Topics

- **Introduction and the Camera Optical System**
- Planar (2D) Projective Geometry
- Spatial (3D) Projective Geometry
- Camera Geometry (3D → 2D Projection)

A powerful tool: SYMMETRY

A SYSTEM IS SYMMETRIC WRT A CERTAIN TRANSFORMATION

IF

APPLYING THAT TRANSFORMATION TO THE SYSTEM,

NO MODIFICATION CAN BE MEASURED

EXAMPLES

- EMPTY SPACE: is symmetric under any motion

IF WE ADD AN ELEMENT TO A SYMMETRIC SYSTEM, SYMMETRY CAN POSSIBLY STILL BE ENJOYED, BUT UNDER A RESTRICED SET OF TRANSFORMATIONS

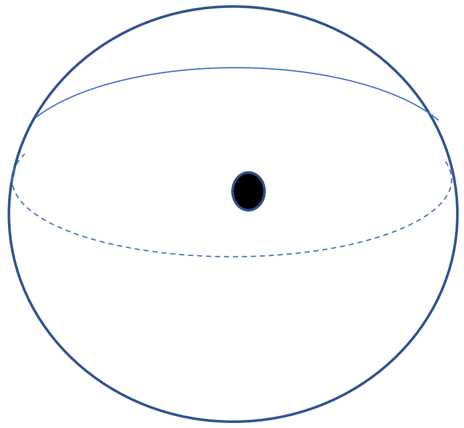
e.g. A SINGLE POINT: is symmetric under any motion preserving the distance from the point → SPHERICAL SYMMETRY

NOTE: the same symmetry is enjoyed by a SPHERE

EXAMPLES

e.g. A SINGLE POINT: is symmetric under any motion preserving the distance from the point $\rightarrow$ SPHERICAL SYMMETRY

NOTE: the same symmetry is enjoyed by a SPHERE



EXAMPLES

LET’S ADD AN ELEMENT TO THE SYSTEM: a second point

EXAMPLES

LET’S ADD AN ELEMENT TO THE SYSTEM: a second point

Let us call AXIS he straight line joining the two points

EXAMPLES

A TWO POINT system: is symmetric under any motion along a circumference orthogonal to the axis, centered on it

→ AXIAL SYMMETRY

EXAMPLES

A TWO POINT system: is symmetric under any motion along a circumference orthogonal to the axis, centered on it

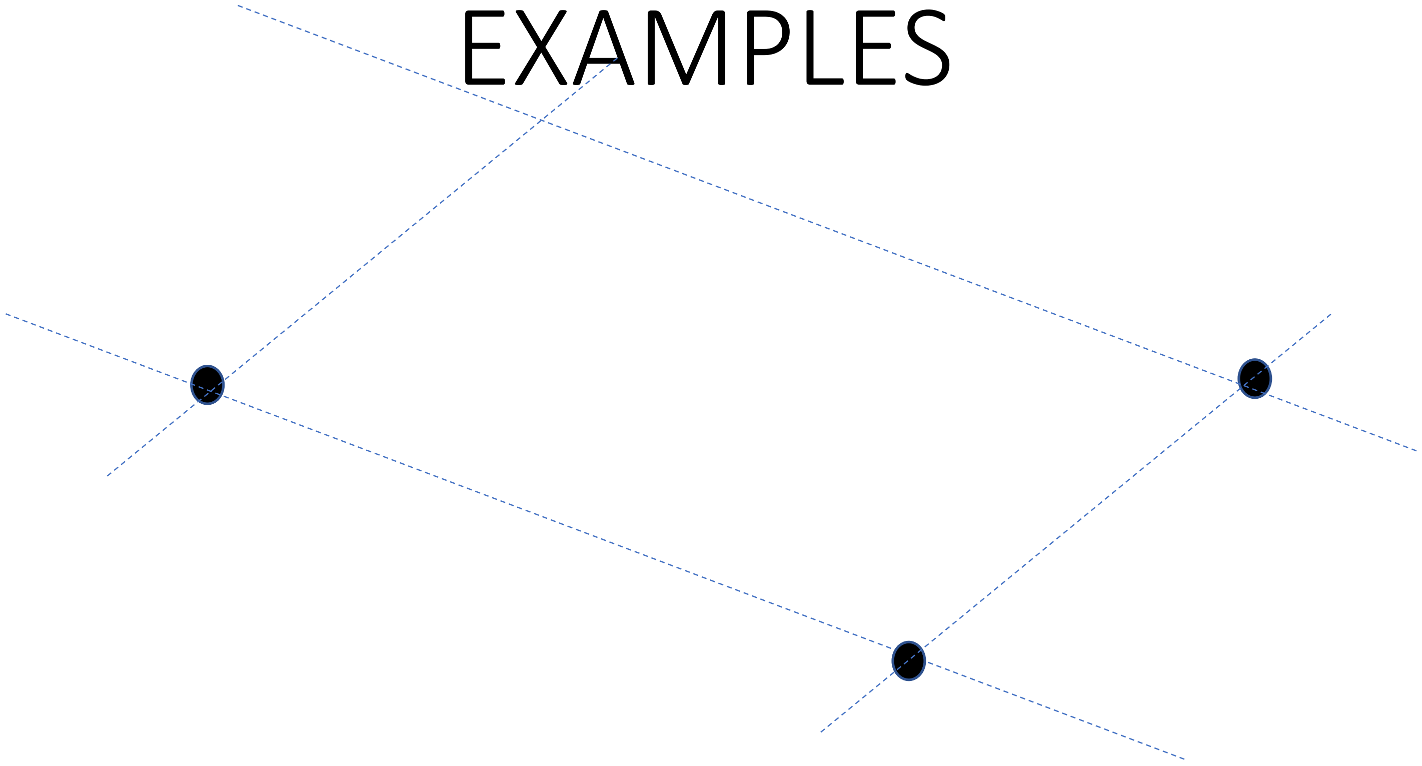
→AXIAL SYMMETRY

NOTE: the same symmetry is enjoyed by a TWO SHPERE system

EXAMPLES

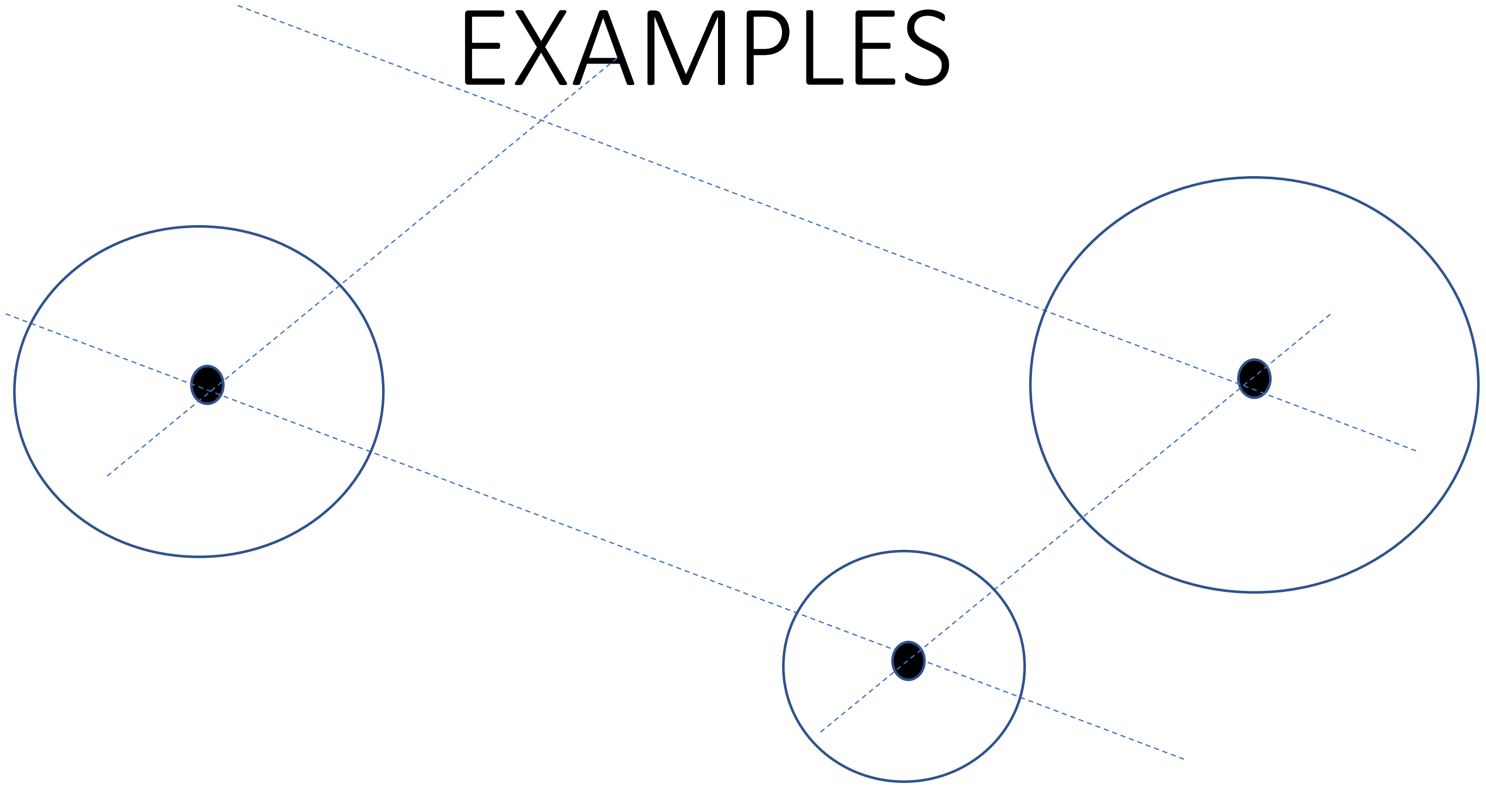
LET’S ADD ONE MORE ELEMENT TO THE SYSTEM: a third point

EXAMPLES



A THREE POINT system: is symmetric under «mirroring» wrt the plane passing through the three points → PLANAR SYMMETRY

EXAMPLES



A THREE POINT system: is symmetric under «mirroring» wrt the plane passing through the three points → PLANAR SYMMETRY

NOTE: the same symmetry is enjoyed by a THREE SHPERE system

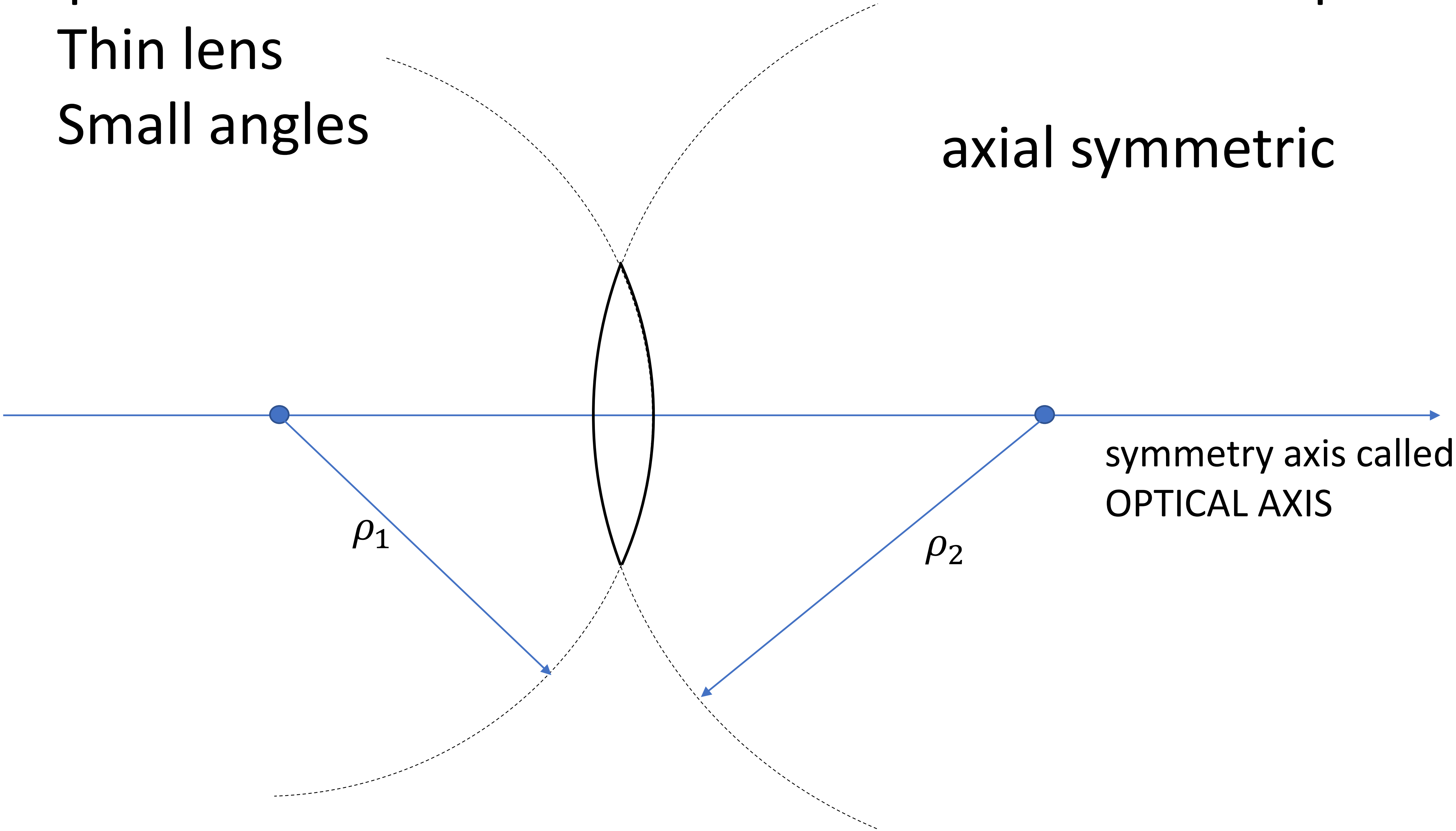
OPTICAL SENSOR: CAMERA

- Screen with (order of 10^6 or more) photosensitive elements called *pixels* (PIXEL = PICture Element)
  - Optical → electric transducers
- **Optical system** to select direction of incoming light at each element
- Electric circuits that collect the signal generated at the pixels

- 30-60 frames per second (even more in some cameras) simplified camera model: lens = intersection of two spheres

- Thin lens
- Small angles

axial symmetric

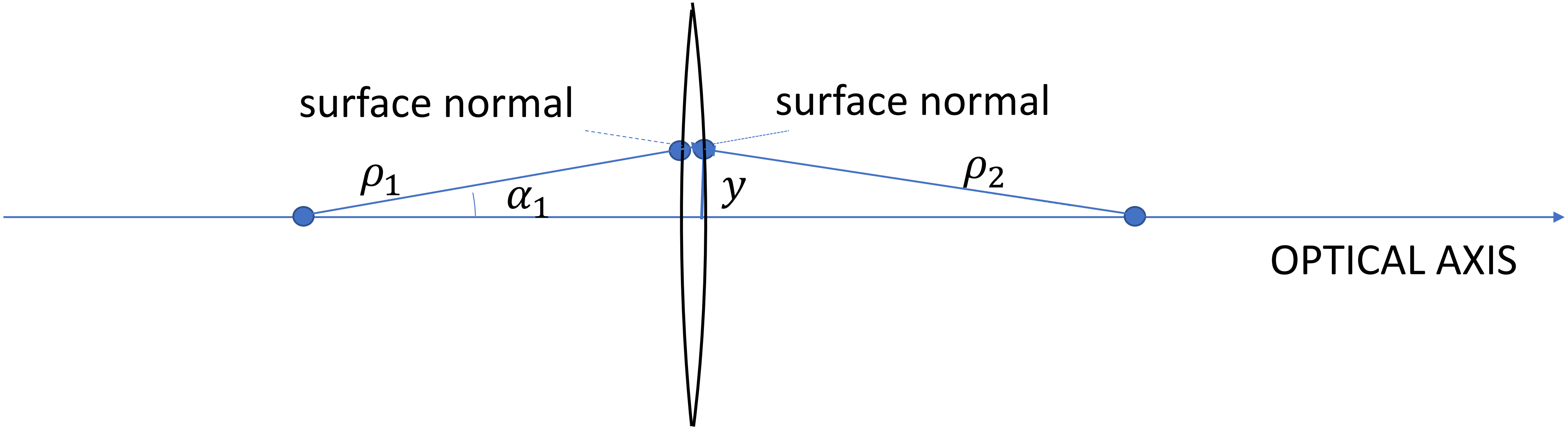


simplified camera model: lens = intersection of two spheres

- **Thin lens**
- **Small angles**

Hp: small angles → $\alpha_1 = y_1/\rho_1$

and also: $\alpha_2 = -y_2/\rho_2$

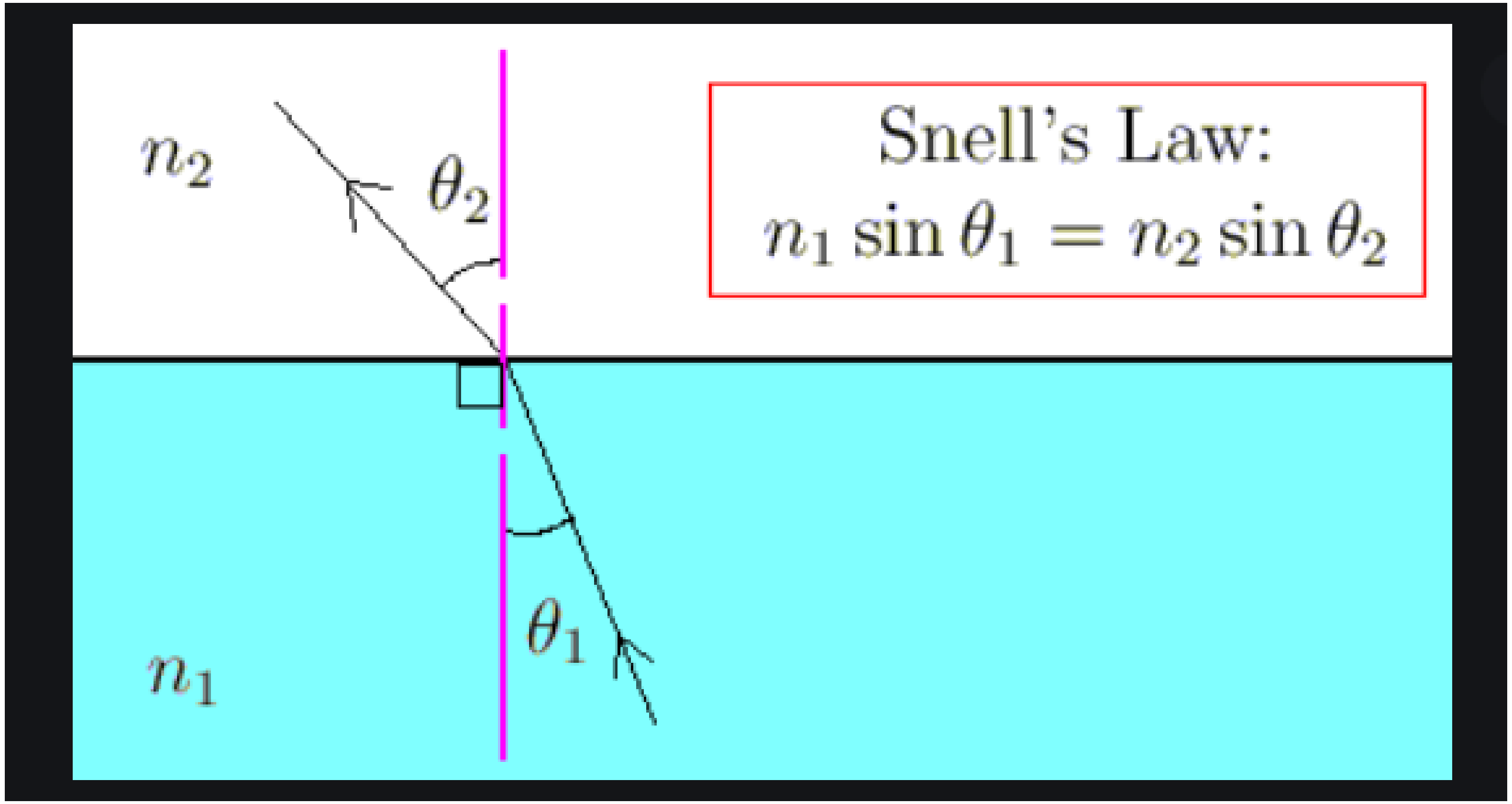


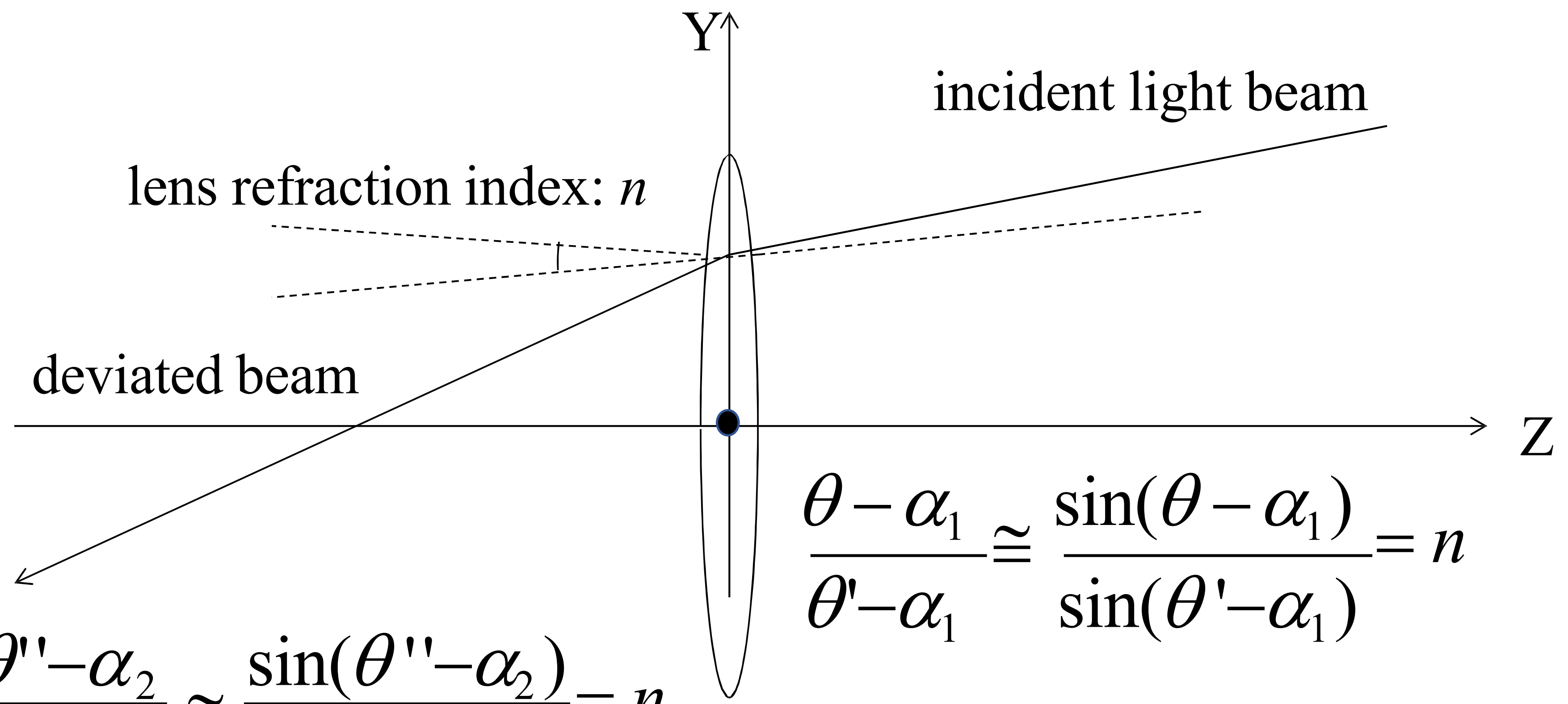
Light ray: 1. air-glass refraction at y_1 2. glass-air refraction at y_2

Hp: thin lens → very short light path within the lens → $y_1 = y_2 = y$

The deviation of a light ray crossing the lens

phenomenon: **refraction**



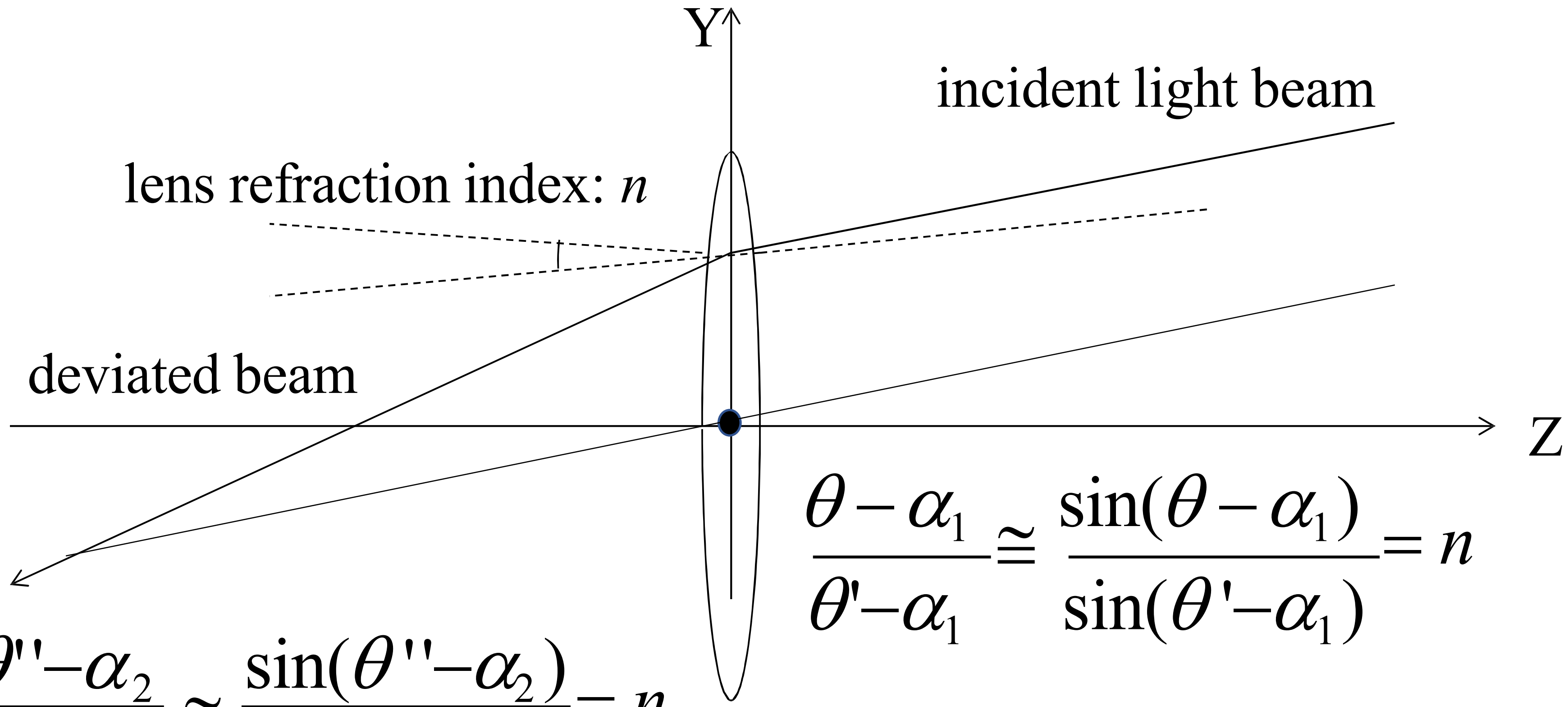


$$\frac{\theta - \alpha_1}{\theta' - \alpha_1} \cong \frac{\sin(\theta - \alpha_1)}{\sin(\theta' - \alpha_1)} = n$$

$$\frac{\theta'' - \alpha_2}{\theta' - \alpha_2} \cong \frac{\sin(\theta'' - \alpha_2)}{\sin(\theta' - \alpha_2)} = n$$

deviation angle ? $\delta\theta = \theta'' - \theta$

$$\delta\theta = (n - 1)Y\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$



$$\frac{\theta-\alpha_1}{\theta'-\alpha_1} \cong \frac{\sin(\theta-\alpha_1)}{\sin(\theta'-\alpha_1)} = n$$

$$\frac{\theta''-\alpha_2}{\theta'-\alpha_2} \cong \frac{\sin(\theta''-\alpha_2)}{\sin(\theta'-\alpha_2)} = n$$

deviation angle ? $\delta\theta = \theta'' - \theta$

$$\delta\theta = (n-1)Y\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

$Y = 0 \rightarrow \delta\theta = 0$ undeviated ray

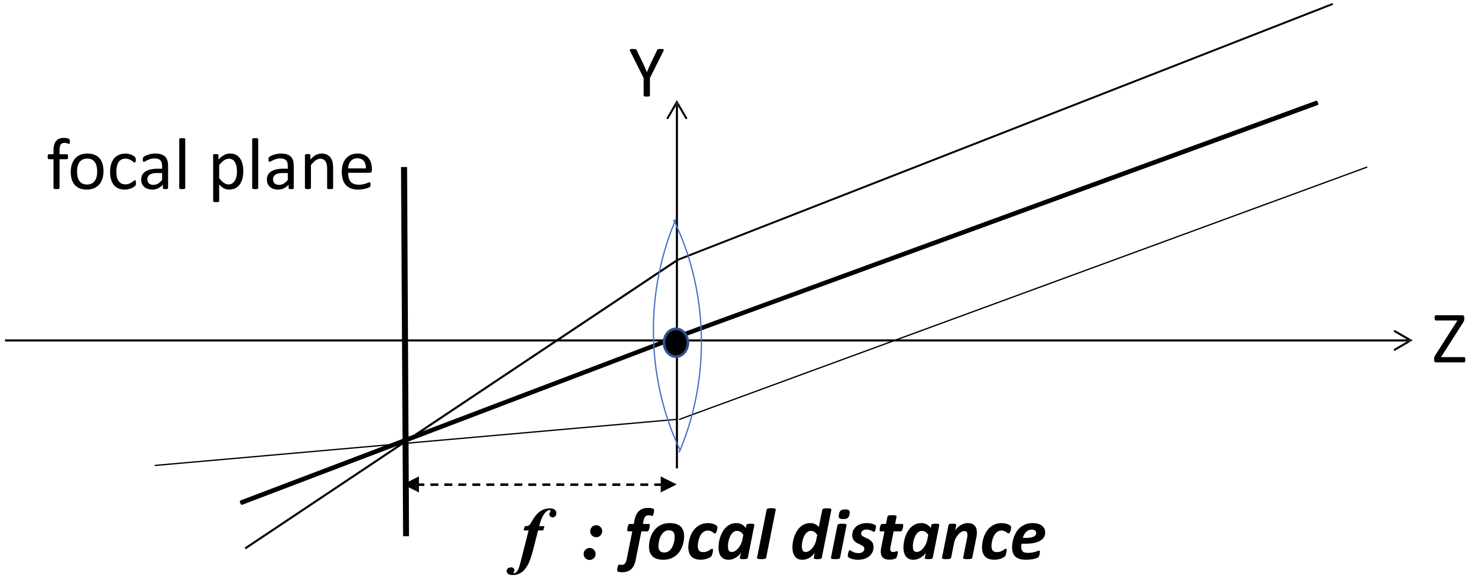
Focalization (convergence) of parallel rays

Focalization of parallel rays

$$f = \frac{1}{(n - 1)(\frac{1}{\rho_1} + \frac{1}{\rho_2})}$$

ALL PARALLEL RAYS CROSSING THE LENS

concur at a common point on the a *focal* plane $Z = -f$

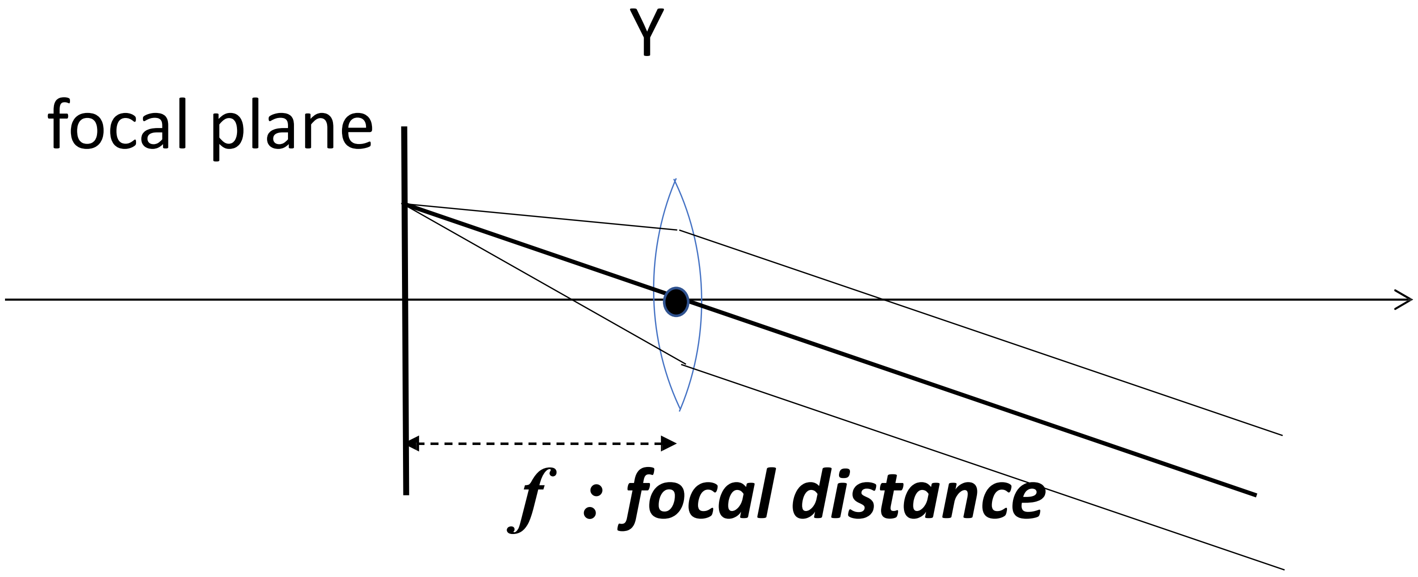


Focalization of parallel rays: same plane for any direction

$$f = \frac{1}{(n-1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}$$

ALL PARALLEL RAYS CROSSING THE LENS

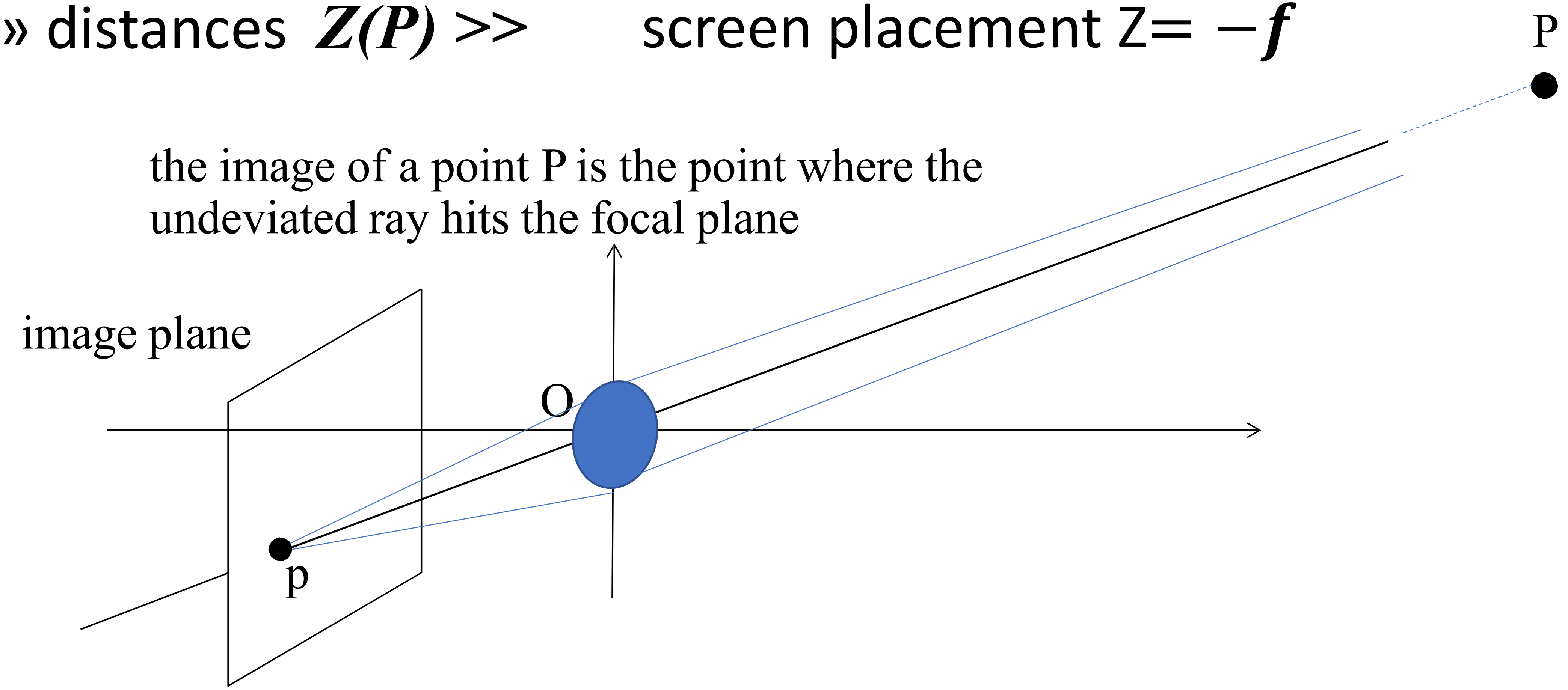
concur at a common point on the a *focal* plane $Z = -f$



Place the screen where parallel rays converge

Hp: «large» distances ***Z(P) >> aperture*** screen placement Z= $-\boldsymbol{f}$

the image of a point P is the point where the undeviated ray hits the focal plane



p = image of P is on the line(P,O) and on the image plane

p = image of P = image plane ∩ line(O,P)

Model: the pin-hole camera model

Our simplified model:

(i) thin spherical lens, (ii) *small angles*, (iii) $\boldsymbol{Z(P) \gg a}$, (iv) $Z = \boldsymbol{f}$

→ **PIN-HOLE CAMERA**

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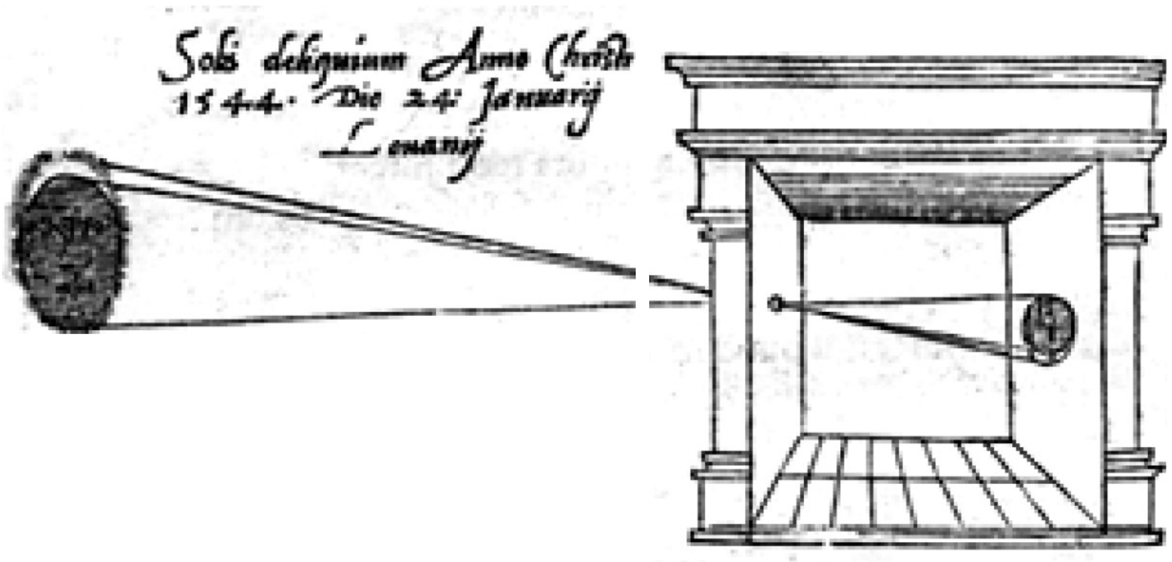

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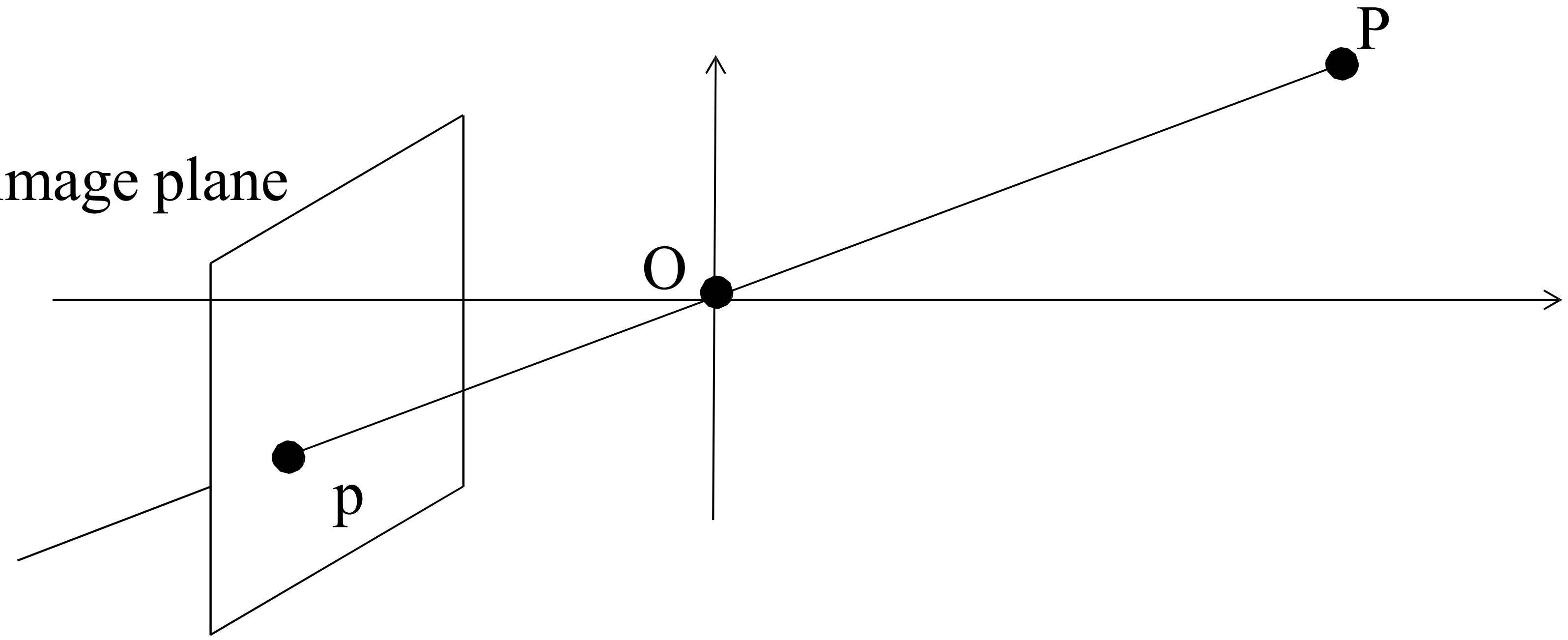
The image of a point P in the scene

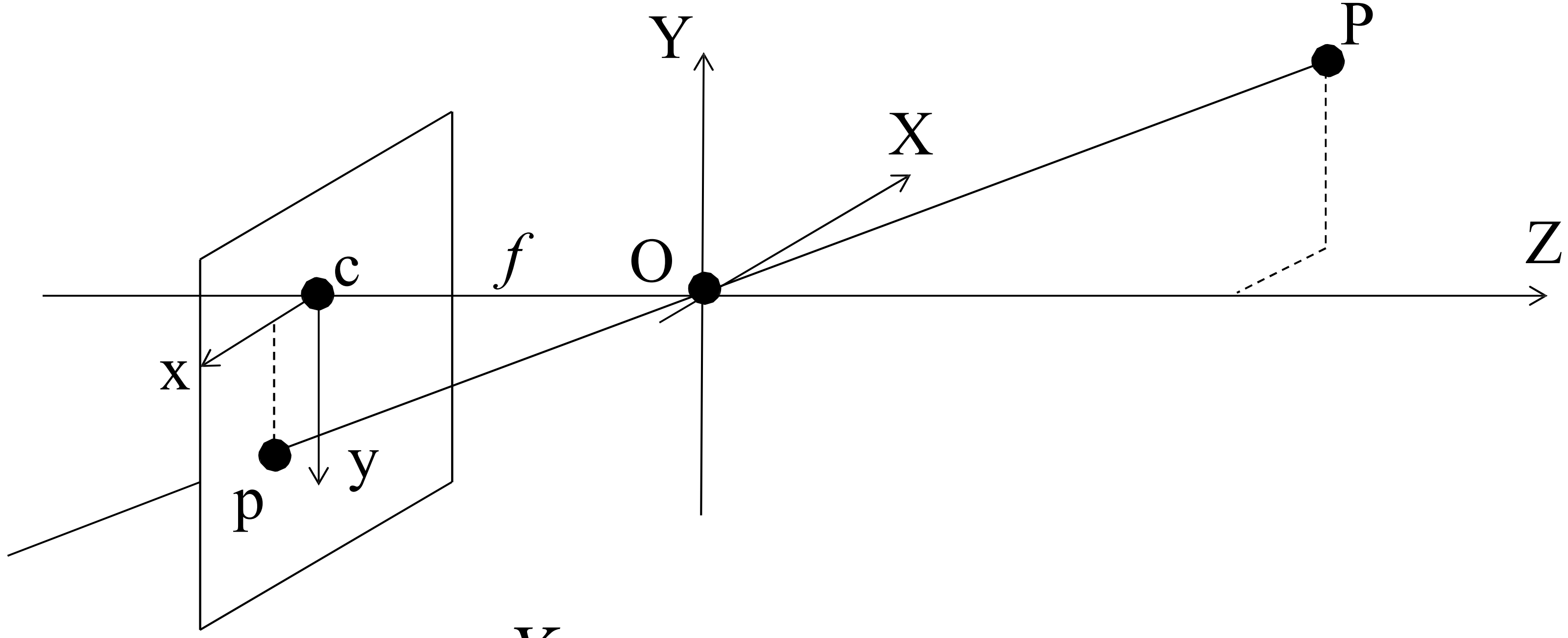
Hp: «large» distances $\boldsymbol{Z(P) >> f}$ screen placement $Z = -\boldsymbol{f}$

the image of a point P belongs to the line (P,O)



p = image of P = is on the image plane and on the line(O,P) → p = image plane ∩ line(O,P)

viewing ray of p: line(O,p) =
locus of the scene points projecting onto image point p



$$x = f\,\frac{X}{Z}$$

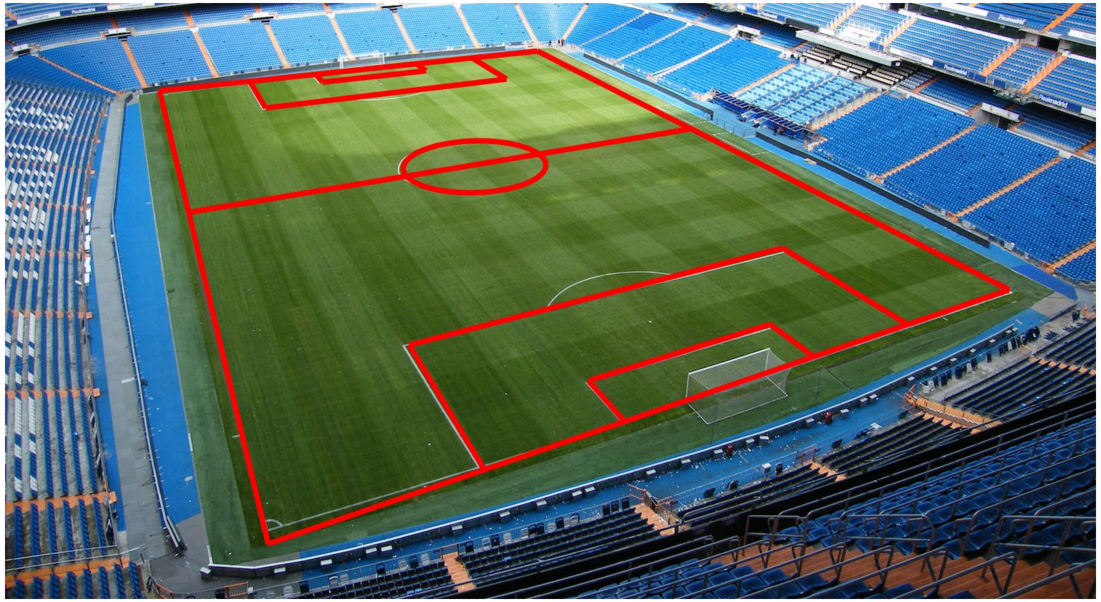
$$y = f\,\frac{Y}{Z}$$

central projection

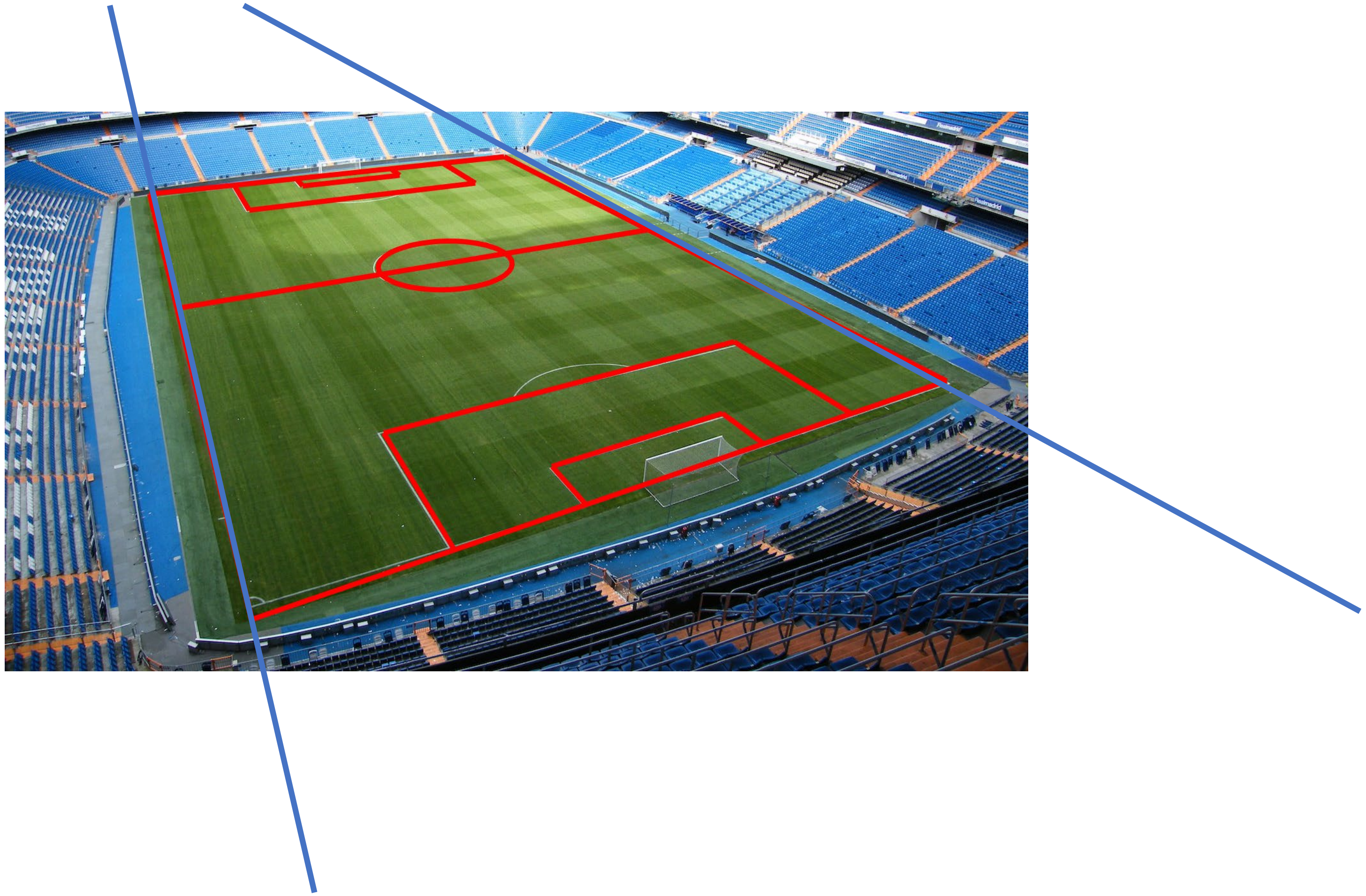
-nonlinear
-not shape-preserving
-not length-ratio preserving

Bad news:
what you see IS NOT what you get

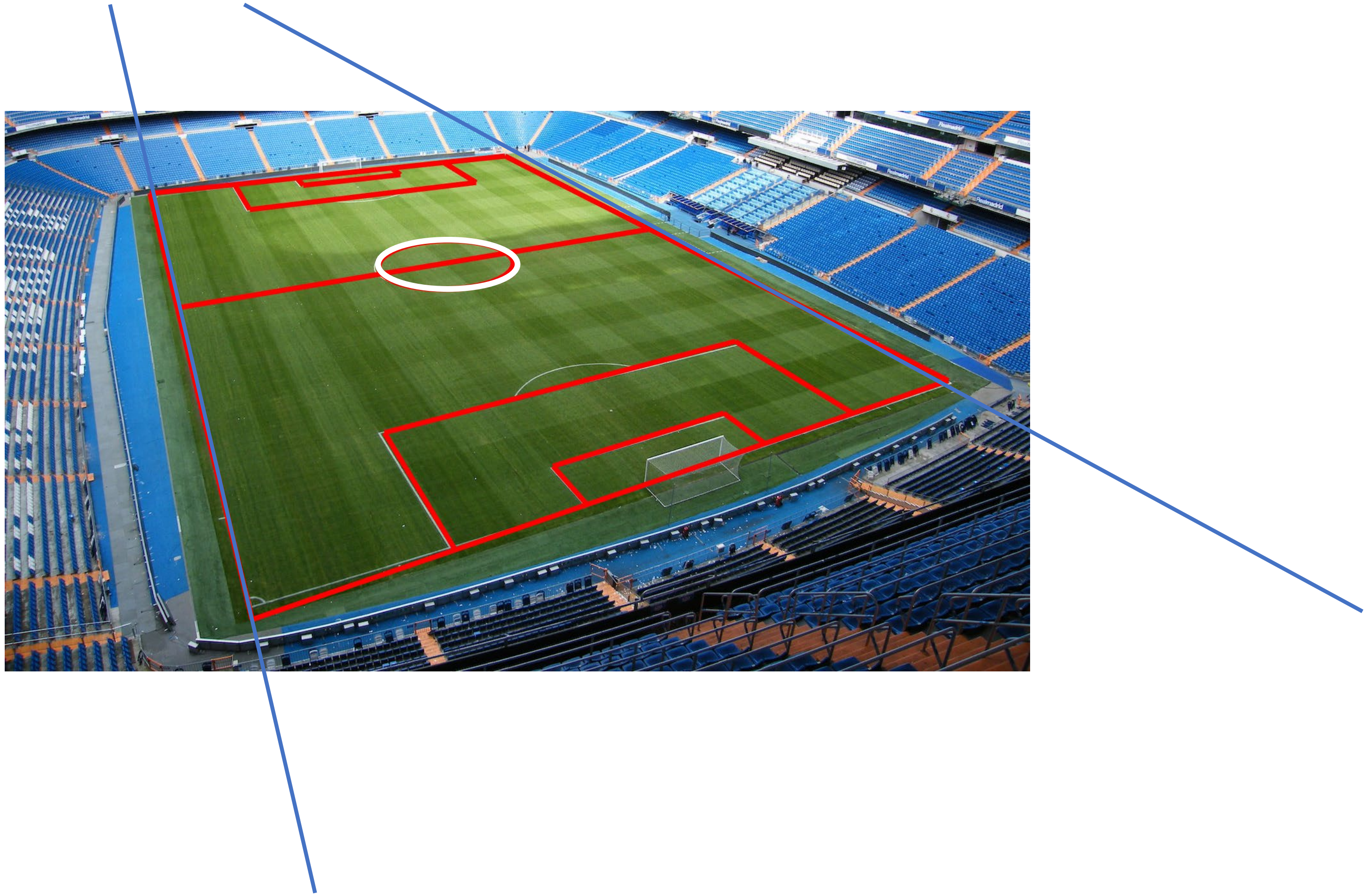
what you see **IS NOT** what you get



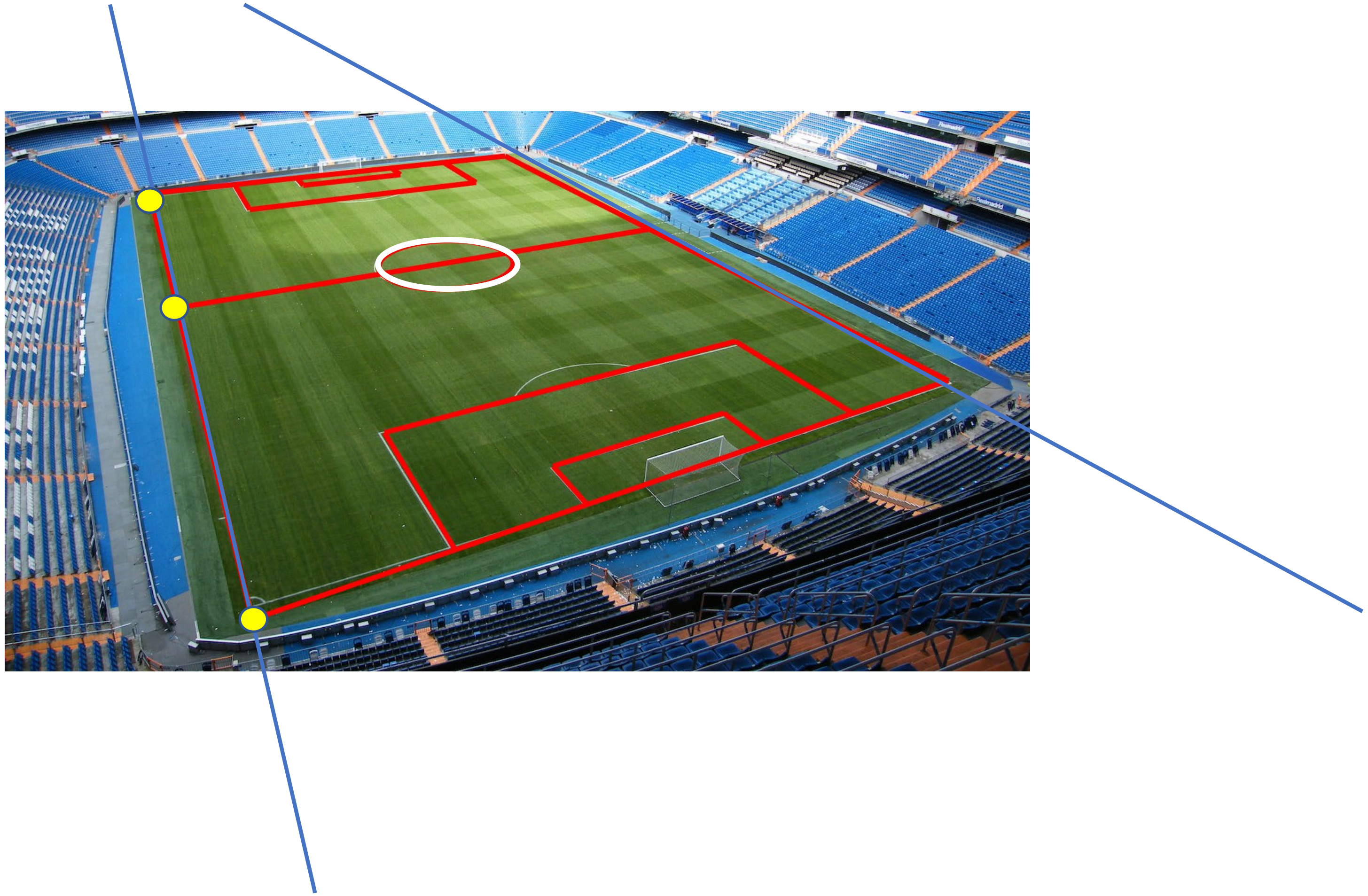
Parallel lines **DO NOT PROJECT ONTO** parallel lines



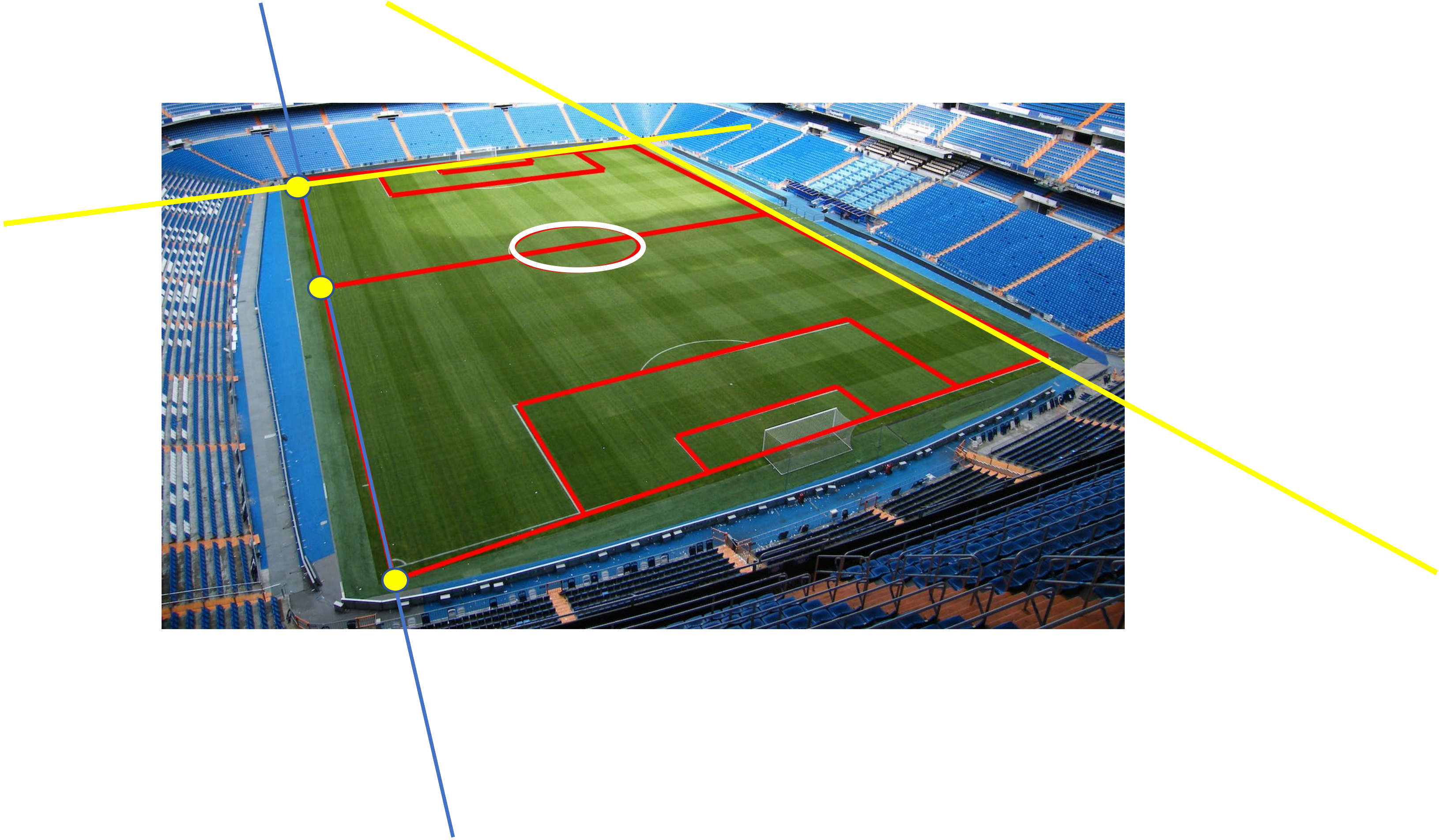
circumpherences **DO NOT PROJECT ONTO** circumferences



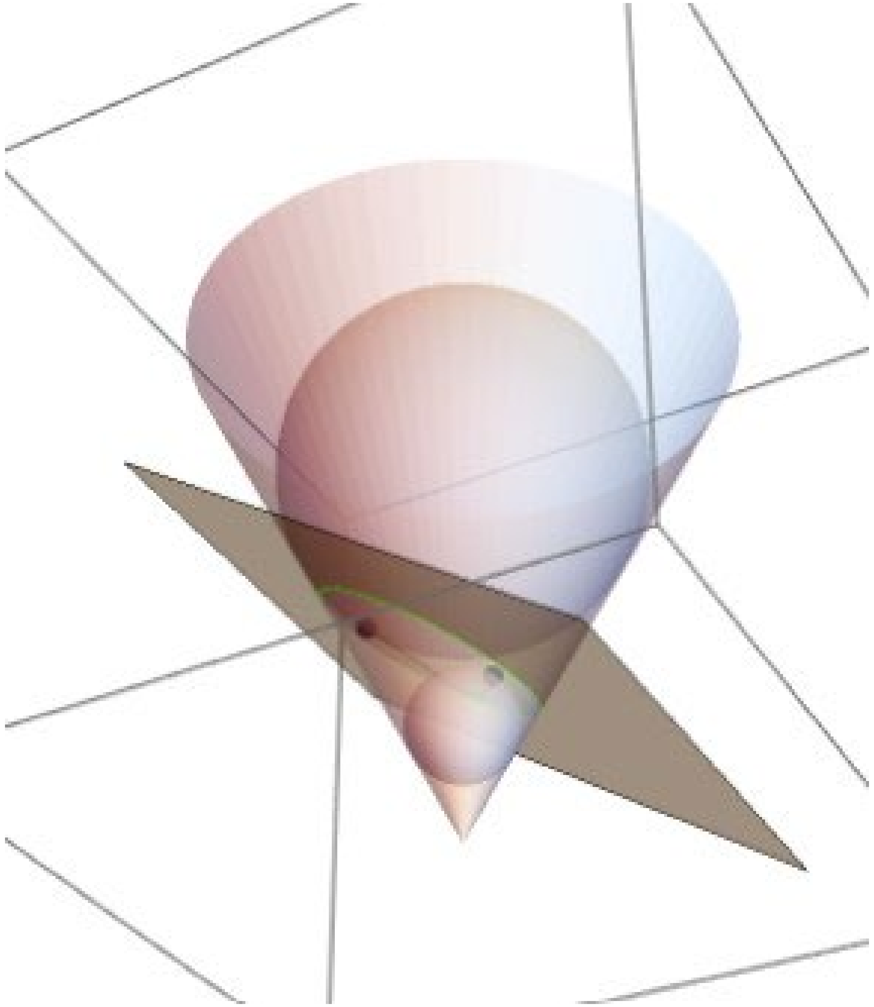
midpoints **DO NOT PROJECT ONTO** midpoints



Orthogonal lines **DO NOT PROJECT ONTO** orthogonal lines

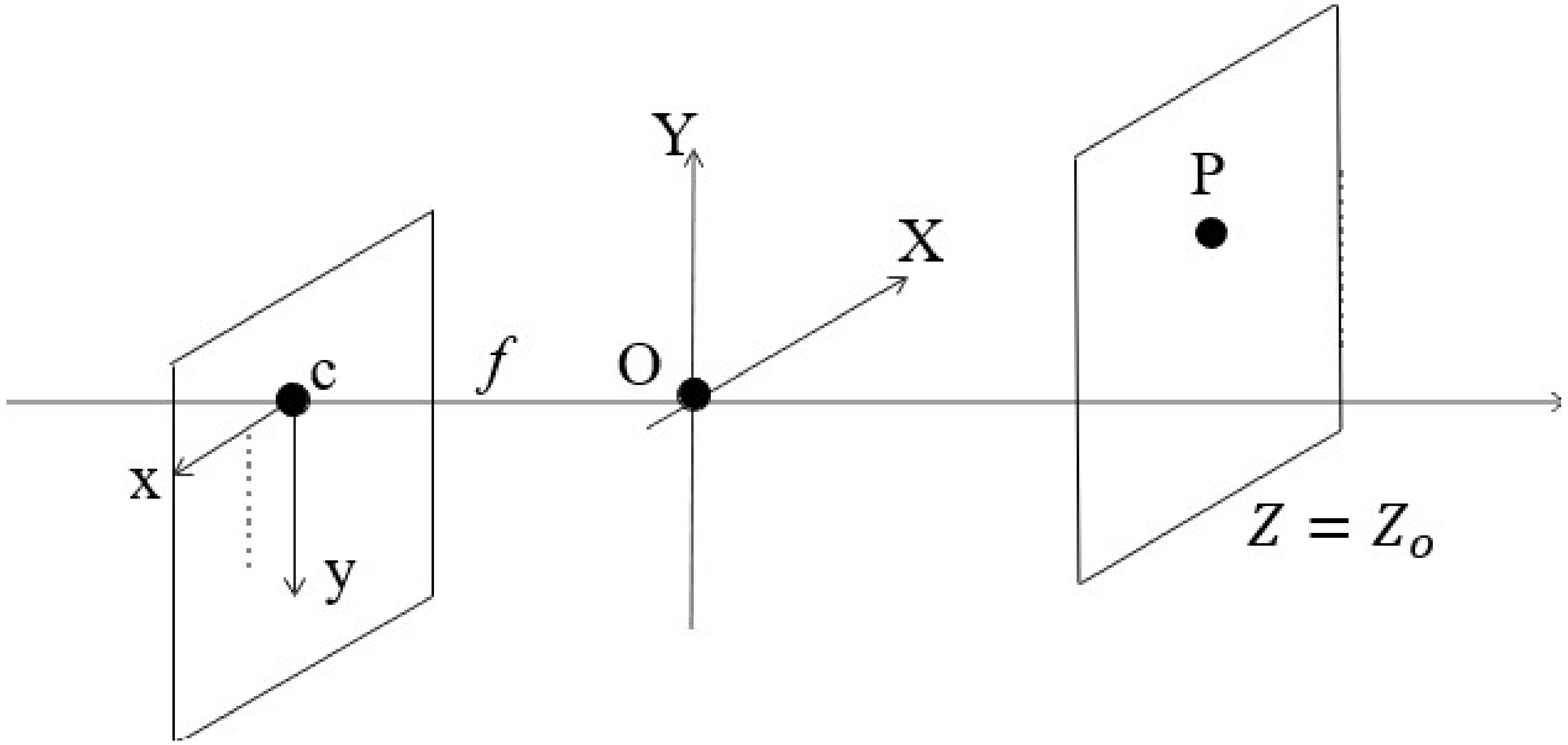


spheres **DO NOT PROJECT ONTO** circles



Exception:
planar scene **parallel** to image plane

$$Z = Z_o = constant$$



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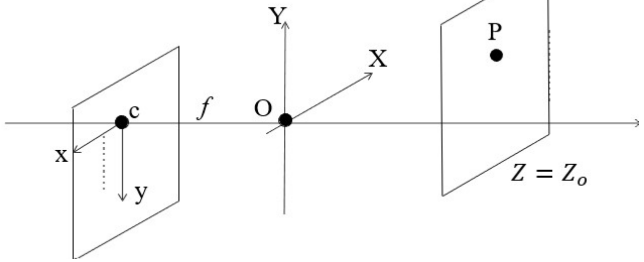
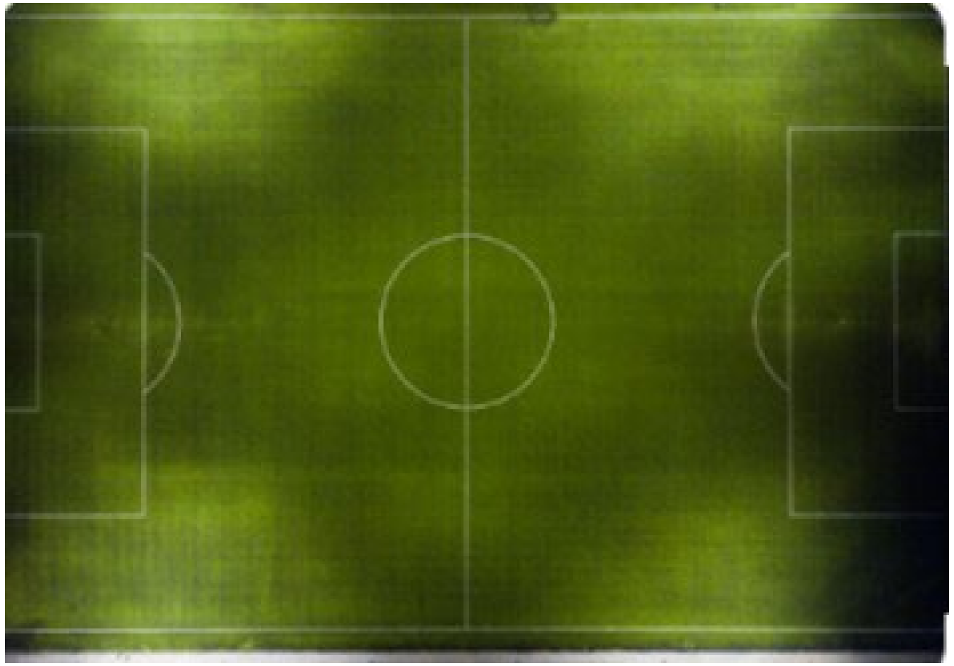


$$x = f\frac{X}{Z_o} = kX$$

$$y = f\frac{Y}{Z_o} = kY$$



The image is a **scaled** version of the scene:

→ **same shape**

↓

angles preserved

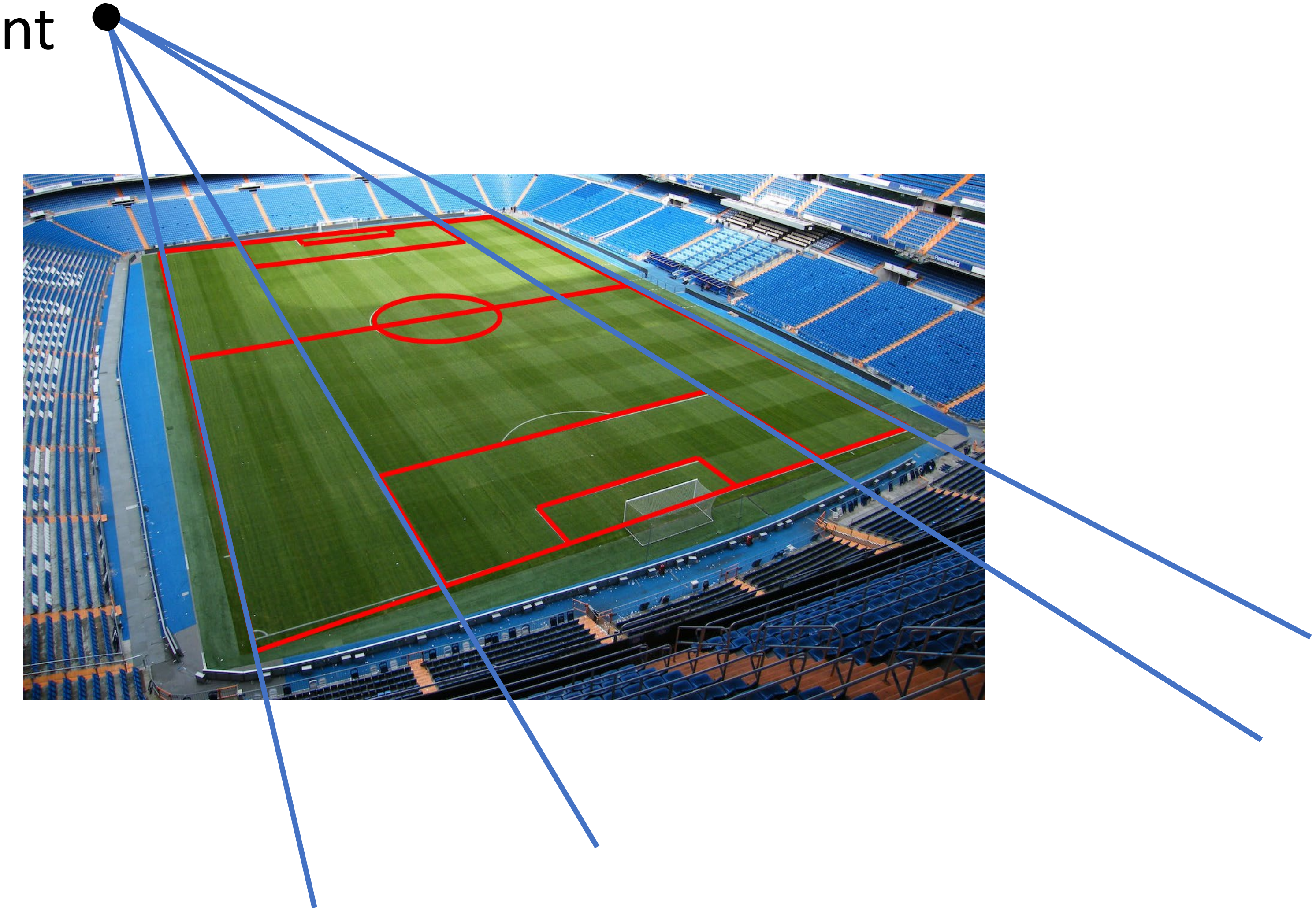
TO UNDERSTAND WHAT WE GET FROM WHAT WE SEE,
WE NEED TO STUDY THE GEOMETRY OF CENTRAL PROJECTION

