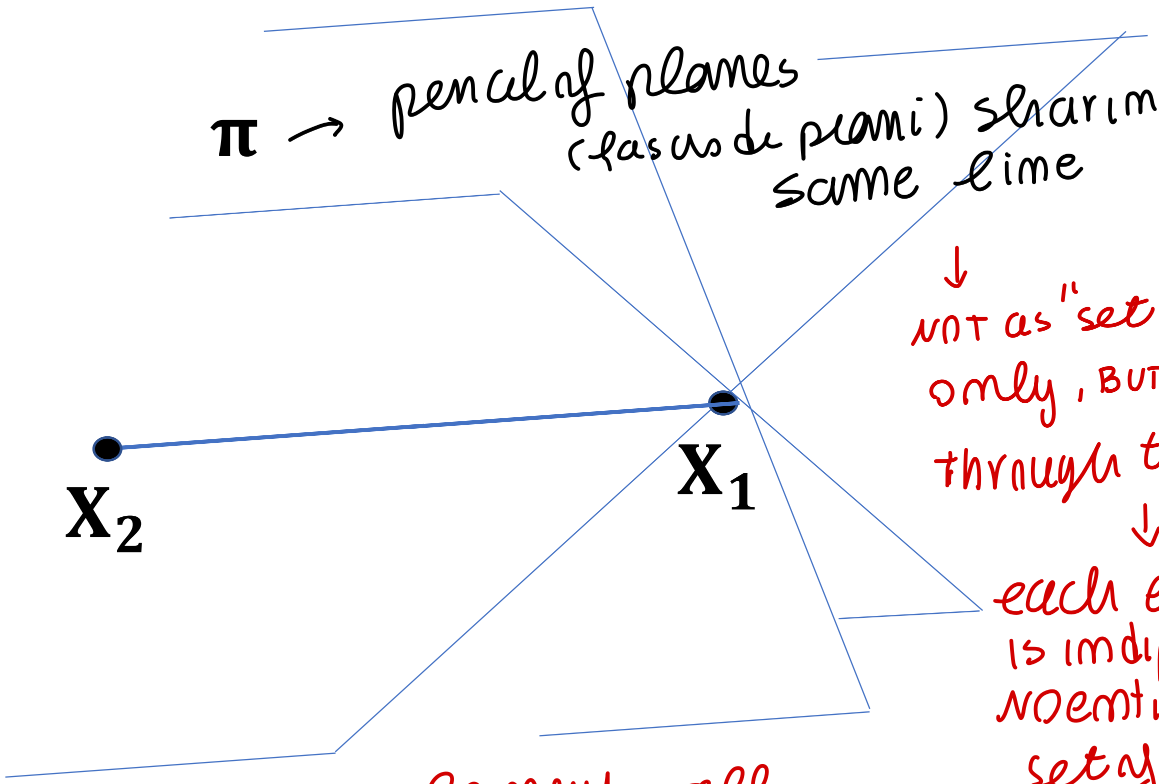
DUAL →

Line: the set of planes $\boldsymbol{\pi}$ through **two** points

line as set of plane!

(because of duality points on 2 planes π_1, π_2) ↗ DUAL

$\boldsymbol{\pi}$ → pencil of planes (fascio di piani) sharing same line



not as "set of plane" only, BUT as through two points

each entity is indipendent, no entity is set of "others"

imposing set π

$$\begin{cases} \mathbf{X_1}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_2}^T \boldsymbol{\pi} = 0 \end{cases}$$

same algebra as plane through two points

↓ MATRIX

$$\underbrace{\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}}_{2 \times 4} \boldsymbol{\pi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{L}^* = \underline{\mathbf{RNS}}\Big(\underbrace{\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}}_{2 \times 4}\Big)$$

in principle ∞^2 solution

1D homogeneity

span of two elements, all linear combination of two planes

span of 2D set of vector solutions: two planes and all their linear combinations

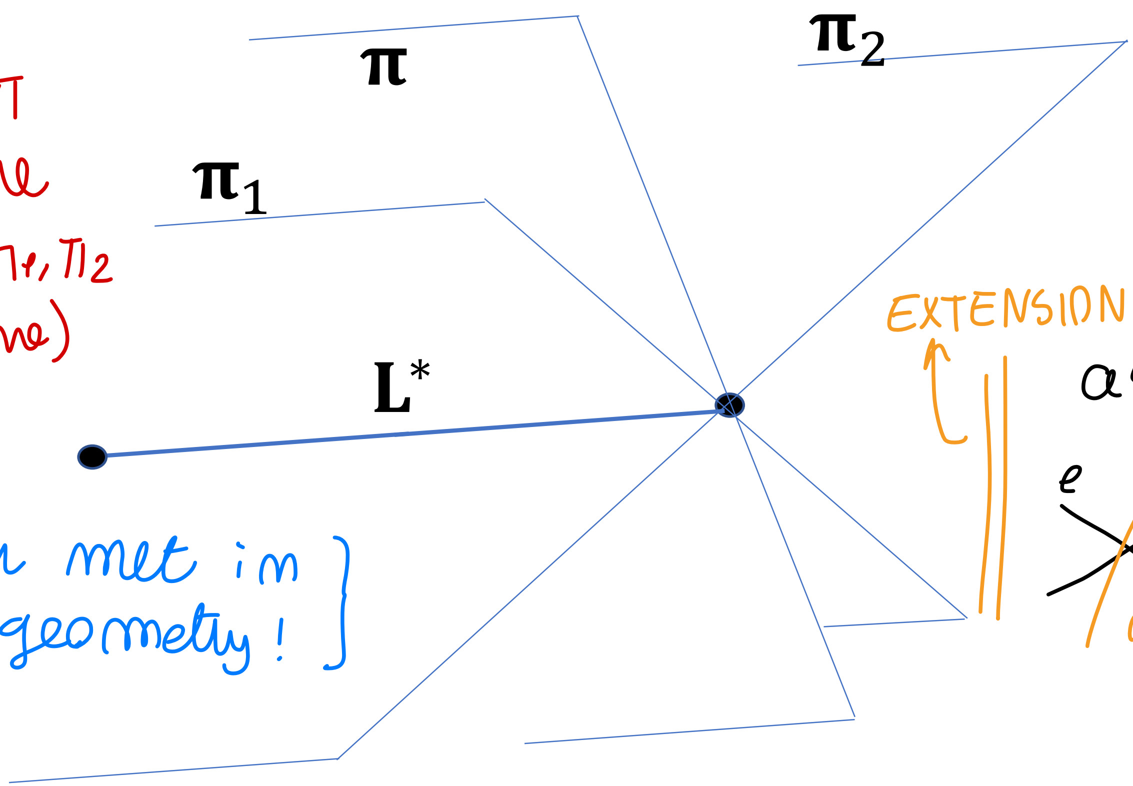
this solution → due to homogeneity: 1D set of planes (parameter: rottion angle)

DUAL: linear combination of two planes

as span, Dual of previous property of $x \in \Pi$ as span... any x_1, x_2 combination is on same line

Dual property: the plane $\boldsymbol{\pi}$, given by the linear combination $\boldsymbol{\pi} = \alpha \boldsymbol{\pi}_1 + \beta \boldsymbol{\pi}_2$ of two planes $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$, goes through the line $\mathbf{L}^*$ on $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$

letting α, β vary, new Π is on same pencil as Π_1, Π_2 (same line)



{never met in 2D geometry!}

⇑ dual property of ↕

$X = \alpha X_1 + \beta X_2 \in L$ through X_1, X_2

EXTENSION...

as in 2D if 2 lines l, m any $\alpha l + \beta m$ is concurrent

$\alpha l + \beta m$

we have dual line $L = RNS\left(\begin{bmatrix} \pi_1^T \\ \pi_2^T \end{bmatrix}\right) \leftrightarrow L^* = RNS\left(\begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix}\right) \Leftarrow$ line is self dual

set of plane set of points

pairs of DUALLY corresponding words

⇓ duality in 3D geometry

- point	→	- plane
- line	line is self → dual	- line
- plane	→	- point
- is on	→	- goes through
- goes through	→	- is on

Important observation...

in 2D geometry we define ℓ_∞

$\Downarrow$

Each plane $\boldsymbol{\pi}$ has its own line at the infinity $\boldsymbol{l_\infty(\pi)}$ and also its own circular points $\boldsymbol{I(\pi)}$ and $\boldsymbol{J(\pi)}$

$\forall$ plane!

and set union of $\ell_\infty(\pi)$ is π_∞ plane at ∞

parallel planes share the same $\boldsymbol{l_\infty}$ and the same circular points $\boldsymbol{I}$ and $\boldsymbol{J}$

in new π_1, you have its own $\ell_{\infty 1}(\pi_1)$ and $I_1(\pi_1)$ $J_1(\pi_1)$

if $\pi_2 // \pi_1 \rightarrow$ common $\ell_{\infty 1}$ and I_1, J_1 share same, being ℓ_∞ representative of directions, any same directions in π_1, π_2

Angle between two 3D directions

how to express direction angle in 3D

E.g.

1) - Angle between the directions normal to two planes

2) - Angle between the directions of two points at the infinity

another way to represent angle

same thing

Angle between two 3D directions

this represents direction!

Angle between the normals to planes $\boldsymbol{\pi_1} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$ and $\boldsymbol{\pi_2} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}$:

what care when computing angles

NOT including

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2 + \cancel{d_1 d_2}}{\sqrt{(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2)}}$$

$$= \frac{\pi_1 \pi_2}{\| \pi_1 \pi_2 \|}$$

but NOT considering π_i !

Angle between two 3D directions

- equivalentely by points at ∞:

represent directions of X_∞

Angle between the directions of points $\mathbf{X}_{\infty\mathbf{1}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{bmatrix}$ and $\mathbf{X}_{\infty\mathbf{2}} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 0 \end{bmatrix}$:

$$\cos\vartheta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{(x_1{}^2 + y_1{}^2 + z_1{}^2)(x_2{}^2 + y_2{}^2 + z_2{}^2)}}$$

before moving to CONICS → QUADRICS
DUAL CONICS → DUAL QUADRICS

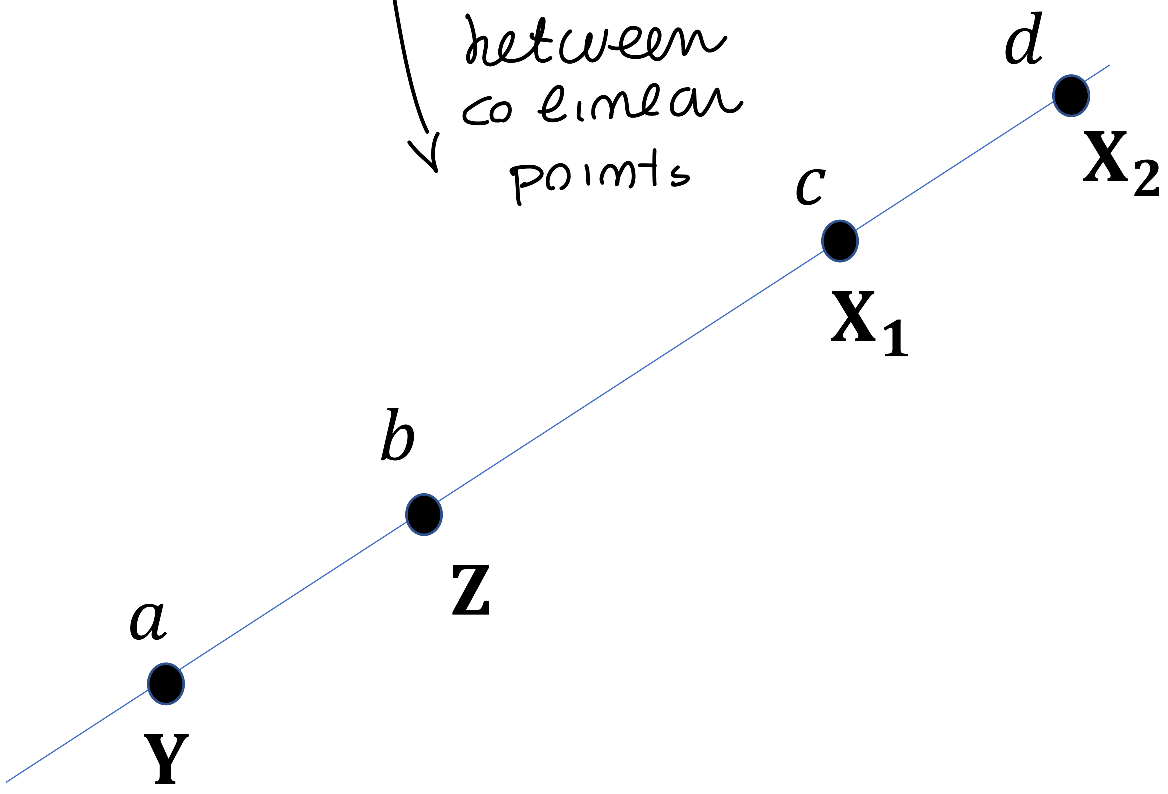
The cross ratio

to be extended in 3D

1D cross ratio of a 4-tuple of colinear points

$$CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

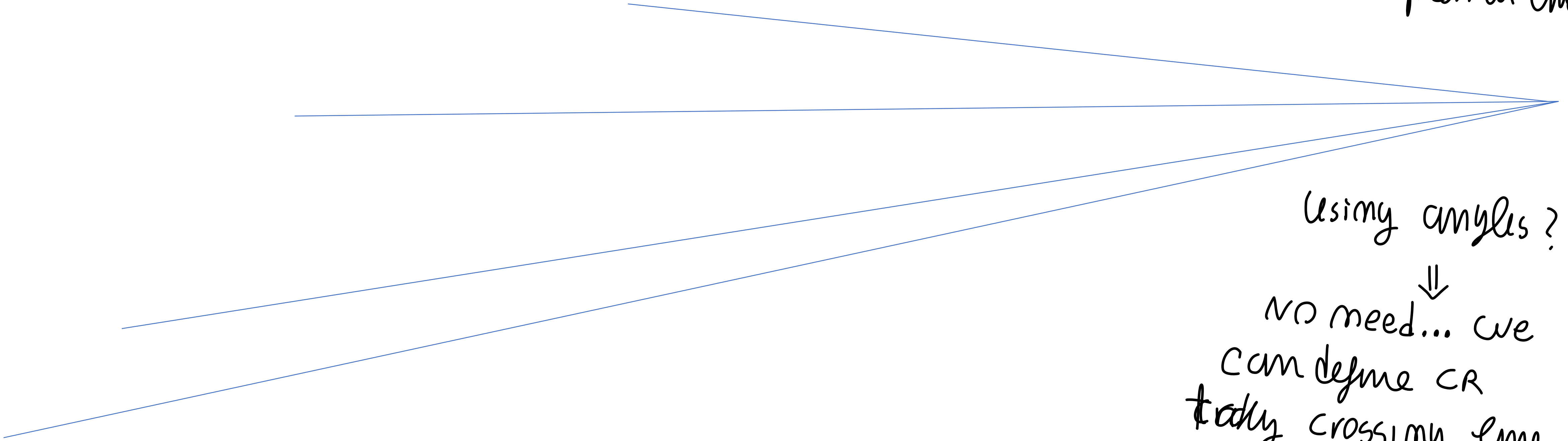
between colinear points



2D cross ratio of a 4-tuple of coplanar, concurrent lines

2D plane) 4 colinear point $\longleftrightarrow$ 4 concurrent lines
(dual on 2D geometry)

we define cross ratio by concurrent coplanar lines!

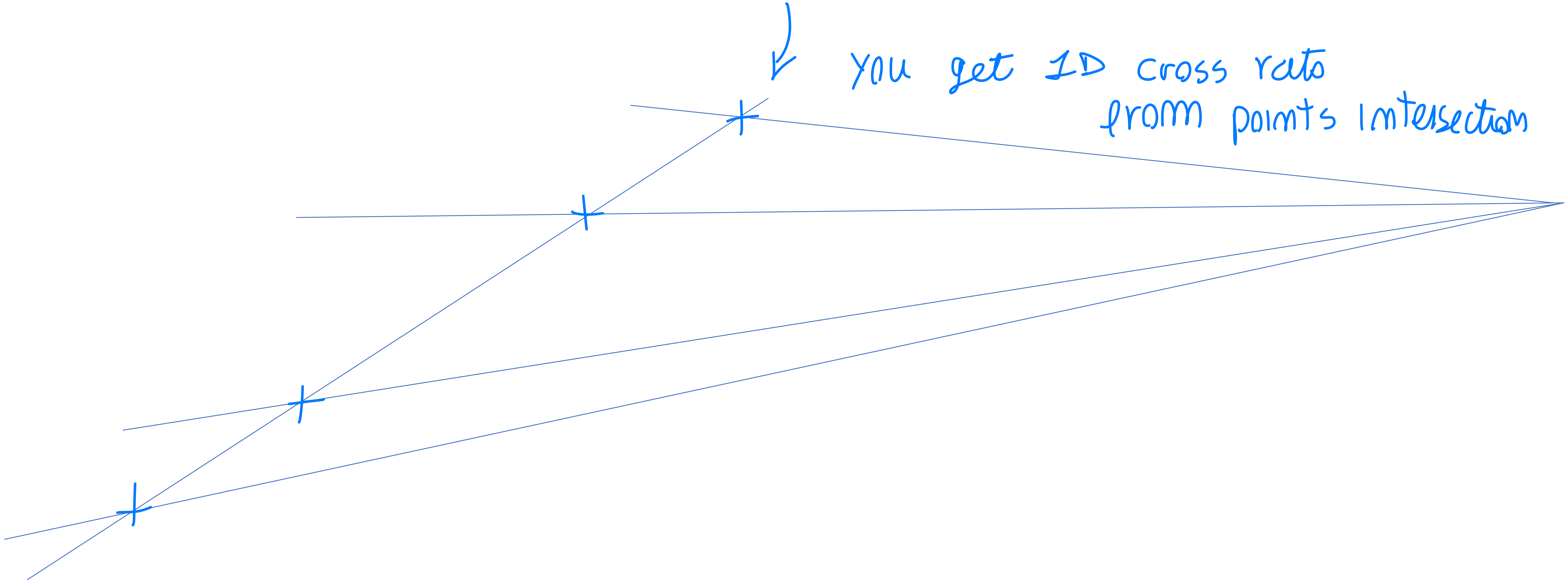


using angles?

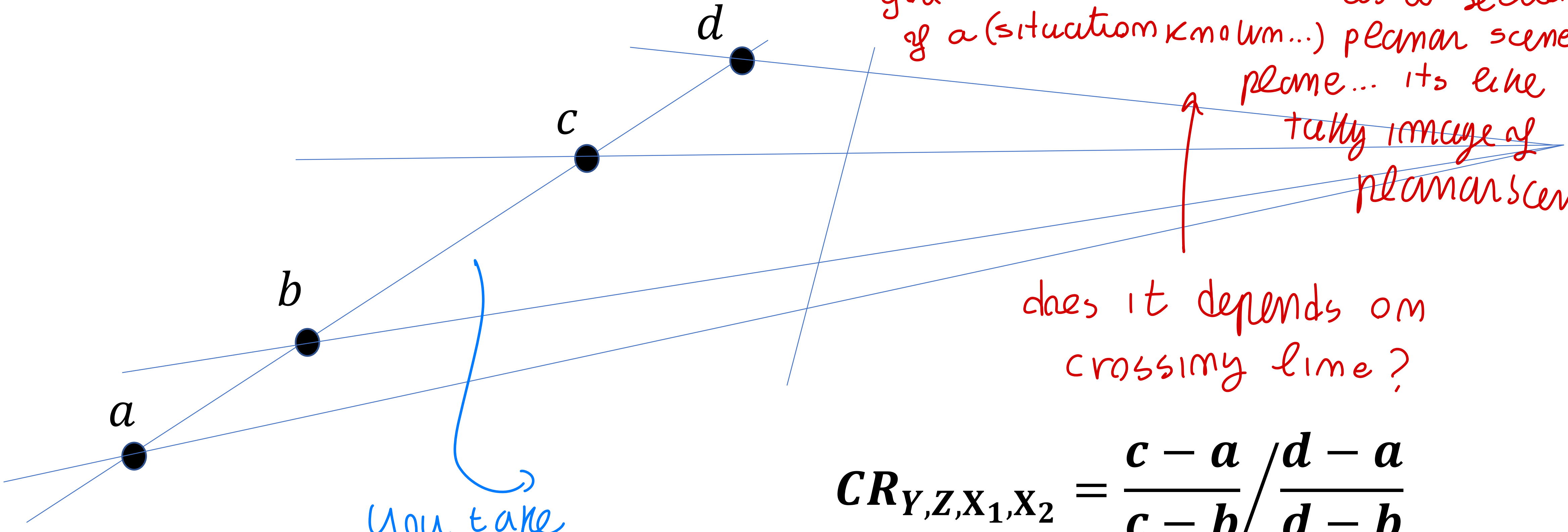
$\Downarrow$

No need... we can define CR taking crossing line $\rightarrow$

2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line …



2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line … compute the 1D cross ratio of intersection points



you can consider this as a section of a (situation known...) planar scene plane... its like taking image of planar scene

does it depends on crossing line?

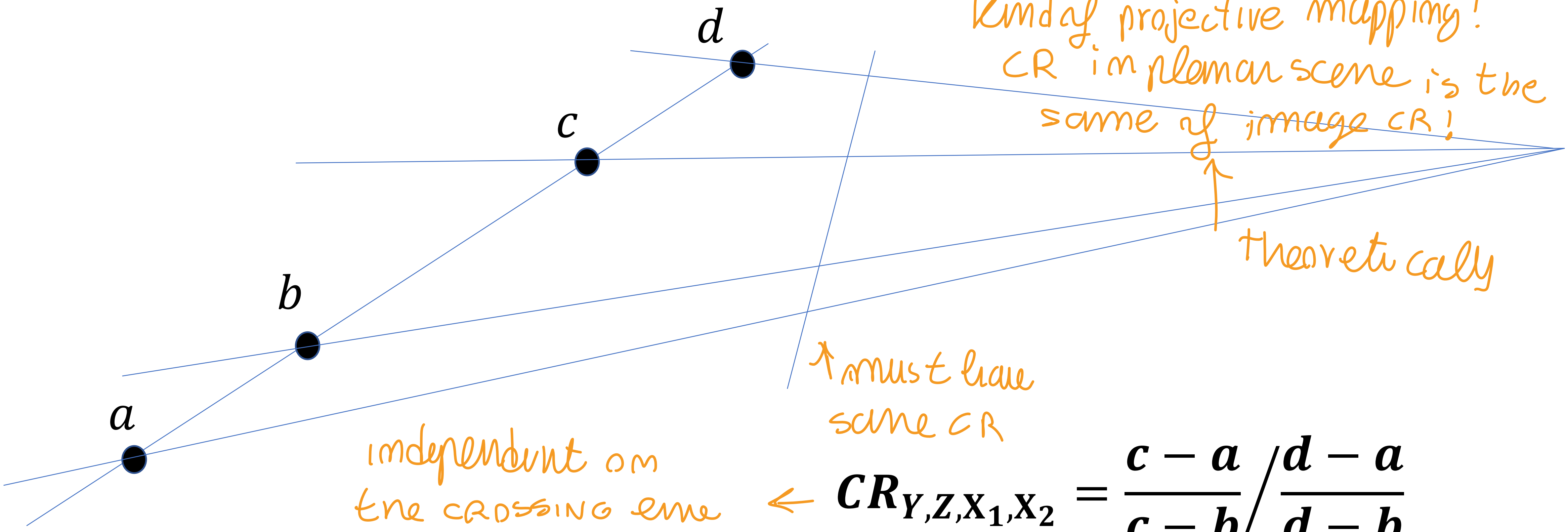
you take crossing line points

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}}$$

2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line … compute the 1D cross ratio of intersection points

Kind of projective mapping! CR in planar scene is the same of image CR!

theoretically



must have same CR

independent on the crossing line chosen only depending on original 4 lines!

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}}$$

3D cross ratio of a 4-tuple of coaxial planes:

⇒ extending further!

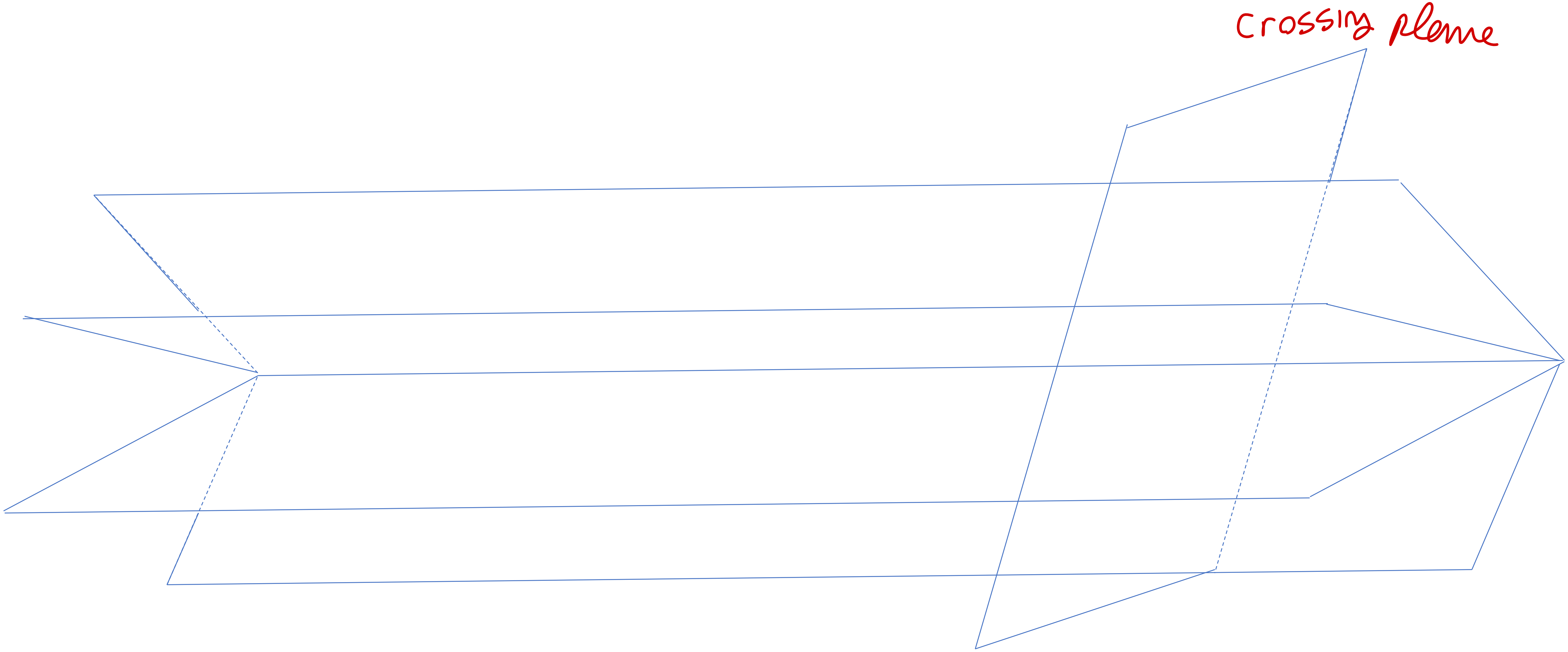
as DUAL of 4 co-linear points in 3D geometry

pencil of 4 planes (coaxial)

colinear ⟷ coaxial ("same axis")

here CR can be computed by crossing plane

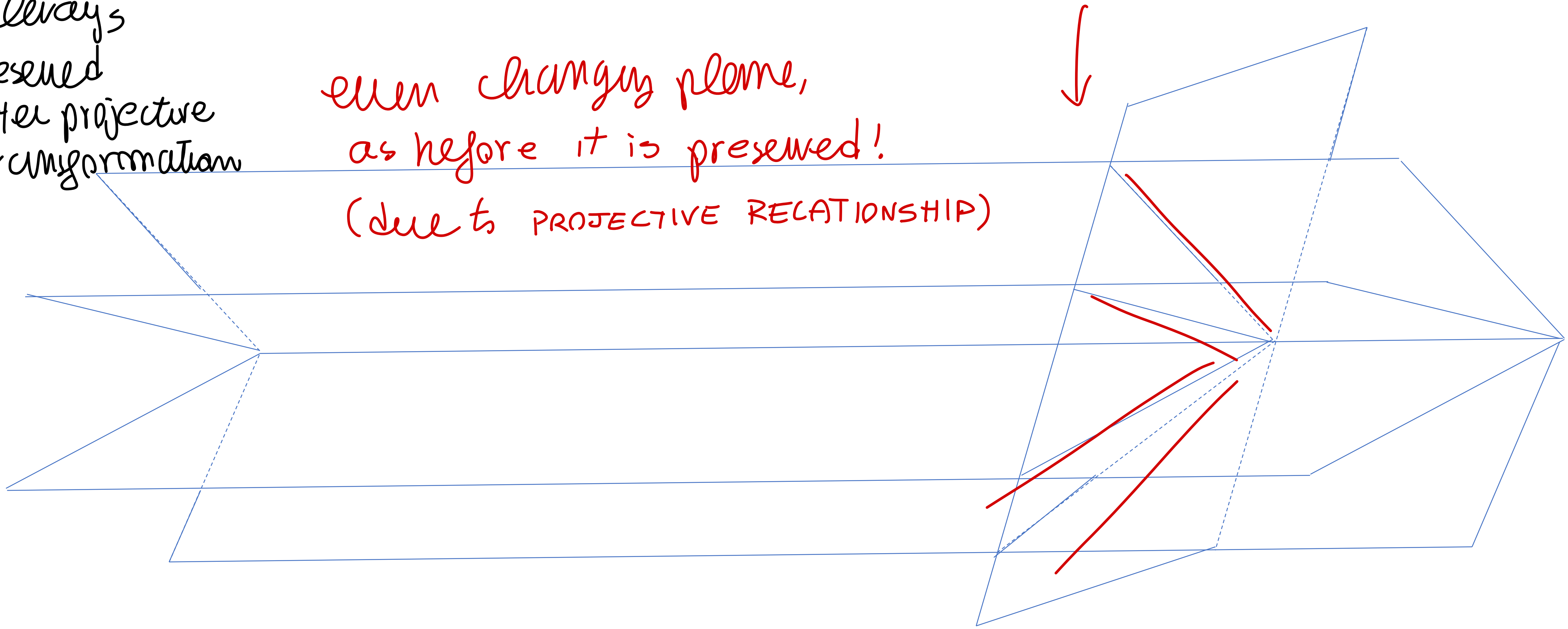
3D cross ratio of a 4-tuple of coaxial planes: take any crossing plane …



