Advanced and Multivariable Control

Pole placement control for SISO systems described with transfer functions

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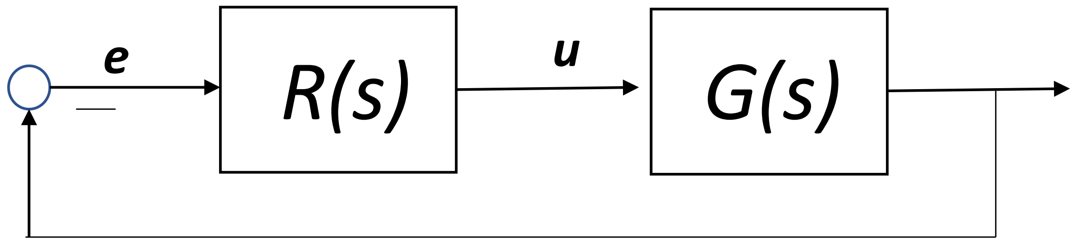
Problem statement

Given a SISO system described in transfer function form

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

We want to find a regulator described by the transfer function

$$R(s) = \frac{F(s)}{\Gamma(s)}$$



Such that the closed-loop system has prescribed poles

Remarks

The problem is solvable if and only if the polynomials ***A(s)*** and ***B(s)*** do not have common factors, i.e. the system is in minimal form (recall that ***G(s)*** only describes the reachable and observable part of the system)

The solution to this poblem can be trivially obtained by using a state space representation of the system, for instance with the reachability canonical form, and then by using the theory previously introduced

Here we want to present a simpler solution for SISO systems, which is computationally non intensive and highly flexible

Main results

The problem can be solved with a regulator of minimal order ***n-1*** ***(n*** is the order of the system)

$$R(s) = \frac{F(s)}{\Gamma(s)} = \frac{f_{n-1}s^{n-1}+f_{n-2}s^{n-2}+\ldots+f_1 s+f_0}{\gamma_{n-1}s^{n-1}+\gamma_{n-2}s^{n-2}+\ldots+\gamma_1 s+\gamma_0}$$

The order ***n-1*** corresponds to using a ***static feedback control law + a reduced order observer***. It is obviously possible to use a regulator of higher order

The characteristic polynomial of the closed-loop system is

$$A(s)\Gamma(s) + B(s)F(s) = P(s)$$

Diophantine equation

The polynomial *P(s)* is of order ***2n-1*** its, roots are the ***desired closed-loop poles***, i.e. the ***free design parameters***

$$P(s) = s^{2n-1} + p_{2n-2}s^{2n-2} + \ldots + p_1 s + p_0$$

The solution to the Diophantine equation is obtained by making equal the coefficients of the powers of ***s*** at the left and right hand side. This corresponds to th solution of a set of linear equations

Example

$$A(s) = s^2 + a_1 s + a_0$$

$$B(s) = b_1 s + b_0$$

$\longrightarrow$ *n=2* $\longrightarrow$

$$F(s) = f_1 s + f_0$$

$$\Gamma(s) = \gamma_1 s + \gamma_0$$

$$P(s) = s^3 + p_2 s^2 + p_1 s + p_0$$

$$A(s)\Gamma(s) + B(s)F(s) = P(s)$$

$$\downarrow$$

$$\gamma_1 s^3 + (\gamma_0 + a_1\gamma_1 + b_1 f_1)s^2 + (a_1\gamma_0 + a_0\gamma_1 + b_1 f_0 + b_0 f_1)s + (a_0\gamma_0 + b_0 f_0) = P(s)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_1 & 0 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ p_2 \\ p_1 \\ p_0 \end{bmatrix}$$

In general, the set of linear equations to be solved is

$$
\begin{array}{c c}
\textit{2n rows} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{n-1} & 1 & 0 & \dots & 0 & b_{n-1} & 0 & \dots & 0 \\
a_{n-2} & a_{n-1} & 1 & \dots & 0 & b_{n-2} & b_{n-1} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_0 & a_1 & a_2 & \dots & 1 & b_0 & b_1 & \dots & b_{n-1} \\
0 & a_0 & a_1 & \dots & a_{n-1} & 0 & b_0 & \dots & b_{n-2} \\
0 & 0 & a_0 & \dots & a_{n-2} & 0 & 0 & \dots & b_{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & a_0 & 0 & 0 & \dots & b_0
\end{bmatrix}
\begin{bmatrix}
\gamma_{n-1} \\ \gamma_{n-2} \\ \gamma_{n-3} \\ \vdots \\ \gamma_0 \\ f_{n-1} \\ f_{n-2} \\ \vdots \\ f_0
\end{bmatrix}
=
\begin{bmatrix}
1 \\ p_{2n-2} \\ p_{2n-3} \\ \vdots \\ p_{n-1} \\ p_{n-2} \\ p_{n-3} \\ \vdots \\ p_0
\end{bmatrix}
\end{array}
$$

n columns *n columns*

The matrix of coefficients (Sylvester matrix) is nonsingular iff ***A(s)*** and ***B(s)*** do not have common factors. However, it should also be well conditioned (no poles and zeros of the system too near each other)