Preview of Geometry: useful properties

1. The vanishing point of a direction

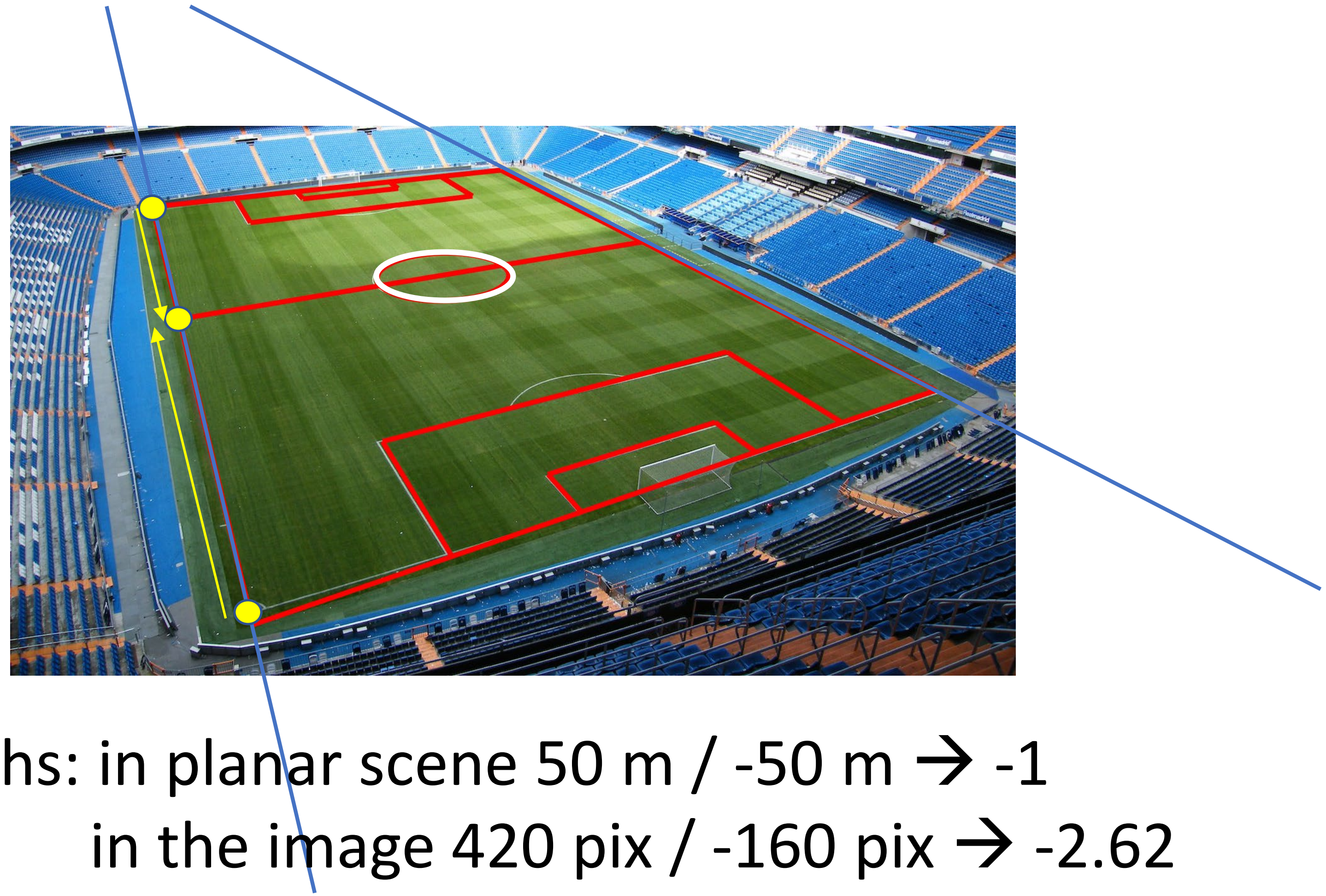
Parallel lines project onto **concurrent** lines

vanishing point
of a direction



Preview of Geometry:
useful properties
2. The cross ratio of four colinear points

ratio of lengths: NOT INVARIANT

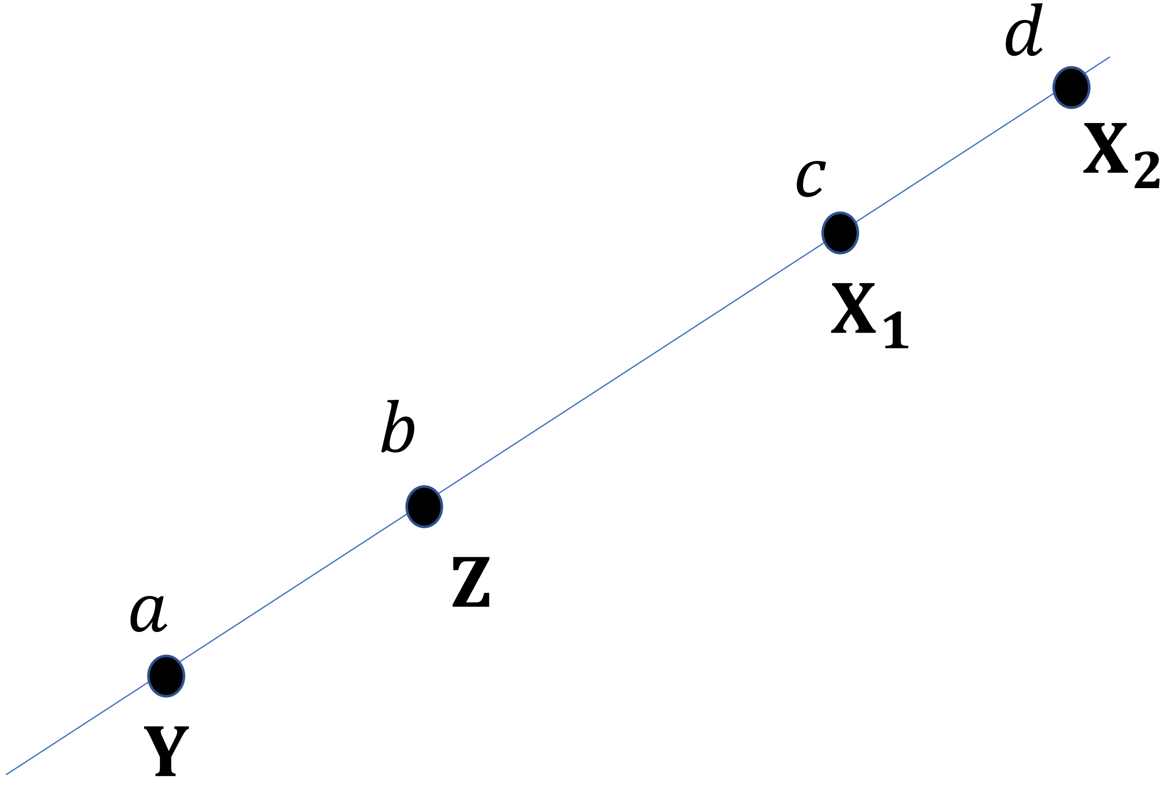


ratio of lengths: in planar scene 50 m / -50 m → -1

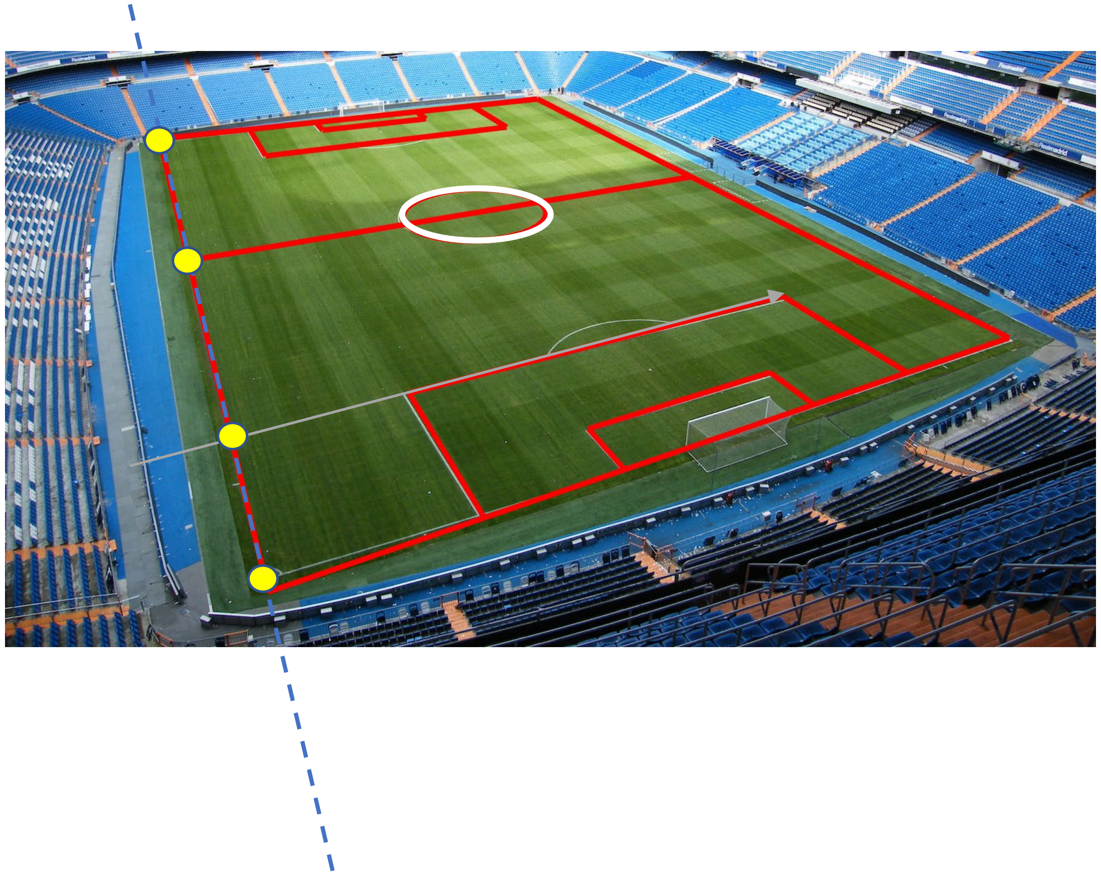
in the image 420 pix / -160 pix → -2.62

Cross ratio of a 4-tuple of colinear points

$$\boldsymbol{CR}_{Y,Z,\mathbf{X}_1,\mathbf{X}_2} = \frac{\boldsymbol{c}-\boldsymbol{a}}{\boldsymbol{c}-\boldsymbol{b}} \bigg/ \frac{\boldsymbol{d}-\boldsymbol{a}}{\boldsymbol{d}-\boldsymbol{b}}$$



the cross ratio, i.e., the ratio between the two ratios, INVARIANT: it is preserved from real word to image



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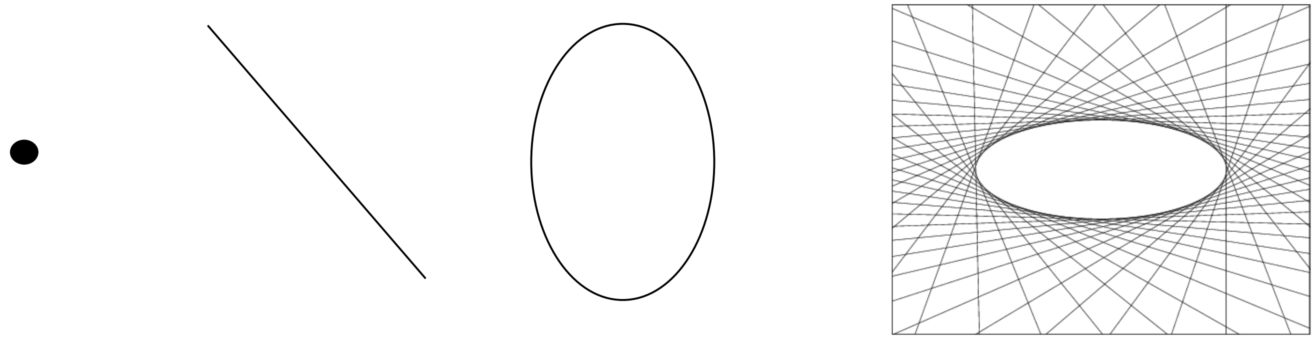
- **Planar (2D) Projective Geometry**
- Spatial (3D) Projective Geometry
- Camera Geometry (3D → 2D Projection)

Planar (2D) Projective Geometry

Planar Projective Geometry

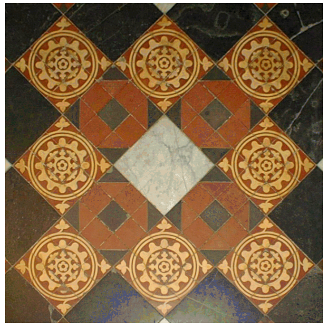
- **Elements**

- **Points**
- Lines
- Conics
- Dual conics



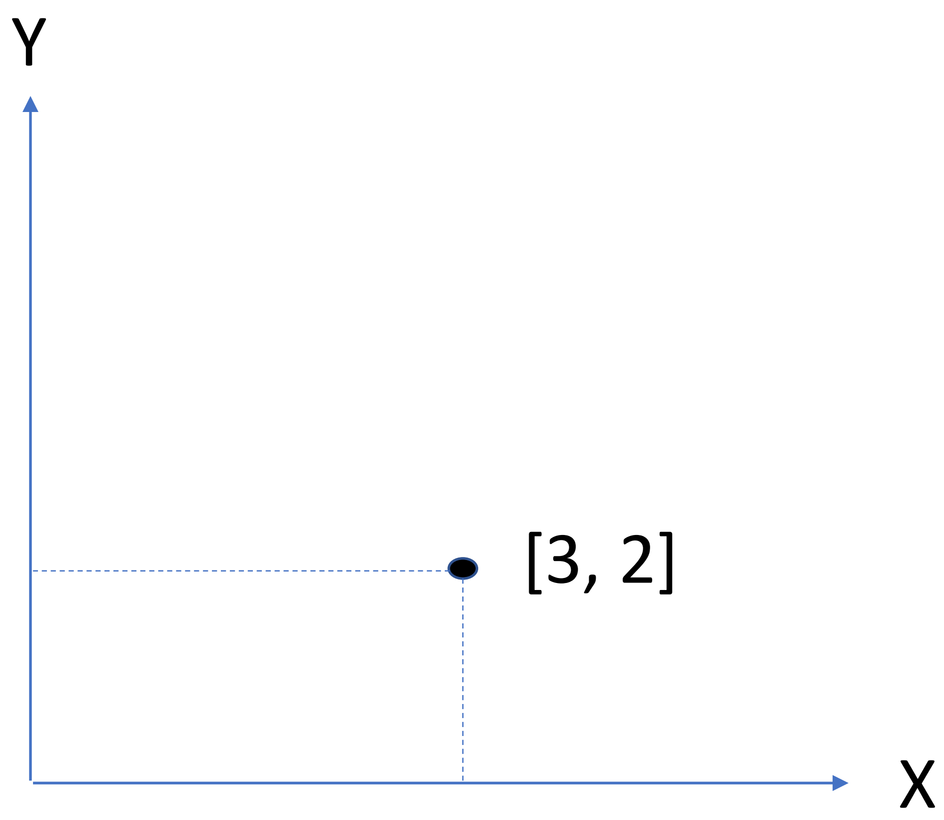
- **Transformations**

- Isometries
- Similarities
- Affinities
- Projectivities



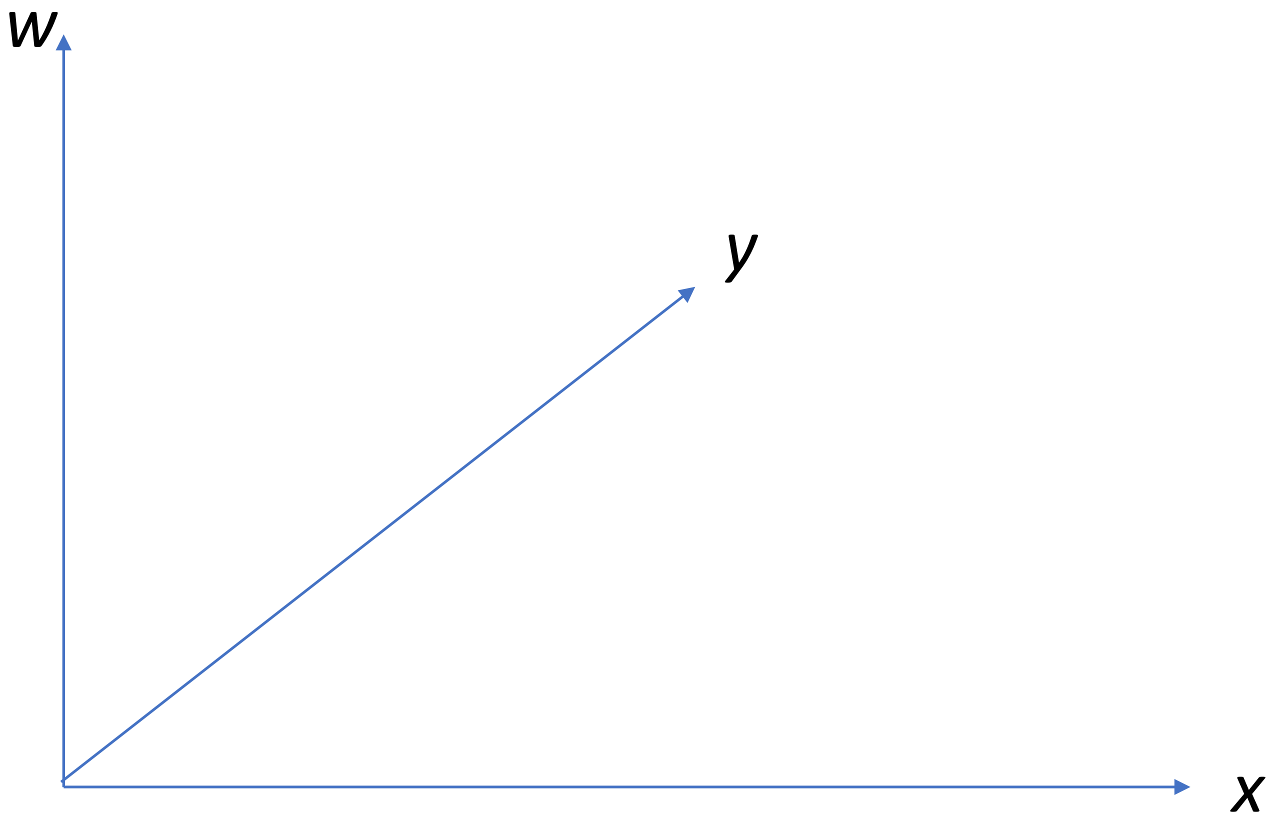
Points in 2D Projective Geometry

Euclidean plane – cartesian coordinates

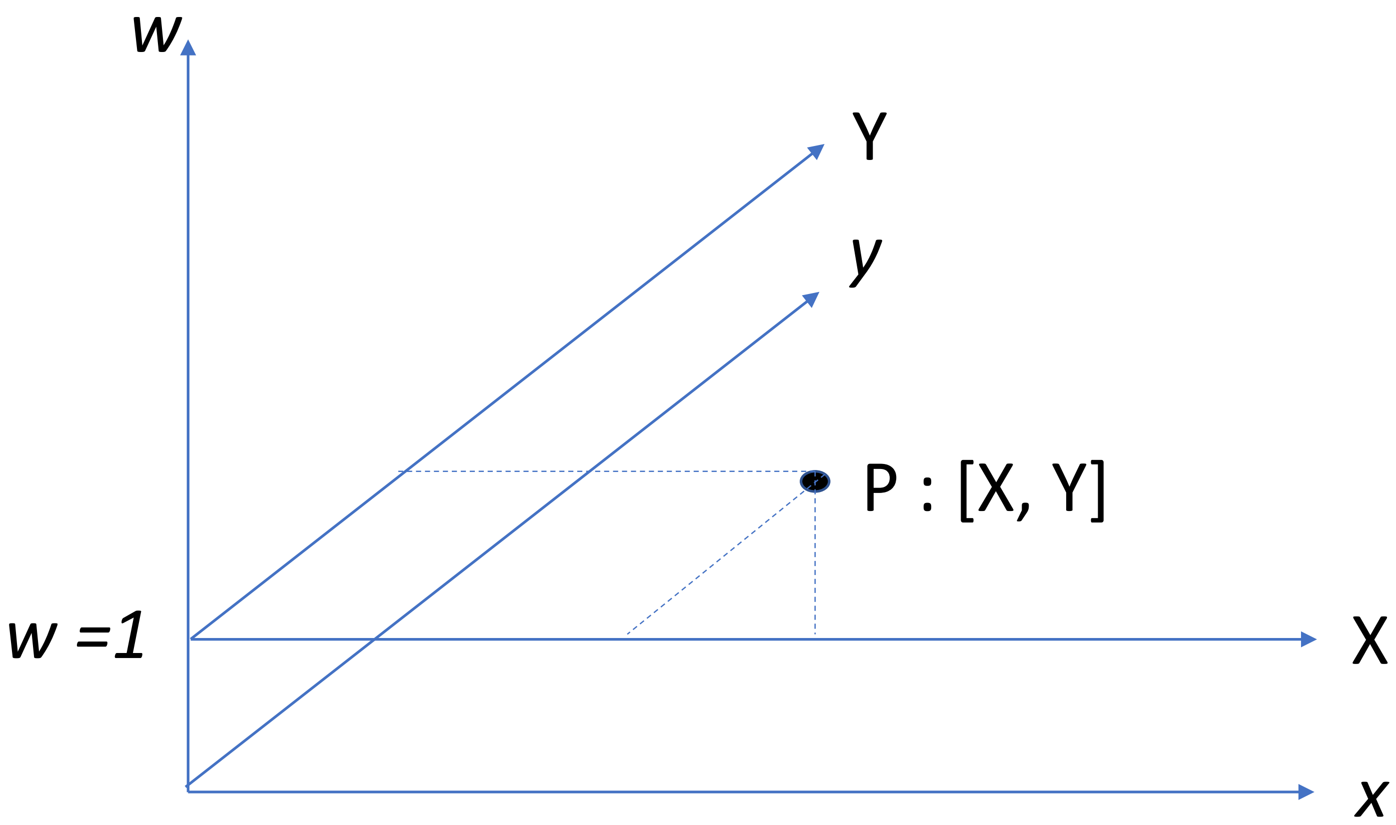


Homogeneous coordinates

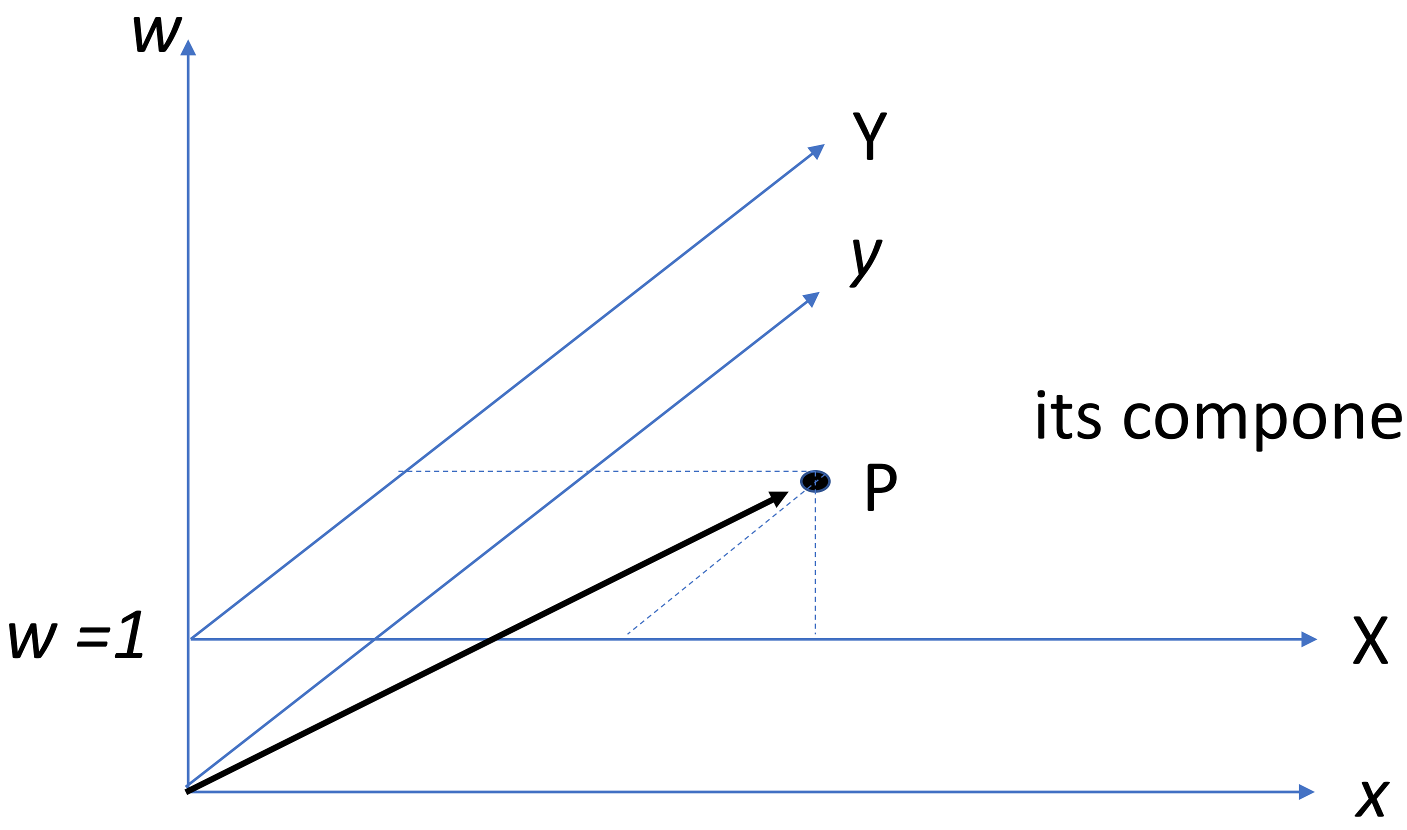
Consider a (3D) space of coordinate vectors



Embed the Euclidean plane into the (3D) space of coordinate vectors as the plane $w = 1$

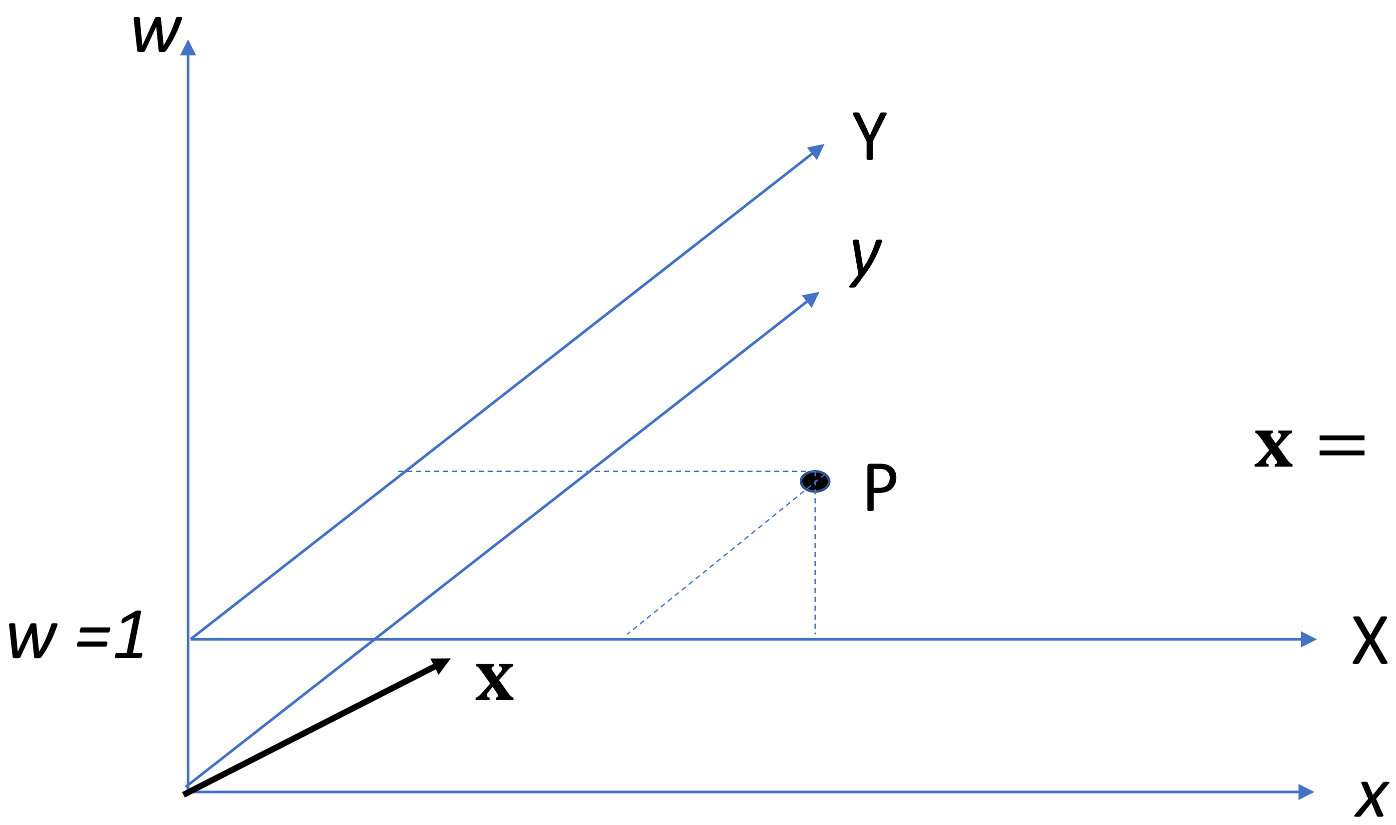


Consider the vector from the origin of the space to the point P



its components are $\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$

the point P is represented by any vector **x** , that is a nonzero multiple of it



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

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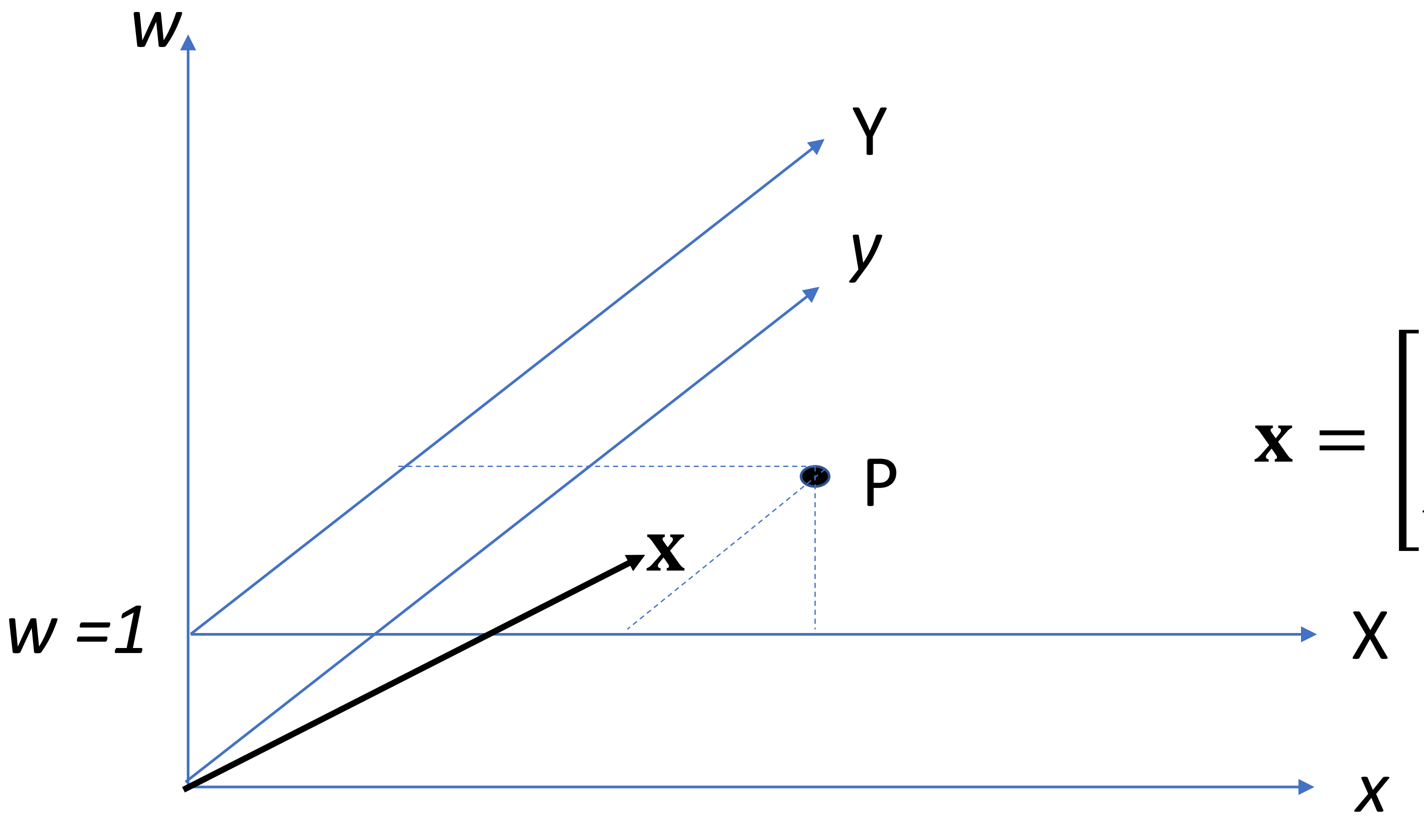

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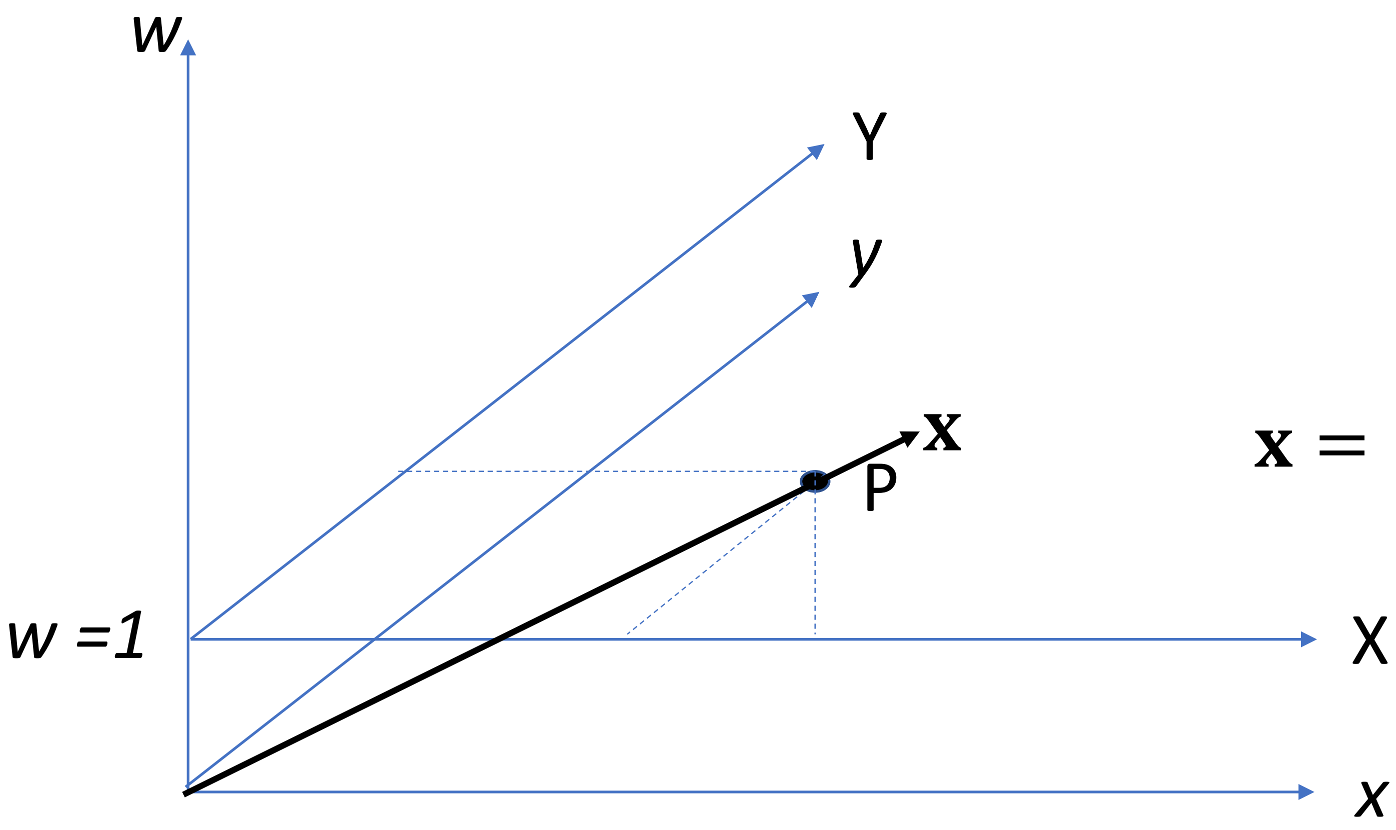

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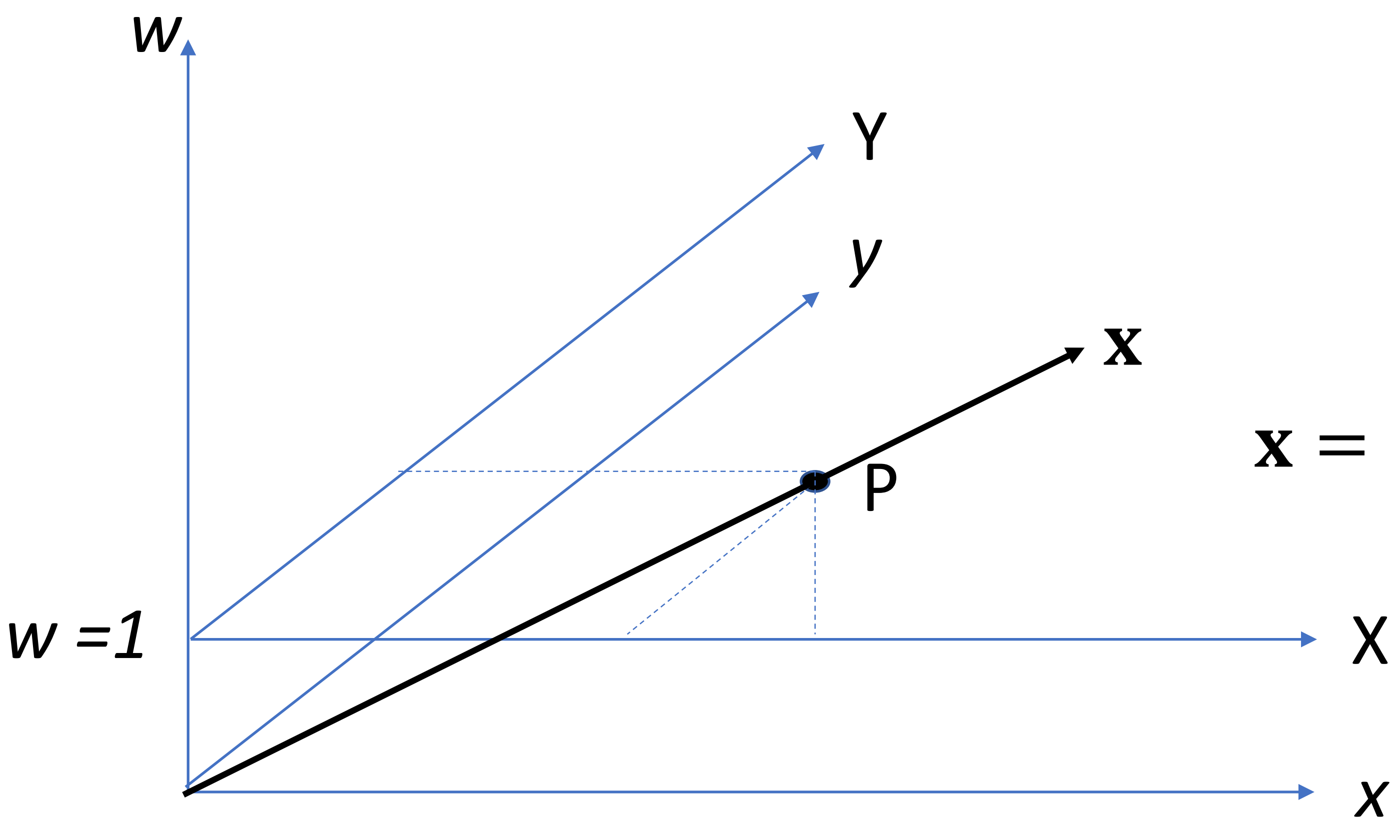

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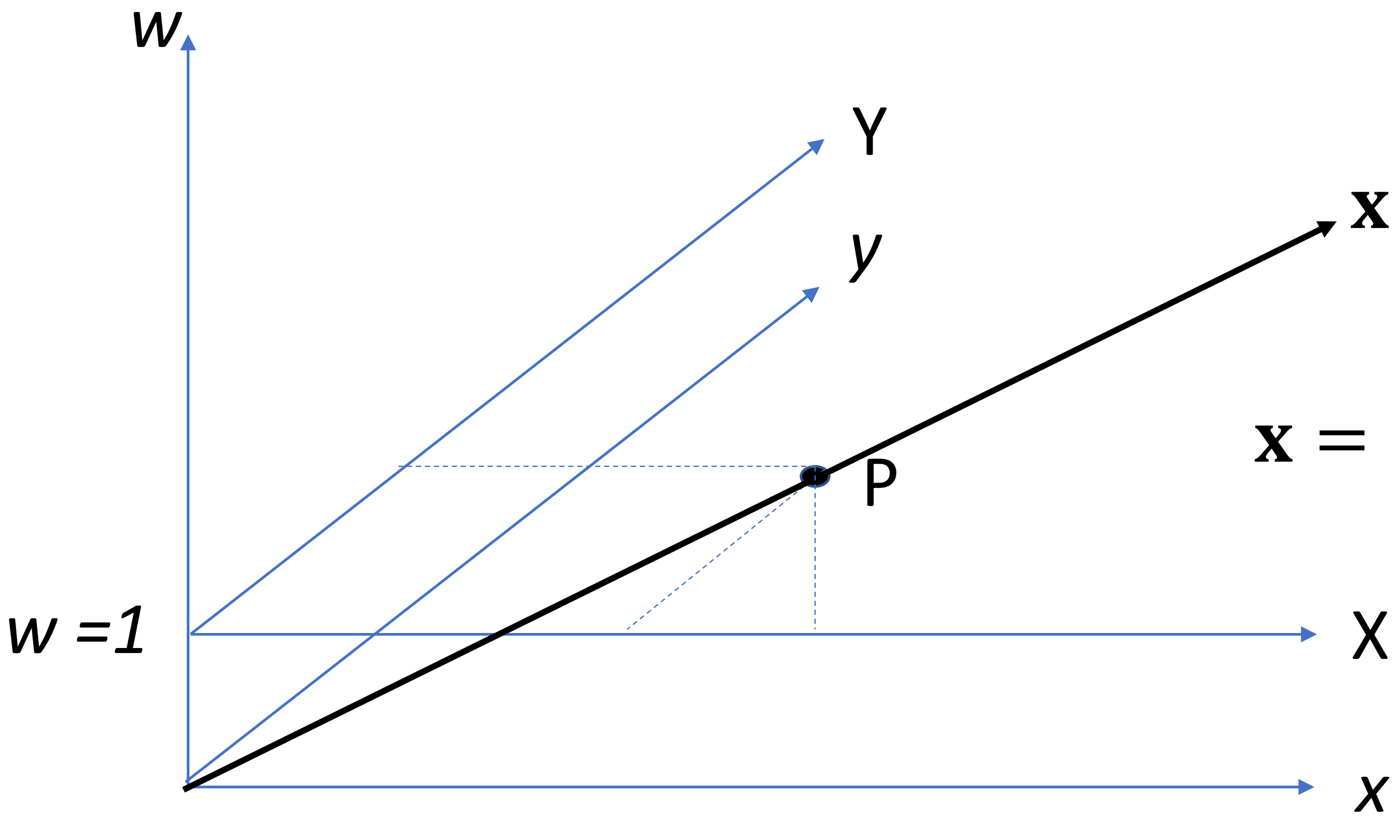


$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

A vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and all its nonzero multiples $\lambda \begin{bmatrix} x \\ y \\ w \end{bmatrix}$, including $\begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$,

represent the point of cartesian coordinates $[X \quad Y] = [x/w \quad y/w]$

on the Euclidean plane

→ homogeneity: any vector $\mathbf{x}$ is equivalent to all its nonzero multiples $\lambda\mathbf{x}$, $\lambda \neq \mathbf{0}$

since they represent the same point

→ $[x \quad y \quad w]$ are **homogeneous** coordinates of the point on the plane

redundancy

3 homogeneous coordinates to represent points in the 2D plane (2 dof)

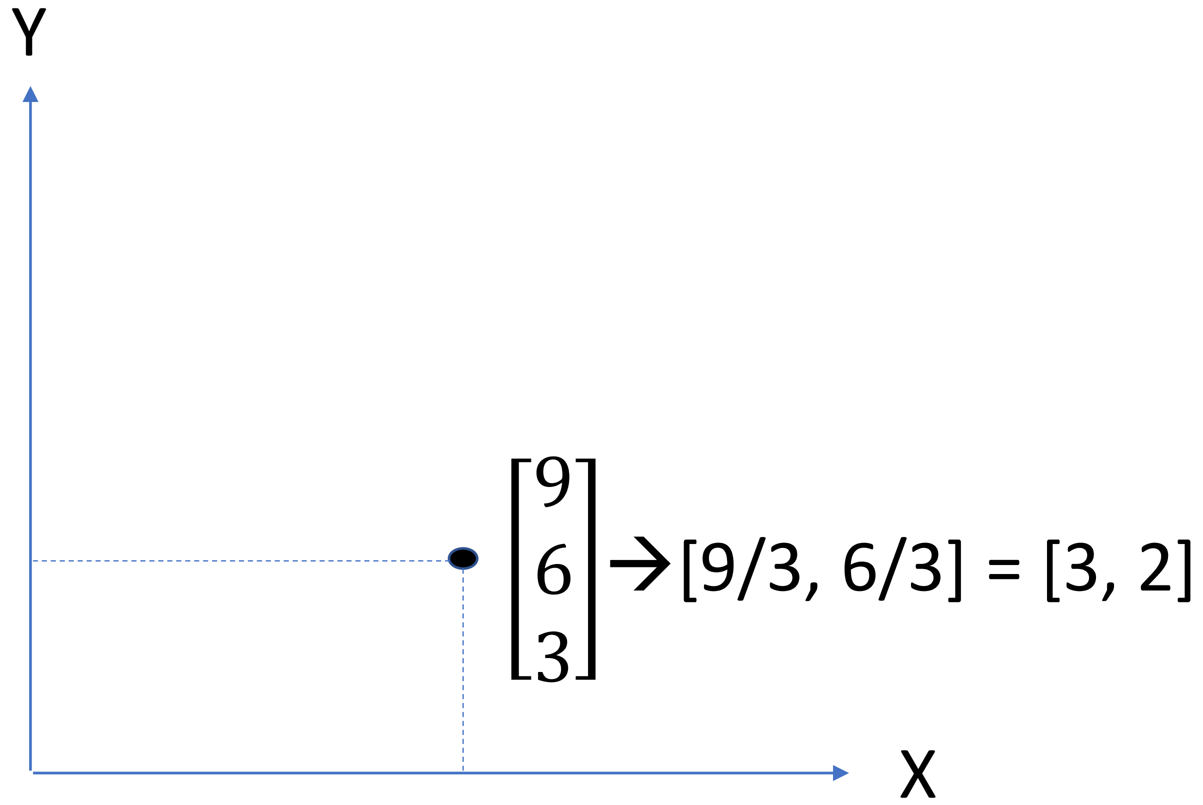
an infinite number of equivalent representations for a single point, namely all nonzero multiples of the vector $[X \quad Y \quad 1]^T$

the null vector $[0 \quad 0 \quad 0]^T$ **does not** represent any point

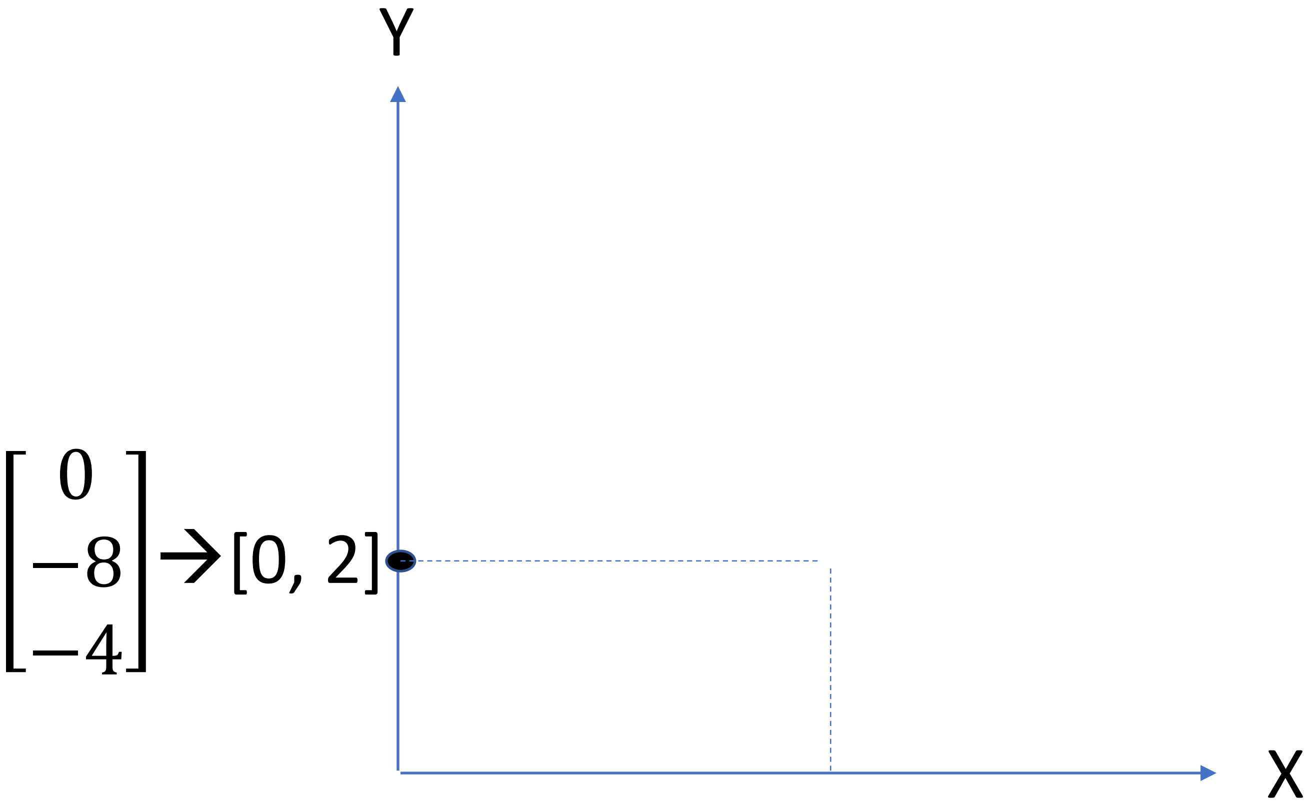
→ Projective plane $\mathbb{P}^2 = \{[x \quad y \quad w]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

→ its two degrees of freedom are the two independent ratios between the three coordinates $x : y : w$

Example: cartesian coordinates vs homogeneous coordinates

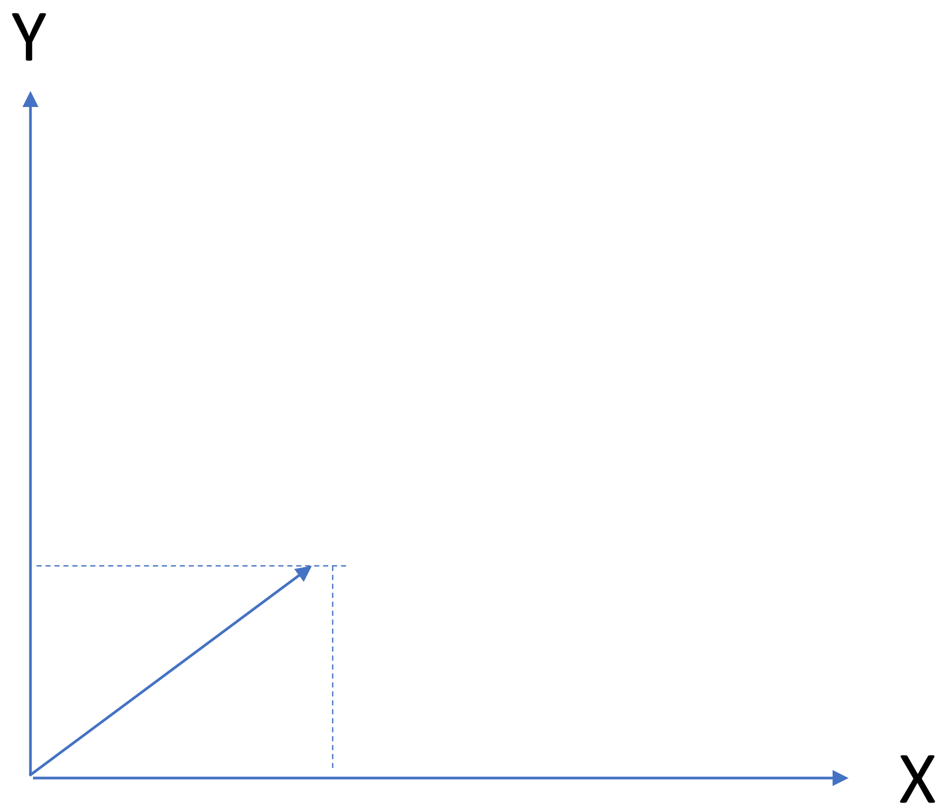


Example: cartesian coordinates vs homogeneous coordinates



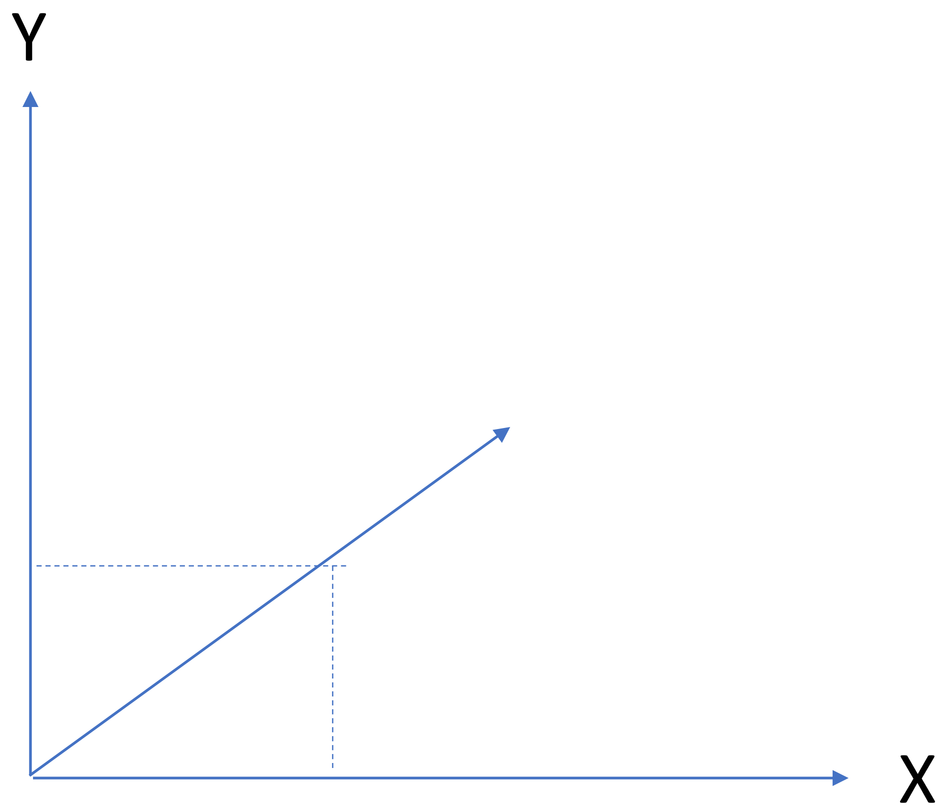
Points at the infinity

points at the infinity



$[3 \quad 2 \quad 0]^T \rightarrow ?$

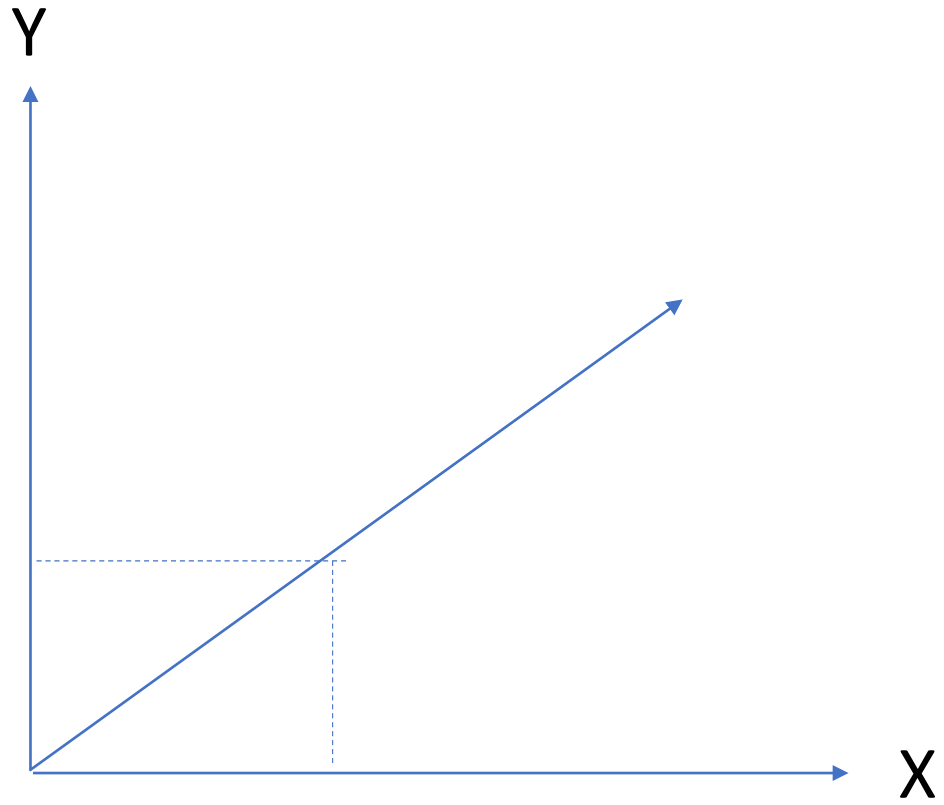
points at the infinity



$$[3 \quad 2 \quad 0]^T$$

$$\rightarrow \lim_{w \to 0} [3/w \quad 2/w]$$

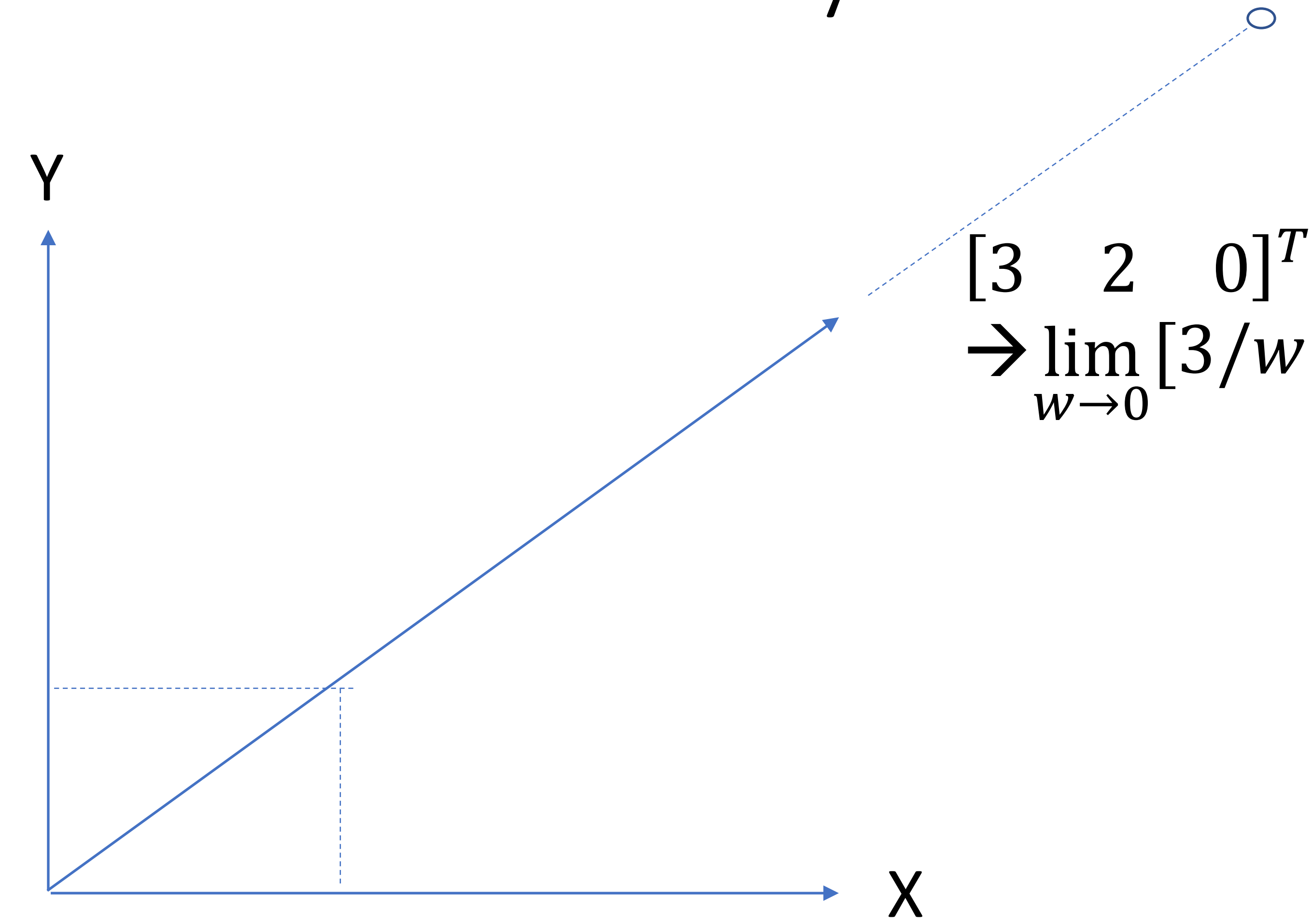
points at the infinity



$[3 \quad 2 \quad 0]^T$

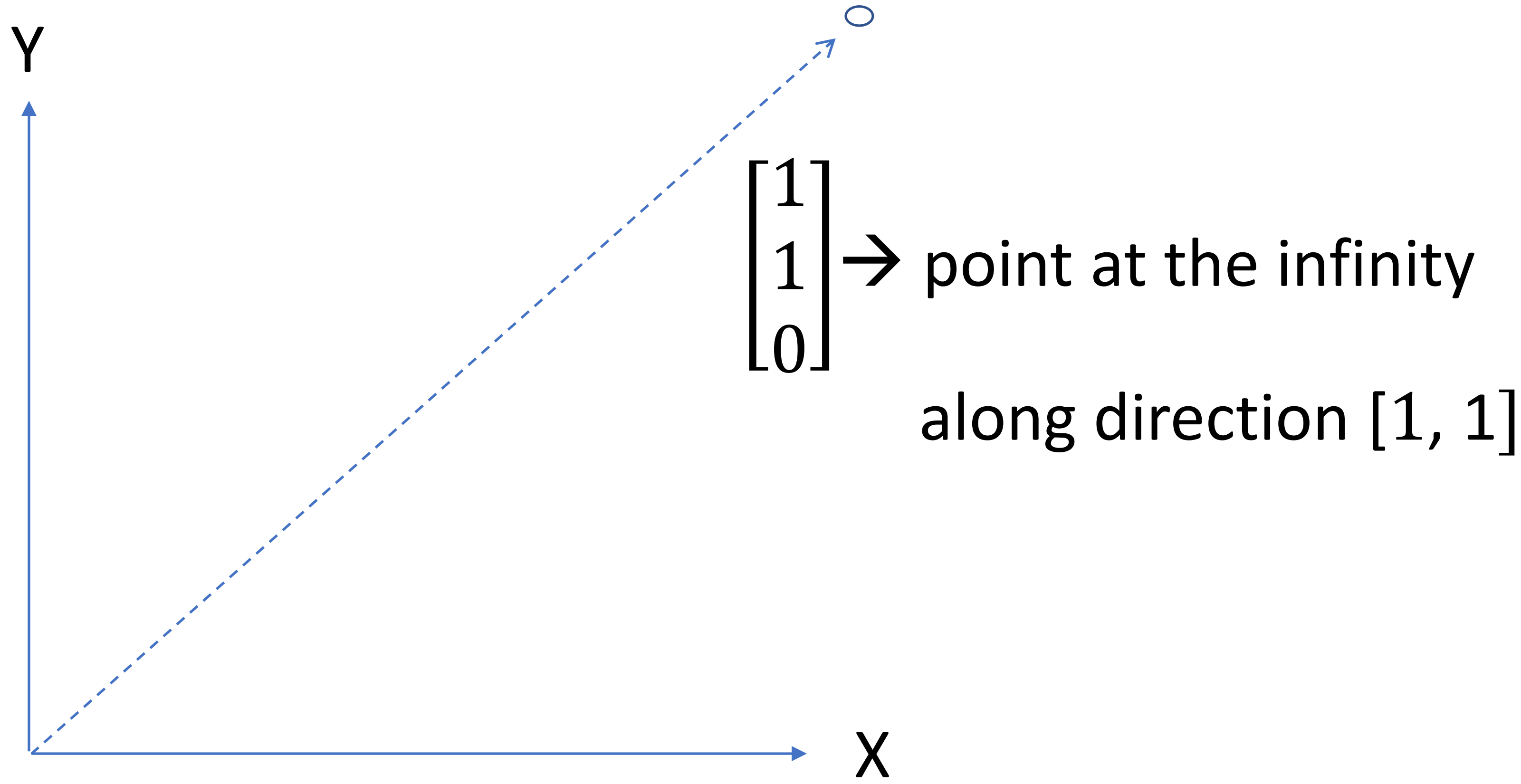
$\rightarrow \lim_{w \to 0} [3/w \quad 2/w]$

points at the infinity



$$[3 \quad 2 \quad 0]^T \rightarrow \lim_{w \to 0} [3/w \quad 2/w]$$

Points at the infinity, who represent directions, are not part of the Euclidean plane: they are extra points, well defined within the Projective plane.


Y
X
$\begin{bmatrix}1\\1\\0\end{bmatrix} \rightarrow$ point at the infinity
along direction [1, 1]

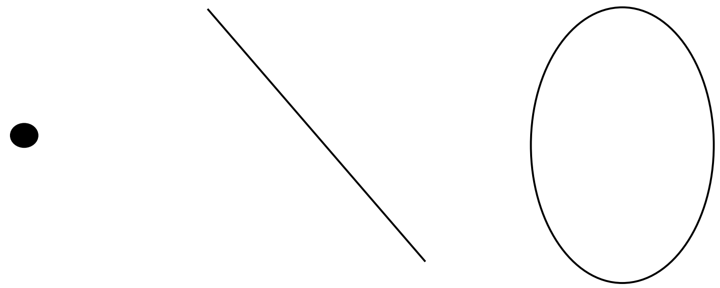
Euclidean plane and Projective plane

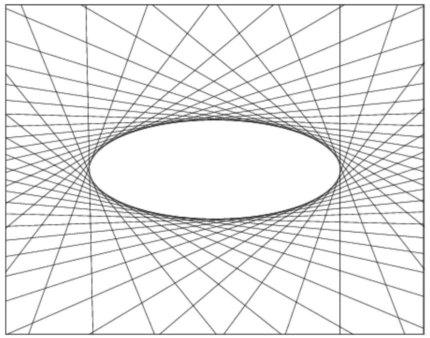
Projective plane $\mathbb{P}^2 = \{[x \quad y \quad w]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

= Euclidean plane ∪ set of the points at the infinity

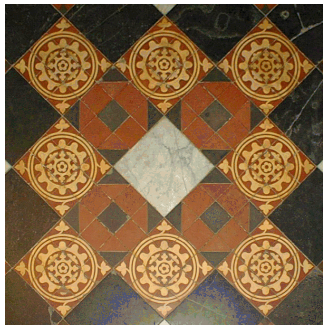
Planar Projective Geometry

- **Elements**
- Points
- **Lines**
- Conics
- Dual conics



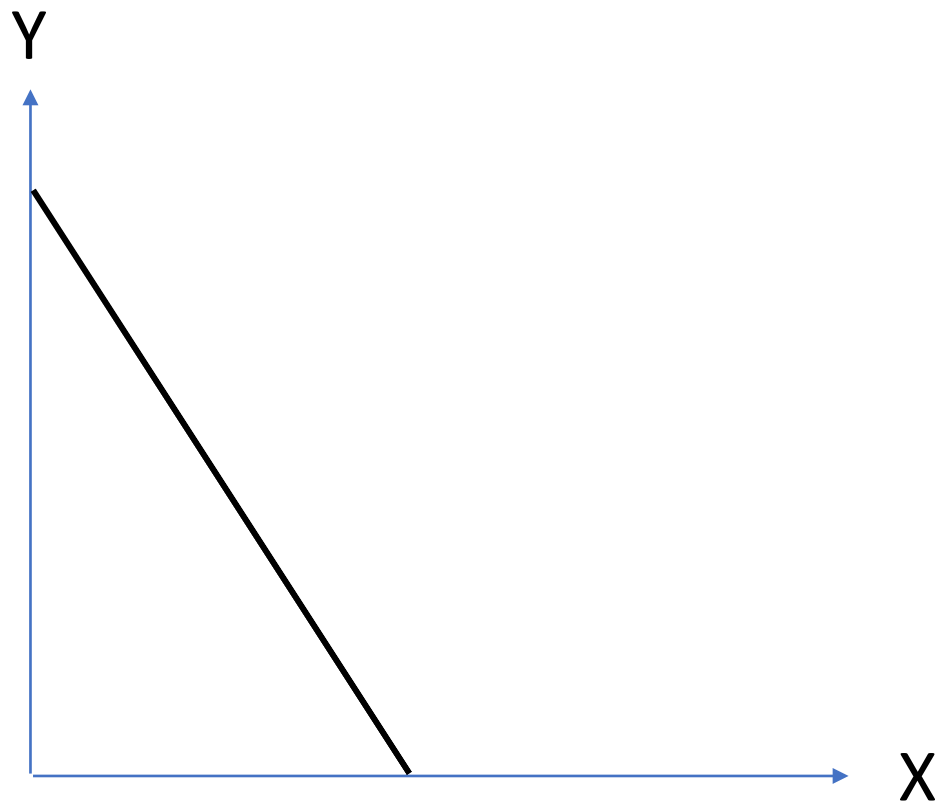


- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



Lines in 2D Projective Geometry

Consider a line on the Euclidean plane



$$aX + bY + c = 0$$

$$a\frac{x}{w} + b\frac{y}{w} + c = 0$$

$$ax + by + cw = 0$$

$$ax + by + cw = 0 \rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

homogeneous, **linear** equation in $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$: $\boldsymbol{l}^T \mathbf{x} = 0$, where the vector $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

and all its nonzero multiples $\lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ represent a line

→ homogeneity: any vector $\boldsymbol{l}$ is equivalent to all its nonzero multiples $\lambda \boldsymbol{l}$, $\lambda \neq \mathbf{0}$ and they represent the same line

→ $\begin{bmatrix} a & b & c \end{bmatrix}$ are **homogeneous** parameters of the line

redundancy

3 homogeneous parameters to represent lines in the 2D plane (2 dof)

an infinite number of equivalent representations for a single line, namely all nonzero multiples of the vector $[a \quad b \quad c]^T$

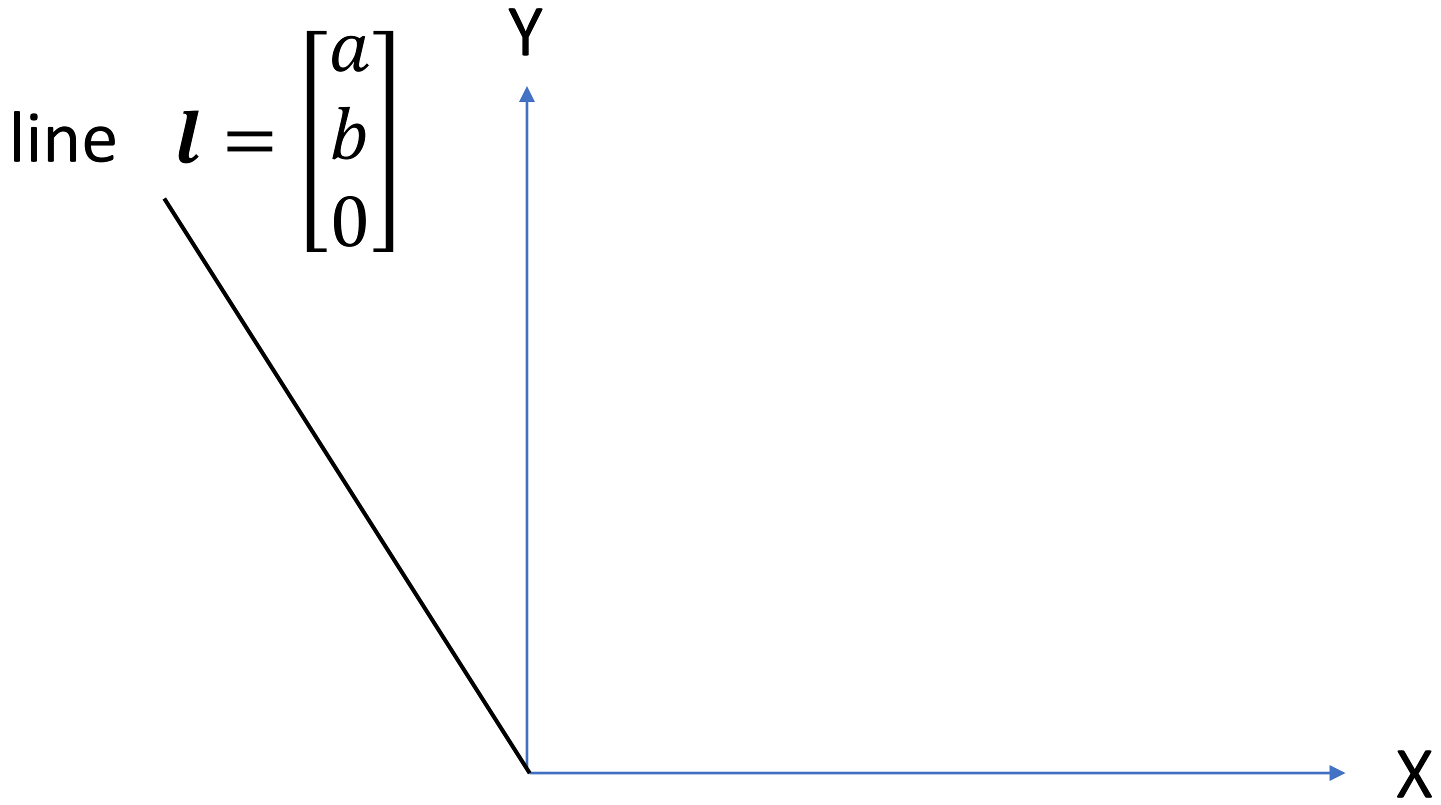
the null vector $[0 \quad 0 \quad 0]^T$ **does not** represent any line

→ Projective «dual» plane $\mathbb{P}^2 = \{[a \quad b \quad c]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

→ its two degrees of freedom are the two independent ratios between the three parameters $a : b : c$

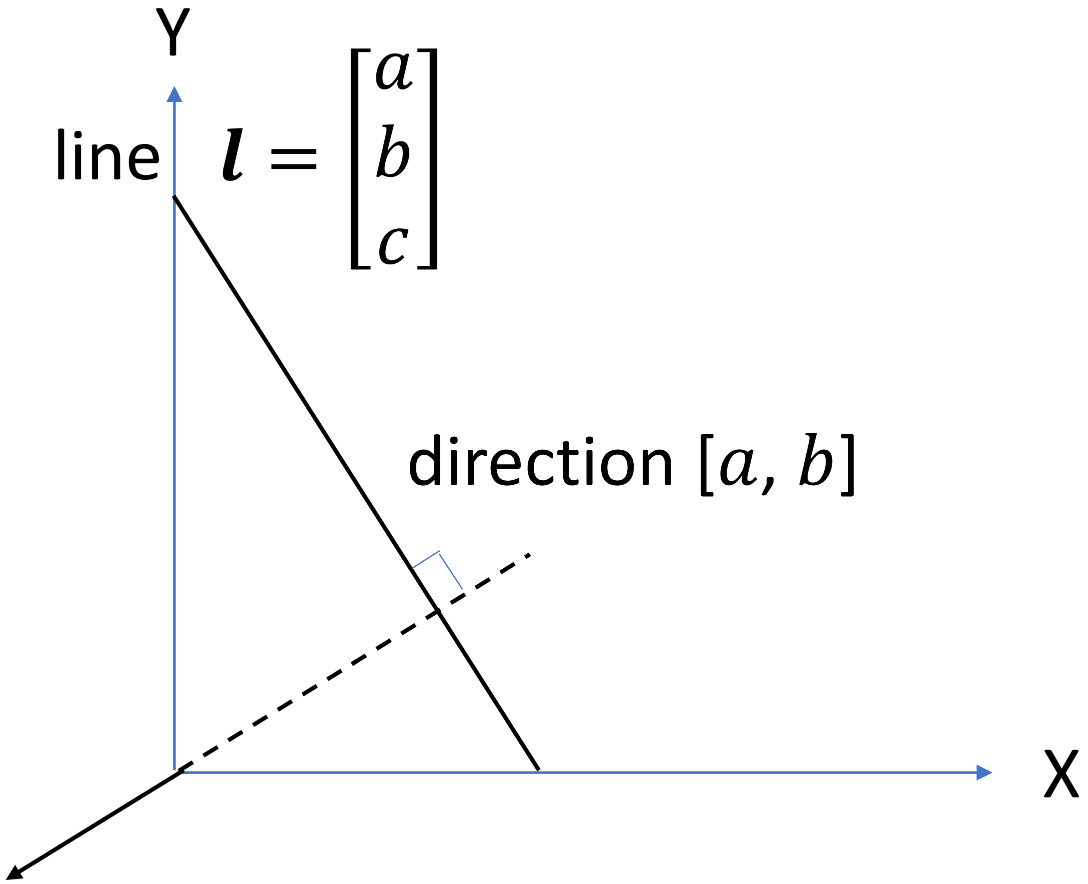
Three remarks

1. If the third parameter is null, $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, then $\boldsymbol{l}$ goes through point $[0,0]$



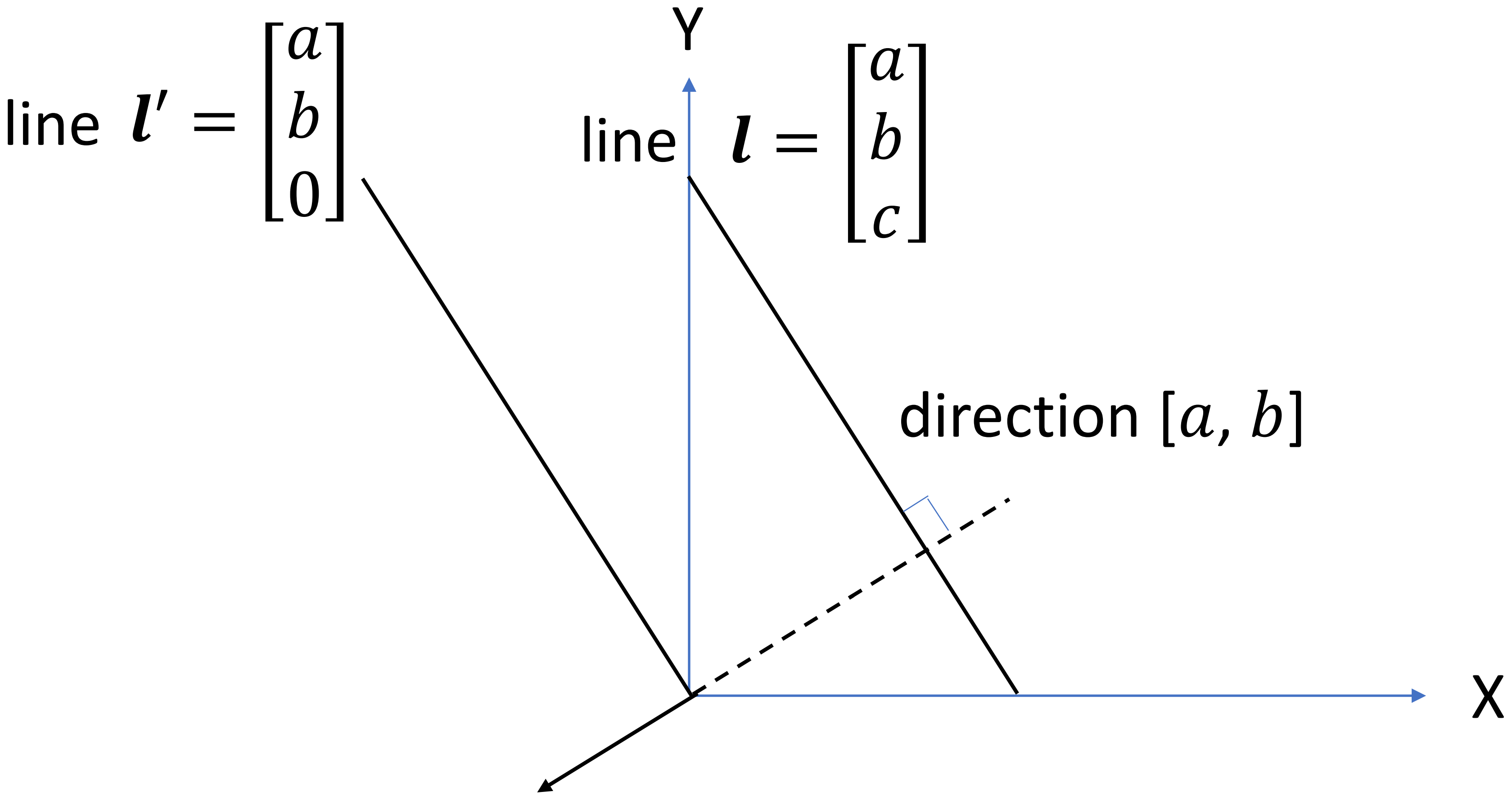
A line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \mathbf{0}]^{\boldsymbol{T}}$ whose third parameter is zero, goes through the origin of the plane

2. the direction $[a, b]$ is normal to the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,



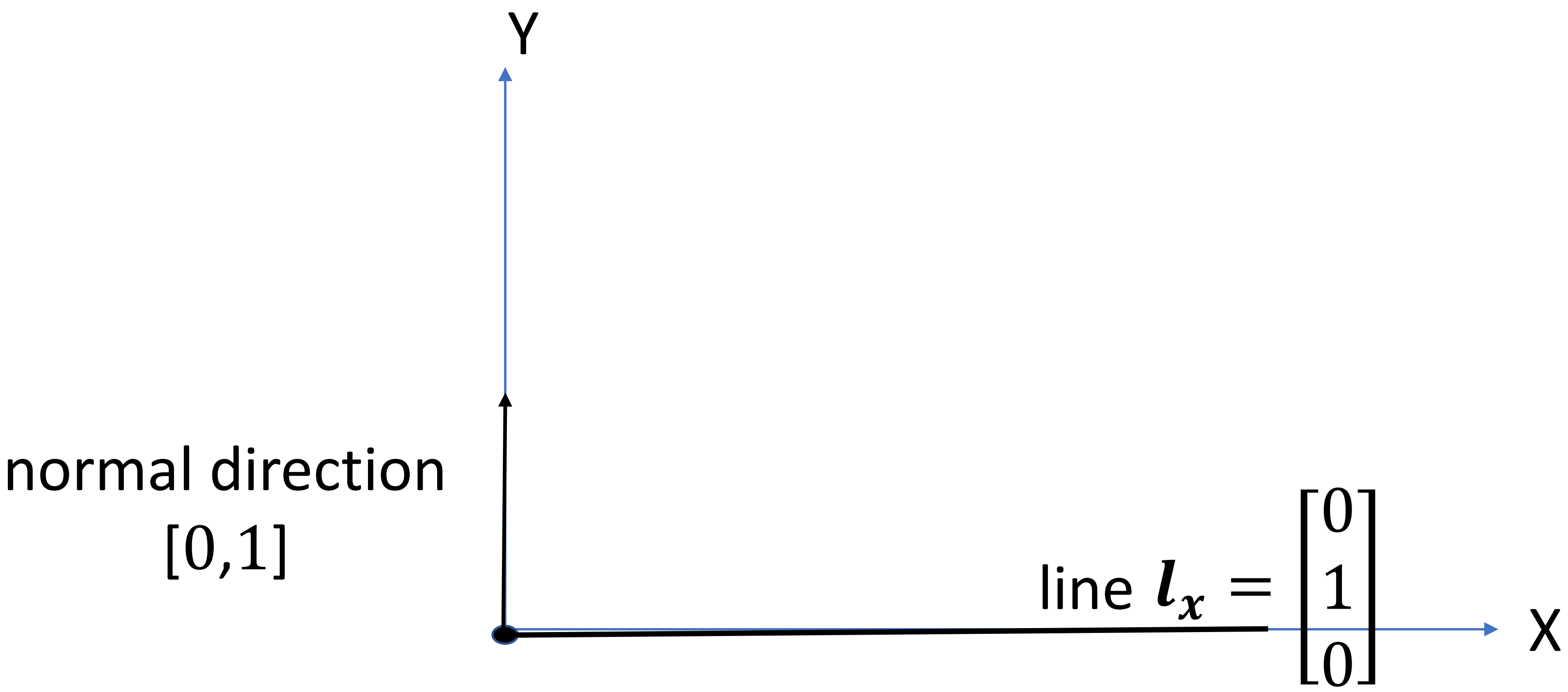
direction $[a, b]$ is normal to the line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]^{\boldsymbol{T}}$

3. the lines $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} a \\ b \\ c' \end{bmatrix}$ are parallel: their common direction is $[b, -a]$

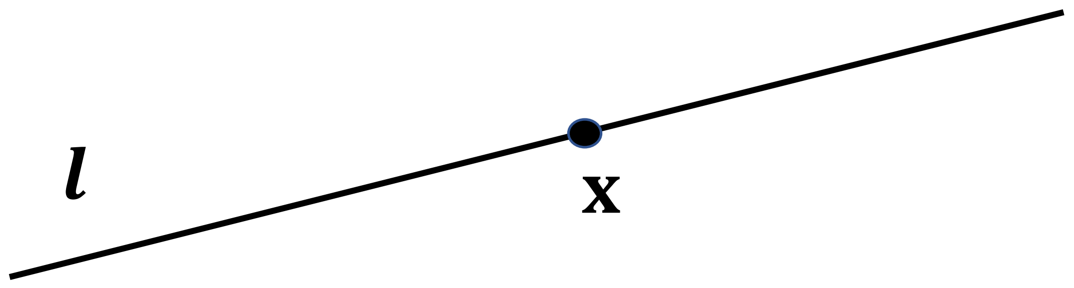


direction $[a, b]$ is normal both to the line $\boldsymbol{l} = [a \quad b \quad c]^{\boldsymbol{T}}$
and to the line $\boldsymbol{l} = [a \quad b \quad 0]^{\boldsymbol{T}}$

Example: the X-axis



The incidence relation
a point is on a line, or a line goes through a point

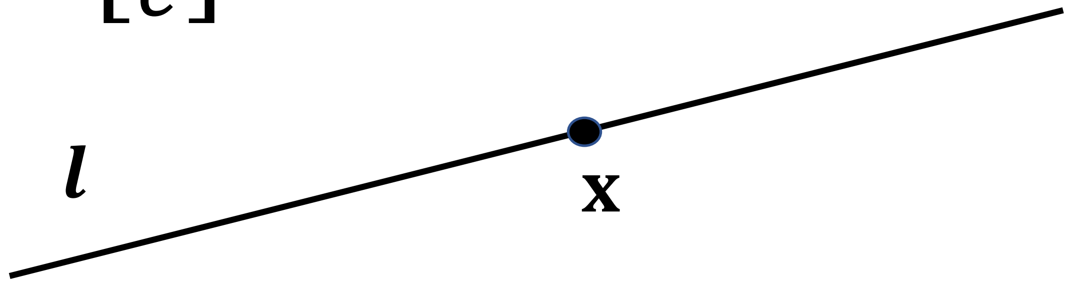


Incidence relation: $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \boldsymbol{l}^T \mathbf{x} = 0$

the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ is on the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

or

the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ goes through the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$



The line at the infinity:
the locus of the points at the infinity

The «locus» of the points at the infinity

As there are infinite points at the infinity (one for each direction), what is the aspect of the set of these points at the infinity?