3D cross ratio of a 4-tuple of coaxial planes: take any crossing plane … compute the 2D cross ratio of intersection lines

always preserved after projective transformation

even changing plane, as before it is preserved! (due to PROJECTIVE RELATIONSHIP)

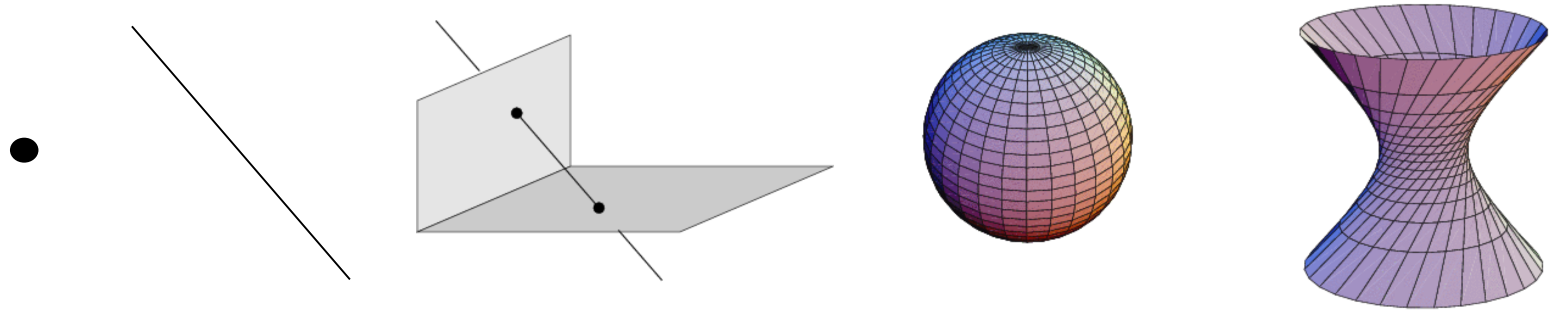


extension of conics

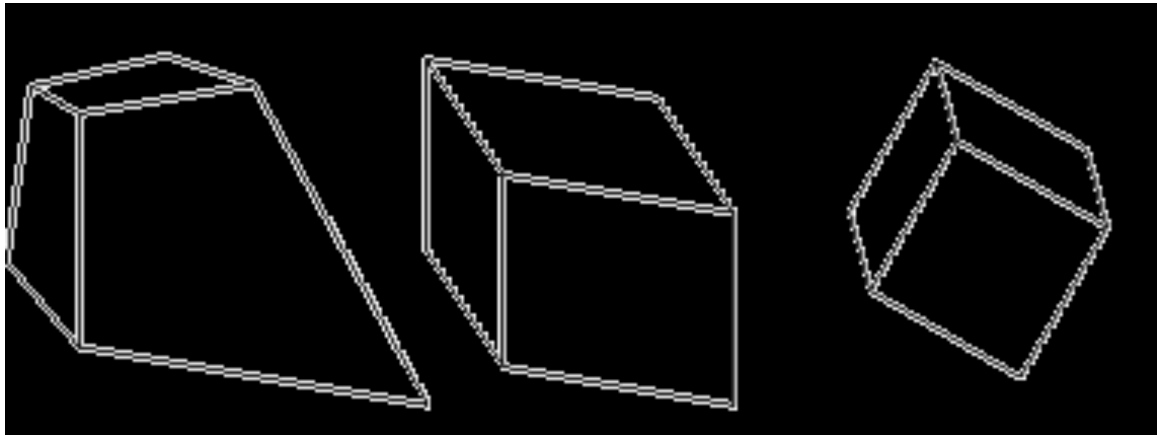
QUADRICS

3D Space Projective Geometry

- **Elements**
  - Points
  - Planes
  - **Quadrics**
  - Dual quadrics



- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities



(extending same concept of conic)

Quadric: a point $\mathbf{X}$ is on a quadric $\boldsymbol{Q}$ if it satisfies a homogeneous *quadratic* equation, namely

dim $\overset{1\times1}{\overset{\shortparallel}{1\times\cancel{4\times4\times4\times4}\times1}}$

$$\mathbf{X}^T \boldsymbol{Q} \mathbf{X} = \mathbf{0}$$

⇐ scalar equation $(\mathbb{R}^4)^T\, 4\times4\ \mathbb{R}^4$

where $\boldsymbol{Q}$ is a 4x4 symmetric matrix.

homogeneous for sure!

10 dof?
homogeneity
⇓
9 dof

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

to fit a quadric we need 9 points eq.

- $\boldsymbol{Q}$ is a homogeneous matrix: $\lambda\boldsymbol{Q} \Leftrightarrow \boldsymbol{Q}$
- 9 degrees of freedom (out of the 10 dof
- 9 points in general positions define a quadric

It is a SURFACE in 3D space ~ curved surface (NOT plane)

Visually a quadric could be:

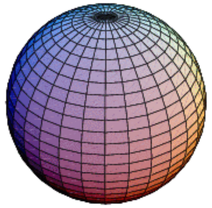
Quadric classification

visually changes between classes

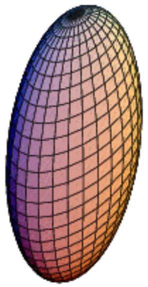
- Projectively equivalent to *sphere:*

extension of hyperbole

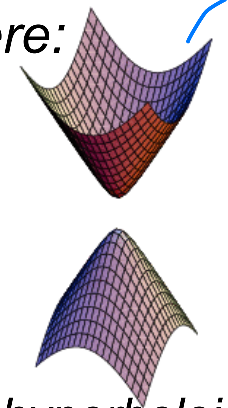
particular ellipsoid



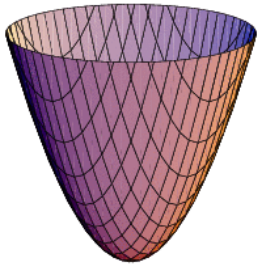
sphere



ellipsoid



hyperboloid of two sheets



paraboloid

"Ruled": even if continuous curved surface.. they contain two set of ∞ straight lines!

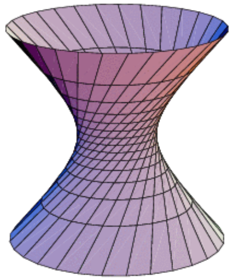
⇓

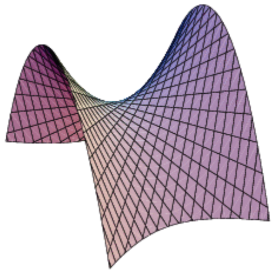
you can work on this surface moving straight in ∞ ways!

- Ruled quadrics:

$\det(Q) \neq 0$

NON SINGULAR ⟹

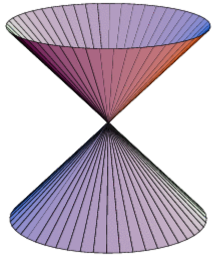




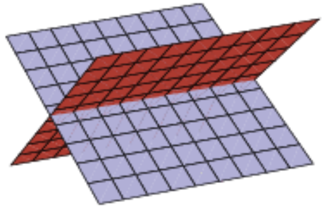
hyperboloids of one sheet

they are made of more family of straight lines! full rank QUADRICS

Degenerate ruled quadrics:



cone

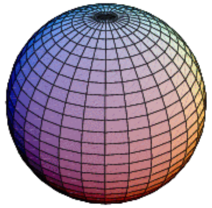


two planes

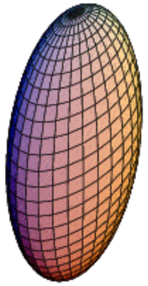
Quadric classification

full rank

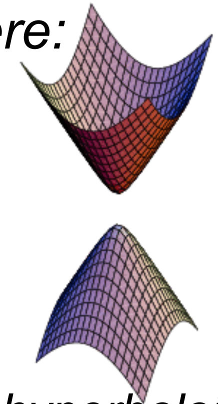
Projectively equivalent to *sphere:*



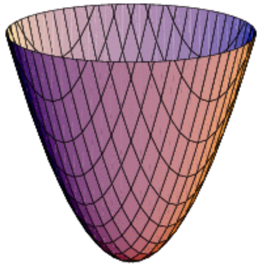
sphere



ellipsoid



hyperboloid of two sheets

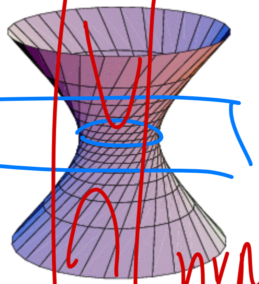


paraboloid

elliptic hyperbolae

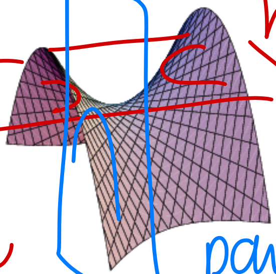
Ruled quadrics:

ellipse



hyperbolae

hyperbolae



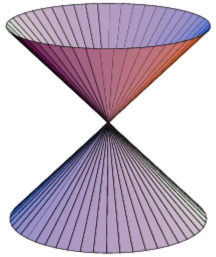
parabolae

hyperboloids of one sheet

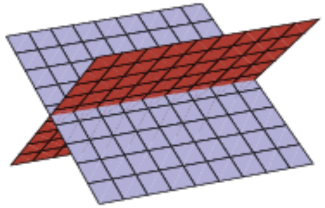
parabolic hyperboloid cutting it in different ways you find hyperbolae or parabolae

det(Q)=0 degenerate NOT full rank

- Degenerate ruled quadrics:



cone

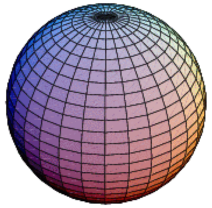


two planes

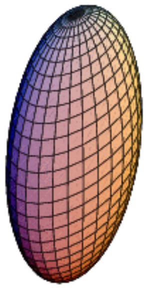
extension of degenerate conics

Quadric classification

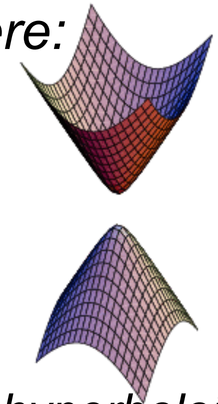
Projectively equivalent to *sphere:*



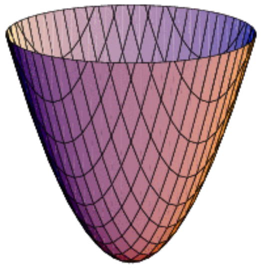
sphere



ellipsoid

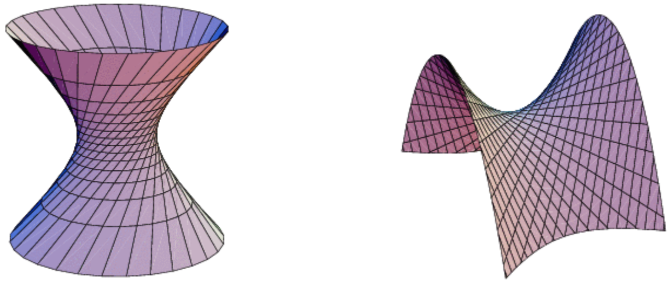


hyperboloid of two sheets



paraboloid

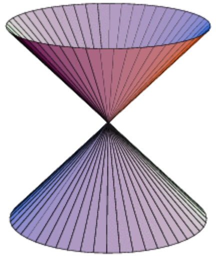
Ruled quadrics:



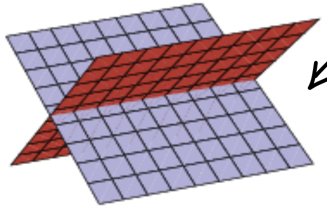
hyperboloids of one sheet

Degenerate ruled quadrics:

degenerate Q of rank = 3



cone



two planes

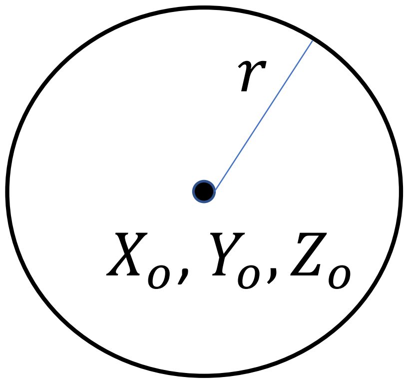
degenerate Q of rank=2

Example: the sphere

⇓ derive Q for a sphere

First in cartesian coordinates:

$$(X - X_o)^2 + (Y - Y_o)^2 + (Z - Z_o)^2 - r^2 = 0$$

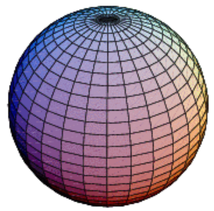


X_o, Y_o, Z_o are the center coordinates, r is the radius.

... then in homogeneous coordinates:

I ← recognize sphere where here there is λI

$$[x \quad y \quad z \quad w]\begin{bmatrix} 1 & 0 & 0 & -X_o \\ 0 & 1 & 0 & -Y_o \\ 0 & 0 & 1 & -Z_o \\ -X_o & -Y_o & -Z_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$



$$\mathbf{X}^\intercal \, \boldsymbol{Q} \, \mathbf{X} = 0$$

$\Longrightarrow$ # Intersection of a plane and a quadric

subset of space

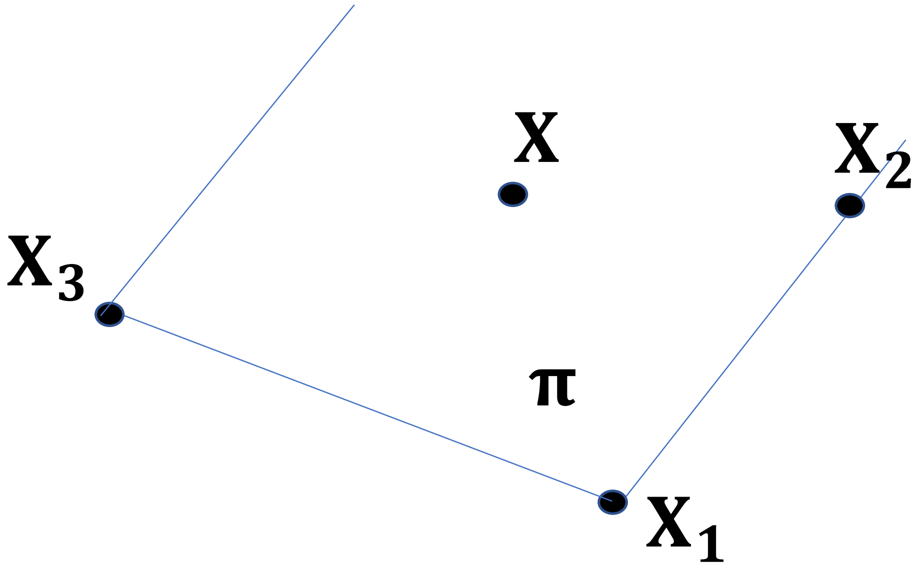
X in space constrained on π plane, by forcing Mx as x = [α β γ]ᵀ

Plane defined by its span: $\boldsymbol{\pi}$: $\boxed{\mathbf{X} = \mathbf{M}\mathbf{x}}$ ($\mathbf{M}$ is a 4x3 matrix)

where vector $\mathbf{x}$ represent homogeneous coordinates within $\boldsymbol{\pi}$

…

remember: the plane as its span



$\mathbf{X}$ is a linear combination $\propto \mathbf{X_1} + \beta \mathbf{X_2} + \gamma \mathbf{X_3}$
→ $\mathbf{X}$ is coplanar to $\mathbf{X_1}, \mathbf{X_2},$ and $\mathbf{X_3}$

i.e. $\mathbf{X} = [\mathbf{X_1} \quad \mathbf{X_2} \quad \mathbf{X_3}] \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix} = \mathrm{M}\mathbf{x}$ where

$\mathbf{x} = \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix}$ can be regarded as homogeneous coordinates within the 2D geometry of plane $\boldsymbol{\pi}$

$$\mathbf{X} = \mathrm{M}\mathbf{x}$$

Intersection of a plane and a quadric

3 columns = 3 points on plane π

Plane defined by its span: $\boldsymbol{\pi}$: $\mathbf{X} = \mathbf{M}\mathbf{x}$ ($\mathbf{M}$ is a 4x3 matrix)

where vector $\mathbf{x}$ represent homogeneous coordinates within $\boldsymbol{\pi}$

Quadric $\mathbf{Q}$: $\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$

intersect!

$C \in \mathbb{R}^{3\times 3}$

Plane-quadric intersection: $\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{x}^T \mathbf{M}^T \boldsymbol{Q} \mathbf{M}\mathbf{x} = \underbrace{\mathbf{x}^T \boldsymbol{C}\, \mathbf{x}} = \mathbf{0}$

x is homogeneus coordinate in 2D plane

… is a **conic $\boldsymbol{C}$**

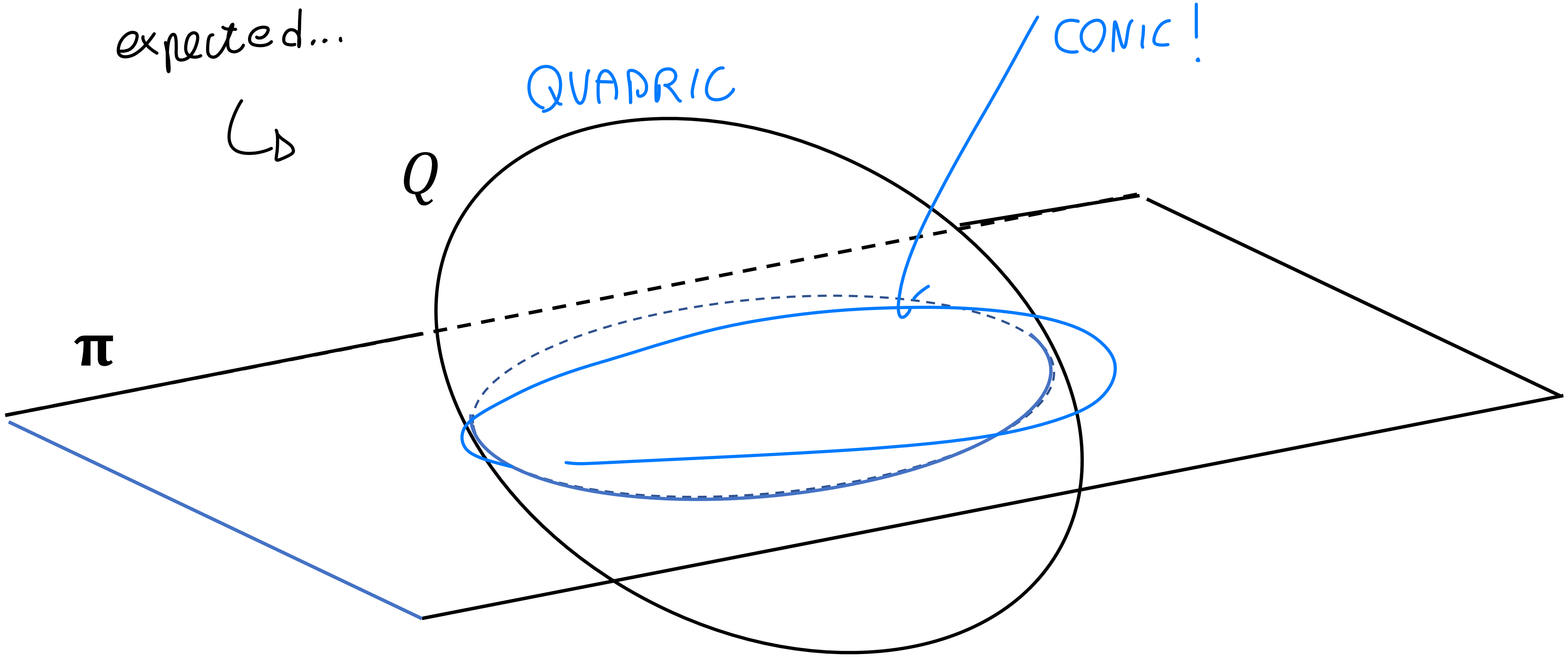
$$\boldsymbol{C} = \mathbf{M}^T \boldsymbol{Q} \mathbf{M}$$

plane – quadric intersection: a conic

.

as expected...

↳



The absolute conic:

an extension of the circular points

particular conic in 3D space!

$\Downarrow$ extension of circular points

$\Uparrow$

primitive elements in 3D = point, plane, quadrics

Why CONIC?

a special one need to be studied for 3D reconstruction! (very usefull) as CIRCULAR POINTS for 2D

as circular point was any circumference intersected by ℓ_∞

$\Downarrow$

extend by sphere and π_∞

D

A noteworthy example:
intersecting a sphere and the plane at the ∞

intersection is a conic!

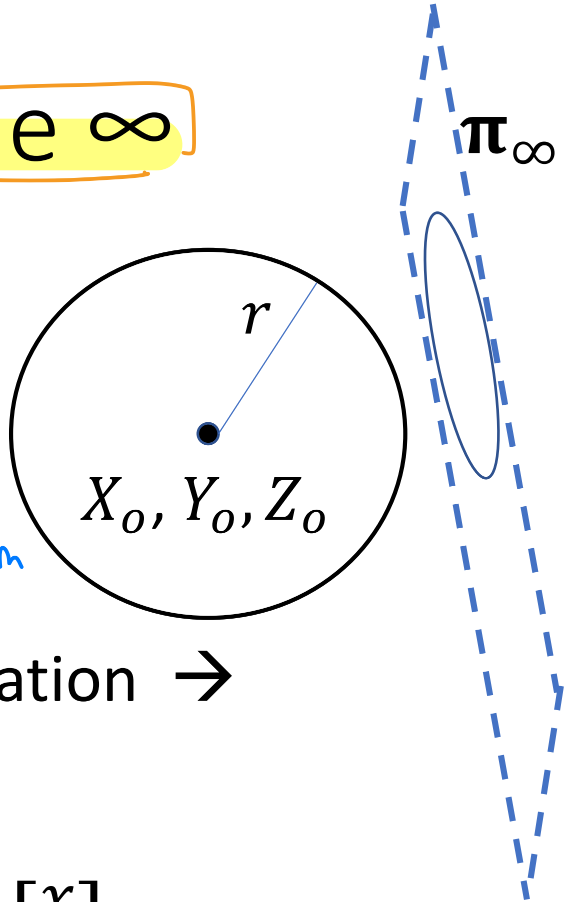
$$\begin{cases}(x - X_o\cancel{w})^2+(y - Y_o\cancel{w})^2+(z - Z_o\cancel{w})^2- \cancel{r^2}w^2 = 0 \\ w = 0\end{cases}$$

$(W=0)$

$$\rightarrow \quad \begin{cases}x^2 + y^2 + z^2 = 0 \\ w = 0\end{cases}$$

same algebra as circular point evaluation

x=y=z=0 is the only Re solution!



The sphere parameters (center and radius) disappear from the equation → the intersection **conic** is the **same for all** spheres:

all points are complex!

Rewriting...

$$x^2 + y^2 + z^2 = \begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x & y & z\end{bmatrix}\Omega_\infty\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

extension of pair of circular points!

identity matrix

A conic within $\boldsymbol{\pi}_\infty$: $\Omega_\infty = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ **ABSOLUTE CONIC**

conic which lives within π_∞ plane at ∞

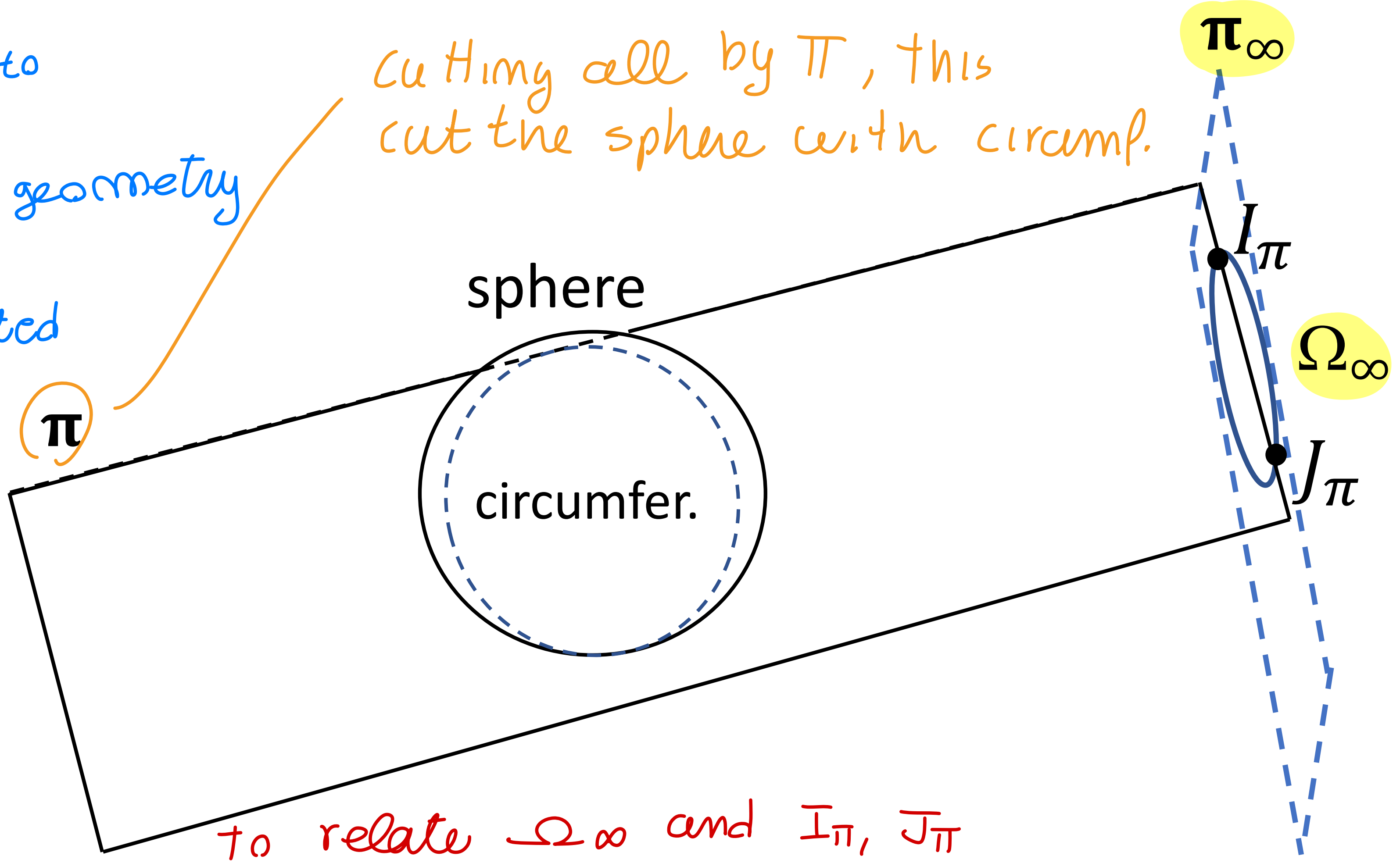
absolute conic: made of (∪) circular points of all planes

Within Π_∞ we go back to planar geometry … in Π_∞ we can stay on 2D geometry

↓

conic in Π_∞ represented by 3×3 matrix

cutting all by Π, this cut the sphere with circumf.



⇒ relationships between $I(\pi), J(\pi)$ and Ω_∞? related somehow?