The equations above can be solved one by one starting from the first one

a very simple design method

Regulator with integral action

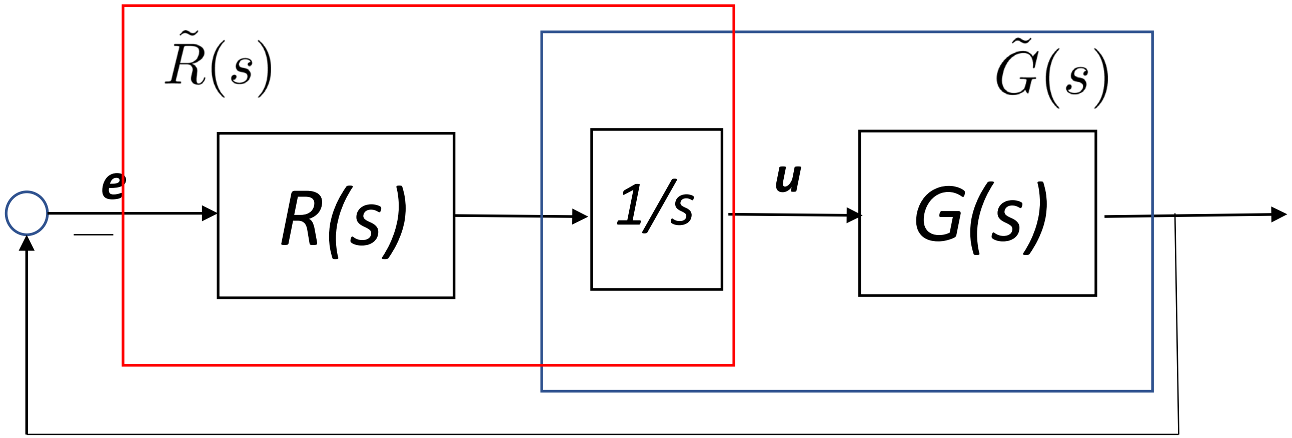
In order to include an integral action into the regulator, one can «enlarge» the the system tranfer function by considering the system with transfer function

$$\tilde{G}(s) = \frac{B(s)}{sA(s)} \text{ of order } n+1$$

For this new system, compute the regulator ***R(s)*** (now of order ***n***) with the theory described to assign ***2n+1*** poles

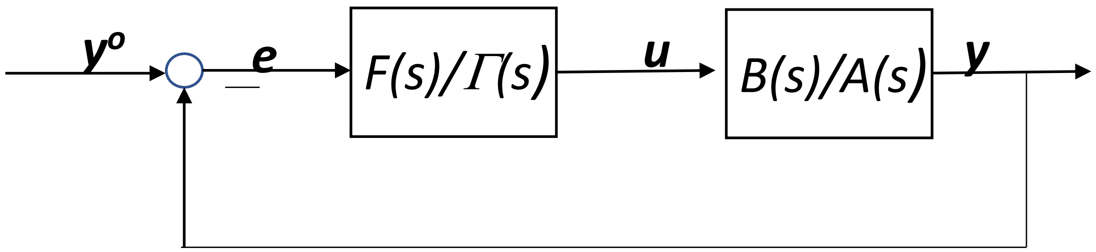
Finally, implement the regulator with transfer function

$$\tilde{R}(s) = \frac{1}{s}R(s)$$



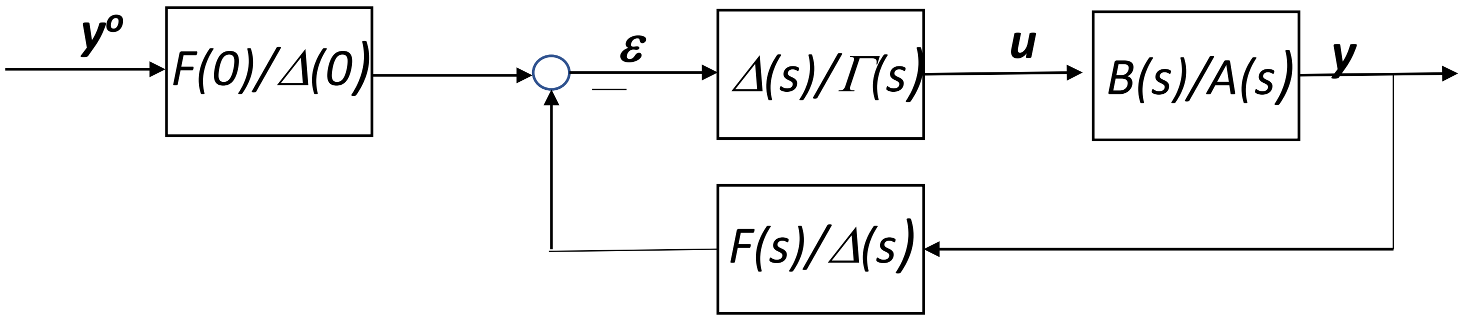
Zeros of the closed-loop system

The roots of the polynomial ***F(s)*** are not design parameters, and the algorithm can set them to values which can seriously deteriorate the closed-loop performance. Remember that «unstable zeros» of ***nonminimum phase*** systems, produce the so-called ***inverse response***



$$Y(s)=\frac{B(s)F(s)}{A(s)\Gamma(s)+B(S)F(s)}Y^o(s)=\frac{B(s)F(s)}{P(s)}Y^o(s)$$

To avoid this problm, define a polynomial $\Delta(s) = s^{n-1} + \delta_{n-2}s^{n-2} + \ldots + \delta_1 s + \delta_0$ with «stable» roots, and implement the scheme



Now $$Y(s) = \frac{B(s)\Delta(s)}{P(s)} \frac{F(0)}{\Delta(0)} Y^o(s)$$

The precompensator is used to guarantee that the signal ε in the scheme at the steady state is proportional to the error ***y^o-y***

There is a cancellation of the roots of ***$\Delta(s)$***, but this is allowed since the polynomial is chosen with asymptotically stable roots

How to cancel poles or zeros?