Simply:

$$w = 0$$

This set is a line: $[\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{w}\end{bmatrix} = 0$, actually $[\mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{w}\end{bmatrix} = \boldsymbol{w} = 0$

namely, **the line at the infinity** $\boldsymbol{l}_\infty = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

The duality principle between points and lines

Since dot product is commutative
→ incidence relation is commutative

$$\boldsymbol{l}^T\mathbf{x} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{w}\end{bmatrix} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\\ \boldsymbol{c}\end{bmatrix} = \mathbf{x}^T\boldsymbol{l} = \mathbf{0}$$

point $\mathbf{x}$ is on line $\boldsymbol{l}$ $\Leftrightarrow$ **point** $\boldsymbol{l}$ is on **line** $\mathbf{x}$

point $\mathbf{x}$ is on line $\boldsymbol{l}$ (i.e. line $\boldsymbol{l}$ goes through point $\mathbf{x}$)

$\leftrightarrow$

point $\boldsymbol{l}$ is on **line** $\mathbf{x}$ (i.e. line $\mathbf{x}$ goes through point $\boldsymbol{l}$)

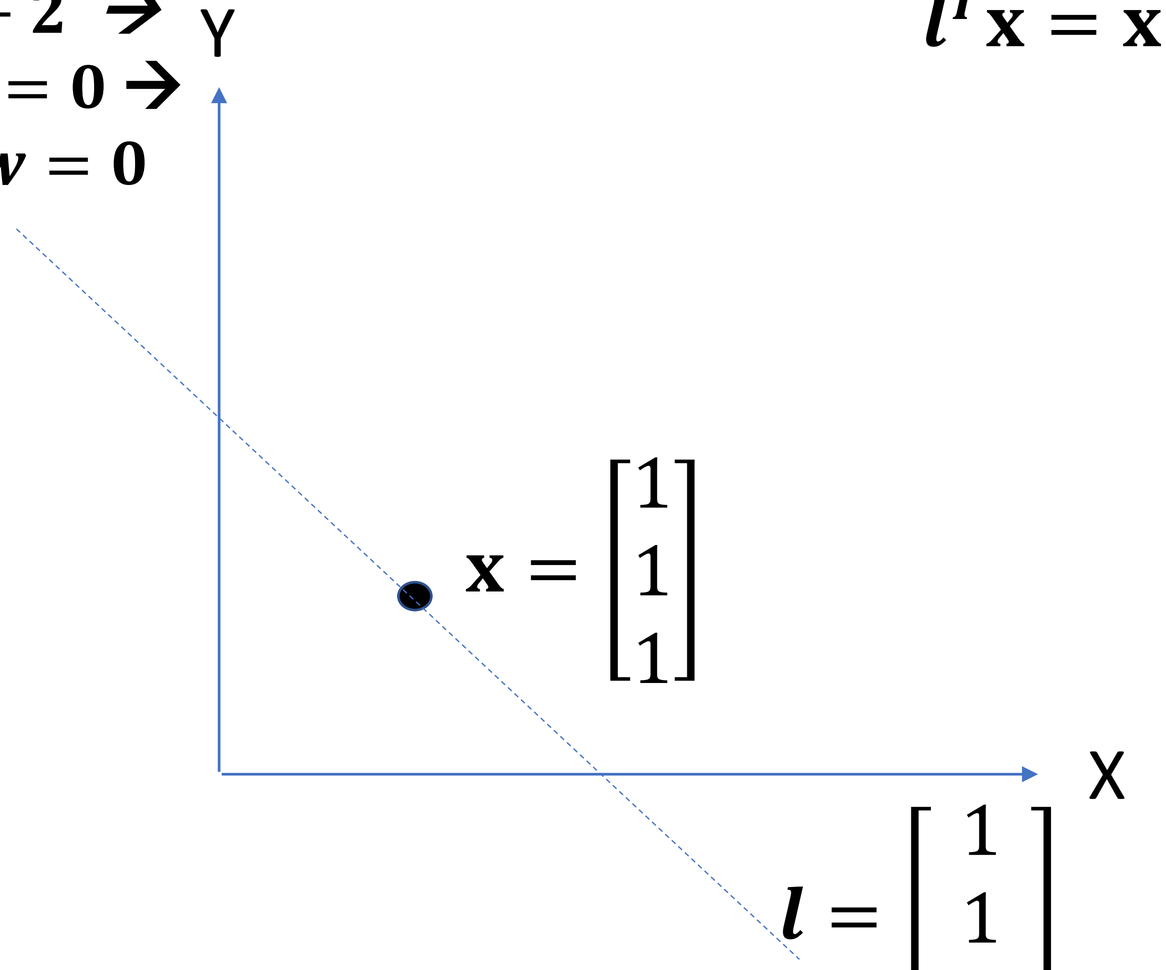
Principle of **duality** between points and lines

(50% discount principle)

point $\mathbf{x}$ is on line $\boldsymbol{l}$ (i.e. line $\boldsymbol{l}$ goes through point $\mathbf{x}$)

$$Y = -X + 2 \rightarrow$$
$$X + Y - 2 = 0 \rightarrow$$
$$\mathbf{x} + y - 2w = 0$$

$$\boldsymbol{l}^T\mathbf{x} = \mathbf{x}^T\boldsymbol{l} = 0$$



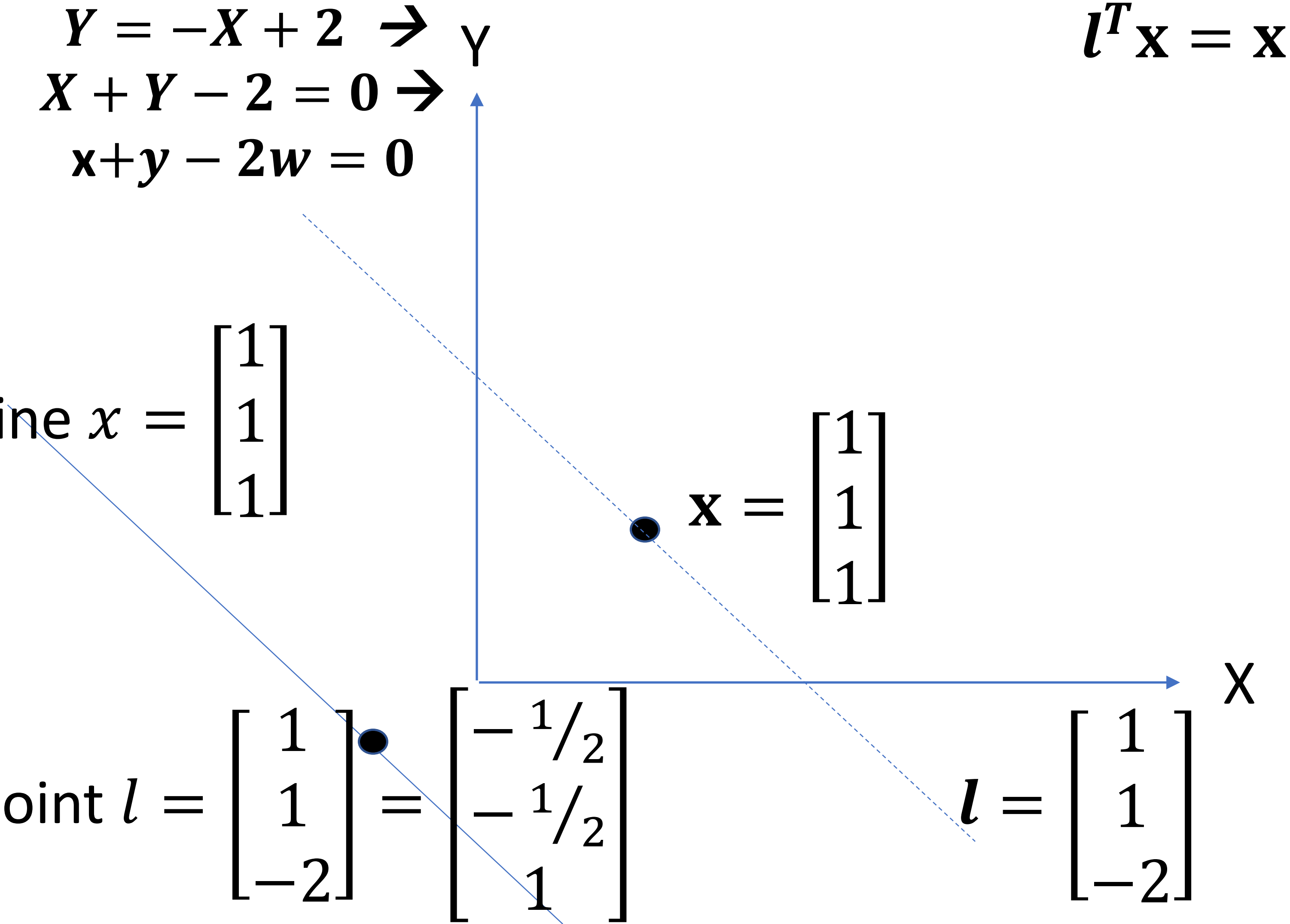
$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

point $\boldsymbol{l}$ is on line $\boldsymbol{x}$ (i.e. line $\boldsymbol{x}$ goes through point $\boldsymbol{l}$)

$$\boldsymbol{Y = -X + 2} \rightarrow$$
$$\boldsymbol{X + Y - 2 = 0} \rightarrow$$
$$\mathbf{x} + \boldsymbol{y} - \mathbf{2}\boldsymbol{w} = \mathbf{0}$$

$$\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$$



line $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

point $l = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -{}^{1}\!/_{2} \\ -{}^{1}\!/_{2} \\ 1 \end{bmatrix}$

$$\boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

For any true sentence containing the items

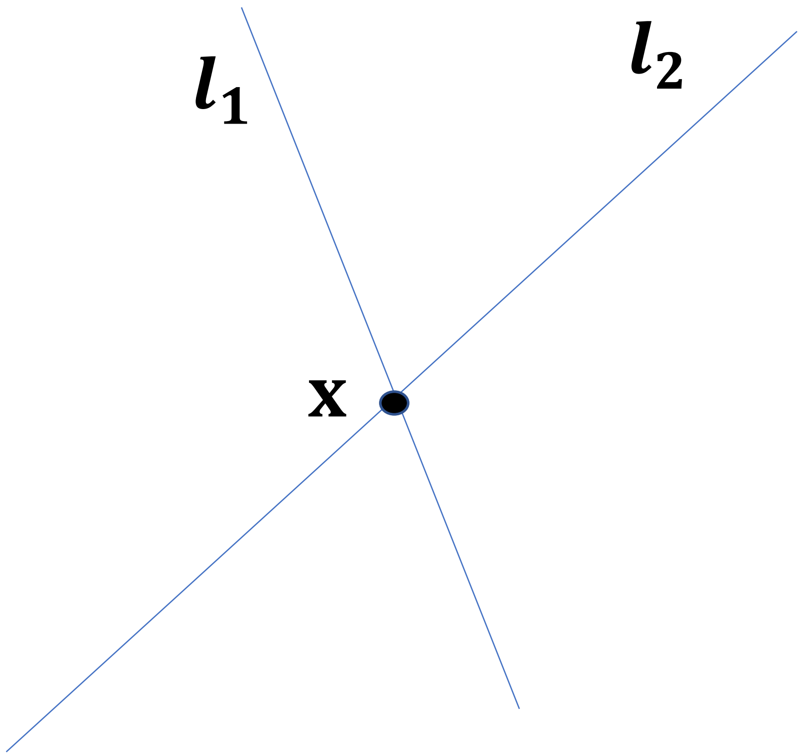
- point
- line
- is on
- goes through

there is a DUAL sentence -also true- obtained by substituting, in the previous one, each occurence of

- point	by	- line
- line	by	- point
- is on	by	- goes through
- goes through	by	- is on

The point on two lines

the point on two lines



$$\begin{cases} \boldsymbol{l_1}^T\mathbf{x} = 0 \\ \boldsymbol{l_2}^T\mathbf{x} = 0 \end{cases}$$

$$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\right)$$

$\mathbf{x}$ is a vector orthogonal to both $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ vectors

$\rightarrow$

$\mathbf{x}$ is (a multiple of) the cross product of $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$: $\mathbf{x} = \boldsymbol{l_1} \times \boldsymbol{l_2}$

Example: intersection of two parallel lines

Suppose that lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ are parallel: this means that

$$l_1 = [a \quad b \quad c_1]^T \text{ and}$$
$$l_2 = [a \quad b \quad c_2]^T$$

The point $\mathbf{x} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]^{\boldsymbol{T}}$ common two these two lines satisfies both

$$ax + by + c_1 w = 0$$

and

$$ax + by + c_2 w = 0$$

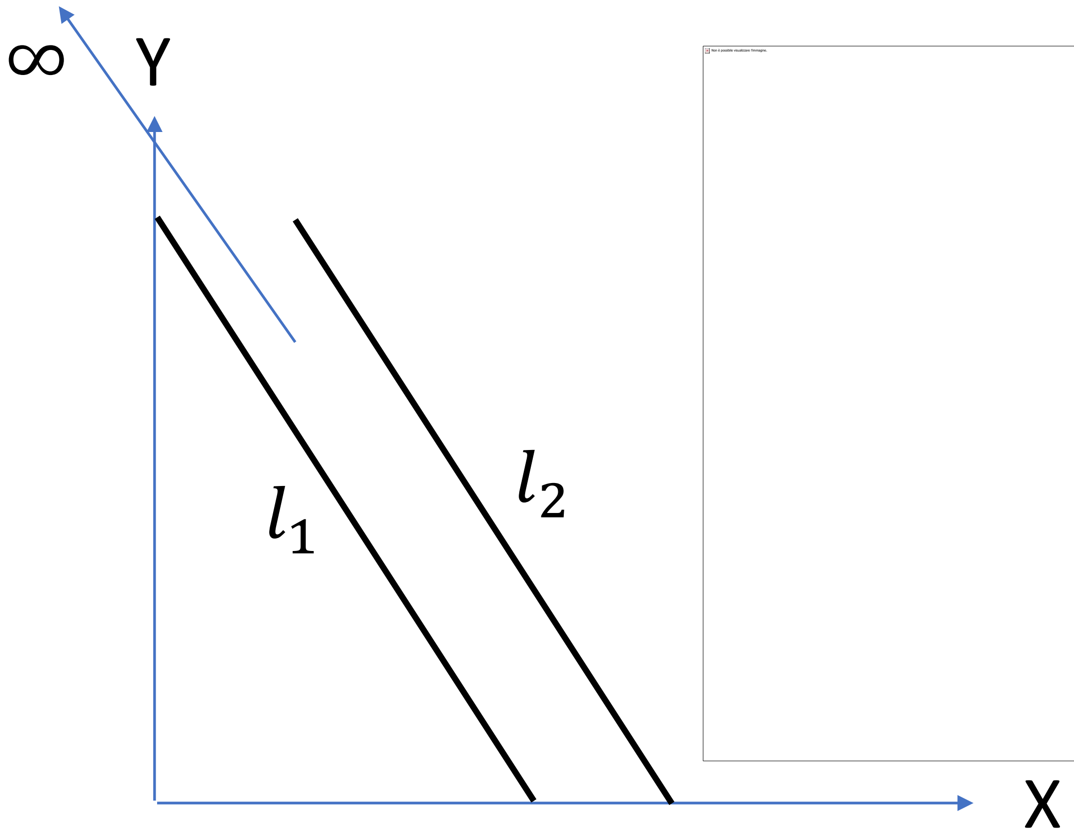
$$\downarrow$$

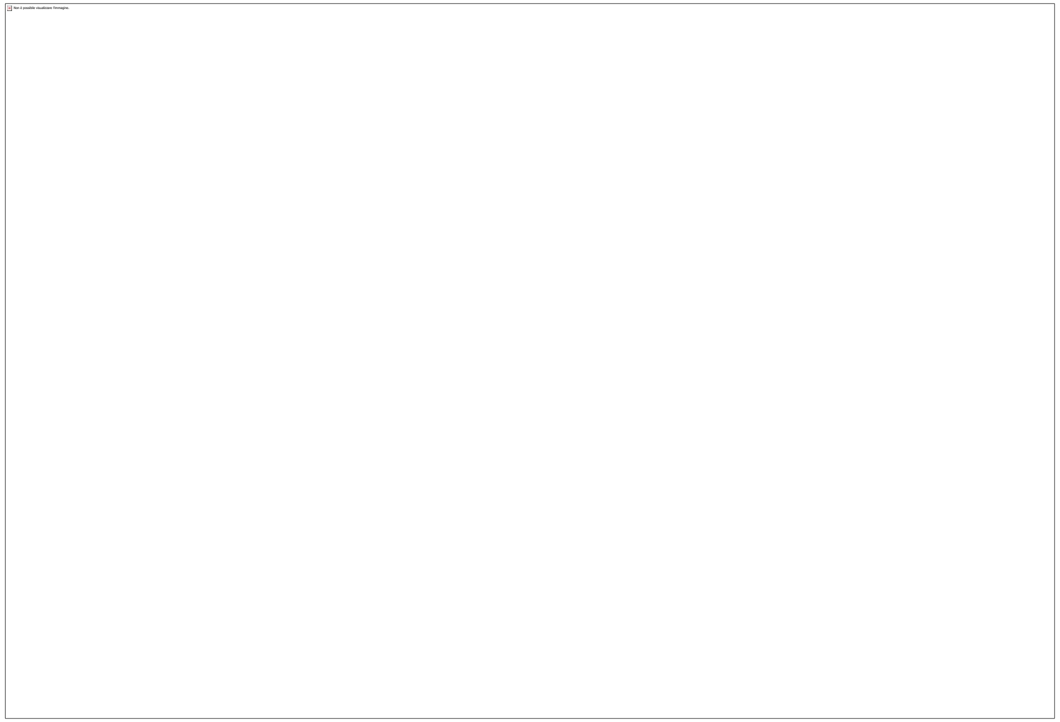
$$\mathbf{x} = [b \quad -a \quad 0]^T$$

Namely, the point at the infinty along the direction of both lines

(remember: [b, -a] is the direction of both lines)

The intersection of two parallel line is the point at the infinity along their common direction



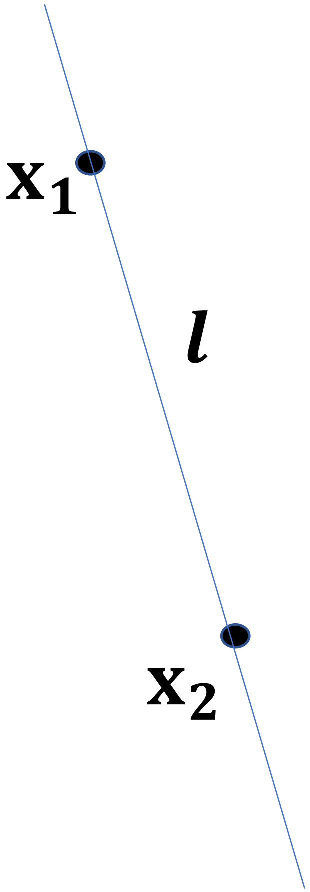


Preview:

The vanishing point is the image of point at the infinity, i.e., the point where parallel lines intersect

the line through two points

Previous: point on two lines
DUAL: line through two points



$$\boldsymbol{l} = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{x_1}^T \\ \mathbf{x_2}^T \end{bmatrix}\right)$$

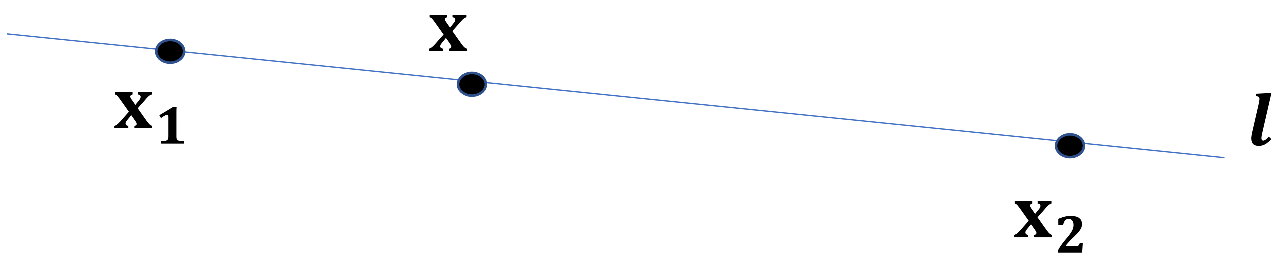
or, just in 2D Proj. Geo.

$$\boldsymbol{l} = \mathbf{x_1} \times \mathbf{x_2}$$

A useful property
and its dual

Example: linear combination of two points

Property: the point $\mathbf{x}$ given by the linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$ (i.e. on the line joining $\mathbf{x}_1$ and $\mathbf{x}_2$) . In less words: $\mathbf{x}$ is COLINEAR to $\mathbf{x}_1$ and $\mathbf{x}_2$



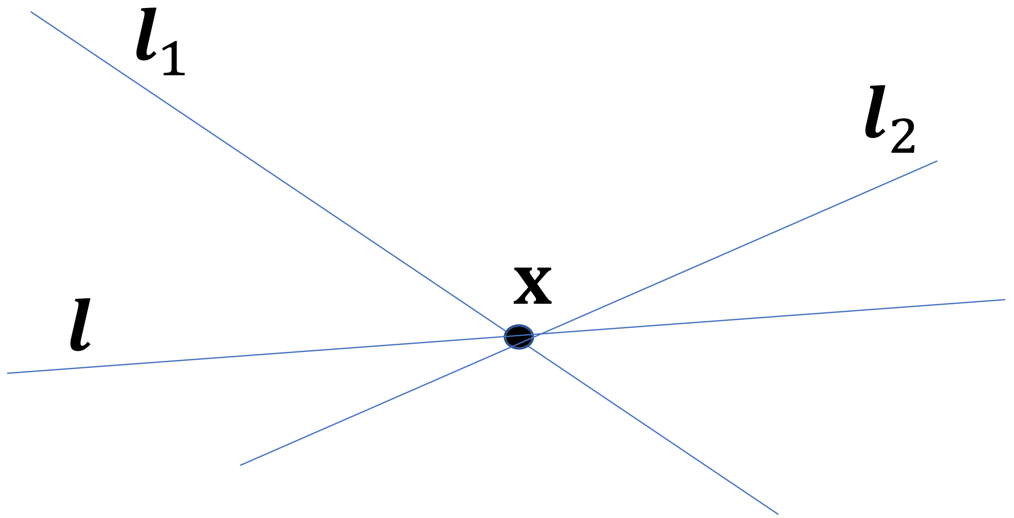
Proof: the line $\boldsymbol{l}$ through both points satisfies $\boldsymbol{l}^T\mathbf{x_1} = 0$ and $\boldsymbol{l}^T\mathbf{x_2} = 0$.
By adding α times the first eqn to β times the second one, we obtain

$$0 = \boldsymbol{l}^T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \boldsymbol{l}^T\mathbf{x} = 0$$

i.e. $\mathbf{x}$ is on the same line joining $\mathbf{x}_1$ and $\mathbf{x}_2$.

DUAL: linear combination of two lines

Property: the point $\mathbf{x}$, given by the linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$, is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$



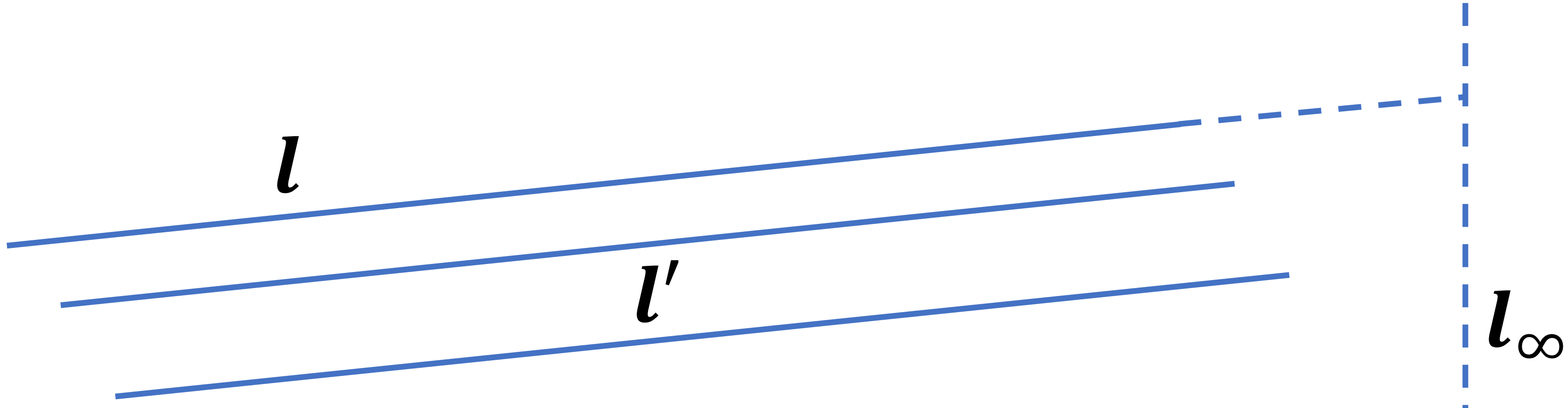
Dual: the line $\boldsymbol{l}$, given by the linear combination $\boldsymbol{l} = \alpha\boldsymbol{l}_1 + \beta\boldsymbol{l}_2$ of two lines $\boldsymbol{l}_1$and $\boldsymbol{l}_2$, goes through the point $\mathbf{x}$ on $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$.

In less words: line $\boldsymbol{l} = \alpha\boldsymbol{l}_1 + \beta\boldsymbol{l}_2$ is CONCURRENT to lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

New dual pair

- point	by	- line
- line	by	- point
- is on	by	- goes through
- goes through	by	- is on
- colinear	by	- concurrent
- concurrent	by	- colinear

A special case: linear combinations of a line $\boldsymbol{l}$ and the line $\boldsymbol{l}_\infty$

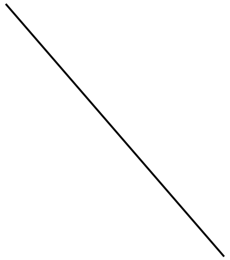


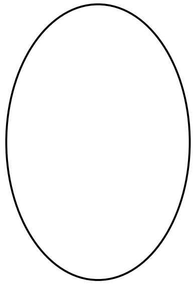
$$\boldsymbol{l}' = \boldsymbol{l} + \lambda \boldsymbol{l}_\infty = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} + \lambda \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c}' \end{bmatrix} \parallel \boldsymbol{l}$$

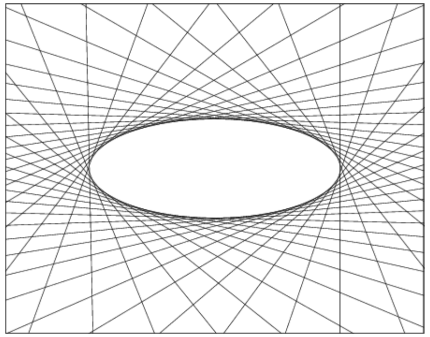
→ the set of all the lines $\boldsymbol{l}'$ parallel to $\boldsymbol{l}$

Planar Projective Geometry

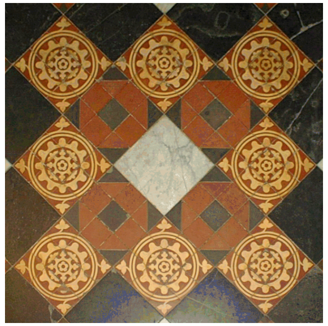
- **Elements**
- Points
- Lines
- **Conics**
- Dual conics





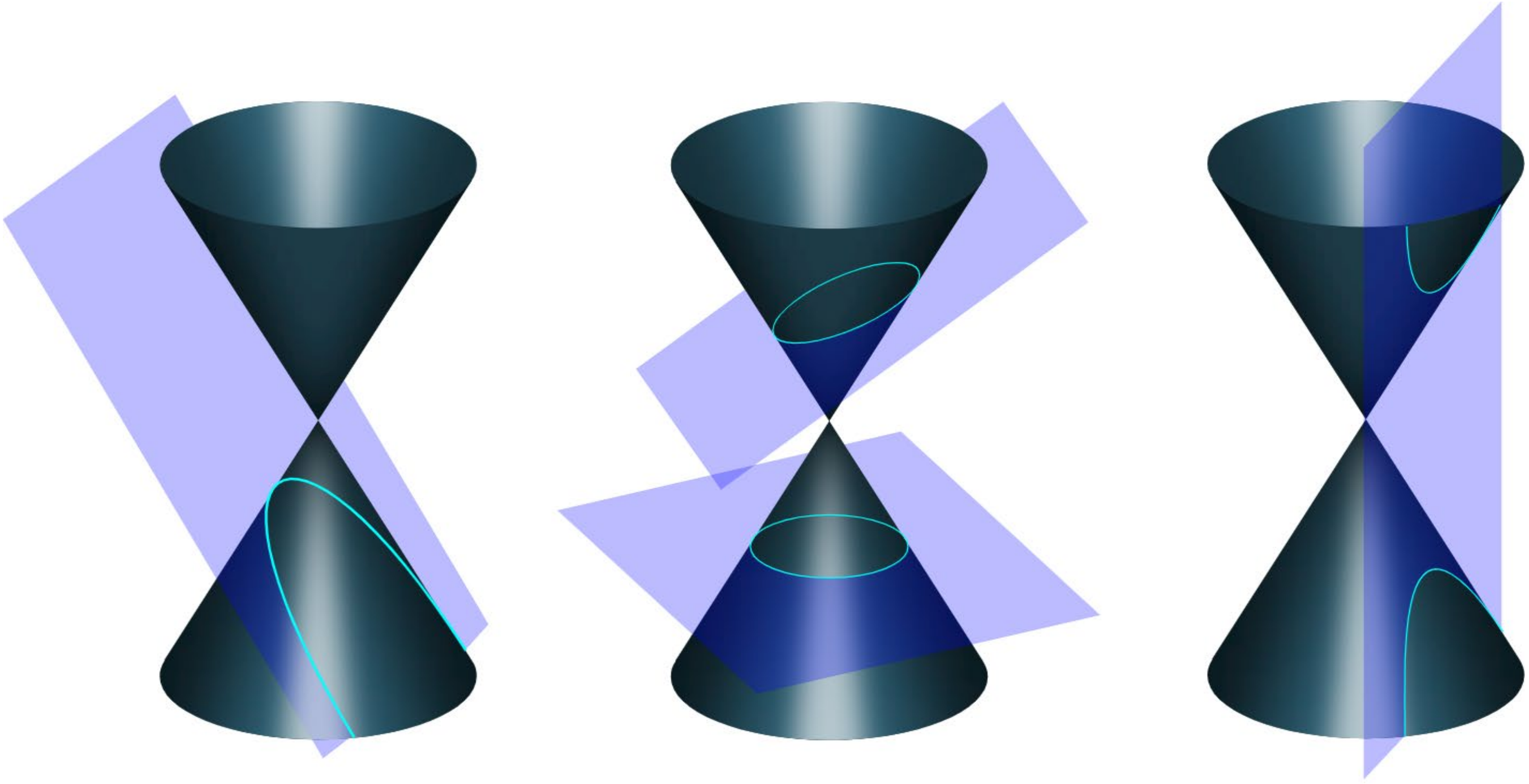


- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



CONICS

Conics name derives from the intersection of cones with planes



PARABOLAE ELLIPSE HYPERBOLAE

Giacomo Boracchi

Lines: a point $\mathbf{x}$ is on a line $\boldsymbol{l}$ if it satisfies a homogeneous ***linear*** equation, namely

$$\boldsymbol{l}^T\mathbf{x} = \mathbf{0}$$

Conics: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}$ if it satisfies a homogeneous ***quadratic*** equation, namely

$$\mathbf{x}^T\boldsymbol{C}\,\mathbf{x} = \mathbf{0}$$

where $\boldsymbol{C}$ is a 3x3 symmetric matrix.

Conics in $\mathbb{P}^2$

- A conic is a curve described by a second-degree equation in the plane.
- In Euclidean coordinates a conic becomes
- $aX^2 + bXY + cY^2 + dX + eY + f = 0$
- i.e. a polynomial of degree 2. “Homogenizing” this by the replacements:
- X→ x/w, Y→ y/w gives
- $ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$
- or in matrix form

$$\mathbf{x}^\top \boldsymbol{C}\mathbf{x} = 0$$

- where the conic coefficient matrix $\boldsymbol{C}$ is given by $\boldsymbol{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$

Conics in $\mathbb{P}^2$

- $\mathbf{x}^\top \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \mathbf{x} = 0$

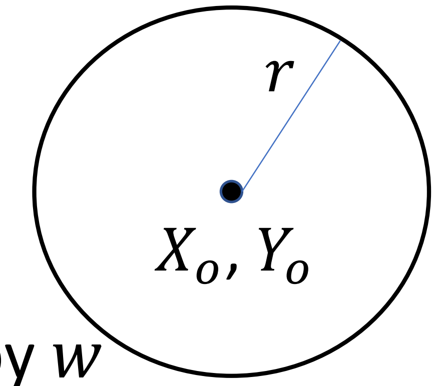
- **Rmk** The conic coefficient matrix is symmetric,
- **Rmk** multiplying $\boldsymbol{C}$ by a non-zero scalar does not change. Only the ratios of the elements in $\boldsymbol{C}$ are important, as for homogeneous points and for lines. → $\boldsymbol{C}$ is a homogeneous matrix
- **Rmk** The conic has five degrees of freedom :
- the ratios {*a* : *b* : *c* : *d* : *e* : *f*} or equivalently
- the six elements of a symmetric matrix minus one for scale.

Example: the circumference

First in cartesian coordinates:

$$(X - X_o)^2 + (Y - Y_o)^2 - r^2 = 0$$

X_o, Y_o are the center coordinates, r is the radius.



then in homogeneous coordinates:

replace X and Y by, respectively, x/w and y/w and multiply by w

$$x^2 - 2X_o w + X_o^2 w^2 + y^2 - 2Y_o w + Y_o^2 w^2 - r^2 w^2 = 0$$

reorder the coefficients

$$\underbrace{[x \quad y \quad w] \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0}_{\mathbf{x}^\top \, \boldsymbol{C} \, \mathbf{x} = 0}$$

Intersection of a line and a conic

Intersection of a line and a conic

Conic: quadratic equation on $\mathbf{x}$, namely $\mathbf{x}^{\top} \boldsymbol{C} \ \mathbf{x} = 0$

Line: linear equation on $\mathbf{x}$, namely $\boldsymbol{l}^{T}\mathbf{x} = \mathbf{0}$

Line-conic intersection leads to a degree 2 equation on $\mathbf{x}$

$\rightarrow$

there are always TWO intersection points between a line and a conic:

they can be real, or complex conjugate, distinct or coincident

[Foundamental Theorem of Algebra]

Line – conic intersection: two real distinct solutions

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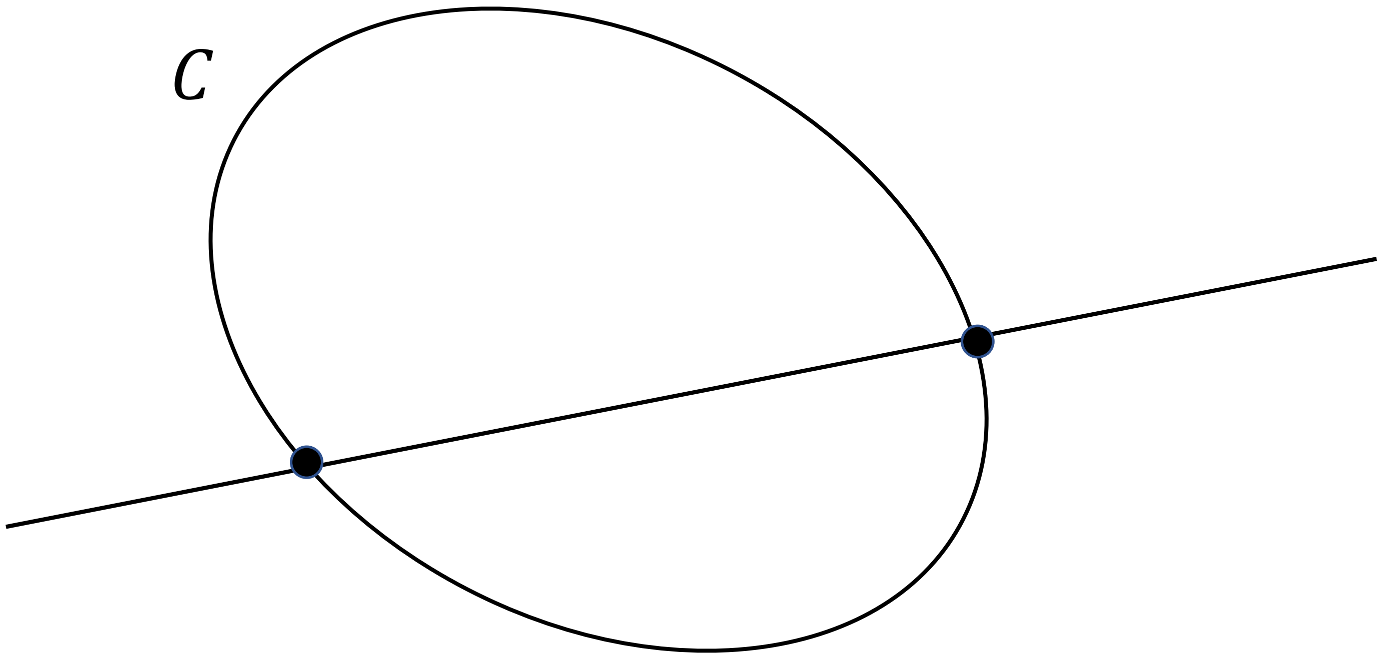
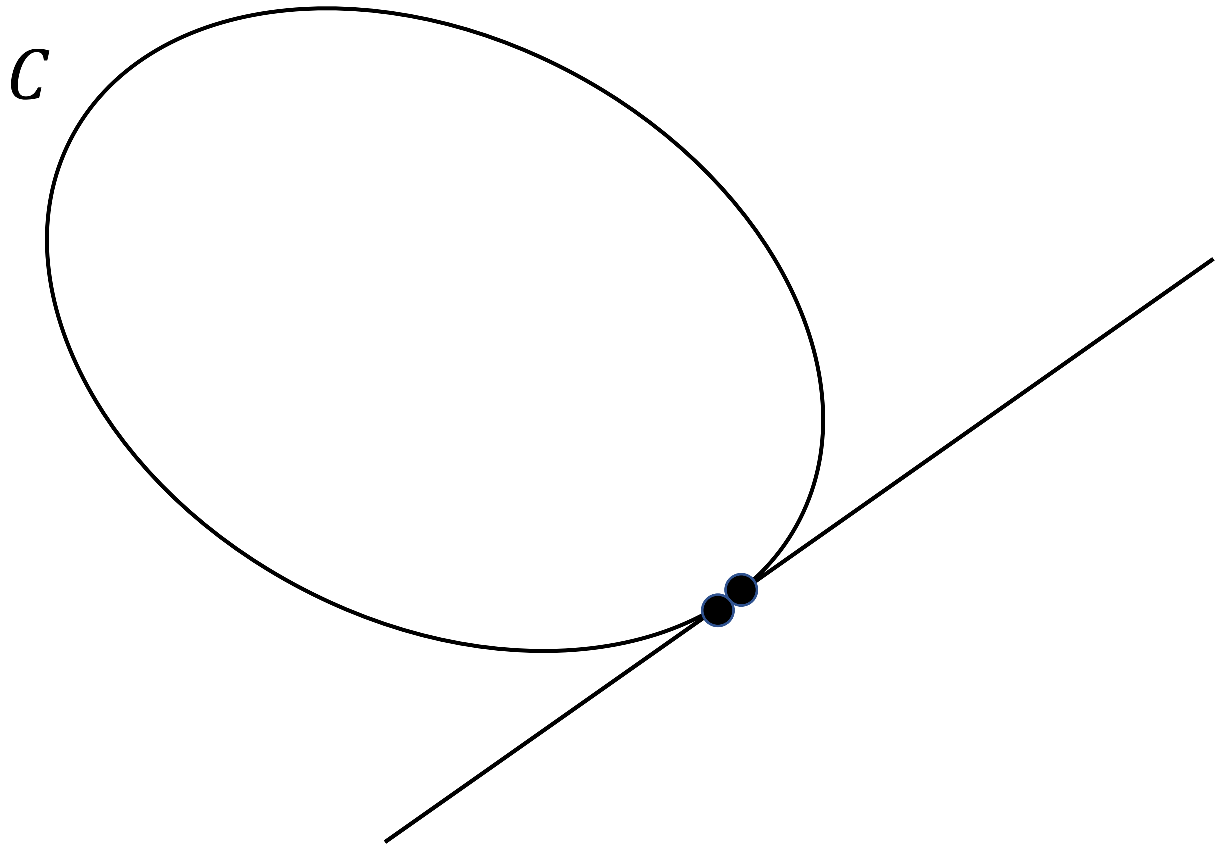


Line – conic intersection: two real coincident solutions → **tangency**

.



Line – conic intersection: two complex-conjugate solutions

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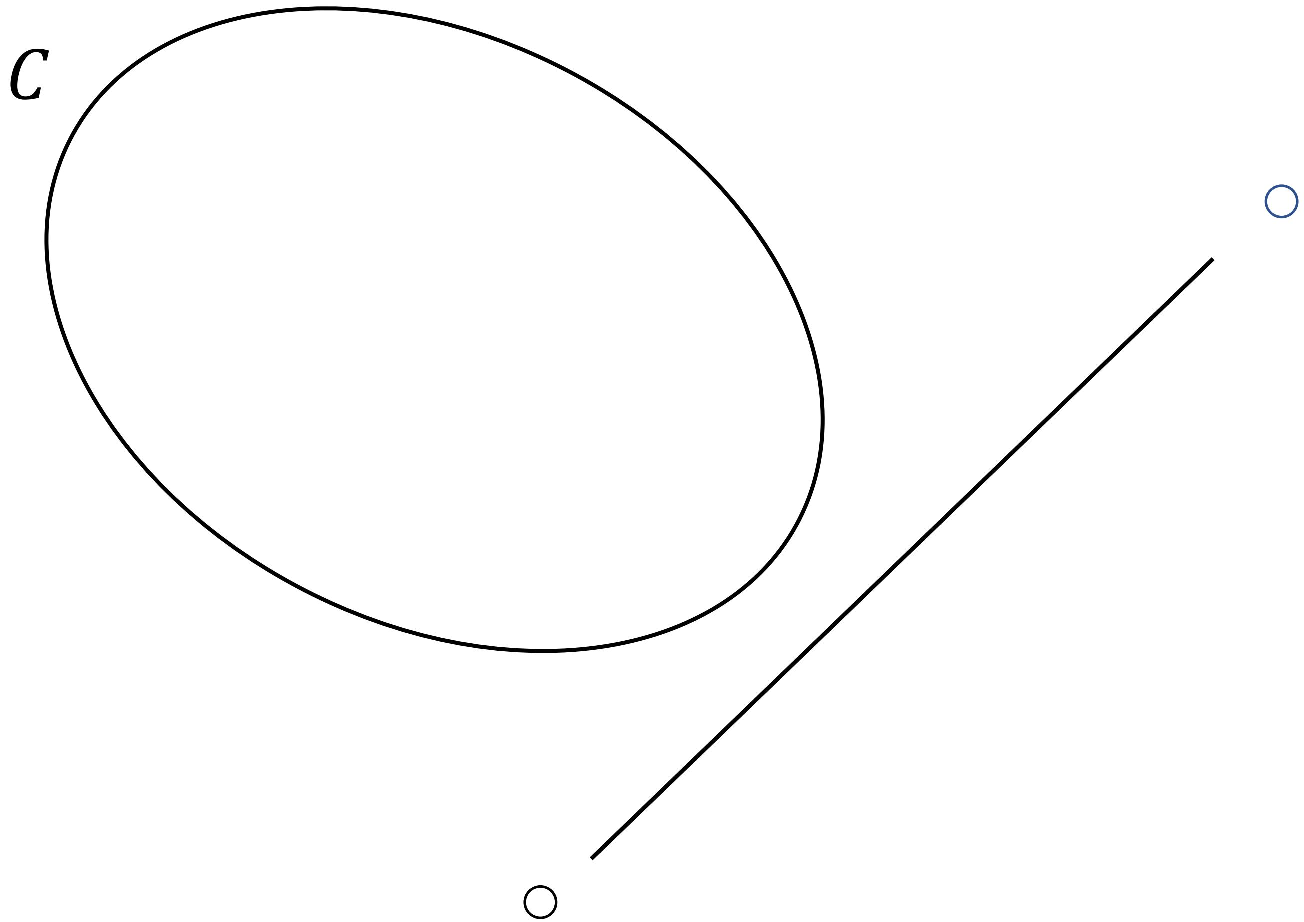
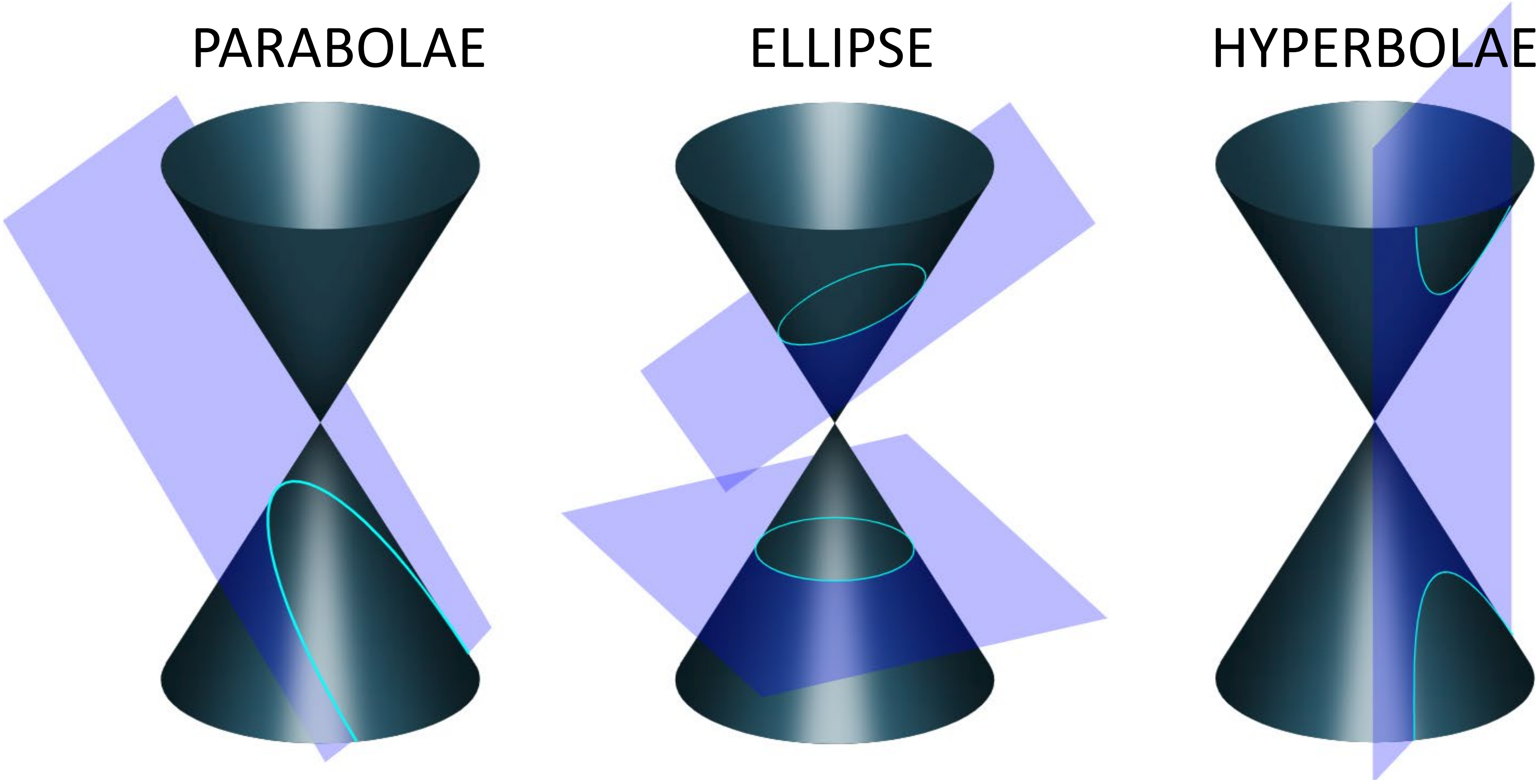


INTERSECTION between the **LINE AT THE ∞** and a CONIC

PARABOLAE ELLIPSE HYPERBOLAE



two coincident solutions: point at the ∞ along axis

two complex-conjugate solutions: no real solution

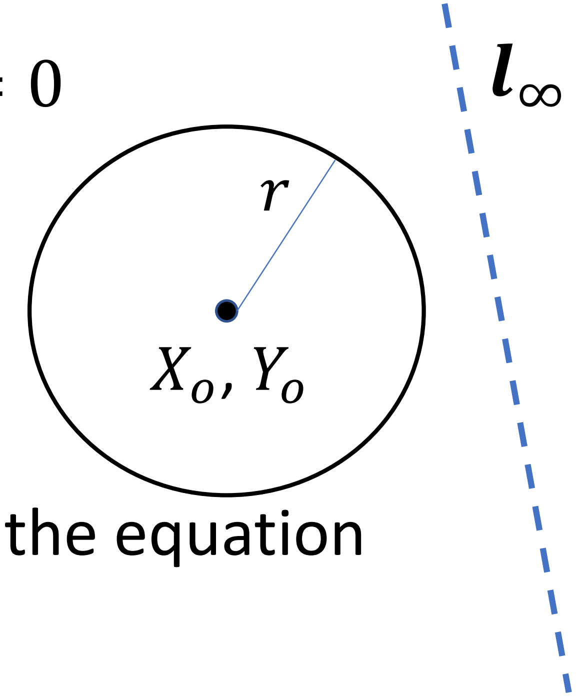
two real distinct solutions: asymptotes

The circular points

- useful in 2D reconstruction

A noteworthy example: intersecting a circumference and the line at the ∞

$$\begin{cases} x^2 - 2X_o w + X_o^2 w^2 + y^2 - 2Y_o w + Y_o^2 w^2 - r^2 w^2 = 0 \\ w = 0 \end{cases}$$



$$\rightarrow$$

$$\begin{cases} x^2 + y^2 = 0 \\ w = 0 \end{cases}$$

The circumference parameters (center and radius) disappear from the equation

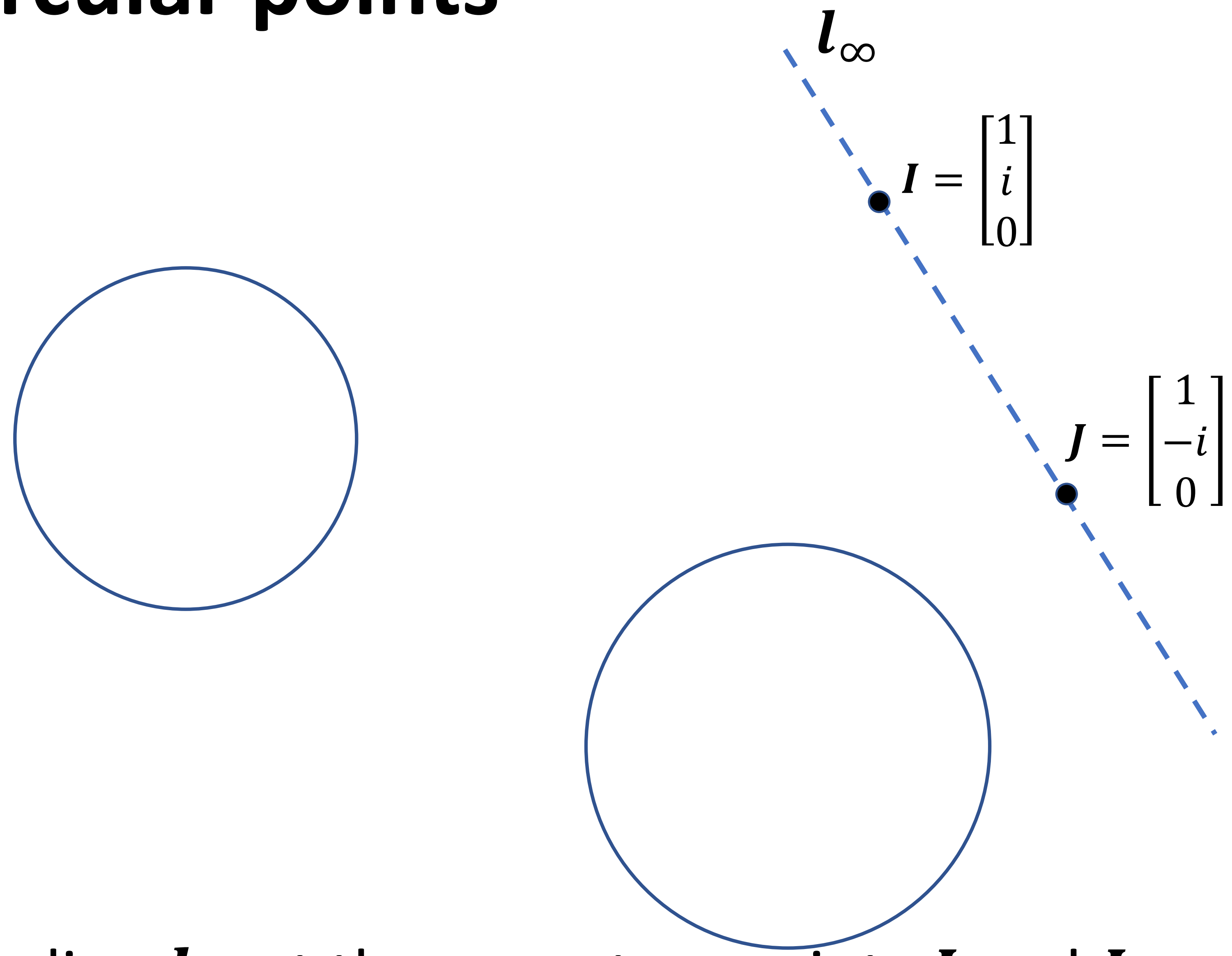
$$\rightarrow$$

the two intersection points are the **same for all** circumferences:

$\boldsymbol{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$ and $\boldsymbol{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$ (i is the imaginary unit number)

These points deserve a special name: the **CIRCULAR POINTS**

The circular points



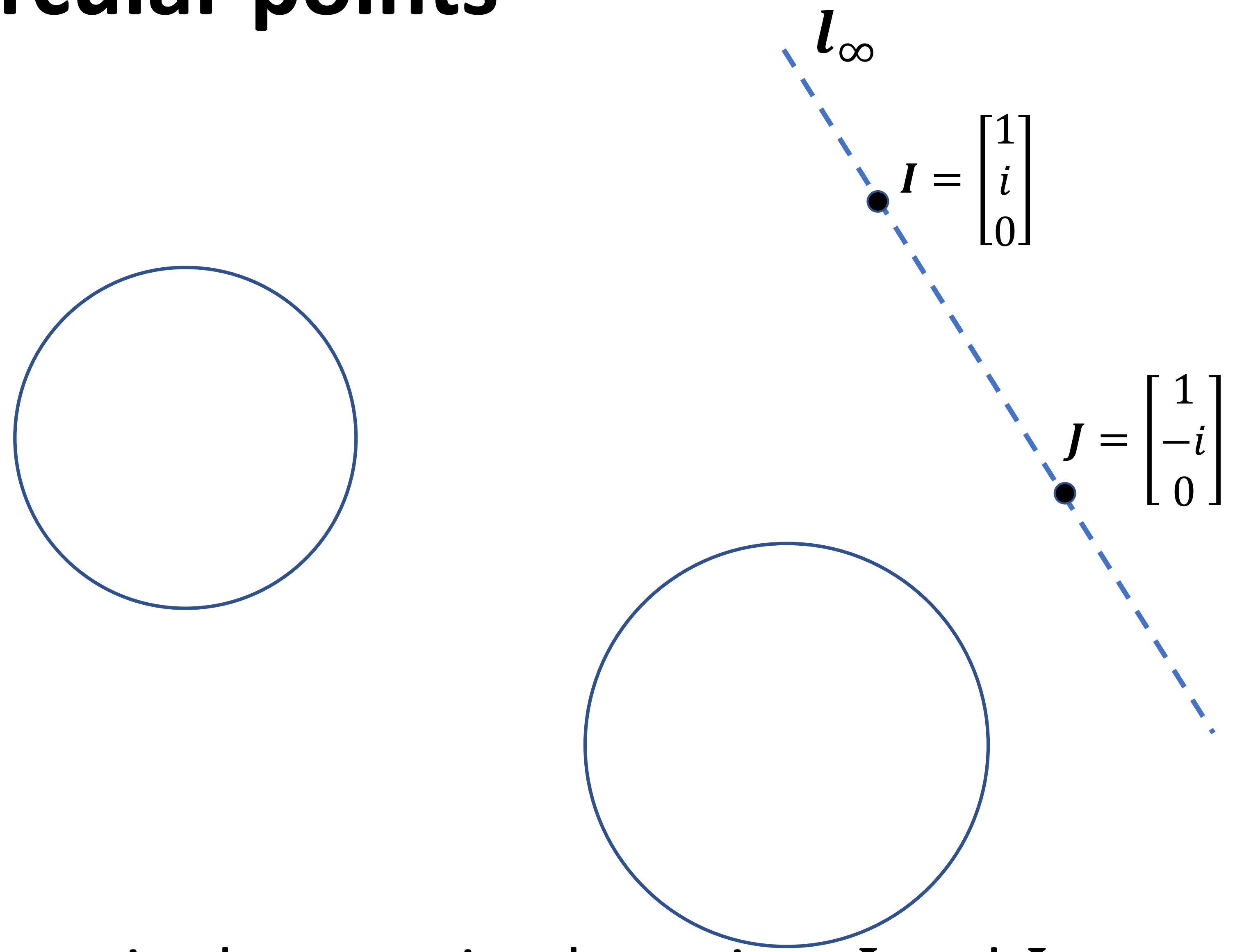
All circumferences cross the line l_∞ at the same two points I and J

The circular points

Remember their coordinates

$$\boldsymbol{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$



All the circumferences contain the two circular points $\boldsymbol{I}$ and $\boldsymbol{J}$

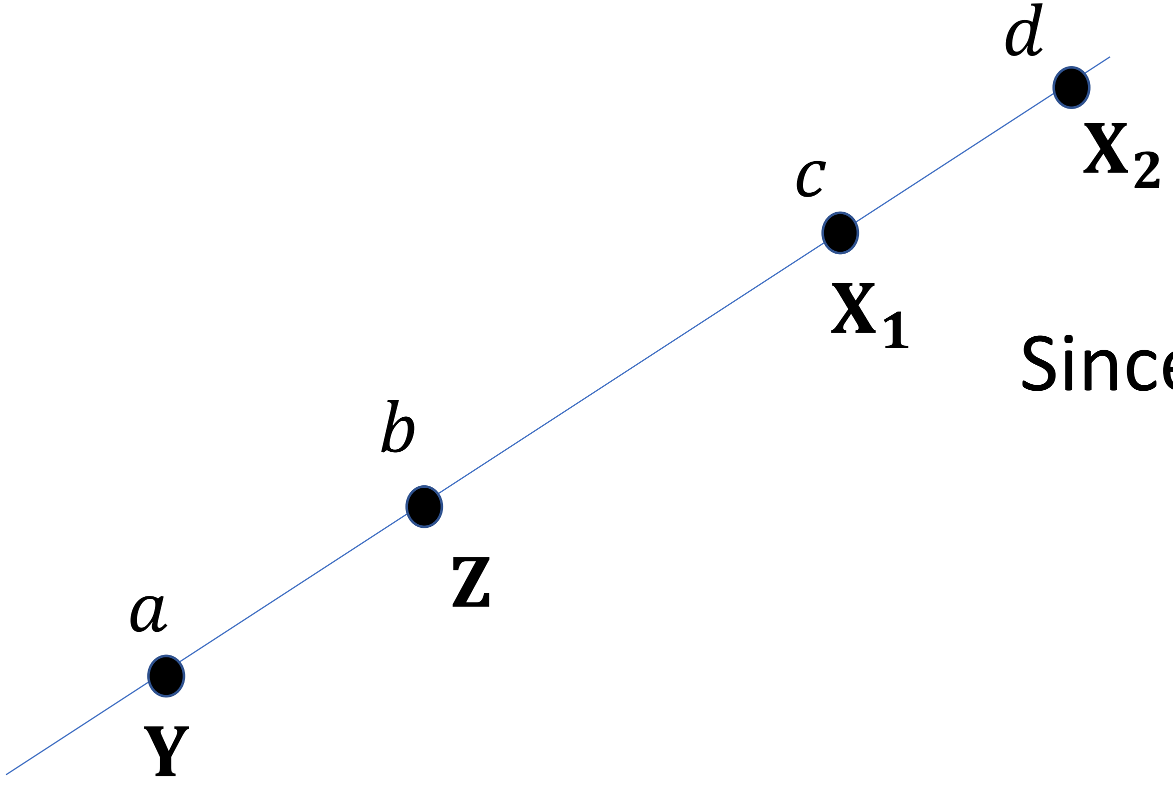
POLARITY

Polarity (conjugacy)
is the projective extension of symmetry

→

useful in the analysis of images of scenes containing symmetric objects (circles, spheres, right cones, right cylinders, planar mirrors)

Cross ratio of a 4-tuple of colinear points $\boldsymbol{CR_{Y,Z,X_1,X_2}} = \frac{\boldsymbol{c-a}}{\boldsymbol{c-b}} / \frac{\boldsymbol{d-a}}{\boldsymbol{d-b}}$ and linear combinations



Since $\mathbf{X_1}$ and $\mathbf{X_2}$ are colinear with $\mathbf{Y}, \mathbf{Z}$

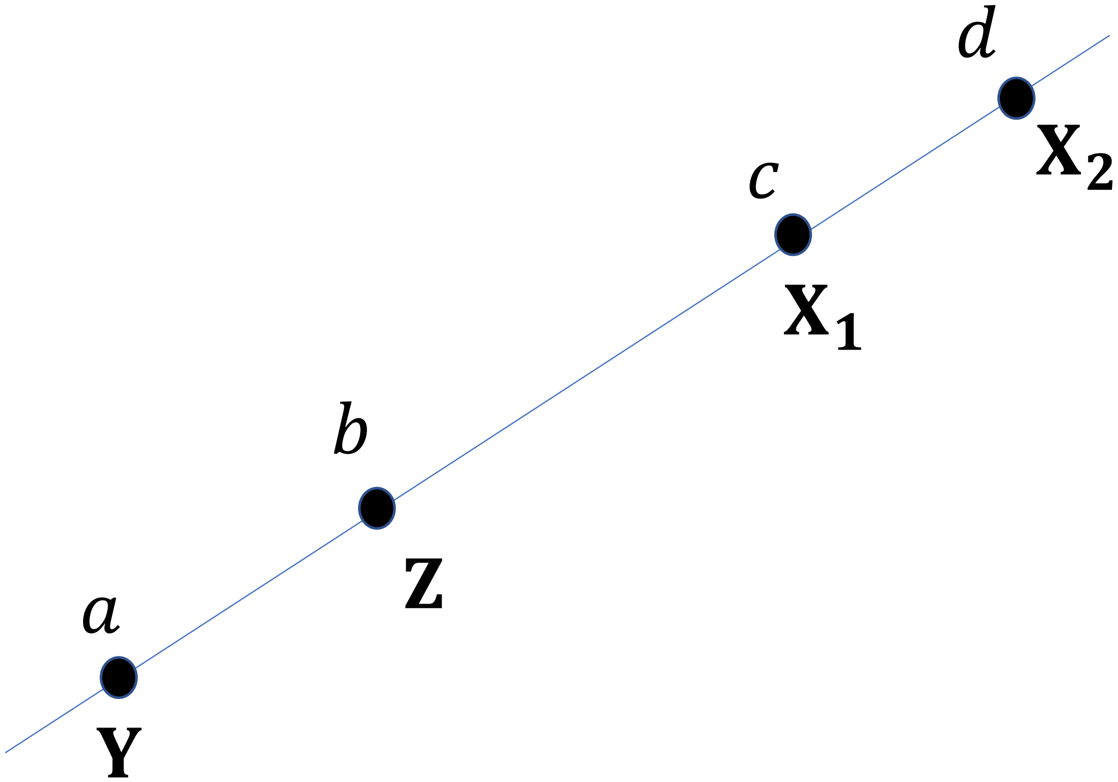
$$\mathbf{X_1} = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

and

$$\mathbf{X_2} = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

Cross ratio and linear combination coefficients: A result

$$\boldsymbol{CR}_{\mathbf{Y},\mathbf{Z},\mathbf{X}_1,\mathbf{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$



Proof: since the abscissae are proportional to, e.g., the X cartesian coordinates, we can replace the abscissae by the X coordinates (let us the same names …)

Let $\mathbf{Y} = \begin{bmatrix} y \\ * \\ v \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} z \\ * \\ w \end{bmatrix}$: then $\boldsymbol{X_1} = \begin{bmatrix} \alpha_1 y + \beta_1 z \\ * \\ \alpha_1 v + \beta_1 w \end{bmatrix}$ and $\boldsymbol{X_2} = \begin{bmatrix} \alpha_2 y + \beta_2 z \\ * \\ \alpha_2 v + \beta_2 w \end{bmatrix}$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Y}$ is

$$c - a = \frac{(\alpha_1 y+\beta_1 z)v-(\alpha_1 v+\beta_1 w)y}{(\alpha_1 y+\beta_1 z)v} = \frac{\beta_1(zv-yw)}{(\alpha_1 y+\beta_1 z)v}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Z}$ is

$$c - b = \frac{(\alpha_1 y+\beta_1 z)w-(\alpha_1 v+\beta_1 w)z}{(\alpha_1 y+\beta_1 z)w} = \frac{-\alpha_1(zv-yw)}{(\alpha_1 y+\beta_1 z)w}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Y}$ is

$$c-a=\frac{(\boldsymbol{\alpha_1 y+\beta_1 z})\boldsymbol{v}-(\boldsymbol{\alpha_1 v+\beta_1 w})\boldsymbol{y}}{(\boldsymbol{\alpha_1 y+\beta_1 z})\boldsymbol{v}}=\frac{\boldsymbol{\beta_1}(\boldsymbol{zv-yw})}{(\boldsymbol{\alpha_1 y+\beta_1 z})\boldsymbol{v}}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Z}$ is

$$c-b=\frac{(\boldsymbol{\alpha_1 y+\beta_1 z})\boldsymbol{w}-(\boldsymbol{\alpha_1 v+\beta_1 w})\boldsymbol{z}}{(\boldsymbol{\alpha_1 y+\beta_1 z})\boldsymbol{w}}=\frac{-\boldsymbol{\alpha_1}(\boldsymbol{zv-yw})}{(\boldsymbol{\alpha_1 y+\beta_1 z})\boldsymbol{w}}$$

The ratio between these differences is $\frac{c-a}{c-b}=-\frac{\boldsymbol{\beta_1}}{\boldsymbol{\alpha_1}}\frac{w}{v}$ and, similarly,

$$\frac{d-a}{d-b}=-\frac{\boldsymbol{\beta_2}}{\boldsymbol{\alpha_2}}\frac{w}{v}$$

Thus the cross ratio of the four colinear points $\boldsymbol{Y},\boldsymbol{Z},\mathbf{X_1},\mathbf{X_2}$ is

$$\boldsymbol{CR_{Y,Z,\mathrm{X}_1,\mathrm{X}_2}}=\frac{\boldsymbol{c-a}}{\boldsymbol{c-b}}\bigg/\frac{\boldsymbol{d-a}}{\boldsymbol{d-b}}=\frac{\boldsymbol{\beta_1}}{\boldsymbol{\alpha_1}}\bigg/\frac{\boldsymbol{\beta_2}}{\boldsymbol{\alpha_2}}$$

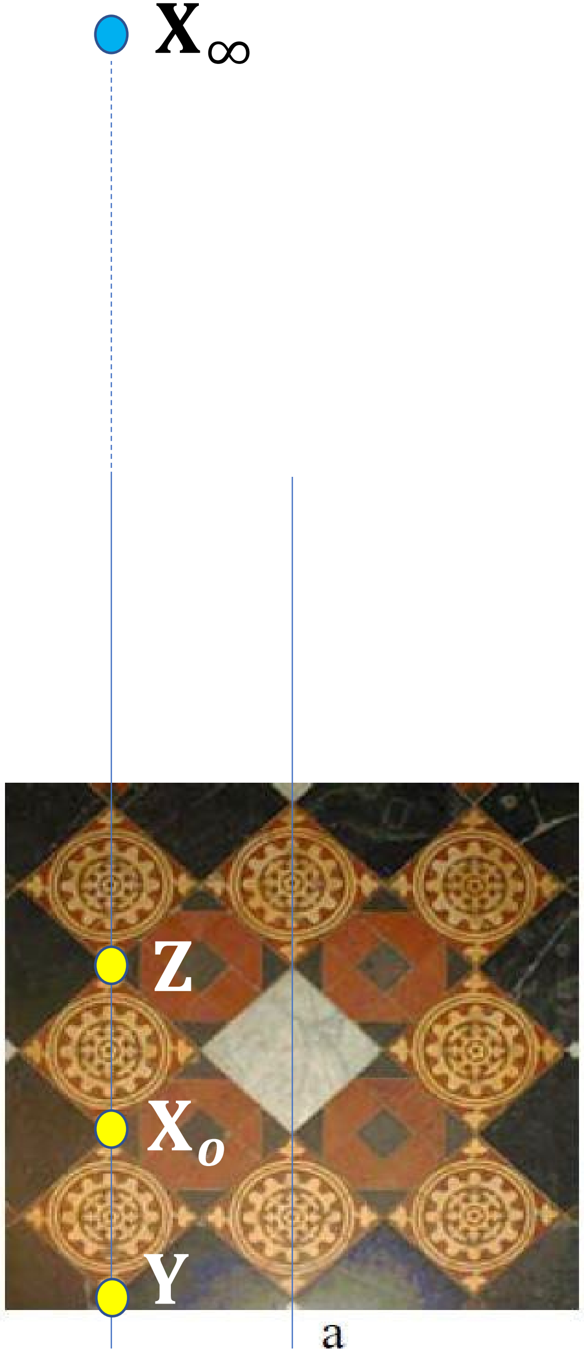
Q.E.D.