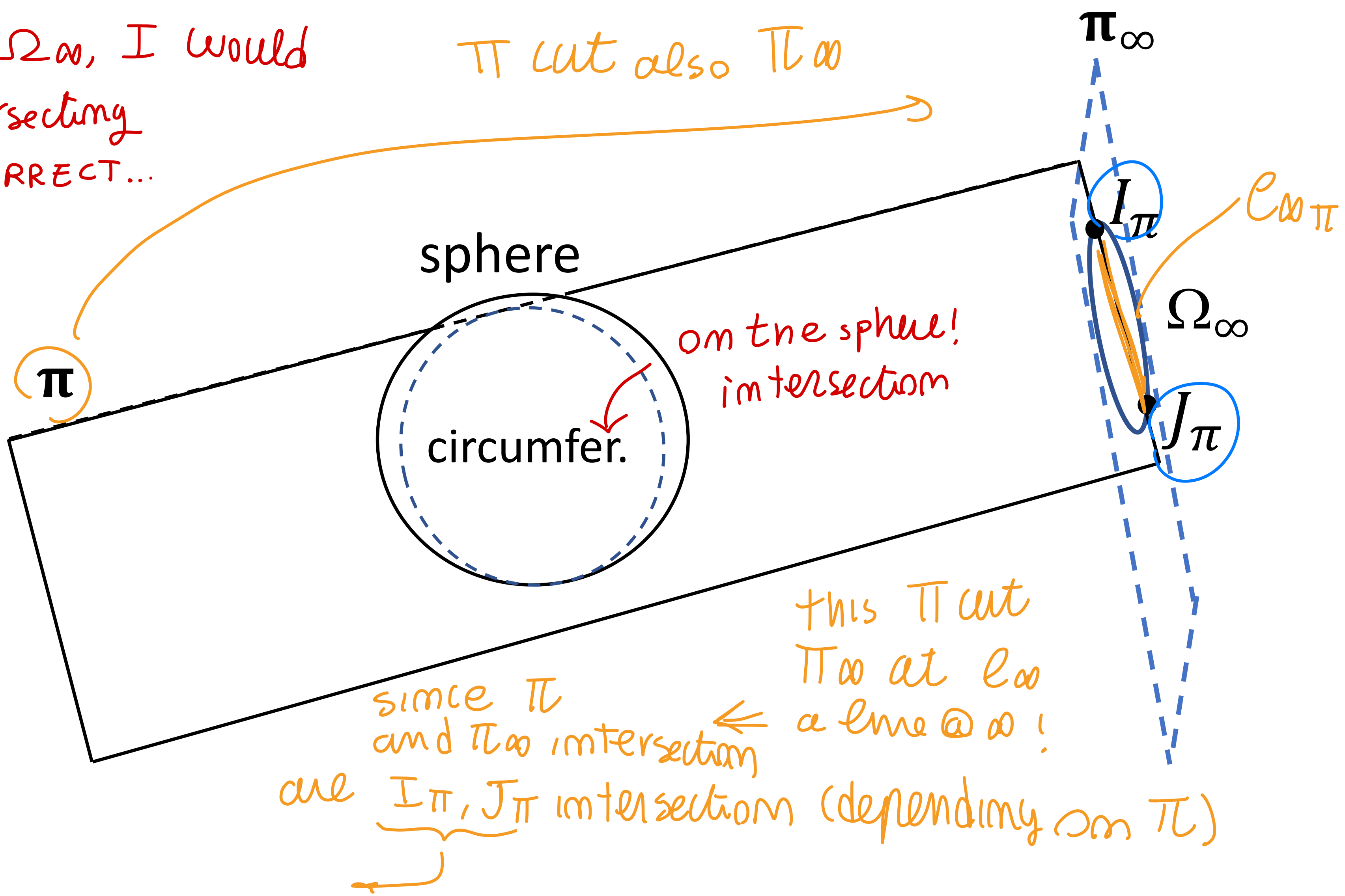
↓

yes!

to relate Ω_∞ and I_π, J_π each π has its own I, J …

absolute conic: made of (∪) circular points of all planes

If they where NOT in Ω_∞, I would have found I, J intersecting π, NOT in Ω_∞ NOT CORRECT... since lines are part of Π_∞ and I_π, J_π must be both part of Π_∞ and Ω_∞!

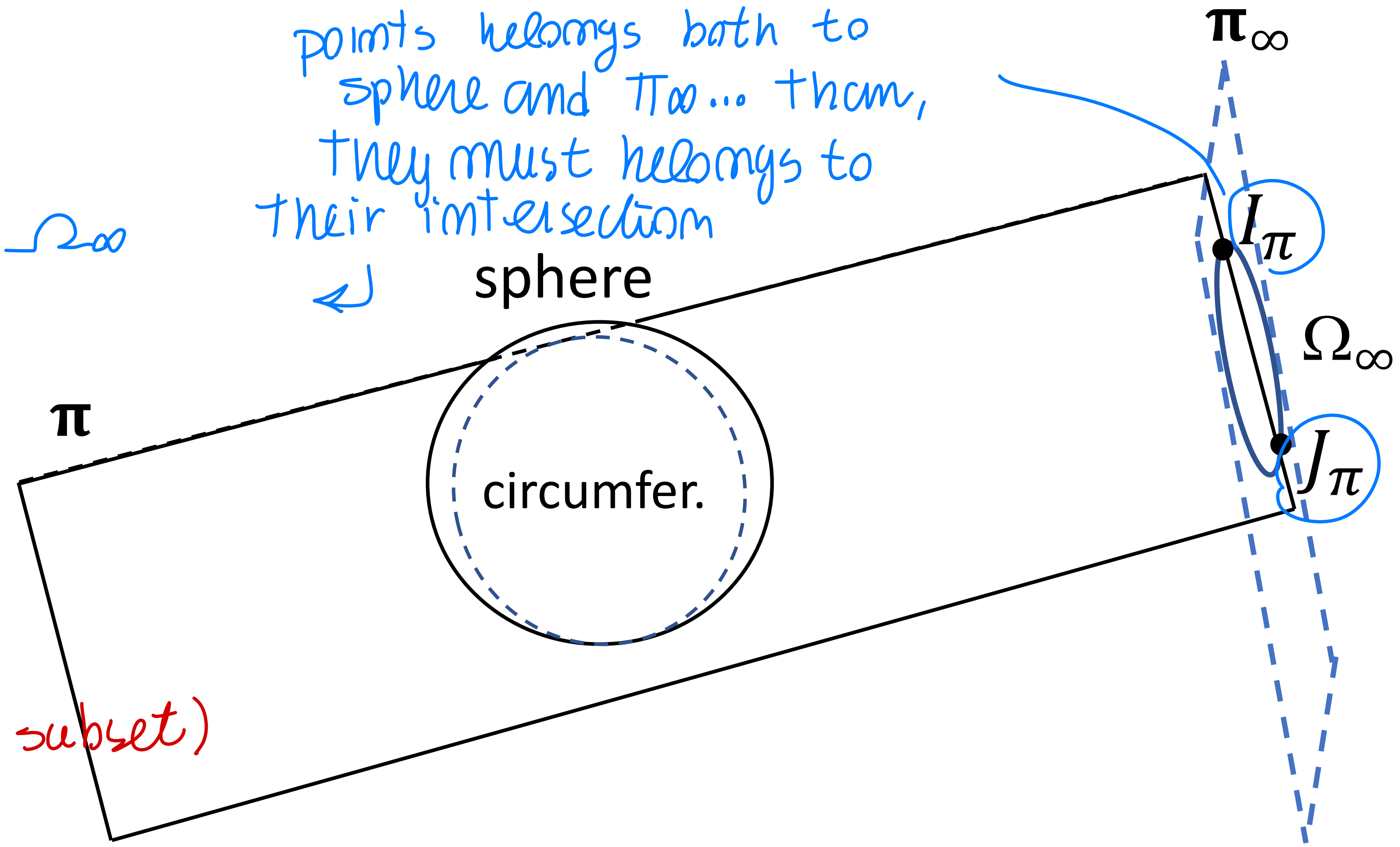


what happens is that $I_\pi, J_\pi \in \Omega_\infty$

absolute conic: made of (∪) circular points of all planes

points belongs both to sphere and π_∞... then, they must belongs to their intersection

so belongs to Ω_∞

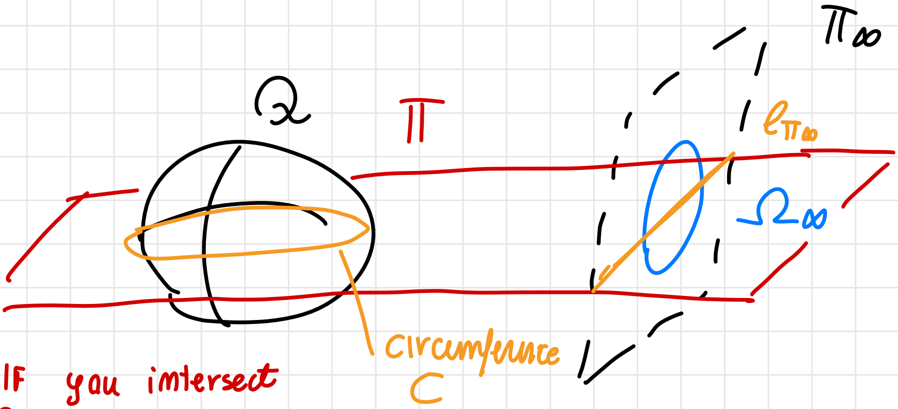


$\Downarrow$

so, any plane π and I_π, J_π, those belongs to Ω_∞

(this is a matter of subset)

conic Ω_∞ is set of all points both on sphere Q and Π_∞



If you intersect Q with Π

$\Downarrow$

and intersection between C and $\ell_{\Pi_\infty} \in \Pi_\infty$

$\downarrow$

must be I_∞, J_∞ pair of circular points, belonging to C and Π_∞

$\leftarrow$ since $I, J \in C$ then $I, J \in Q$

being $C \subseteq Q$ subset circumference, also $I, J \in \Pi_\infty$

$\downarrow$

therefore $I, J \in C \cap Q$ intersect

and $I, J \in Q \cap \Pi_\infty$

$\| \; \Omega_\infty$ they must belong to Ω_∞

$\Downarrow$

so, Ω_∞ is made of $I_\Pi \; J_\Pi$ of all planes Π !

Union set of all $\underbrace{I_\Pi, J_\Pi}_{\forall \Pi \text{ in space}}$ create Ω_∞

$\hookrightarrow$ span of I_Π, J_Π of planes Π

THE POLAR PLANE

polar line of conic wrt conic $\longrightarrow$ (3D)
(2D)

Polar plane of a point wrt a quadric

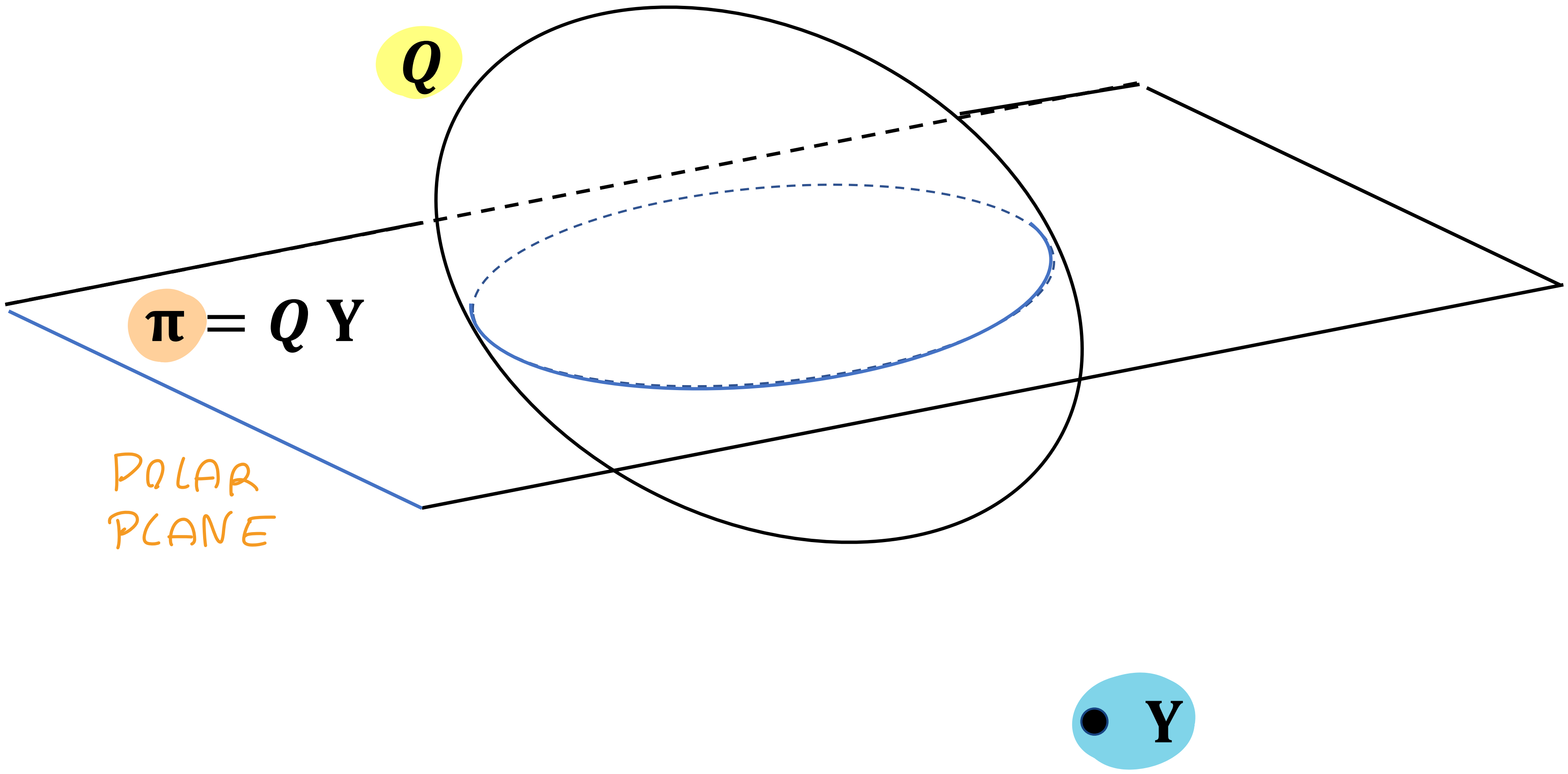
Given a point $\mathbf{Y}$ and a quadric $\boldsymbol{Q}$, the plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ is called the *polar plane* of point $\mathbf{Y}$ with respect to the quadric $\boldsymbol{Q}$.

$\mathbb{R}^4$ vector ~ plane!
vector collecty
polar plane parameters

extension!

Polar plane of a point wrt a quadric

Given a point $\mathbf{Y}$ and a quadric $\boldsymbol{Q}$, the plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ is called the *polar plane* of point $\mathbf{Y}$ with respect to the quadric $\boldsymbol{Q}$.



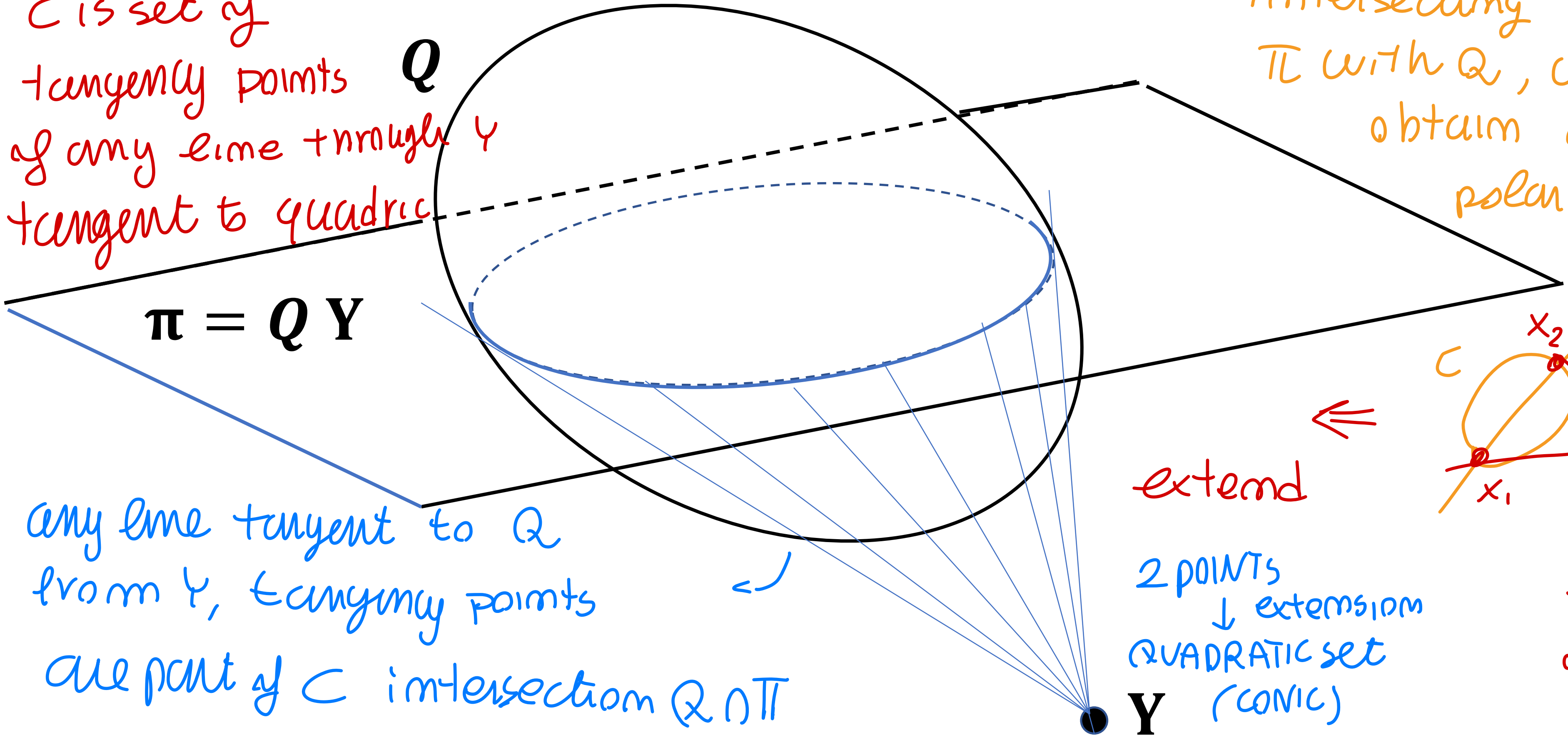
Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

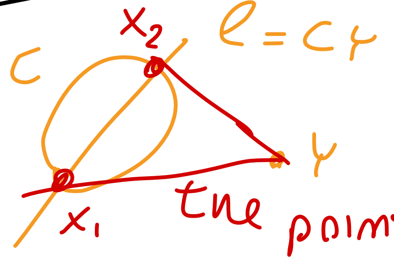
The intersection conic $\boldsymbol{\pi} \cap \boldsymbol{Q}$ is the set of tangency points of the planes through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$ (or the lines through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$)

intersecting polar plane π and Q → conic C

C is set of tangency points of any line through Y tangent to quadric

as in 2D geometry ... intersecting π with Q, we obtain as in 2D polar line ∩ conic





⇐ extend

the points x_1, x_2 where tangency points of l, m

any line tangent to Q from Y, tangency points are part of C intersection $Q \cap \pi$

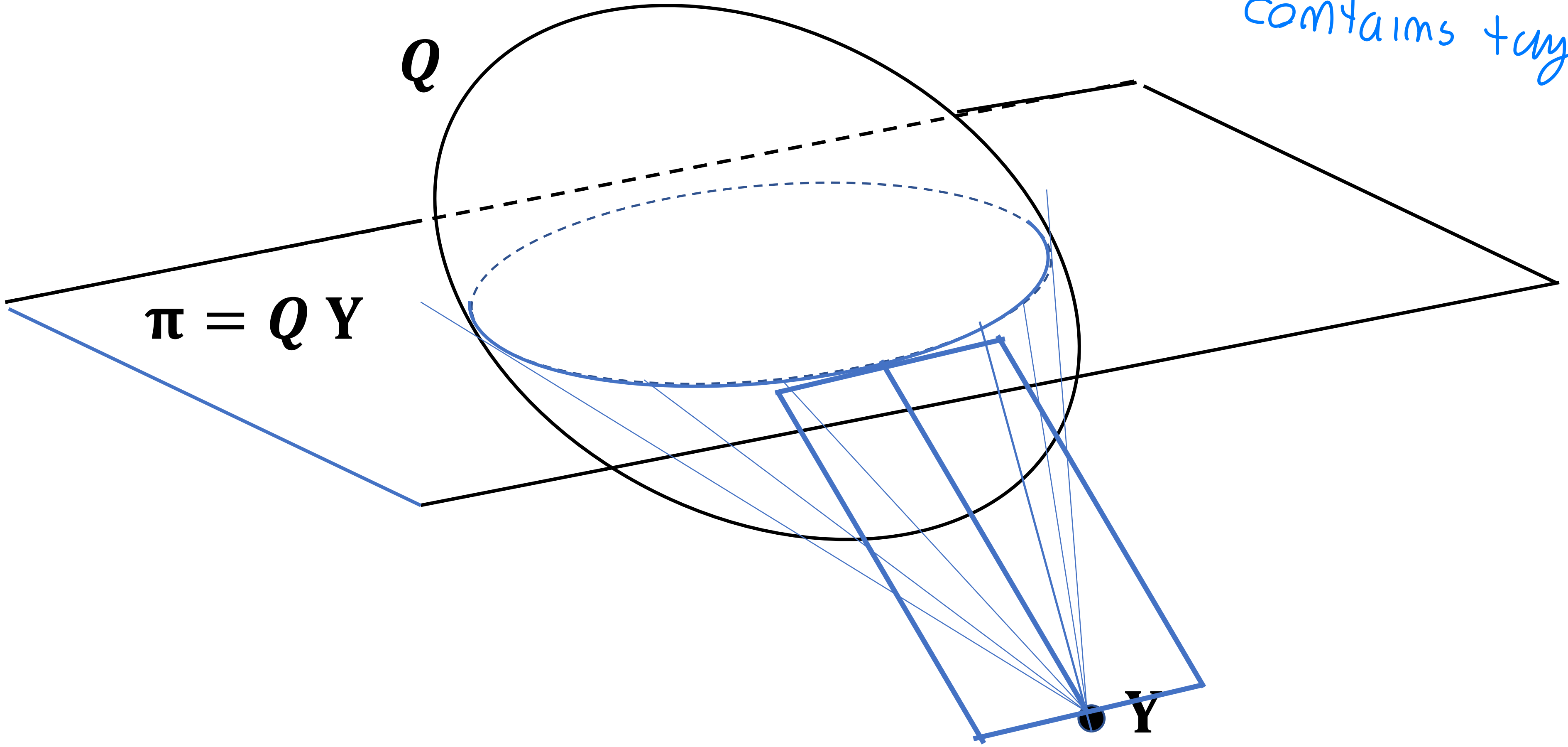
2 POINTS ↓ extension QUADRATIC set (CONIC)

Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

The intersection conic $\boldsymbol{\pi} \cap \boldsymbol{Q}$ is the set of tangency points of the planes through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$ (or the lines through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$)

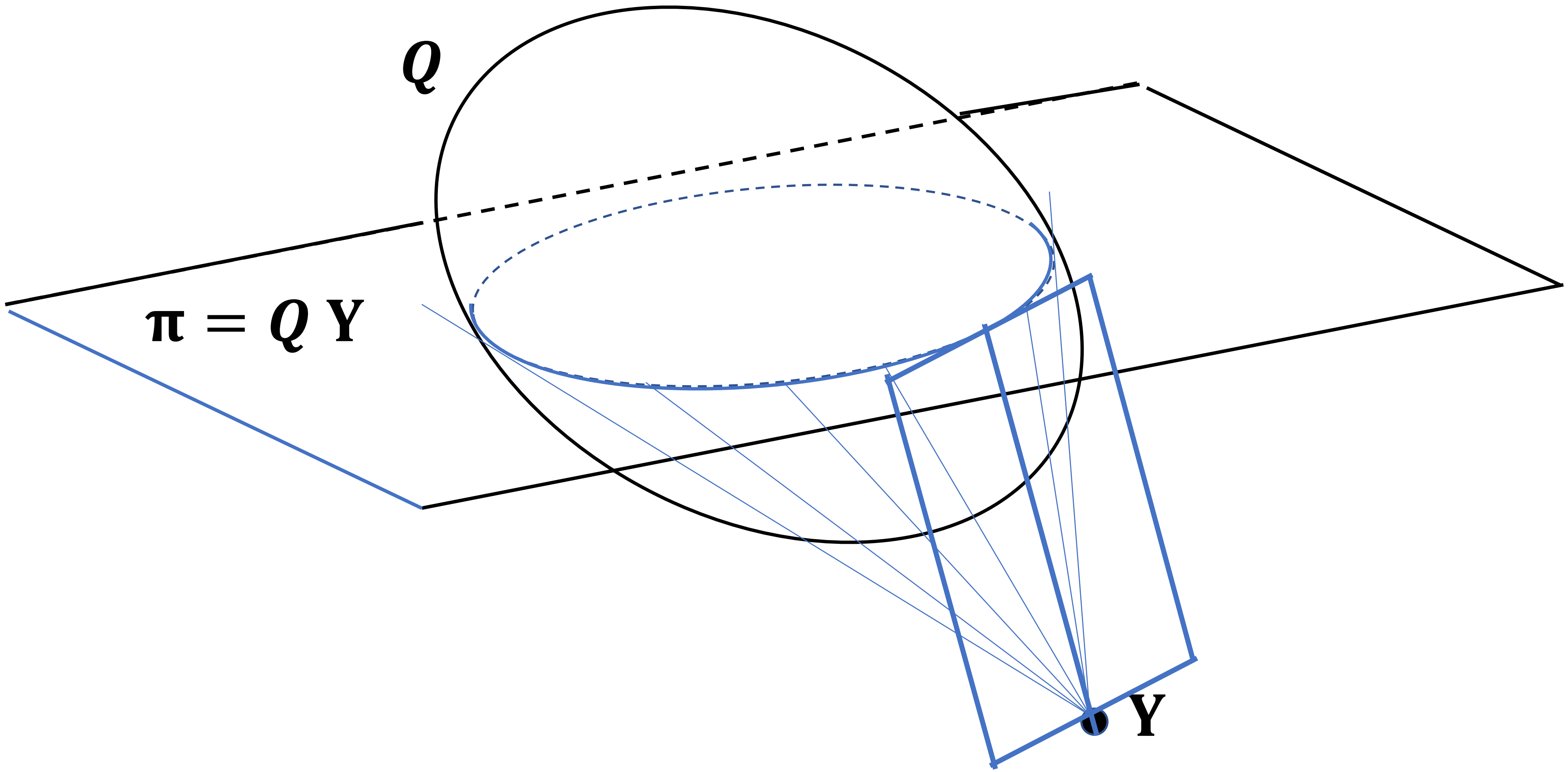
you need to talk about tangent plane in 3D

each tangent plane contains tangent line



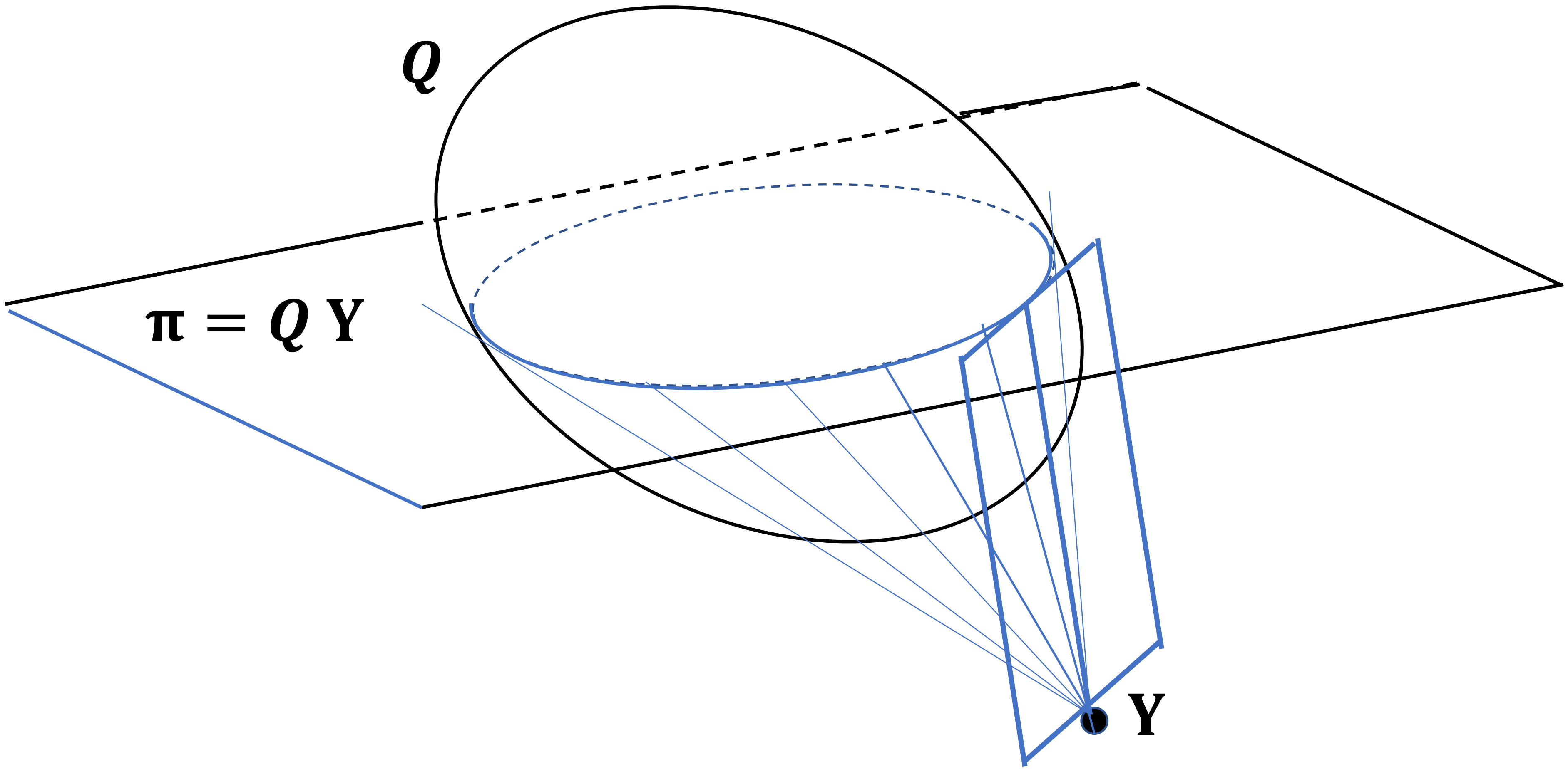
Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

The intersection conic $\boldsymbol{\pi} \cap \boldsymbol{Q}$ is the set of tangency points of the planes through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$ (or the lines through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$)



Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

The intersection conic $\boldsymbol{\pi} \cap \boldsymbol{Q}$ is the set of tangency points of the planes through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$ (or the lines through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$)



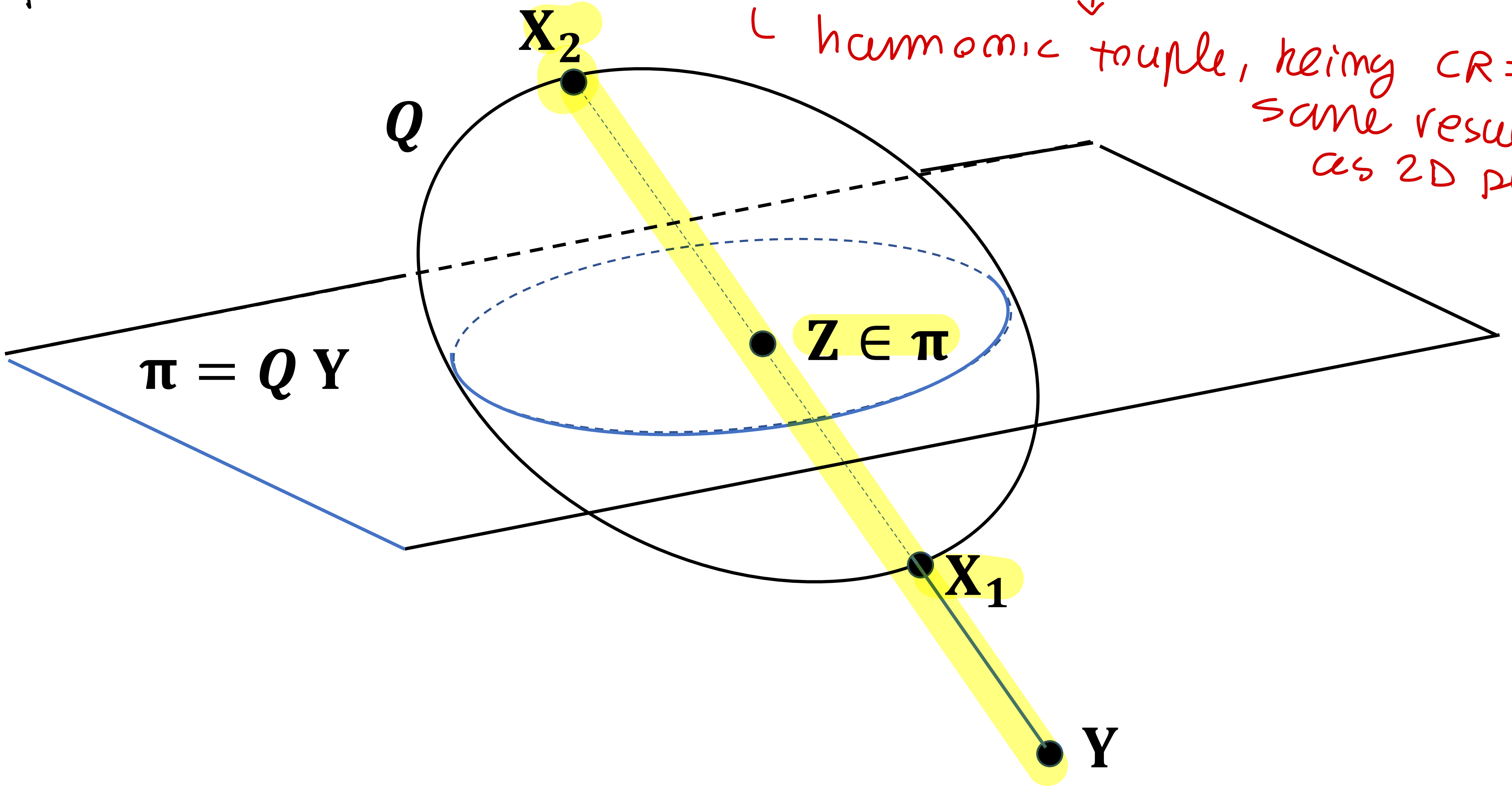
Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

another property as extension of what holds on 2D geometry

intersecting, you find 4-touple of points

Any 4-tuple $(\mathbf{Y}, \mathbf{Z}, \mathbf{X_1}, \mathbf{X_2})$ is harmonic (CR = -1)

harmonic touple, being CR=-1 same result as 2D planar geom.