Assume that $$A(s) = (s+a)A'(s)$$

and we want to design a regulator that cancels the pole in ***s=-a***

It is sufficient to choose $$P(s) = (s+a)P'(s)$$

In fact, the Diophantine equation becomes

$$(s+a)A'(s)\Gamma(s) + B(s)F(s) = (s+a)P'(s) \longrightarrow B(s)F(s) = (s+a)[P'(s) - A'(s)\Gamma(s)]$$

This means that ***(s+a)*** must be a factor of ***B(S)F(s)***, so that it will be automatically included into ***F(s)***

and a cancellation occurs between the terms ***(s+a)*** in ***A(s)*** and ***F(s)***

The same procedure can be followed to cancel zeros of ***B(s)***

What about discrete time systems?

Nothing changes in the algorithm save for the fact that

1. ***s --> z***
2. ***1/s --> z/(z-1) or 1/(z-1)***
3. ***s=0 --> z=1***

Exercise

Given the system $G(s) = \frac{1}{s-1}$ design a regulator with integral action such that the closed-loop poles are in $s=-1$

Use 3 different approaches: a) with state space formulation, b) use pole placement based on the transfer function approach, c) a PI regulator tuned with the root locus

Solution a - state space

$$G(s) = \frac{1}{s-1} \longrightarrow \begin{cases} \dot{x} = x + u \\ y = x \end{cases}$$

Enlarged state with integrator

$$\begin{vmatrix} \dot{x} \\ \dot{v} \end{vmatrix} = \overbrace{\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}}^{\bar{A}} \begin{vmatrix} x \\ v \end{vmatrix} + \overbrace{\begin{vmatrix} 1 \\ 0 \end{vmatrix}}^{\bar{B}} u$$

Easy to verify that $(\bar{A}, \bar{B})$ is reachable and the invariant zeros (if any) of the original system are not at the origin (obvious, $G(s)$ does not have zeros)

For the enlarged system we compute a pole placement control law

$$u = -\bar{K} \begin{vmatrix} x \\ v \end{vmatrix}$$

(note that the overall state is measurable)

We can ~~use~~ use the Ackermann's formula or, simply, set

$$\bar{K} = | k_x \;\; k_v |$$

$$\bar{A} - \bar{B}\bar{K} = \begin{vmatrix} 1-k_x & -k_v \\ -1 & 0 \end{vmatrix} \rightarrow \det(\bar{A}-\bar{B}\bar{K}) = s^2 + (k_x - 1)s - k_v$$

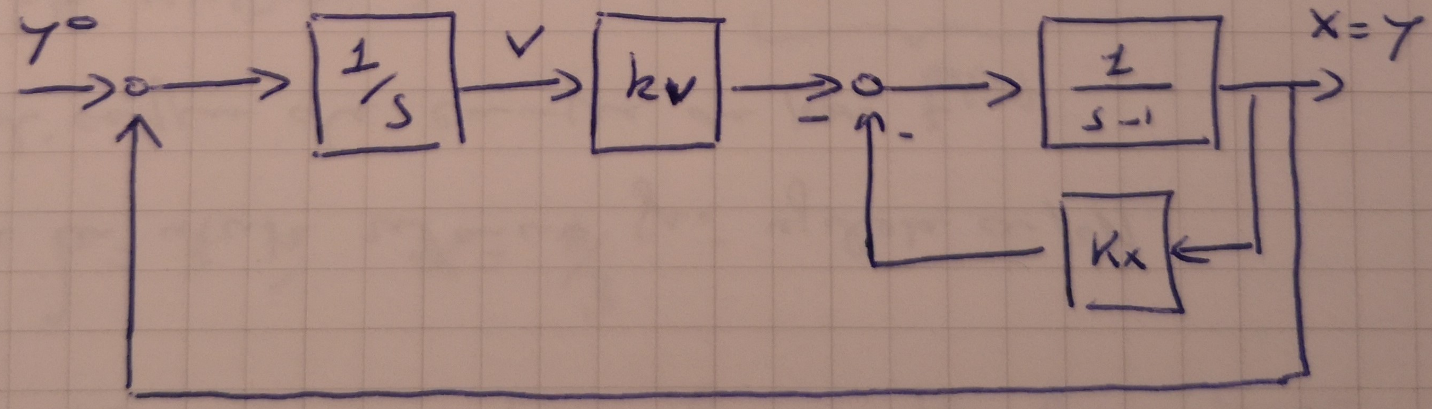
We set $\det(sI - (\bar{A} - \bar{B}\bar{K})) = s^2 + 2s + 1 = (s+1)^2$