CROSS RATIO SPECIAL VALUE: -1 (*harmonic* 4-tuple)

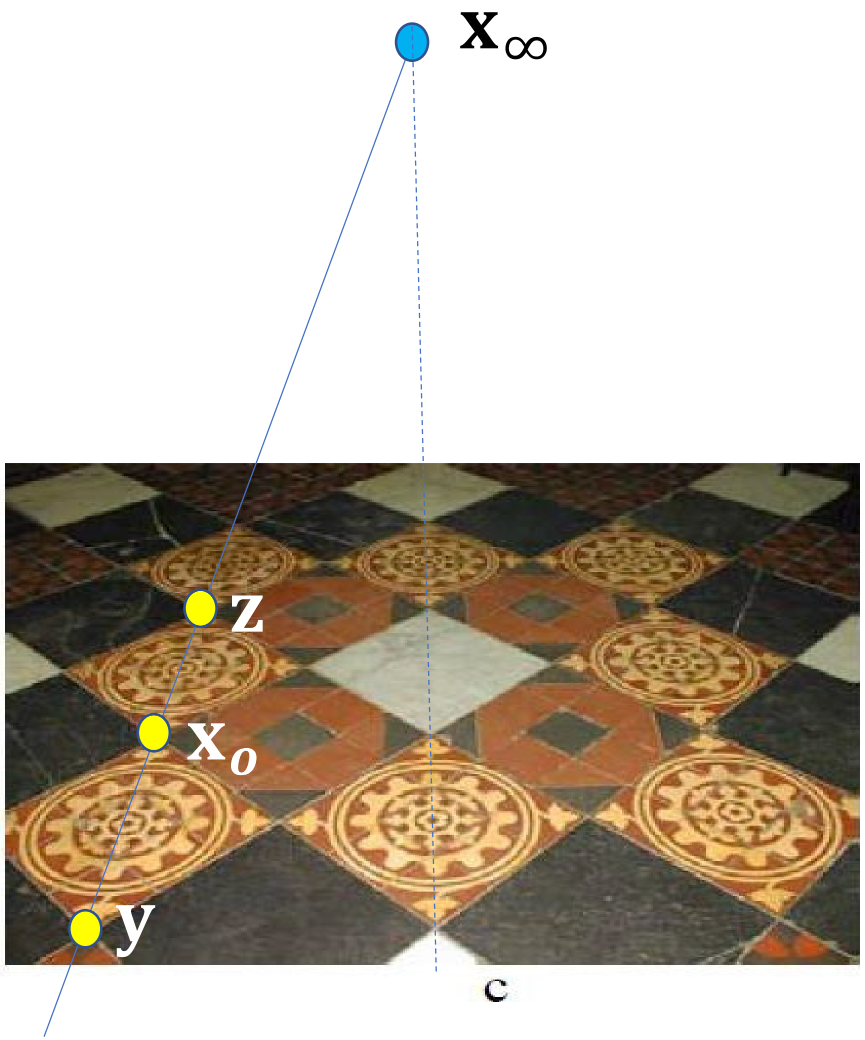
When $\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X_1},\mathbf{X_2}} = \mathbf{-1}$, $\mathbf{y}$ and $\mathbf{z}$ are said to be **conjugate** wrt $\mathbf{x}_1$ and $\mathbf{x}_2$

Given the segment $\boldsymbol{Y},\boldsymbol{Z}$, an instance of two conjugate points wrt $\boldsymbol{Y},\boldsymbol{Z}$ is pair ($\mathbf{X_0}$ = midpoint of $\boldsymbol{Y},\boldsymbol{Z}$, $\mathbf{X}_\infty$ = point at the infinity along $\boldsymbol{Y},\boldsymbol{Z}$)

Other instances: since the cross ratio of 4 colinear points is invariant under image projection, other instances of conjugate points are images $\mathbf{y},\mathbf{z}$ of segment endpoints, image $\mathbf{x}_{\boldsymbol{o}}$ of midpoint and vanishing point $\mathbf{x}_\infty$ (image of point at the infinity)



a



c

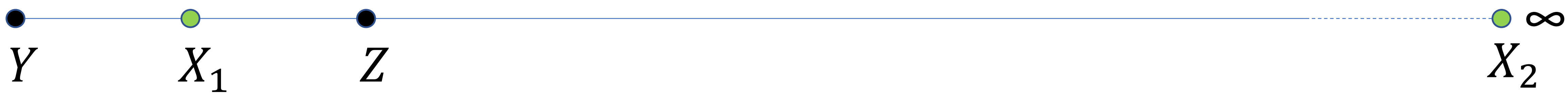
Harmonic 4-tuples and conjugate points

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = -1}$$

$\boldsymbol{X_1}$ **and** $\boldsymbol{X_2}$ **are conjugate wrt** $(\boldsymbol{Y}, \boldsymbol{Z})$

$\boldsymbol{Y}$ **and** $\boldsymbol{Z}$ **are conjugate wrt** $(\boldsymbol{X_1}, \boldsymbol{X_2})$



best-known case:

- two points Y and Z are the endpoints of a segment

$-$ X_1 and X_2 are the midpoint of (Y, Z) and the point at the ∞

Harmonic 4-tuples and conjugate points

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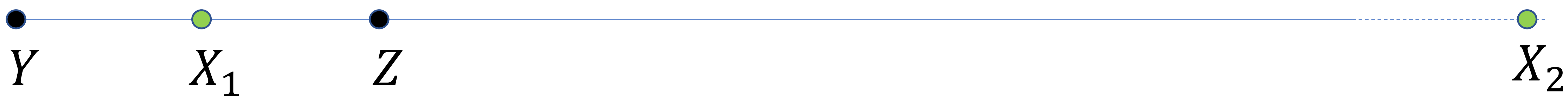

Other case: to preserve the value of the cross ratio = -1:
the «far» point X_2 approaches a little bit
and the other point X_1 also moves to get a bit closer to X_2

Harmonic 4-tuples and conjugate points

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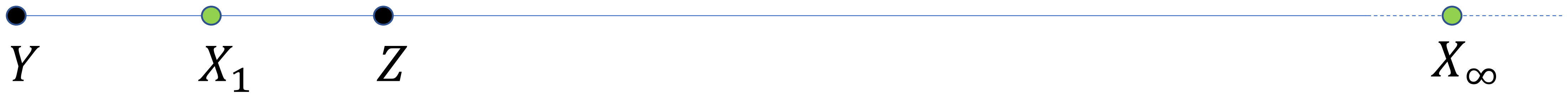

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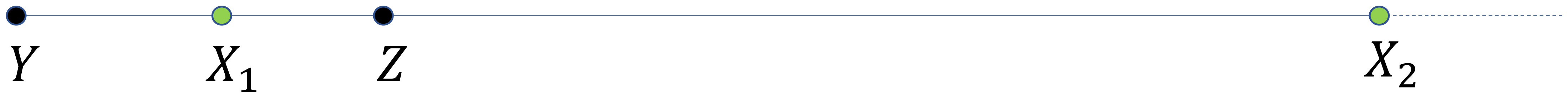

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Harmonic 4-tuples and conjugate points

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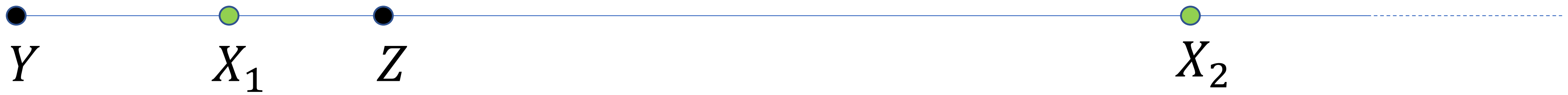

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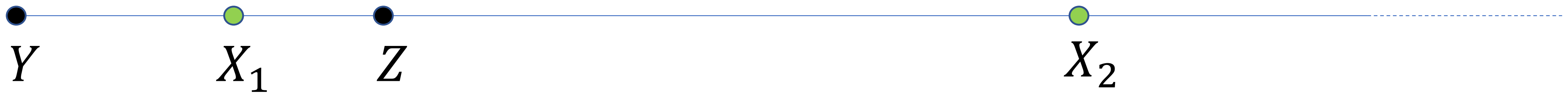

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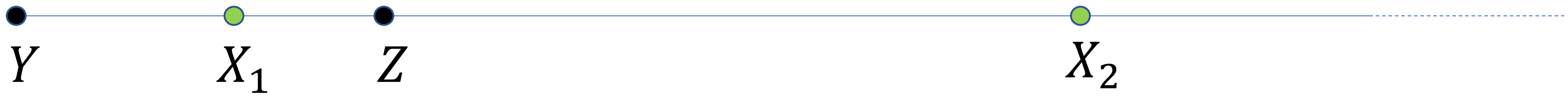

If we revert X_1 and X_2 , the cross ratio inverts its value:
but the inverse of -1 is itself!

→

In a harmonic tuple we can revert the roles of X_1 and X_2

→

X_1 and X_2 are **conjugate** points wrt Y and Z, namely both
X_1 is **conjugate** to X_2 wrt Y and Z, and X_2 is **conjugate** to X_1 wrt Y and Z

Harmonic 4-tuples and conjugate points

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = -1}$$

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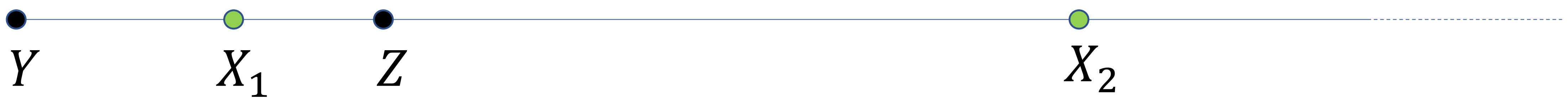


NOTE:

if X_1 and X_2 are **conjugate** points wrt Y and Z

since the cross ratio = -1 has a negative value, the simple ratios

$\boldsymbol{\frac{c-a}{c-b}}$ and $\boldsymbol{\frac{d-a}{d-b}}$ have opposite signs: i.e. one is positive and the other is negative

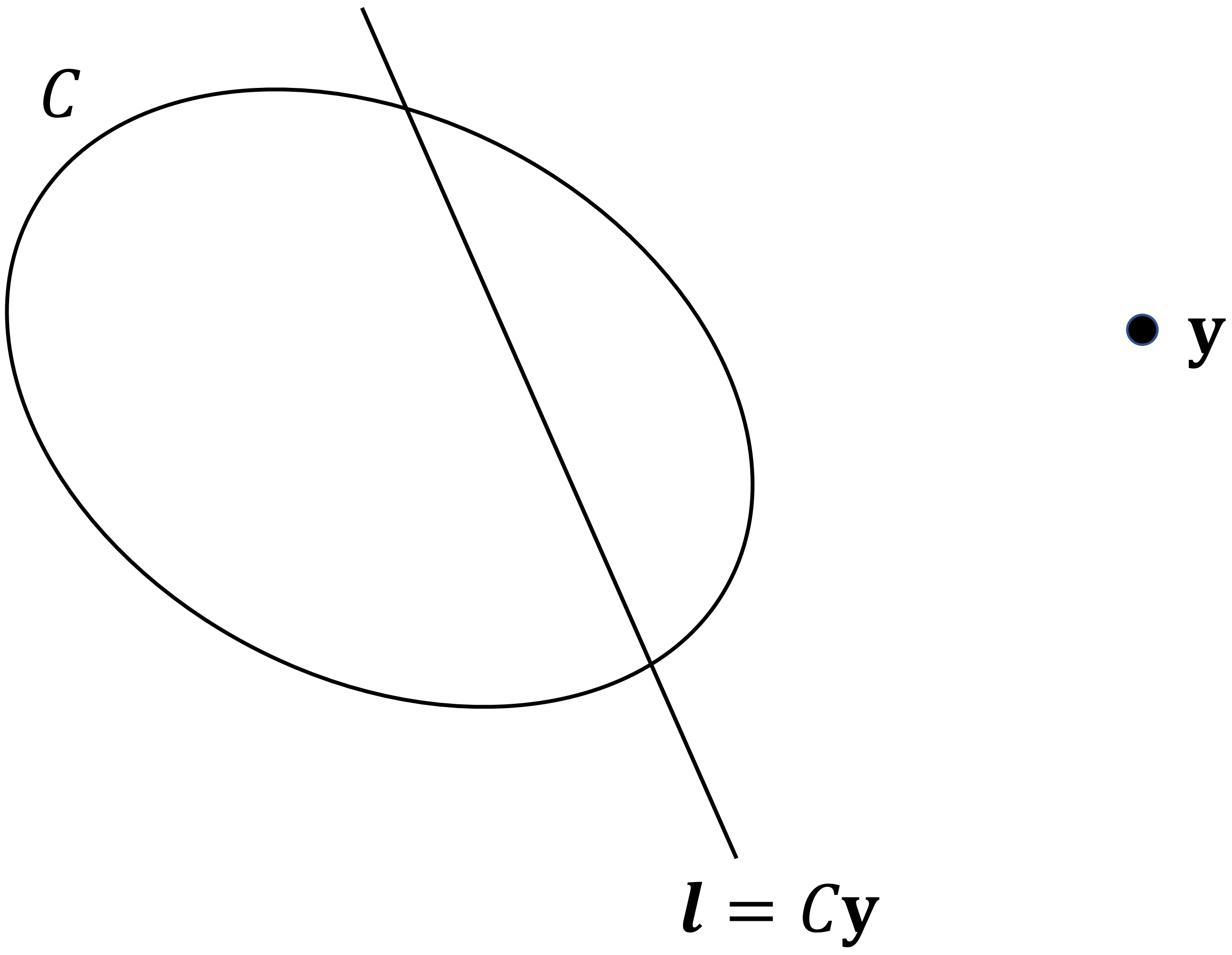
$\rightarrow$

one among X_1 and X_2 is **internal** to the segment YZ, while the other is **external**

The polar line of a point wrt a conic

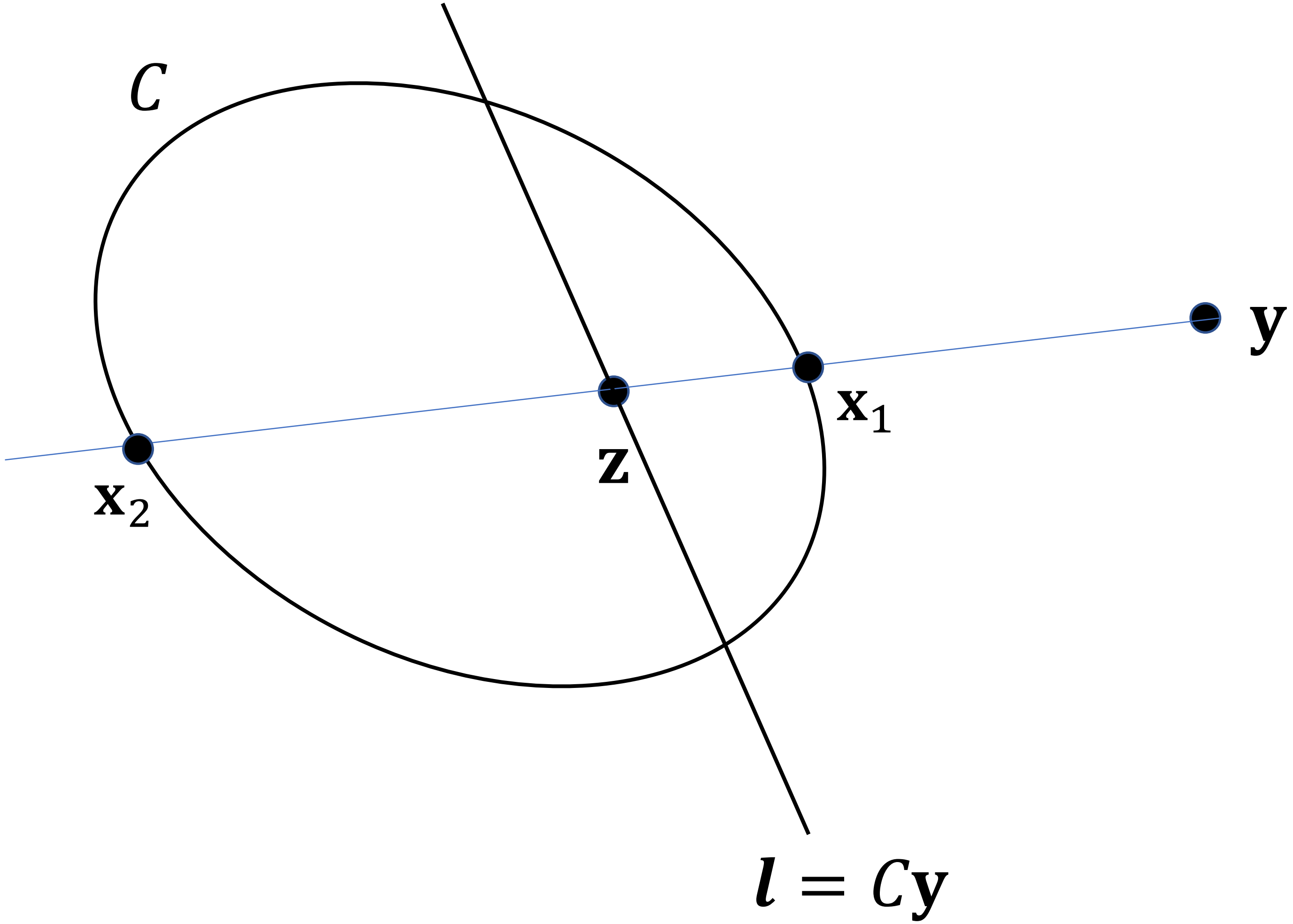
Polar line of a point wrt a conic

Given a point $\mathbf{y}$ and a conic C in the plane, the line $\boldsymbol{l} = C\mathbf{y}$ is called the *polar line* of point $\mathbf{y}$ with respect to the conic C.



Polar line and cross ratio (= -1)

Theorem. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points where the line through $\mathbf{y}$ and $\mathbf{z}$ crosses C: cross ratio $\boldsymbol{CR_{Y,Z,X_1,X_2}} = \mathbf{-1}$ ($\mathbf{y}$ and $\mathbf{z}$ are said to be **conjugate** wrt $\mathbf{x}_1$ and $\mathbf{x}_2$)



Exercise: prove it!

(or see next page)

Theorem. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points where the line through $\mathbf{y}$ and $\mathbf{z}$ crosses C:

$\mathbf{y}$ and $\mathbf{z}$ are **conjugate** with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$, i.e. their cross ratio = -1.

Proof sketch. $\mathbf{x}_1$ and $\mathbf{x}_2$ are the two solutions (for $\mathbf{x}$) to the intersection problem:

$\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$ (since $\mathbf{x}$ is on line $(\mathbf{y}, \mathbf{z})$) AND $\mathbf{x}^T C\,\mathbf{x} = \mathbf{0}$ (since $\mathbf{x}$ is on conic C). Namely, $(\alpha\mathbf{y} + \beta\mathbf{z})^T C(\alpha\mathbf{y} + \beta\mathbf{z}) = 0$, with $\mathbf{z}$ on line $\boldsymbol{l} = C\mathbf{y}$, i.e. $\mathbf{z}^T C\,\mathbf{y} = 0 = \mathbf{y}^T C\mathbf{z}$ (remember C is symmetric). The remaining terms are therefore $\alpha^2\mathbf{y}^T C\mathbf{y} + \beta^2\mathbf{z}^T C\mathbf{z} = \mathbf{0}$. Thus there are two opposite solutions for the ratio β/α:

$$\beta/\alpha = \pm\sqrt{-\mathbf{y}^T C\mathbf{y}/\mathbf{z}^T C\mathbf{z}} \qquad \text{i.e.} \ \frac{\beta_2}{\alpha_2} = -\frac{\beta_1}{\alpha_1}$$

Remember we are solving the intersection problem:

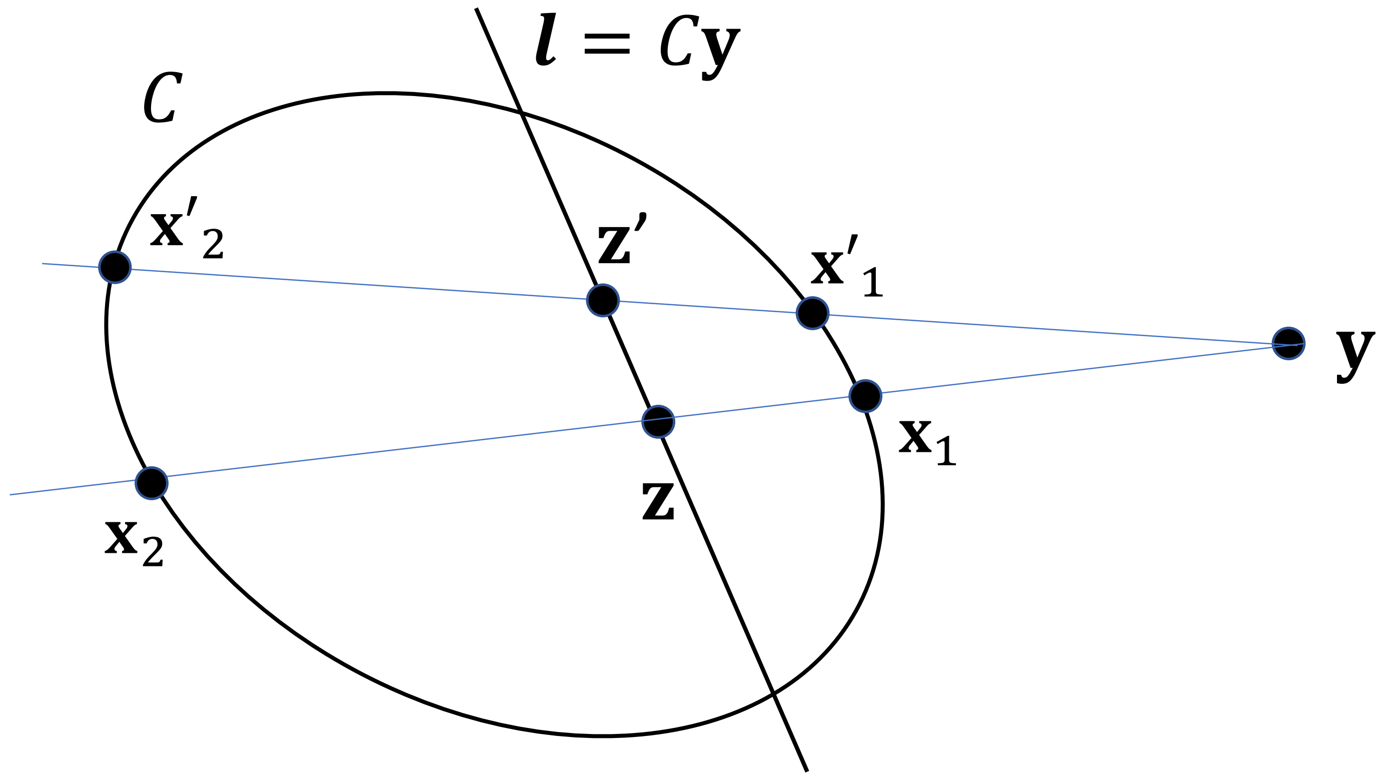
→ these two solutions correspond to the points $\mathbf{x}_1$ and $\mathbf{x}_2$

Hence $\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X}_1,\mathbf{X}_2} = \frac{\boldsymbol{\beta}_1}{\boldsymbol{\alpha}_1}\Big/\frac{\boldsymbol{\beta}_2}{\boldsymbol{\alpha}_2} = \mathbf{-1}$

The polar line $\boldsymbol{l} = C\mathbf{y}$ is the locus of points $\mathbf{z}$ conjugate of $\mathbf{y}$ wrt conic C (more precisely, conjugate wrt to the intersection points of C with any line through $\mathbf{y}$)

Polar line and cross ratio (= -1)

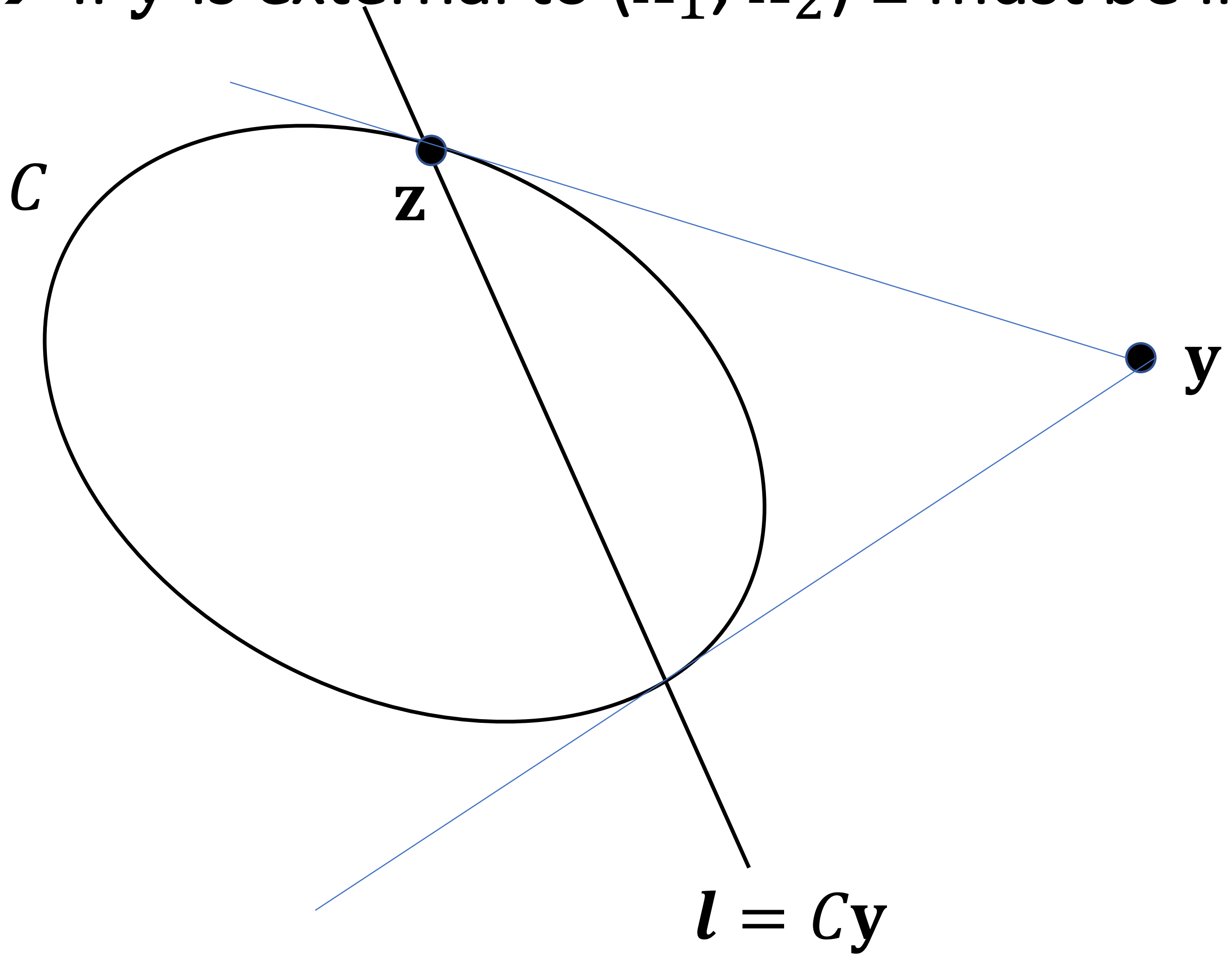
Theorem. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points where a line through $\mathbf{y}$ and $\mathbf{z}$ crosses C: cross ratio $\boldsymbol{CR_{Y,Z,X_1,X_2}} = \mathbf{-1}$ ($\mathbf{y}$ and $\mathbf{z}$ are said to be **conjugate** wrt $\mathbf{x}_1$ and $\mathbf{x}_2$)



Therefore, the polar line $\boldsymbol{l}$ of $\mathbf{y}$ wrt C is the set of points $\mathbf{z}$, that are conjugate to $\mathbf{y}$ wrt the intersection points $\mathbf{x}_1$ and $\mathbf{x}_2$ between C and any line through $\mathbf{y}$.

Polar line and tangency points

Property:
The polar line $\boldsymbol{l} = C\mathbf{y}$ goes through the tangency points from $\mathbf{y}$ to C
Geometric/Qualitative Proof: $\mathbf{y}$ and $\mathbf{z}$ conjugate wrt $\mathbf{x}_1$ and $\mathbf{x}_2$ → negative cross ratio → if $\mathbf{y}$ is external to $(\mathbf{x}_1, \mathbf{x}_2)$ $\mathbf{z}$ must be internal



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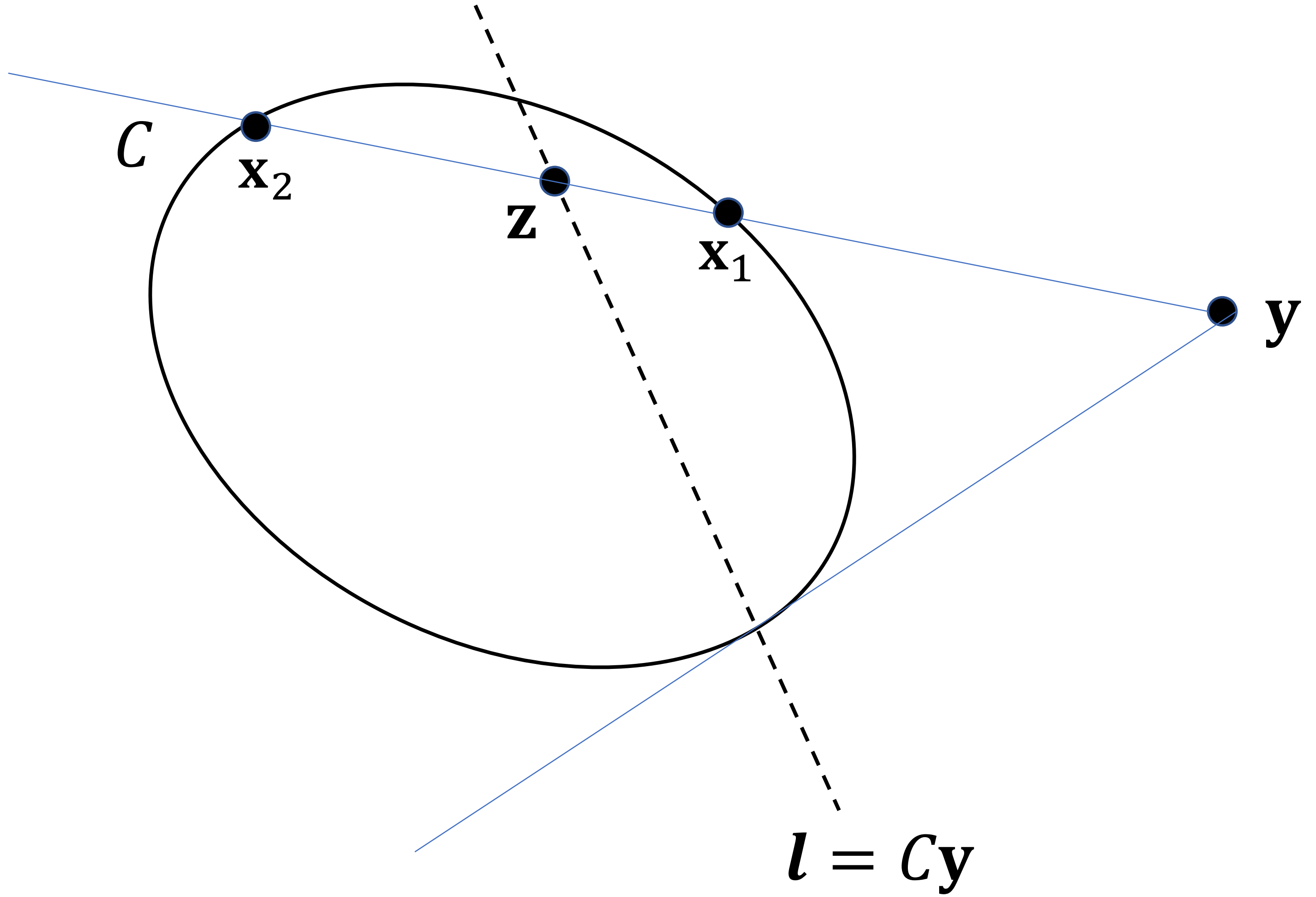

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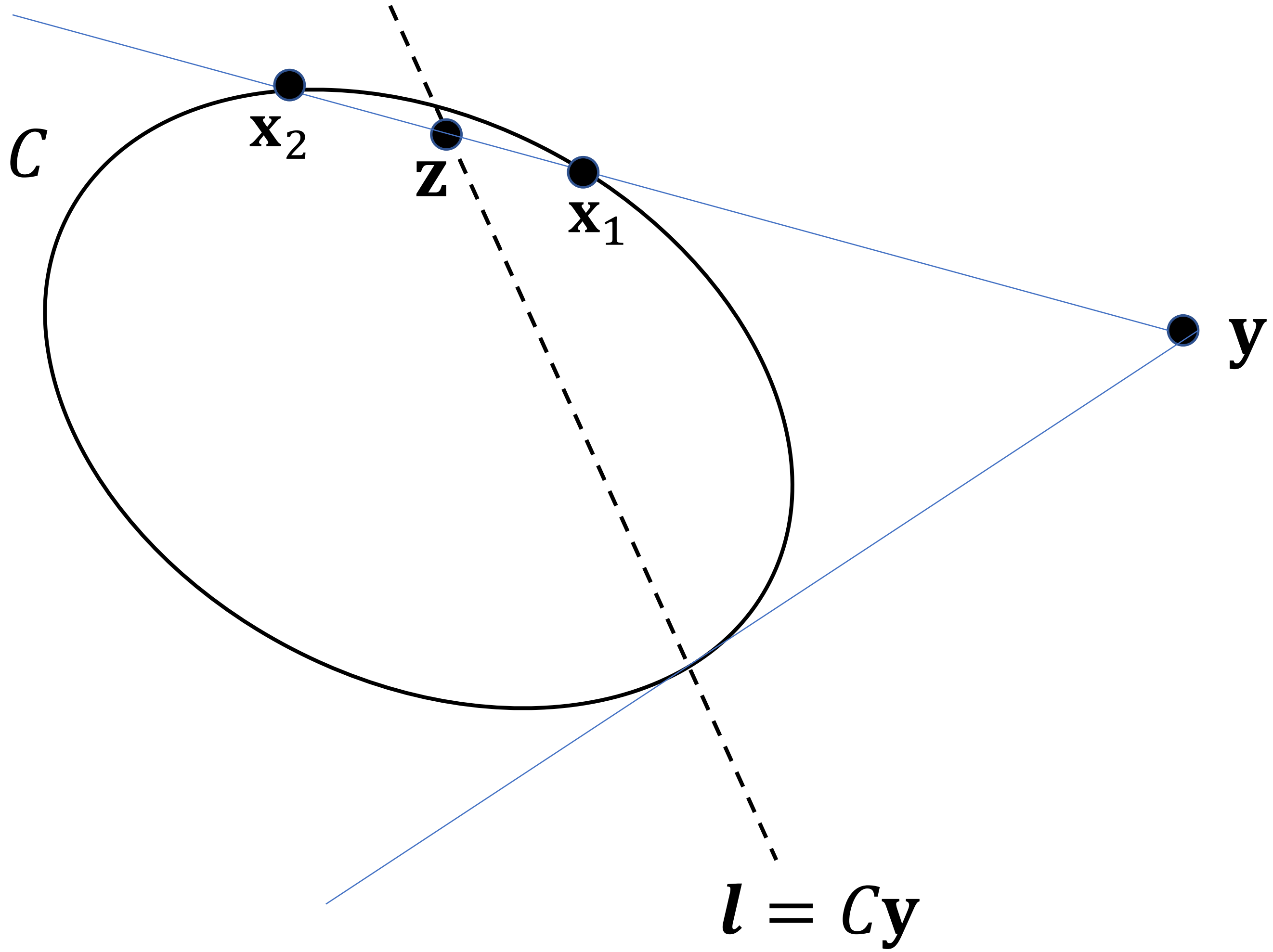

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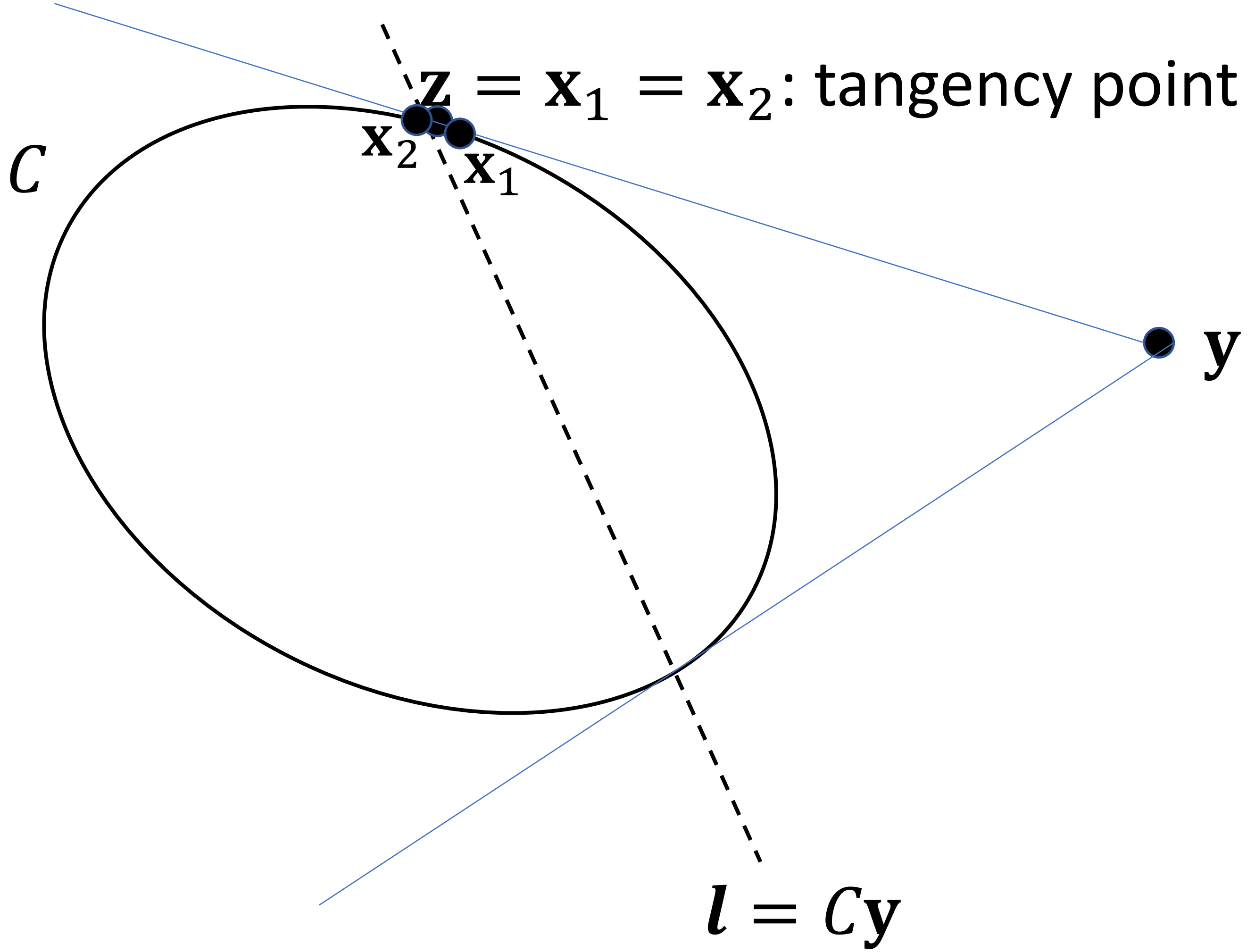
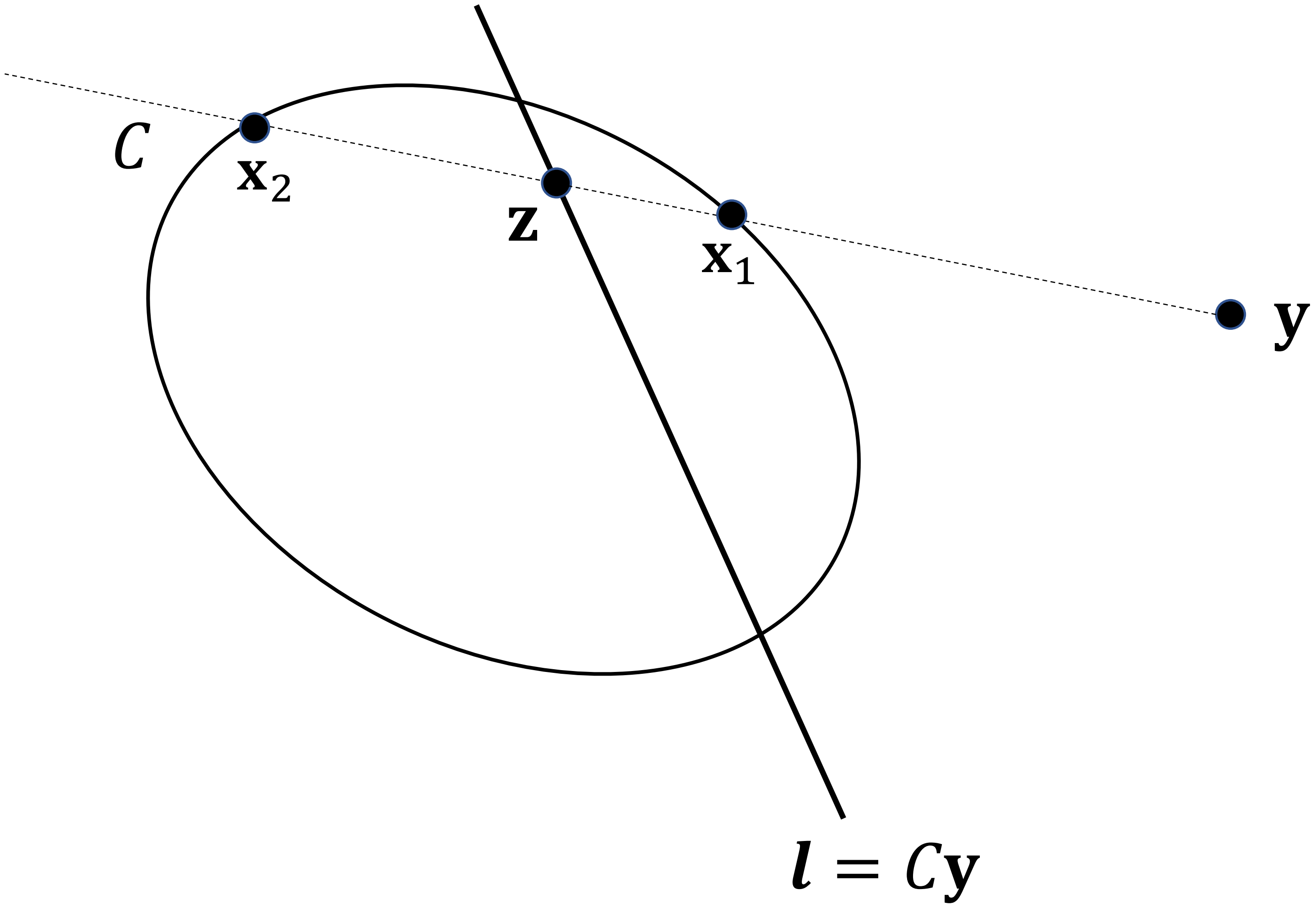


Property:

The polar line $\boldsymbol{l} = C\mathbf{y}$ goes through the tangency points from $\mathbf{y}$ to C

Algebraic Proof:

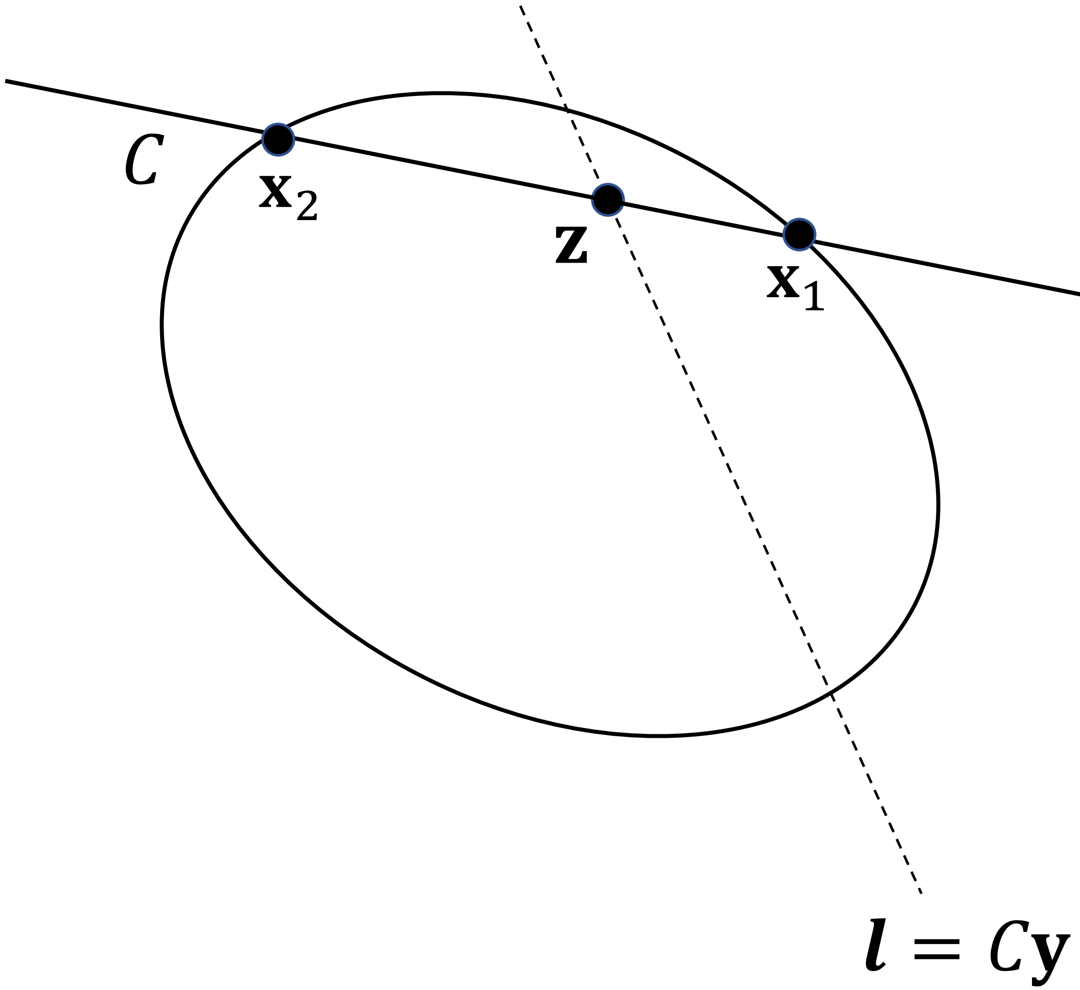
$\mathbf{z} \in \boldsymbol{l} = C\mathbf{y} \rightarrow \boldsymbol{l}^T\mathbf{z} = \mathbf{y}^T C\mathbf{z} = \mathbf{0}$



Algebraic Proof: Intersect polar line $\boldsymbol{l}$ with line joining $\mathbf{y}$ and $\mathbf{z}$

$\mathbf{z} \in \boldsymbol{l} = C\mathbf{y} \rightarrow \boldsymbol{l}^T\mathbf{z} = \mathbf{y}^T\boldsymbol{C}\mathbf{z} = \mathbf{0}$ and $\mathbf{z} + \alpha\mathbf{y} \in \boldsymbol{C} \rightarrow (\mathbf{z} + \alpha\mathbf{y})^T\boldsymbol{C}(\mathbf{z} + \alpha\boldsymbol{y}) = \mathbf{0}$

$\rightarrow \mathbf{z}^T\boldsymbol{C}\mathbf{z} + \alpha^2\, \mathbf{y}^T\boldsymbol{C}\mathbf{y} = \mathbf{0}$ two solutions $\alpha = \pm\sqrt{-\mathbf{z}^T\boldsymbol{C}\mathbf{z}/\mathbf{y}^T\boldsymbol{C}\mathbf{y}}$



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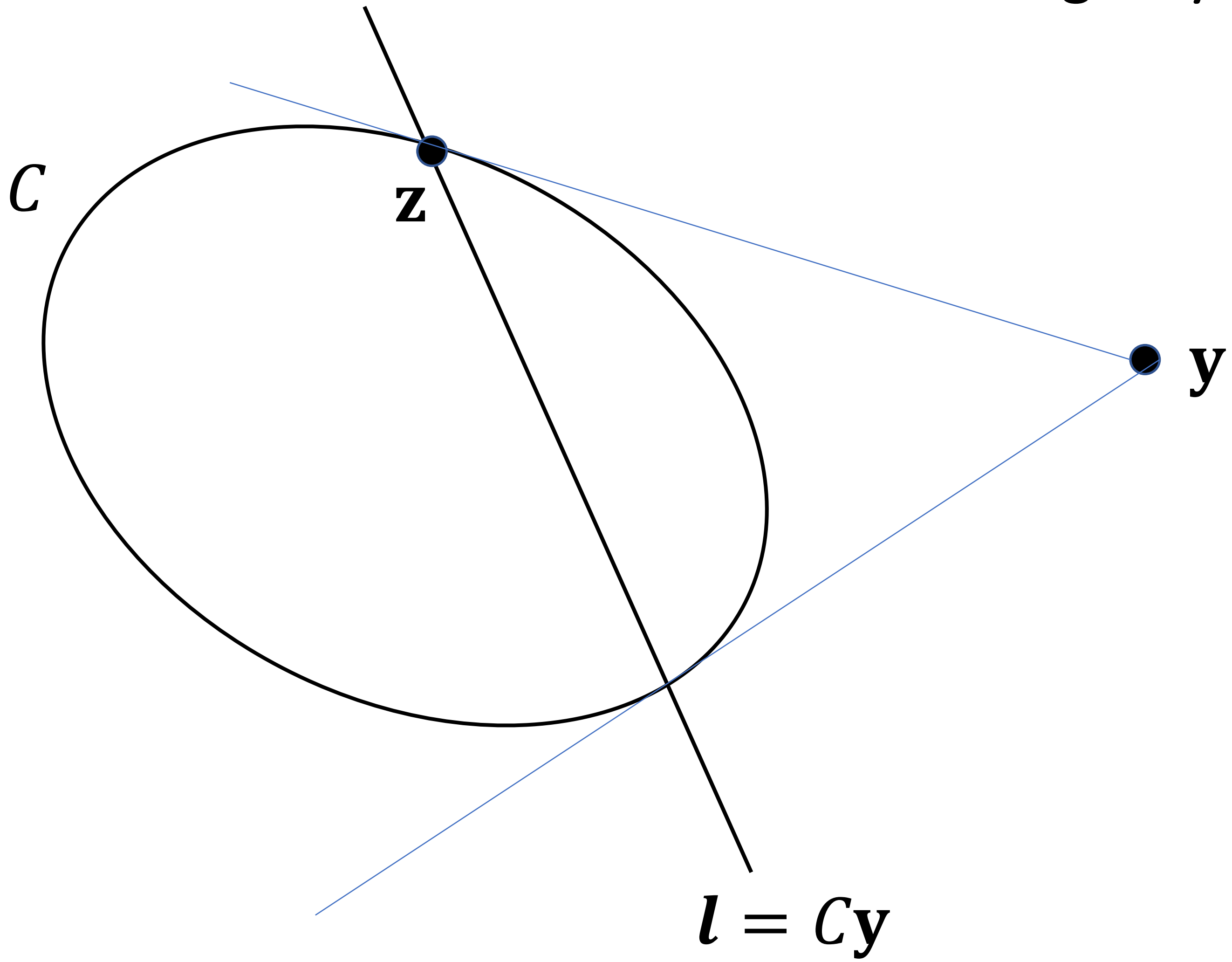
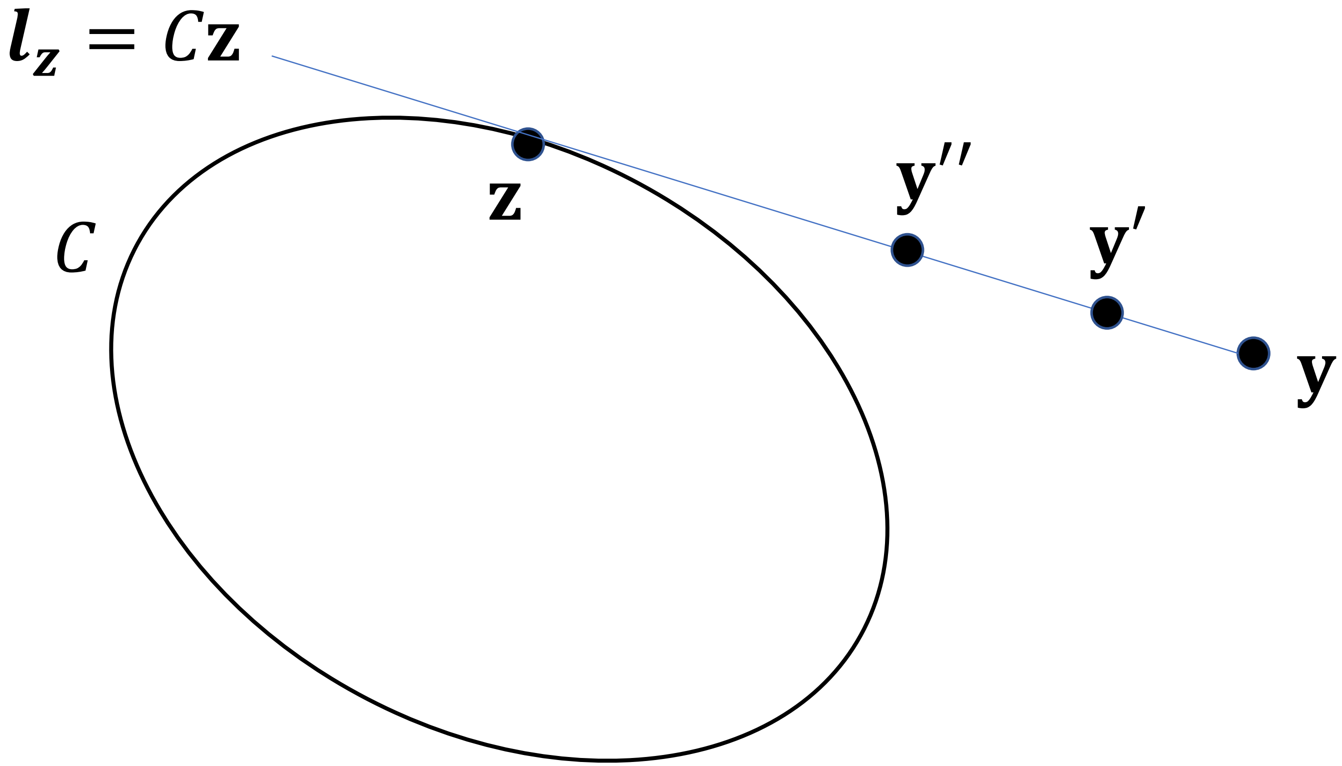
$\rightarrow \mathbf{z}^T\boldsymbol{C}\mathbf{z} + \alpha^2\, \mathbf{y}^T\boldsymbol{C}\mathbf{y} = \mathbf{0}$ two solutions $\alpha = \pm\sqrt{-\mathbf{z}^T\boldsymbol{C}\mathbf{z}/\mathbf{y}^T\boldsymbol{C}\mathbf{y}}$

tangency: one double solution when $-\mathbf{z}^T\boldsymbol{C}\mathbf{z} = \mathbf{0}$ i.e. $\mathbf{z} \in \boldsymbol{C} \rightarrow$ tangency point from y



The polar line of a point **ON** the conic C

The polar line $\boldsymbol{l_z} = C\mathbf{z}$ of a point the conic C is the tangent line to C through $\mathbf{z}$

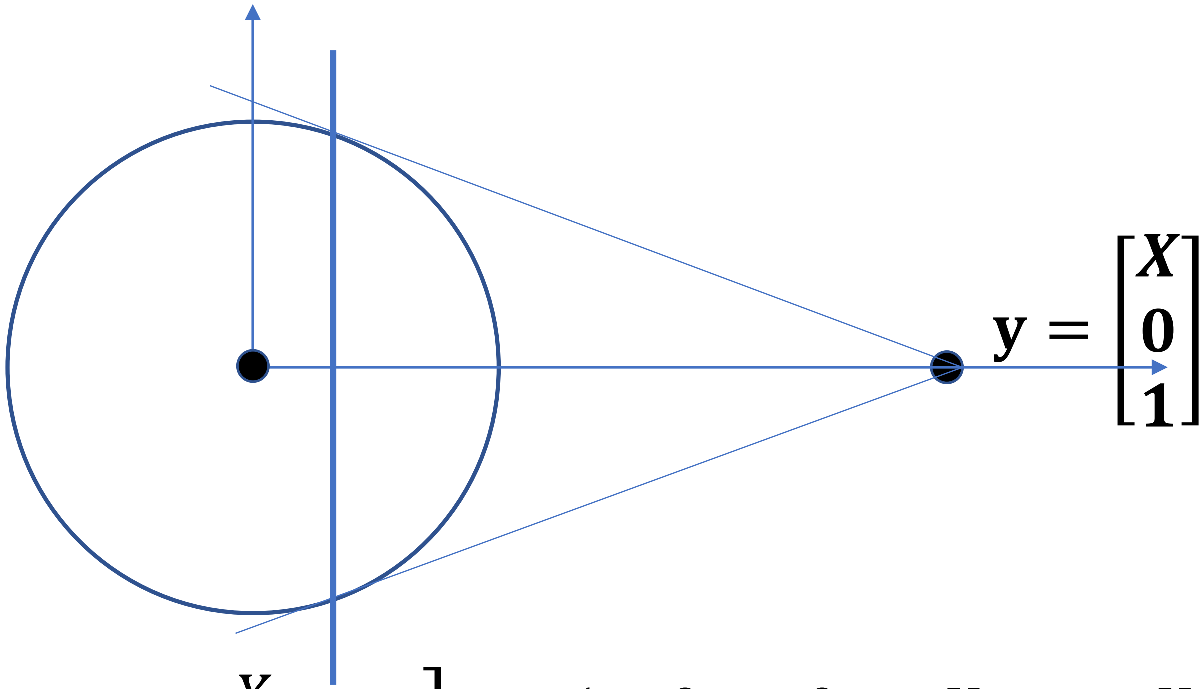


In fact, $\mathbf{z}$ on C is conjugate to any point $\mathbf{y}$ on the tangent to C through $\mathbf{z}$

Examples: the polar line with respect to a circumference

the polar of a point $\mathbf{y}$ wrt a circumference is a line perpendicular to the segment (*center*, $\mathbf{y}$)

Simple case: center on the origin, and point $\mathbf{y}$ on the X-axis

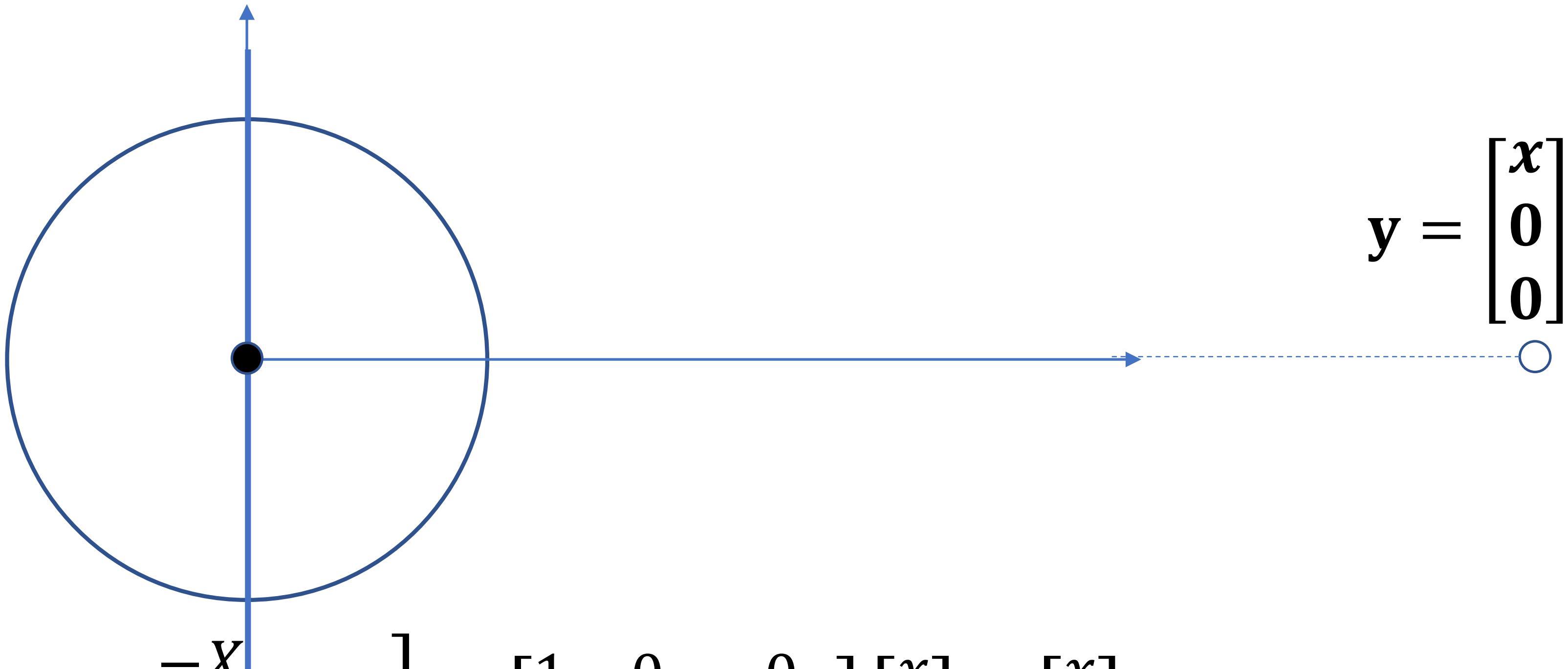


polar line: $\boldsymbol{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ -r^2 \end{bmatrix}$

the polar line (cartesian) equation X X $-r^2 = 0$ → X $= r^2/X$: a vertical line

the polar of a point ***at the*** ∞ wrt a circumference

Particular case: point $\mathbf{y}$ at the ∞



$$\mathbf{y} = \begin{bmatrix} \boldsymbol{x} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

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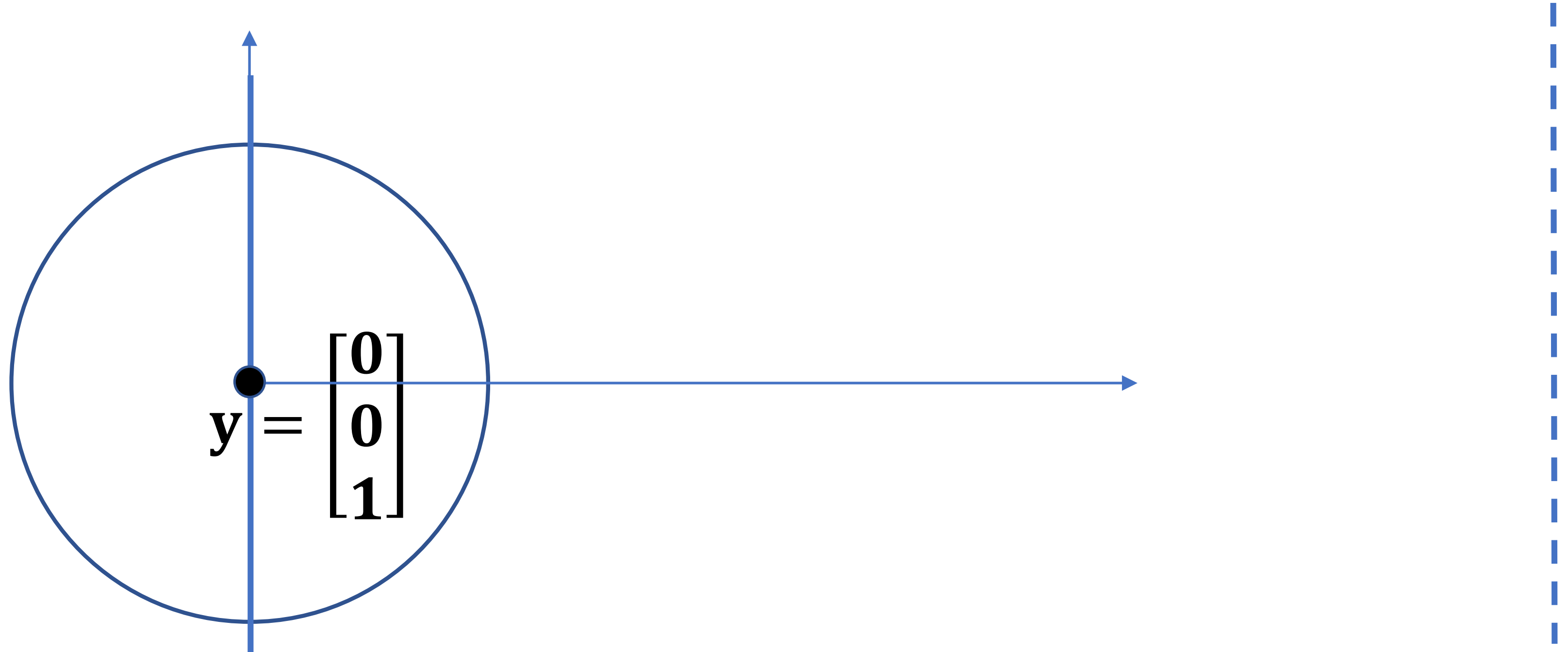
the polar line (cartesian) equation $\mathrm{X} = 0$ is the **diameter** ⊥ direction of the point $\mathbf{y}$

the polar line of the center of a circumference

Particular case: point $\mathbf{y}$ is the origin



polar line: $\boldsymbol{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & -X_o \\ 0 & 1 & -Y_o \\ -X_o & -Y_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r^2 \end{bmatrix}$

the polar line equation is $-r^2 w = 0$ i.e. $w = 0$ i.e. the line at the infinity $\boldsymbol{l}_\infty$

Degenerate conics

Degenerate conics

Conics: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}$ if it satisfies a homogeneous *quadratic* equation, namely $\mathbf{x}^{\boldsymbol{T}}\boldsymbol{C}\,\mathbf{x} = \mathbf{0}$, where $\boldsymbol{C}$ is a 3x3 symmetric matrix.

Nondegenerate conics: matrix $\boldsymbol{C}$ is nonsingular, i.e. $|\boldsymbol{C}| \neq 0$, $\mathrm{rank}(\boldsymbol{C}) = 3$

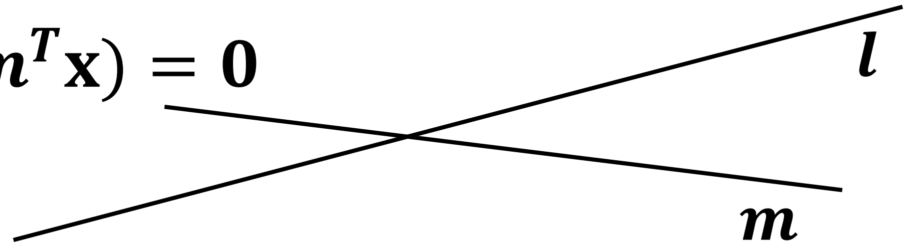
Degenerate conics: matrix $\boldsymbol{C}$ is singular, i.e. $|\boldsymbol{C}| = 0$, $\mathrm{rank}(\boldsymbol{C}) < 3$

two cases:

- rank $(C) = 2$, $C = \boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}} + \boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}$ with $\boldsymbol{l}$ and $\boldsymbol{m}$ are column vectors

$\rightarrow$ $\mathbf{x}^{\boldsymbol{T}}\boldsymbol{C}\,\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\left(\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}} + \boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}\right)\mathbf{x} = \mathbf{2}\mathbf{x}^{\boldsymbol{T}}\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}}\mathbf{x} = \mathbf{2}(\mathbf{x}^{\boldsymbol{T}}\boldsymbol{l})(\boldsymbol{m}^{\boldsymbol{T}}\mathbf{x}) = \mathbf{0}$

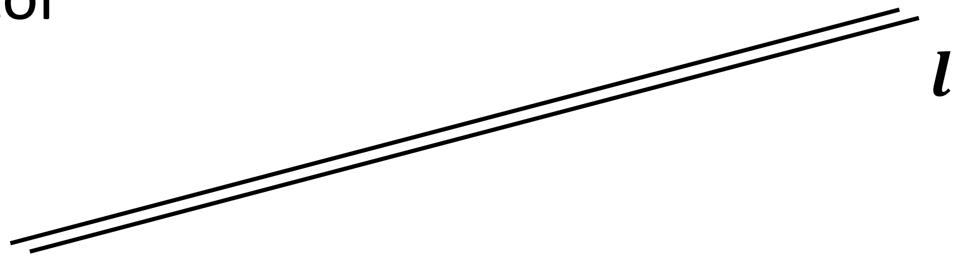
$\mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$ OR $\boldsymbol{m}^{\boldsymbol{T}}\mathbf{x} = \mathbf{0}$



- rank $(C) = 1$, $C = \boldsymbol{l}\boldsymbol{l}^{\boldsymbol{T}}$ with $\boldsymbol{l}$ column vector

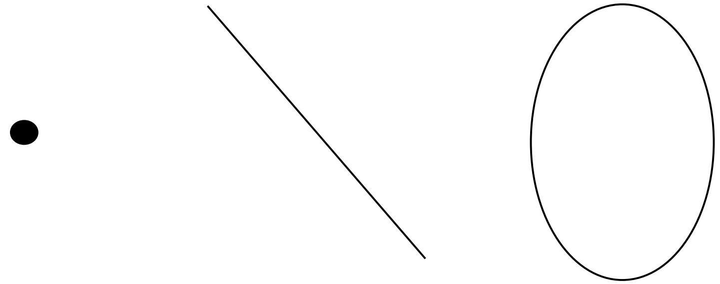
$\rightarrow$ $\mathbf{x}^{\boldsymbol{T}}\boldsymbol{C}\,\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\boldsymbol{l}\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x} = (\mathbf{x}^{\boldsymbol{T}}\boldsymbol{l})(\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x}) = \mathbf{0}$

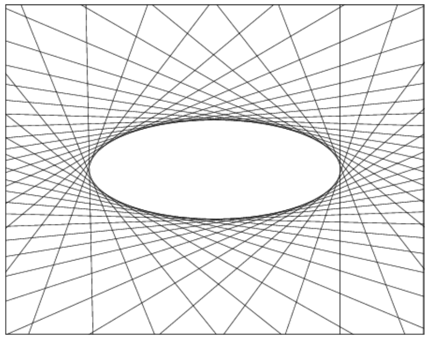
$\mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$ counted 2 times



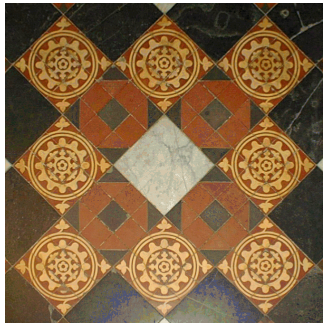
Planar Projective Geometry

- **Elements**
- Points
- Lines
- Conics
- **Dual conics**





- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



DUAL CONICS

Conics: a conic is a set of points $\mathbf{x}$ that satisfy equation

$$\mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{0}$$

where $\boldsymbol{C}$ is a 3x3 symmetric matrix.

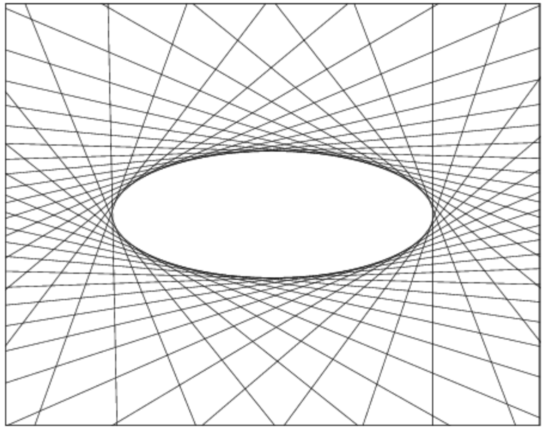
its dual:

Dual conics: a *dual conic* is a set of *lines* $\boldsymbol{l}$ that satisfy equation

$$\boldsymbol{l}^T \boldsymbol{C}^* \boldsymbol{l} = \mathbf{0}$$

where $\boldsymbol{C}^*$ is a 3x3 symmetric matrix.

Nondegenerate dual conics: a nondegenerate dual conic is a dual conic whose matrix $\boldsymbol{C}^*$ is NONSINGULAR, i.e., . $|\boldsymbol{C}^*| \neq 0, \mathrm{rank}(\boldsymbol{C}^*) = 3$



A nondegenerate dual conic $\boldsymbol{C}^*$ is the set of lines TANGENT to a nondegenerate conic $\boldsymbol{C}$

What is the relation between matrixes $\boldsymbol{C}^*$ and $\boldsymbol{C}$?

$$\boldsymbol{C}^* = \boldsymbol{C}^{-1}$$

degenerate dual conics

Degenerate dual conics

Dual Conics: a point $\mathbf{x}$ is on a conic $\boldsymbol{C}^*$ if it satisfies a homogeneous *quadratic* equation, namely $\boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \mathbf{0}$, where $\boldsymbol{C}^*$ is a 3x3 symmetric matrix.