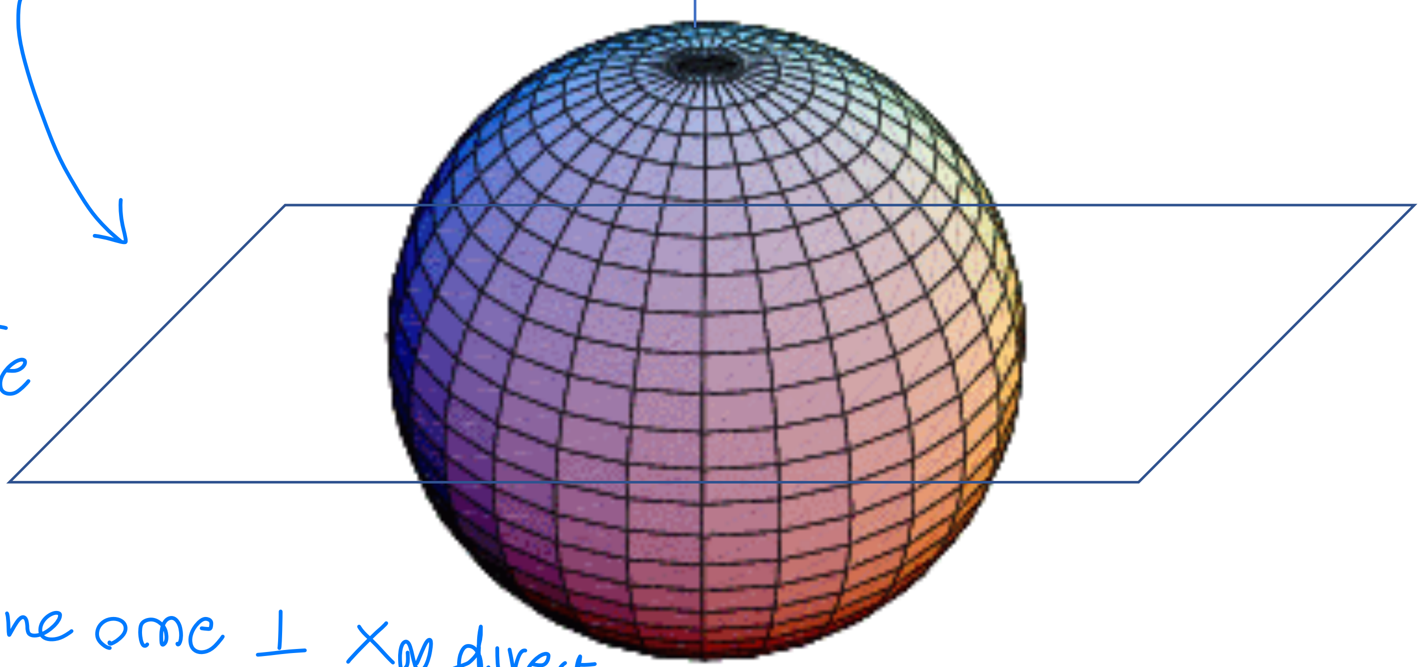


Example: polar of a point at the infinity wrt a sphere

$\mathbf{X}_\infty$

↳ chose a sphere, and
X_∞ as polar plane particular point

equatorial plane orthogonal to the direction of $\mathbf{X}_\infty$



goes through sphere center! $= \Pi_e$

$\Downarrow$

we find the one $\perp$ X_∞ direction what if:

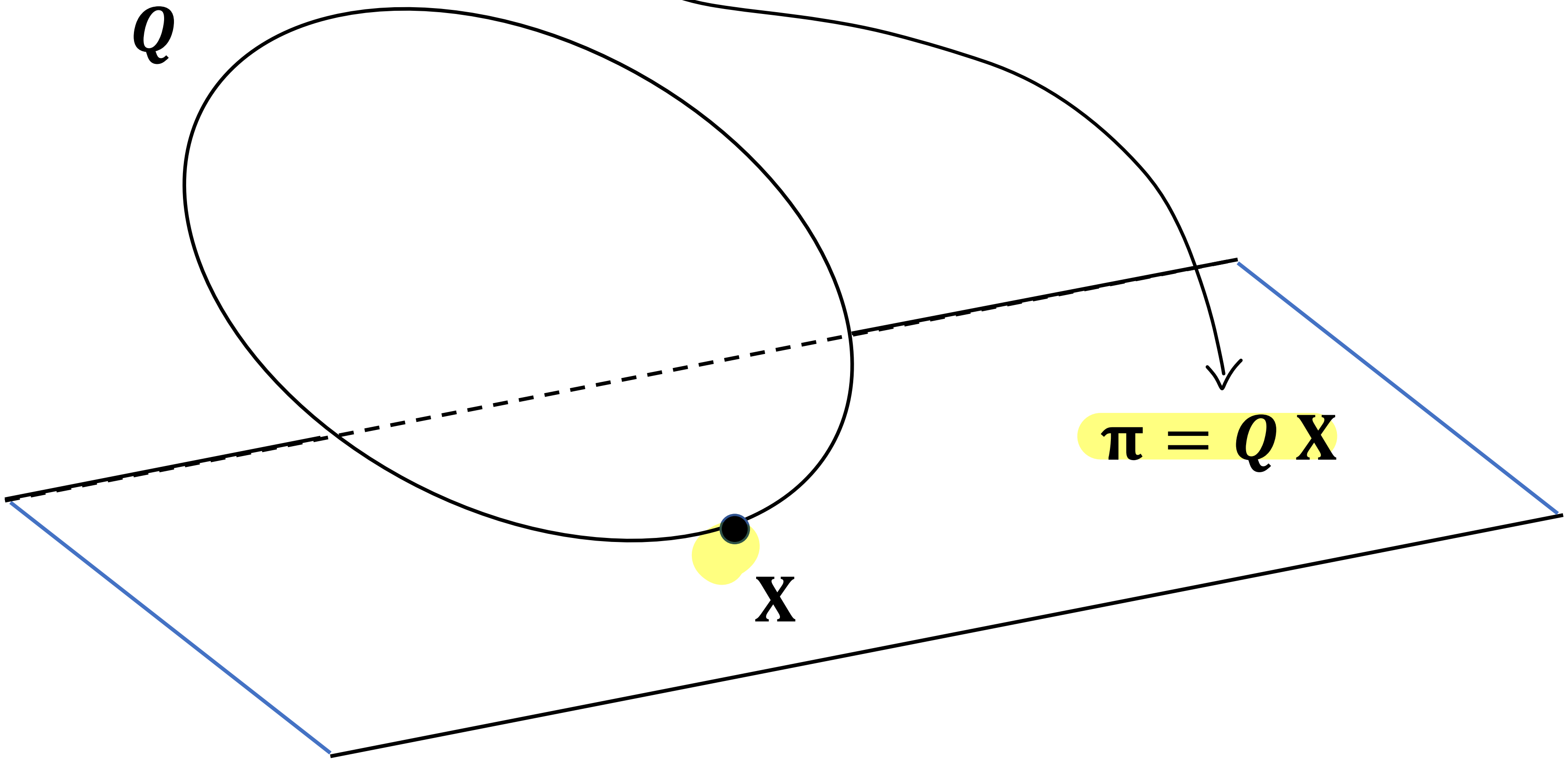
the polar plane of a point **ON** the quadric $\boldsymbol{Q}$

as we obtained for 2D geometry,

when $y \in C$, yC was tangent to C at y $\longrightarrow$ atension! $\Rightarrow$

the polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{X}$ of a point $\mathbf{X}$ ON the quadric $\boldsymbol{Q}$

… the plane tangent to $\boldsymbol{Q}$ through $\mathbf{X}$



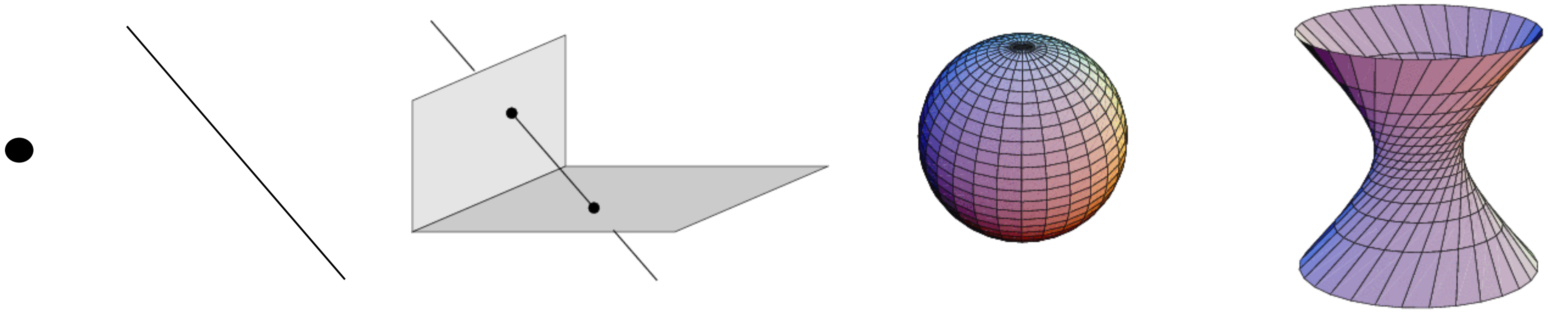
(NONDEGENERATE) DUAL QUADRICS

← thats tangency is usefull for Dual quadrics computation

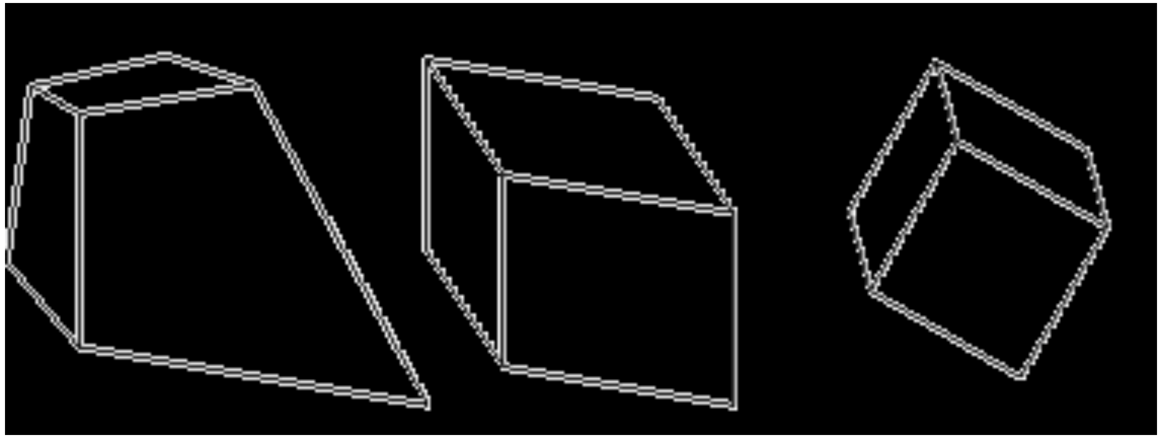
↳

3D Space Projective Geometry

- **Elements**
- Points
- Planes
- Quadrics
- **Dual quadrics**



- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



set of planes π satisfying quadratic homogeneus equations

Dual quadric: a plane $\boldsymbol{\pi}$ is on a dual quadric $\boldsymbol{Q}^*$ if it satisfies a homogeneous *quadratic* equation, namely

$$\boldsymbol{\pi}^T \boldsymbol{Q}^* \boldsymbol{\pi} = \mathbf{0}$$

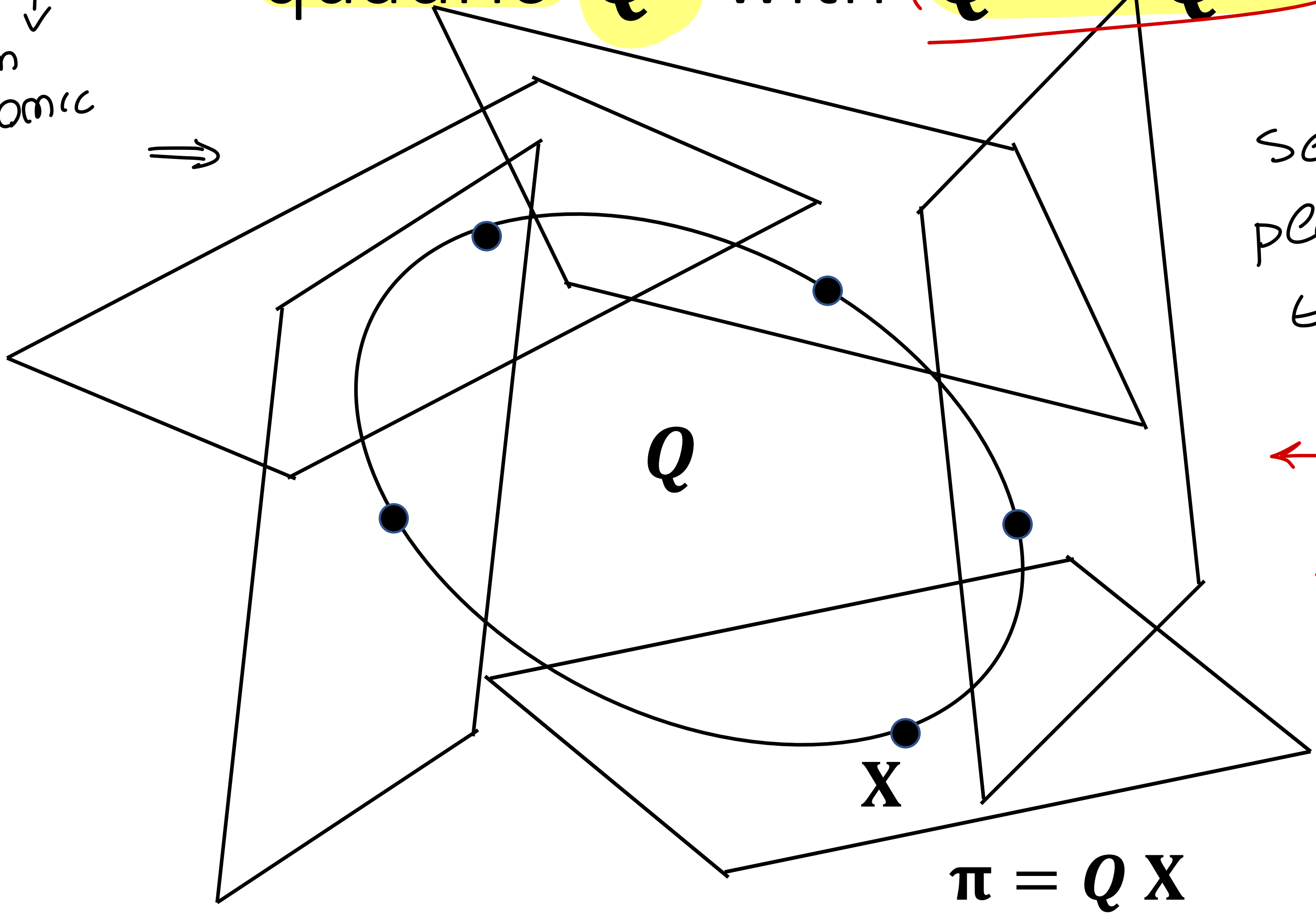
where $\boldsymbol{Q}^*$ is a 4x4 symmetric matrix.

$\boldsymbol{Q}^*$ is a set of **planes**

- $\boldsymbol{Q}^*$ is a homogeneous matrix: $\lambda \boldsymbol{Q}^* \Leftrightarrow \boldsymbol{Q}^*$
- 9 degrees of freedom

the set of planes tangent to a quadric $\boldsymbol{Q}$ is the dual quadric $\boldsymbol{Q}^*$ with $\boldsymbol{Q}^* = \boldsymbol{Q}^{-1}$

extension of dual conic of 2D... ⇒



set of planes tangent to Q

↵ layout, you start from Q^*, then if $\exists (Q^*)^{-1}$ NONdg.

⇓

$Q = (Q^*)^{-1} \equiv$ QVADRIC

↓

you take all π tangent to Q → dual quadric

DEGENERATE QUADRICS

classifyed accordingly to rank of Q

$$\mathbf{X}^T Q\, \mathbf{X} = 0$$

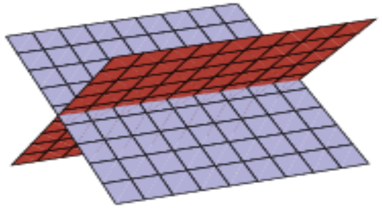
when Q NOT FULL RANK

- rank $Q = 1$ → $Q = \mathbf{A}\mathbf{A}^T$ repeated plane **A** (as in 1D was repeated line)

(decomposing...)

- rank $Q = 2$ → $Q = \mathbf{A}\mathbf{B}^T + \mathbf{B}\mathbf{A}^T$

* A, B vectors



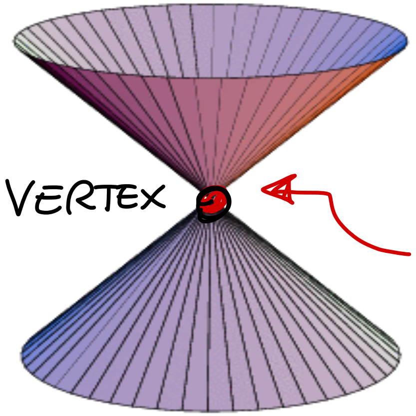
two planes **A** and **B**

⇓

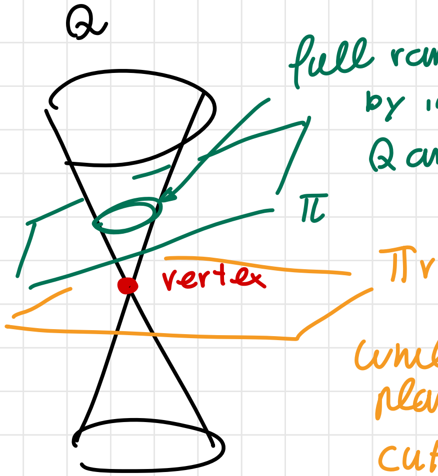
extension of union set of 2 lines in 2D geometry

- rank $Q = 3$ a **cone**

↳ new object! NOT full rank Q



quadric with only one singularity point → the set of points belonging to this cone, all linear combination of VERTEX and points is a line in Q

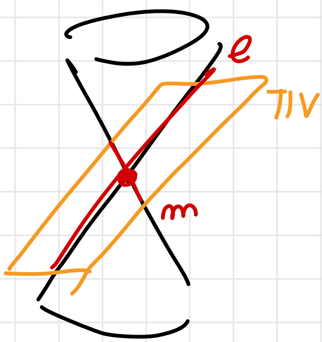


full rank conic
by intersecting
Q and π

since rank(Q) = 3
↓
there exists π
s.t. intersecting
π ∩ Q we find
full conic
(NOT degenerate)

while a
plane through vertex Πv
cutting Q find a degenerate
conic of rank=2, which is
2 lines!

you find
2 lines as
intersection



in other intersection
you may find hyperbolae, parabolae...

DEGENERATE QUADRICS

when rank(Q) = 3
the RNS(Q) has a 1D solution ⟹ point
this is one point!

$$\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$$

$$\text{rank } \boldsymbol{Q} = \mathbf{3}$$

when rank = 3
you know that many info can be derived analyzing RNS of Q

Consider $\boldsymbol{V} = \text{RNS}(\boldsymbol{Q}) : \boldsymbol{Q}\,\boldsymbol{V} = \mathbf{0}$

IMPORTANT POINT!

- $\boldsymbol{V} \in \boldsymbol{Q}$ $\quad [\mathbf{V}^T \boldsymbol{Q}\, \mathbf{V} = \mathbf{V}^T\ \mathbf{0} = \mathbf{0}]$ ⟵ VERTEX of CONIC

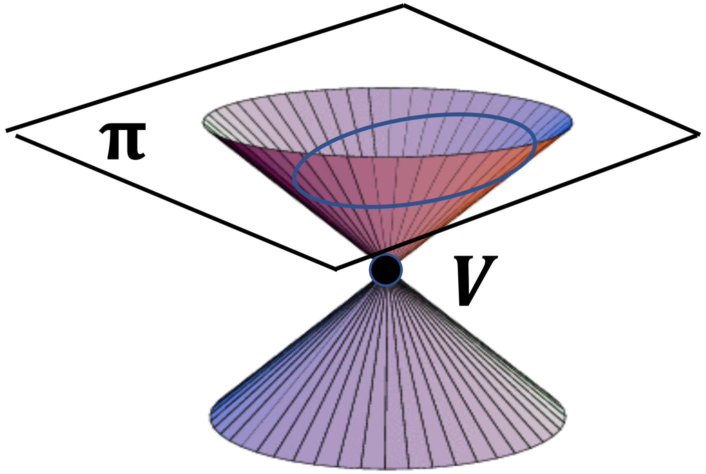
spanning lines joining V, X

- $\forall \boldsymbol{X} \in \boldsymbol{Q}$, any point $\boldsymbol{Y}$ colinear with $\mathbf{X}$ and $\boldsymbol{V}$, namely $\boldsymbol{Y} = \alpha \boldsymbol{X} + \boldsymbol{V}$, is also $\in \boldsymbol{Q}$

  in fact $\boldsymbol{Y}^T \boldsymbol{Q}\, \boldsymbol{Y} = (\alpha \boldsymbol{X} + \boldsymbol{V})^T \boldsymbol{Q}\, (\alpha \boldsymbol{X} + \boldsymbol{V}) = \alpha^2 \boldsymbol{X}^T \boldsymbol{Q} \boldsymbol{X} = \mathbf{0}$

- A generic plane $\boldsymbol{\pi}$ crosses $\boldsymbol{Q}$ in a conic

any point on the line joining VERTEX with any other points on Q is on the quadratic
⇓
highlight shape of quadric as a cone



DEGENERATE QUADRICS

$$\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$$

$$\text{rank } \boldsymbol{Q} = \mathbf{3}$$

Consider $\boldsymbol{V} = \text{RNS}(\boldsymbol{Q}) : \boldsymbol{Q}\, \boldsymbol{V} = \mathbf{0}$

- $\boldsymbol{V} \in \boldsymbol{Q}$ $\quad [\mathbf{V}^T \boldsymbol{Q}\, \mathbf{V} = \mathbf{V}^T\ \mathbf{0} = \mathbf{0}]$

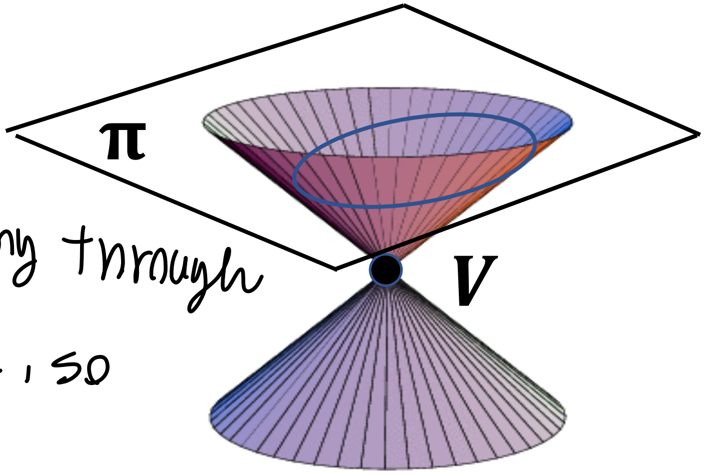
- $\forall \boldsymbol{X} \in \boldsymbol{Q}$, any point $\boldsymbol{Y}$ colinear with $\mathbf{X}$ and $\boldsymbol{V}$, namely $\boldsymbol{Y} = \alpha \boldsymbol{X} + \boldsymbol{V}$, is also $\in \boldsymbol{Q}$

  in fact $\boldsymbol{Y}^T \boldsymbol{Q}\, \boldsymbol{Y} = (\boldsymbol{\alpha X} + \boldsymbol{V})^T \boldsymbol{Q}\, (\alpha \boldsymbol{X} + \boldsymbol{V}) = \boldsymbol{\alpha}^2 \boldsymbol{X}^T \boldsymbol{Q} \boldsymbol{X} = \mathbf{0}$

- A generic plane $\boldsymbol{\pi}$ crosses $\boldsymbol{Q}$ in a conic

↑ this highlight how Q is CONE...

because all line is ∈ Q ⇒ as union set of lines going through V, but also because cutting π, Q → we find conic C, so we know is a CONE!



08/11/2023

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^T Q^* \boldsymbol{\pi} = 0$$

dual of what discussed for quadrics...

(dual of repeated plane) ⇆ repeated point

- rank $Q^* = 1 \rightarrow Q^* = XX^T$ repeated point: planes through point X

(set of planes going through X repeated twice)

- rank $Q^* = 2 \rightarrow Q^* = XY^T + YX^T$ two points: planes through X or Y

dual quadrics as union set of two points ≡ union of set of planes through points X, Y

- rank $Q^* = 3$ the **dual** of a cone

⇓

strange element...

(we can try to derive what is this according to duality)

being CONE ≡ set of points as linear combination of vertex V and any point on conic

dual! BUT conic is NOT primitive... we have to reformulate

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T} \boldsymbol{Q}^{*} \boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone

to define it according to DUALITY, we have to define in terms of primitive!

- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} = \mathrm{RNS}(\boldsymbol{Q})$ and any point on a conic

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V}$ = RNS($\boldsymbol{Q}$) and any point on a conic

NOT a primitive element in 3D geometry

planes, quadrics, points

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \boldsymbol{0}$$

- rank $\boldsymbol{Q} = \boldsymbol{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} = \mathrm{RNS}(\boldsymbol{Q})$ and any point on a conic (conic = an intersection $\boldsymbol{\pi} \cap \boldsymbol{Q_o}$ of a plane and a quadric)

reduce conic (non primitive)
as a PRIMITIVE elements operation

NOT a primitive element
in 3D geometry

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\overbrace{\boldsymbol{V} = \mathrm{RNS}(\boldsymbol{Q})}^{\text{VERTEX}}$ and any point that $\underbrace{\text{(i) belongs to a quadric } \boldsymbol{Q_o} \text{ and (ii) is on a plane } \boldsymbol{\pi}}_{\text{is on a CONE}}$.