(closed-loop poles in $s=-1$

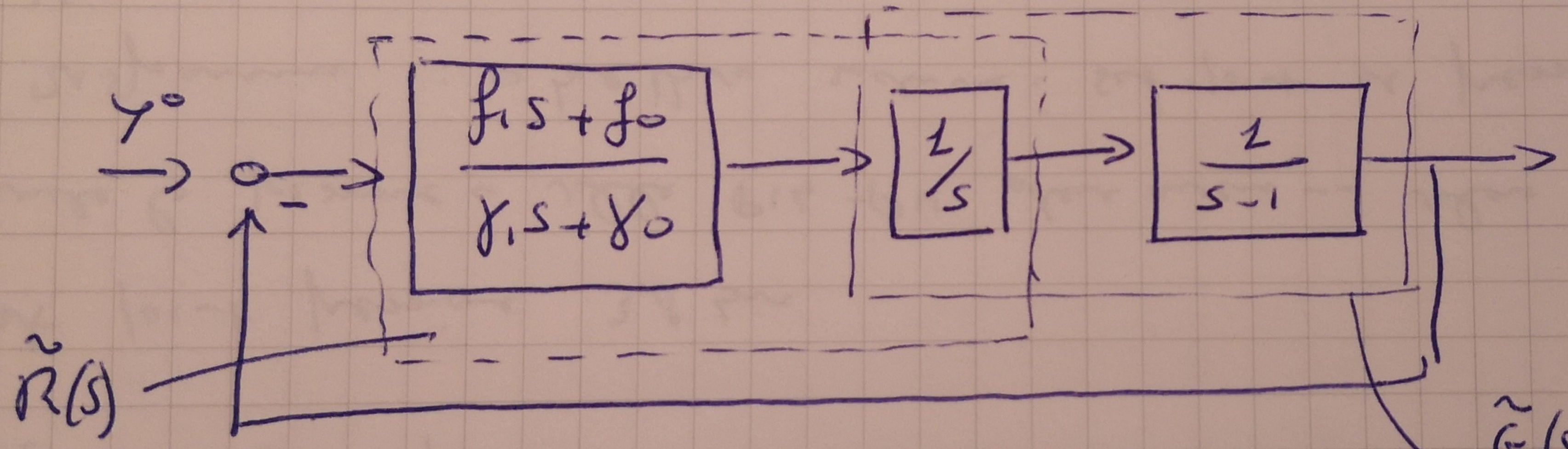
we have $k_x - 1 = 2 \rightarrow k_x = 3$

$-k_v = 1 \rightarrow k_v = -1$

Overall scheme



Solution 2 - pole placement with transfer functions



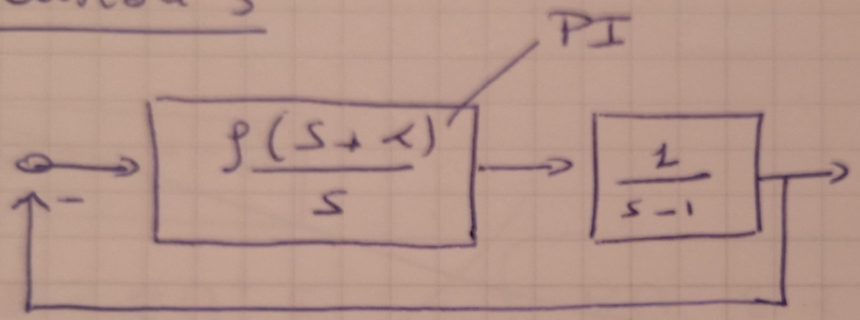
$$\tilde{G}(s) = \frac{1}{s(s-1)}$$

$$A(s) = s^2 - s \;, \quad B(s) = 1 \;, \quad P(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{vmatrix} = \begin{vmatrix} 1 \\ 3 \\ 3 \\ 1 \end{vmatrix} \rightarrow \begin{array}{l} \gamma_1 = 1 \\ \gamma_0 = 4 \\ f_1 = 7 \\ f_0 = 1 \end{array}$$