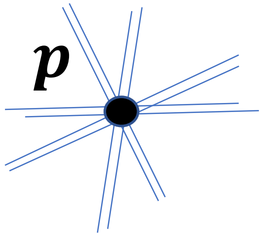
Nondegenerate conics: matrix $\boldsymbol{C}^*$ is nonsingular, i.e. $|\ \boldsymbol{C}^*| \neq 0$, rank $\boldsymbol{C}^* = 3$
Degenerate conics: matrix $\boldsymbol{C}^*$ is singular, i.e. $|\ \boldsymbol{C}^*| = 0$, rank $\boldsymbol{C}^* < 3$

two cases:
- rank $\boldsymbol{C}^* = 2$, $\boldsymbol{C}^* = \boldsymbol{p}\boldsymbol{q}^T + \boldsymbol{q}\boldsymbol{p}^T$ with $\boldsymbol{p}$ and $\boldsymbol{q}$ are column vectors

$\rightarrow \boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \boldsymbol{l}^T\left(\boldsymbol{p}\boldsymbol{q}^T + \boldsymbol{q}\boldsymbol{p}^T\right)\boldsymbol{l} = \mathbf{2}\boldsymbol{l}^T\boldsymbol{p}\boldsymbol{q}^T\boldsymbol{l} = \mathbf{2}(\boldsymbol{l}^T\boldsymbol{p})(\boldsymbol{q}^T\boldsymbol{l}) = \mathbf{0}$

$\boldsymbol{l}^T\boldsymbol{p} = \mathbf{0}$ OR $\boldsymbol{q}^T\boldsymbol{l} = \mathbf{0}$

- rank $\boldsymbol{C}^* = 1$, $\boldsymbol{C}^* = \boldsymbol{p}\boldsymbol{p}^T$ with $\boldsymbol{p}$ column vector



$\rightarrow \boldsymbol{l}^T\ \boldsymbol{C}^*\boldsymbol{l} = \boldsymbol{l}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{l} = (\boldsymbol{l}^T\boldsymbol{p})(\boldsymbol{p}^T\boldsymbol{l}) = \mathbf{0}$

$\boldsymbol{l}^T\boldsymbol{p} = \mathbf{0}$ counted 2 times

A special dual conic

The (degenerate) conic-dual to the circular points

what is the degenerate dual conic $\boldsymbol{C}^{*}=\boldsymbol{p}\boldsymbol{q}^{\boldsymbol{T}}+\boldsymbol{q}\boldsymbol{p}^{\boldsymbol{T}}$
when points $\boldsymbol{p}$ and $\boldsymbol{q}$ are the circular points?

$$\boldsymbol{C}_{\infty}^{*}=\boldsymbol{I}\boldsymbol{J}^{\boldsymbol{T}}+\boldsymbol{J}\boldsymbol{I}^{\boldsymbol{T}}=\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}$$

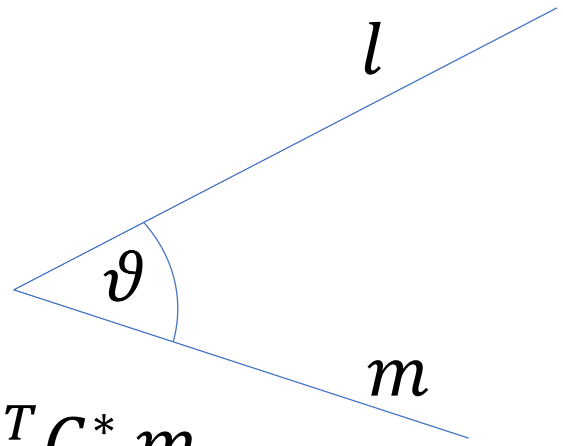
$\boldsymbol{C}_{\infty}^{*}$ is called the conic dual to the circular points:
it will be useful in the 2D reconstruction of planar scenes (or, **image rectification**)

Angle between two lines via the conic $\boldsymbol{C}_{\infty}^{*}$ dual to the circular points

Angle between two lines $\boldsymbol{l}$ and $\boldsymbol{m}$

The angle ϑ between $l = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $m = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is the angle between their normals $[a_1 \quad b_1]$ and $[a_2 \quad b_2]$:

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1{}^{\mathbf{2}} + b_1{}^{\mathbf{2}})(a_2{}^{\mathbf{2}} + b_2{}^{\mathbf{2}})}}$$

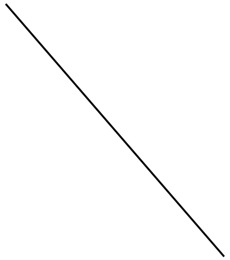
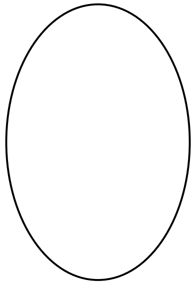
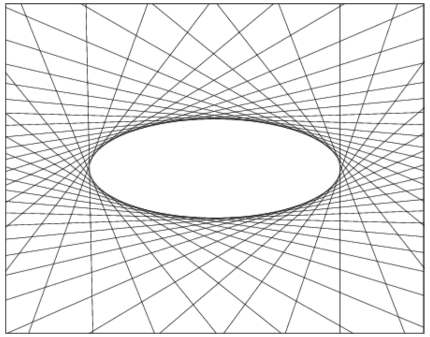


But, e.g., $a_1 a_2 + b_1 b_2 = [a_1 \quad b_1 \quad c_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = l^T C_\infty^* m$

$$\rightarrow \cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

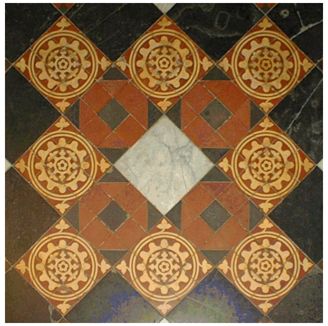
Planar Projective Geometry

- **Elements**
- Points
- Lines
- Conics
- Dual conics

- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities





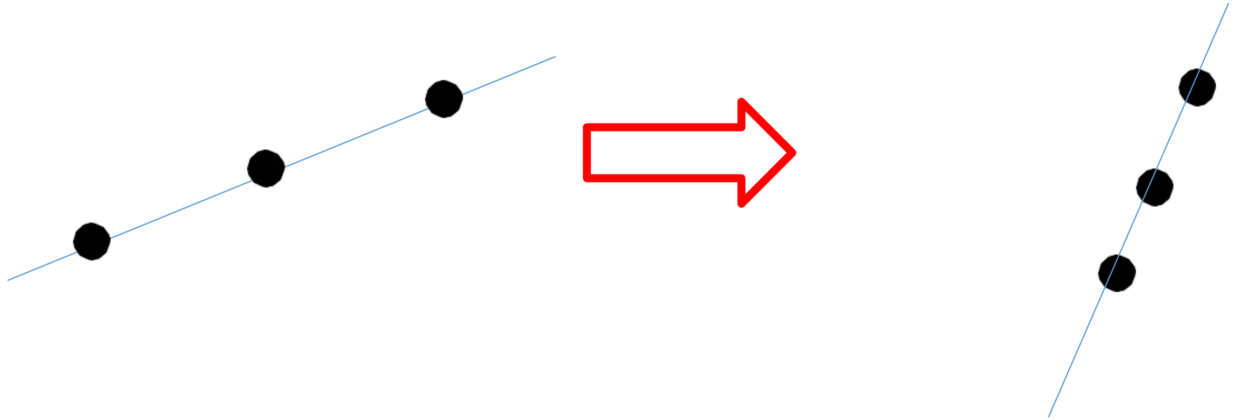
Projective mappings

Def. *A **projective mapping** between a projective plane $\mathbb{P}^2$ and an other projective plane $\mathbb{P}'^2$ is an **invertible** mapping which preserves colinearity:*

$$h\colon \mathbb{P}^2 \rightarrow \mathbb{P}'^2, \mathbf{x}' = h(\mathbf{x}),\ \mathbf{x}_1,\ \mathbf{x}_2, \mathbf{x}_3 \text{ are colinear}$$

$$\leftrightarrow$$

$$\mathbf{x}'_1 = h(\mathbf{x}_1),\ \mathbf{x}'_2 = h(\mathbf{x}_2), \mathbf{x}'_3 = h(\mathbf{x}_3) \text{ are colinear}$$

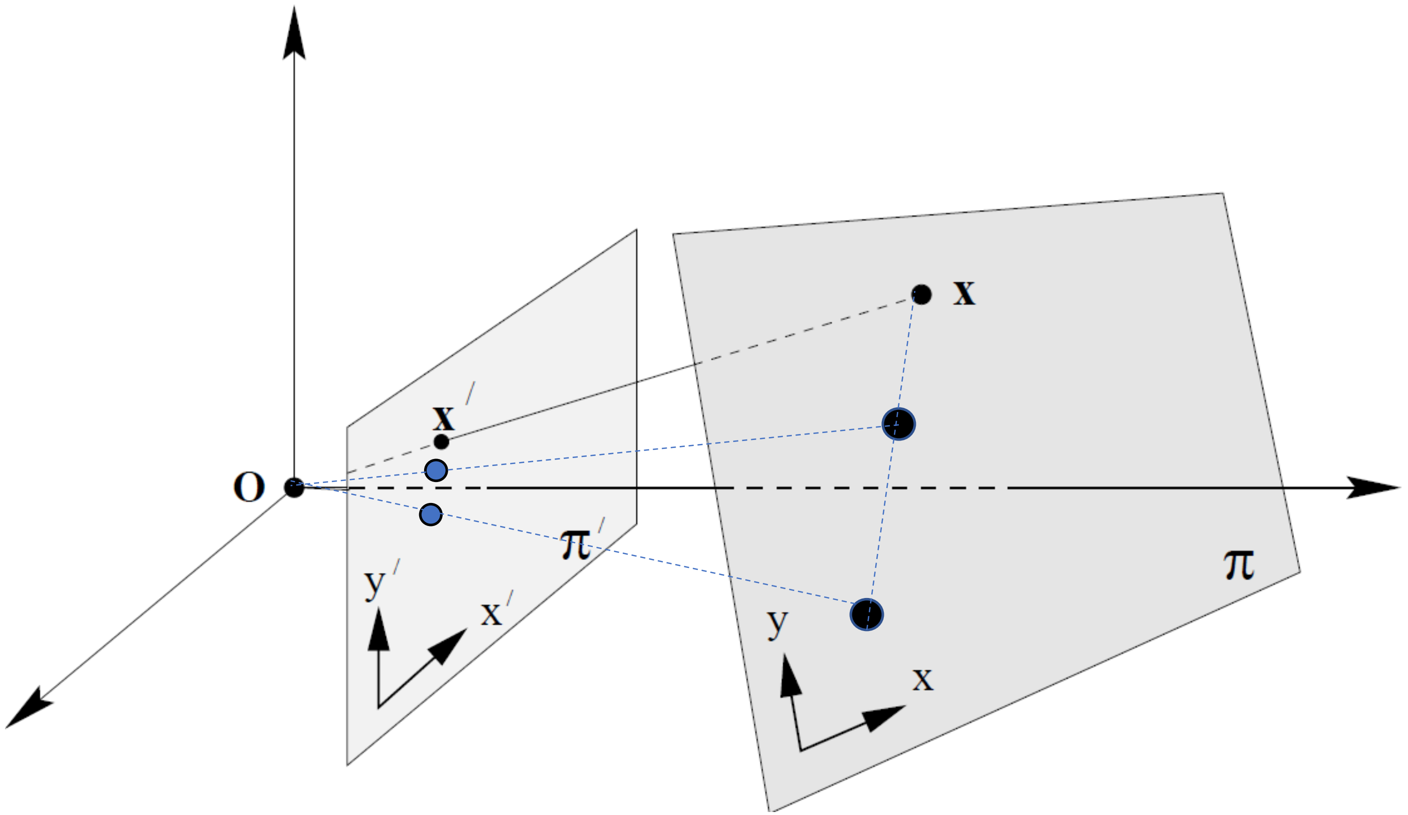


Alternative names:

- *Projectivity*
- *Homography*

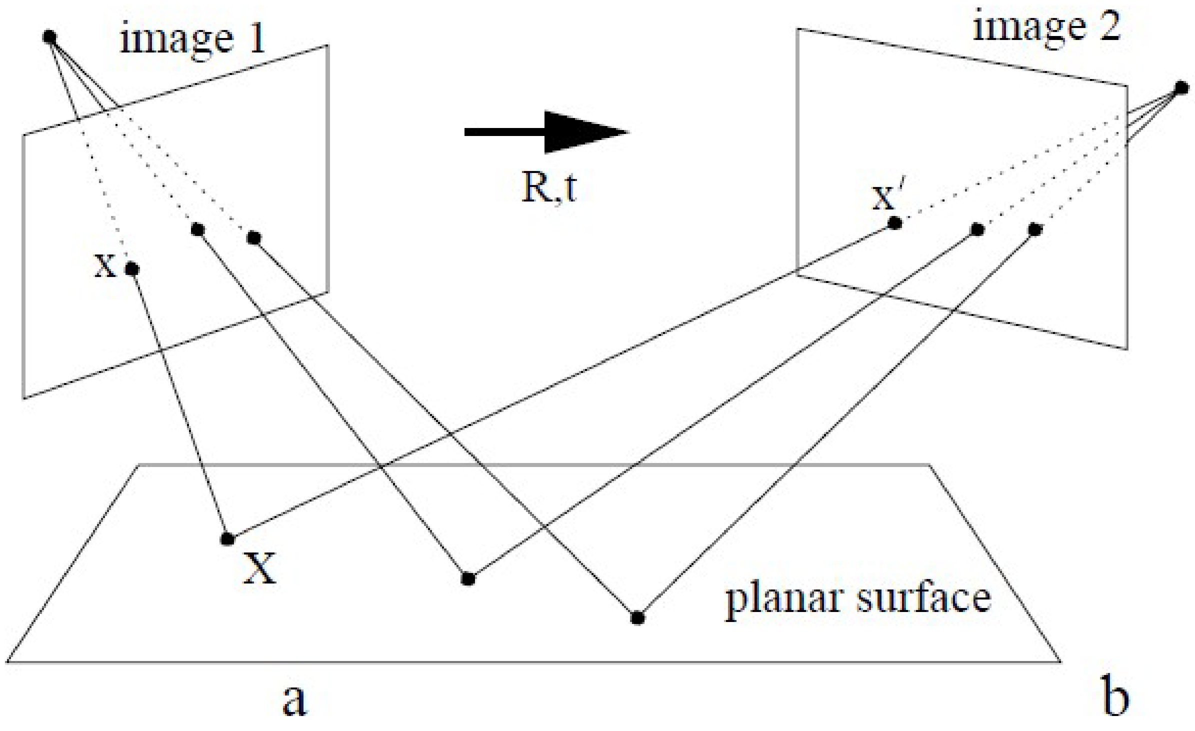
Examples of projective mappings

Mappings between two planes induced by central projection are projective, since they preserve colinearity



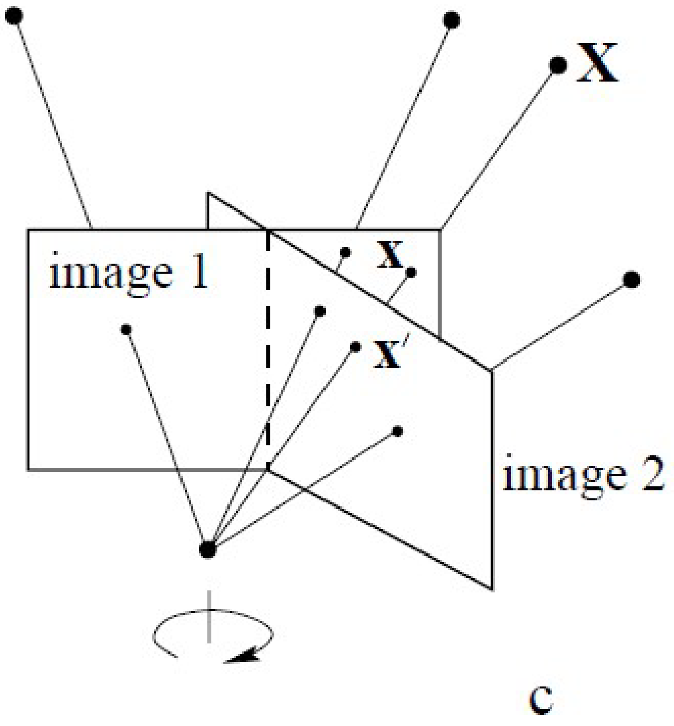
Examples of projective mappings

Mapping between two images of a planar scene is a homography a composition of the 1° image → scene homography and the scene → 2° image homography ⇨ it is a **homography**



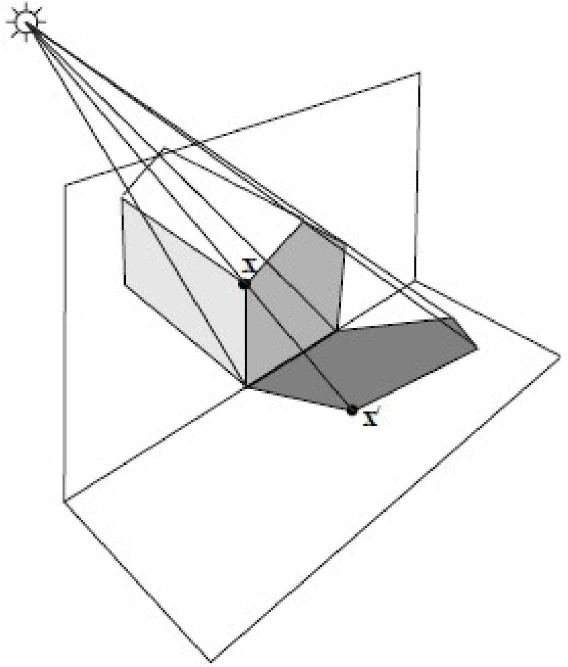
Examples of projective mappings

Two images of a 3D scene, taken by a camera rotating around its center are related by a **homography**, since the 2° image can be regarded as a central projection of the 1° image



Examples of projective mappings

The shadow cast by a **planar** silhouette onto a **ground plane** is a projective transformation of the planar silhouette, since they are related by a central projection



Fundamental Theorem of Projective Geometry

Theorem: *A mapping* $h : \mathbb{P}^2 \rightarrow \mathbb{P}'^2$ *is projective if and only if there exists an invertible 3x3 matrix* H *such that for any point in* $\mathbb{P}^2$ *represented by the vector* $\mathbf{x}$ *, is* $h(\mathbf{x}) = H\,\mathbf{x}$

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

i.e. projective mappings are LINEAR in the homogeneous coordinates

(they are not linear in cartesian coordinates)

Homography: 8 degrees of freedom

From the theorem

$$h(\mathbf{x}) = \mathbf{x}' = H\,\mathbf{x}$$

Therefore, if we multiply the matrix H by any nonzero scalar λ, the relation is satisfied by the same points

$$\mathbf{x}' = \lambda H\,\mathbf{x}$$

Thus any nonzero multiple of the matrix H represents the same projective mapping as H.

Hence H is a homogeneous matrix: in spite of its 9 entries,

H has only 8 degrees of freedom, namely the ratios between its elements.

Homography estimation

H has only 8 degrees of freedom, namely the ratios between its elements. E.g.

$$H = \begin{bmatrix} A_{11} & A_{12} & t_1 \\ A_{21} & A_{22} & t_2 \\ v_1 & v_2 & 1 \end{bmatrix}$$

Therefore, it can be estimated by just FOUR point correspondences,

since each point correspondence $\mathbf{x}' = H\,\mathbf{x}$ yelds **two** independent equations

Transformation of points, lines, conics, dual conics

Transformation rules for the plane elements

A homography transforms **each point** $\mathbf{x}$ into a point $\mathbf{x}'$ such that:

$$\mathbf{x} \rightarrow H\mathbf{x} = \mathbf{x}'$$

A homography transforms **each line** $\boldsymbol{l}$ into a line $\boldsymbol{l}'$ such that:

$$\boldsymbol{l} \rightarrow H^{-T}\boldsymbol{l} = \boldsymbol{l}'$$

A homography transforms **each conic** C into a conic C' such that:

$$C \rightarrow H^{-T}CH^{-1} = C'$$

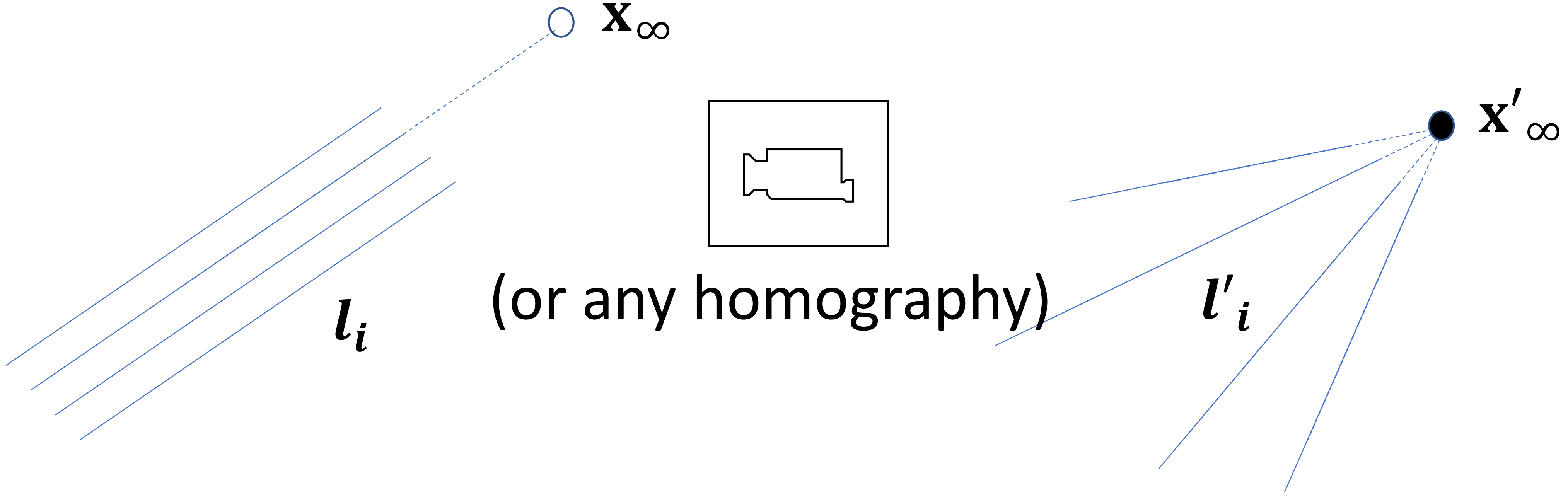
A homography transforms **each dual conic** C^* into a dual conic $C^{*\prime}$

$$C^* \rightarrow HC^*H^T = C^{*\prime}$$

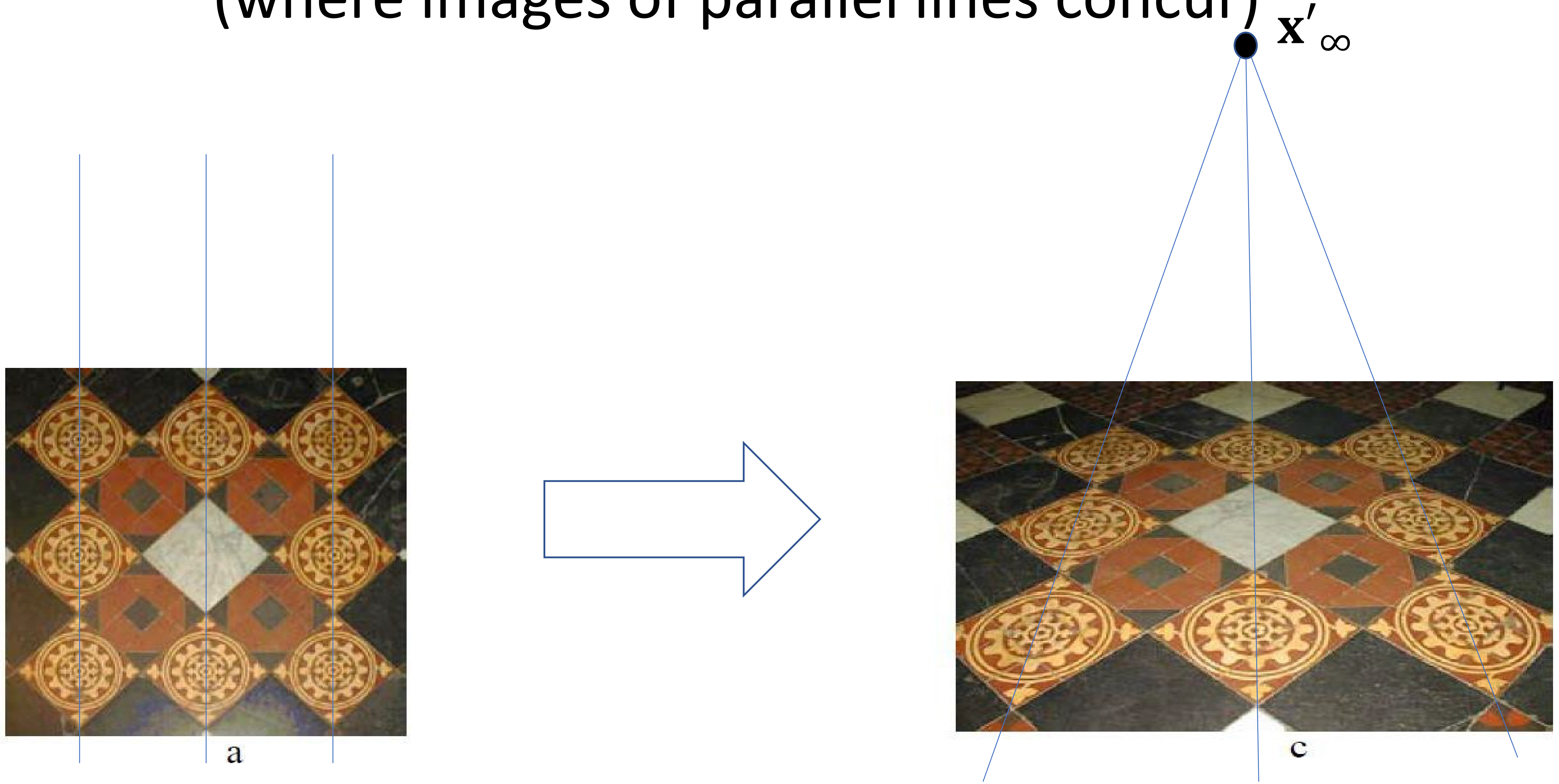
Vanishing points

vanishing point

Theorem: *the image of a set of parallel lines* $\boldsymbol{l_i}$ *is a set of lines* $\boldsymbol{l'}_i$ ***concurrent*** *at a common point* $\boldsymbol{x}'$ *called the vanishing point of the direction of lines* $\boldsymbol{l_i}$

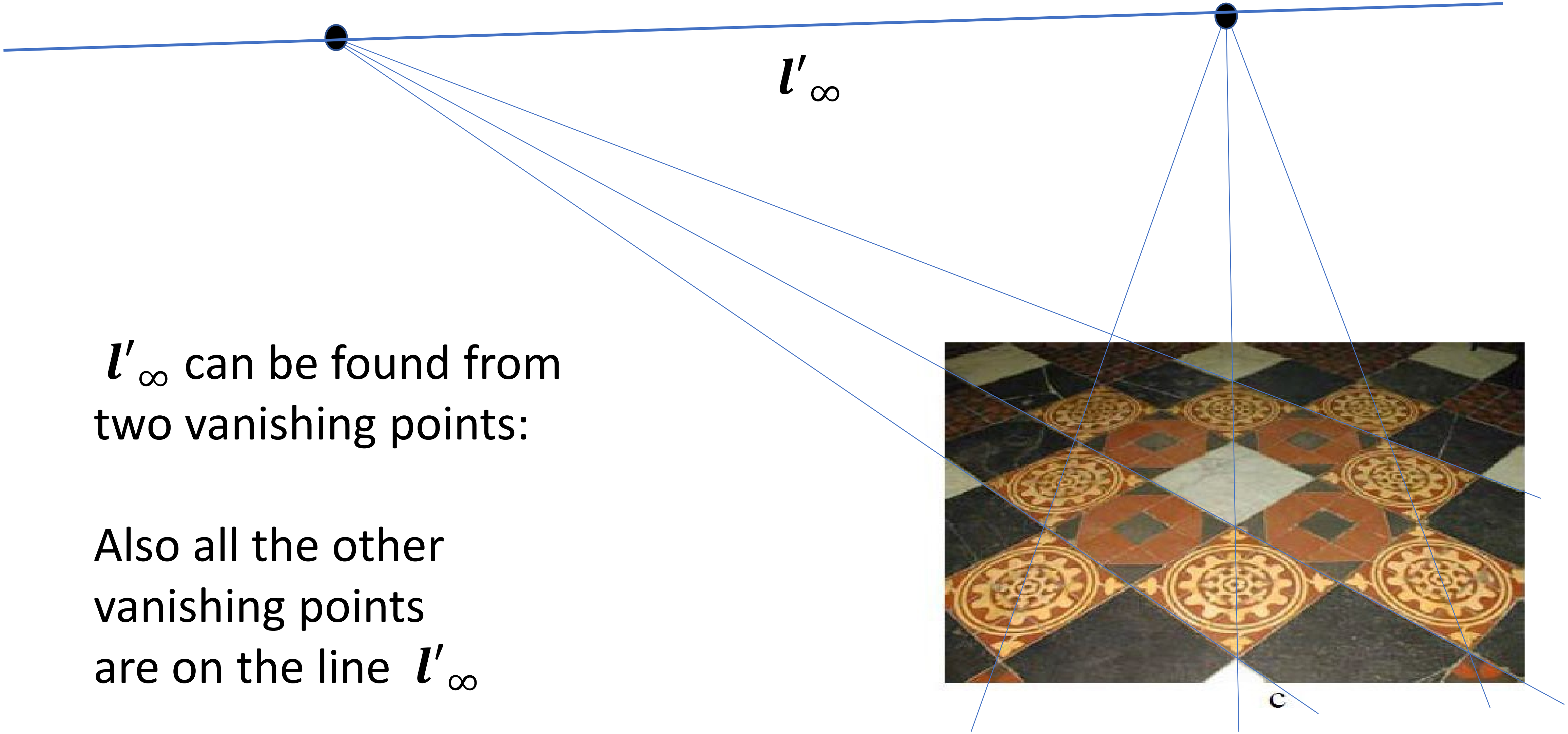


Vanishing point: image of a point at the ∞, (where images of parallel lines concur)



The vanishing line

vanishing line $\boldsymbol{l}'_\infty$ *or horizon* = image of the set of the points at the ∞ = the image of $\boldsymbol{l}_\infty$ (must be a line: why?)



$\boldsymbol{l}'_\infty$ can be found from two vanishing points:

Also all the other vanishing points are on the line $\boldsymbol{l}'_\infty$

Polarity is preseved under projective mapping

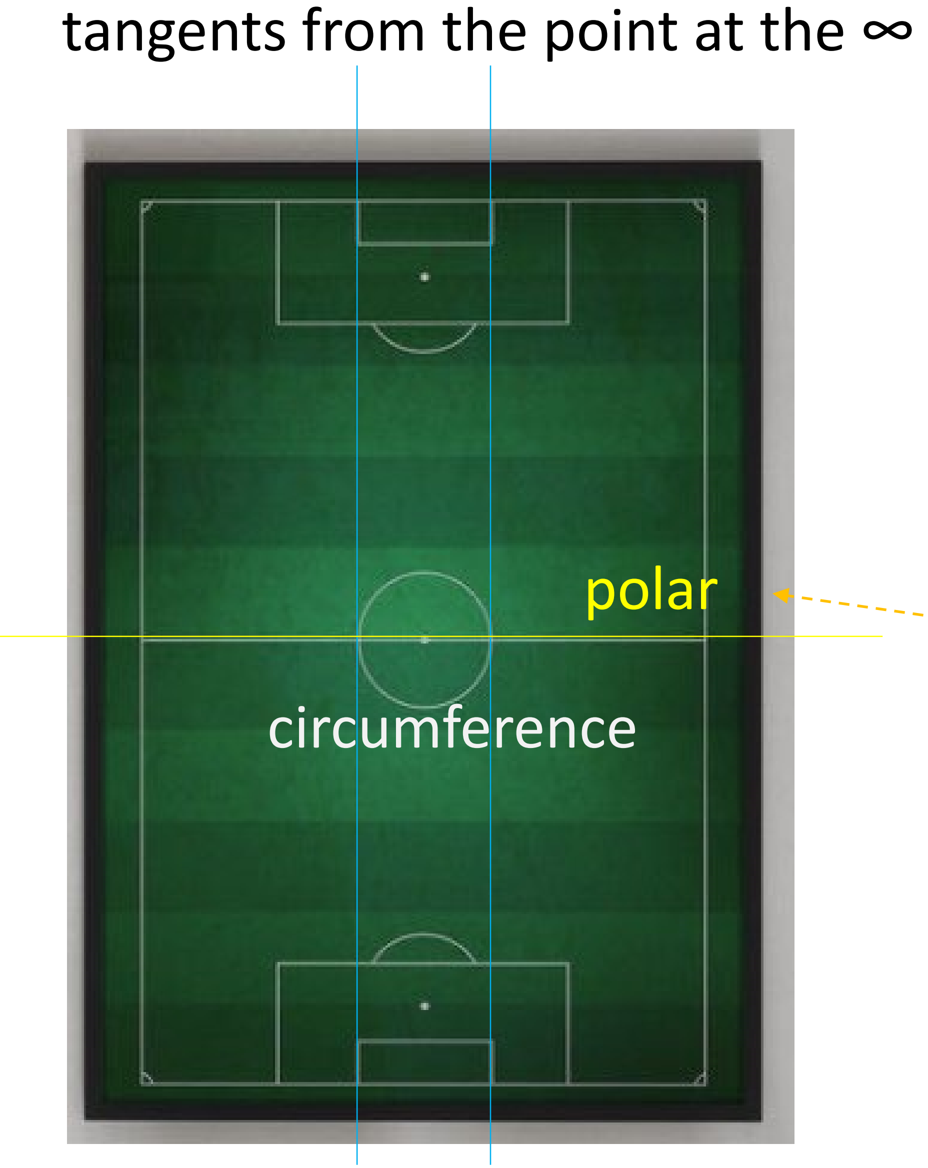
i.e.

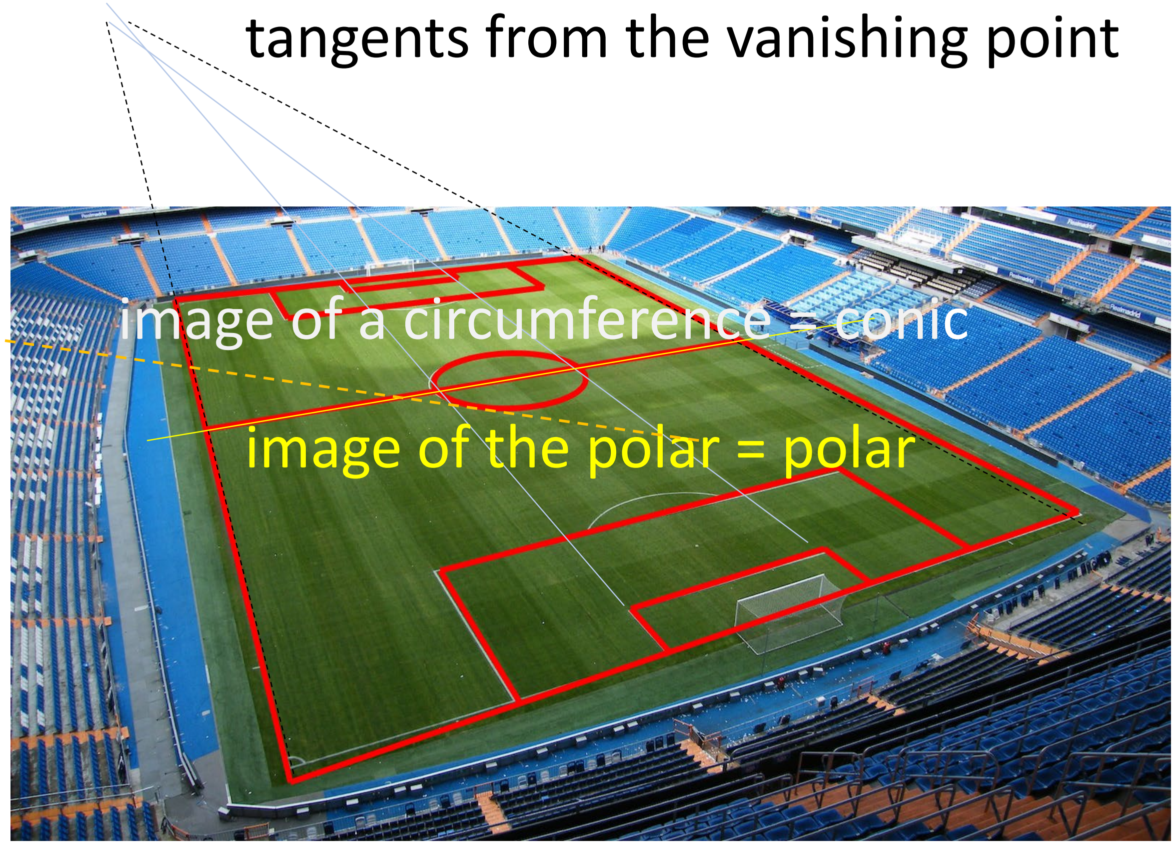
$$\boldsymbol{l} = C\mathbf{y} \Rightarrow \boldsymbol{l}' = C'\mathbf{y}'$$

Proof. From transformation rules for conics, lines and points:

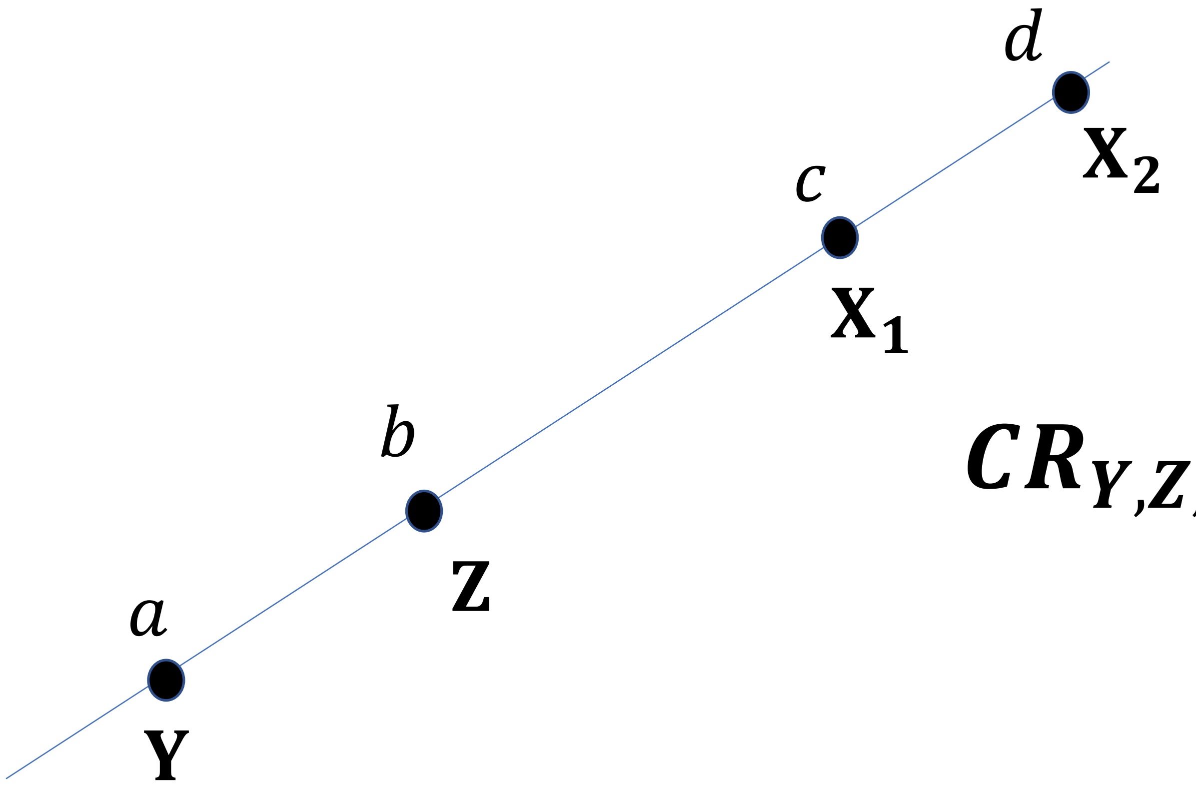
$$C'\mathbf{y}' = H^{-T}CH^{-1}\mathbf{y} = H^{-T}CH^{-1}H\mathbf{y} = H^{-T}C\mathbf{y} = H^{-T}\boldsymbol{l} = \boldsymbol{l}'$$

Polarity is preserved under projective mappings





Cross ratio: a projective invariant



$$CR_{Y,Z,\mathbf{X}_1,\mathbf{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

Cross ratio invariance under projective mappings

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = CR_{Y',Z',X'_1,X'_2}}$$

Proof:

- applying any homography H, the coefficients of the linear combinations remain the same

$$\mathbf{X}_1 = \alpha_1\mathbf{Y} + \beta_1\mathbf{Z}$$

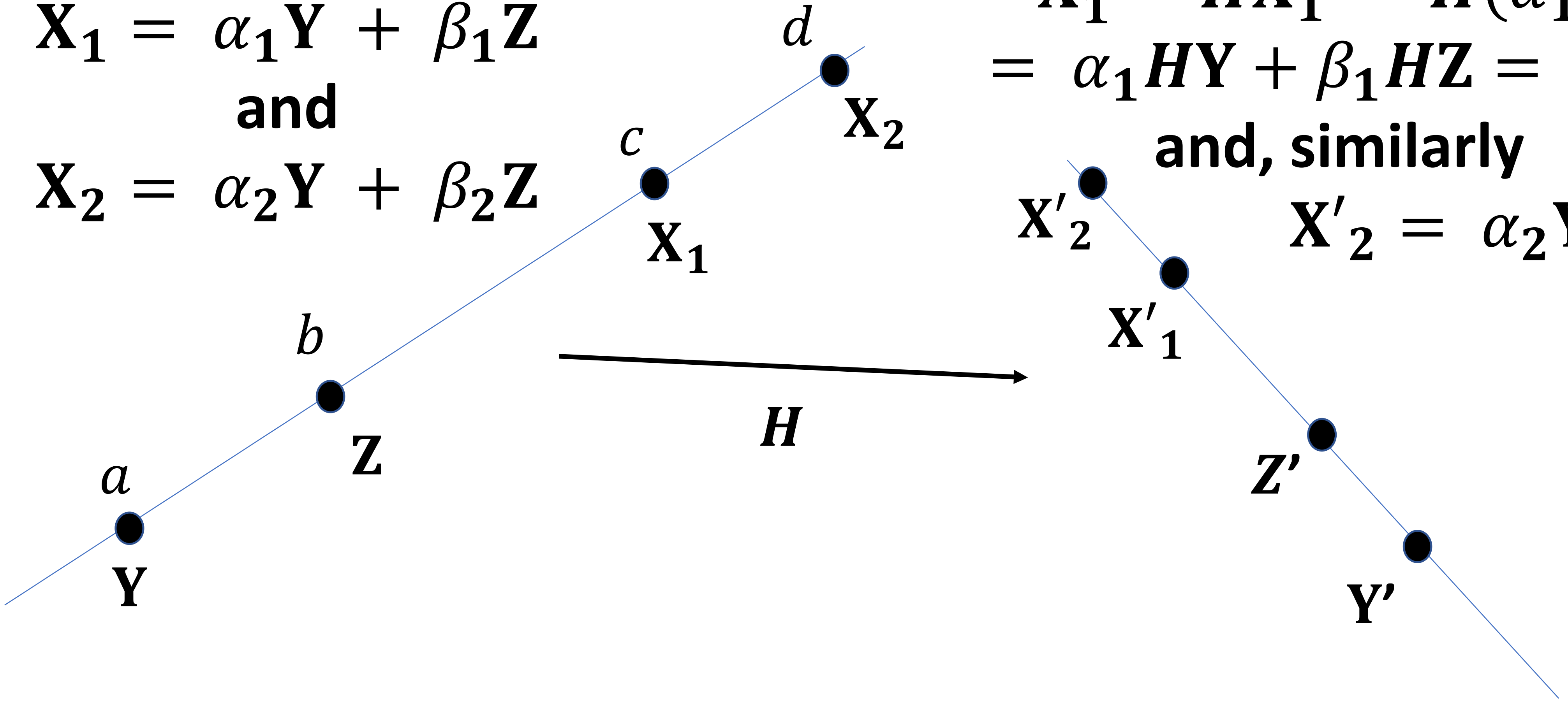
and

$$\mathbf{X}_2 = \alpha_2\mathbf{Y} + \beta_2\mathbf{Z}$$

$$\mathbf{X}_1' = \boldsymbol{H}\mathbf{X}_1 = \boldsymbol{H}(\alpha_1\mathbf{Y} + \beta_1\mathbf{Z})$$
$$= \alpha_1\boldsymbol{H}\mathbf{Y} + \beta_1\boldsymbol{H}\mathbf{Z} = \alpha_1\mathbf{Y}' + \beta_1\boldsymbol{Z}'$$

and, similarly

$$\mathbf{X}'_2 = \alpha_2\mathbf{Y}' + \beta_2\mathbf{Z}'$$



and, from result

$$\boldsymbol{CR}_{Y,Z,\mathbf{X}_1,\mathbf{X}_2} = \frac{\boldsymbol{c-a}}{\boldsymbol{c-b}} \Big/ \frac{\boldsymbol{d-a}}{\boldsymbol{d-b}} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$

$$\mathbf{X}_1 = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

and

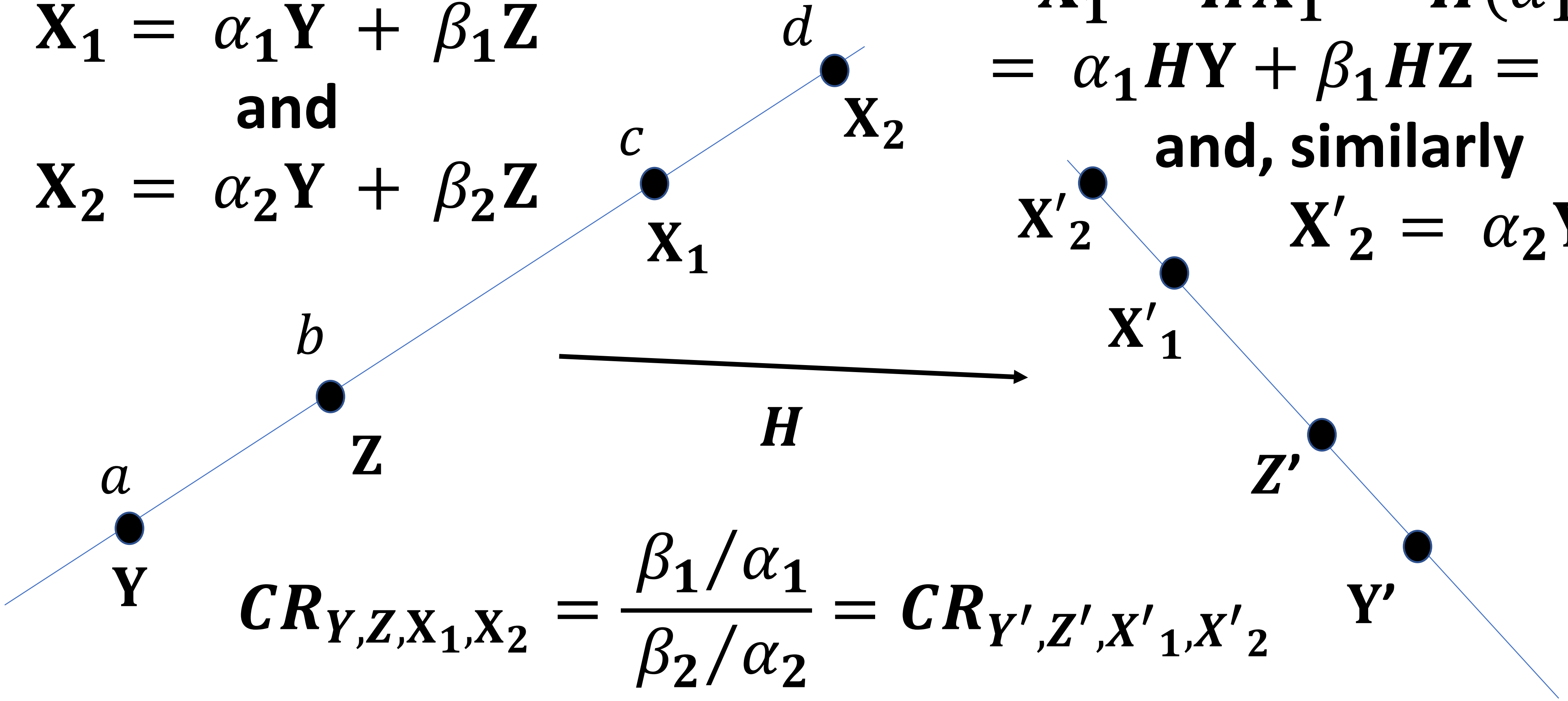
$$\mathbf{X}_2 = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

$$\mathbf{X}'_1 = \boldsymbol{H}\mathbf{X}_1 = \boldsymbol{H}(\alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z})$$

$$= \alpha_1 \boldsymbol{H}\mathbf{Y} + \beta_1 \boldsymbol{H}\mathbf{Z} = \alpha_1 \mathbf{Y}' + \beta_1 \boldsymbol{Z}'$$

and, similarly

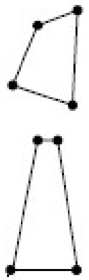
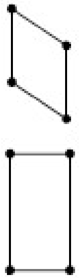
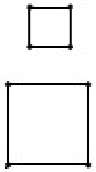
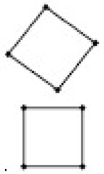
$$\mathbf{X}'_2 = \alpha_2 \mathbf{Y}' + \beta_2 \mathbf{Z}'$$



$$\boldsymbol{CR}_{Y,Z,\mathbf{X}_1,\mathbf{X}_2} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2} = \boldsymbol{CR}_{Y',Z',X'_1,X'_2}$$

Hierarchy of projective transformations

Hierarchy of projective transformations

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$. +
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3). + +
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area + + +

Isometries (or Euclidean mappings)

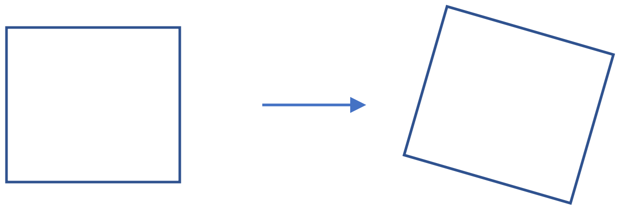
$$H_I = \begin{bmatrix} R_\perp & & \boldsymbol{t} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$R_\perp$ is an orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^T$

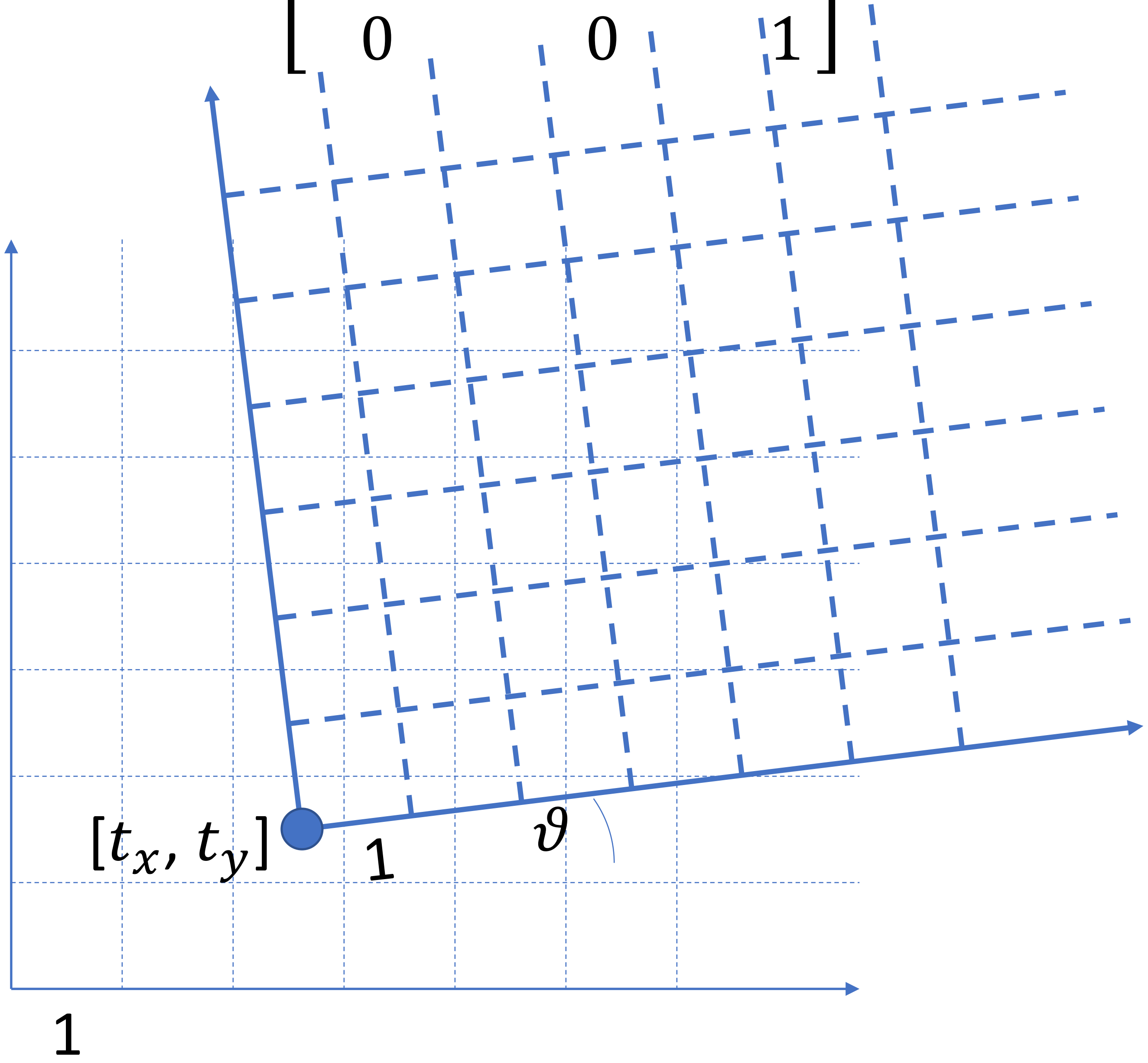
$\det {R_\perp}^{-1} = 1$ planar rigid displacement (-1 for reflection)

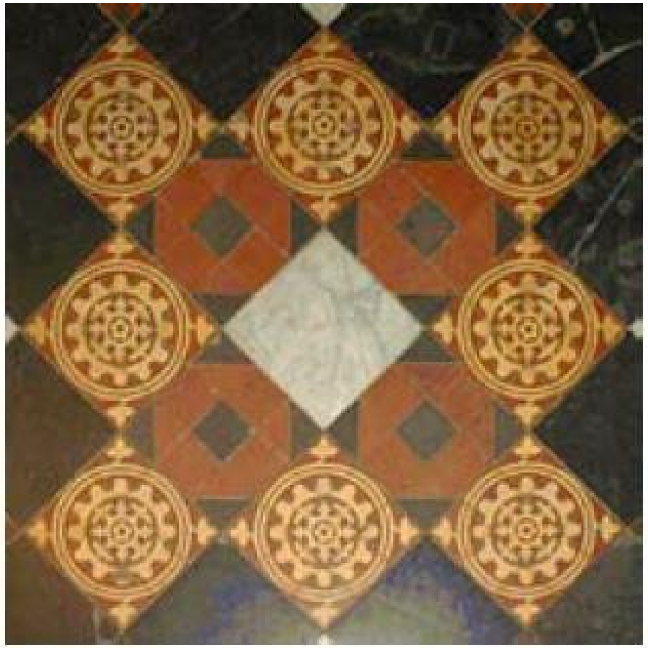
3 dofs: translation $\boldsymbol{t}$ + rotation angle ϑ

Invariants: lenghts, distances, areas → shape and size → relative positions



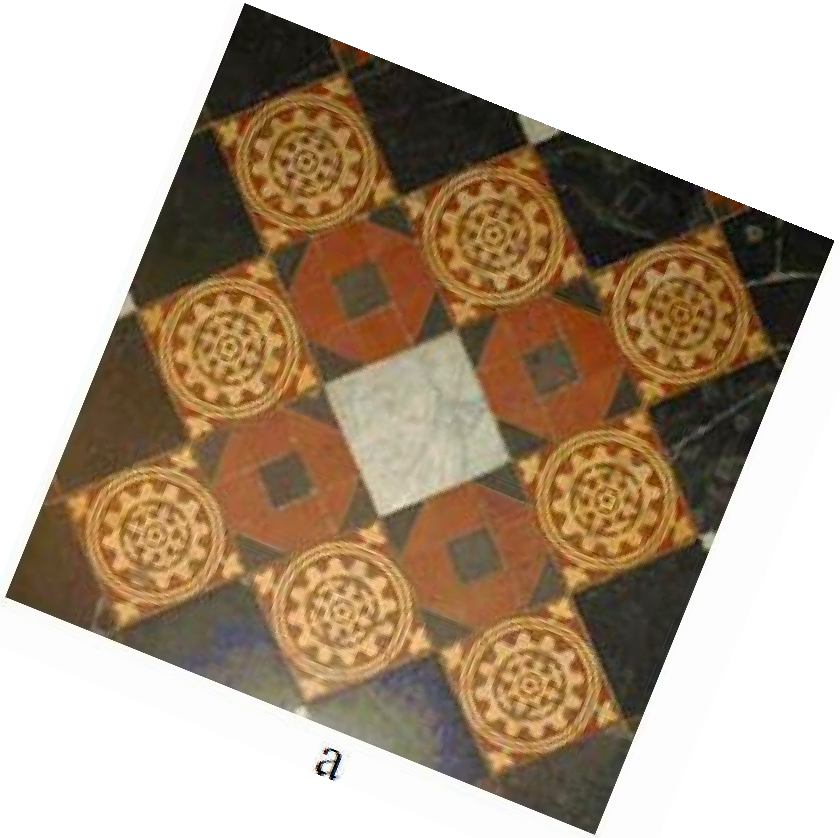
Isometry $H_I = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$: 3 degrees of freedom





a

isometry



a

Similarities

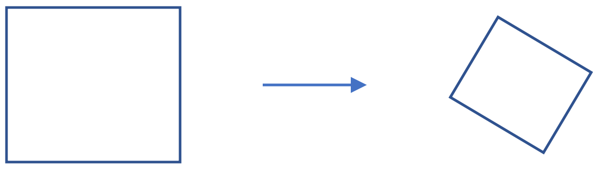
$$H_S = \begin{bmatrix} s\,R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{s}\cos\vartheta & -\,\mathrm{s}\sin\vartheta & t_x \\ s\ \sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$R_\perp$ is an orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^T$

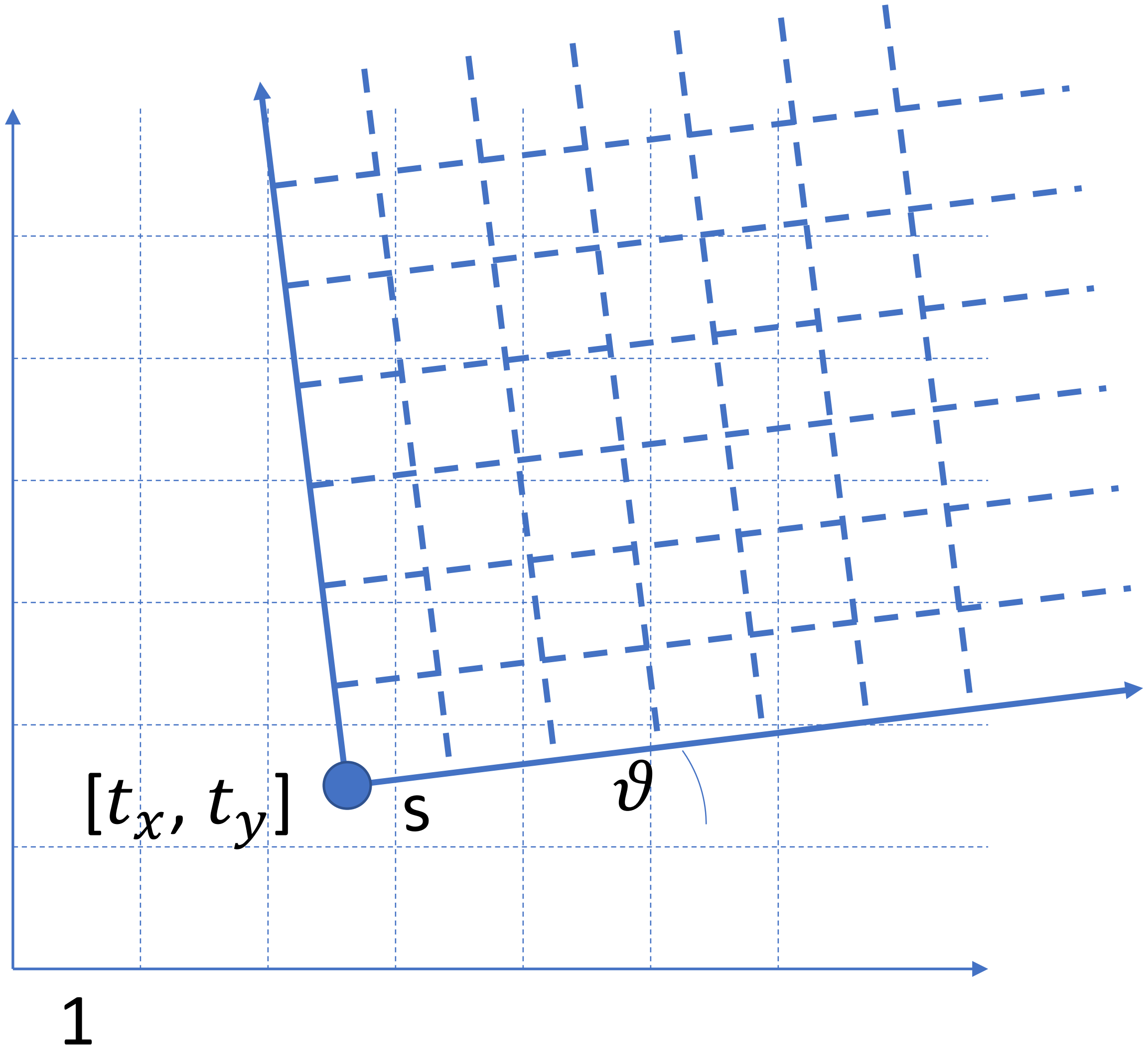
4 dofs: rigid displacement + *scale*

Invariants: ratio of lenghts, angles → shape (not size)

<u>the circular points ***I*** and ***J***</u>



$$\text{Similarity } H_S = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix} : \text{4 degrees of freedom}$$

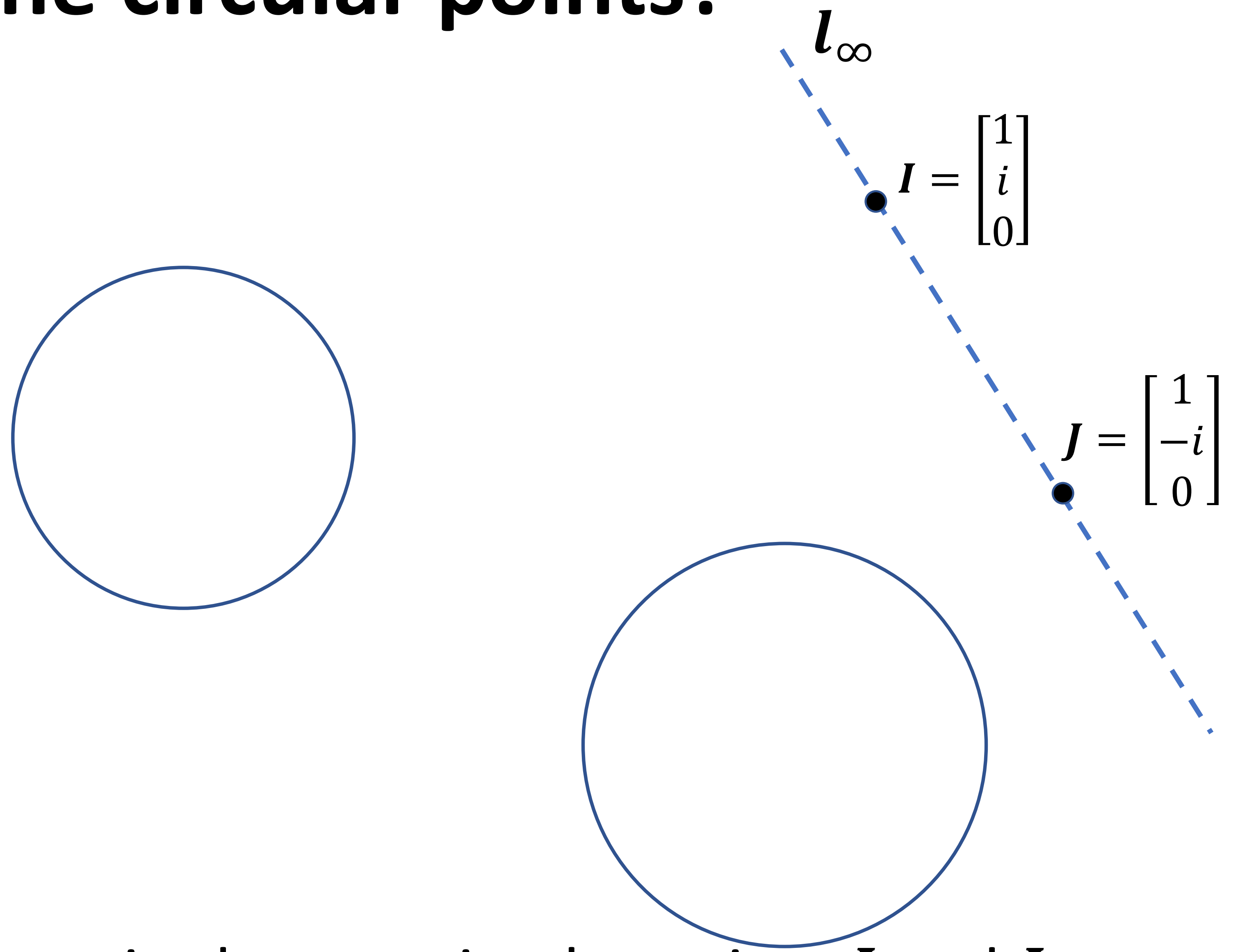


remember **the circular points?**

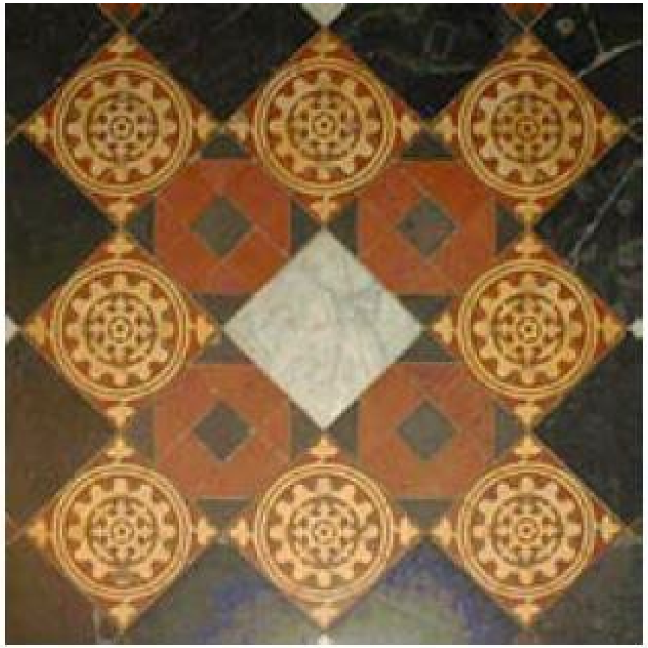
Remember their coordinates

$$\boldsymbol{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

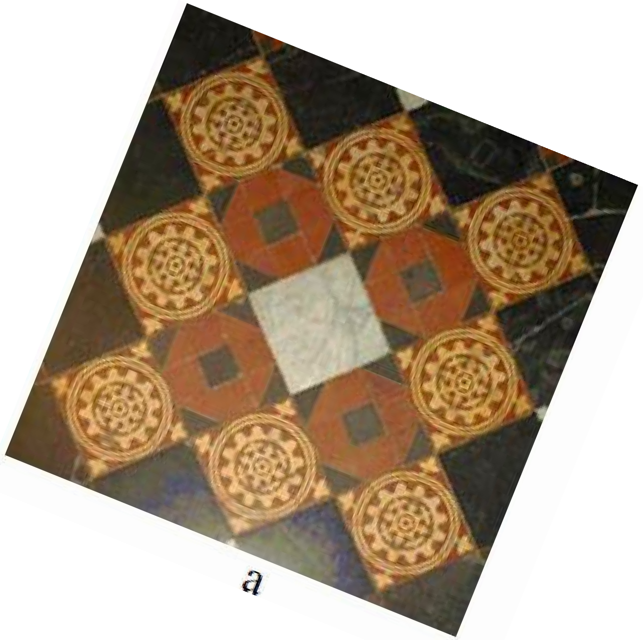


All the circumferences contain the two circular points $\boldsymbol{I}$ and $\boldsymbol{J}$



a

similarity



Affinities (or affine mappings)

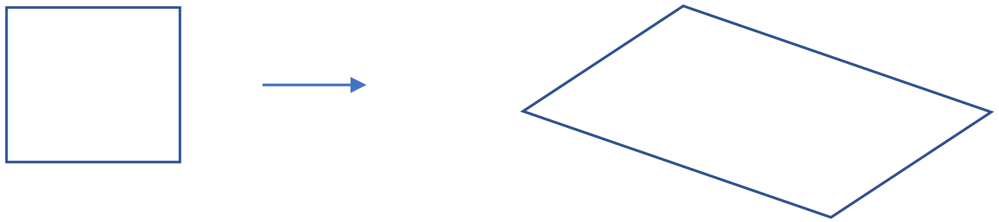
$$H_A = \begin{bmatrix} A & & \boldsymbol{t} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

A is any 2x2 rank-2 matrix

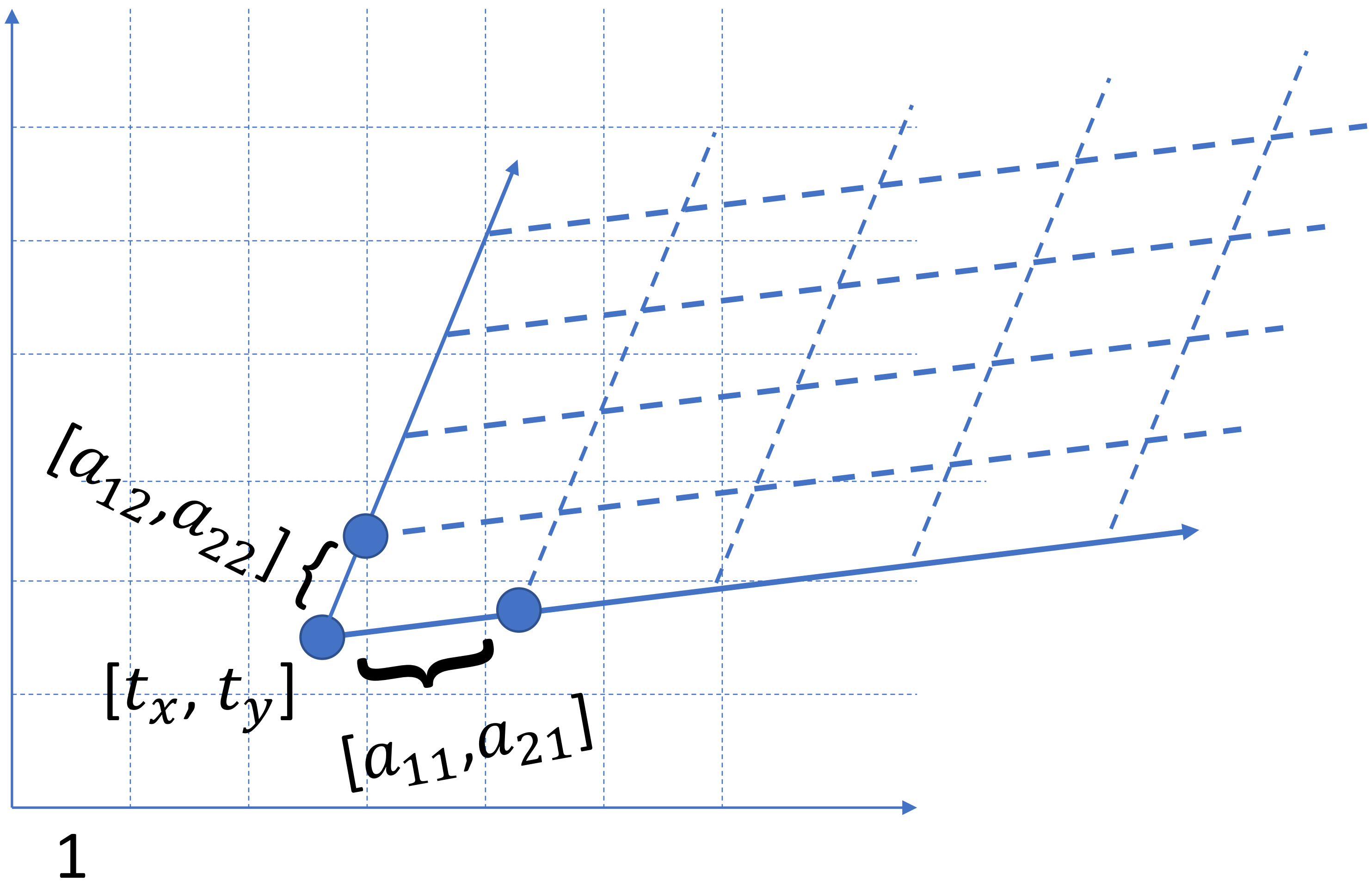
6 dofs: $A + \boldsymbol{t}$

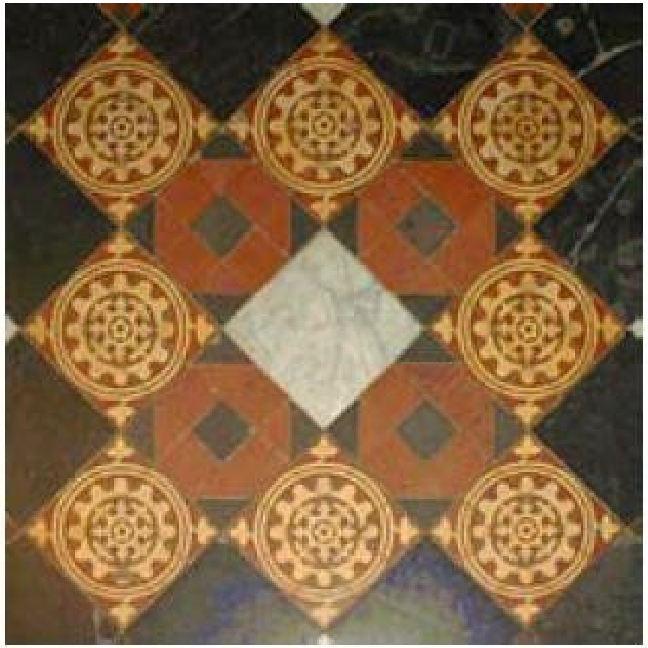
Invariants: parallelism, ratio of parallel lenghts, ratio of areas

<u>the line at the infinity</u> $\boldsymbol{l}_\infty = [\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{1}]^T$



$$\text{Affinity } H_A = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} : \text{6 degrees of freedom}$$





a

affinity

b

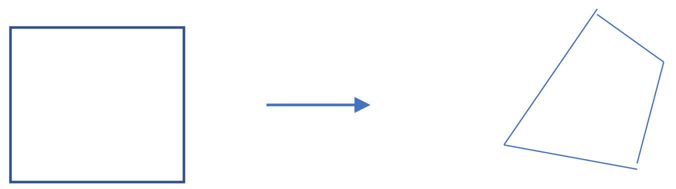
Projectivities (or projective mappings, or homographies)

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$$

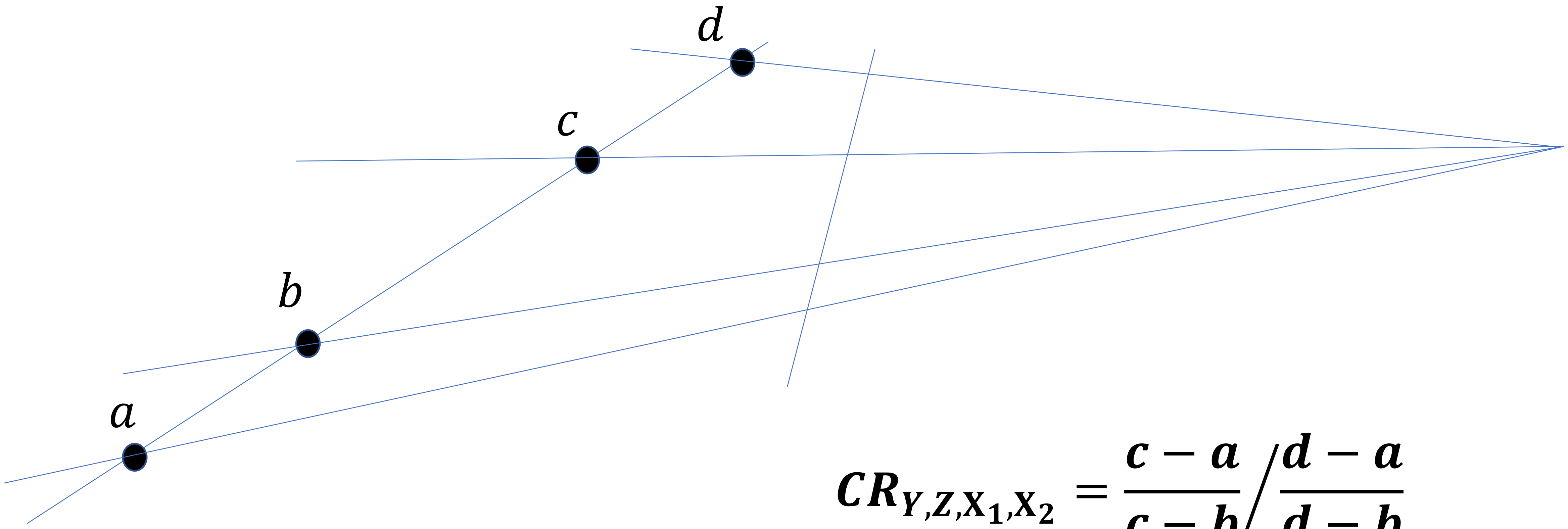
A is any 2x2 rank-2 matrix

8 dofs: $A + \boldsymbol{v} + \boldsymbol{t}$

Invariants: colinearity, incidence, order of contact (crossing, tangency, inflections), <u>the cross ratio</u>