DUAL...

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V}$ = RNS($\boldsymbol{Q}$) and any point that (i) belongs to a quadric $\boldsymbol{Q_o}$ and (ii) is on a plane $\boldsymbol{\pi}$.

  $\updownarrow$ dual of V

  ↓ DUALITY
- **Dual cone**: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) belongs to a dual quadric $\boldsymbol{Q_o}^{*}$ and (ii) goes through a point $\boldsymbol{X}$.

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{\boldsymbol{T}} \boldsymbol{Q}^{*} \boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone
- **Dual cone**: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) **belongs to a dual quadric** $\boldsymbol{Q}_{\boldsymbol{o}}{}^{*}$ and (ii) goes through a point $\boldsymbol{X}$.

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone

Dual cone: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) **is tangent to a quadric** $\boldsymbol{Q_o} = \boldsymbol{Q_o}^{*-\mathbf{1}}$and (ii) goes through a point $\boldsymbol{X}$.

DEGENERATE DUAL QUADRICS

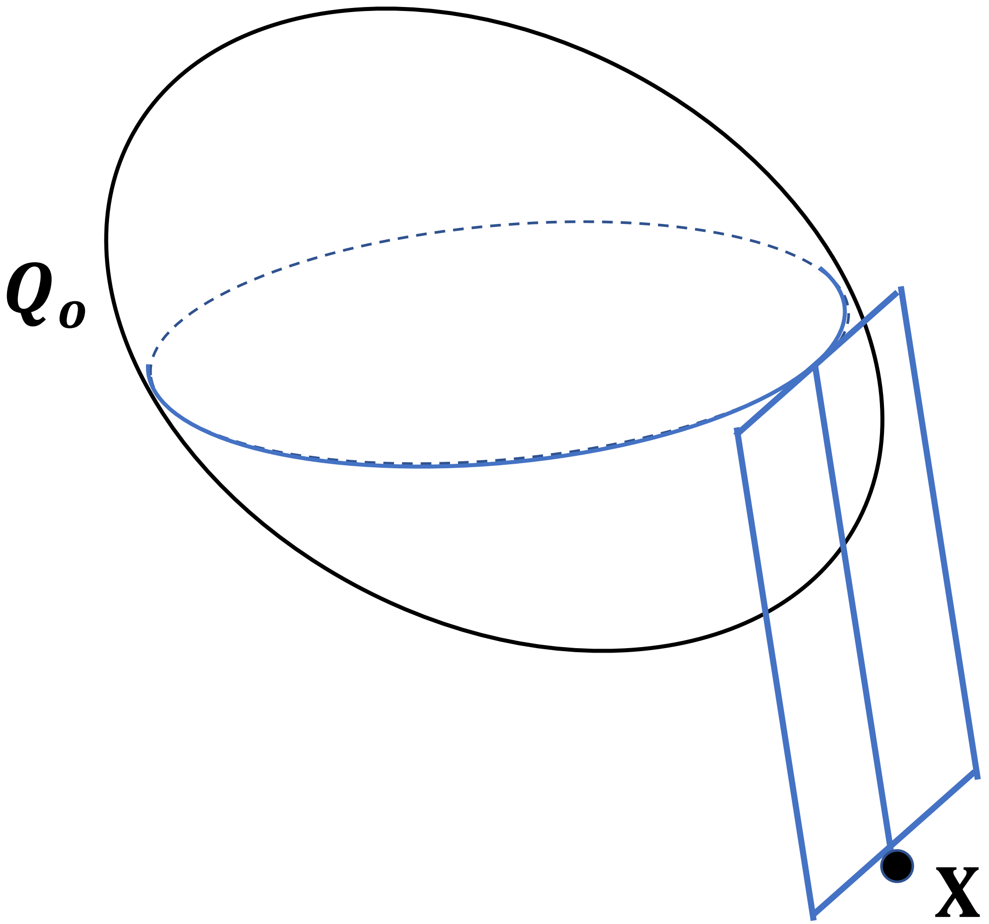
$$\boldsymbol{\pi}^{T}\, \boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone

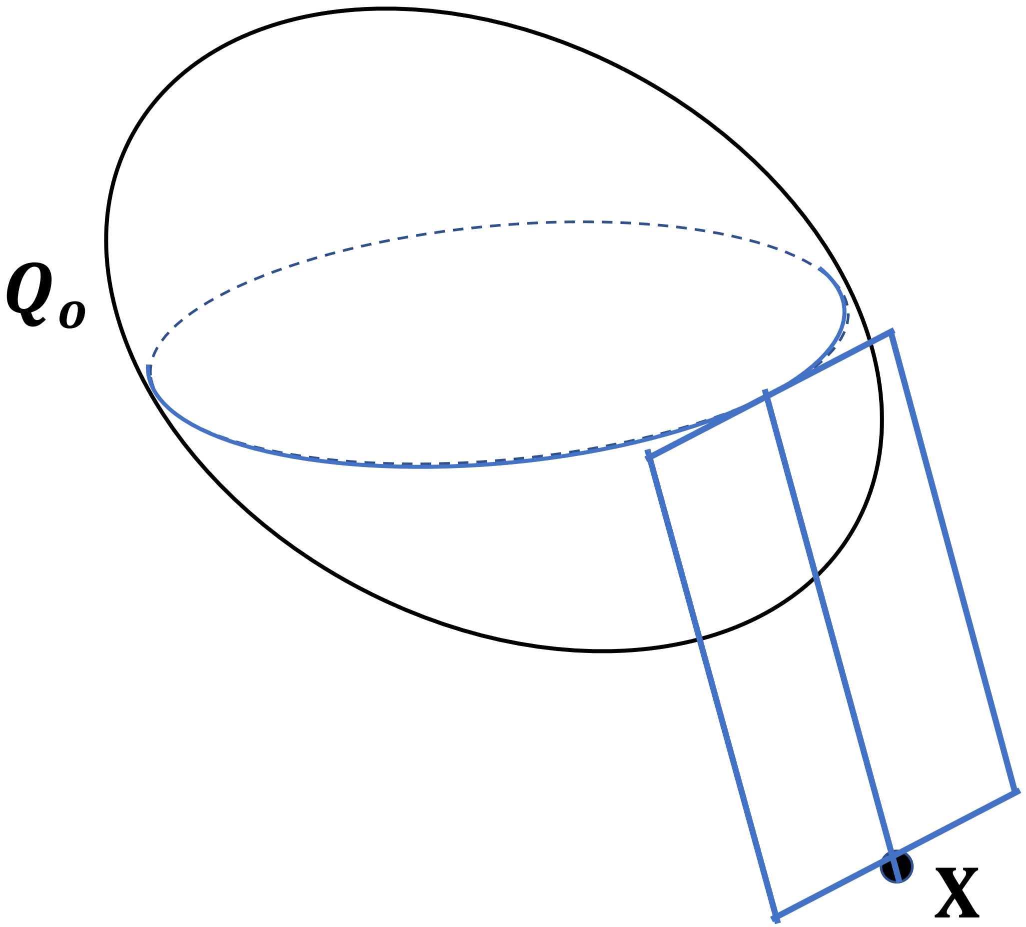
Dual cone: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) **is tangent to a quadric** $\boldsymbol{Q_o} = \boldsymbol{Q_o}^{*-\mathbf{1}}$ and (ii) goes through a point $\boldsymbol{X}$.

what is the locus of such planes?

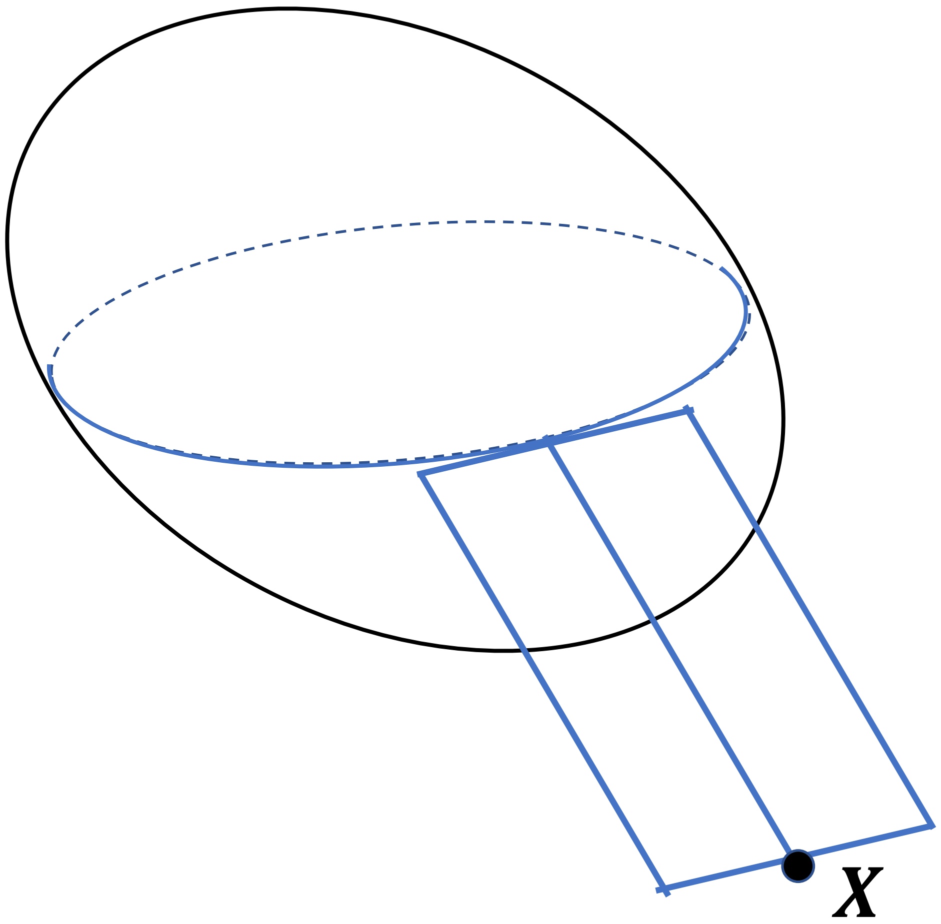
Planes through point $\boldsymbol{X}$ tangent to quadric $\boldsymbol{Q_o}$



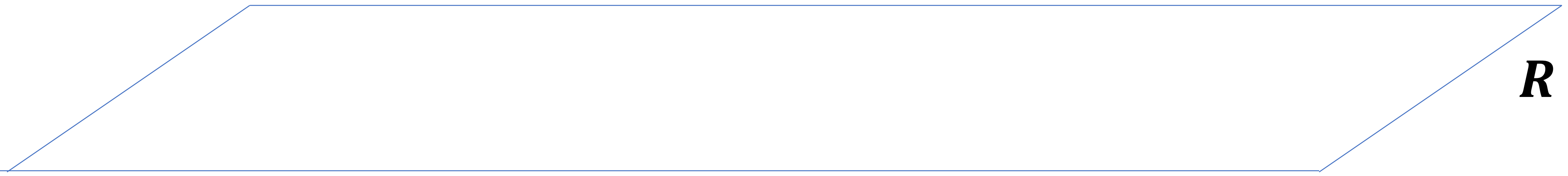
Planes through point $\boldsymbol{X}$ tangent to quadric $\boldsymbol{Q_o}$



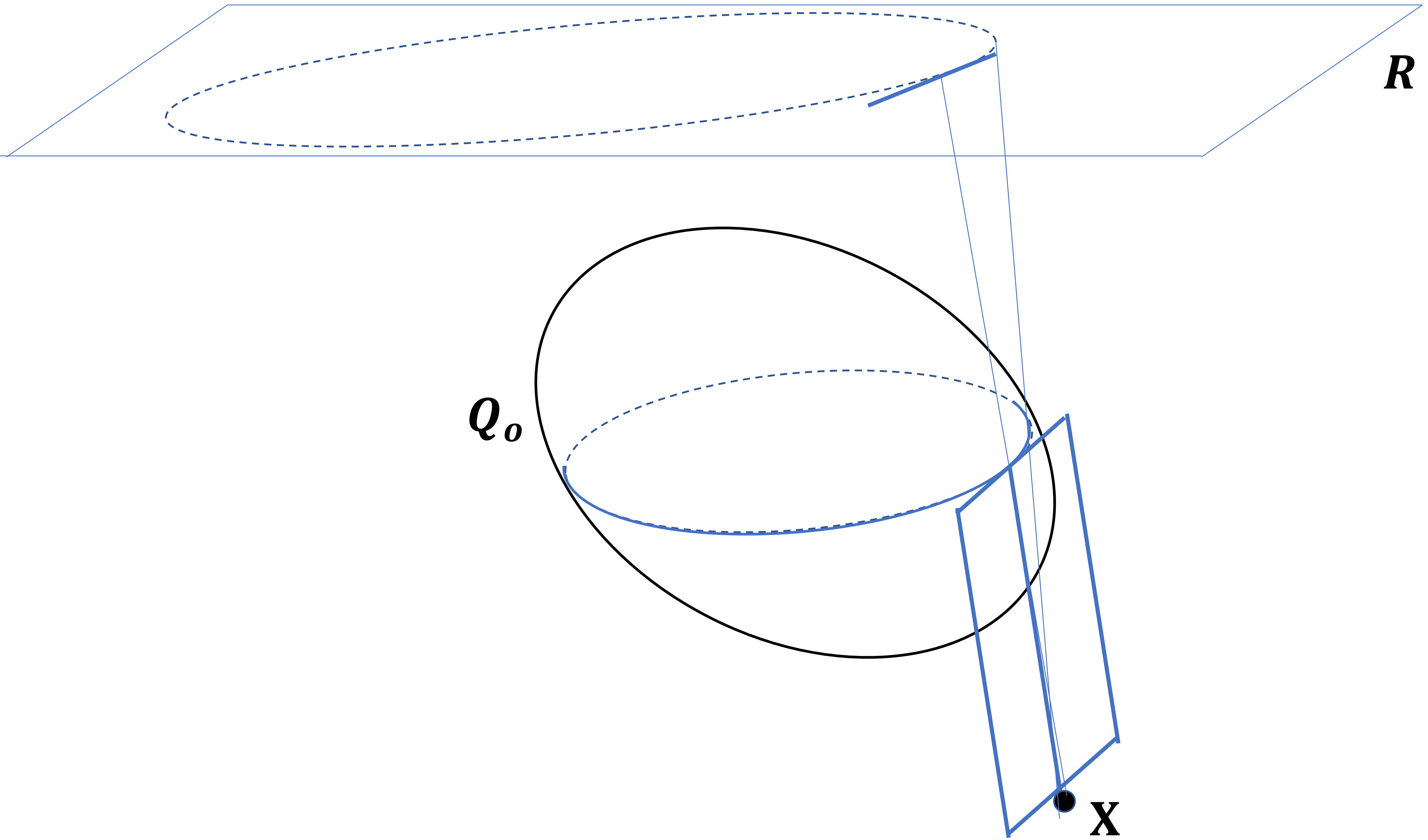
Planes through point $\boldsymbol{X}$ tangent to quadric $\boldsymbol{Q_o}$



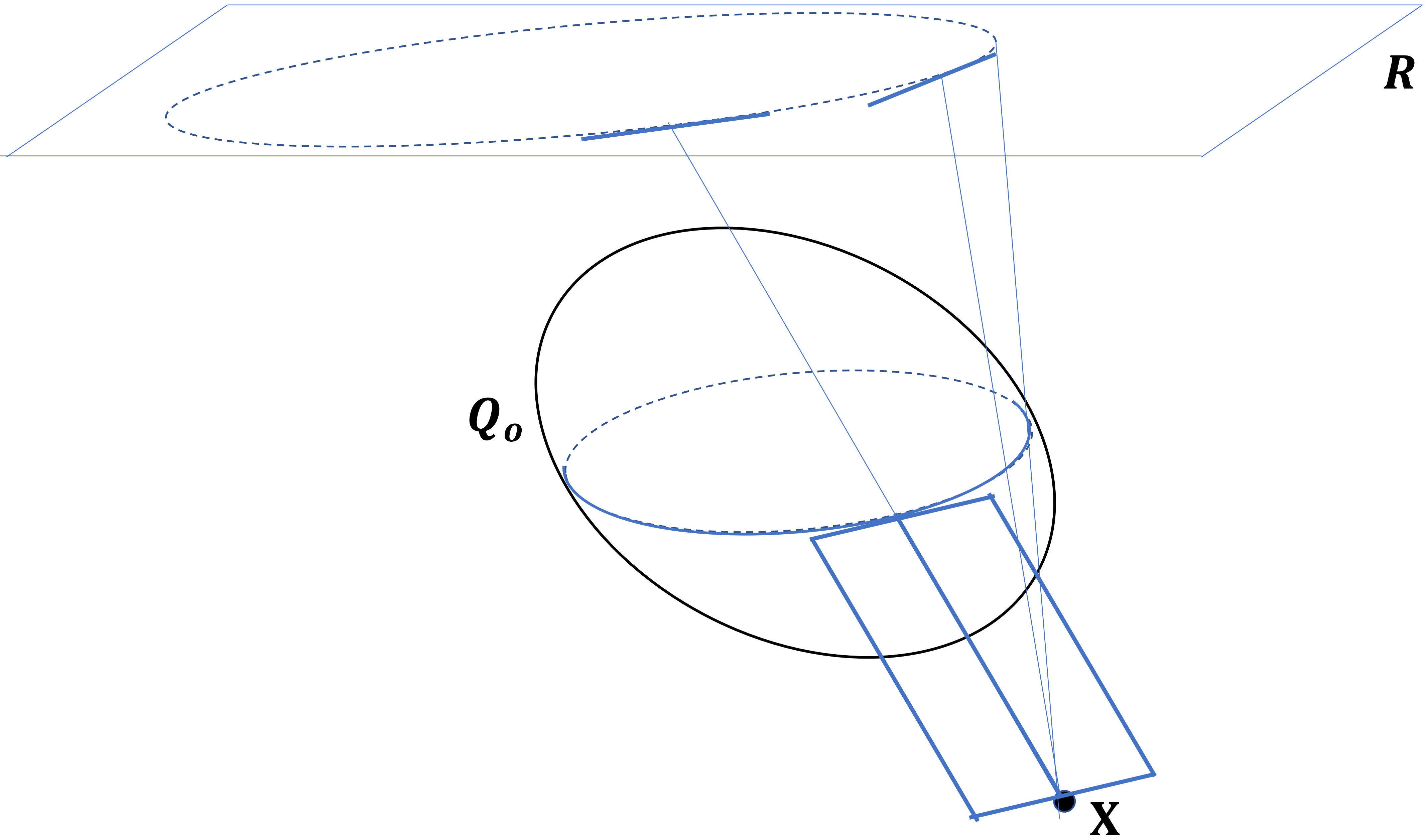
Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



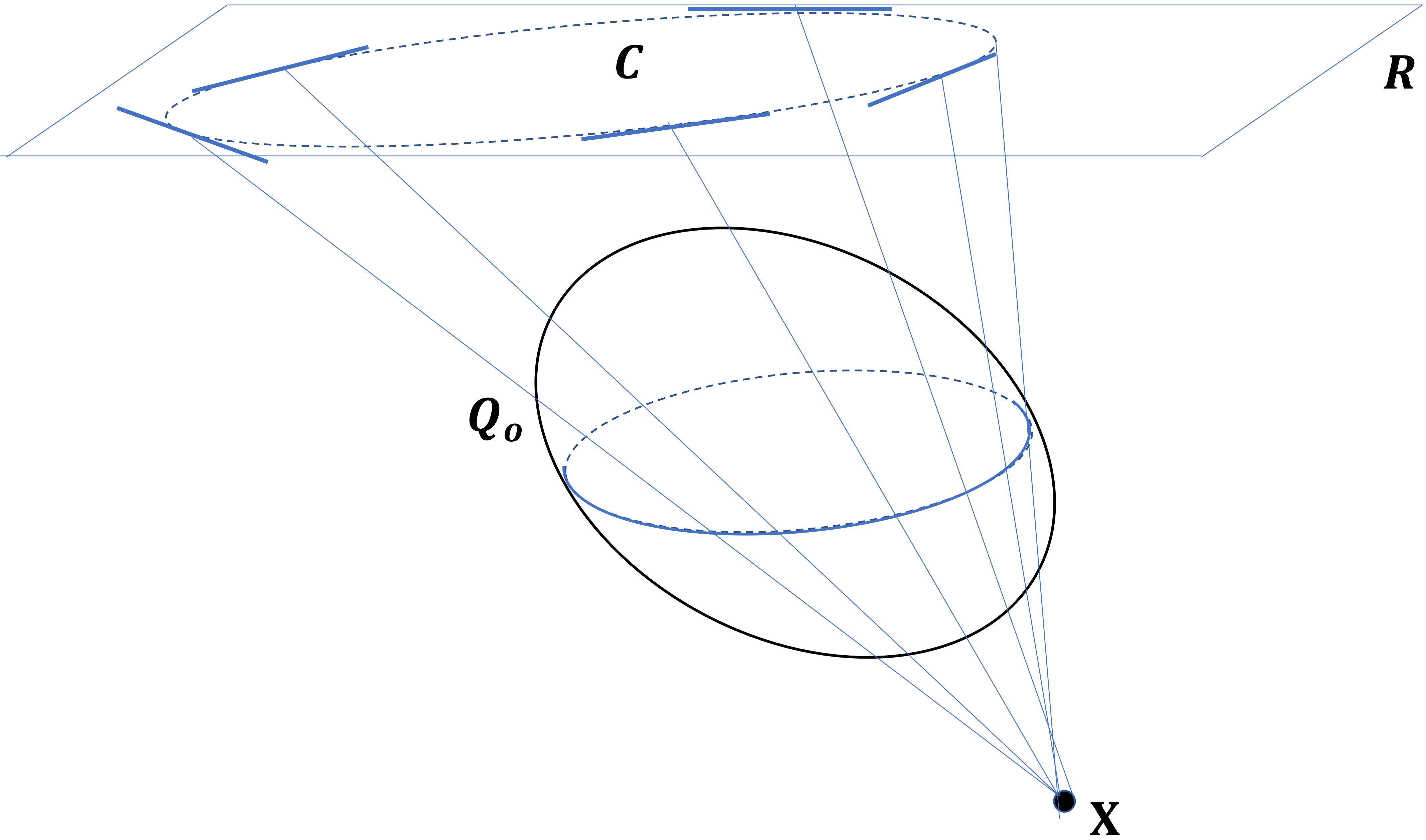
Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$)



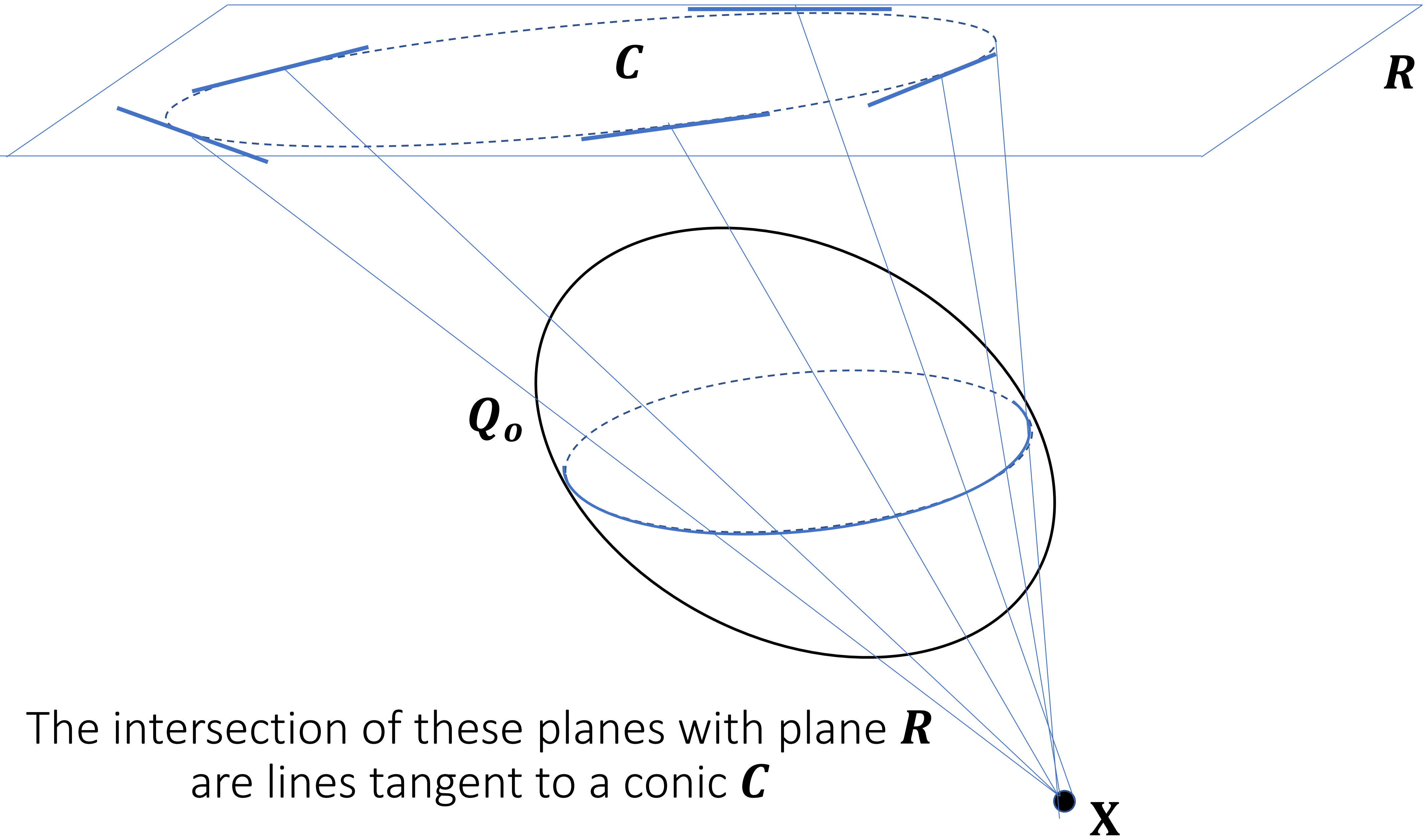
Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)

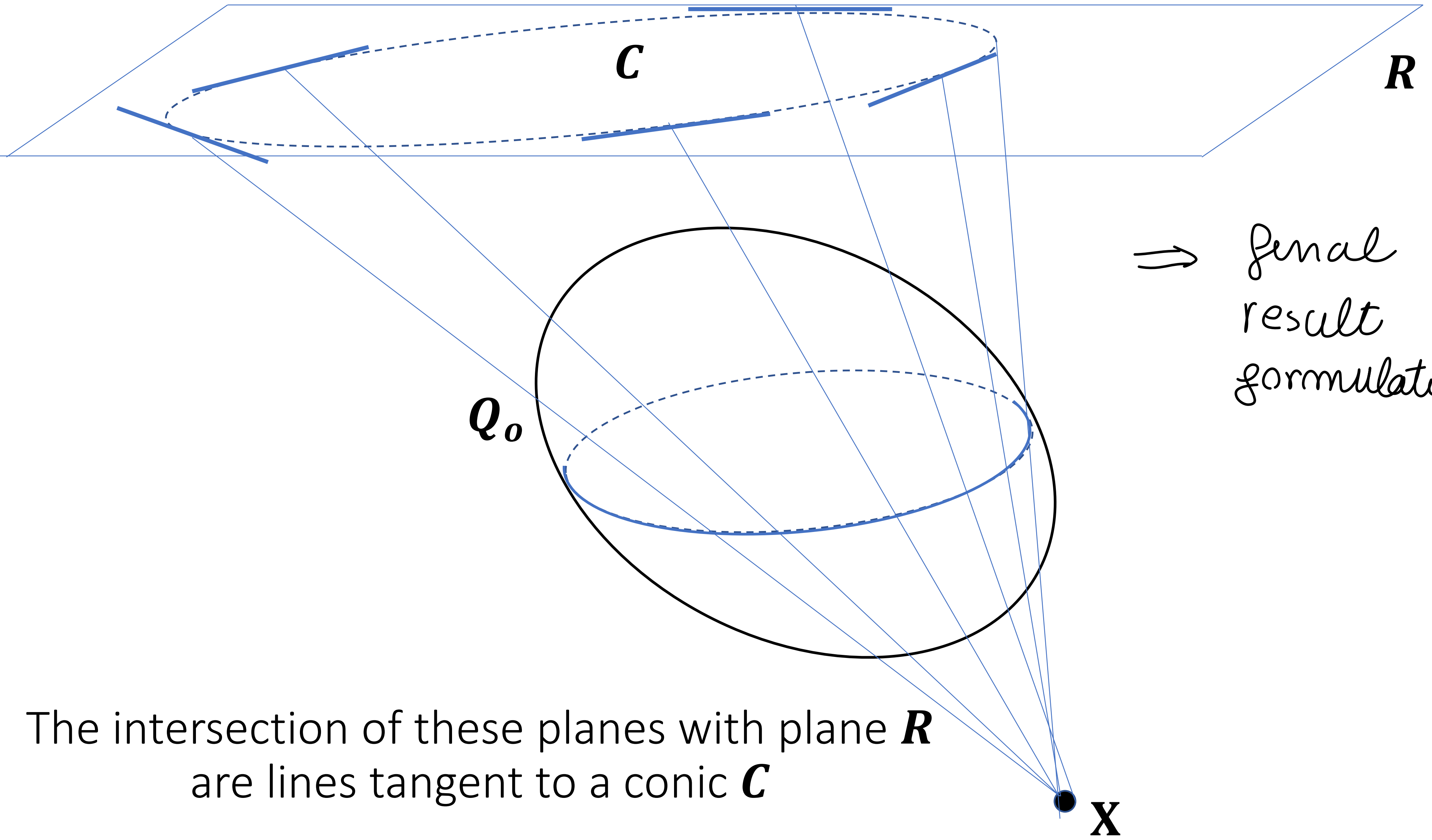


Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



The intersection of these planes with plane $\boldsymbol{R}$ are lines tangent to a conic $\boldsymbol{C}$

Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



$\Rightarrow$ final result formulation...