$$\tilde{R}(s) = \frac{7(s + 1/7)}{s(s+4)}$$

Solution 3



Characteristic equation

$$s(s-1)+\rho(s+\alpha)=\underbrace{s^2+2s+1}_{P(s)}$$

$$\rho = 3 \,, \quad \alpha = 1/3$$

possible approach only possible for simple systems

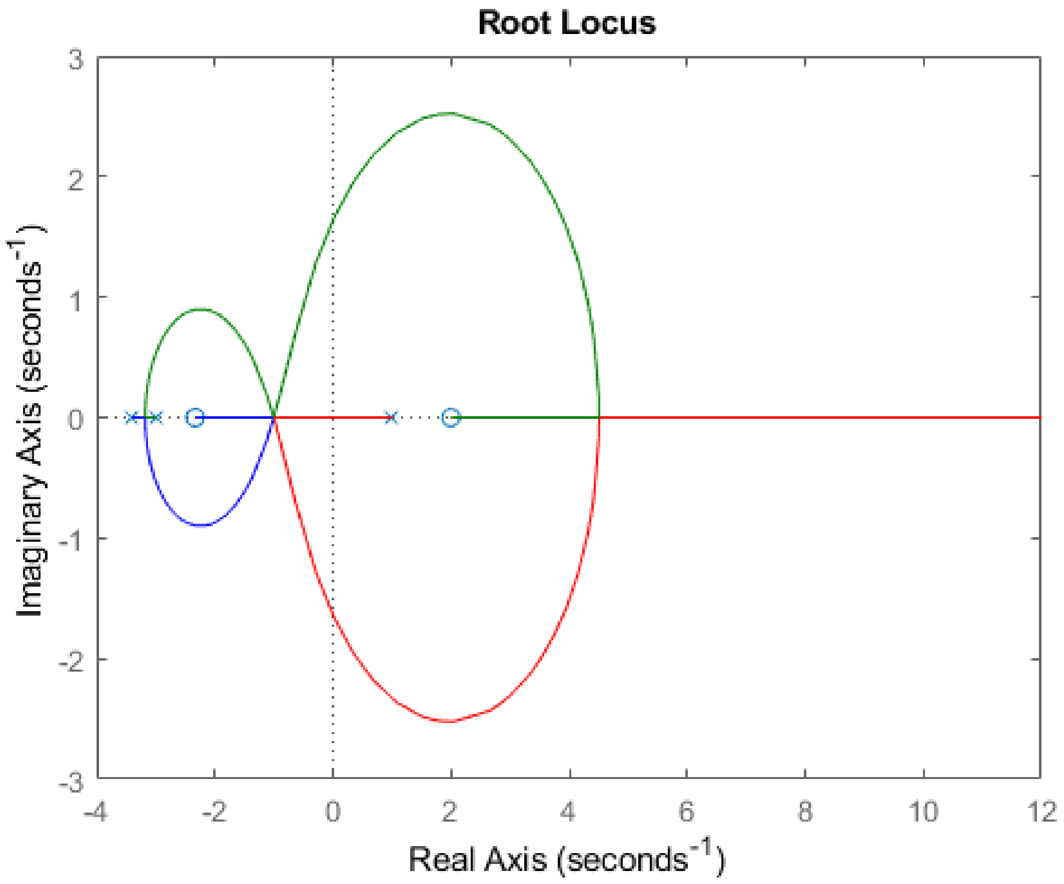
New Example

$$G(s)=\frac{s-2}{(s-1)(s+3)}$$ unstable and nonminimum phase

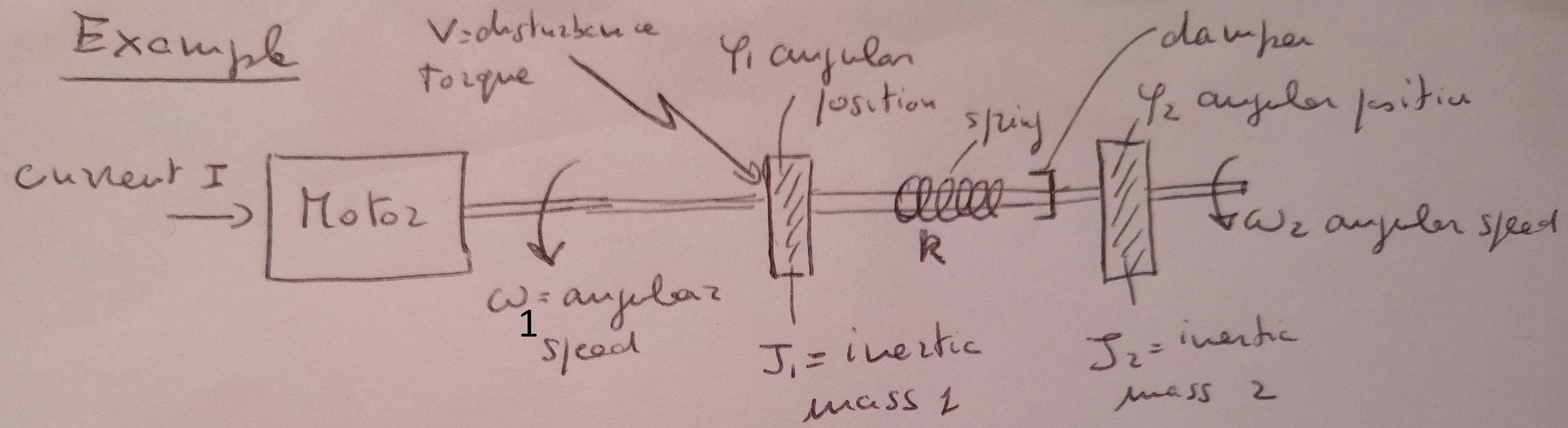
Goal: assign poles in -1. $R(s)=\dfrac{f_1 s+f_0}{\gamma_1 s+\gamma_0}$, $P(s)=(s+1)^3$

$$G(s)=\frac{s-2}{(s^2+2s-3)}, \quad P(s)=s^3+3s^2+3s+1$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -3 & 2 & -2 & 1 \\ 0 & -3 & 0 & -2 \end{vmatrix} \begin{vmatrix} \gamma_1 \\ \gamma_0 \\ f_1 \\ f_0 \end{vmatrix} = \begin{vmatrix} 1 \\ 3 \\ 3 \\ 1 \end{vmatrix} \longrightarrow R(s)=\frac{-2.4s-5.6}{s+3.4}$$


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
3
2
1
0
-1
-2
-3
-4
-2
0
2
4
6
8
10
12

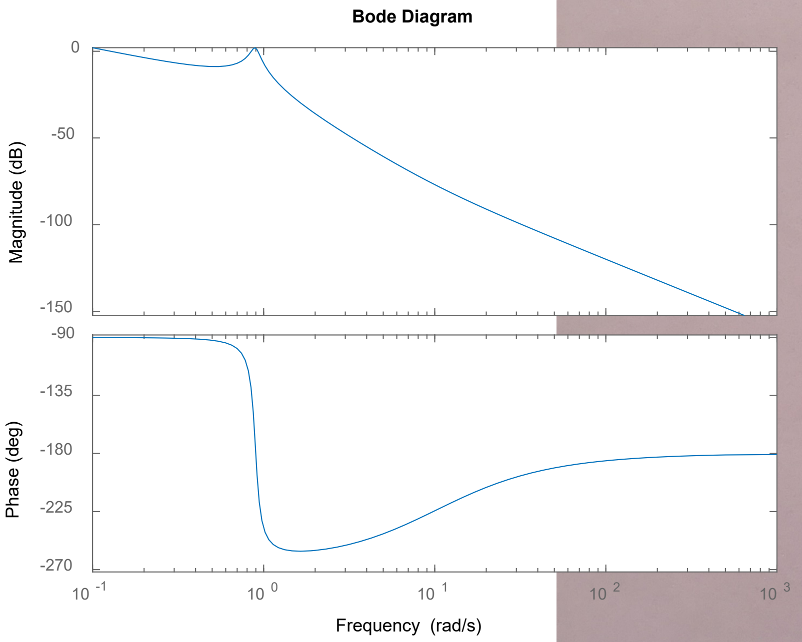
Example



Model

$$
\begin{cases}
\dot{\varphi}_1 - \dot{\varphi}_2 = \omega_1 - \omega_2 \\
J_1 \dot{\omega}_1 = d(\omega_2 - \omega_1) + k(\varphi_2 - \varphi_1) + k_I \cdot I + V \\
J_2 \dot{\omega}_2 = d(\omega_1 - \omega_2) + k(\varphi_2 - \varphi_1)
\end{cases}
$$

$$
\begin{cases}
x_1 = \varphi_1 - \varphi_2 \\
x_2 = \omega_1 \\
x_3 = \omega_2
\end{cases}
, \quad u = \begin{vmatrix} I \\ V \end{vmatrix} \text{ disturbance} , \quad y = \omega_2
$$