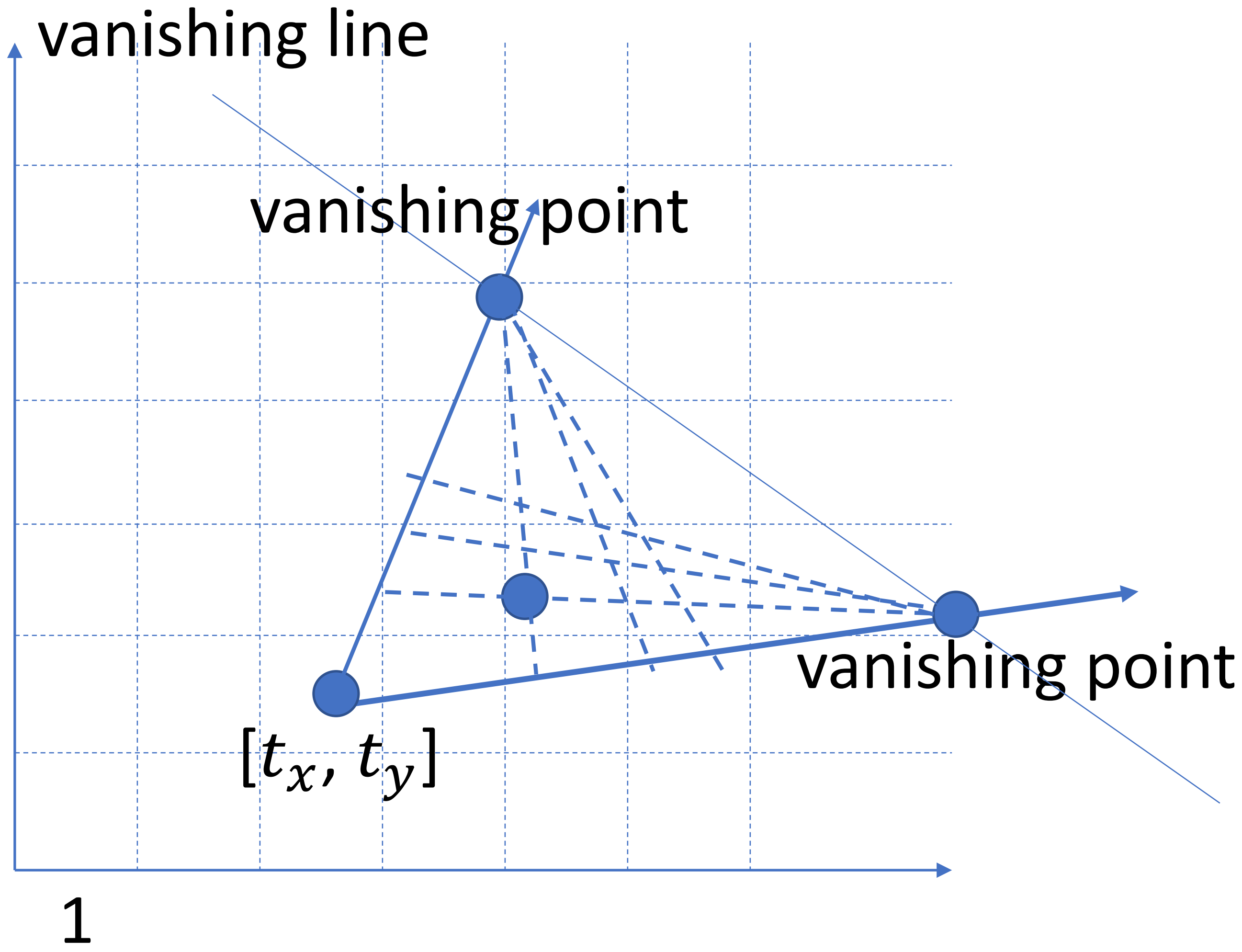


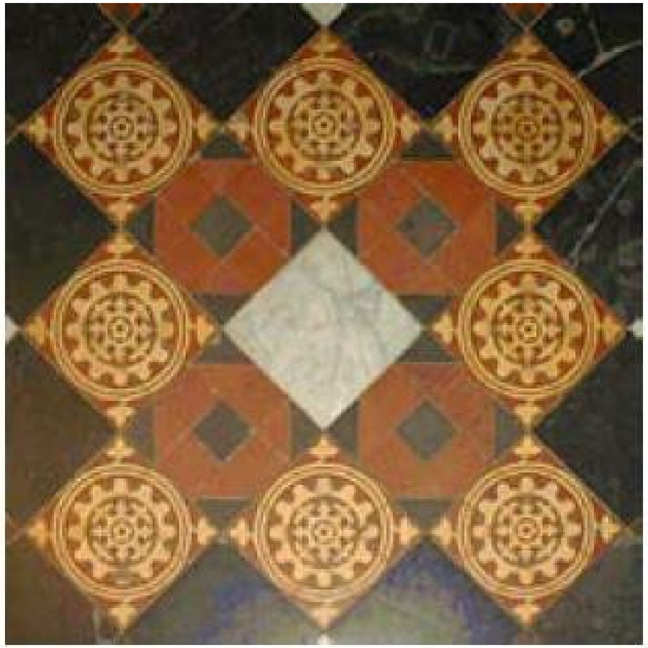
2D cross ratio of a 4-tuple of coplanar, concurrent lines: the dual of the 1D cross ratio of 4 colinear points. Take any crossing line … compute the 1D cross ratio of intersection points



$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = \frac{\boldsymbol{c-a}}{\boldsymbol{c-b}} \Big/ \frac{\boldsymbol{d-a}}{\boldsymbol{d-b}}$$

General Projective mapping





a

projectivity

c

2D reconstruction (image rectification)

original image

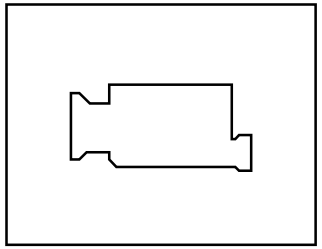
rectified image

2D reconstruction (image rectification): adopted framework

planar scene

image

a



projectivity H

c

- planar scene: unknown
- relative pose of scene and camera: unknown
- camera parameters (e.g. focal distance): unknown

→

scene-to-image projective **mapping** H : **unknown**

2D reconstruction (or image rectification) problem formulation

given: an image of an **unknown** planar scene

recover a ***model*** of the scene

Scene: a set of **unknown** points $\mathbf{x}_i$ on a plane

Image: a projective transformation of the scene → $\mathbf{x'}_i = H\mathbf{x}_i$

the image points $\mathbf{x'}_i$ are **known**, but H is **unknown**

Difficulty: the projective mapping H is **unknown**

→ we can not simply invert the mapping H

2D reconstruction problem: utopistic formulation

unknown
planar scene

unknown
projective
map $H??$

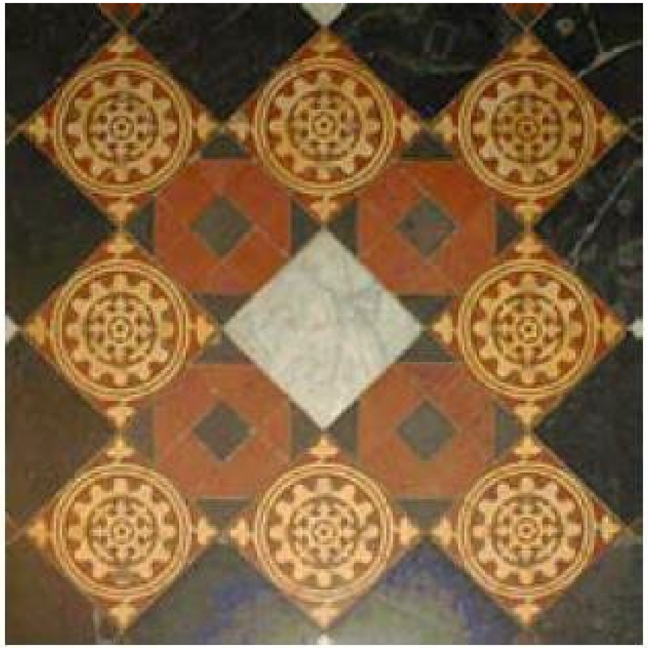
known image

design a
mapping
H_R

recovered
(reconstructed)
scene

combined mapping: an identity $\mathbf{I}_3$

if only the image is given → too many unknowns (8)



a

?

c

Combined strategy:

1. reduce unknowns

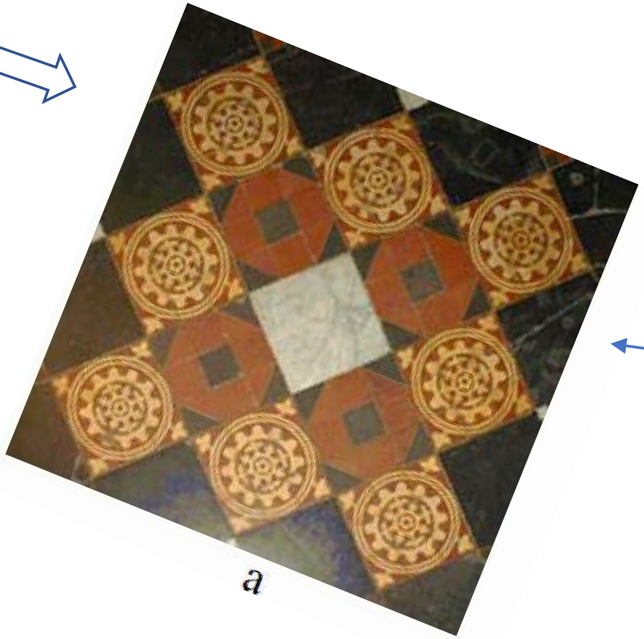
2. add constraints

1. Reduce unknowns

shape reconstruction: 4 unkns = 8 - 4

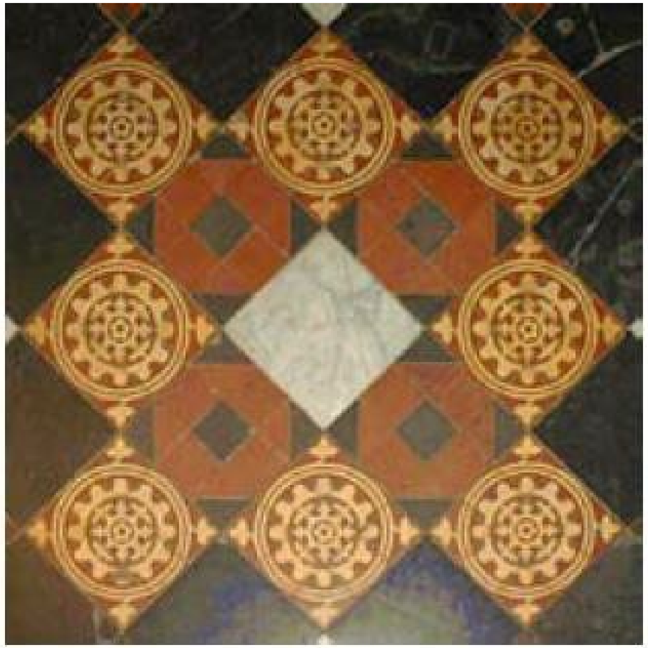
c

given image



reconstructed scene:
same shape as the original,
but different size and pose

similarity



a

2D reconstruction: 1. reduce unknowns

unknown projective map $H??$

known image

design a mapping H_R

unknown original scene

recovered (reconstructed) scene: same **shape** as original

combined mapping: a **similarity** H_S
not an identity

1. Reduce unknowns
(less satisfactory) affine reconstruction:
often used as an intermediate step towards shape reconstruction

affine reconstruction: 2 unkns = 8 - 6

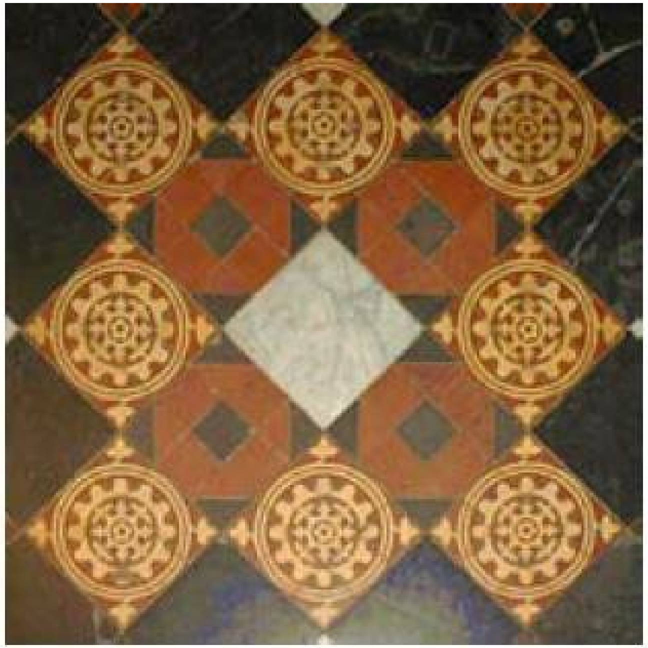
c

given image

b

reconstructed scene:
parallelism preserved
different shape, pose and size

affinity



a

2D reconstruction: 1. reduce unknowns

unknown projective map $H??$

unknown original scene

known image

design a mapping H_R

recovered (reconstructed) scene: affine to the original

combined mapping: an affinity H_A

2. Add constraints

Add constraints (from additional information): two cases

- **affine** reconstruction

  (reconstructed scene is an affine mapping of the original one)

- **shape** reconstruction

  (reconstructed scene is a similarity mapping of the original one)

affine reconstruction

2D affine reconstruction problem

unknown projective map $H??$

unknown original scene

known image

design a mapping H_{AR}

recovered (reconstructed) scene: affine to the original

combined mapping: an affinity H_A

A theorem on an affine invariant

Theorem. *A projective transformation H maps the line at the infinity $\boldsymbol{l}_\infty$ onto itself (i.e., $\boldsymbol{l}_\infty$ is invariant under the projective transformation H)*

$$\Updownarrow$$

H *is **affine***

Proof: Any point at the infinity $\mathbf{x}_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is mapped onto a point $\mathbf{x}' = H\mathbf{x}_\infty$ also at the infinity if and only if the third coordinate of $\mathbf{x}'$ is = 0, i.e.,

$$\forall(x, y)\ [v_1 \quad v_2 \quad 1] \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0 \to [v_1 \quad v_2 \quad 1] = [0 \quad 0 \quad 1]$$

namely,

$$H = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$ is **affine**

Application to affine reconstruction

The given image is a general projective mapping of the original scene

→

the vanishing line $\boldsymbol{l'}_{\infty}$(i.e. the image of the line at the infinity $\boldsymbol{l}_{\infty}$ of the scene) is in general $\neq \boldsymbol{l}_{\infty}$!!

Use $\boldsymbol{l'}_{\infty}$ as **additional information**: if we apply to the image a new projective mapping $H_{\boldsymbol{AR}}$ that maps $\boldsymbol{l'}_{\infty}$ back to $\boldsymbol{l}_{\infty}$, we obtain a new, modified image

The (new) image of the line at the infinity $\boldsymbol{l}_{\infty}$ is again $\boldsymbol{l}_{\infty}$ (itself)

→

From the theorem, the obtained model (i.e. the new image) is an **affine** mapping of the original scene

→

The obtained model is an **affine reconstruction** of the scene

l'_∞

H_{AR}

l_∞

unknown projective map H? ?

unknown original scene

known: Image + l'_∞

design a mapping H_{AR}

recovered (reconstructed) scene model: affine to the original

combined mapping: affine mapping H_A

Affine rectification from the image $\boldsymbol{l}'_\infty$ of the vanishing line $\boldsymbol{l}_\infty$

- Image of points at the infinity = vanishing points $\mathbf{v_1}, \mathbf{v_2}$ → vanishing line $\boldsymbol{l}'_\infty$

$$\boldsymbol{l}'_\infty = \mathbf{v_1} \times \mathbf{v_2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- Affine rectification matrix

$$H_{AR} = \begin{bmatrix} * & * & * \\ * & * & * \\ & \boldsymbol{l}'^{\boldsymbol{T}}_\infty & \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ a & b & c \end{bmatrix}$$

WHY?

- Affine reconstructed model $M_A = H_{AR} * \text{Image}$

Shape (Euclidean) reconstruction

2D shape reconstruction problem

unknown
original scene

unknown
projective
map $H??$

known image

design a
mapping
H_{SR}

recovered
(reconstructed)
scene model: same
shape as original

combined mapping: a **similarity** $\boldsymbol{H_S}$

A theorem on an invariant under similarities

Theorem. *A projective transformation H maps the **circular points** $\boldsymbol{I}$ and $\boldsymbol{J}$ onto themselves (i.e., $\boldsymbol{I}$ and $\boldsymbol{J}$ are invariant under the projective transformation H)*

$$\Updownarrow$$

*H is **a similarity***

Proof: Multiplying a similarity matrix H_S by circular point $\boldsymbol{I}$, a multiple of $\boldsymbol{I}$ is obtained. Analogous result is obtained for the other circular point $\boldsymbol{J}$

more simply

$\{\boldsymbol{I}, \boldsymbol{J}\}$ = circle ∩ $\boldsymbol{l}_\infty$ ------- H_S ------> other circle ∩ $\boldsymbol{l}_\infty$ = $\{\boldsymbol{I}, \boldsymbol{J}\}$

Application to shape reconstruction

The given image is a general projective mapping of the original scene

→

the image $(\boldsymbol{I}', \boldsymbol{J}')$ of circular points $(\boldsymbol{I}, \boldsymbol{J})$ of the scene plane is in general $\neq (\boldsymbol{I}, \boldsymbol{J})$.

Use $(\boldsymbol{I}', \boldsymbol{J}')$ as **additional information**: if we apply to the image a new projective mapping H_{SR} that maps $(\boldsymbol{I}', \boldsymbol{J}')$ back to $(\boldsymbol{I}, \boldsymbol{J})$, we obtain a new, modified image

The (new) image of the circular points $(\boldsymbol{I}, \boldsymbol{J})$ is again $(\boldsymbol{I}, \boldsymbol{J})$ (themselves)

→

From the theorem, the obtained model (new image) is *similar* to the original scene

→

The obtained model is a **shape reconstruction** of the scene

2D shape reconstruction problem

unknown
projective
map $H??$

unknown
original scene

known:
image + $(\boldsymbol{I}', \boldsymbol{J}')$

design a
mapping
H_{SR}

reconstructed
scene model:
same shape as
original

combined mapping: a **similarity** H_S

Shape (Euclidean) reconstruction
via Singular Value Decomposition

Shape (Euclidean) reconstruction

Instead of using the image $\boldsymbol{I}'$ and $\boldsymbol{J}'$ of the circular points, one can use the image $C_\infty^{*\prime} = \boldsymbol{I}'\,\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}}$ of the conic dual to the circular points C_∞^*

$$C_\infty^* = \boldsymbol{I}\boldsymbol{J}^{\boldsymbol{T}} + \boldsymbol{J}\boldsymbol{I}^{\boldsymbol{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}} \neq C_\infty^*$$

Singular value decomposition

Why $\mathbf{C}_\infty^*{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^\mathsf{T}$?

Normally: SVD (Singular Value Decomposition)

$$\mathbf{C}_\infty^*{}' = \mathbf{V}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\mathbf{U}^\mathsf{T}$$ with **U** and **V** orthogonal

But $\mathbf{C}_\infty^*{}'$ is symmetric → $\mathbf{C}_\infty^*{}'^\mathsf{T} = \mathbf{UDV}^\mathsf{T} = \mathbf{VDU}^\mathsf{T} = \mathbf{C}_\infty^*{}'$

and SVD is unique → $\mathbf{U} = \mathbf{V}$

Observation : **H=U** orthogonal (3x3): not a $\mathbf{P}^2$ isometry

Use of $C_\infty^{*\prime}$ in shape reconstruction

finding a projectivity H_{SR} that maps $(\boldsymbol{I}', \boldsymbol{J}')$ back to $(\boldsymbol{I}, \boldsymbol{J})$
reduces to finding a projectivity H_{SR} that maps $C_\infty^{*\prime}$ back to C_∞^{*}
Using the transformation rule for dual conics under projective mappings we get

$$C_\infty^{*} = H_{SR}\, C_\infty^{*\prime}\, {H_{SR}}^T$$

solving this for $C_\infty^{*\prime}$ yields

$$C_\infty^{*\prime} = {H_{SR}}^{-1} C_\infty^{*}\, {H_{SR}}^{-T} = {H_{SR}}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ({H_{SR}}^{-1})^T$$

but, from SVD applied to the symmetric matrix $C_\infty^{*\prime}$

$$(SVD)C_\infty^{*\prime} = U_\perp \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} {U_\perp}^T \rightarrow {H_{SR}}^{-1} = U_\perp \rightarrow H_{SR} = {U_\perp}^{-1} = {U_\perp}^T$$

i.e., one of the possible ∞^4 solutions

There is a numerical issue in image rectification

Accuracy issue in image rectification

- In principle, given $C_\infty^{*\prime}$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^{*} H_{rect}^{-T}$$

- But, due to noise and numerical errors, SVD gives

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^{*} H_{rect}^{-T}$$

- In principle, given $C_\infty^{*\prime}$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

- But, due to noise and numerical errors, SVD output is

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = U \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

$$\rightarrow H_{rect}^{-1} = U \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow H_{rect} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

To sum up: image rectification from the image $(\boldsymbol{I}', \boldsymbol{J}')$ of the circular points $(\boldsymbol{I}, \boldsymbol{J})$

- Image of the circular points → image of the conic dual to the circular points

$$C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}}$$

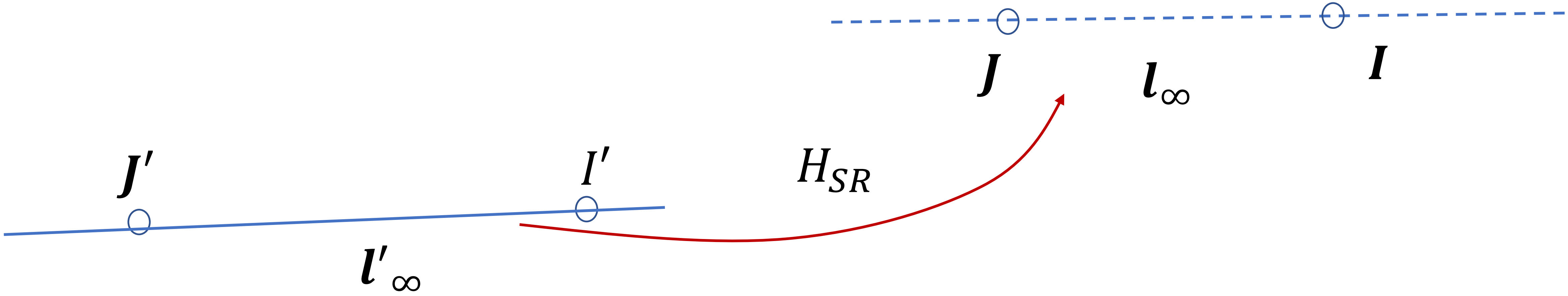
- Singular value decomposition

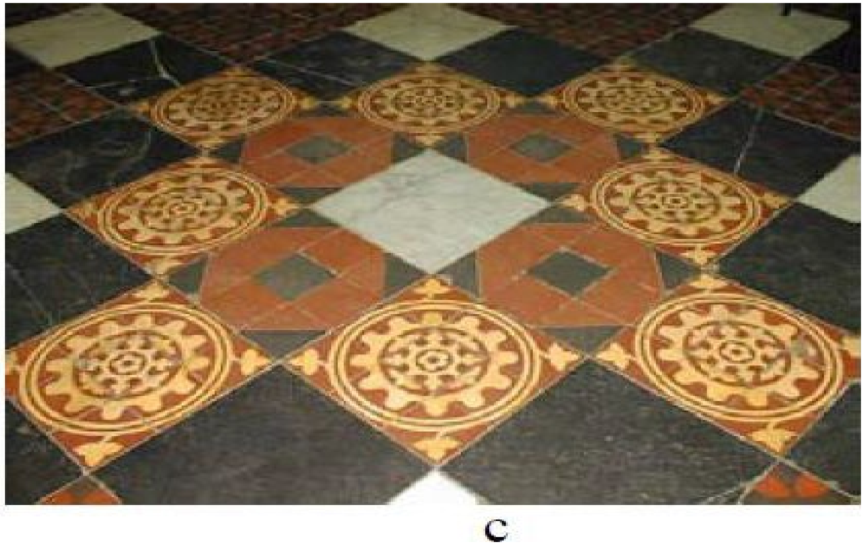
$$\mathrm{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^{*} H_{SR}^{-T}$$

- Rectifying transformation (from svd output U)

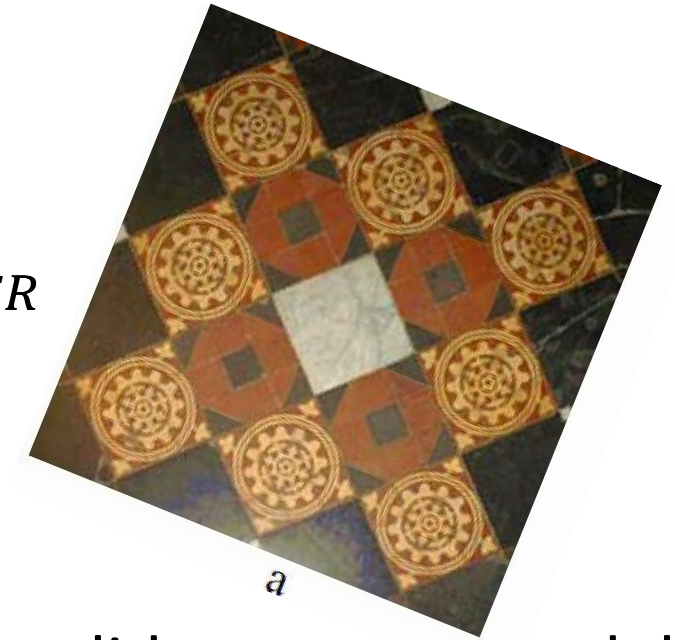
$$H_{SR} = \begin{bmatrix} {}^{1}/_{\sqrt{a}} & 0 & 0 \\ 0 & {}^{1}/_{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} * \mathrm{Image}$





shape reconstruction H_{SR}



image

euclidean scene model

How to find $(\boldsymbol{I}', \boldsymbol{J}')$ (or $\boldsymbol{C}_{\infty}^{*\prime}$) in practical cases?

How to find $(\boldsymbol{I'},\boldsymbol{J'})$ (or $\boldsymbol{C_\infty^{*\prime}}$) in practical cases?

In 2D rectification we use $(\boldsymbol{I'},\boldsymbol{J'})$, or equivalently $\boldsymbol{C_\infty^{*\prime}}$,

as additional information

How can we find $(\boldsymbol{I'},\boldsymbol{J'})$ (or $C_\infty^{*\prime}$) **?**

from information on the observed scene we derive constraints on $C_\infty^{*\prime}$

a. known angles between lines

b. known shape of objects, e.g., circumferences

c. combinations of a. and b.

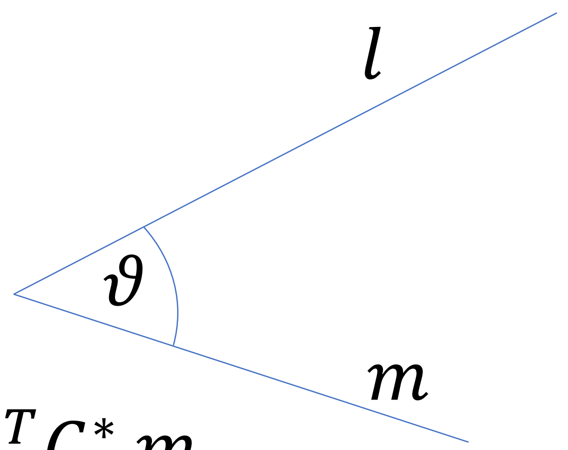
c. observation of rigid planar motion

a. known angles between lines

REMEMBER: Angle between two lines l and m

The angle ϑ between $l = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $m = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is the angle between their normals $[a_1 \quad b_1]$ and $[a_2 \quad b_2]$:

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1{}^{\mathbf{2}} + b_1{}^{\mathbf{2}})(a_2{}^{\mathbf{2}} + b_2{}^{\mathbf{2}})}}$$



But, e.g., $a_1 a_2 + b_1 b_2 = [a_1 \quad b_1 \quad c_1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = l^T C_\infty^* m$

$$\rightarrow \cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

Express the angle ϑ between two lines **in the scene** in terms of elements **in the image**: $l', m', C_\infty^{*\prime}$

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}} = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

in view of transformation rules

$$\text{e.g., } l^T C_\infty^* m = l'^T H H^{-1} C_\infty^{*\prime} H^{-T} H^T m' = l'^T C_\infty^{*\prime} m'$$

$\rightarrow$

Express the angle ϑ between two lines **in the scene** in terms of elements **in the image**: $l', m', C_\infty^{*\prime}$

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1{}^2 + b_1{}^2)(a_2{}^2 + b_2{}^2)}} = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

in view of transformation rules

e.g., $l^T C_\infty^* m = l'^T H H^{-1} C_\infty^{*\prime} H^{-T} H^T m' = l'^T C_\infty^{*\prime} m'$

$\rightarrow$

$$\cos\vartheta = \frac{l'^T C_\infty^{*\prime} m'}{\sqrt{(l'^T C_\infty^{*\prime} l')(m'^T C_\infty^{*\prime} m')}}$$

Express the angle ϑ between two lines **in the scene** in terms of elements **in the image**: l', m', $C_\infty^{*\prime}$

$$\cos\vartheta = \frac{l'^T C_\infty^{*\prime} m'}{\sqrt{(l'^T C_\infty^{*\prime} l')(m'^T C_\infty^{*\prime} m')}}$$

Here, l' and m' are extracted from the image (see later), whereas $\boldsymbol{C_\infty^{*\prime}}$ is the **unknown** matrix we want to estimate

Known angle ϑ between two **scene lines** → nonlinear eqn on $C_\infty^{*\prime}$

if the scene lines are perpendicular, $\cos\vartheta = 0$ → $l'^T C_\infty^{*\prime} m' = 0$ **linear**

Knowledge of l'_∞ → constraints on $C^{*\prime}_\infty$

$$C^{*\prime}_\infty l'_\infty = (I'J'^T + J'I'^T)l'_\infty = I'(\underbrace{J'^T l'_\infty}_{0}) + \mathrm{J}'(\underbrace{I'^T l'_\infty}_{0}) = \mathbf{0}$$

$$C^{*\prime}_\infty l'_\infty = 0$$

2 constraints

line at the infinity l'_∞ = RNS($C^{*\prime}_\infty$)

How many constraints are needed to find $C^{*\prime}_{\infty}$?

Matrix $C^{*\prime}_{\infty}$ is symmetric, homogeneous, and **singular**

→ 4 degrees of freedom (e.g. $\boldsymbol{I'}, \boldsymbol{J'}$)

→ 4 **independent** constraints are needed

however

singularity is in general a nonlinear constraint

→use 5 linear constraints (orthogonal lines)

or

if l'_{∞} can be derived from parallelism (or two pairs of orthogonal lines with one line in common), use

$l'_{\infty} = RNS(C^{*\prime}_{\infty})$: 2 linear constraints which implies singularity

+ 2 linear constraints from pairs of orthogonal lines

Are the constraints associated to the 5 indicated pairs of orthogonal lines independent? NO: l'_∞ can be derived

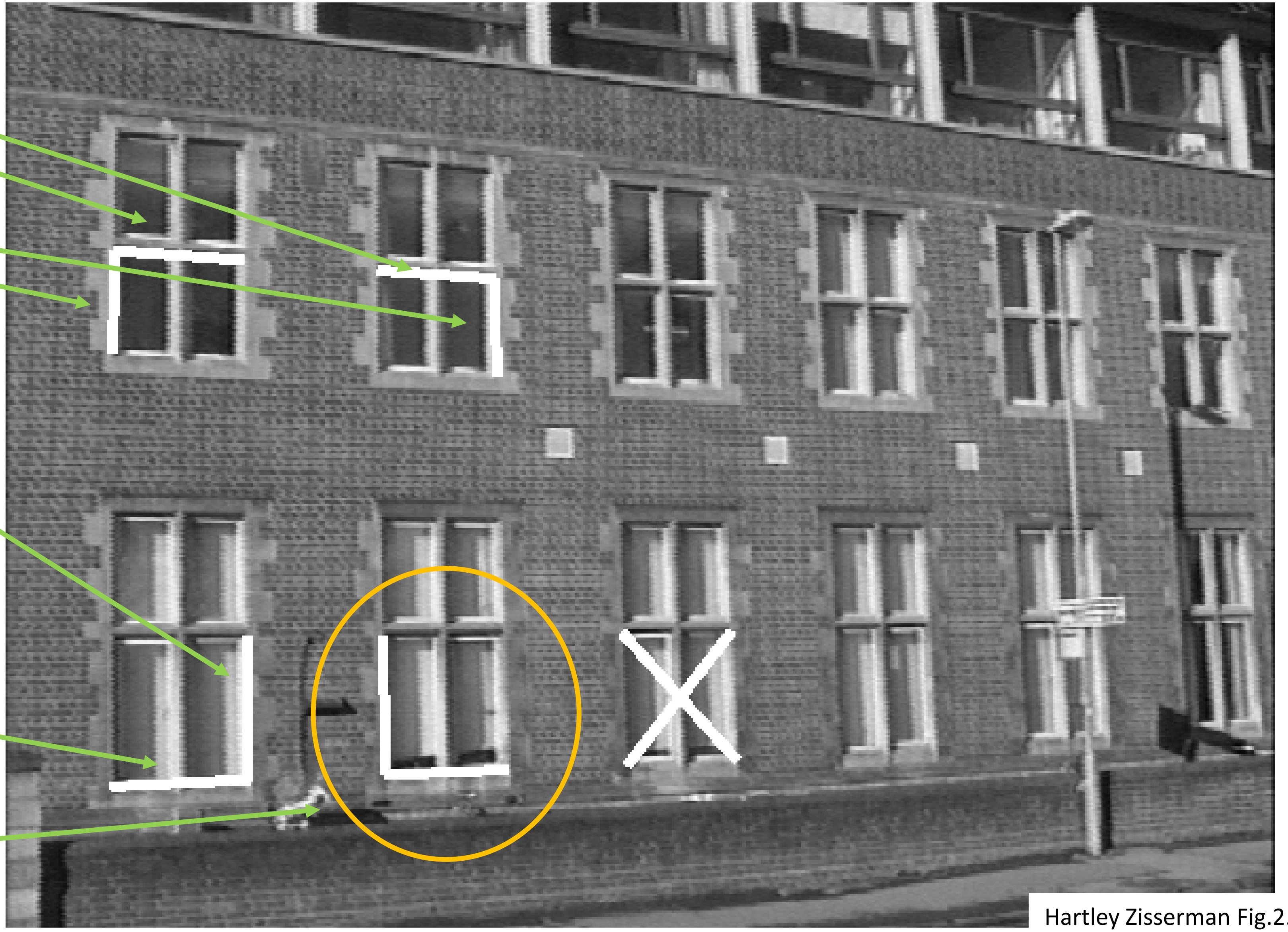
colinear

→ parallel

concurrent at same vertical vanish. point

→ must concur at same horizontal vanish. point

→ derivable from previous contraints

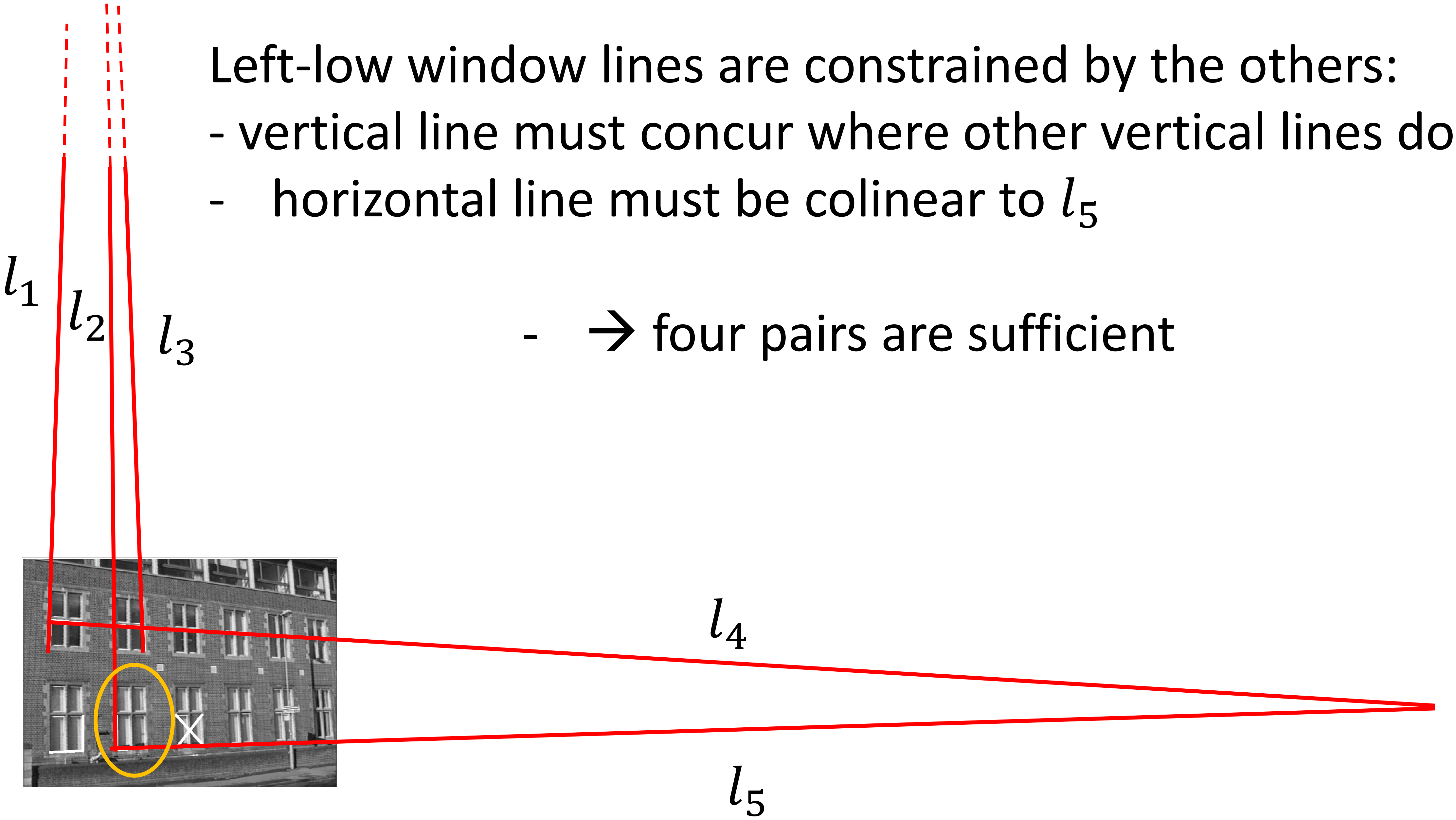


Hartley Zisserman Fig.2.17

Left-low window lines are constrained by the others:

- vertical line must concur where other vertical lines do
- horizontal line must be colinear to l_5

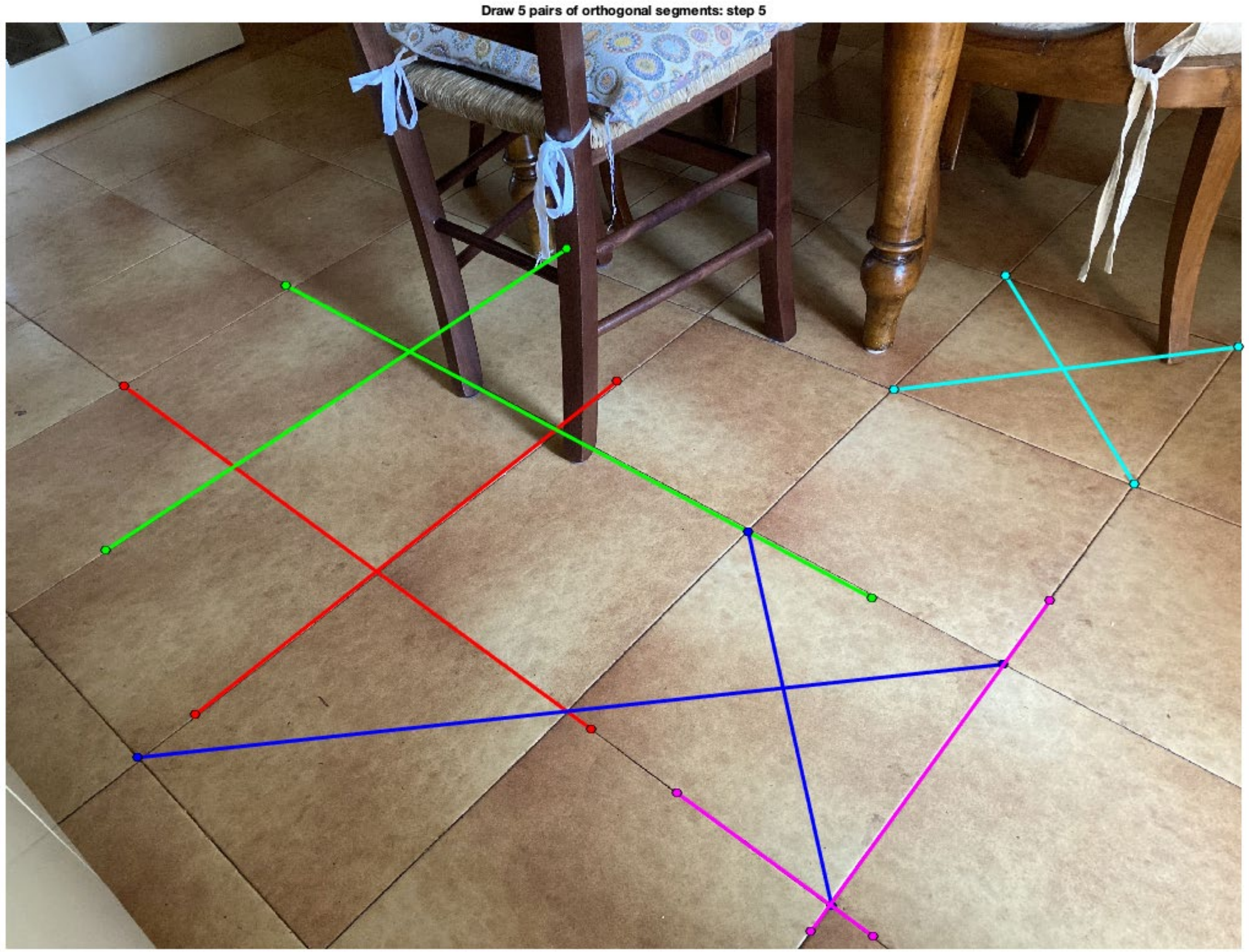
- → four pairs are sufficient



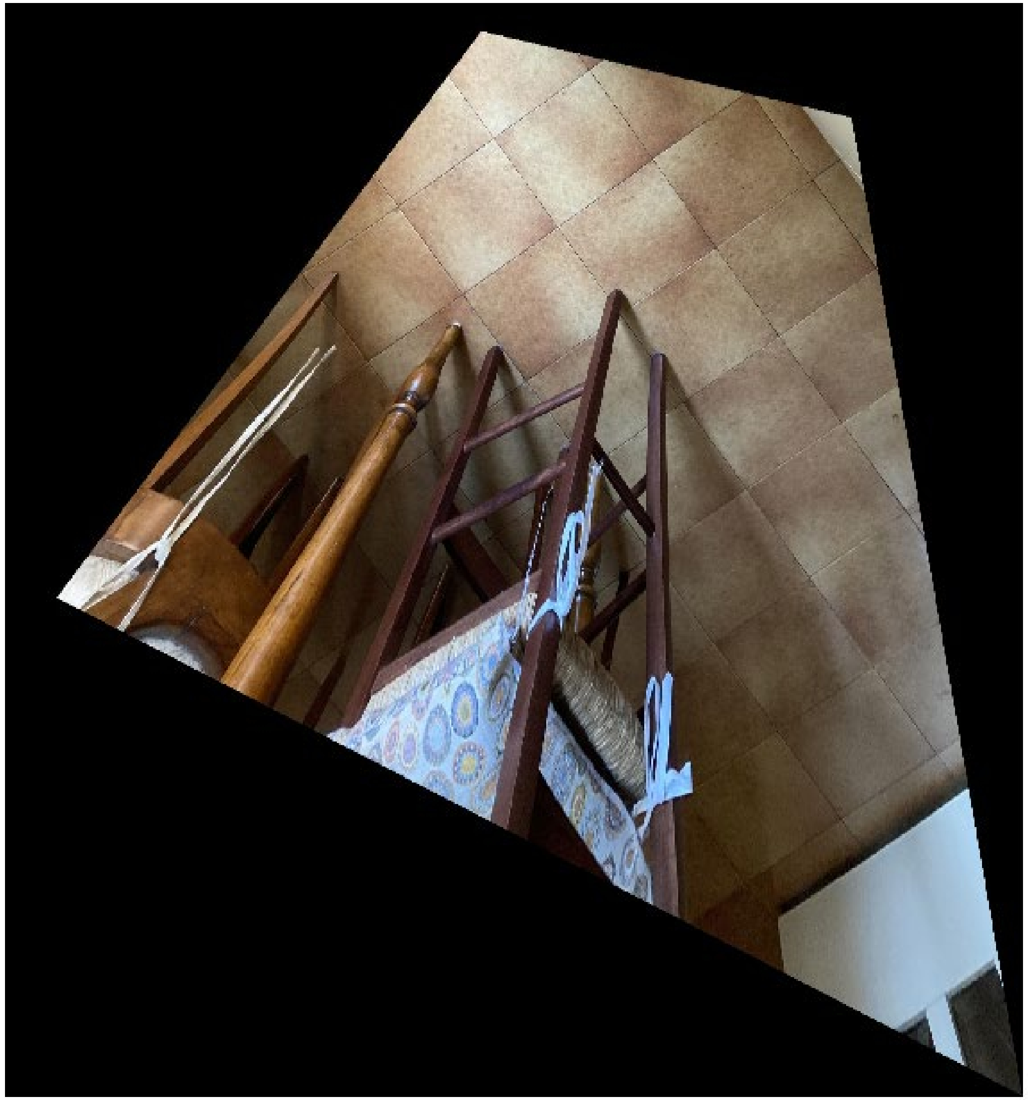
a. rectification from pairs of orthogonal lines

Hartley Zisserman Fig.2.17

Are they all independent?



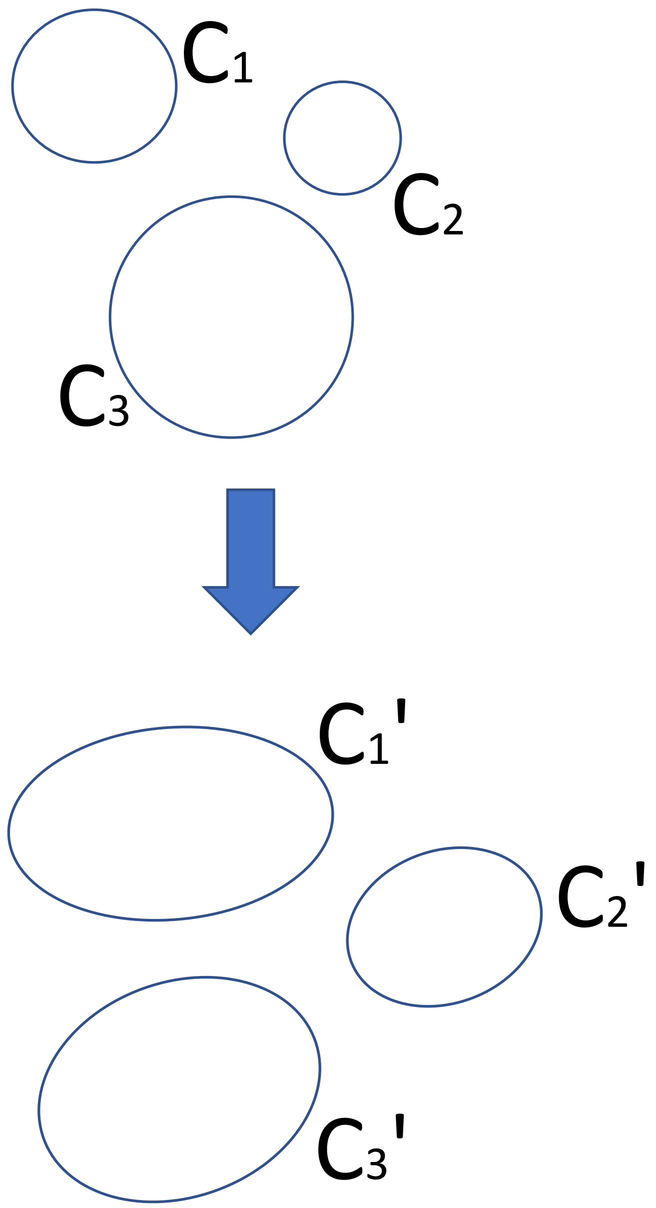
rectified image



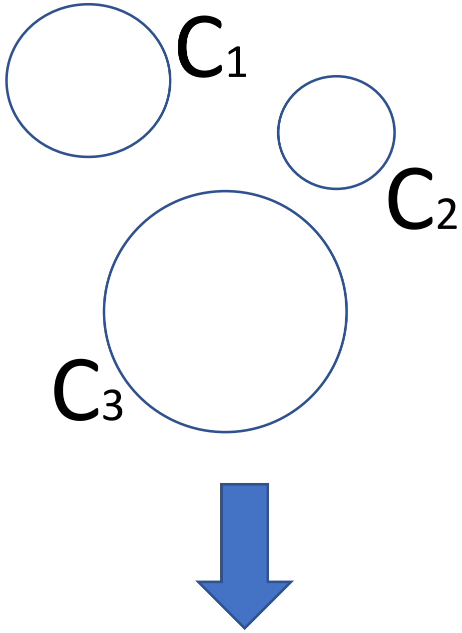
credits: Luca Magri

b. Example: image of circumferences

Example: image of circumferences



Example: image of circumferences



intersect the images of circumferences
→ image of circular points

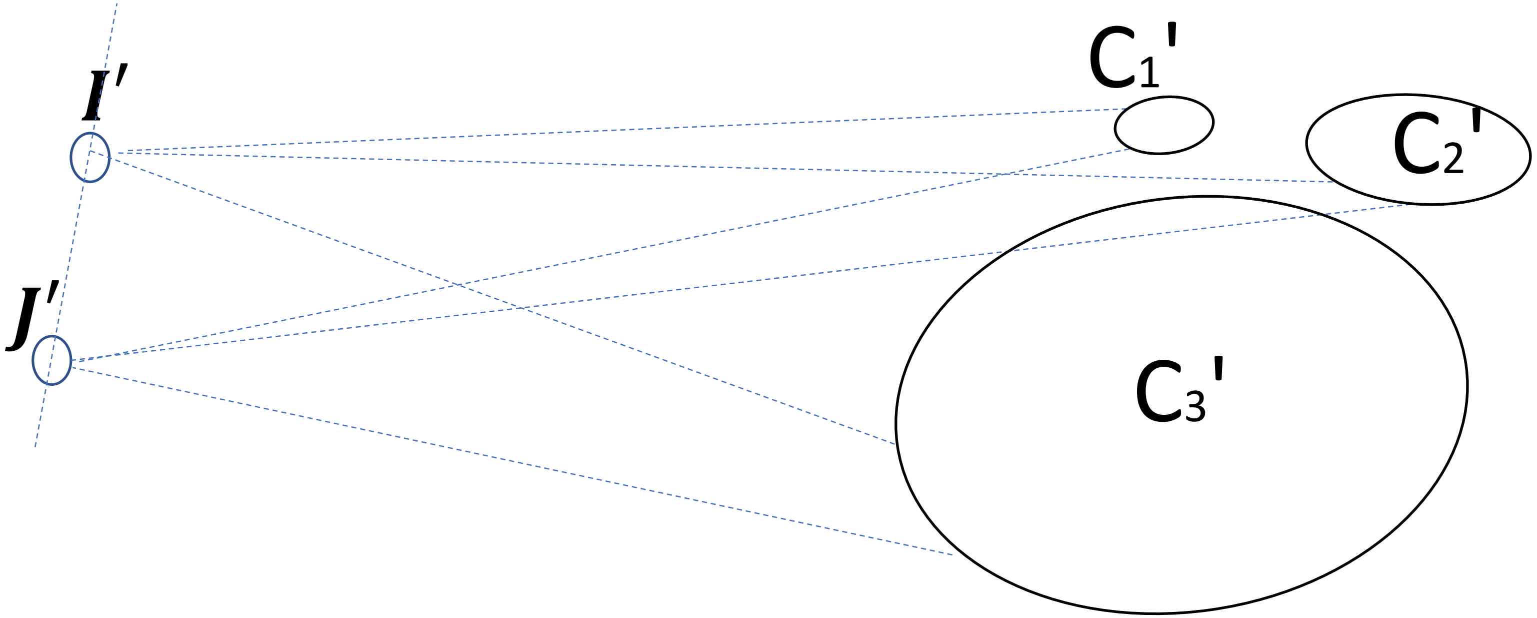


Image rectification from the image of circular points: {I’, J’} = $C_1' \cap C_2' \cap C_3'$

- Image of the circular points → image of the conic dual to the circular points

$$C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}}$$

- Singular value decomposition

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^* H_{SR}^{-T}$$

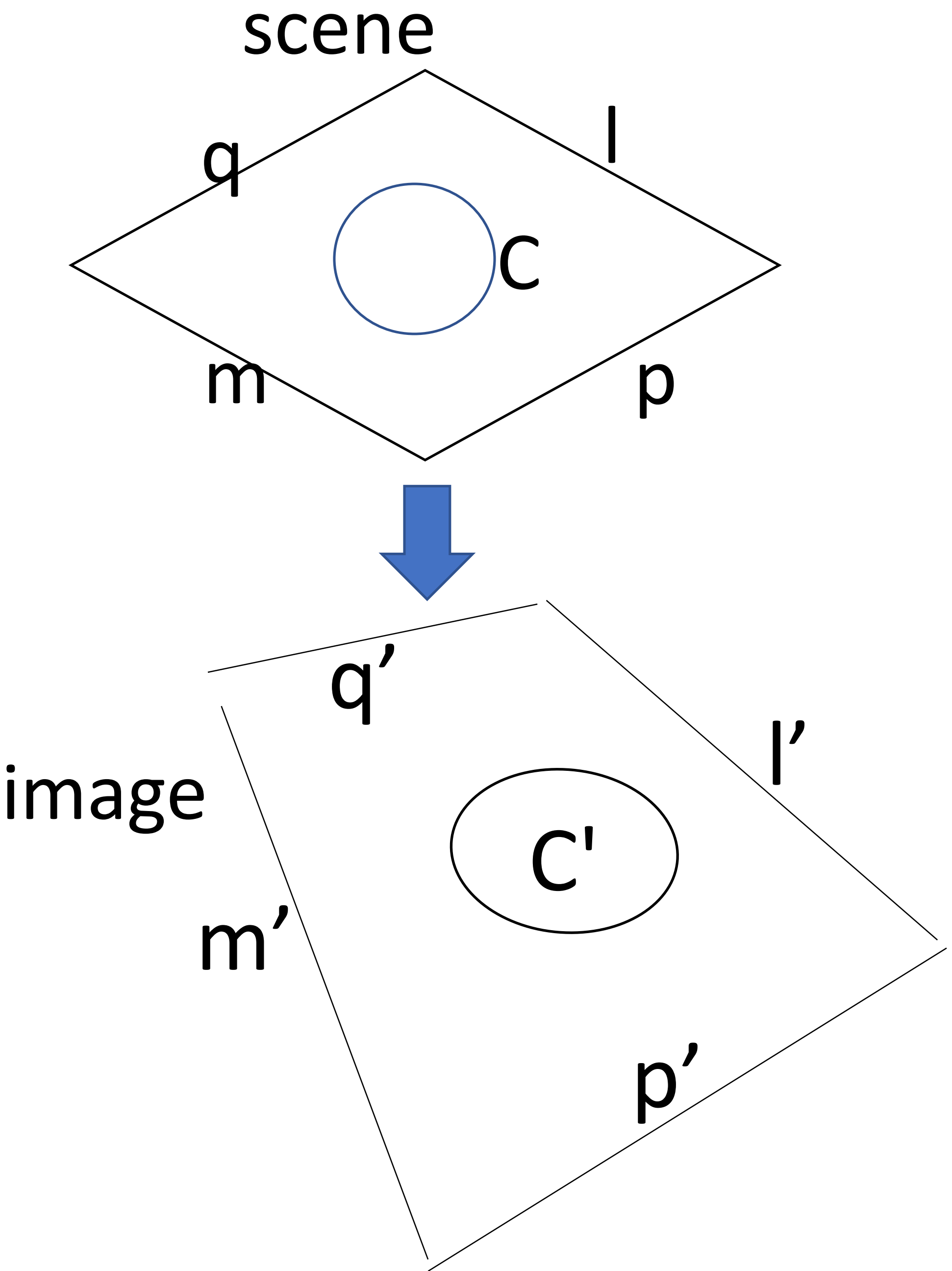
- Rectifying transformation (from svd output U)

$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} \ * \text{Image}$

c. Example: circumference + parallelogram

Example



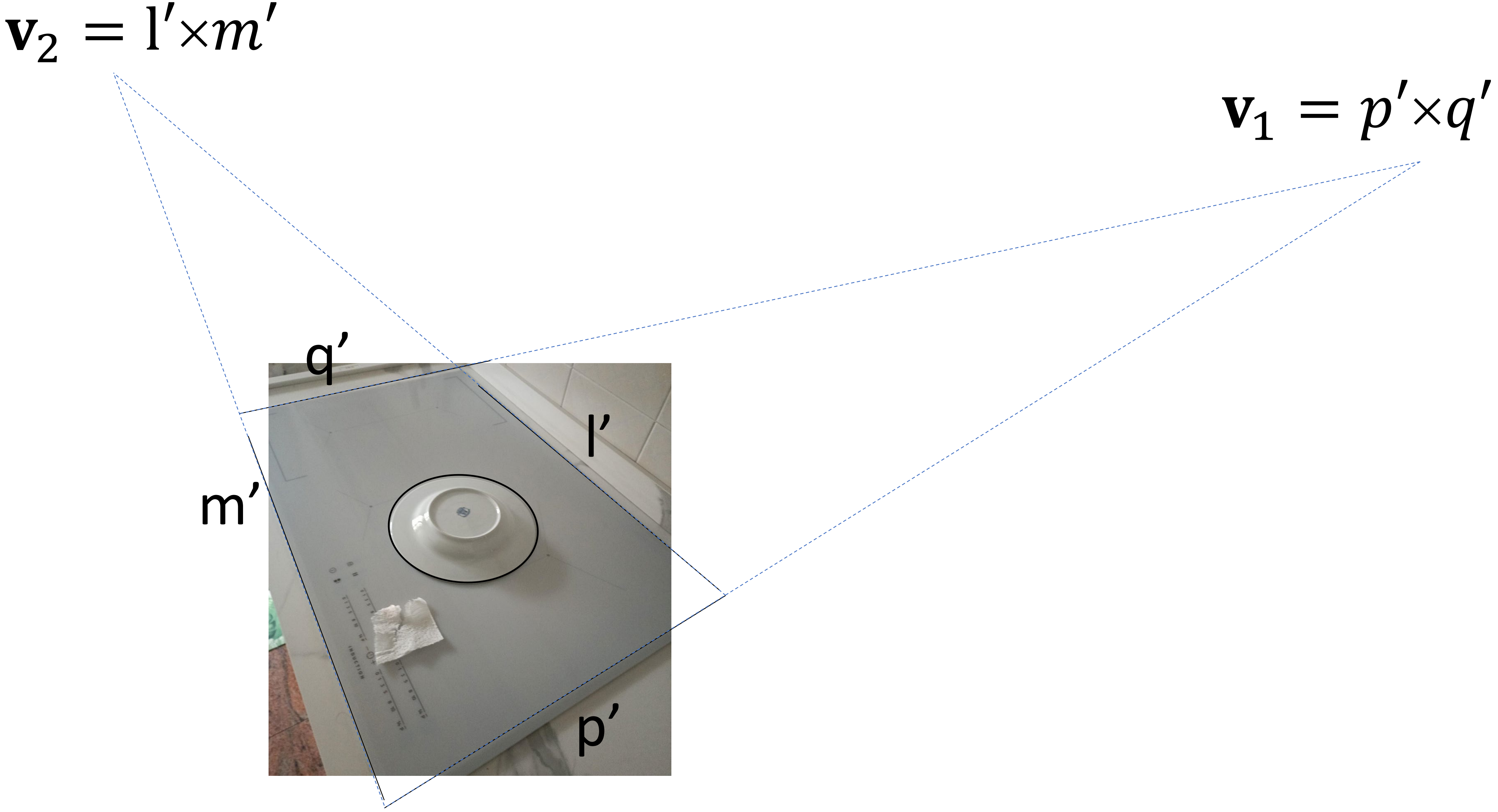

$\mathbf{v}_2 = l' \times m'$
$\mathbf{v}_1 = p' \times q'$
q’
l’
m’
p’

image of the line at the infinity

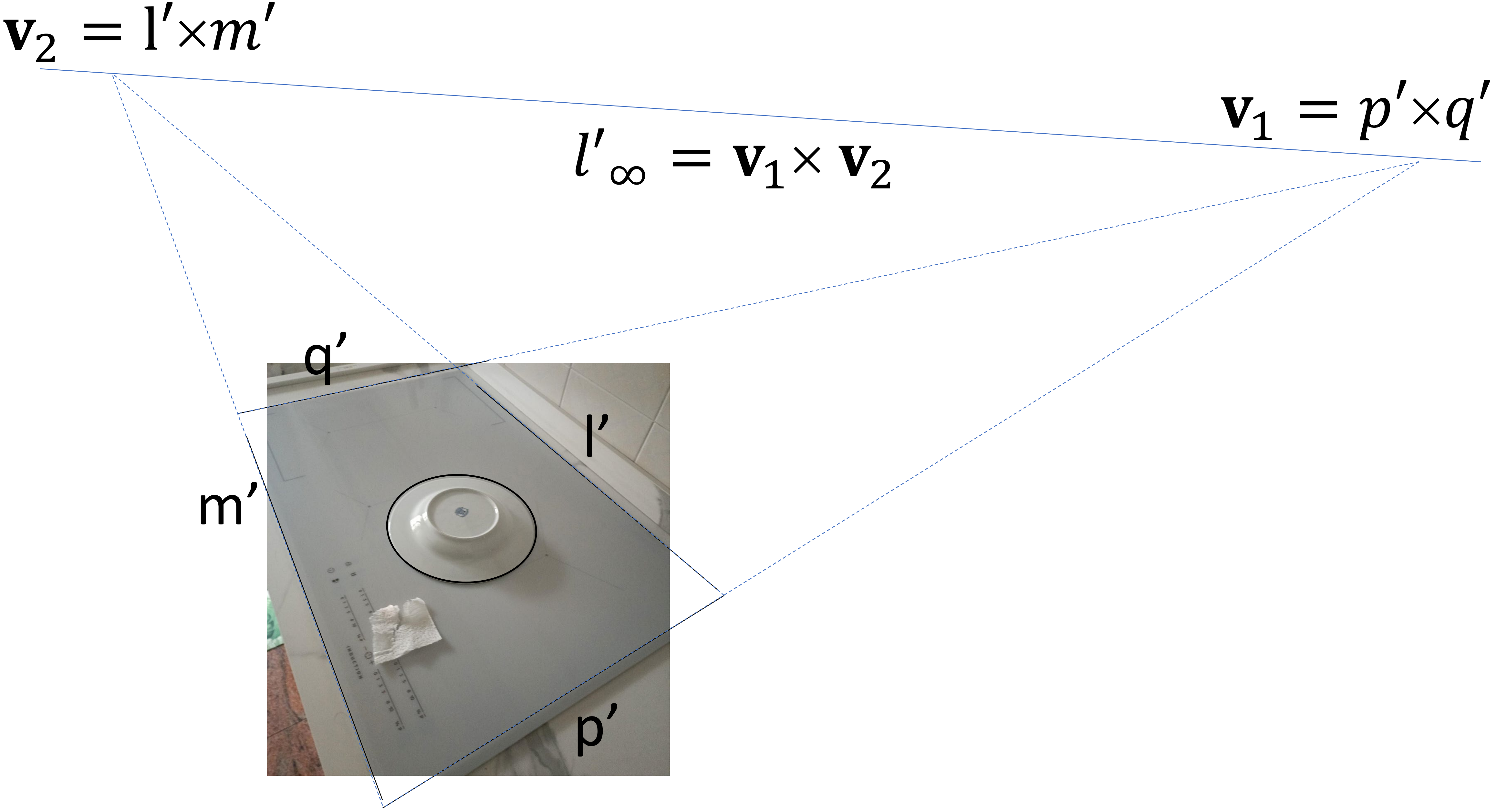
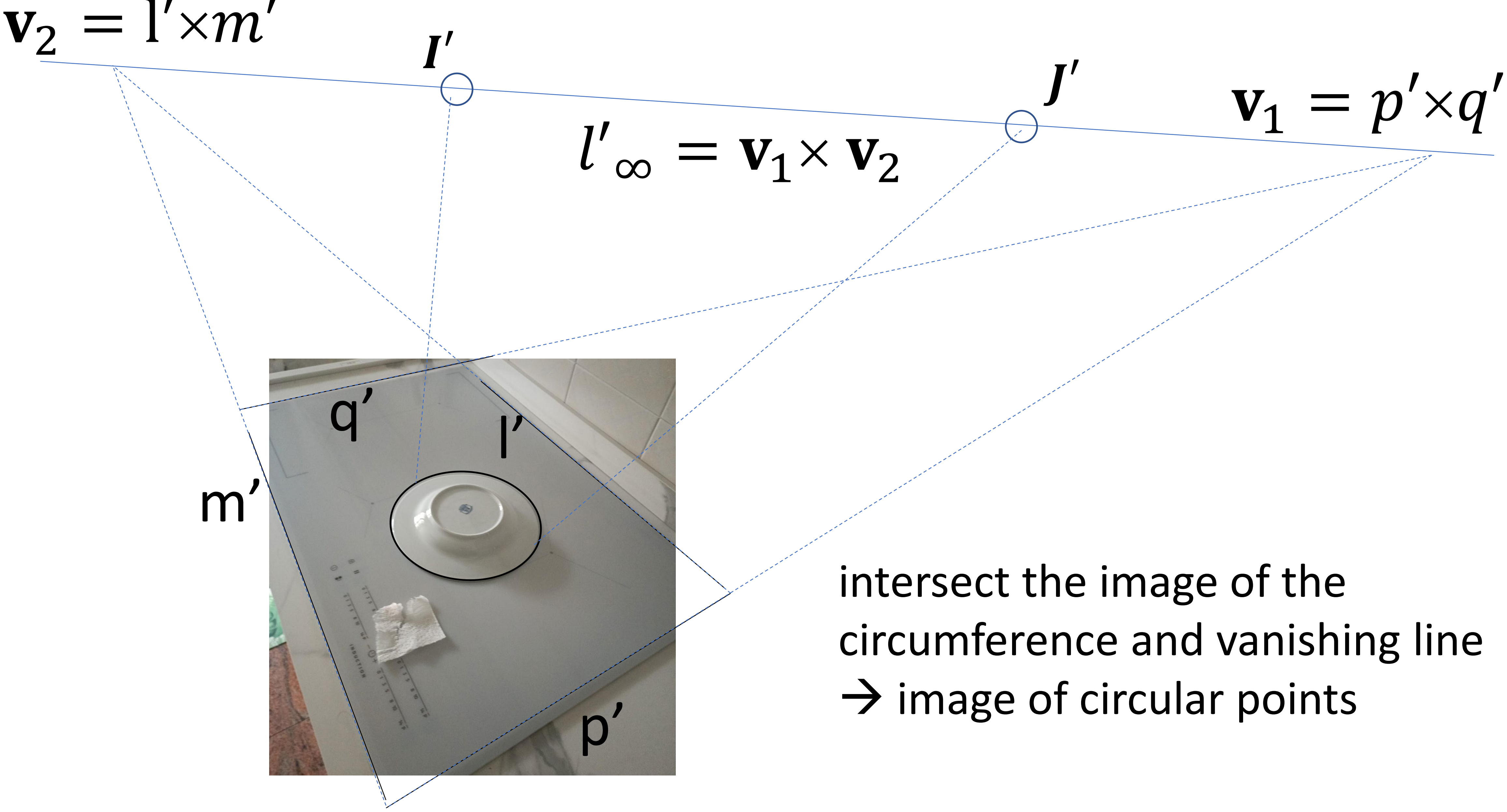


Image of circular points: {I', J'} = $l'_\infty \cap C'$



intersect the image of the circumference and vanishing line → image of circular points

Image rectification from the image of circular points: {I’, J’} = $l'_\infty \cap C'$

- Image of the circular points → image of the conic dual to the circular points
$$C_\infty^{*\prime} = \boldsymbol{I'J'^T} + \boldsymbol{J'I'^T}$$
- Singular value decomposition
$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^* H_{SR}^{-T}$$
- Rectifying transformation (from svd output U)
$$H_{SR} = \begin{bmatrix} {}^1/_{\sqrt{a}} & 0 & 0 \\ 0 & {}^1/_{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$
- Euclidean reconstructed model $M_S = H_{SR} \, * \, \text{Image}$

d. Example: rectification from planar displacement

Image of a rigid planar motion

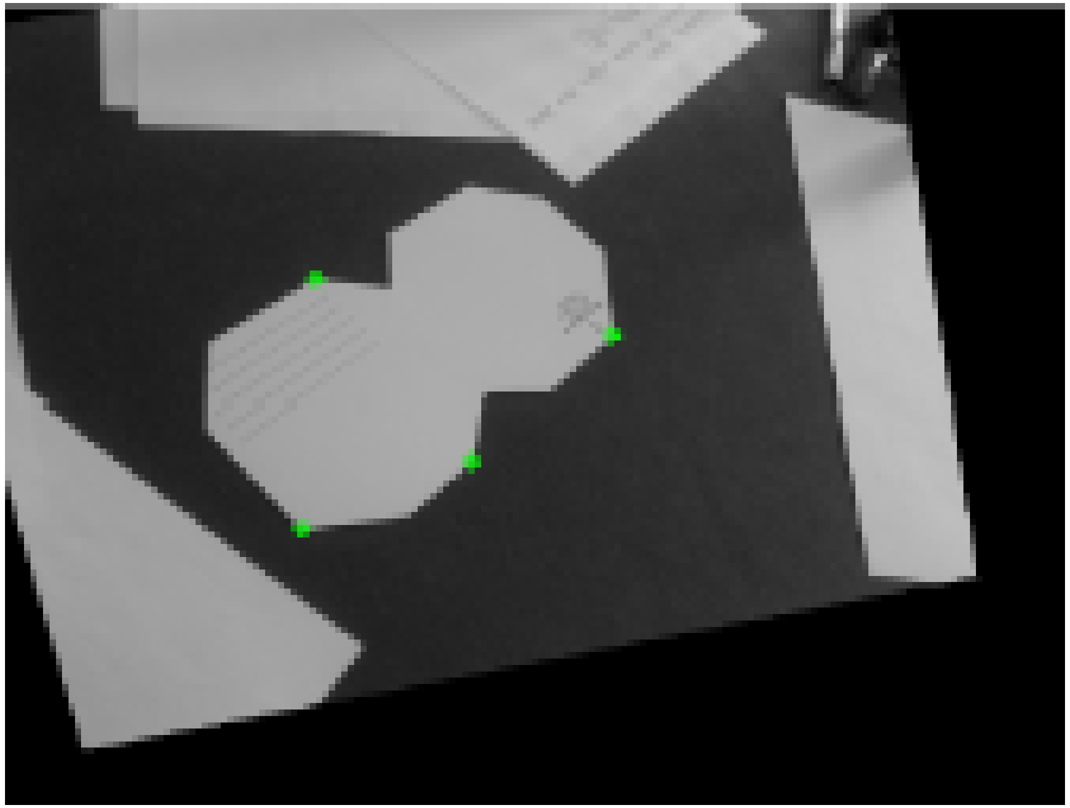


image before displacement

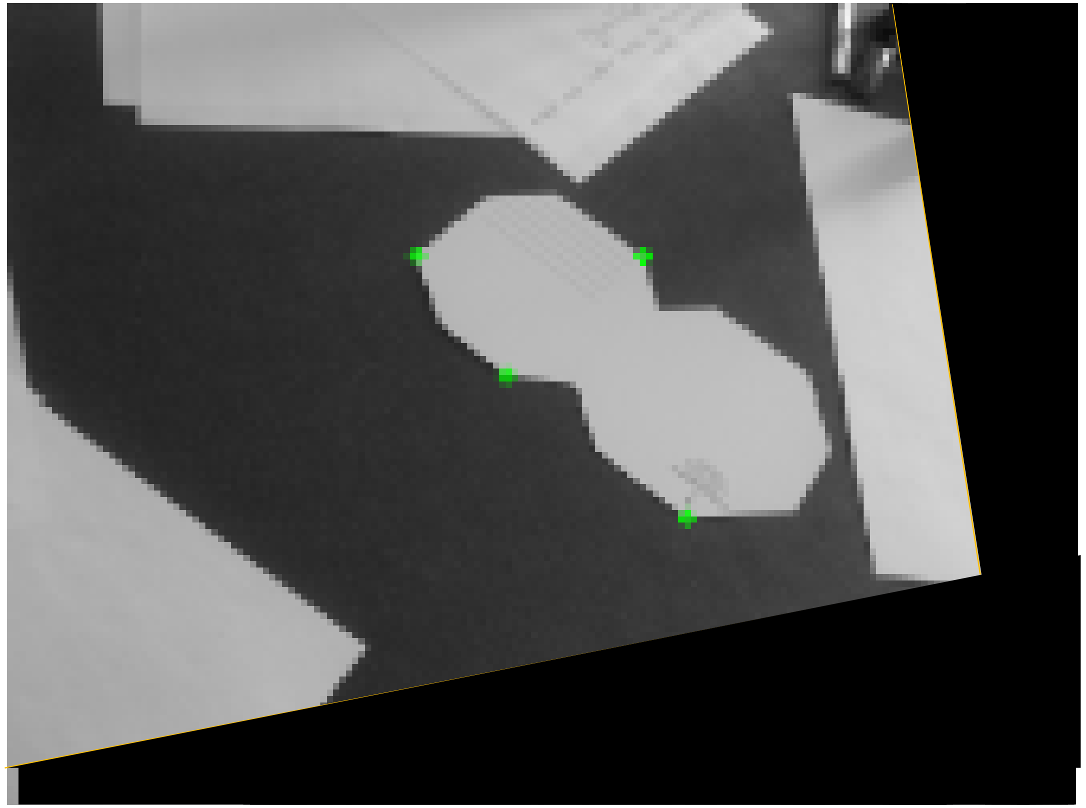
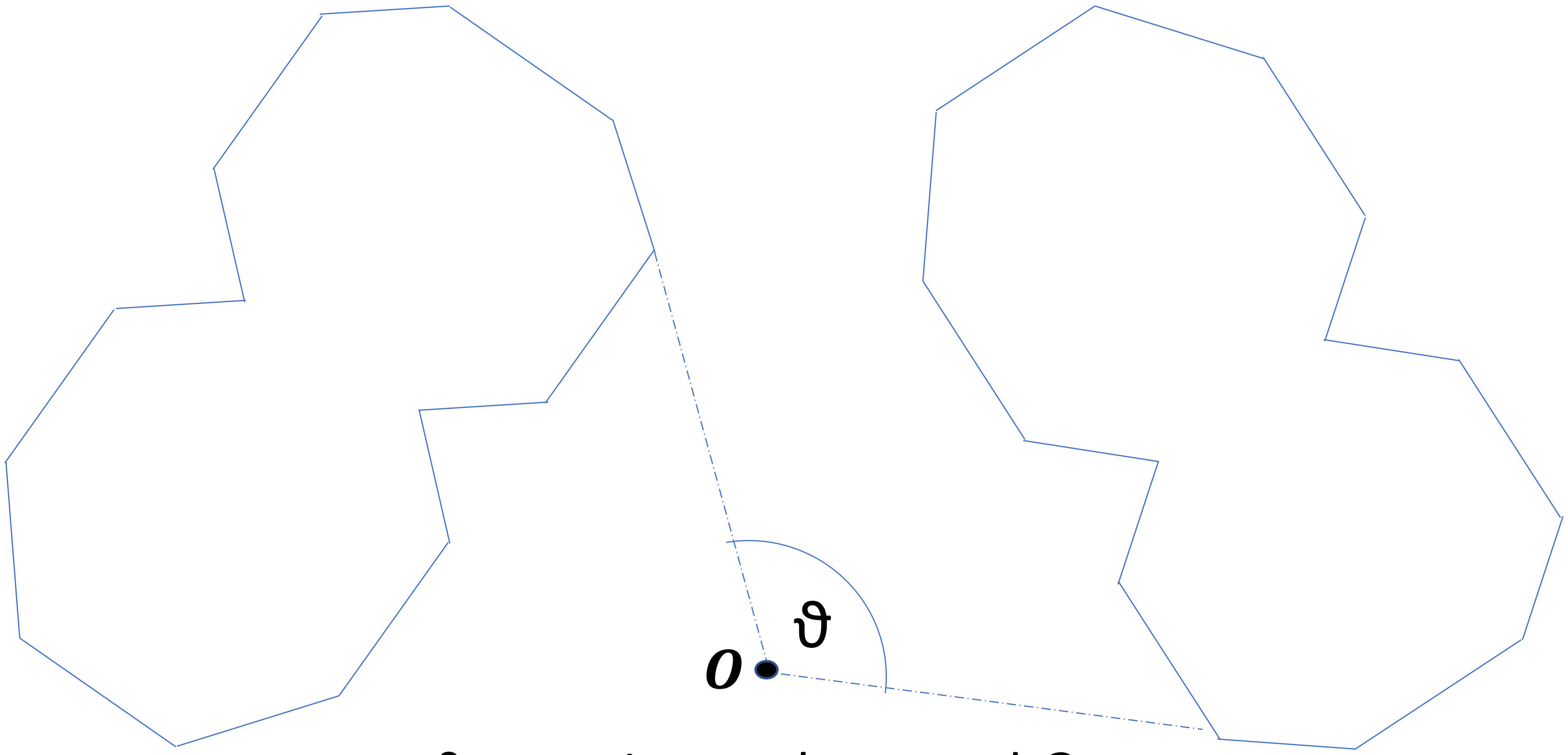


image after displacement

Property of rigid planar motion:
any roto-translation is a **pure rotation**



ϑ : rotation angle around $\boldsymbol{O}$

rigid planar motion

- Planar rigid motion: isometry → is a similarity → **invariant** circ. pts $\boldsymbol{I}, \boldsymbol{J}$
- any (planar) rigid displacement is a pure rotation: 3 dofs X_O, Y_O, ϑ