The intersection of these planes with plane $\boldsymbol{R}$ are lines tangent to a conic $\boldsymbol{C}$

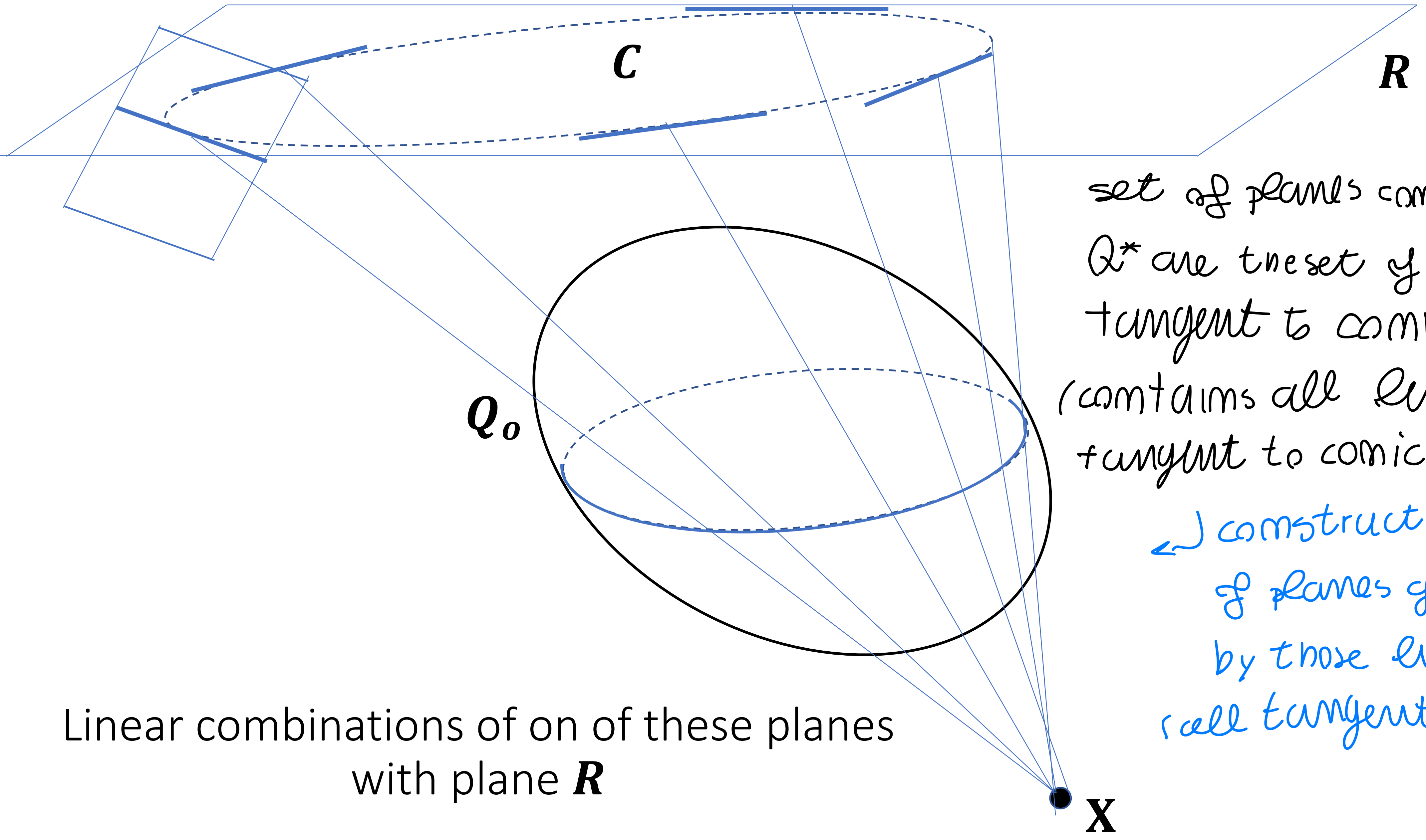
DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\boldsymbol{Q}^{*}\boldsymbol{\pi}=\mathbf{0}$$

Dual cone: set of **planes that are linear combinations of the plane $\boldsymbol{R}$** $=$RNS($\boldsymbol{Q}^{*}$) **and any plane** that (i) is tangent to a quadric $\boldsymbol{Q_o}=\boldsymbol{Q_o}^{*-\mathbf{1}}$and (ii) goes through a point $\boldsymbol{X}$.

Now that we know these planes, we have to construct –for each of them- the set of linear combinations with plane $\boldsymbol{R}$. This set of linear combinations is a pencil of planes: the set of coaxial planes, whose axis is the intersection line of the plane and plane $\boldsymbol{R}$. This axis is a tangent to conic $\boldsymbol{C}$.

Linear combinations with plane $\boldsymbol{R}$

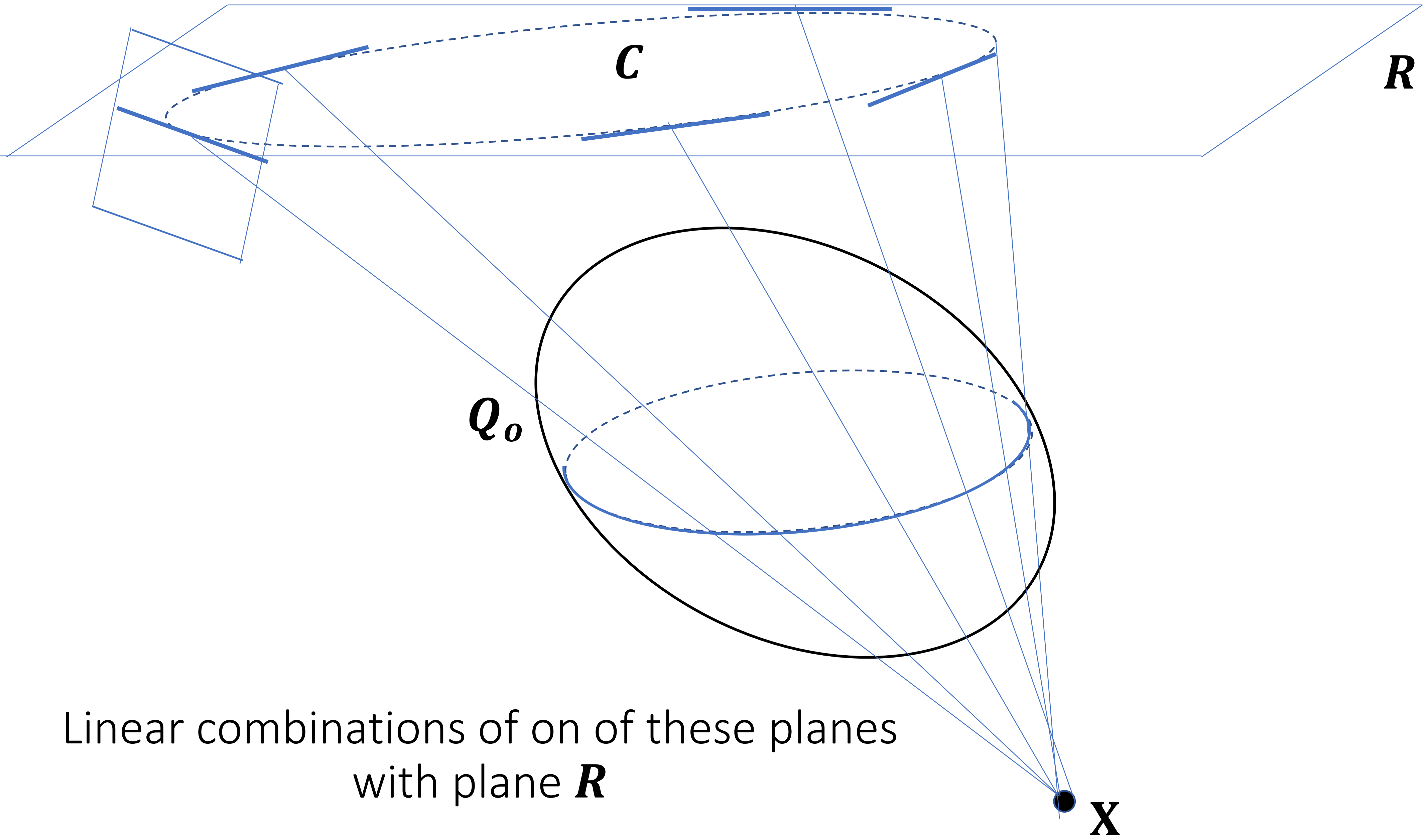


set of planes constituting Q^* are the set of planes tangent to conic! (contains all line tangent to conic)

construct pencil of planes generated by those lines... (all tangent to conic)

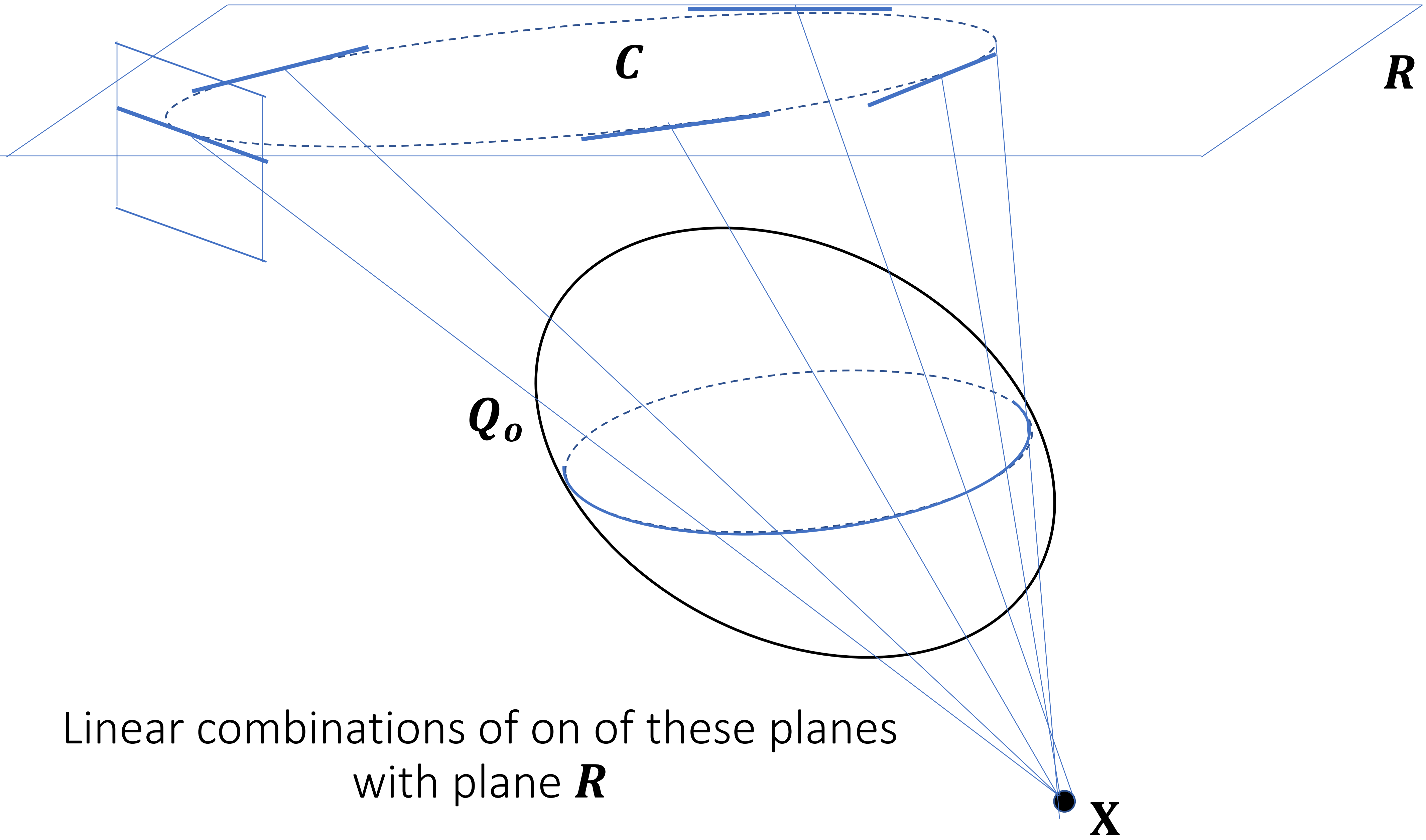
Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



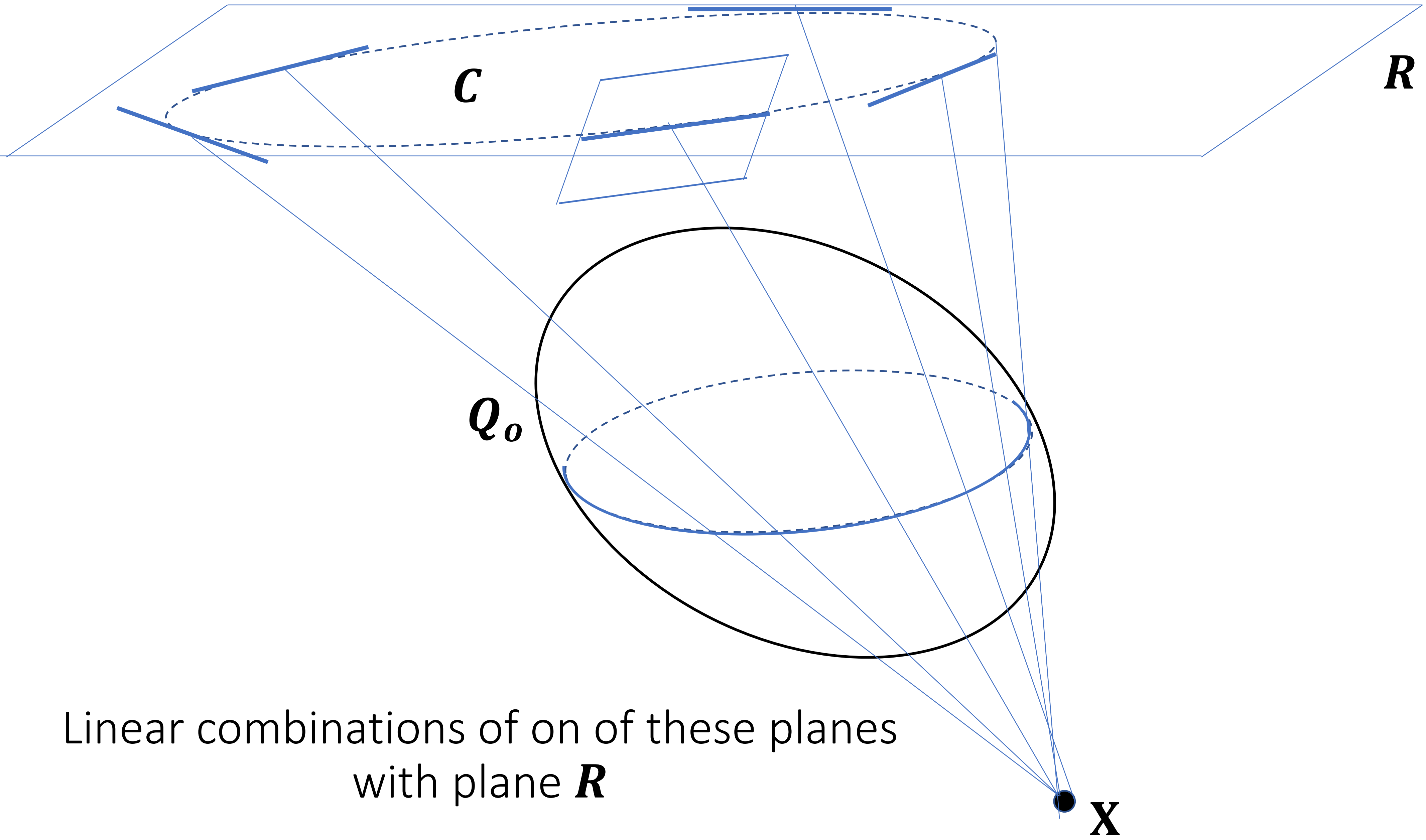
Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



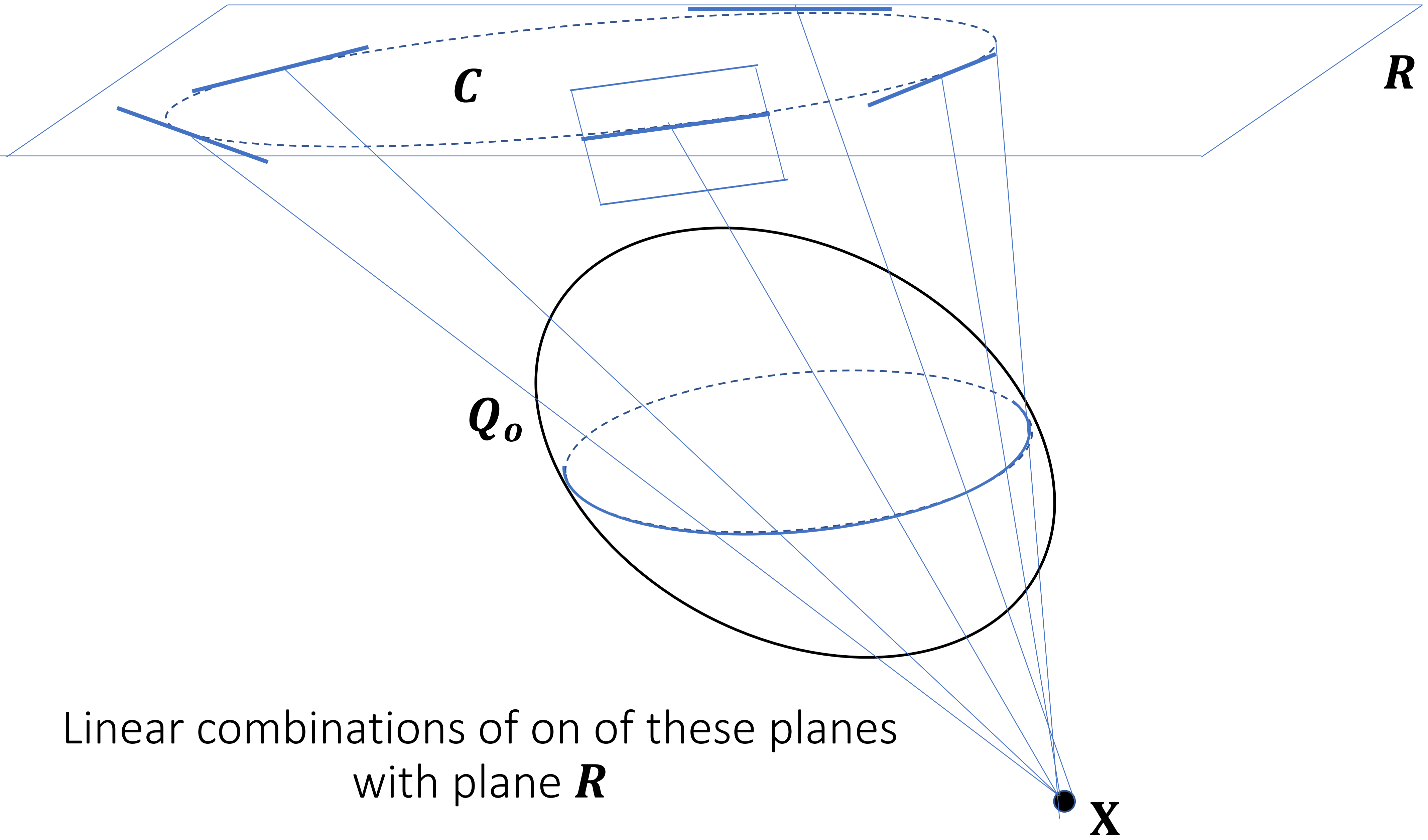
Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



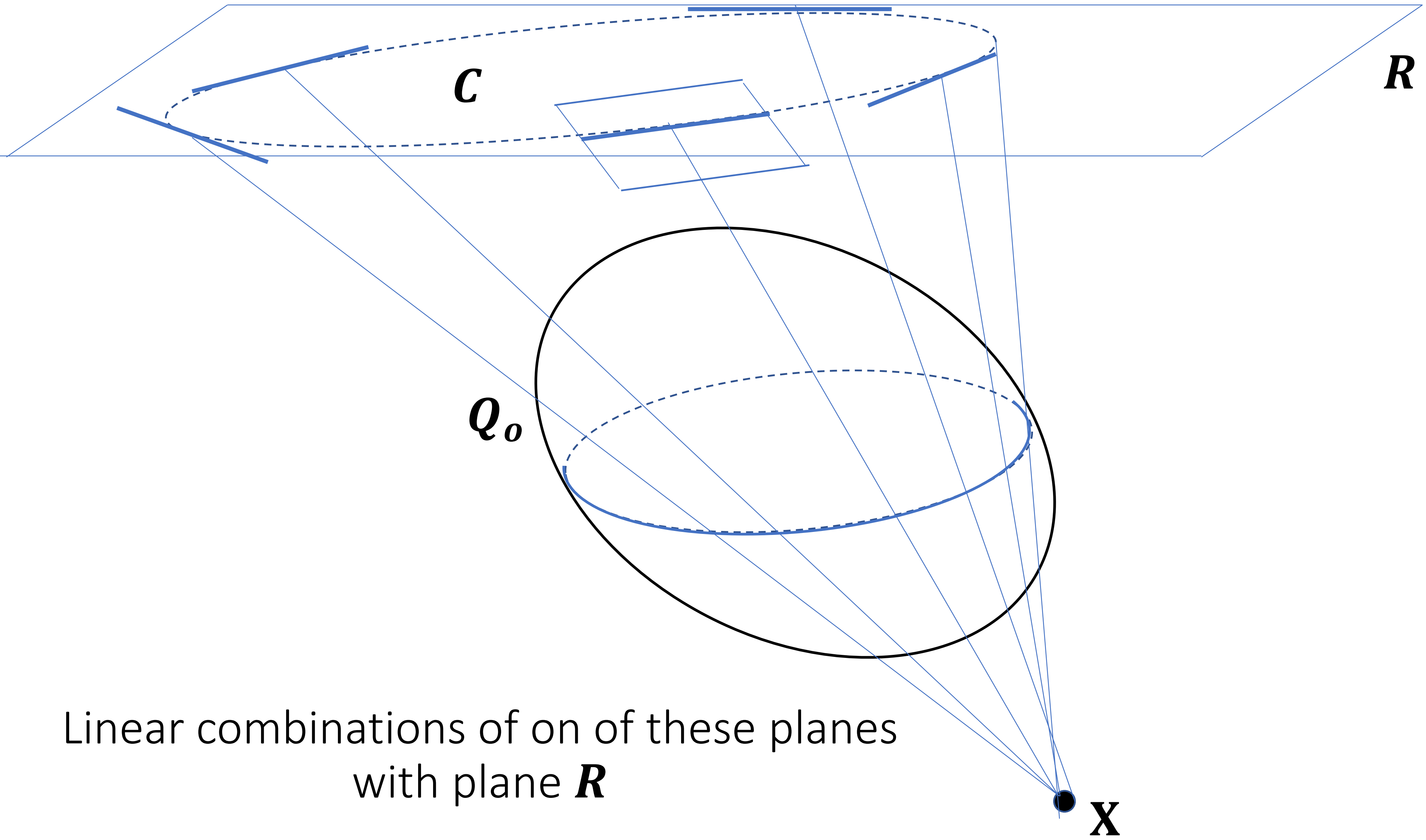
Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



Linear combinations of on of these planes
with plane $\boldsymbol{R}$

DEGENERATE DUAL QUADRICS

what if we take as conic this absolute conic

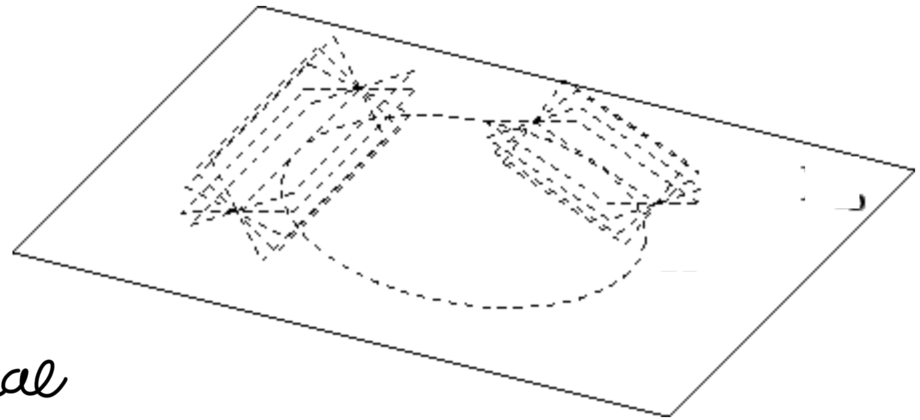
$$\boldsymbol{\pi}^T \boldsymbol{Q}^* \boldsymbol{\pi} = \mathbf{0}$$

any set of planes tangent to conic

- rank $\boldsymbol{Q}^* = \mathbf{3}$ the **dual** of a cone
- Dual of a cone: → The set of planes that are tangent to a conic

taking absolute conic... the set of planes tangent to absolute conic?

→ special dual quadric



from important 3D conic ⟹ (absolute conic = intersection between sphere and Π_∞) → this is an IMPORTANT POINT, as all I, J of all π

A noteworthy example: THE ABSOLUTE (dual) QUADRIC

primitive element on plane

Ω_∞ absolute conic was I_n as conic @ Π_∞

$$\pi^T Q^*_\infty \pi = 0$$

The set of planes that are tangent to the absolute conic is the quadric Q^*_∞ with

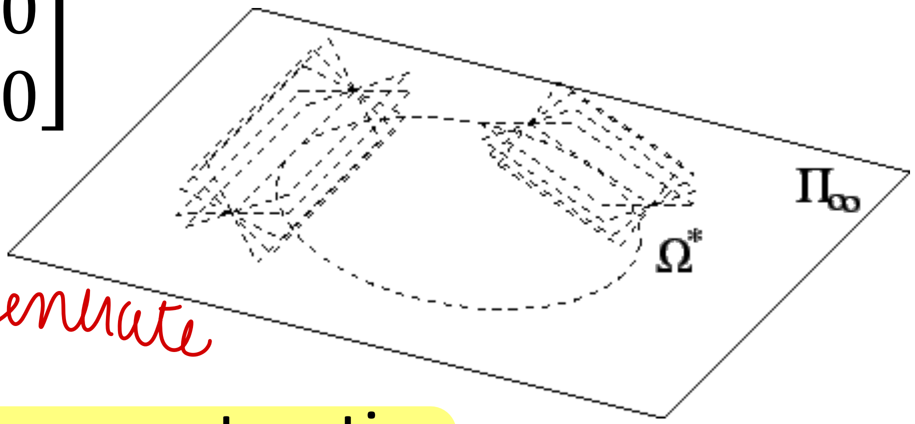
degenerate dual quadric of rank = 3

I_3 ZEROS

rank = 3

$$Q^*_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

comes out that this degenerate dual quadric given by set of planes tangent to absolute conic has nice aspect



≡ ABSOLUTE dual QUADRIC → degenerate

The absolute dual quadric Q^*_∞ is useful in the 3D reconstruction

Proof: it can be proven! helps to formulate plane intersection

dual of absolute conic: Absolute conic

Dual absolute conic $\Omega_\infty^* = \Omega_\infty^{-1}$ in $\boldsymbol{\pi}_\infty$: set of lines tangent to $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⇓ build set of planes: dual! ≈

A plane through one of such lines: $\boldsymbol{\pi} = [a \quad b \quad c \quad d]^T$, which intersects π_∞ at a line l: $\begin{cases} ax + by + cz + dw = 0 \\ w = 0 \end{cases} \rightarrow ax + by + cz = 0 = l^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

with $l = [a \quad b \quad c]^T$, where $l \in \Omega_\infty^*$, i.e. $l^T \Omega_\infty^* l = 0$. But

$$l^T \Omega_\infty^* l = [a \quad b \quad c]^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a \quad b \quad c \quad d]^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Hence

$$\boldsymbol{\pi}^T \boldsymbol{Q}^*_\infty \boldsymbol{\pi} = \mathbf{0}$$

Proof:

Dual absolute conic $\Omega^*_\infty = \Omega_\infty^{-1}$ in $\boldsymbol{\pi}_\infty$: set of lines tangent to $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↓ to represent it mathematically.

generic plane

A plane through one of such lines: $\boldsymbol{\pi} = [a \quad b \quad c \quad d]^T$, which intersects π_∞ at a line l: $\begin{cases} ax + by + cz + dw = 0 \\ w = 0 \end{cases} \rightarrow ax + by + cz = 0 = l^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

this is a line on plane at infinity

with $l = [a \quad b \quad c]^T$, where $l \in \Omega^*_\infty$, i.e. $l^T \Omega^*_\infty l = 0$. But

we impose e = dual of abs conic

imposing tangent to Ω_∞

DUALLY: as plane condition

$$l^T \Omega^*_\infty l = [a \quad b \quad c]^T \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Omega^*_\infty = \Omega_\infty^{-1} = I_3^{-1} = I_3} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a \quad b \quad c \quad d]^T \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{in homogeneous coord}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

the tangent to Ω_∞, must be part of Ω^*_∞

as plane constrain

Hence

$$\boldsymbol{\pi}^T \boldsymbol{Q}^*_\infty \boldsymbol{\pi} = \mathbf{0}$$

Proof:

Dual absolute conic $\Omega_\infty^* = \Omega_\infty^{-1}$ in $\boldsymbol{\pi}_\infty$: set of lines tangent to $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A plane through one of such lines: $\boldsymbol{\pi} = [a \quad b \quad c \quad d]^T$, which intersects π_∞ at a line l: $\begin{cases} ax + by + cz + dw = 0 \\ w = 0 \end{cases} \rightarrow ax + by + cz = 0 = l^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

with $l = [a \quad b \quad c]^T$, where $l \in \Omega_\infty^*$, i.e. $l^T \Omega_\infty^* l = 0$. But

rank 3

$$l^T \Omega_\infty^* l = [a \quad b \quad c]^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{[a \quad b \quad c \quad d]^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}} = 0$$

linear homogeneous equation on a plane.
→ QUADRIC (dual)

Hence

set of planes tangent to absolute conic:
= Abs dual Quadric

$$\boldsymbol{\pi}^T \boxed{\boldsymbol{Q}^*_\infty} \boldsymbol{\pi} = \boldsymbol{0}$$

A property of the THE ABSOLUTE (dual) QUADRIC

Notice: Absolute conic and absolute dual quadric are equivalent! from Ω_∞ straight forward to get Q^*_∞ and viceversa from Q^*_∞ is easy to find Ω_∞!