Numerical values

$J_1 = \frac{10}{9}$, $J_2 = 10$, $k_e = 1$, $d = 0.1$, $K_I = 1$

$$G_2 = \frac{0.01s + 0.1}{s^3 + 0.1s^2 + 0.8s}$$

$\omega_p = 0.894$

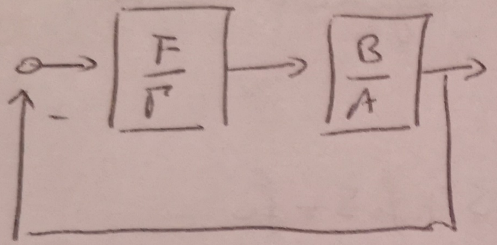
$\xi_p = 0.0559$

poles $s = 0$

$s = -0.05 \mp j0.89$ (poorly damped)

zero $s = -10$ (high frequency)

Order of the system $n=3$, order of the regulator $= 2$



$$F = f_2 s^2 + f_1 s + f_0$$

$$\Gamma = \gamma_2 s^2 + \gamma_1 s + \gamma_0$$

Desired closed loop polynomial $P(s)$ of order $2n-1 = 5$

Project 1

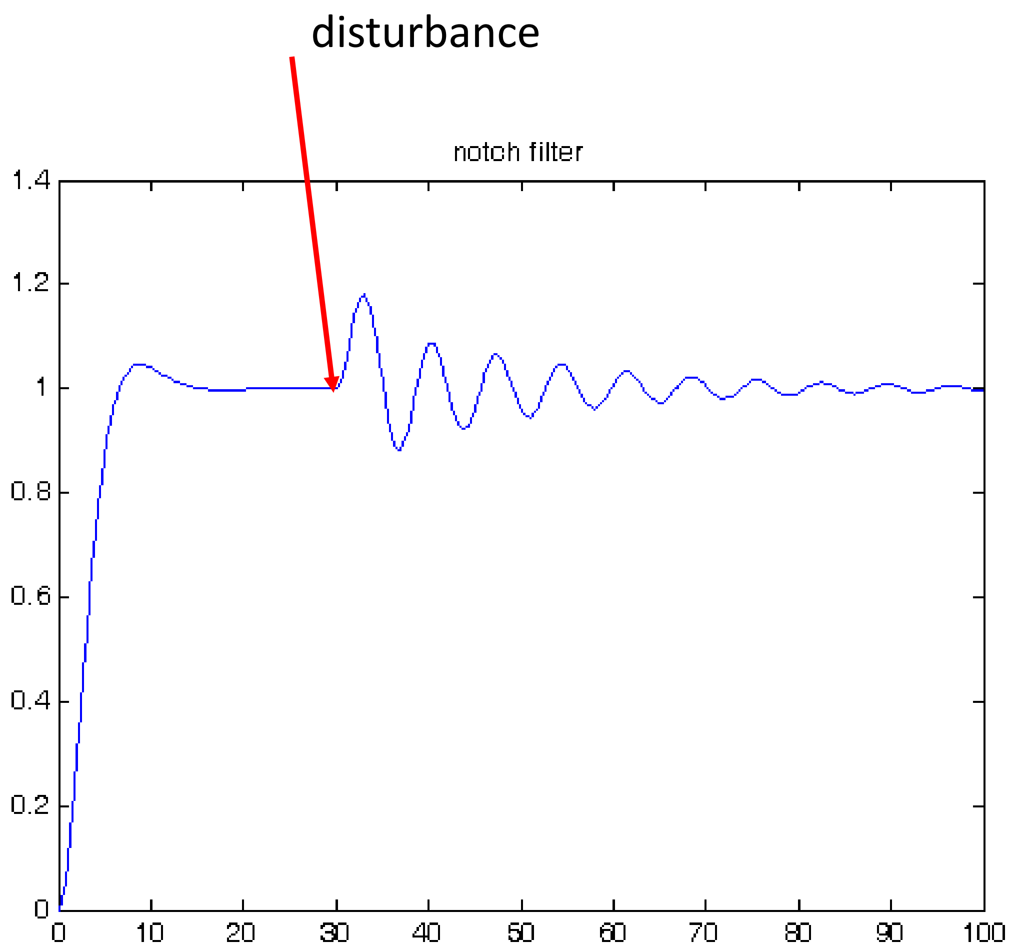
$$P(s) = \underbrace{(s^2 + 2\xi_n \omega_n s + \omega_n^2)}_{\omega_n = 0.5,\ \xi = 0.7} (s^2 + 2\xi_p \omega_p s + \omega_p^2)(s+10)$$

$\xi_p = 0.056$

$\omega_p = 0.83$

↑ these are also open-loop poles. Cancellation!

Project 1: good response to the reference signal, poor response to disturbance variations. Why?



Project 1

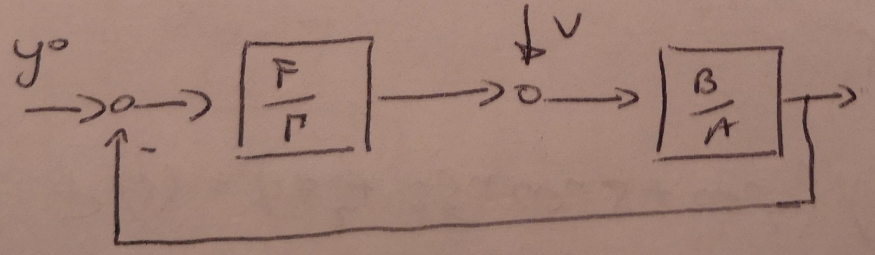
$$A(s) = \underbrace{A'(s)}_{\substack{\text{Poles with} \\ \omega_p = 0.89 \\ \xi_p = 0.056}} s$$

If $P(s) = A'(s)\,P'(s) \rightarrow A(s)\Gamma(s) + B(s)F(s) = P(s)$

$$\underbrace{A'(s)\cdot(s+10)}_{A(s)}\,\Gamma(s) + B(s)\,F(s) = A'(s)\,P'(s)$$

$$\Downarrow$$

$F(s)$ contains the term $A'(s)$, $F(s) = \bar{f}\,A'(s)$

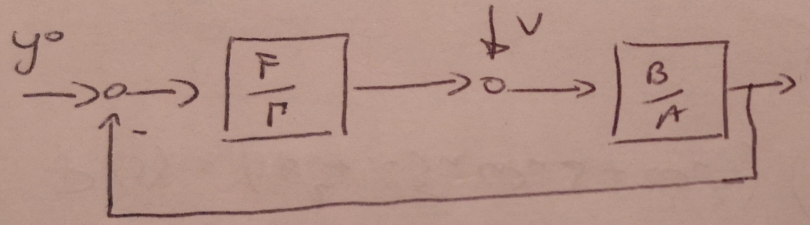


If $P(s) = A'(s)\,P'(s) \rightarrow A(s)\Gamma(s) + B(s)F(s) = P(s)$

$$\underbrace{A'(s)\cdot(s+10)}_{A(s)}\,\Gamma(s) + B(s)F(s) = A'(s)\,P'(s)$$

$$\Downarrow$$

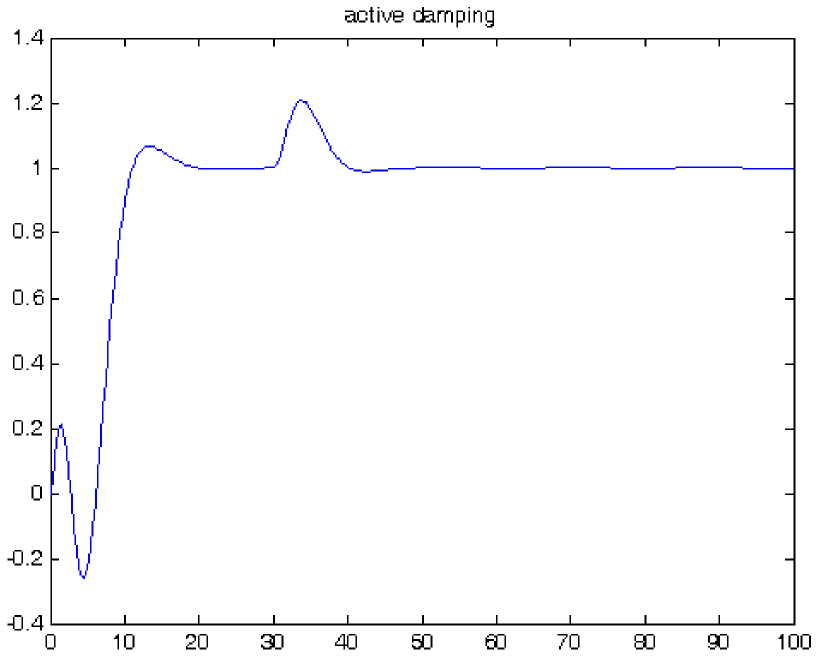
$F(s)$ contains the term $A'(s)$, $F(s) = \bar{f}\,A'(s)$