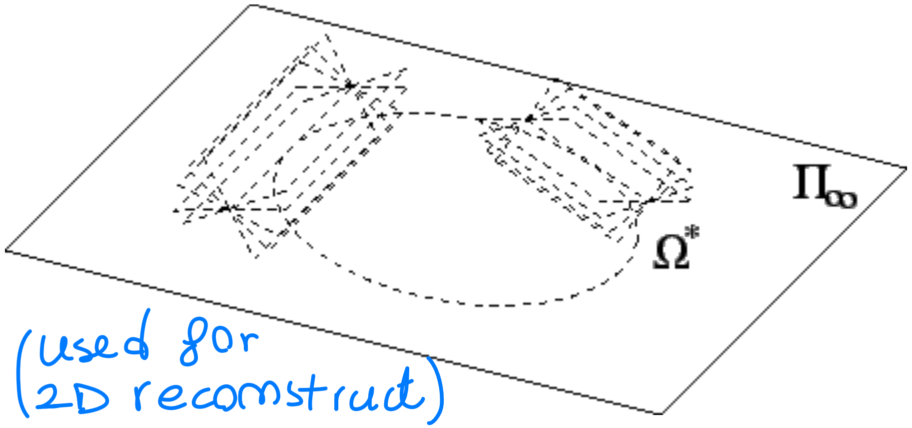
it is usefull because Ω_∞ is NOT a primitive element of 3D geometry
↓
instead Q^*_∞ is a PRIMITIVE element in 3D single matrix represent it in our space

$$\boldsymbol{\pi}^T \boldsymbol{Q}^* \boldsymbol{\pi} = \mathbf{0}$$

→The set of planes that are tangent to the absolute conic:

but Ω_∞ lays on Π_∞, NOT in 3D space, it is NOT PRIMITIVE element! require 2 elements to be specifyed

$$\boldsymbol{Q}^*_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



As in 2D geometry we had I, J set of two points → conic dual to CIRCULAR POINT is usefull in 2D being PRIMITIVE! (used for 2D reconstruct)

Property: the RNS of absolute dual quadric $\boldsymbol{Q}^*_\infty$ is the plane at the infinity $\boldsymbol{\pi}_\infty$

$$\boldsymbol{Q}^*_\infty \boldsymbol{\pi}_\infty = 0 \rightarrow \text{RNS}(\boldsymbol{Q}^*_\infty) = \boldsymbol{\pi}_\infty$$

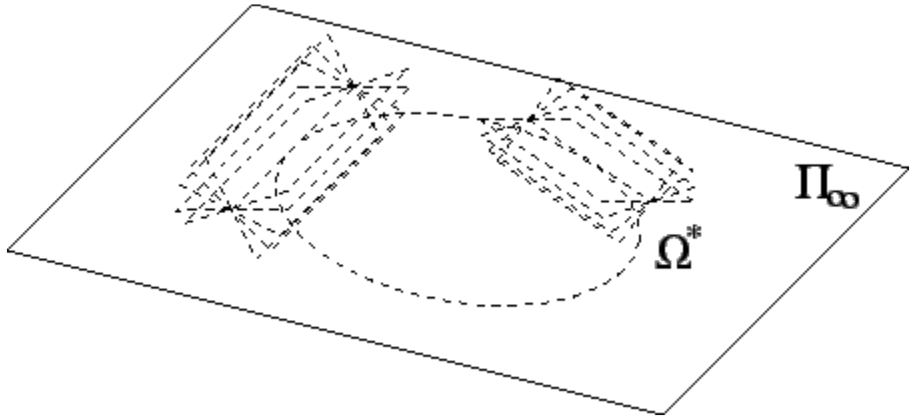
A property of the THE ABSOLUTE (dual) QUADRIC

$$\boldsymbol{\pi}^T \boldsymbol{Q}^* \boldsymbol{\pi} = \mathbf{0}$$

↳ this will be usefull in 3D reconstruction solution. from $(Q^*_\infty)'$ we map to Ω^*_∞ somehow shape reconstruction

→The set of planes that are tangent to the absolute conic:

$$\boldsymbol{Q}^*_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Property: the RNS of absolute dual quadric $\boldsymbol{Q}^*_\infty$ is the plane at the infinity $\boldsymbol{\pi}_\infty$

$$\boldsymbol{Q}^*_\infty \boldsymbol{\pi}_\infty = 0 \rightarrow \text{RNS}(\boldsymbol{Q}^*_\infty) = \boldsymbol{\pi}_\infty$$

Projective 3D Geometry: Projective Transformations

↳ we know the primitives!
let's analyze TRANFORM between those elements

Projective mappings

(as in 2D!)

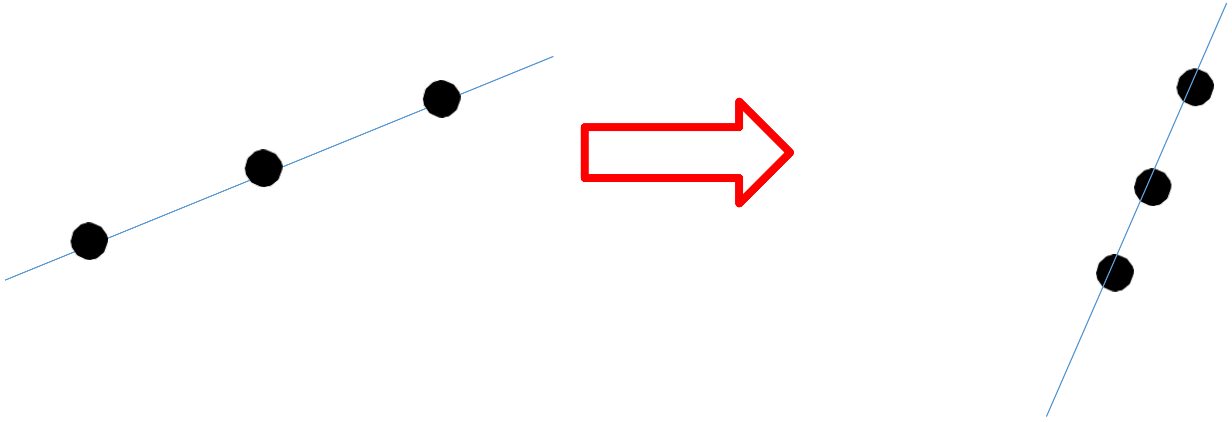
Def. *A* ***projective mapping*** *between a projective space* $\mathbb{P}^3$ *and an other projective space* $\mathbb{P}'^3$ *is an* ***invertible*** *mapping which preserves colinearity:*

$$h: \mathbb{P}^3 \rightarrow \mathbb{P}'^3, \mathbf{X}' = h(\mathbf{X}), \ \mathbf{X}_1, \ \mathbf{X}_2, \mathbf{X}_3 \text{ are colinear}$$

(must remain colinear)

$$\leftrightarrow$$

$$\mathbf{X}'_1 = h(\mathbf{X}_1), \ \mathbf{X}'_2 = h(\mathbf{X}_2), \mathbf{X}'_3 = h(\mathbf{X}_3) \text{ are colinear}$$



Alternative names:

- *Projectivity*

(NOT "HOMOGRAPHY")

which is related to planar stuff

also there holds:

Fundamental Theorem of Projective Geometry

Theorem: *A mapping h : $\mathbb{P}^3 \rightarrow \mathbb{P}'^3$ is projective if and only if there exists an invertible 4x4 matrix H such that for any point in* $\mathbb{P}^3$ *represented by the vector* $\mathbf{X}$, *is* $h(\mathbf{X}) = H\,\boldsymbol{X}$

↑ TRANSFORMED POINT
represented by linear transformation in hom. coord

i.e. projective mappings are LINEAR in the homogeneous coordinates

(they are not linear in cartesian coordinates)

(as 2D proj. transf)

H: 4×4=16 BUT HOMOGENEUS!

Projectivity: 15 degrees of freedom

From the theorem

$$h(\mathbf{X}) = \mathbf{X}' = H\,\mathbf{X}$$

Therefore, if we multiply the matrix H by any nonzero scalar λ, the relation is satisfied by the same points

$$\mathbf{X}' = \boxed{\lambda H}\,\mathbf{X}$$

Thus any nonzero multiple of the matrix H represents the same projective mapping as H.

Hence H is a homogeneous matrix: in spite of its 16 entries,

H has only 15 degrees of freedom, namely the ratios between its elements.

a lot! (in 2D geometry we had 8 dof)

general formulation

Projectivity estimation

H has only 15 degrees of freedom, namely the ratios between its elements.
E.g.

to FIT the mapping...

$$H = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^T & 1 \end{bmatrix}$$

3×3 sub matrix

3×1 component vector

this can be any $\lambda \neq 0$

1×3 row vector

Therefore, it can be estimated by just FIVE point correspondences,

since each point correspondence $\mathbf{X}' = H\,\mathbf{X}$ yelds **three** independent equations

OF COURSE INDIPENDENT points!

NOT colinear/all coplanar!

AT most 2 colinear/3 coplanar

each point X has 3 dof

(4 homogeneus coord, but 3 indipendent)

(IF 4 coplanar, the fourth X_4 is combination of X_1, X_2, X_3!)

Transformation of points, planes, quadrics, dual quadrics

rules to
transform primitive elements...
SAME AS in planar geometry! =D

Transformation rules for the space elements

- A homography transforms **each point $\boldsymbol{X}$** into a point $\boldsymbol{X}'$ such that:

$$\boldsymbol{X} \rightarrow H\boldsymbol{X} = \boldsymbol{X}'$$

plane

- A homography transforms **each plane π** into a ~~line~~ $\boldsymbol{\pi}'$ such that:

$$\boldsymbol{\pi} \rightarrow H^{-T}\boldsymbol{\pi} = \boldsymbol{\pi}'$$

- A homography transforms **each quadric $\boldsymbol{Q}$** into a quadric Q′ such that:

$$Q \rightarrow H^{-T} Q H^{-1} = Q'$$

same as 2D geometry!

- A homography transforms **each dual quadric** Q^* into a dual quadric $Q^{*\prime}$

$$Q^* \rightarrow H Q^* H^T = Q^{*\prime}$$

Vanishing points and vanishing line

of course it exists also in 3D

remember: intersection of two parallel lines on a plane

Suppose that lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ are parallel: this means that

$$\left.\begin{array}{l} l_1 = [a \quad b \quad c_1]^T \text{ and} \\ l_2 = [a \quad b \quad c_2]^T \end{array}\right\} \text{ they intersect @ } \infty$$

The point $\mathbf{x} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]^{\boldsymbol{T}}$ common two these two lines satisfies both

$$ax + by + c_1 w = 0$$

and

$$ax + by + c_2 w = 0$$

→

$$\mathbf{x} = [b \quad -a \quad 0]^T$$

Namely, the point at the infinty along the direction of both lines

(remember: [a, b] is the direction **normal** to both lines)

intersection of **many** parallel lines in the 3D space

Suppose that lines are parallel to direction $[a \quad b \quad c]^T$:

All these lines cross at the common point

$\rightarrow$

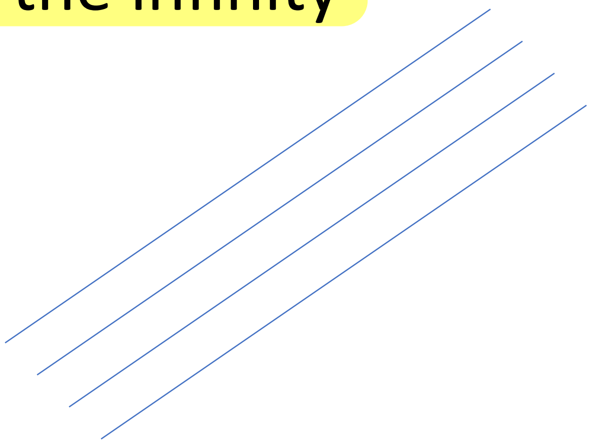
$$\mathbf{X}_\infty = [a \quad b \quad c \quad 0]^T$$

point @ ∞

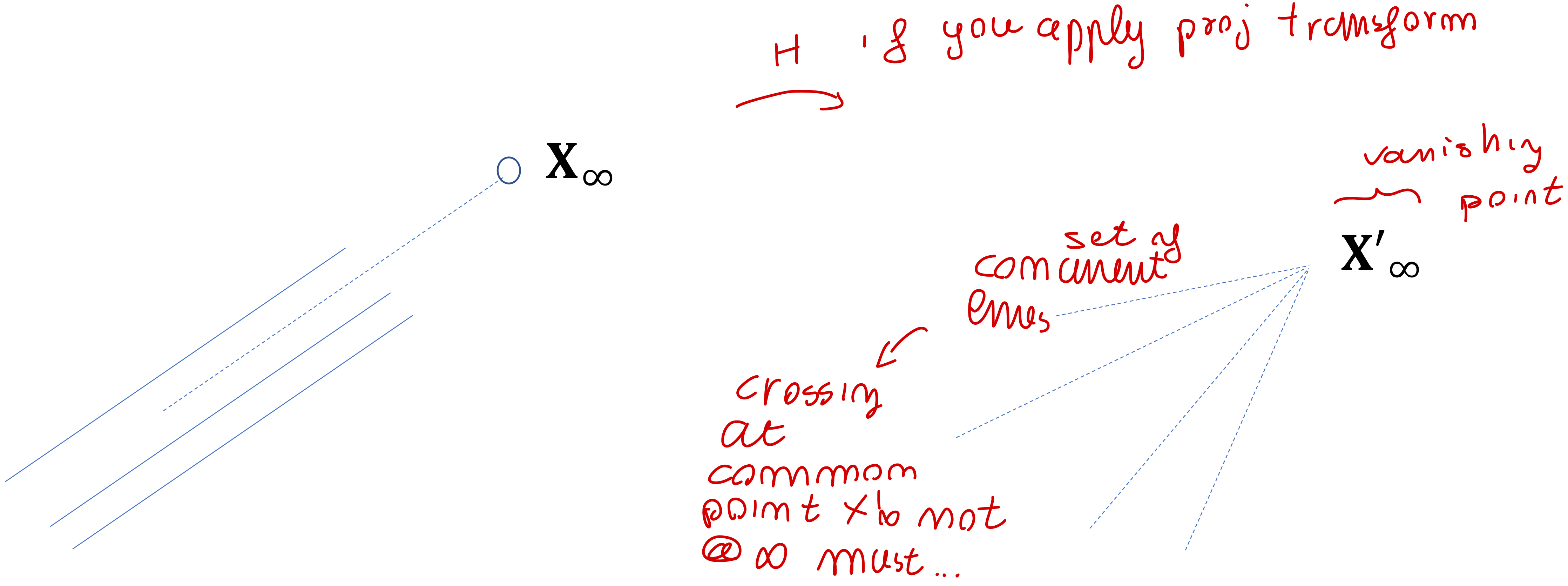
↑

○ $\mathbf{X}_\infty$

This point is at the infinity



Applying a projective transformation (e.g. an image) to all the above parallel lines we obtain concurrent lines. The common point (at the infinity) $\mathbf{X}_\infty$ to all parallel lines is mapped onto a point $\mathbf{X}'_\infty = H\mathbf{X}_\infty$ where all mapped lines concur.



EXTENSION of parallel lines...

intersection of **many** parallel planes in the 3D space

letting $d_i \in \mathbb{R}$ vary... we get set of parallel planes...

All parallel planes $[a \quad b \quad c \quad d_i]^T$ contain a common line

π_i

$\begin{cases} X_1 \in \pi \\ X_2 \in \pi \end{cases}$ defined s.t. $X^T \pi_i = 0$ even if $d_i \in \mathbb{R}$

line common to all parallel planes

they contain common line

$$\mathbf{L}^* = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}\right) = \mathbf{RNS}\left(\begin{bmatrix} b & -a & 0 & 0 \\ -c & 0 & a & 0 \end{bmatrix}\right) \rightarrow$$

points on this plane, orthogonal to π

This line is a linear combination of two points at the infinity, e.g.,

independently of d_i, those $X_1, X_2 \in \pi_i$

$$\begin{bmatrix} b \\ -a \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -c \\ 0 \\ a \\ 0 \end{bmatrix}$$

one possible formulation of line!

↳ dual version of line ⟹

intersection of **many** parallel planes in the 3D space

dually: all parallel planes $[a \quad b \quad c \quad d_i]^T$ contain a common line

$\rightarrow$

$$\mathbf{L} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{\pi}_1^T \\ \boldsymbol{\pi}_2^T \end{bmatrix}\right) = \mathbf{RNS}\left(\begin{bmatrix} a & b & c & d_1 \\ a & b & c & d_2 \end{bmatrix}\right)$$

taking π_1, π_2 two planes...

taking parallel plane

This line is a linear combination of two parallel planes, e.g.,

$$\begin{bmatrix} a \\ b \\ c \\ d_1 \end{bmatrix} \text{ and } \begin{bmatrix} a \\ b \\ c \\ d_2 \end{bmatrix}$$

pencil of parallel planes...

with line at ∞

CROSS RATIO preserved etc.

Polarity is preserved under projective mappings

The polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{X}$ of a point $\mathbf{X}$ wrt a quadric $\boldsymbol{Q}$ is mapped onto the polar plane $\boldsymbol{\pi}' = \boldsymbol{Q}'\mathbf{X}'$ of the transformed point $\mathbf{X}' = H\mathbf{X}$ wrt the transformed quadric $\boldsymbol{Q}' = H^{-T}\mathbf{Q}\,H^{-1}$

projective Transform both X, Q

In fact,

POLAR PLANE

$$\boldsymbol{\pi}' = H^{-T}\boldsymbol{\pi} = H^{-T}\mathbf{Q}\mathbf{X} = H^{-T}\mathbf{Q}\,H^{-1}H\mathbf{X} = \mathbf{Q}'\mathbf{X}'$$

PROJ MAPPING

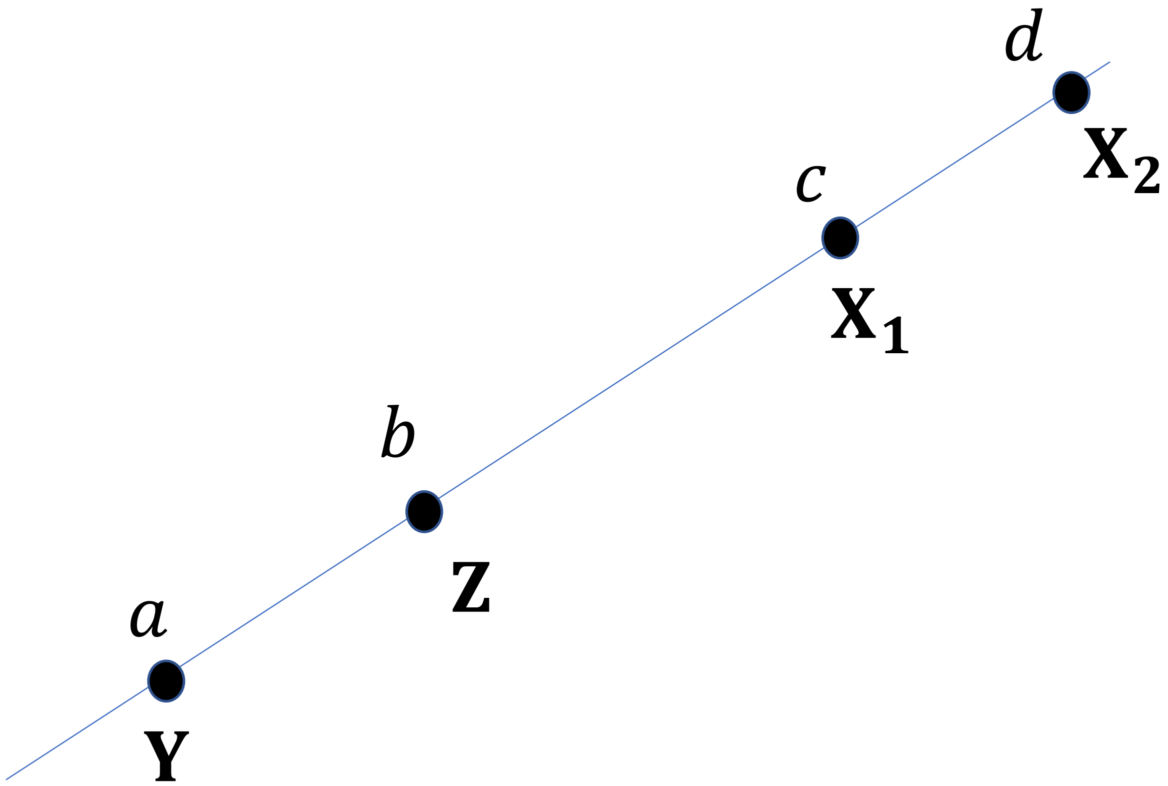
POLARITY in projective space

polar plane of transformed point X' wrt transformed quadr Q'

Cross ratios: invariant under projective mappings

1D cross ratio of a 4-tuple of colinear points

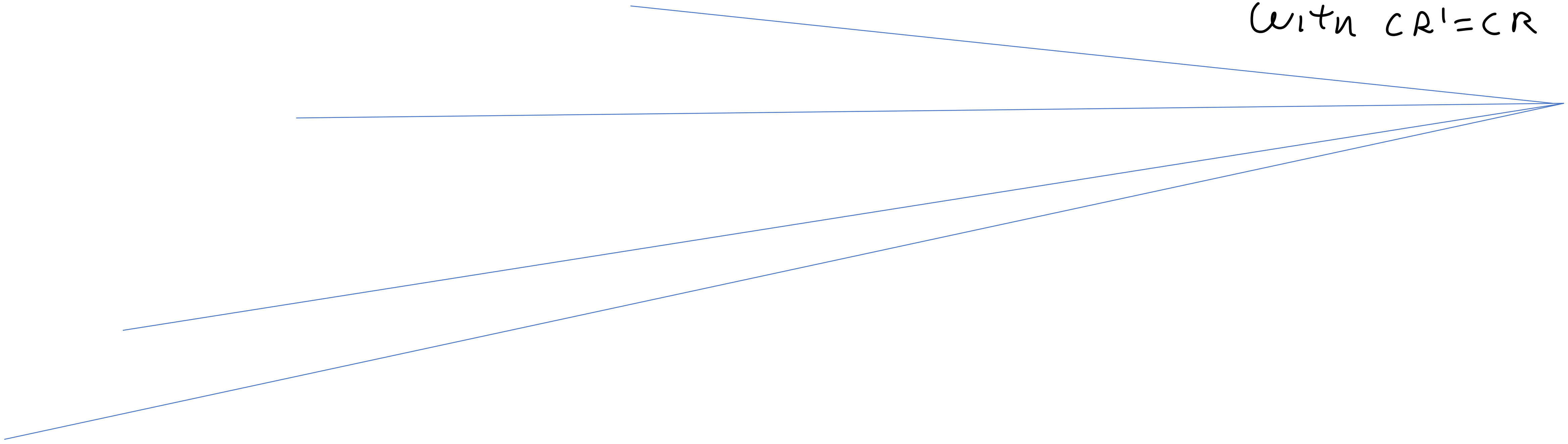
$$CR_{Y,Z,\mathrm{X}_1,\mathrm{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$



2D cross ratio of a 4-tuple of coplanar, concurrent lines

Preserved!

you find new concurrent line with CR'=CR



3D cross ratio of a 4-tuple of coaxial planes:

share common line!

Hierarchy of projective transformations

as in 2D we have an

hierarchy of more general

classes of proj transformations ...

from simpler most constrained

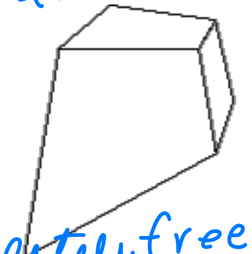
Hierarchy of transformations

less constraind

Projective
15dof
(3 more)

purely free!

$$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$



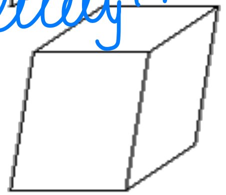
Intersection and tangency

↑

Affine
12dof
(5 more)

3×3 → completely free, 9 dof on A!

$$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$



Parallellism of planes,
Volume ratios, centroids,
The plane at infinity π_∞

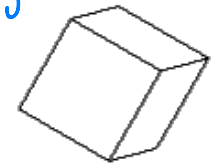
↑

Similarity
7dof
(1 more)

resize, any $s \neq 0$ NON ZERO ⟹ SCALING FACTOR! additional DOF s

$\begin{cases} s > 1 : \text{bigger} \\ s < 1 : \text{smaller} \end{cases}$

$$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$



The absolute conic Ω_∞

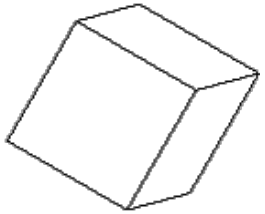
↑

most constrained

Euclidean
6dof

$$\begin{array}{cc} 3\times3 & 3\times1 \\ \end{array}$$

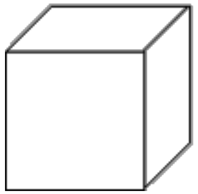
$$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$



Volume ← apply from vertex, same cube, same size! ROTOTRASLATED but no shape change

≡ ROTOTRANSLATION in space

3×3 orthogonal matrix, impose 6 constraints!
9-6=3 dof ← Rotation (3 dof) + translation (3 dof) ∀ point



TRANSFORM

Hierarchy of transformations

PRESERVED

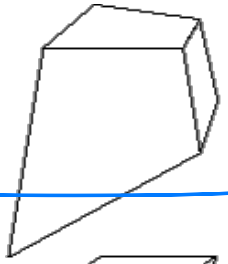
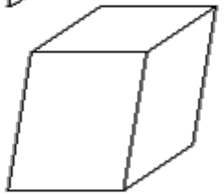
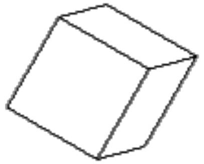
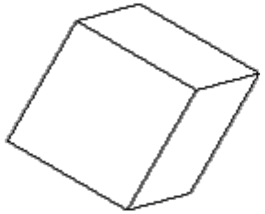
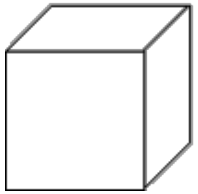
colinearity coplanarity etc.

NON linearity in cartesian

not in general PROJ

linear also in cartesian coordinate

few stuff preserved :(

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Intersection and tangency (from this, CR preserve; only CROSS RATIO)
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		maintain parallel! Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞** (you don't preserve shape, NO ANGLES preserved)
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		shape preserved! Angles, shape, ratio between lens; **The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Volume 

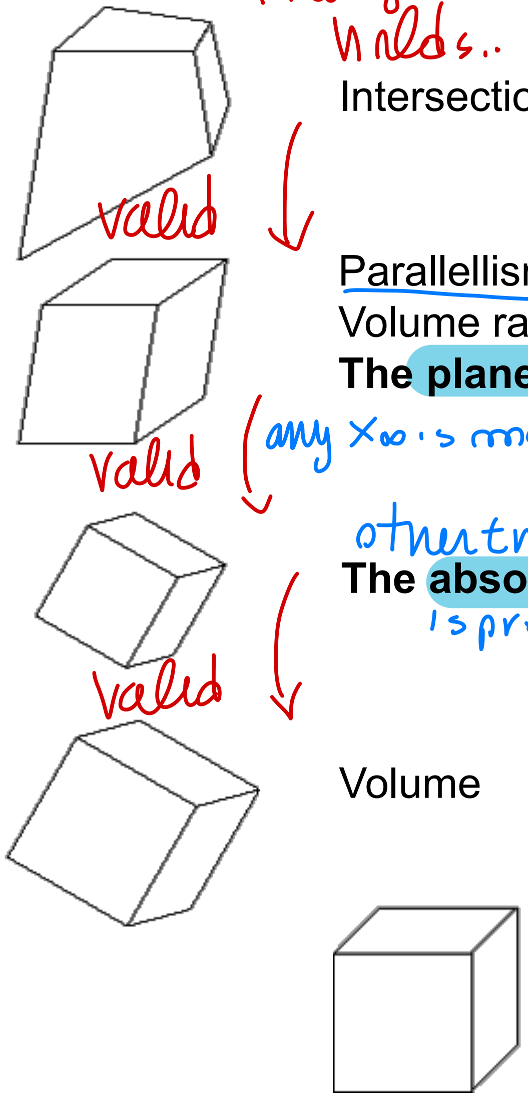
this 0^T at last ROW, which multiply w, keep linearity in cartesian coordinate!

we will find invariant special elements under this classes of transformation

Hierarchy of transformations

transform below holds..

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$	Intersection and tangency	
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$	Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**	
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$	**The absolute conic Ω_∞**	
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$	Volume	



valid

valid

valid

became 2 lines parallel remain parallel!

parallelism is preserved! so also π_∞

all x_∞ mapped to new infinity point! you preserve π_∞!

(any x_∞ is mapped onto x'_∞ again at ∞ by affine → so π_∞ preserved

other than shape, The absolute conic Ω_∞ is preserved!