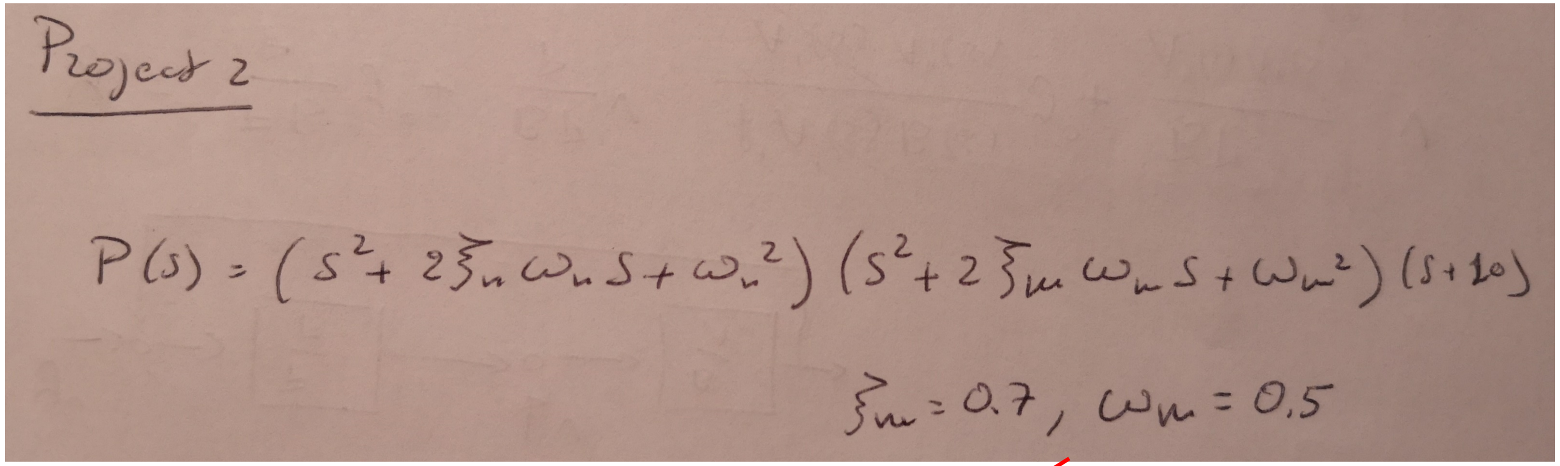


$$y = \frac{FB}{P}y^o + \frac{B\Gamma}{P}v = \frac{f'\cancel{A'(s)}B(s)}{\cancel{A'(s)}\,P'(s)}y^o + \frac{B\Gamma}{A'(s)\,P'(s)}v$$

cancellation no cancellation

Unacceptable «inverse response» due to the zeros of the polynomial ***F(s)***

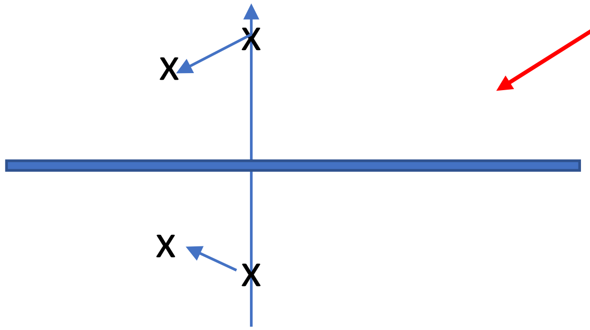




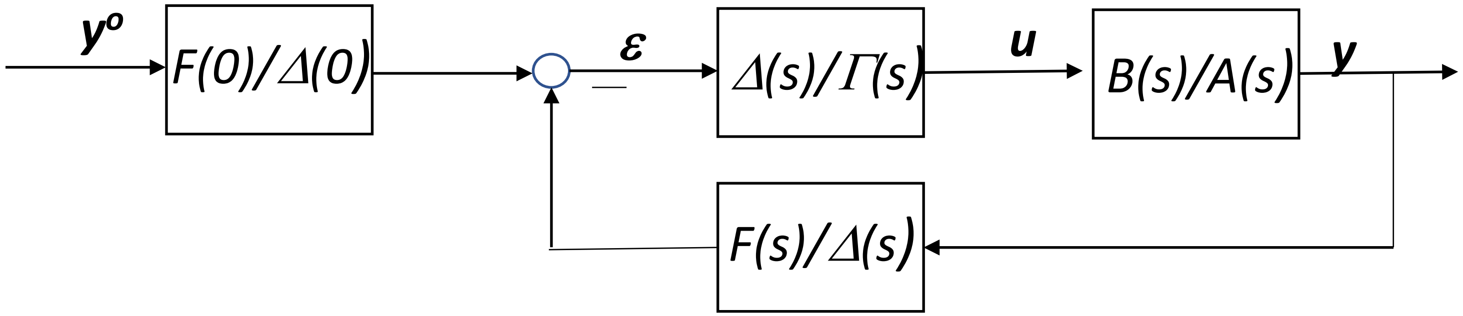
Project 2

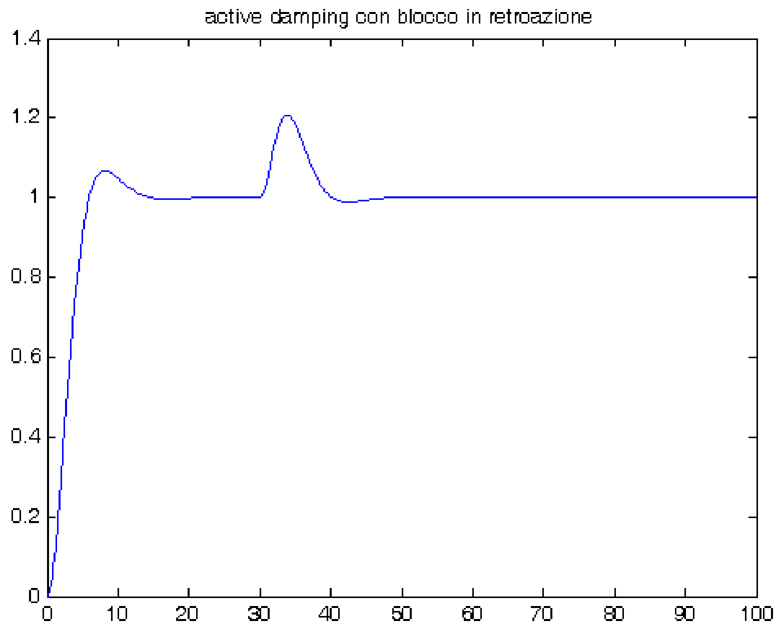
$$P(s) = (s^2 + 2\xi_n \omega_n s + \omega_n^2)(s^2 + 2\xi_m \omega_m s + \omega_m^2)(s+10)$$

$\xi_m = 0.7, \quad \omega_m = 0.5$



Project 2 with the scheme





Excellent result