Ω'_∞ mapped on itself

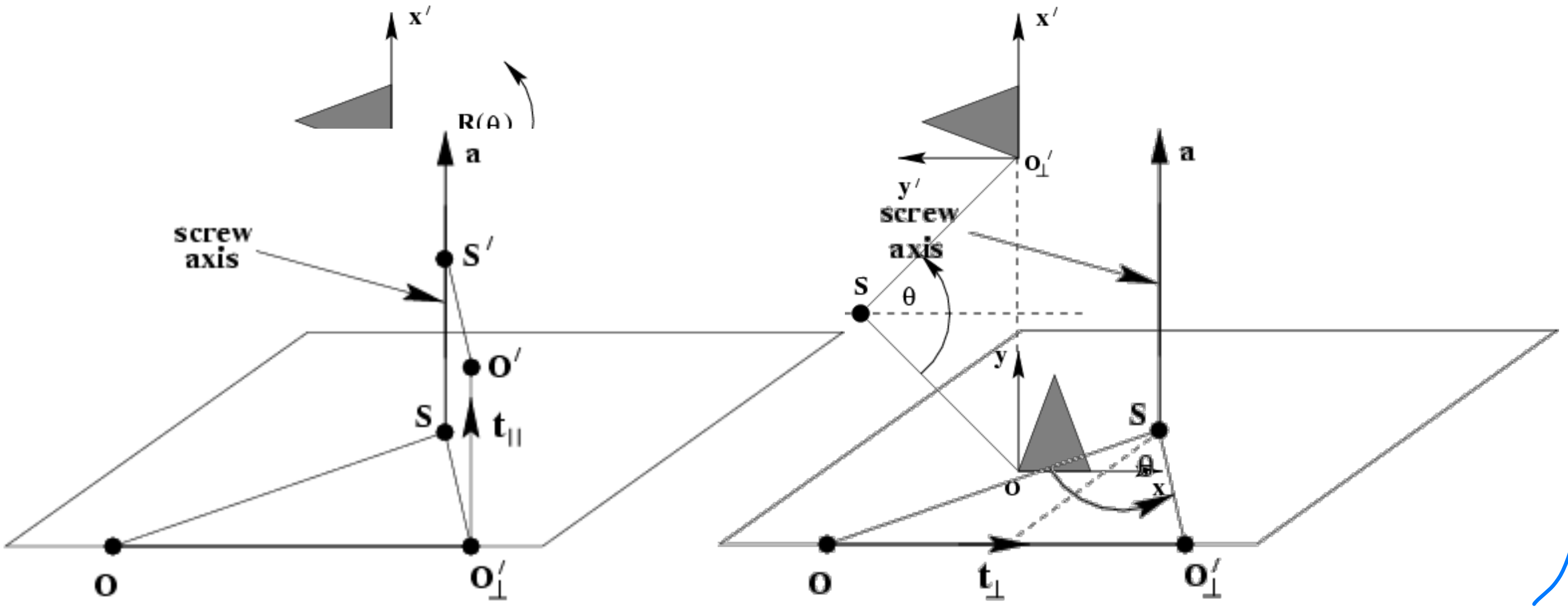
PROPERTY in 3D makes analysis more complex than in 2D!

A 3D rototranslation **is not** a pure rotation: screw decomposition

while in 2D any planar rototranslation could be reduced to simple rotation
↓
in 3D NO!
never reduce to pure rotation
↓
you can find CANONICAL decomposition of displacement by SCREW decomposition
|||
ROTATION + TRANSLATION
↳ translation & rot. s.t

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.

translate and rotate around taking same axis in two steps



screw axis // rotation axis

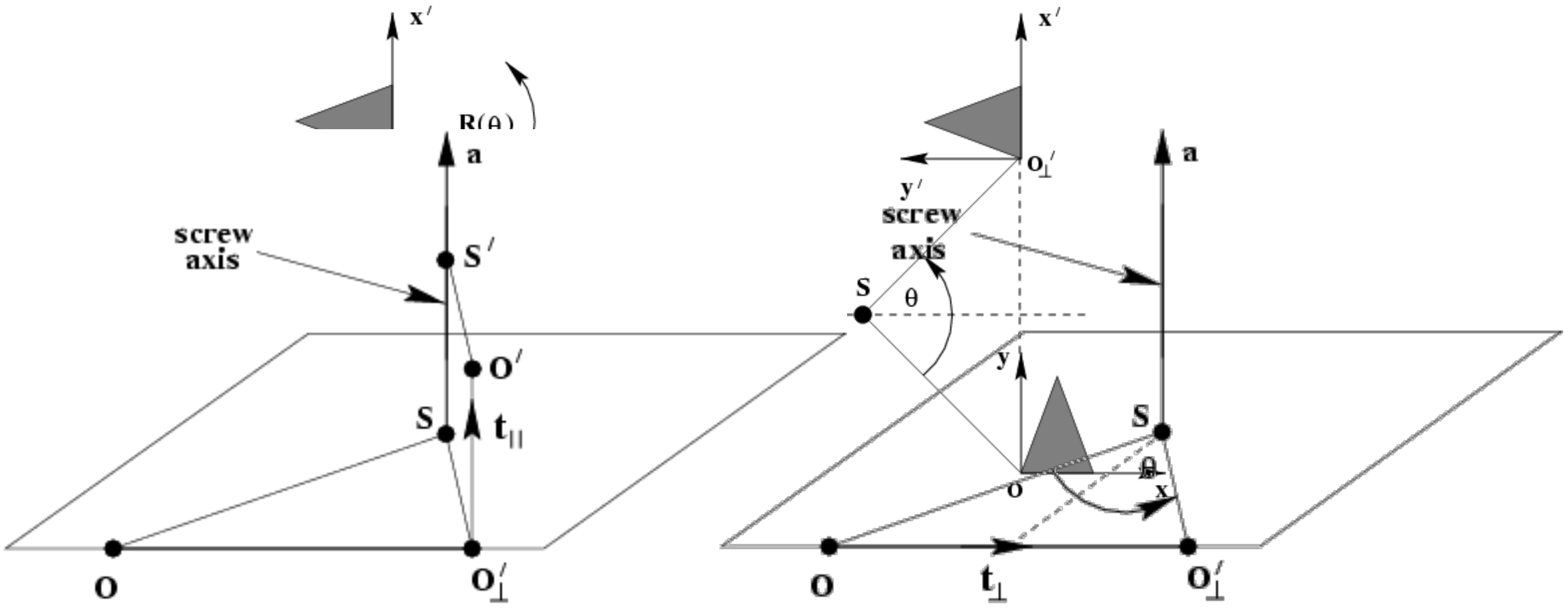
$t = t_{//} + t_{\perp}$

translation direction parallel to rotation axis direction

A 3D rototranslation **is not** a pure rotation: screw decomposition

this replicate the motion of a "screw", BUT NOT as pure rotation as 2D!

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.



screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

Isometries (or Euclidean mappings)

ROTATION

TRANSLATION

$$H_I = \begin{bmatrix} R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix}$$

det (R⊥) = 1

6 do constraints

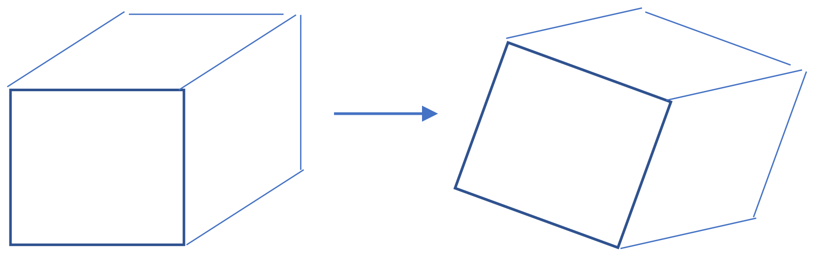
$R_\perp$ is a 3x3 orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^{T}$

det ${R_\perp}^{-1} = 1$ planar rigid displacement (-1 for reflection)

6 dofs: translation $\boldsymbol{t}$ + Euler angles $\vartheta, \varphi,$ ▯

Invariants: lenghts, distances, areas → shape and size → relative positions

you keep on reference plane the relative positions, same coordinates



Similarities

a similarity in PROJ SPACE, moves all but Ω_∞ remain the same!

Q^*_∞ equivalently is preserved!

$$H_S = \begin{bmatrix} s\,R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix}$$

det$(H_S) \neq 1$

$R_\perp$ is a 3x3 orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^T$

(ROTOTRANSLATE)

7 dofs: rigid displacement + *scale*

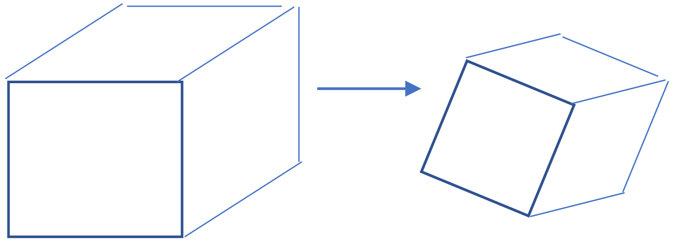
move in space with 6 dof + scale 1 dof

Invariants: ratio of lenghts, angles → shape (not size)

the absolute conic Ω_∞

equivalent information

and the absolute dual quadric $\boldsymbol{Q}^*_\infty$



Affinities (or affine mappings)

9 elements free, s.t A^{-1} exist!

$$H_A = \begin{bmatrix} & A & & \boldsymbol{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

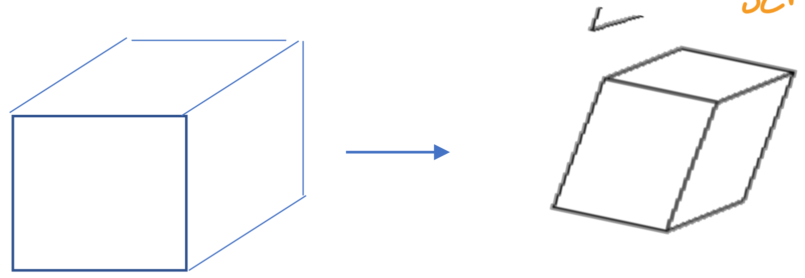
A is any 3x3 invertible matrix

12 dofs: $A + \boldsymbol{t}$

Invariants: parallelism, ratio of parallel lenghts, $\underbrace{\text{ratio of areas}}$ because of parallelism!

the plane at the infinity $\boldsymbol{\pi}_\infty$ (since parallel lines preserved, you preserve X_∞, than π_∞)

as a global set this is preserved, all X_∞ as set is preserved globally! ← [[preserved entity!]]



3d parallelogram!
parallelism preserved

MOST GENERAL...

Projectivities (or projective mappings)

all free elements

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix}$$

generic scalar

A is any 3x3 invertible matrix

15 dofs: $A + \boldsymbol{v} + \boldsymbol{t}$

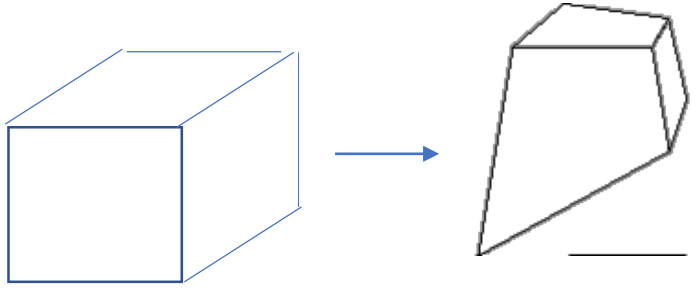
(weak invariants)

NOT strong preservation

ord 1 ord 2

Invariants: colinearity, incidence, order of contact (crossing, tangency, inflections), the 1D cross ratio, the 2D cross ratio, the 3D cross ratio

order 3 solutions, surface contacts!



$\Downarrow$ MOST IMPORTANT theorems
for Affine Reconstruction
and shape Reconstruction

The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{A}\,\mathrm{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canical position $\pi_\infty = (0,0,0,1)^\mathsf{T}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^\mathsf{T}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

A theorem on an affine invariant

Theorem. *A projective transformation H maps the plane at the infinity* $\boldsymbol{\pi}_\infty$ *onto itself (i.e.,* $\boldsymbol{\pi}_\infty$ *is invariant under a projective transformation)*

$$\Updownarrow$$

H is ***affine***

in general π_∞
goes to π_∞', but
if affine, it maps
on itself, and
read as IFF!

3D reconstruction problem

Affine / Shape Reconstruction

⇓

(3D)

Unknown original scene = set of points in the 3D space

→

(NOT an "image" because image reduce 3D into 2D! NOT image here!)

An unknown 3D projective mapping is applied to them

preserve CR, colinearity etc.

↑ how to discuss about 3D to 2D reduction, later!

→

Suppose that the transformed 3D scene can be observed

→

from (3D)' transformed scene back to model of (3D) closer to original scene than the transformed one!

From the observed scene (different from the original)

recover a model of the original scene

Reconstruction

3D reconstruction problem

"image analysis" we want to recover 3D data from image? image are 2D, later we see HOW!

Unknown original scene = set of points in the 3D space

→

An unknown 3D projective mapping is applied to them

→

Suppose that the transformed 3D scene can be observed

→

From the observed scene (different from the original)

recover a model of the original scene

HOW? Images are 2D, not 3D: **issues to be addressed later**

SIMILAR Reconstruction (same shape!)

AFFINE Reconstruction (less interesting, But still good)

Reason as in 2D geometry!

Application to affine reconstruction

Given 3D points obtained by an uknown projective mapping of an unknown original scene (set of points in 3D space)

→ π'_∞ map of π_∞, because generic transformation

the plane $\boldsymbol{\pi}'_\infty$ (i.e. the transformed $\boldsymbol{\pi}_\infty$) is in general $\neq \boldsymbol{\pi}_\infty$!!

we find a way to obtain π'_∞ transformed π_∞ and map back!

Use $\boldsymbol{\pi}'_\infty$ as additional information: if we apply to the transfomed set a second mapping H_{AR} which sends $\boldsymbol{\pi}'_\infty$ back to $\boldsymbol{\pi}_\infty$, we obtain a new, reconstructed model

The composed mapping of $\boldsymbol{\pi}_\infty$ is again $\boldsymbol{\pi}_\infty$ →

we define H_{AR} to map π'_∞ back to π_∞!

$\pi_\infty = [0 \ 0 \ 0 \ 1]$ known where we want to map

From the theorem, the obtained model is an affine mapping of the original scene

$H^* = H \cdot H_{AR}$ composed mapping →

The obtained model is an **affine reconstruction** of the scene

maps back π'_∞ to π_∞!

1. Use of $\boldsymbol{\pi}'_\infty$ in affine reconstruction

....

apply to the transformed point set a second projective mapping $\boldsymbol{H_{AR}}$

that maps $\boldsymbol{\pi}'_\infty$ back to $\boldsymbol{\pi}_\infty$,

how can we find such a projective mapping H_{AR} ?

chosen to be H_{AR}^{-1} exist!

$$H_{AR} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & \boldsymbol{\pi}'^{T}_{\infty} & & \end{bmatrix},$$

last row as

$\Pi'_\infty = [a' \quad b' \quad c' \quad d']$

such that H_{AR} it is invertible

construct points we know were at infinity!

To sum up: affine rectification from $\boldsymbol{\pi}'_\infty$

those represents $X_{\infty i}$ projection!

- Find three points $\boxed{\mathbf{X}'_1, \mathbf{X}'_2, \mathbf{X}'_3}$ that result from mapping three points at the infinity

- Fit the transformed plane $\boldsymbol{\pi}'_\infty$ to them: $\boldsymbol{\pi}'_\infty = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X}'^T_1 \\ \mathbf{X}'^T_2 \\ \mathbf{X}'^T_3 \end{bmatrix}\right)$

as plane passing through those points

- Affine rectification matrix

$$H_{AR} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & \boldsymbol{\pi}'^T_\infty & & \end{bmatrix}$$

- Affine reconstructed model $M_A = H_{AR}$ given_points for SHAPE RECONSTRUCTION, Ω_∞ is invariant under similarity being it the intersection between any sphere and Π_∞... by apply similarity, we find same Ω_∞,

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $(X_1, X_2, X_3)\mathrm{I}(X_1, X_2, X_3)^{\mathsf{T}}$

(with no real points)

The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

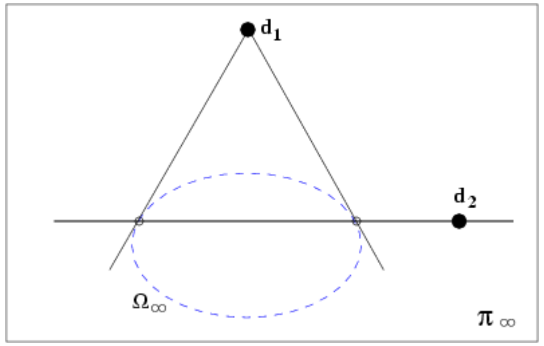
1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two points
3. Spheres intersect π_∞ in Ω_∞

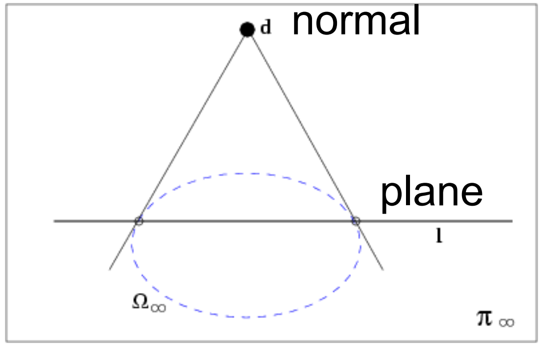
The absolute conic

Euclidean: $$\cos\theta = \frac{\left(d_1^T d_2\right)}{\sqrt{\left(d_1^T d_1\right)\left(d_2^T d_2\right)}}$$

Projective: $$\cos\theta = \frac{\left(d_1^T \Omega_\infty d_2\right)}{\sqrt{\left(d_1^T \Omega_\infty d_1\right)\left(d_2^T \Omega_\infty d_2\right)}}$$

$$d_1^T \Omega_\infty d_2 = 0$$ (orthogonality=conjugacy)





A theorem on an invariant under similarities

Theorem. *A projective transformation H maps the **absolute conic** Ω_∞ onto itself (i.e., Ω_∞ is invariant under a projective transformation)*

$$\Updownarrow$$

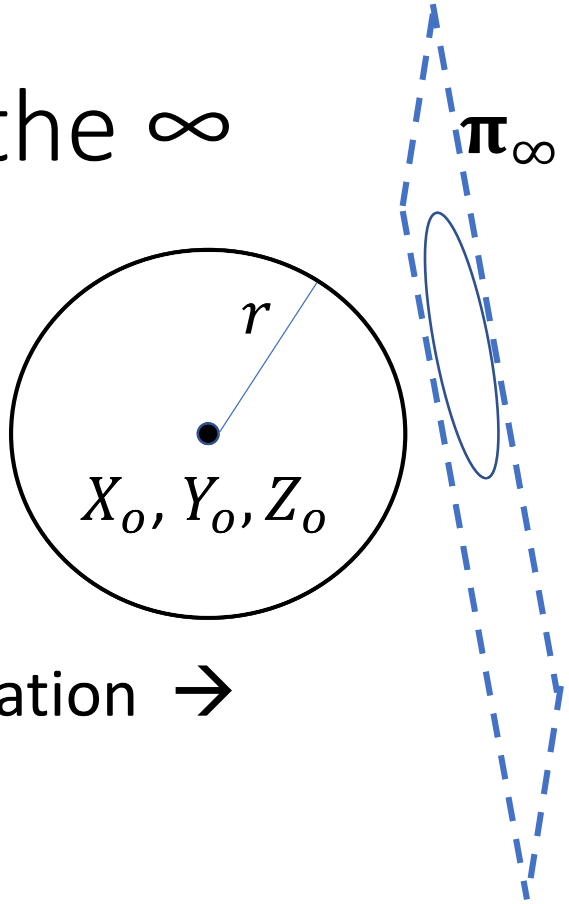
H *is **a similarity***

The absolute conic: intersection of a sphere and the plane at the ∞

$$\begin{cases}(x - X_o w)^2 + (y - Y_o w)^2 + (z - Z_o w)^2 - r^2 w^2 = 0 \\ w = 0\end{cases}$$

$$\rightarrow$$

$$\begin{cases}x^2 + y^2 + z^2 = 0 \\ w = 0\end{cases}$$



The sphere parameters (center and radius) disappear from the equation →

the intersection **conic** is the **same for all** spheres:

$$x^2 + y^2 + z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

A conic within $\boldsymbol{\pi}_\infty$: $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **ABSOLUTE CONIC**

«proof» without computation

- $\Omega_\infty \ = \ \boldsymbol{\pi}_\infty \cap$ any sphere (Keep shape!)
- Similarity maps sphere onto sphere' (new sphere)
- Similarity maps $\boldsymbol{\pi}_\infty$ onto $\boldsymbol{\pi}_\infty$ (because similarity ⊂ Affine, then Π_∞!)
- Similarity maps sphere ∩ $\boldsymbol{\pi}_\infty$ onto sphere' ∩ $\boldsymbol{\pi}_\infty$
- Similarity maps Ω_∞ onto Ω_∞ ← Ω_∞ is any sphere intersection with Π_∞
- → absolute conic Ω_∞ is **invariant** under similarity

this holds also for Ω^*_∞ absolute dual quadric, which is equivalent to Ω_∞

Application to 3D shape reconstruction

to use this information...

observable

Given 3D points obtained by an uknown projective mapping of an unknown original scene (set of points in 3D space)

$\rightarrow$

generic projective mapping

The absolute conic Ω_∞ is mapped onto a conic $\Omega'_\infty \neq \Omega_\infty$!!

$\Downarrow$ if we can discover Ω'_∞, we can use this to map back to Ω_∞

Use Ω'_∞ as additional information: if we apply to the transfomed set a second mapping H_{SR} which sends Ω'_∞ back to Ω_∞, we obtain a new, reconstructed model

The composed mapping of Ω_∞ is again $\Omega_\infty \rightarrow$

From the theorem, the obtained model is a similarity of the original scene

$\rightarrow$

The obtained model is a **shape reconstruction** of the original scene

A noteworthy example:
THE ABSOLUTE (dual) QUADRIC

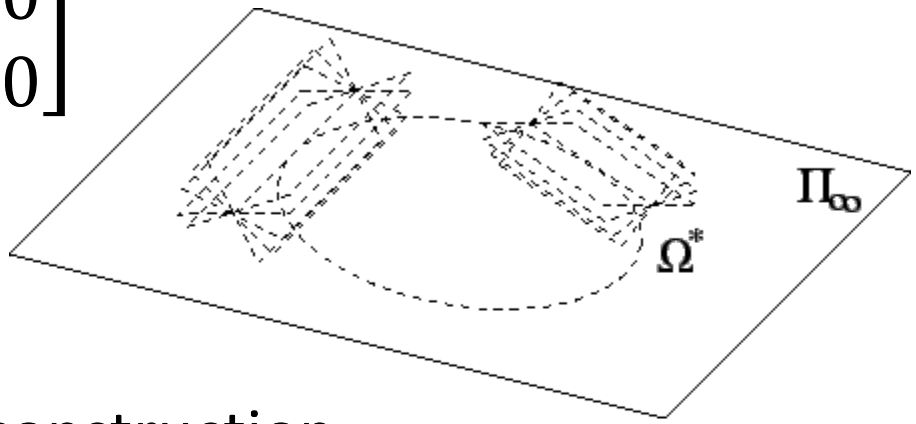
$$\boldsymbol{\pi}^{T}\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

instead of using Ω_∞, conics are NOT primitive in 3D!

we use primitives in 3D by Ω^*_∞

→The set of planes that are tangent to a the absolute conic:

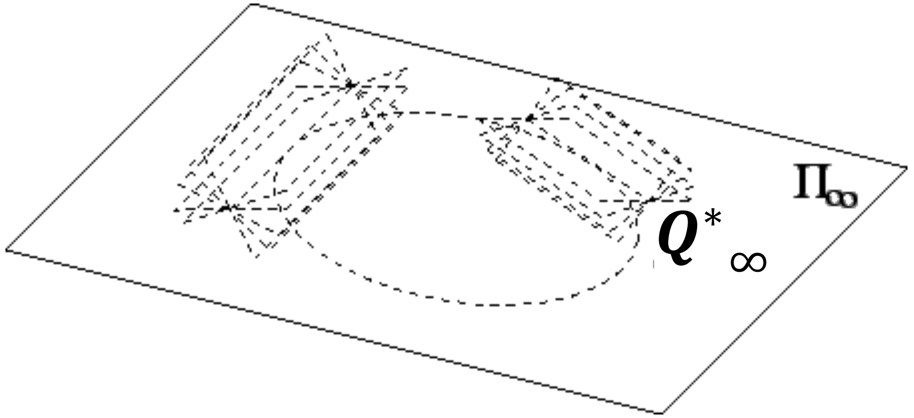
$$\boldsymbol{Q}^{*}{}_{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



The absolute dual quadric $\boldsymbol{Q}^{*}{}_{\infty}$ is useful in the 3D reconstruction

The absolute dual quadric

$$\boldsymbol{Q}^*_\infty = \begin{bmatrix} \mathrm{I} & 0 \\ 0^\mathsf{T} & 0 \end{bmatrix}$$



The absolute conic Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the null vector of $\boldsymbol{Q}^*_\infty$
3. Angles:

A theorem on an invariant under similarities

Theorem. *A projective transformation* H *maps the* ***absolute dual quadric*** $\boldsymbol{Q}^*_\infty$ *onto itself (i.e.,* $\boldsymbol{Q}^*_\infty$ *is invariant under a projective transformation)*

$$\Updownarrow$$

H *is* ***a similarity***

1. Use of $\boldsymbol{Q'^*}_\infty$ in shape reconstruction

→ better algebraically

finding a projectivity H_{SR} which maps $\boldsymbol{Q'^*}_\infty$ back to $\boldsymbol{Q^*}_\infty$

reduces to finding a projectivity H_{SR} that maps Ω'_∞ back to Ω_∞

transformation rule

$$\boldsymbol{Q^*}_\infty = H_{SR}\,\boldsymbol{Q'^*}_\infty\,H_{SR}{}^T \rightarrow$$

$$\boldsymbol{Q'^*}_\infty = H_{SR}{}^{-1}\,\boldsymbol{Q^*}_\infty\,H_{SR}{}^{-T} = H_{SR}{}^{-1}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}(H_{SR}{}^{-1})^T$$

SVD to find shape reconstruct (as 2D geom)

$$\mathrm{SVD}(\boldsymbol{Q'^*}_\infty) = U_\perp\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}U_\perp{}^T$$

→ one of the ∞^8 solutions is $H_{SR} = U_\perp{}^{-1} = U_\perp{}^T$

To sum up: shape reconstruction from $\boldsymbol{Q'}^*_\infty$ the transformed absolute dual quadric $\boldsymbol{Q}^*_\infty$

- Find the transformed absolute dual quadric $\boldsymbol{Q'}^*_\infty$

- Singular value decomposition

$$\mathrm{SVD}(\boldsymbol{Q'}^*_\infty) = U_\perp \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {U_\perp}^T = H_{SR}^{-1}\, \boldsymbol{Q}^*_\infty H_{SR}^{-T}$$

- Reconstructing transformation (from svd output U)

$H_{SR} = U^T$ ← notice: $H_{SR} = U^T$ is orthogonal 4x4, not a $\mathbb{P}^3$ isometry

- Euclidean reconstructed model $M_S = H_{SR}$ given_points

2. How to find Ω'_∞ (or $\boldsymbol{Q'^*}_\infty$) in practical cases?

estimate ?

how to find it being complex valued numbers..
abstract entity?

In 3D reconstruction we use Ω'_∞, or equivalently $\boldsymbol{Q'^*}_\infty$, as additional information

how can we find Ω'_∞ or equivalently $\boldsymbol{Q'^*}_\infty$?

provide equations
on this
conic!

from information on the observed scene
additional constraints are derived:

using known stuff

observing transformations, angles

a. known angles between plane normals

b. known shape of objects, e.g., spheres

c. combinations of a. and b.

any sphere intersected with Π_∞ is absolute conic'

from this you recover $Q^{*\prime}_\infty$

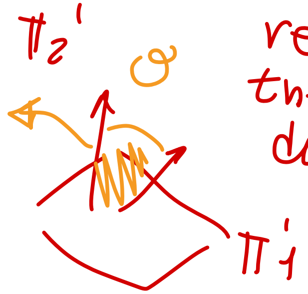
Angle ϑ between two planes in the original scene in terms of the mapped elements: $\underbrace{\boldsymbol{\pi}'_1, \boldsymbol{\pi}'_2}$, $\boldsymbol{Q}'^*_\infty$

usefull result for reconstruction

planes vector $a'\ b'\ c'$ represent the orthogonal direction

considering two planes. and you know the angle ϑ

$$\cos\vartheta = \frac{\boldsymbol{\pi}_1'^T \boldsymbol{Q}'^*_\infty \boldsymbol{\pi}'_2}{\sqrt{(\boldsymbol{\pi}_1'^T \boldsymbol{Q}'^*_\infty \boldsymbol{\pi}'_1)(\boldsymbol{\pi}_2'^T \boldsymbol{Q}'^*_\infty \boldsymbol{\pi}'_2)}}$$



Here, $\boldsymbol{\pi}'_1, \boldsymbol{\pi}'_2$ are extracted from the mapped scene (see later), whereas $\boldsymbol{Q}'^*_\infty$ is the required information we want to find

Known angle ϑ between two **scene planes** → nonlinear eqn on $\boldsymbol{Q}'^*$

planes orthogonal in original scene which I can observe then projection π_1', π_2' eq.

if the scene planes are perpendicular, $\cos\vartheta = 0$ → $\boxed{\boldsymbol{\pi}_1'^T \boldsymbol{Q}'^*_\infty \boldsymbol{\pi}'_2 = 0}$ **linear**