→ Center of rotation $\boldsymbol{O}$ is also invariant

→3 **invariants**: circular points $\boldsymbol{I}, \boldsymbol{J}$ and center of rotation $\boldsymbol{O}$

these 3 points, namely, $\boldsymbol{I}, \boldsymbol{J}$ and $\boldsymbol{O}$ remain fixed during motion

→

also their images $\boldsymbol{I}', \boldsymbol{J}'$ and $\boldsymbol{O}'$ remain fixed during motion image points $\boldsymbol{I}'$, $\boldsymbol{J}'$ and $\boldsymbol{O}'$ remain fixed

corresponding features → estimate homography H

$\boldsymbol{I}'$, $\boldsymbol{J}'$ and $\boldsymbol{O}'$ are invariant under mapping H

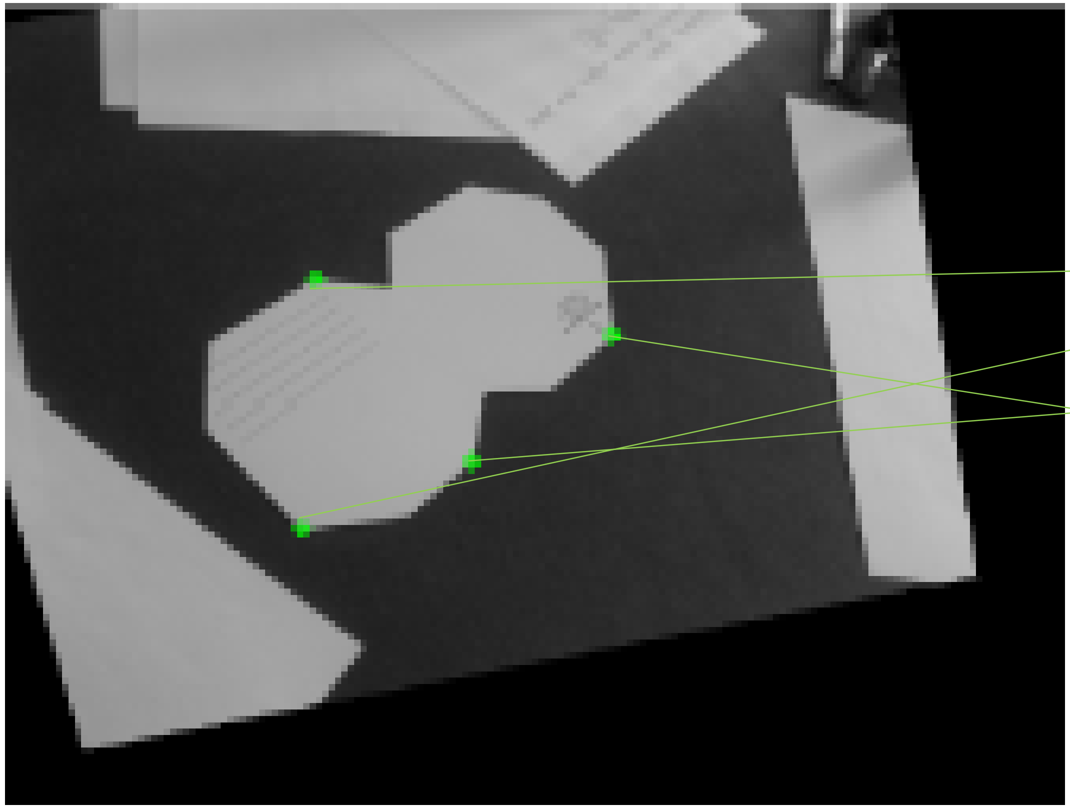


image before displacement

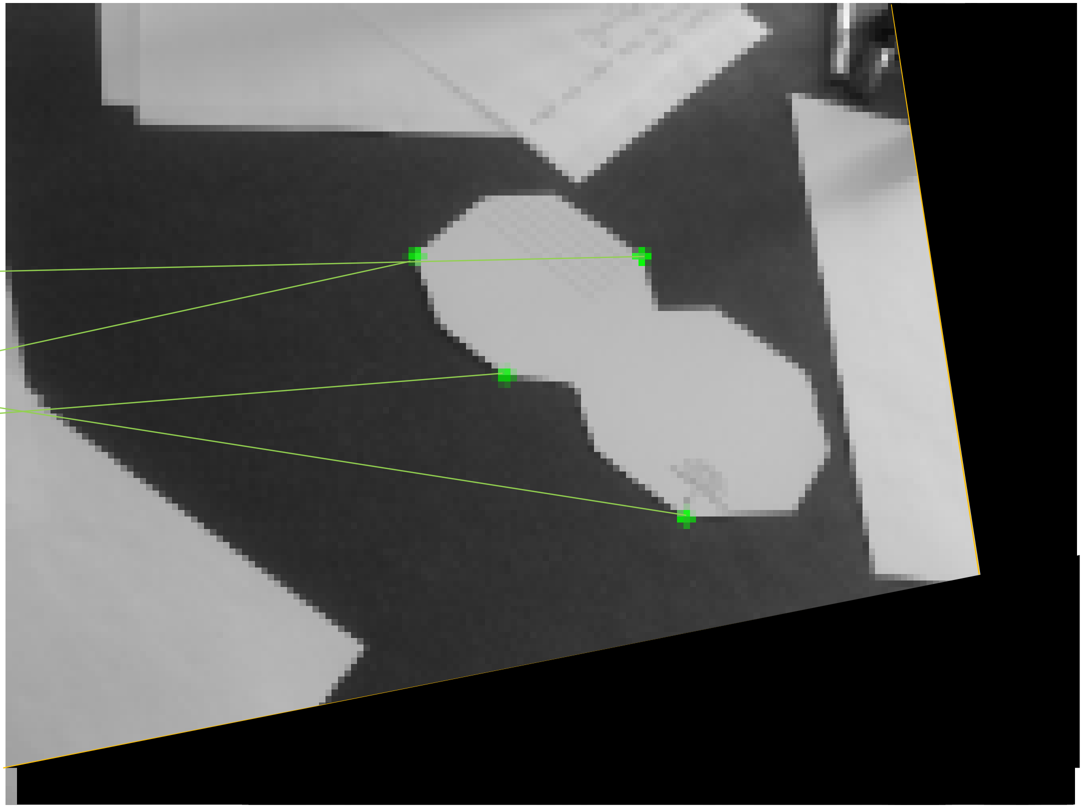


image after displacement

invariants under a projective mapping H

Homogeneous coordinates:

invariants He = λe ←→ **eigenvectors**

it can be shown that:

- eigenvectors $\boldsymbol{I}', \boldsymbol{J}'$ correspond to complex eigenvalues
- the phase of their eigenvalues is the rotation angle
- eigenvector $\boldsymbol{O}'$ correspond to the real eigenvalue

Rectification from planar motion

- find eigenvector-eigenvalues of H :
- eigenvalues are proportional to $\lambda' = 1, \lambda'' = e^{i\theta}, \lambda''' = e^{-i\theta}$
- eigenvector e' associated to $\lambda' = 1$ is the image of the C.O.R. $\boldsymbol{O}$
- angle θ is the rotation angle
- eigenvectors e'' and e''' associated to λ'' and λ''' are the images $\boldsymbol{I}', \boldsymbol{J}'$ of the circular points $\boldsymbol{I}, \boldsymbol{J}$
- thus $C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}}$
- apply singular value decomposition svd$(C_\infty^{*\prime}) = U C_\infty^* U^T$
- we obtain the rectification matrix $H_{SR} = U^T$

Remember: angle between scene lines expressed in terms of image lines

- From transformation rules for lines and dual conics:

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)\ (m^T C_\infty^* m)}} = \frac{l\prime^T C_\infty^{*\prime} m\prime}{\sqrt{(l\prime^T C_\infty^{*\prime} l\prime)\,(\,m'^T C'^*_\infty m\prime)}}$$

known angle → equation on $C_\infty^{*\prime}$

e.g., for perpendicular lines $l'^T C_\infty^{*\prime} m' = 0$ **linear**

Example 1.
rectification from two planar rectangles

Image rectification from two coplanar rectangles

or … rectification from vanishing points

Direct method

Direct method

1. Find $C_{\infty}^{*\prime}$
2. Compute H_{rect}

- 4 dof for $C_{\infty}^{*\prime}$:

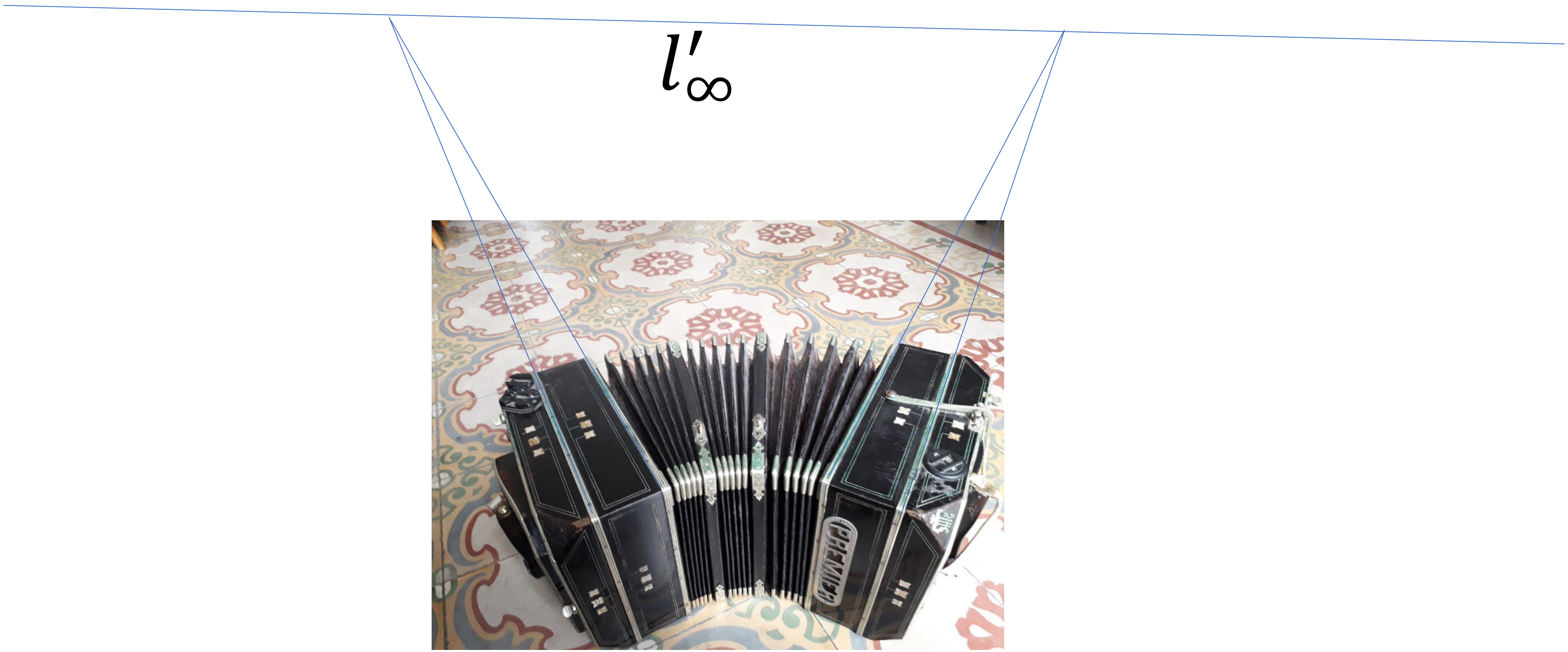
  9 elements minus

  - 3x3 symmetric → 3 constraints;
  - homogeneous matrix → 1 constraint;
  - singular matrix → 1 constraint

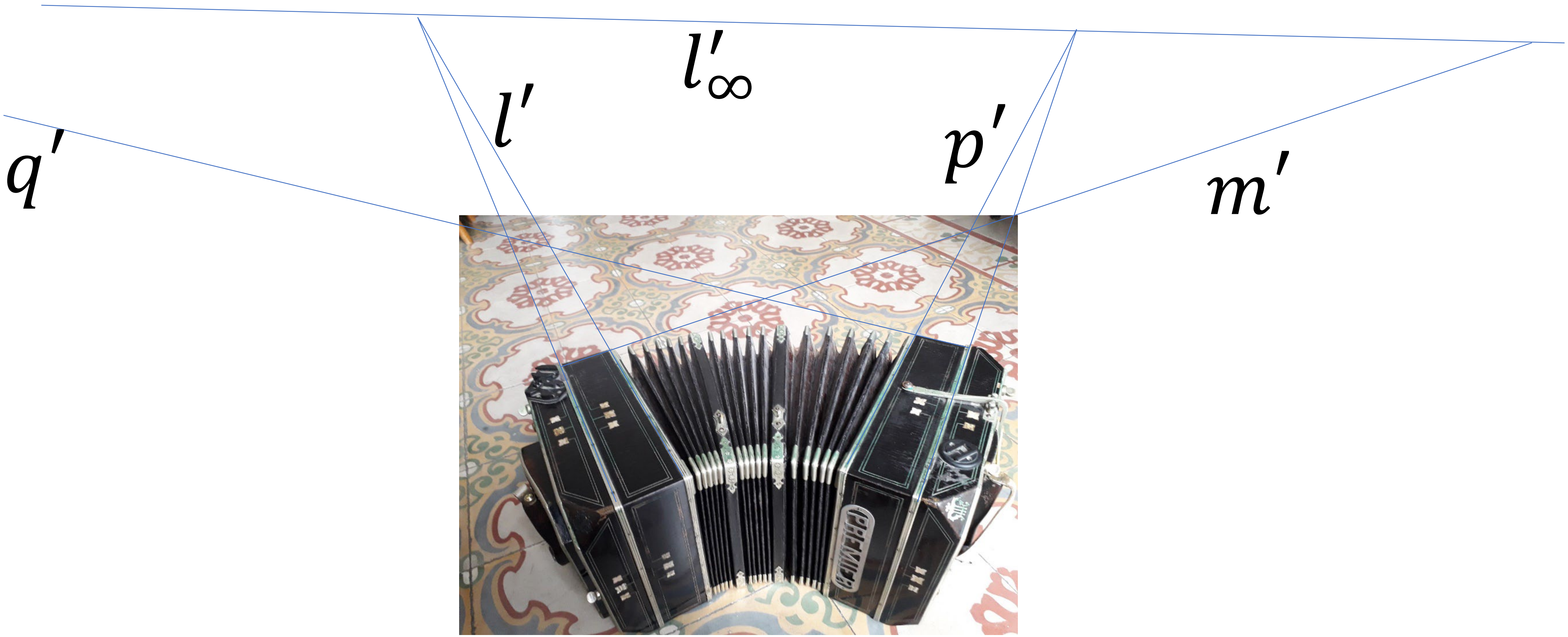
direct method to reconstruct the upper face

vanishing points

image of the line at the infinity



known angles (pairs of perpendicular lines)



4 linear constraints on $C_\infty^{*\prime}$

$$l'^T C_\infty^{*\prime} m' = 0 \quad \text{(1 constr)}$$

$$p'^T C_\infty^{*\prime} q' = 0 \quad \text{(1 constr)}$$

$$0 = C_\infty^{*\prime} l'_\infty = (I' J'^T + J' I'^T) l'_\infty =$$
$$= I'(\underbrace{J'^T l'_\infty}_{0}) + \mathrm{J}'(\underbrace{I'^T l'_\infty}_{0}) = \mathbf{0} \quad \text{(2 constr)}$$

line at the infinity l'_∞ = RNS($C_\infty^{*\prime}$)

Image rectification

- From the above constraints → find $C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}}$
- Singular value decomposition

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^* H_{SR}^{-T}$$

- Rectifying transformation (from svd output U)

$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} * \text{Image}$

reconstruction of the upper face

Stratified method

Stratified method

1. First step: affine reconstruction - from projective to affine

then

2. Second step: shape reconstruction - from affine to metric
(= image rectification or metric or euclidean reconstruction)

- sometimes reduces numerical errors

Stratified method

1. affine reconstruction - from projective to affine

Images of pairs of parallel lines

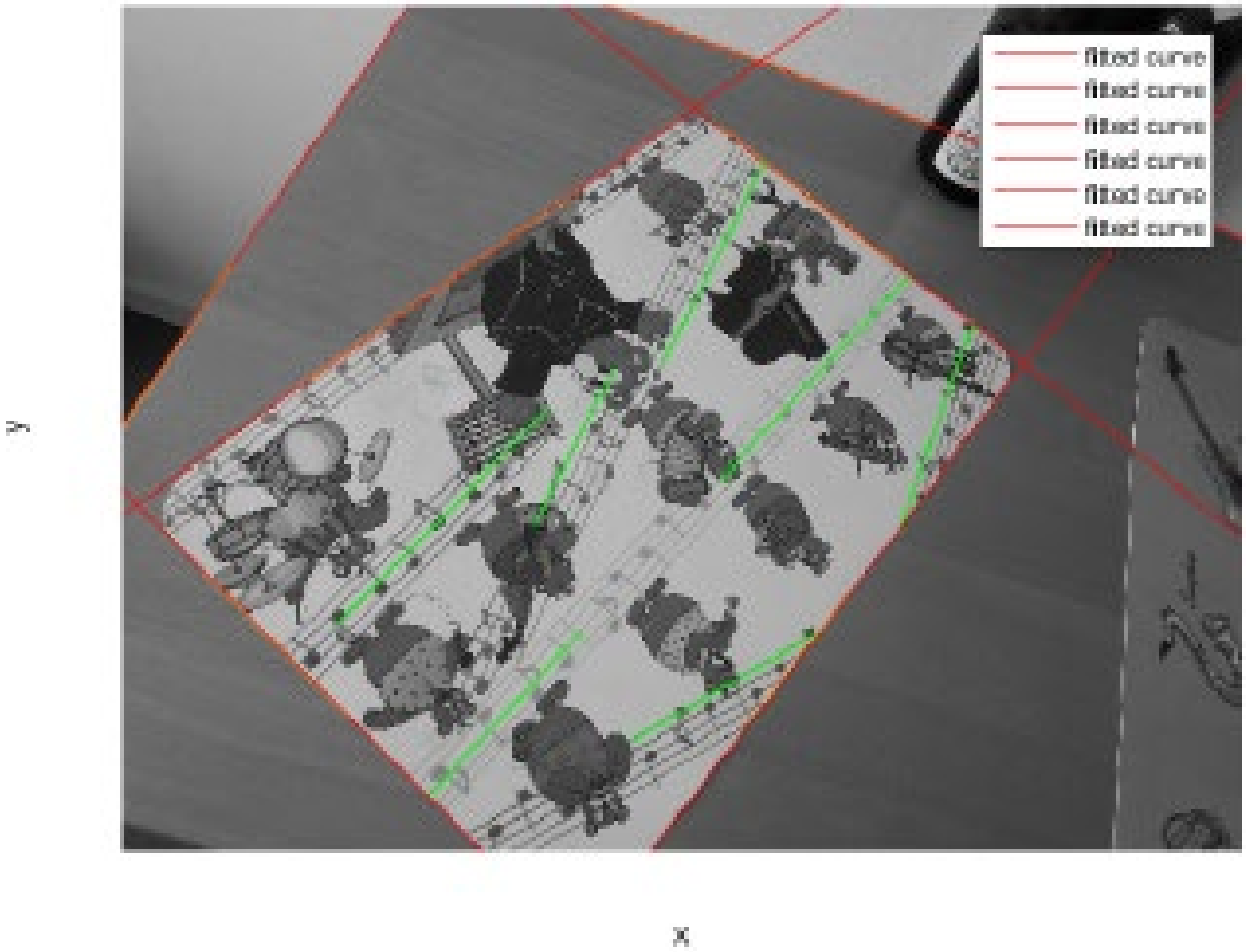
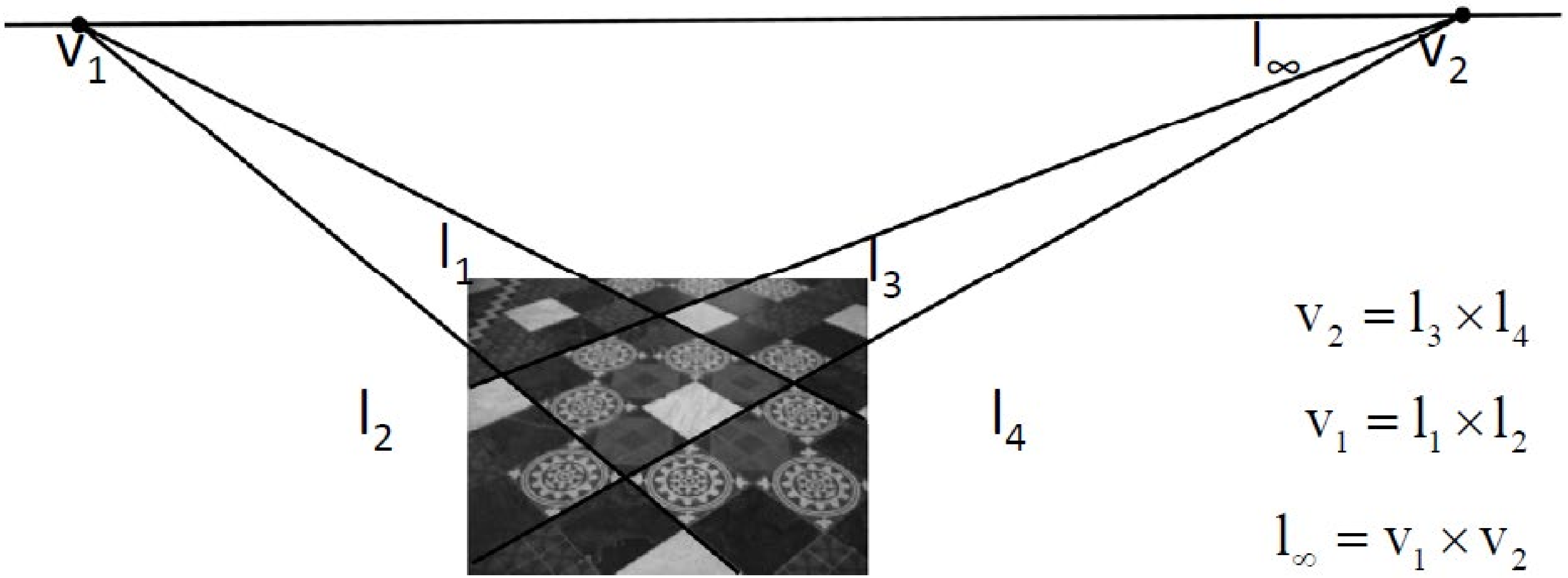


Image of **two** pairs of parallel lines

Affine rectification

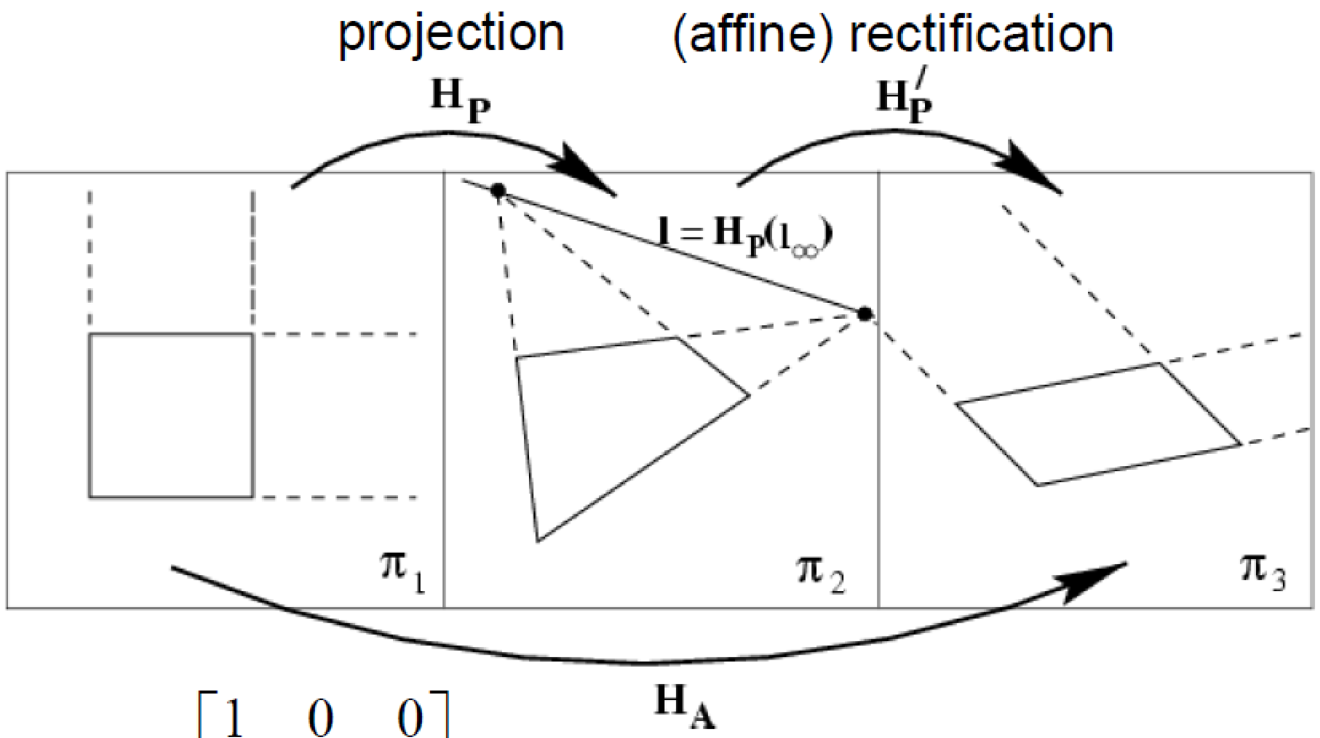


$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$

Affine properties from images



$$\mathbf{H'}_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \mathbf{H}_A \qquad \mathbf{l}'_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\mathsf{T}, l_3 \neq 0$$

in fact, any point x on l'_∞ is mapped to a point at the ∞

Affine rectification

- Apply the above mapping $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{vmatrix}$ to image lines and points

- while following transformation rules for lines and points respectively

In this phase, no need to use information about conic dual to circular points

Affine rectification

direct rectification versus stratified rectification

Metric properties from images

$$
\begin{aligned}
\mathbf{C}_\infty^{*}{}' &= (\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)\mathbf{C}_\infty^{*}(\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)^{\mathsf{T}} \\
&= (\mathbf{H}_P\mathbf{H}_A)\mathbf{H}_S\mathbf{C}_\infty^{*}\mathbf{H}_S^{\mathsf{T}}(\mathbf{H}_P\mathbf{H}_A)^{\mathsf{T}} \\
&= (\mathbf{H}_P\mathbf{H}_A)\mathbf{C}_\infty^{*}(\mathbf{H}_P\mathbf{H}_A)^{\mathsf{T}} \\
&= \begin{bmatrix} \mathbf{K}\mathbf{K}^{\mathsf{T}} & \mathbf{K}^{\mathsf{T}}\mathrm{v} \\ \mathrm{v}^{\mathsf{T}}\mathbf{K} & \mathrm{v}^{\mathsf{T}}\mathrm{v} \end{bmatrix}
\end{aligned}
$$

Rectifying transformation from SVD

$$
\mathbf{C}_\infty^{*}{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^{\mathsf{T}} \qquad \mathbf{H} = \mathbf{U}^{\mathsf{T}}
$$

after affine rectification

- Relationship between

original conic dual to the circular points $C_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and

affine(ly rectified) image of the conic dual to the circular points $C_\infty^{*\prime}$: **affine**

$$C_\infty^{*\prime} = H_A C_\infty^* H_A^T = \begin{bmatrix} G & \boldsymbol{t} \\ \boldsymbol{0}^{\boldsymbol{T}} & 1 \end{bmatrix}. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \begin{bmatrix} G^T & \boldsymbol{0} \\ \boldsymbol{t}^T & 1 \end{bmatrix} = \begin{bmatrix} GG^T & 0 \\ \boldsymbol{0}^{\boldsymbol{T}} & 0 \end{bmatrix}$$

after affine rectification

- **Important Property** after an affine rectification, ${C_\infty^*}'$ can be written as

$$
{C_\infty^*}' = \begin{bmatrix} GG^\top & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}
$$

where K is a 2×2 invertible matrix (cfr HZ example 2.26)

- this is the image of the canonical C_∞^* through an affine transformation since similarities do not change C_∞^*, and projective transformation has been removed by the affine rectification

Stratified method

1. First step: affine reconstruction - from projective to affine

then

2. Shape reconstruction - from affine to metric

Remember: costraints from known angles

$$\cos\theta = \frac{l^T C_\infty^* m}{\sqrt{l^T C_\infty^* l \;\; m^T C_\infty^* m}} = \frac{l\prime^T C_\infty^{*\prime} m\prime}{\sqrt{l\prime^T C_\infty^{*\prime} l\prime \;\; m'^T C\prime_\infty^* m\prime}}$$

known angle → equation on $C_\infty^{*\prime}$

and for perpendicular lines $l'^T C_\infty^{*\prime} m' = 0$ **linear**

image rectification from orthogonal lines

- If $\boldsymbol{l}'$ and $\boldsymbol{m}'$ are images two orthogonal lines $\boldsymbol{l}$ and $\boldsymbol{m}$ (in the 3D world), then

$$\boldsymbol{l}'^{\top} C_{\infty}^{*}{}' \boldsymbol{m}' = 0$$

- Let us use this information to compute $C_{\infty}^{*}{}'$, thus compute H
- Remember that, after an affine rectification

$$C_{\infty}^{*}{}' = \begin{bmatrix} GG^{\top} & \mathbf{0} \\ \mathbf{0}^{\top} & 0 \end{bmatrix}$$

- and that $S = GG^{\top}$ **is a symmetric homogeneous matrix** $\left((GG^{\top})^{\top} = GG^{\top}\right)$, thus there are only 2 unknowns to identify $C_{\infty}^{*}{}'$
- Each pair of orthogonal lines yield a single equation

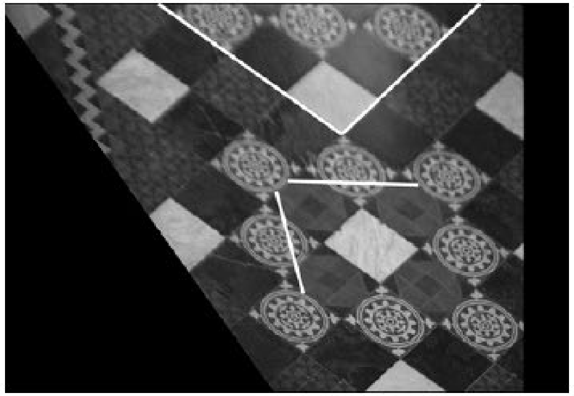
$$\boldsymbol{l}'(1:2)^{\top} S\, \boldsymbol{m}'(1:2) = 0$$

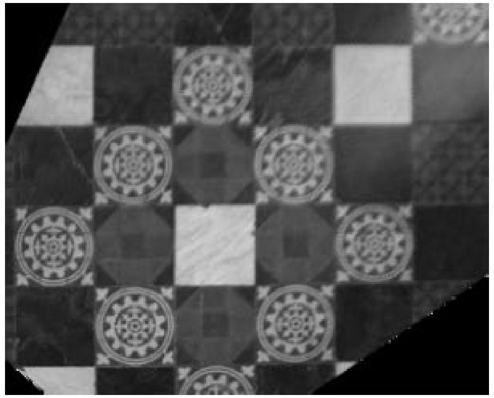
Metric from affine

l' and ***m'*** are lines in the affinely rectified image; they are image of orthogonal lines

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{K}\mathbf{K}^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

$$\left(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2\right)\left(k_{11}^2 + k_{12}^2, k_{11}k_{12}, k_{22}^2\right)^\mathsf{T} = 0$$





from two pairs
→ estimate GG^T
(homogeneous)

image rectification from orthogonal lines

- If $\boldsymbol{l}'$ and $\boldsymbol{m}'$ are images two orthogonal lines $\boldsymbol{l}$ and $\boldsymbol{m}$ (in the 3D world), then

$$\boldsymbol{l}'^{\top} {C_{\infty}^{*}}' \boldsymbol{m}' = 0$$

- Let us use this information to compute ${C_{\infty}^{*}}'$, thus compute H
- Remember that, after an affine rectification

$${C_{\infty}^{*}}' = \begin{bmatrix} GG^{\top} & \mathbf{0} \\ \mathbf{0}^{\top} & 0 \end{bmatrix}$$

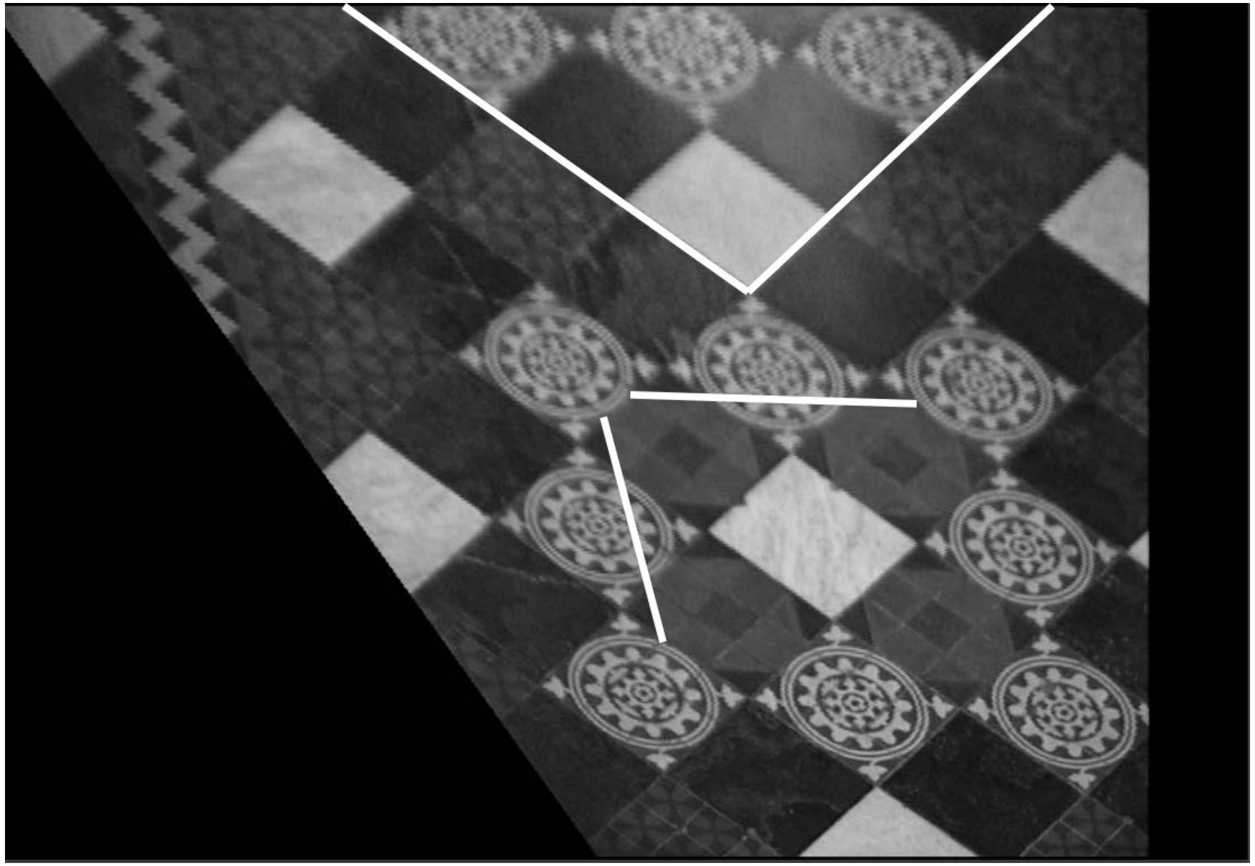
- and that $S = GG^{\top}$ **is a symmetric homogeneous matrix** $(GG^{\top} = G^{\top}G)$, thus there are only 2 unknowns to identify ${C_{\infty}^{*}}'$
- Each pair of orthogonal lines yields a single equation

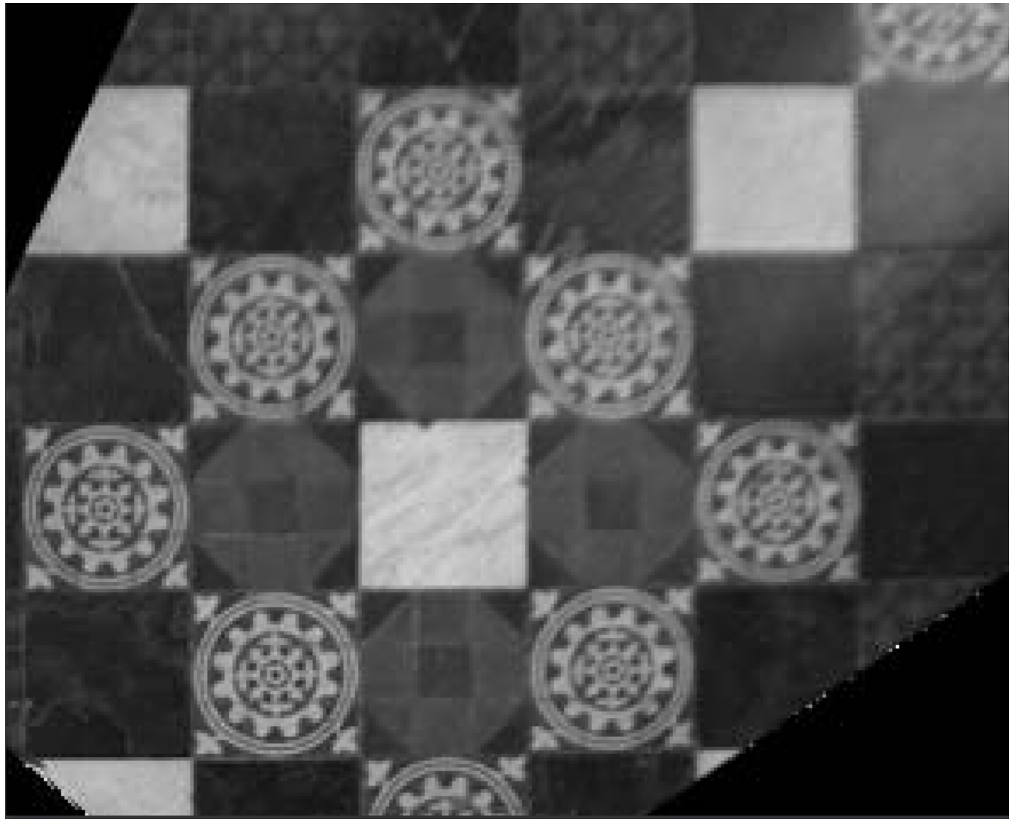
$$\boldsymbol{l}'(1{:}2)^{\top} S\, \boldsymbol{m}'(1{:}2) = 0$$

Two pairs of orthogonal lines are enough to identify $\boldsymbol{C_{\infty}^{*}}'$

Stratified Rectification from Orthogonal Lines

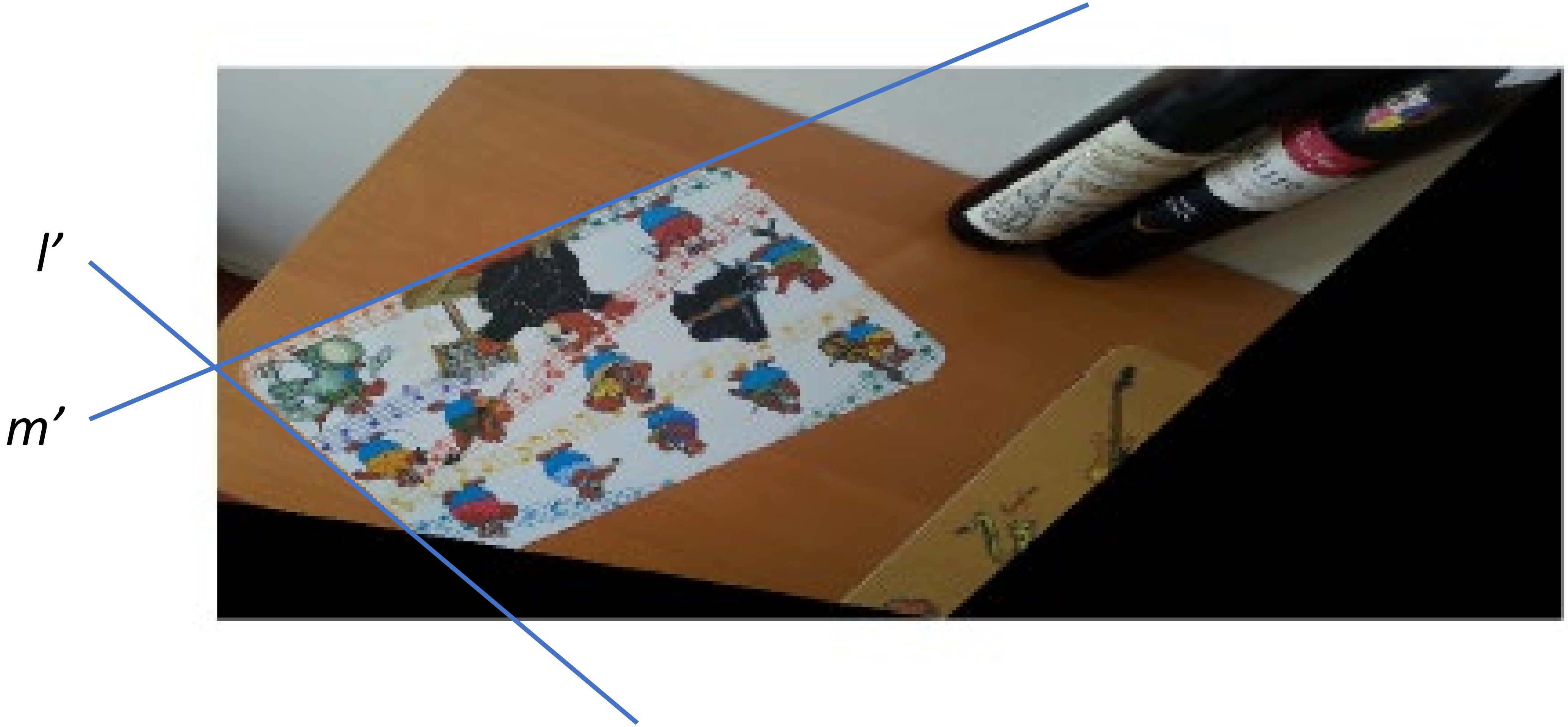
- $\boldsymbol{l}'(1{:}2)^{\top} S\, \boldsymbol{m}'(1{:}2) = 0$
- Two (different, i.e. not parallel) pairs of orthogonal lines are enough to estimate $S = GG^{\top}$, (thus G through Choleski factorisation)





Hartley Zisserman Fig.2.17

$$l'^T \begin{bmatrix} GG^T & 0 \\ 0 & 0 \end{bmatrix} m' = 0$$



$$p'^T \begin{vmatrix} GG^T & 0 \\ 0 & 0 \end{vmatrix} q' = 0$$

Use image of pairs of orthogonal lines to estimate GG^T

remember: just 2 pairs are needed

Apply Cholesky factorisation to find G
(an upper triangular matrix G is provided)

NOTE: **no SVD is needed**

From affine reconstruction to metric reconstruction

From $C_\infty^{*\prime} = \begin{bmatrix} GG^T & 0 \\ 0 & 1 \end{bmatrix} = H_A C_\infty^* H_A^T = \begin{bmatrix} G & \boldsymbol{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G^T & \mathbf{0} \\ \boldsymbol{t}^T & 1 \end{bmatrix}$

and $H_{rect} = H_A^{-1}$

$$H_{rect} = \begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix}^{-1}$$

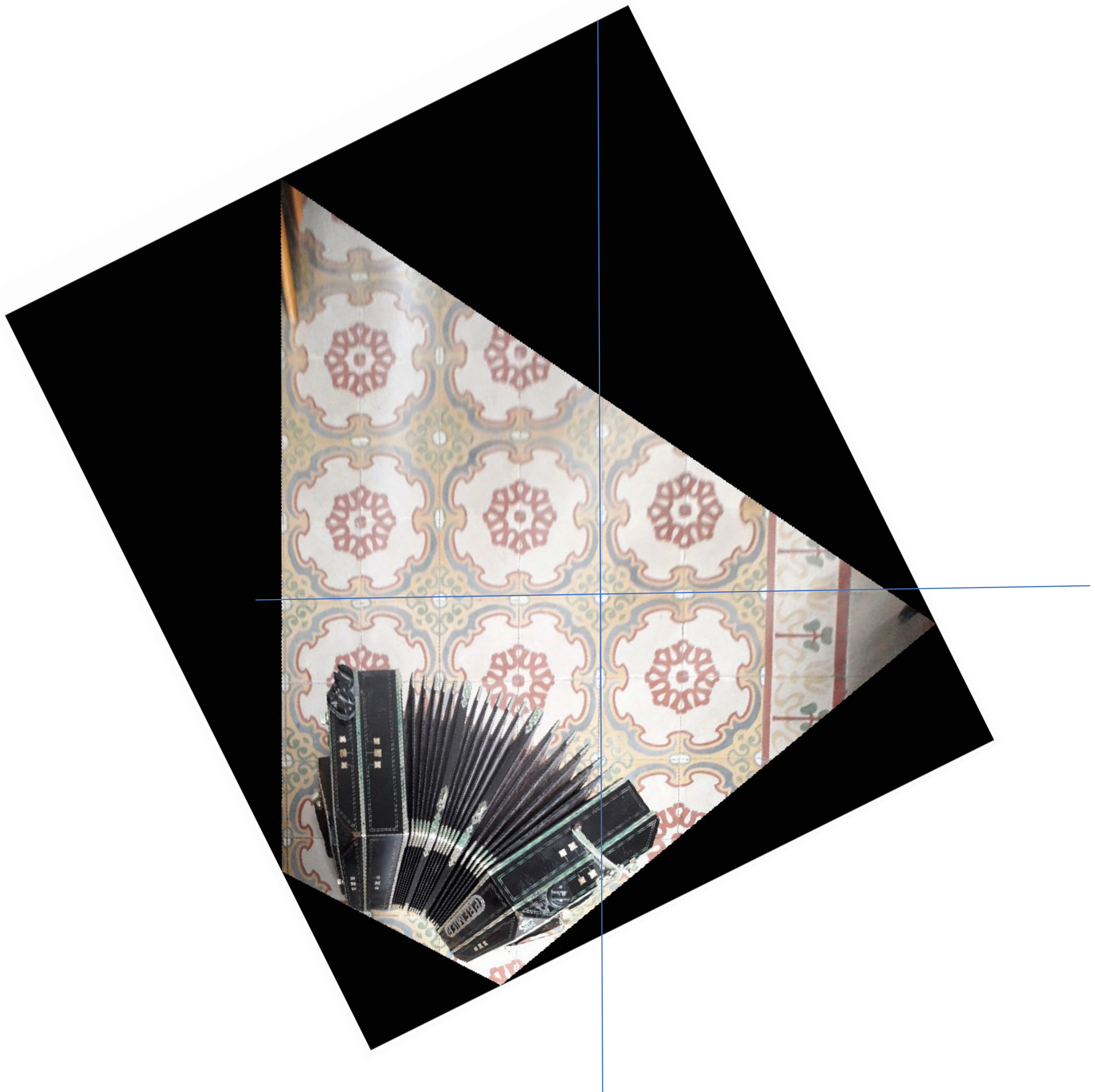
where

t is a free (but useless) vector

Result of metric (shape) reconstruction

same stratified approach for

reconstructed upper face



Some accuracy issues in image rectification

Some accuracy issues in image rectification

1. Noise and numerical errors (already seen)

$$\mathrm{svd}(C_{\infty}^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

- In principle, given $C_\infty^{*\prime}$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

- But, due to noise and numerical errors, SVD output is

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{rect}^{-1} C_\infty^* H_{rect}^{-T}$$

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = U \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

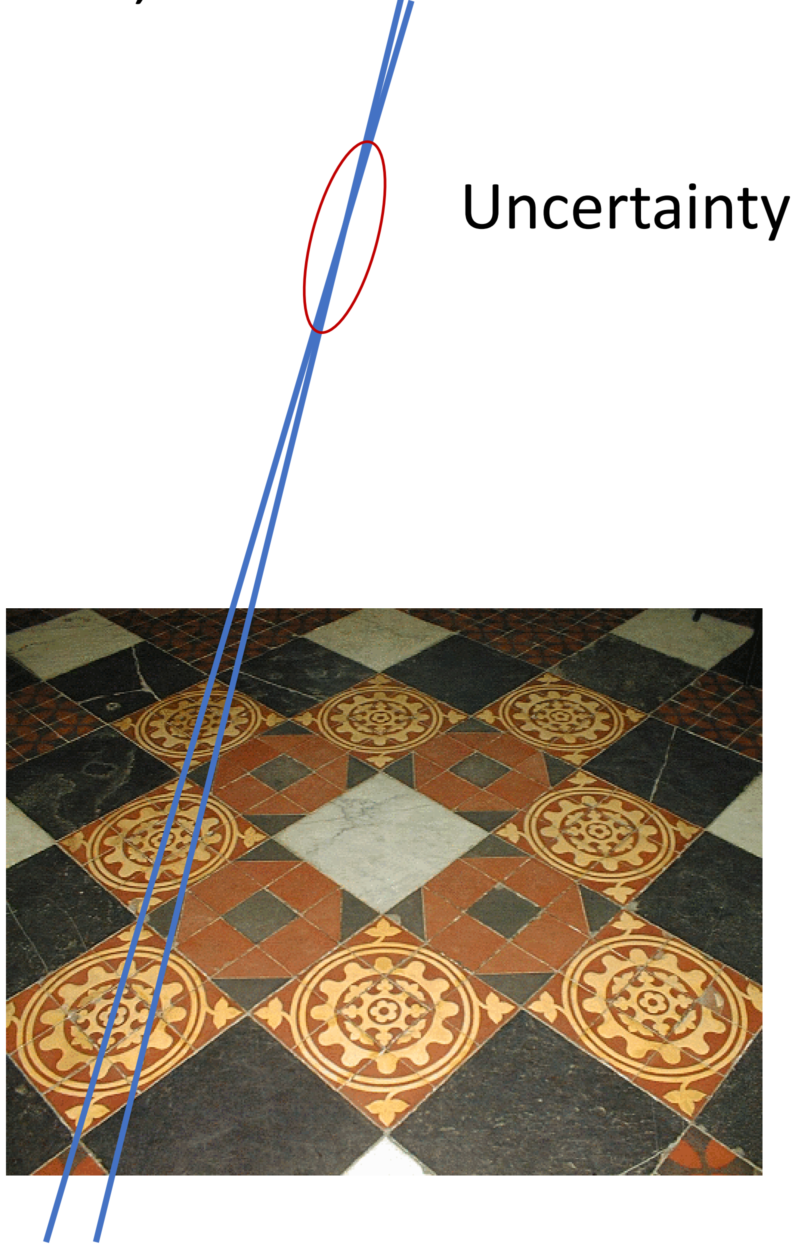
$$\rightarrow H_{rect}^{-1} = U \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow H_{rect} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

Some accuracy issues in image rectification

2. Poor estimation:

e.g., intersection of two lines, that are too close to eachother

intersection of two lines, that are too close to eachother

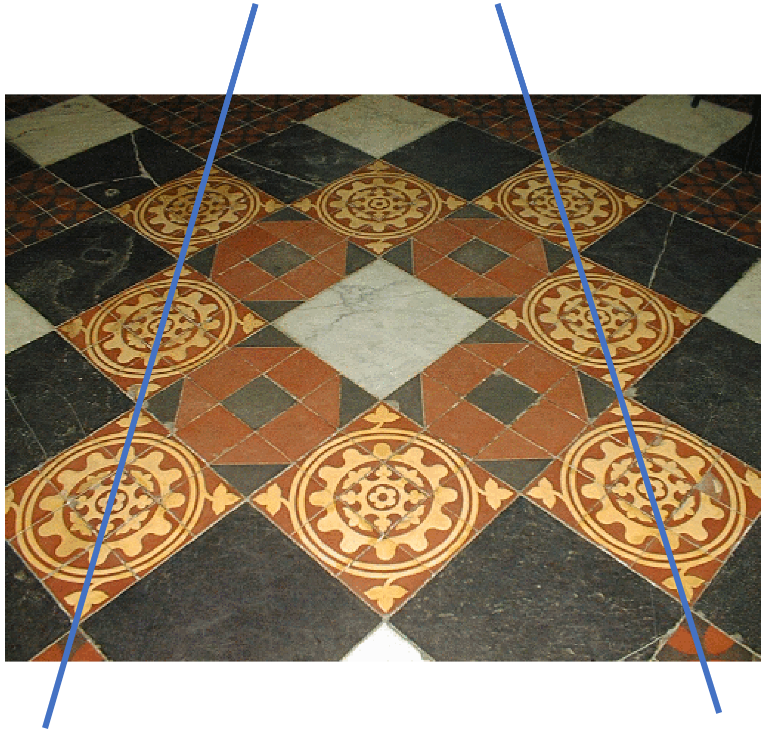


Uncertainty in the intersection point

Some accuracy issues in image rectification

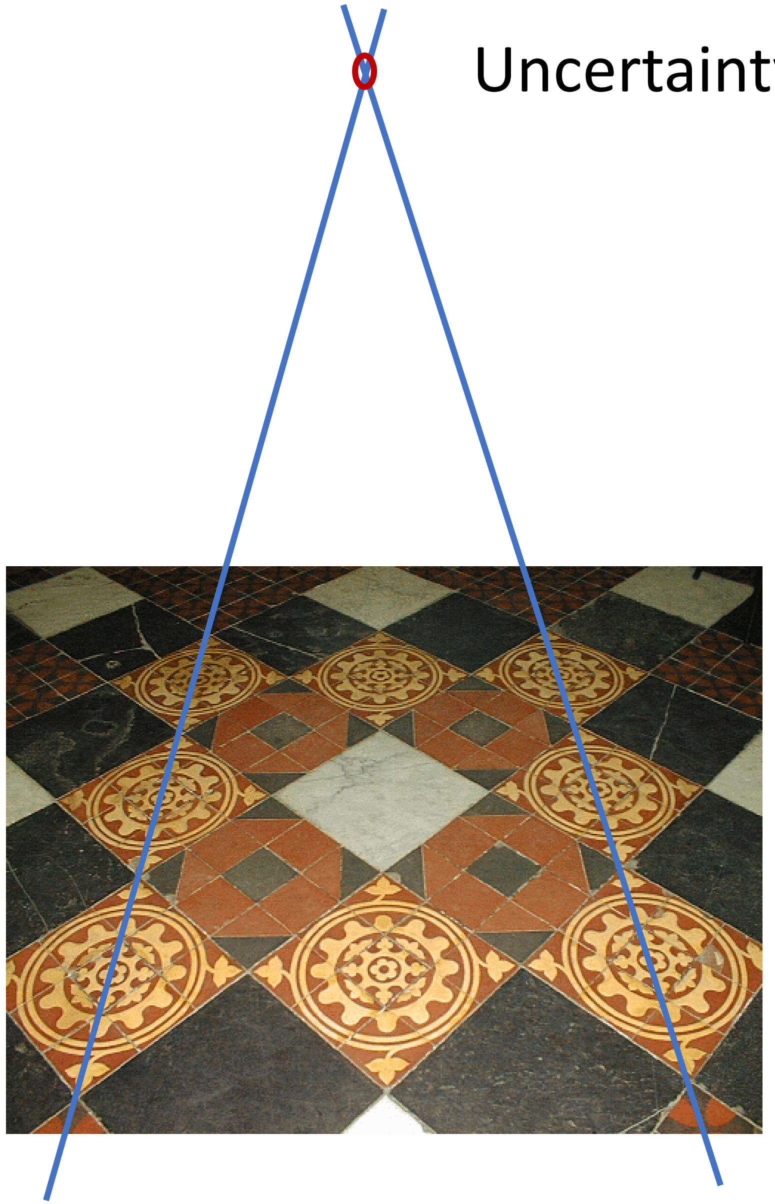
2. Poor estimation:

e.g., intersection of two lines, better **use lines that are far from eachother**



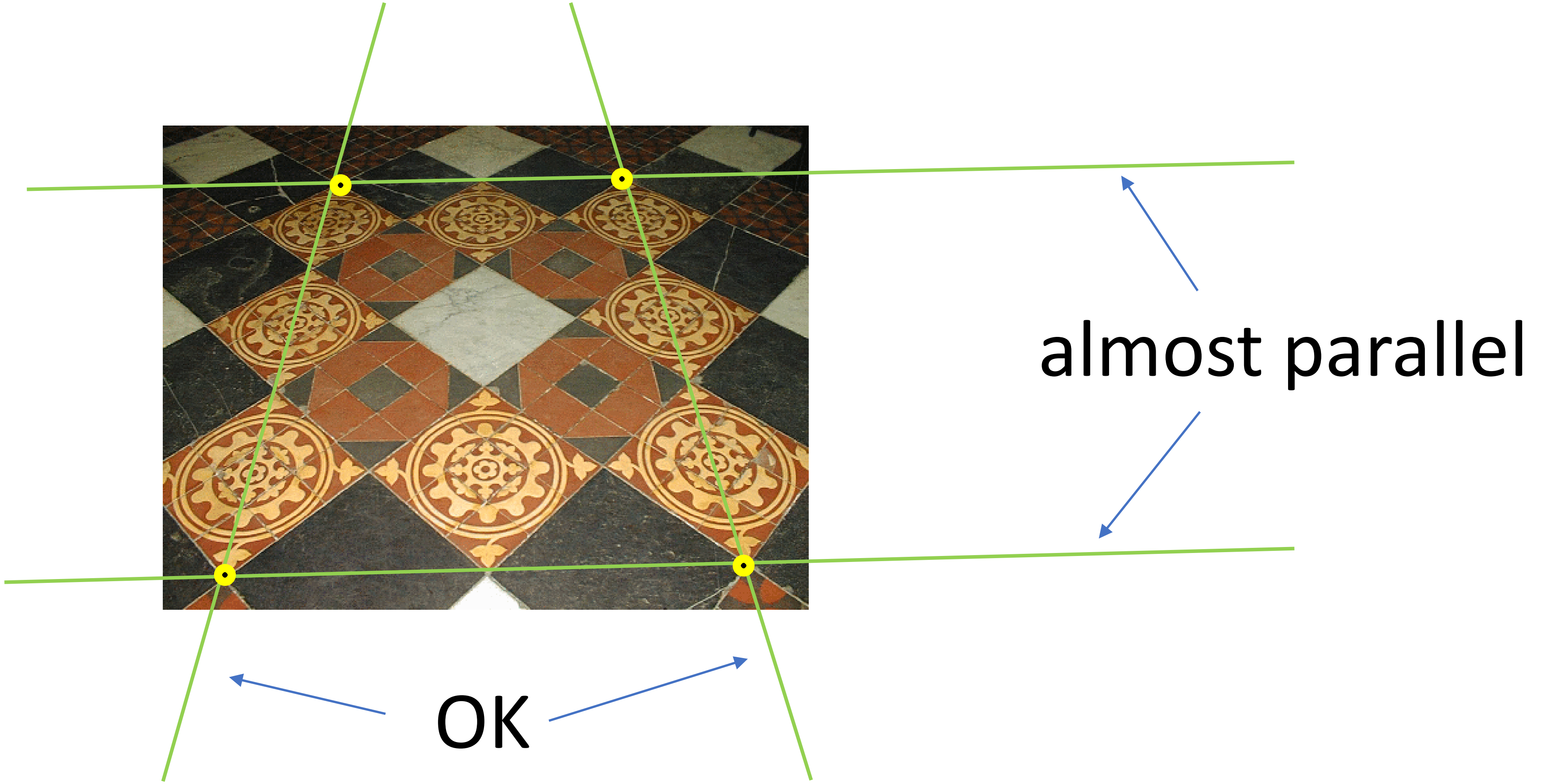
... better **use lines that are far from eachother**

Uncertainty in the intersection point



Some accuracy issues in image rectification

3. Vanishing point almost at the ∞



Vanishing points towards the ∞

When the images of parallel lines are almost parallel

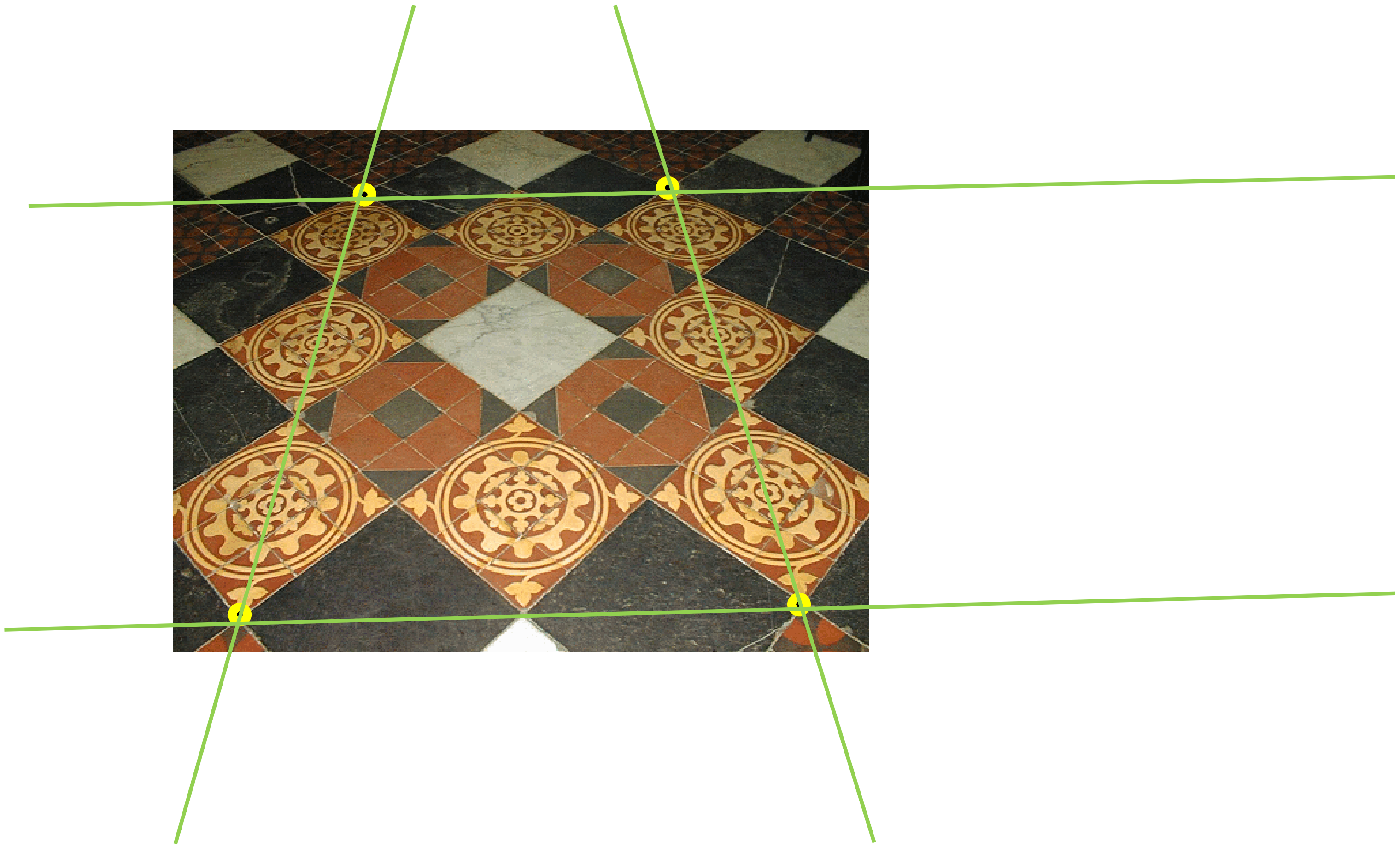
→ Estimate of vanishing points: not accurate

→ estimate of l'_∞ (image of l_∞): not accurate

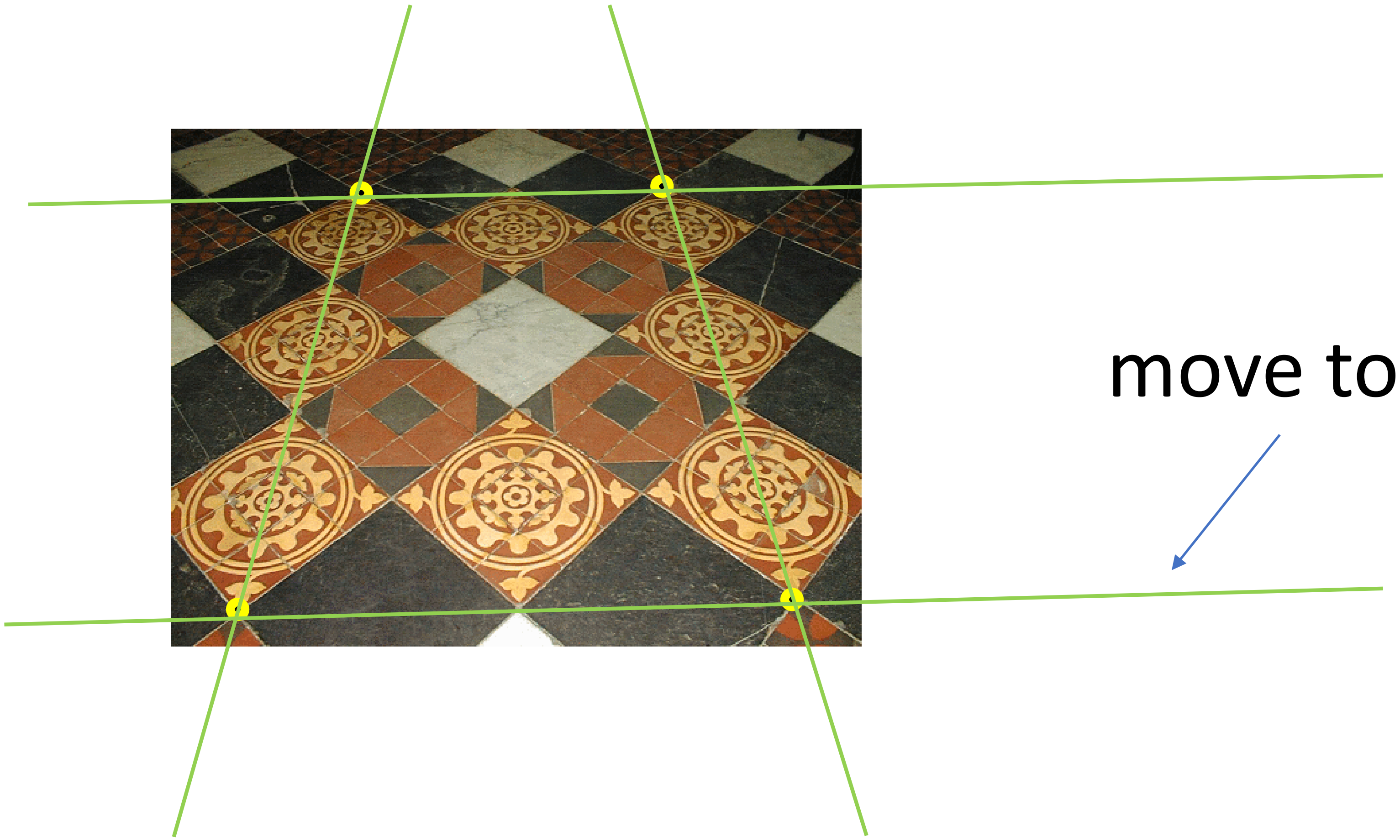
→ affine rectification: not accurate

Stratified «geometric» method

Stratified «geometric» method

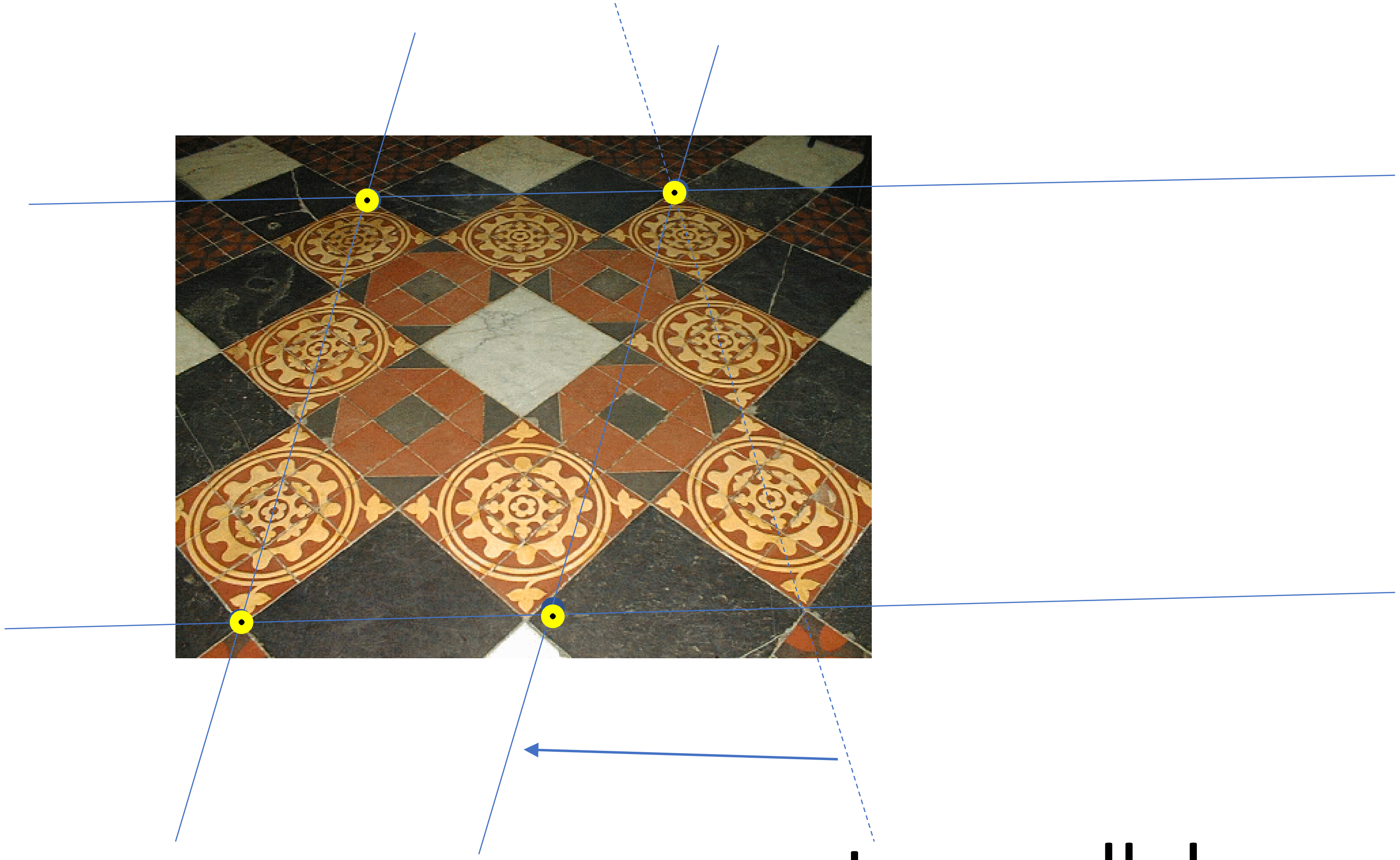


1° step: affine rectification



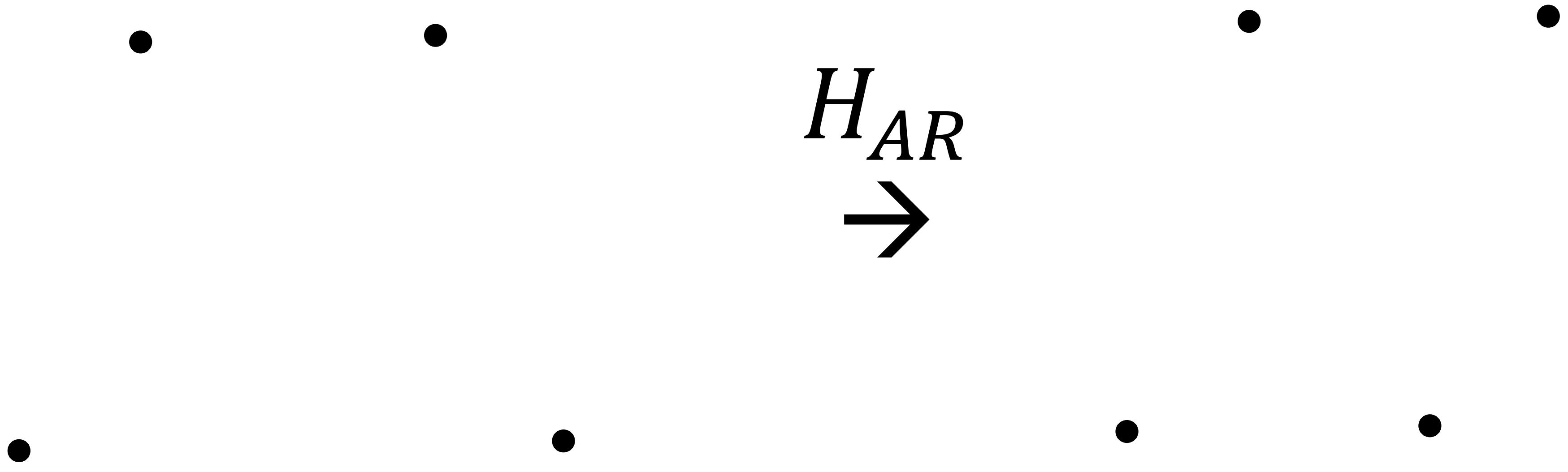
move to parallel

1° step: affine rectification

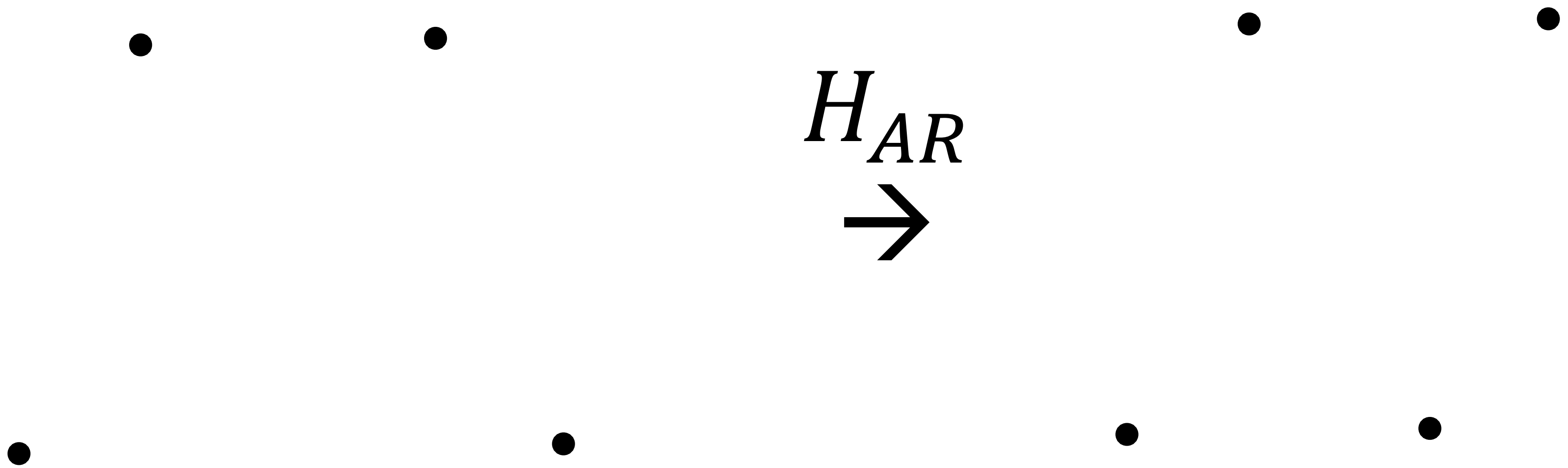


move to parallel

4 points → 4 points HOMOGRAPHY



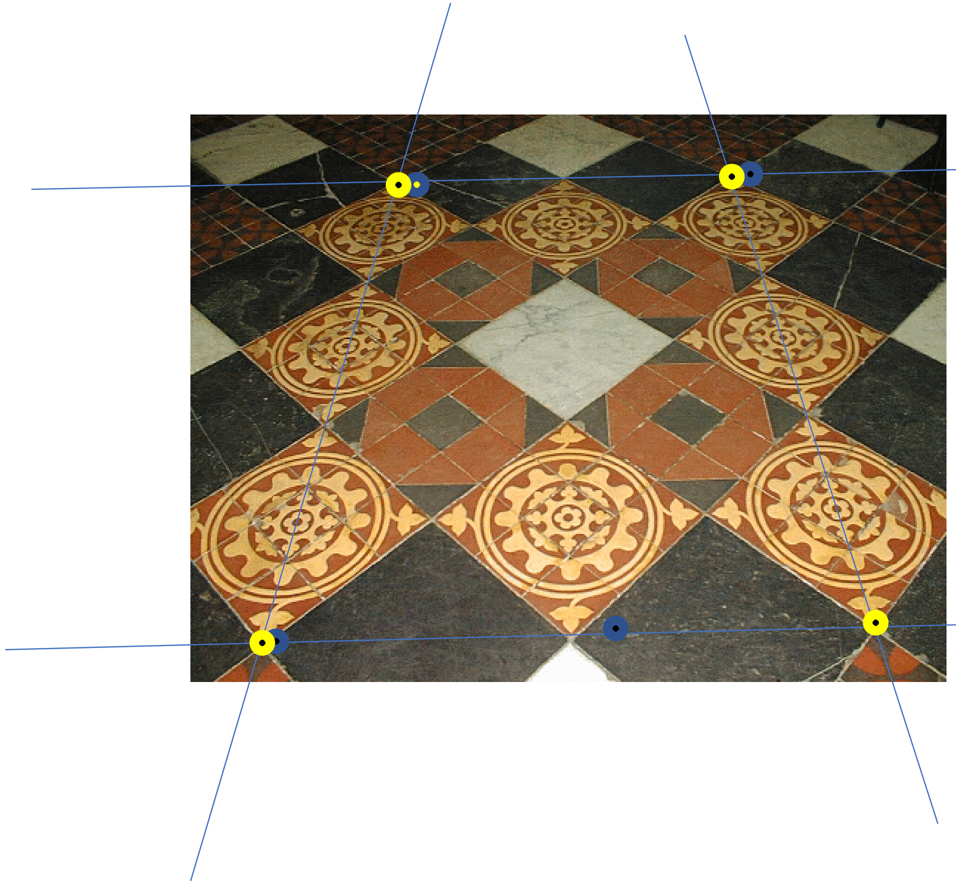
4 points → 4 points HOMOGRAPHY



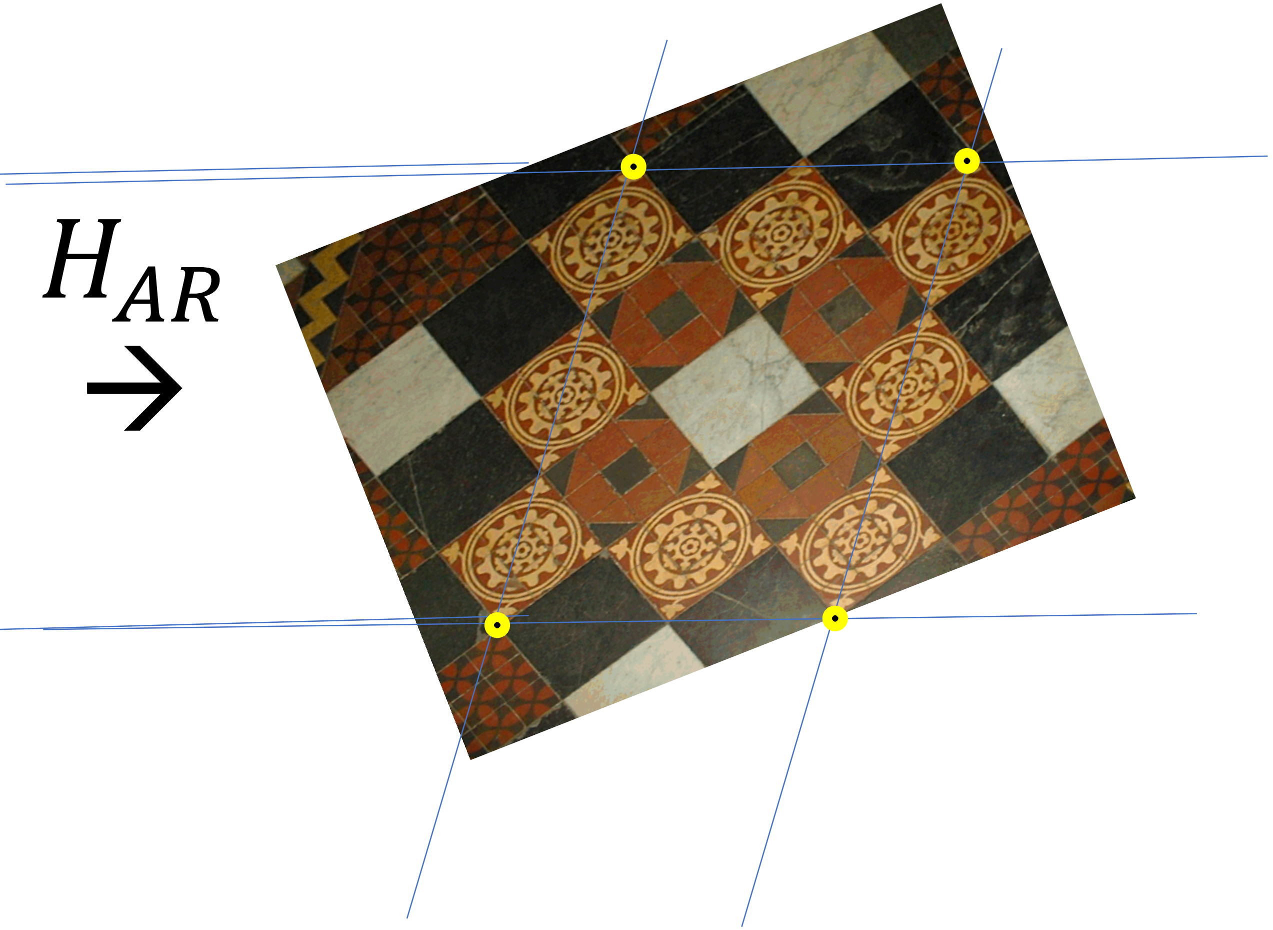
where H_{AR} is the solution of

- $A' = H_{AR}\mathrm{A}$
- $B' = H_{AR}\mathrm{B}$ (homogeneous coordinates)
- $C' = H_{AR}\mathrm{C}$
- $D' = H_{AR}\mathrm{D}$

affine reconstruction



$\xrightarrow{H_{AR}}$



2° step: affine → metric (Euclidean)

two INDEPENDENT constraints:

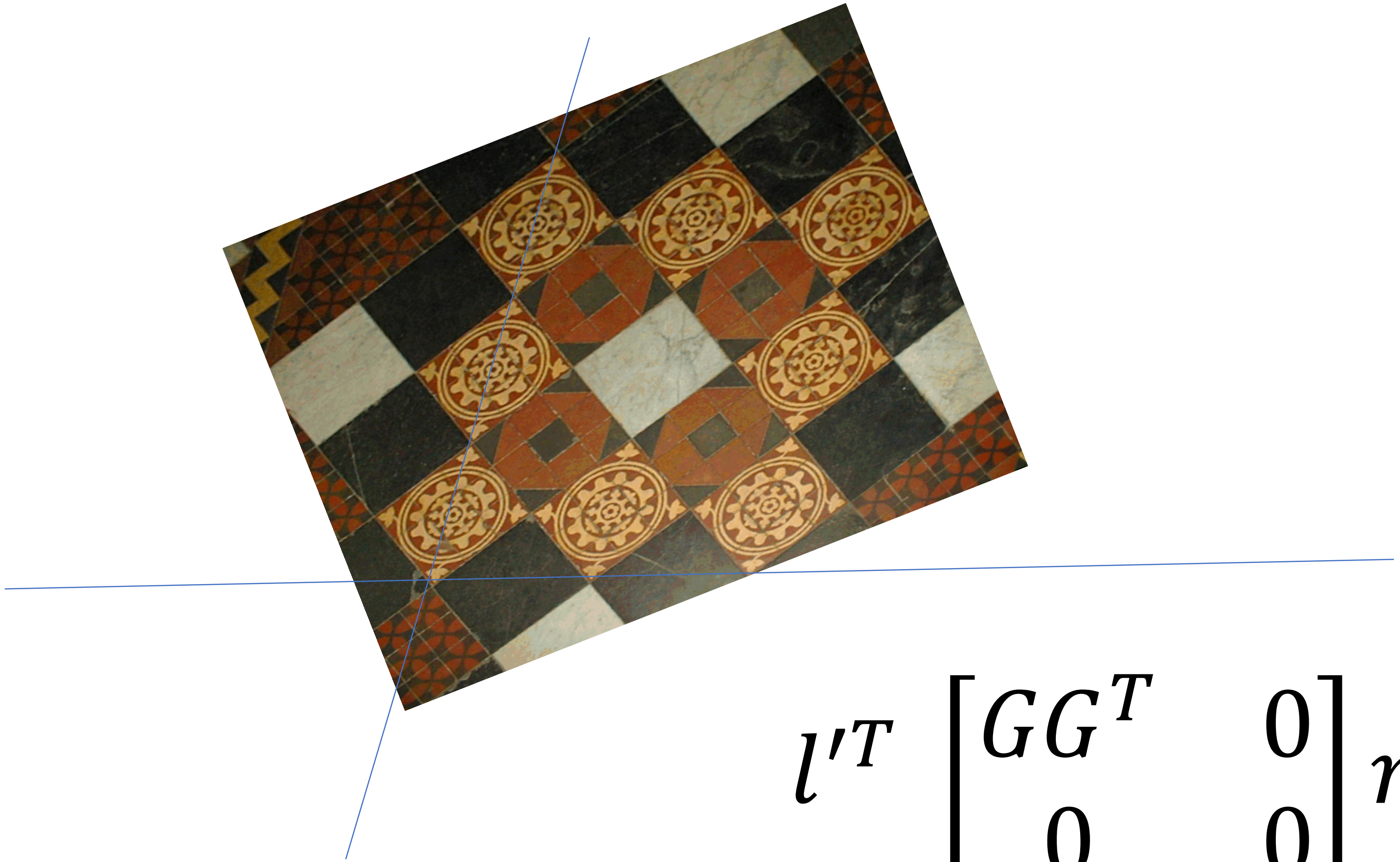
- two pairs of image lines, that are images of orthogonal lines

$$l'^T C_\infty^{*\prime} m' = 0$$

$$p'^T C_\infty^{*\prime} q' = 0$$

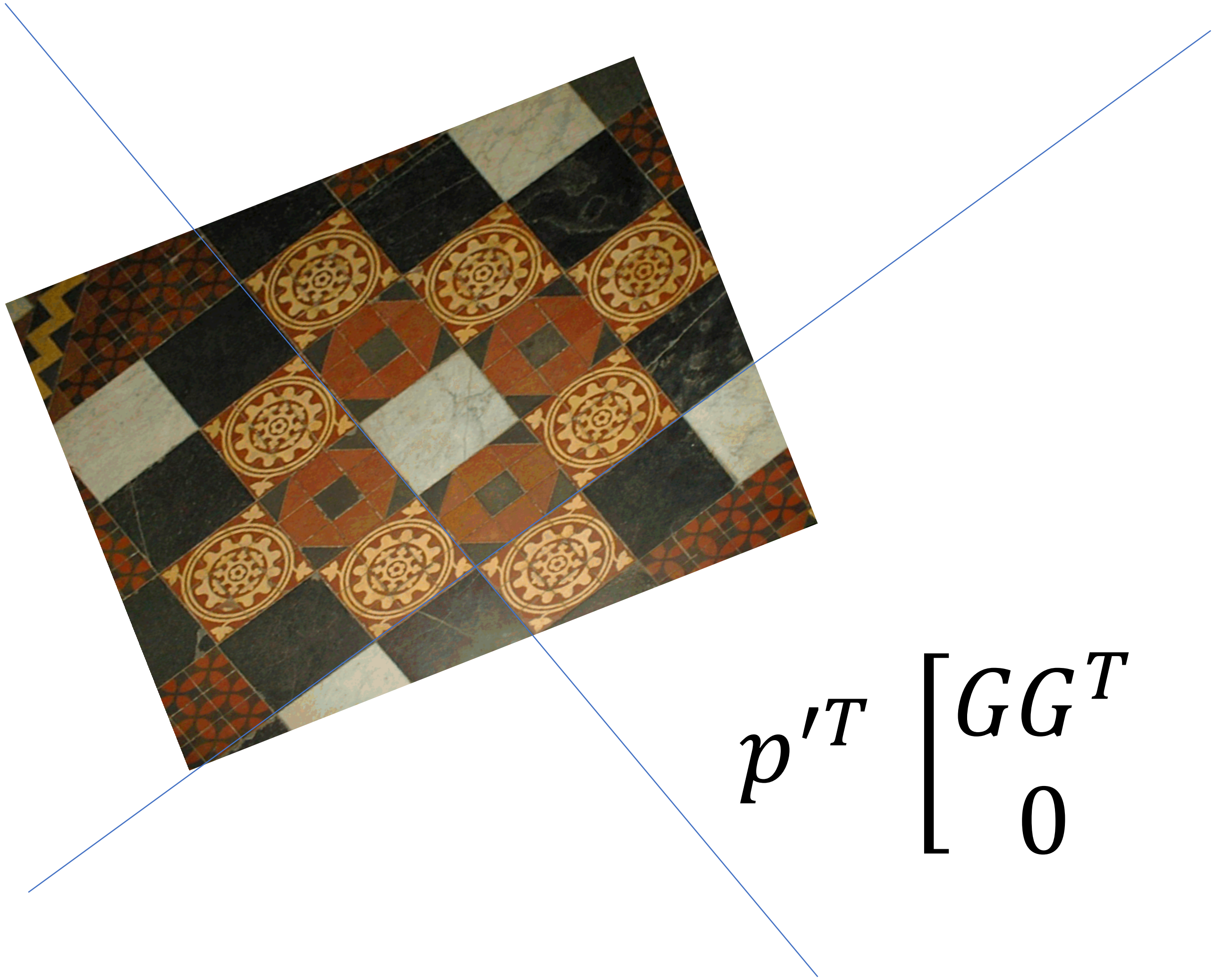
these two pairs lead to the same constraint
→ they are not independent

→ use only one of the two pairs as first pair



$$l'^T \begin{bmatrix} GG^T & 0 \\ 0 & 0 \end{bmatrix} m' = 0$$

→ use a new pair as a second pair



$$p'^T \begin{bmatrix} GG^T & 0 \\ 0 & 0 \end{bmatrix} q' = 0$$

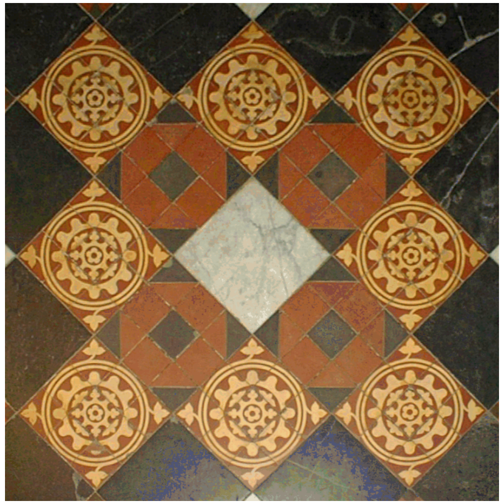
Once GG^T has been estimated, find G by Cholesly Factorisation

then apply rectifying transformation

$$H_{rect} = \begin{bmatrix} G & t \\ 0 & 1 \end{bmatrix}^{-1}$$

Metric rectification from affine

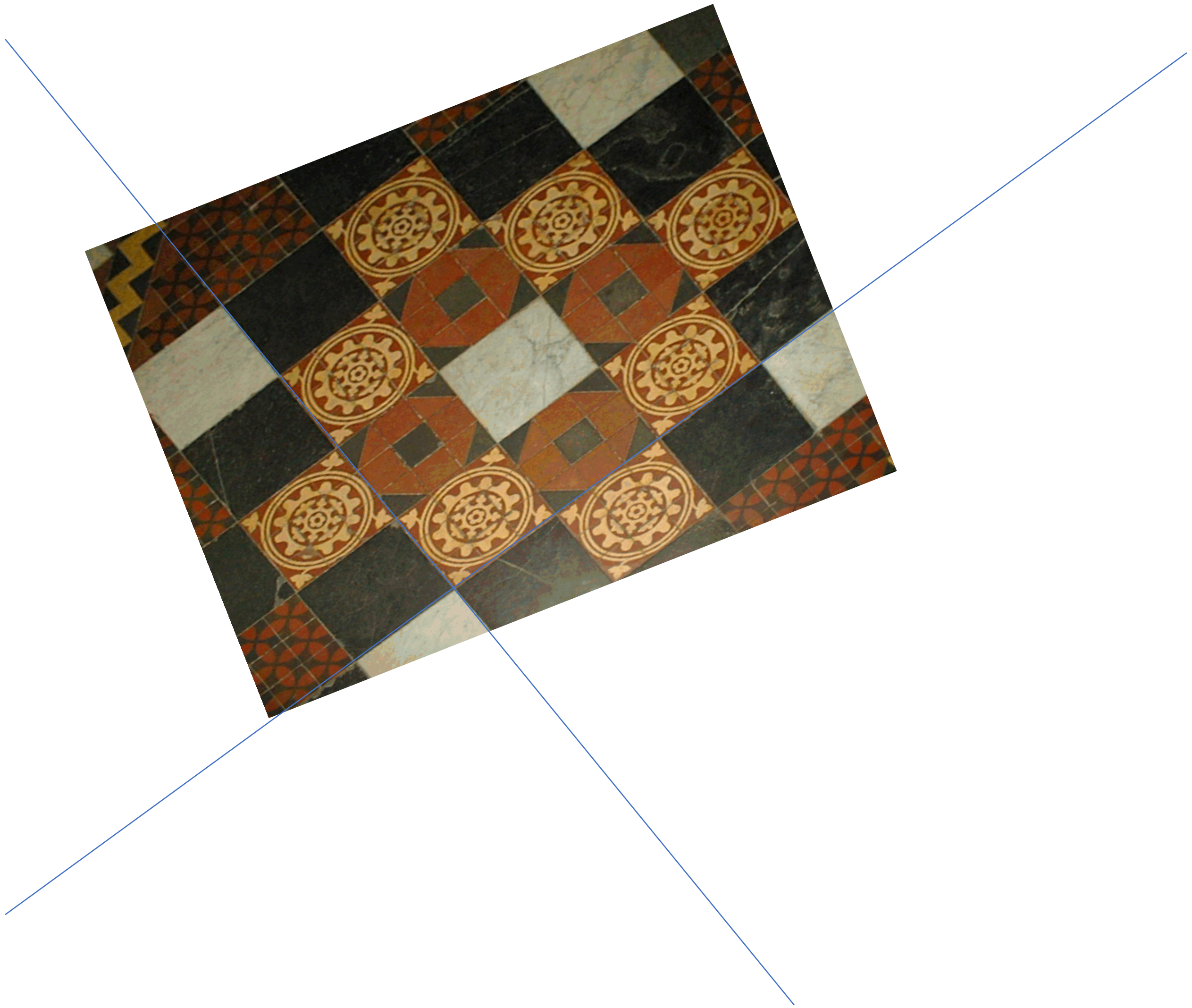
H_{rect}



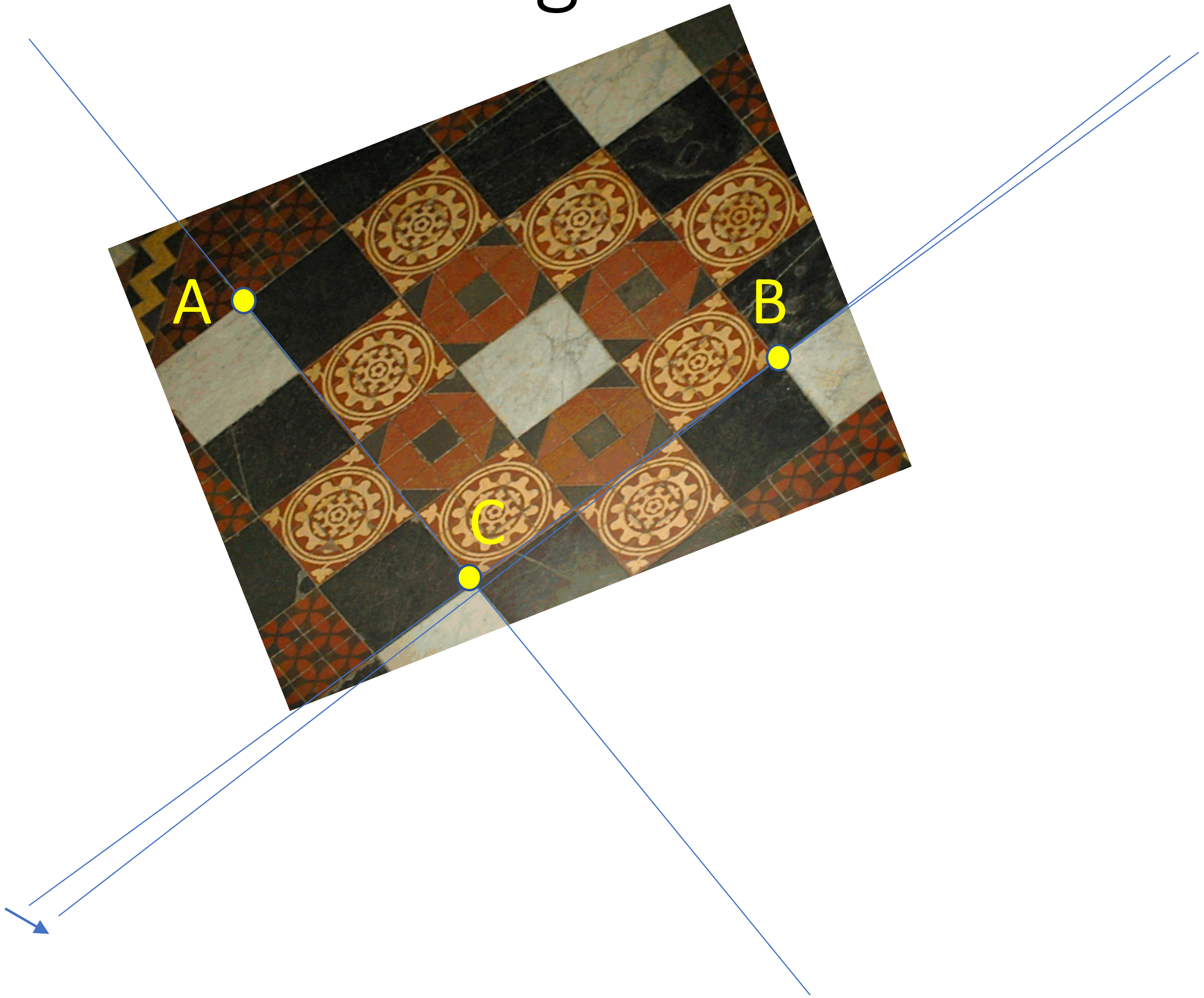
affine rectification

metric rectification

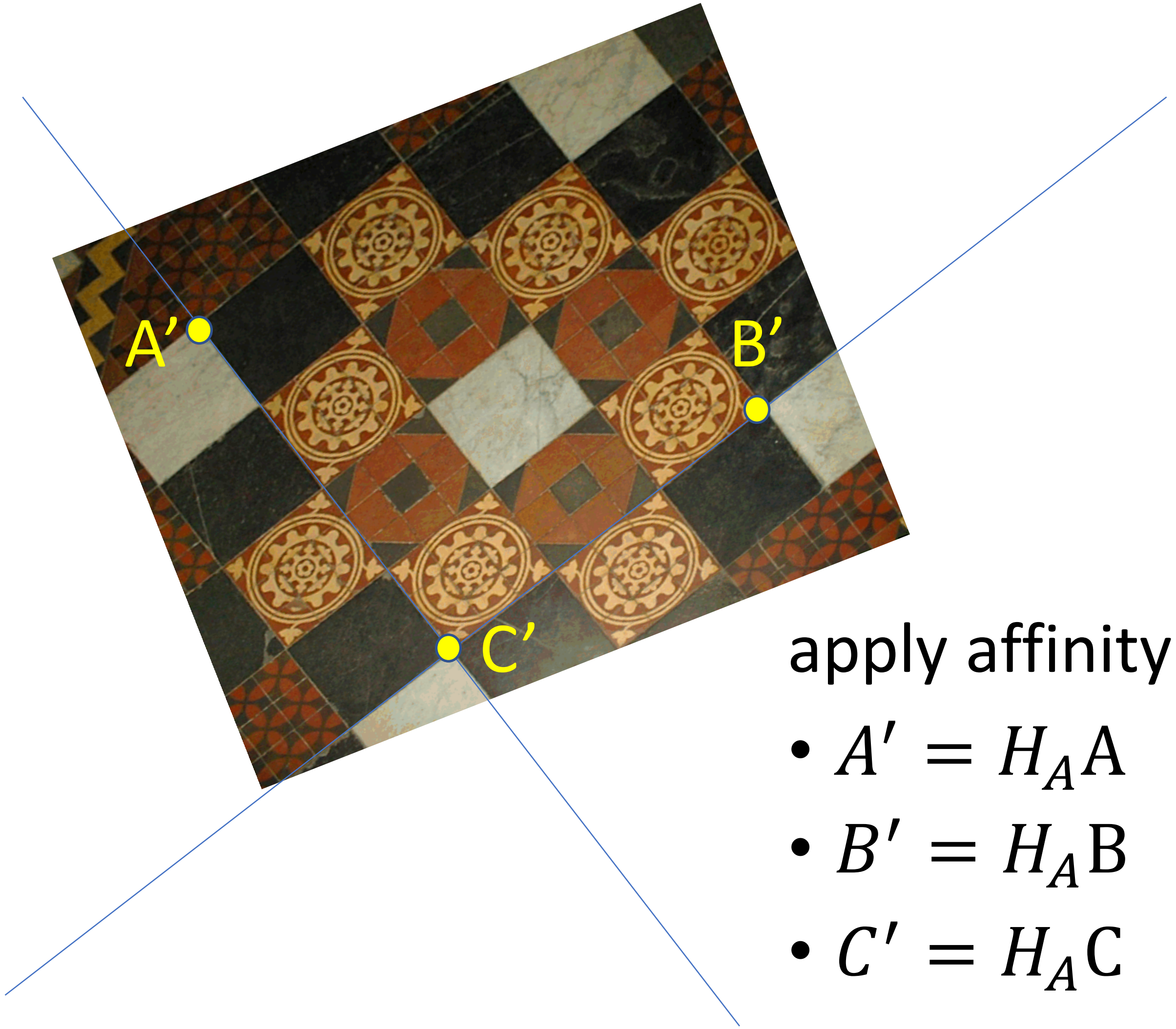
from affine to metric: **geometric** method with two pairs of orthogonal lines



First pair of orthogonal lines: apply a homography (in fact, an **affinity**) that maps the second line onto a line orthogonal to the first line



First pair of orthogonal lines: apply an affinity that maps the second line onto a line orthogonal to the first line

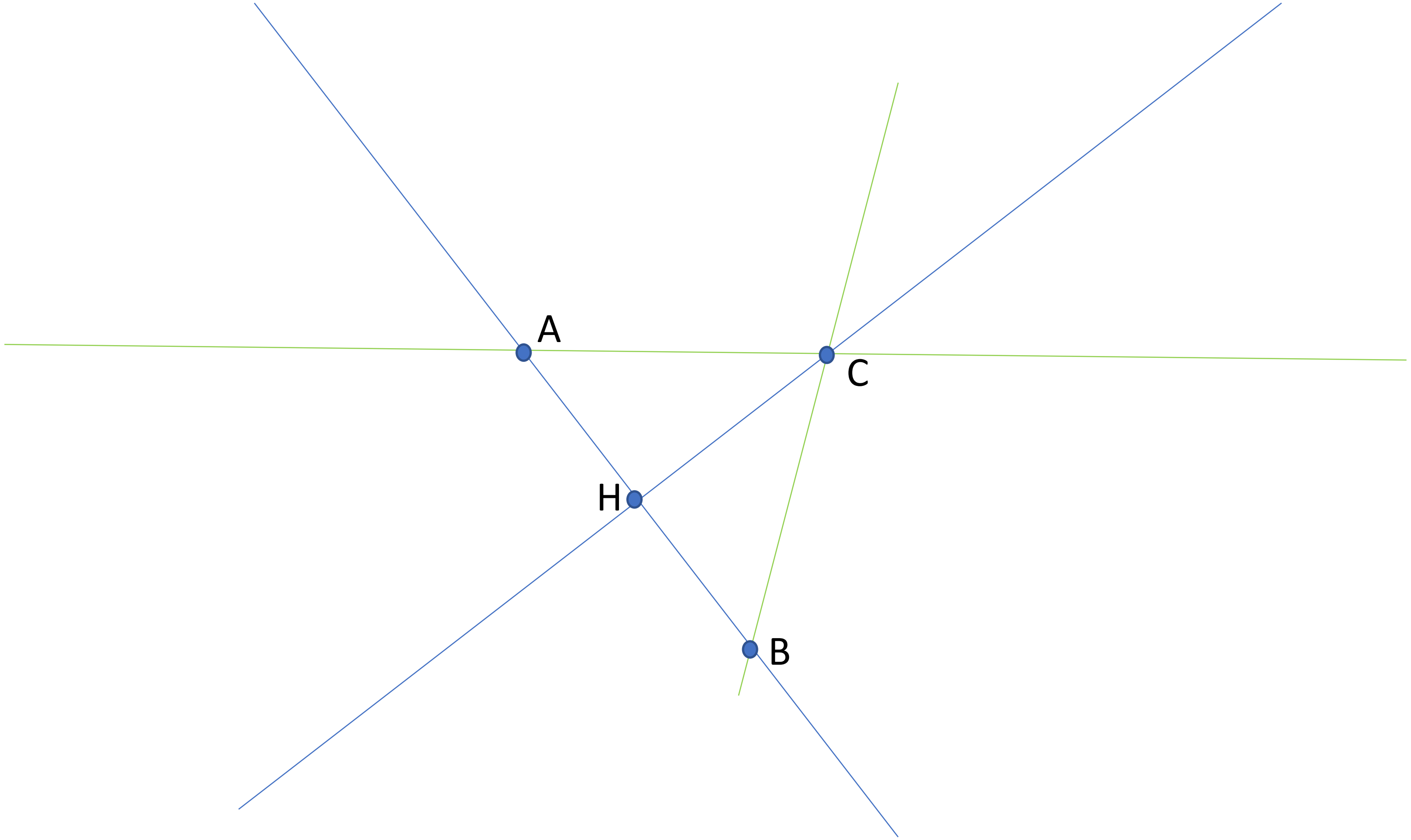


apply affinity H_A, which solves

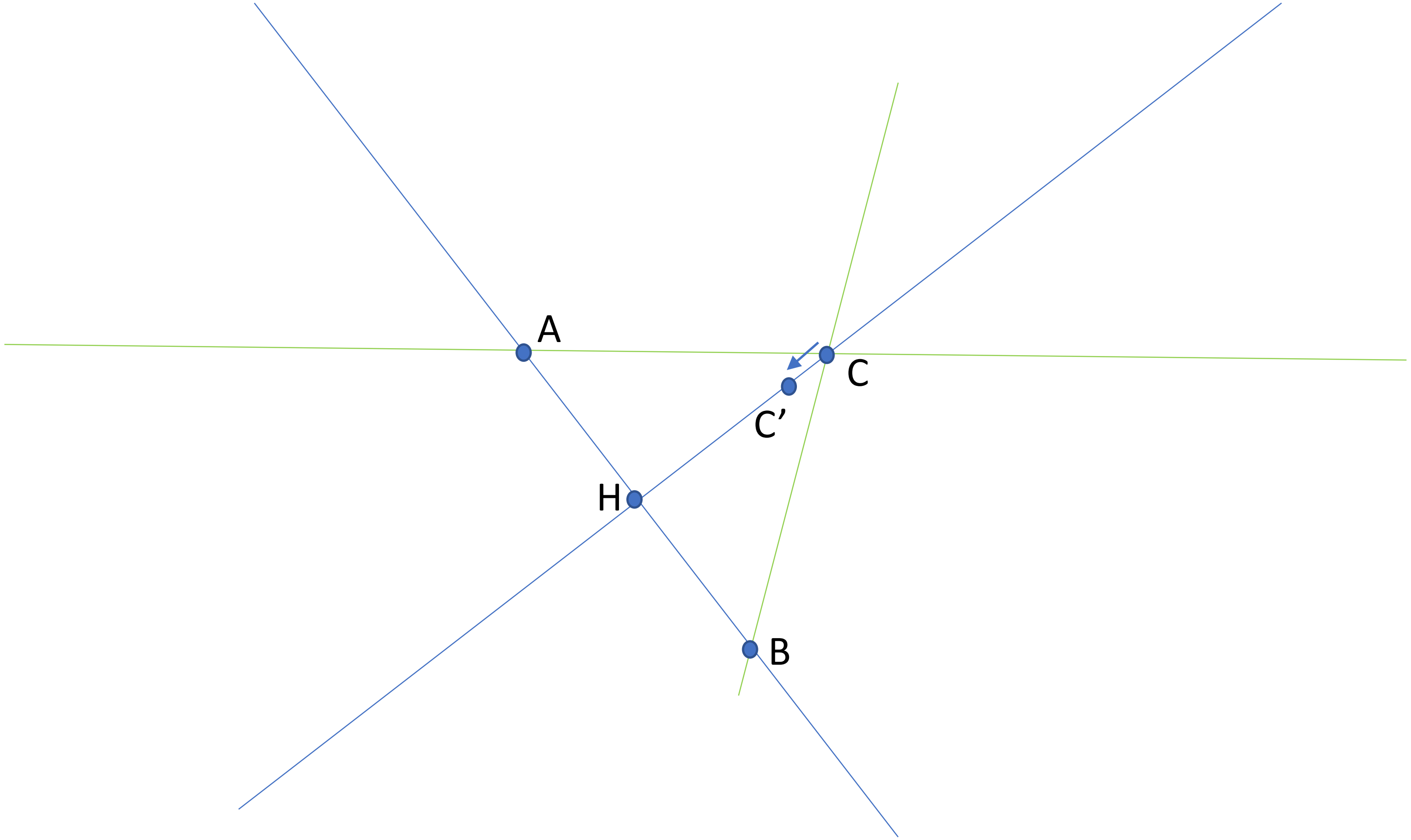
- $A' = H_A \mathrm{A}$
- $B' = H_A \mathrm{B}$
- $C' = H_A \mathrm{C}$

Second pair of images of orthogonal lines

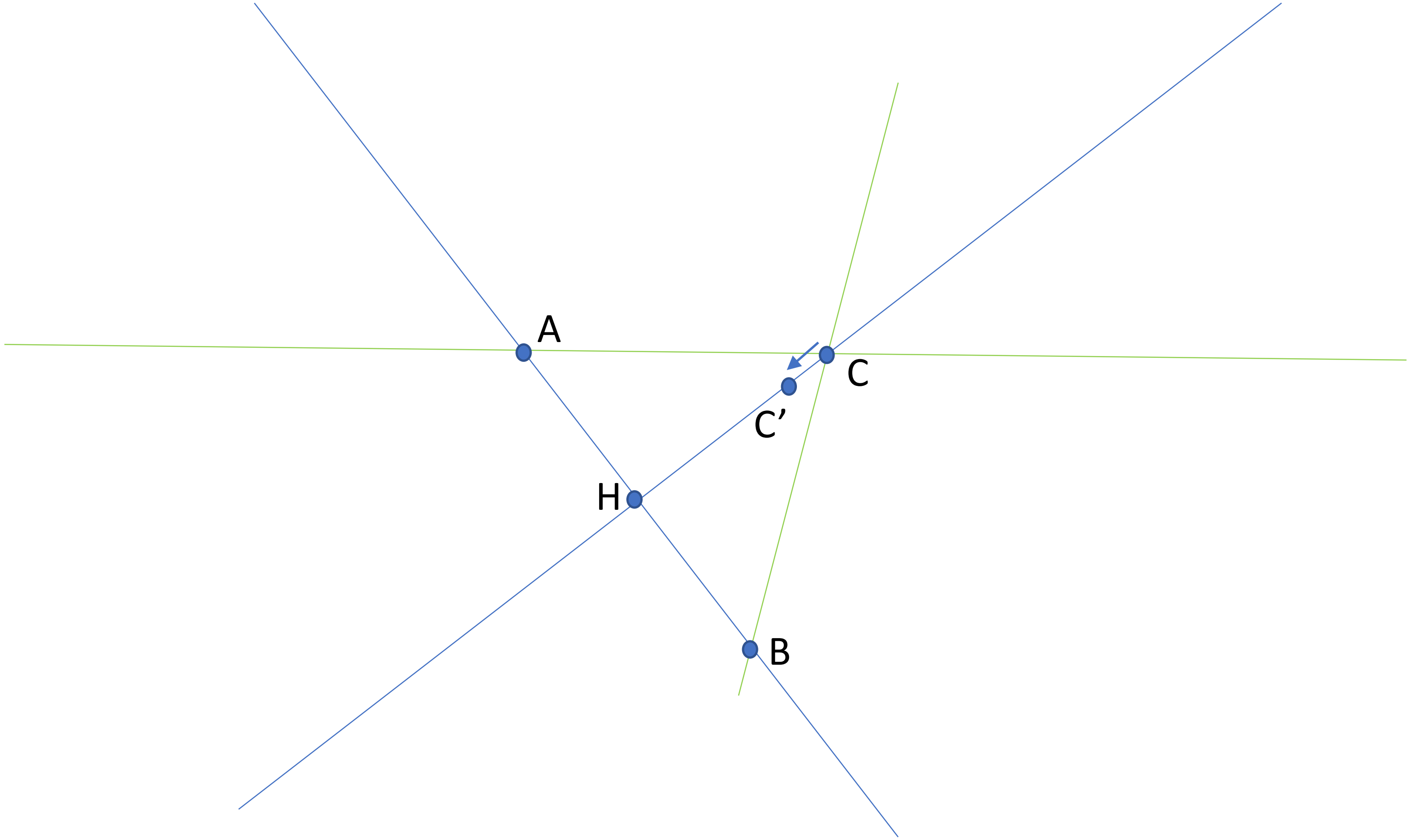
Second pair of image lines must be mapped onto orthogonal lines while keeping the first pair orthogonal



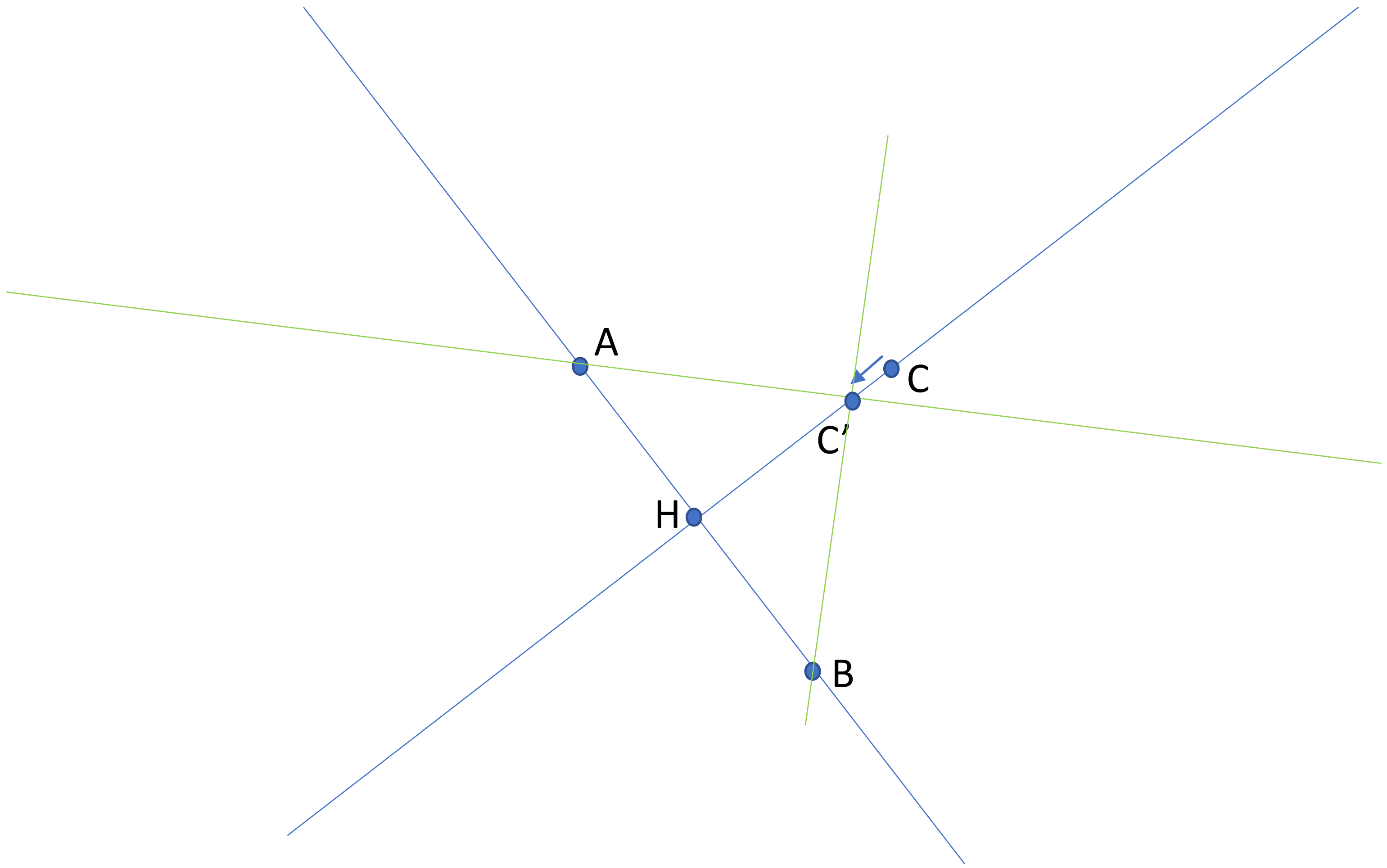
Map point C onto a point C' still on the line HC such that line HC' is orthogonal to line AB



Euclide's theorem: ABC' is a rectangular triangle iff HC' = $\sqrt{AH\ HB}$

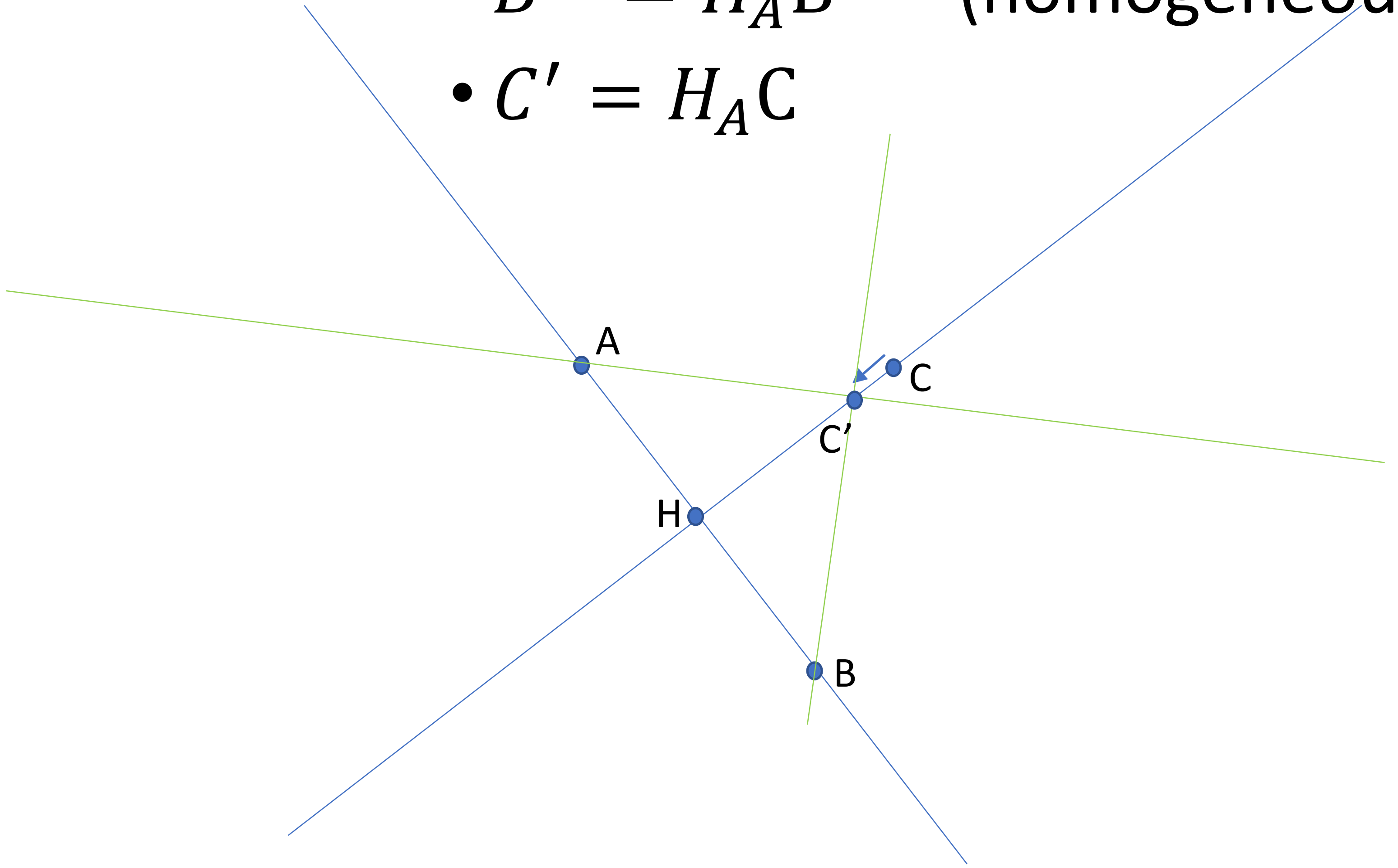


Euclide’s theorem: ABC’ is a rectangular triangle iff HC’ = $\sqrt{AH\ HB}$

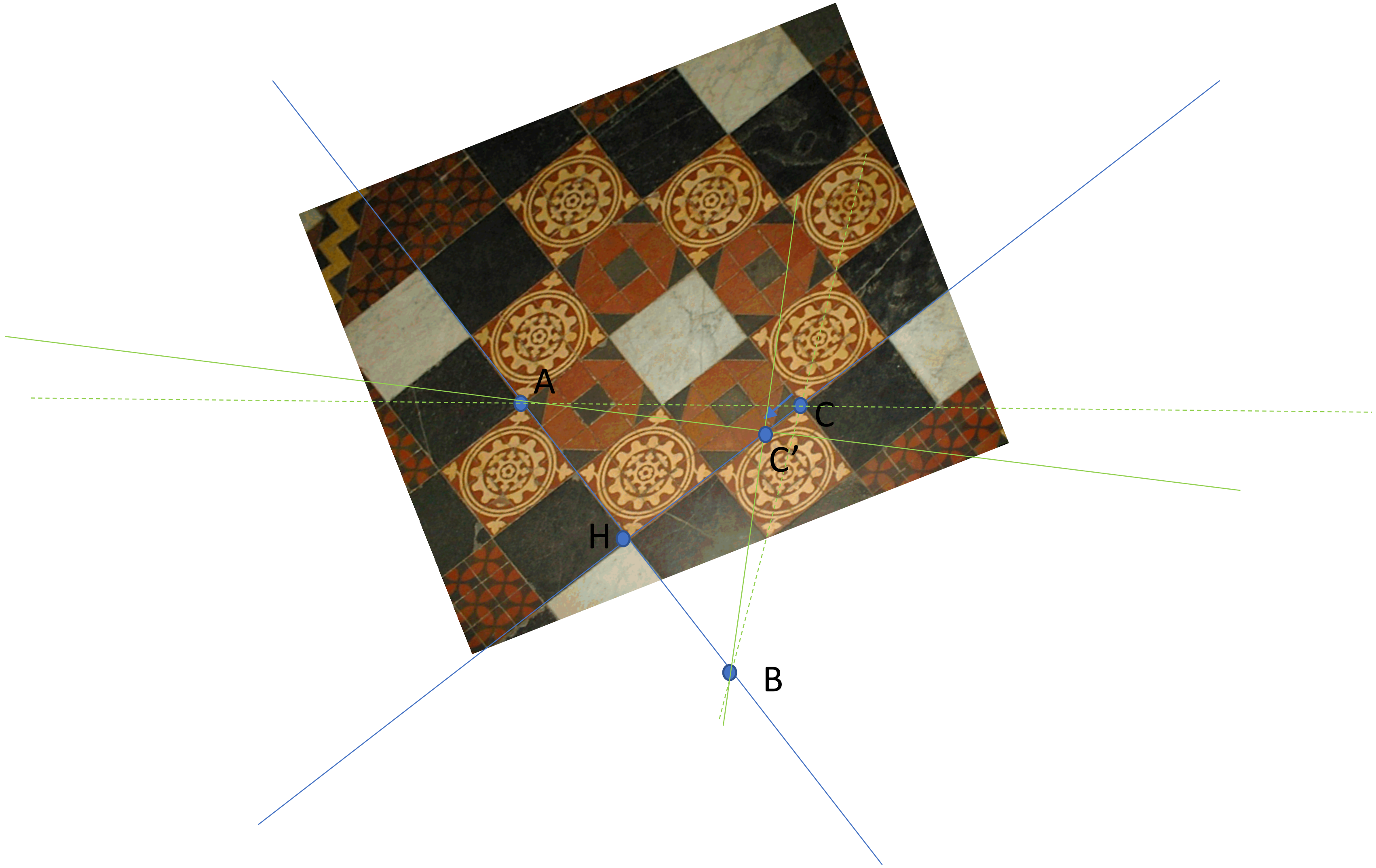


apply affinity H_A, where H_A is the solution of

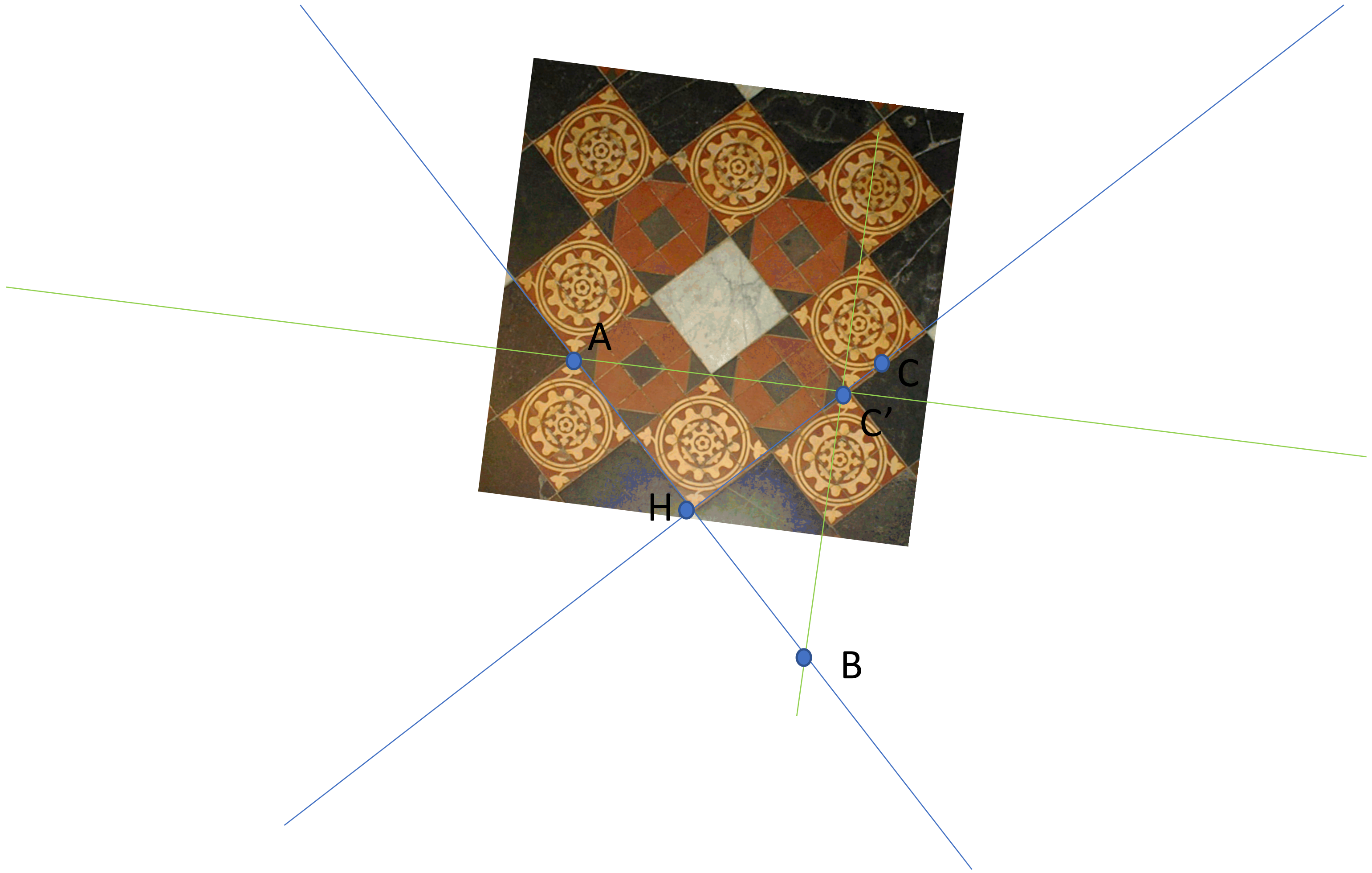
- $A \quad = H_A \mathrm{A}$
- $B \quad = H_A \mathrm{B}$ (homogeneous coordinates)
- $C' = H_A \mathrm{C}$



Apply affinity H_A

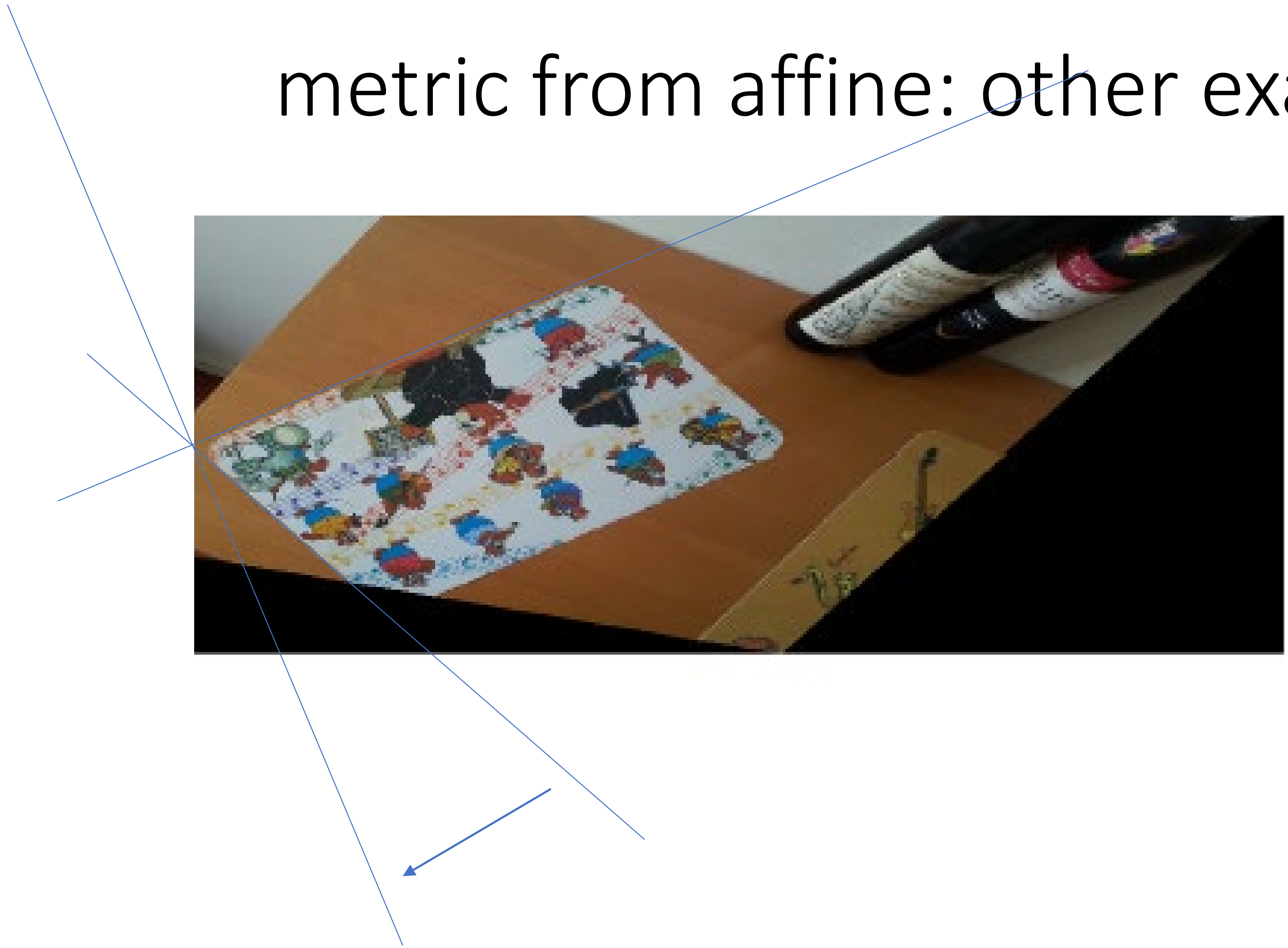


Apply affinity H_A
→ reconstructed shape



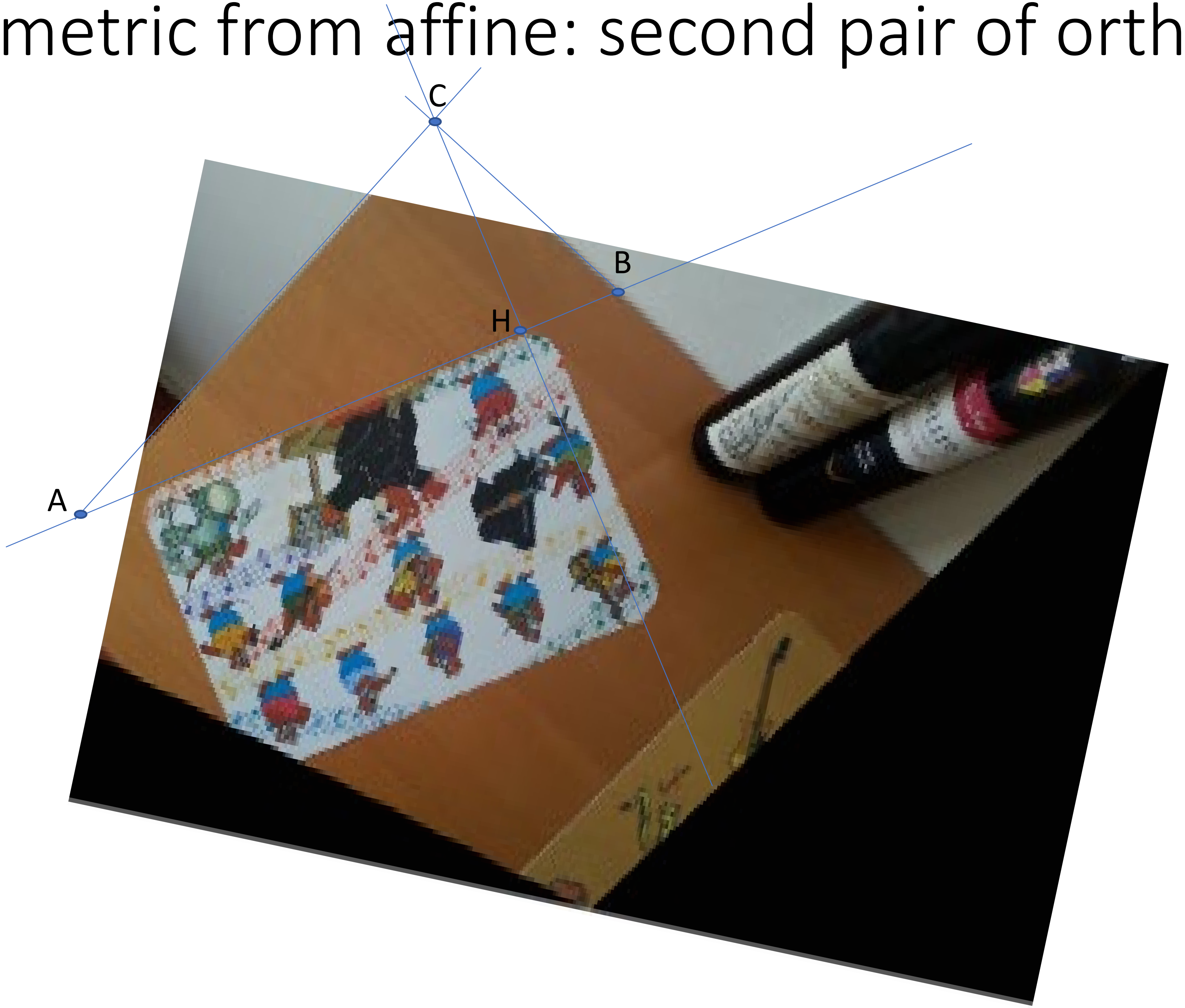
metric from affine: other example

metric from affine: other example



metric from affine: first pair of orthogonal lines → make images orthogonal too

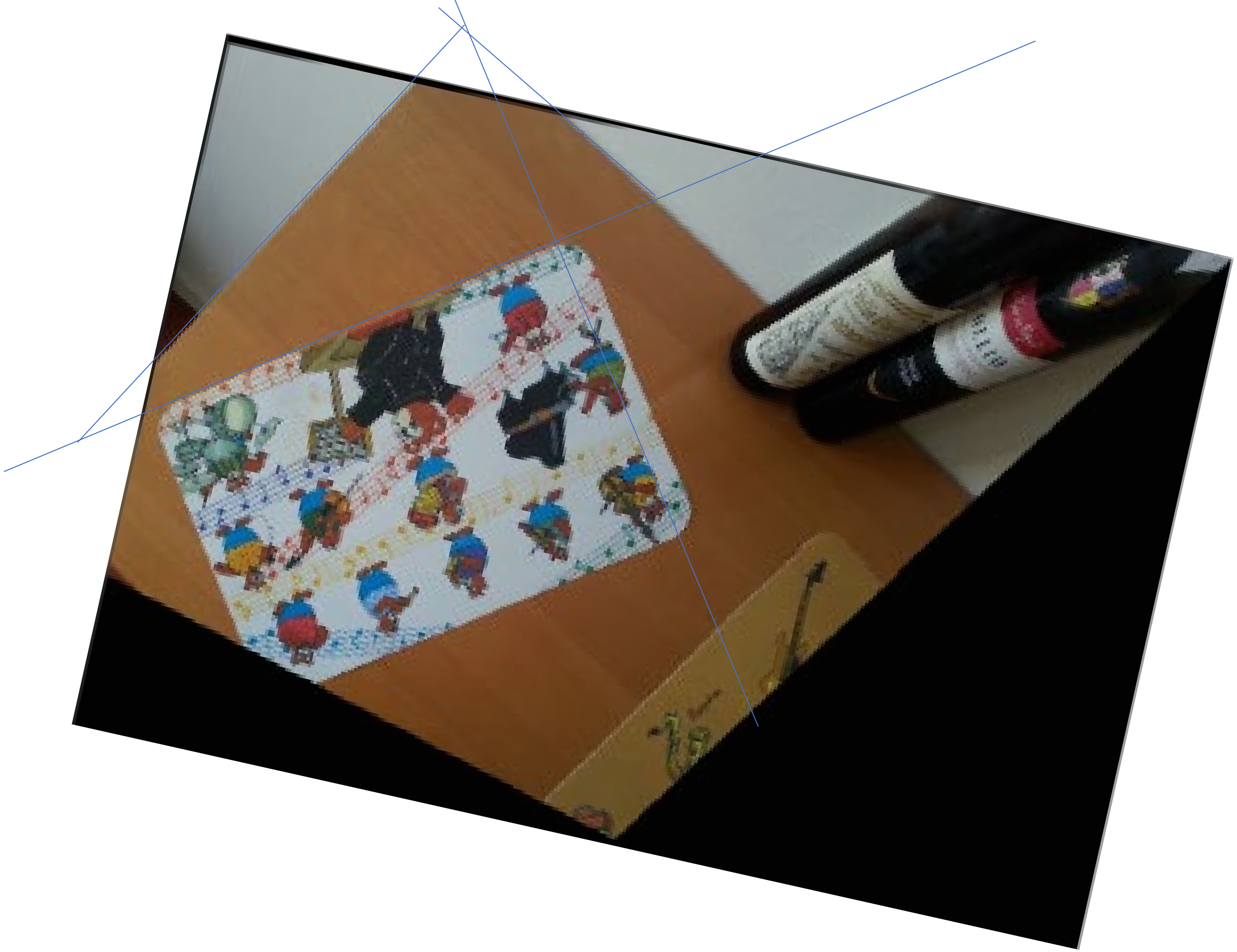
metric from affine: second pair of orthogonal lines



second pair of orthogonal lines $HC'^2 = AH \times BH$

Euclides Theorem

rectified image (2D shape reconstructed)

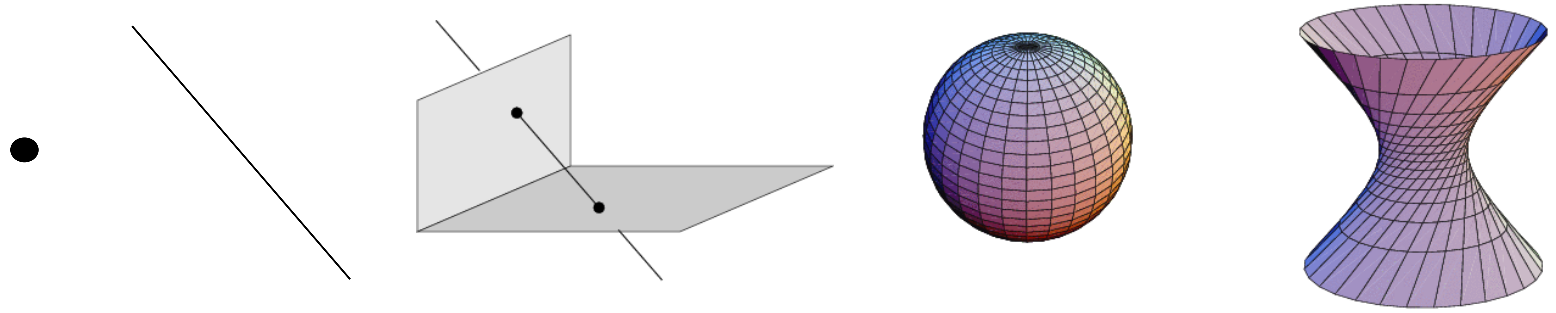


- Introduction and the Camera Optical System
- Planar (2D) Projective Geometry
- **Space (3D) Projective Geometry**
- Camera Geometry (3D → 2D Projection)

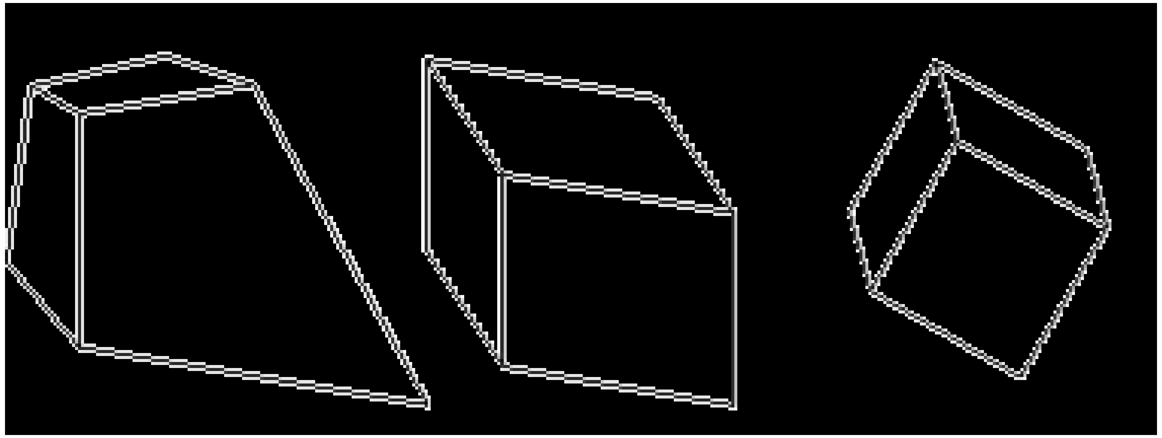
Space (3D) Projective Geometry

3D Space Projective Geometry

- **Elements**
  - **Points**
  - Planes
  - Quadrics
  - (Dual quadrics)

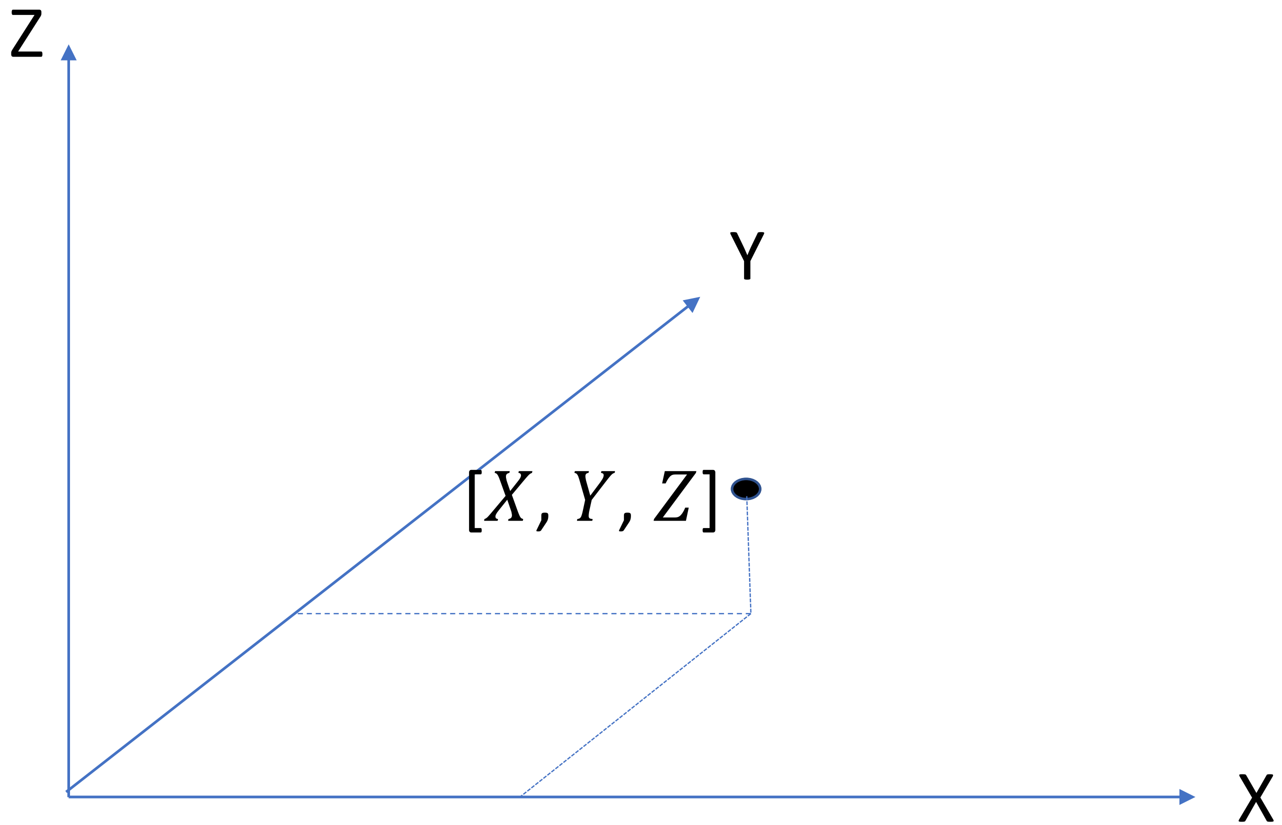


- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities

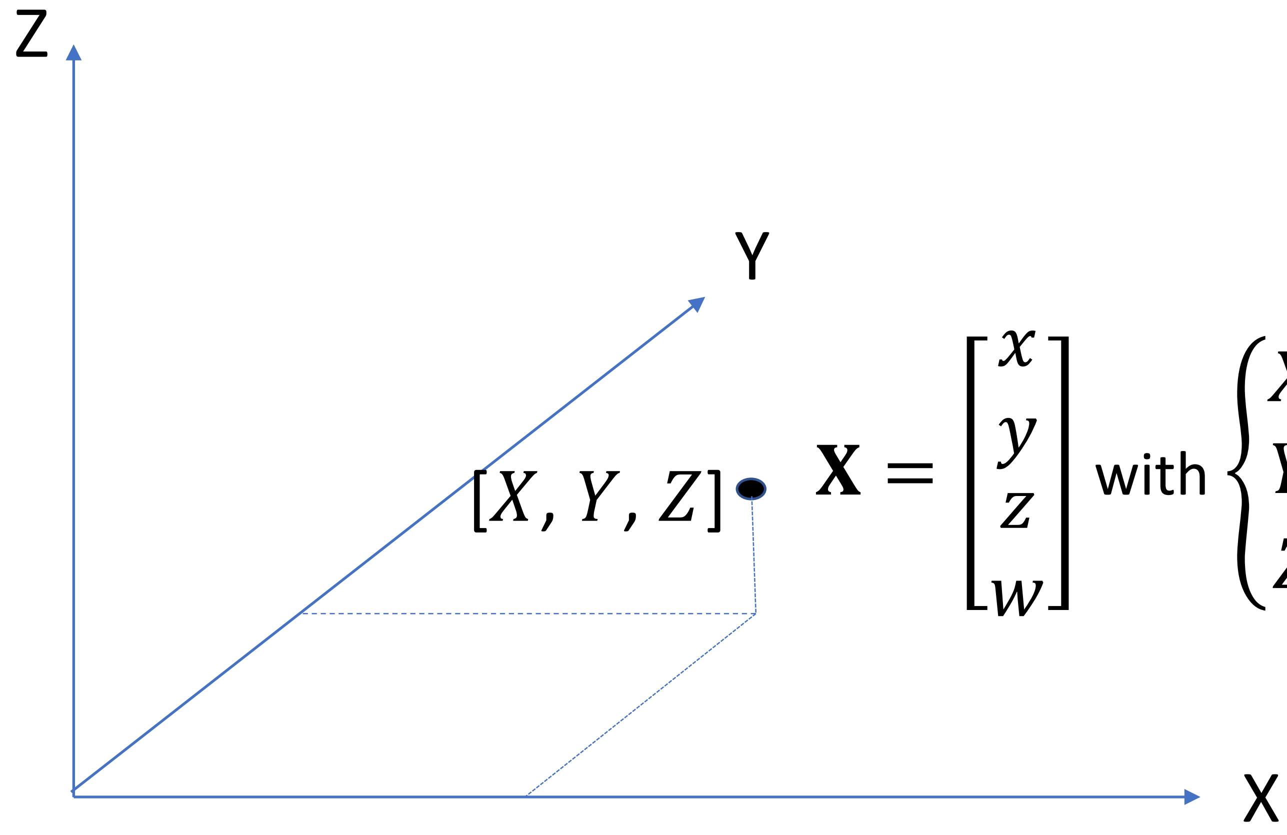


Points in the projective space

Euclidean space (3D) cartesian coordinates

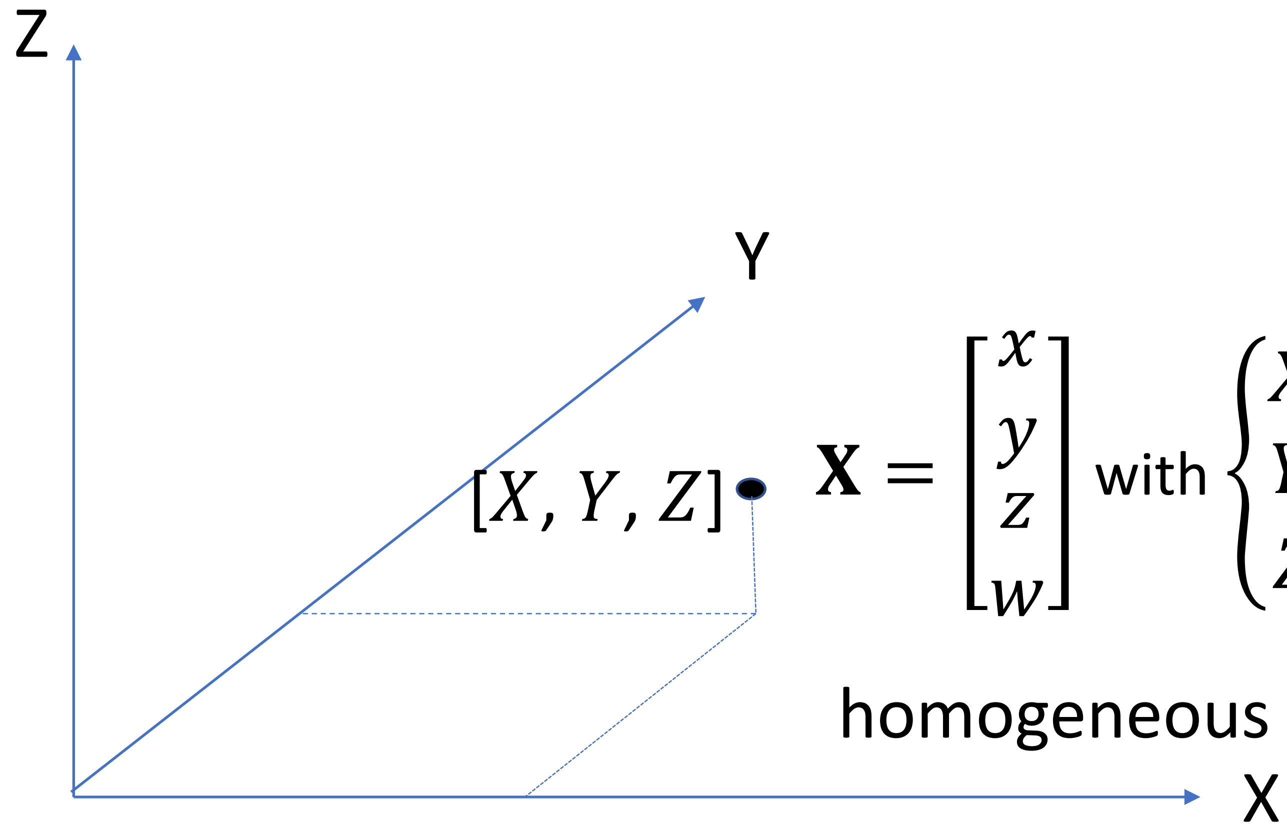


Projective space (3D)
4 homogeneous coordinates



$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

Projective space (3D)
4 homogeneous coordinates

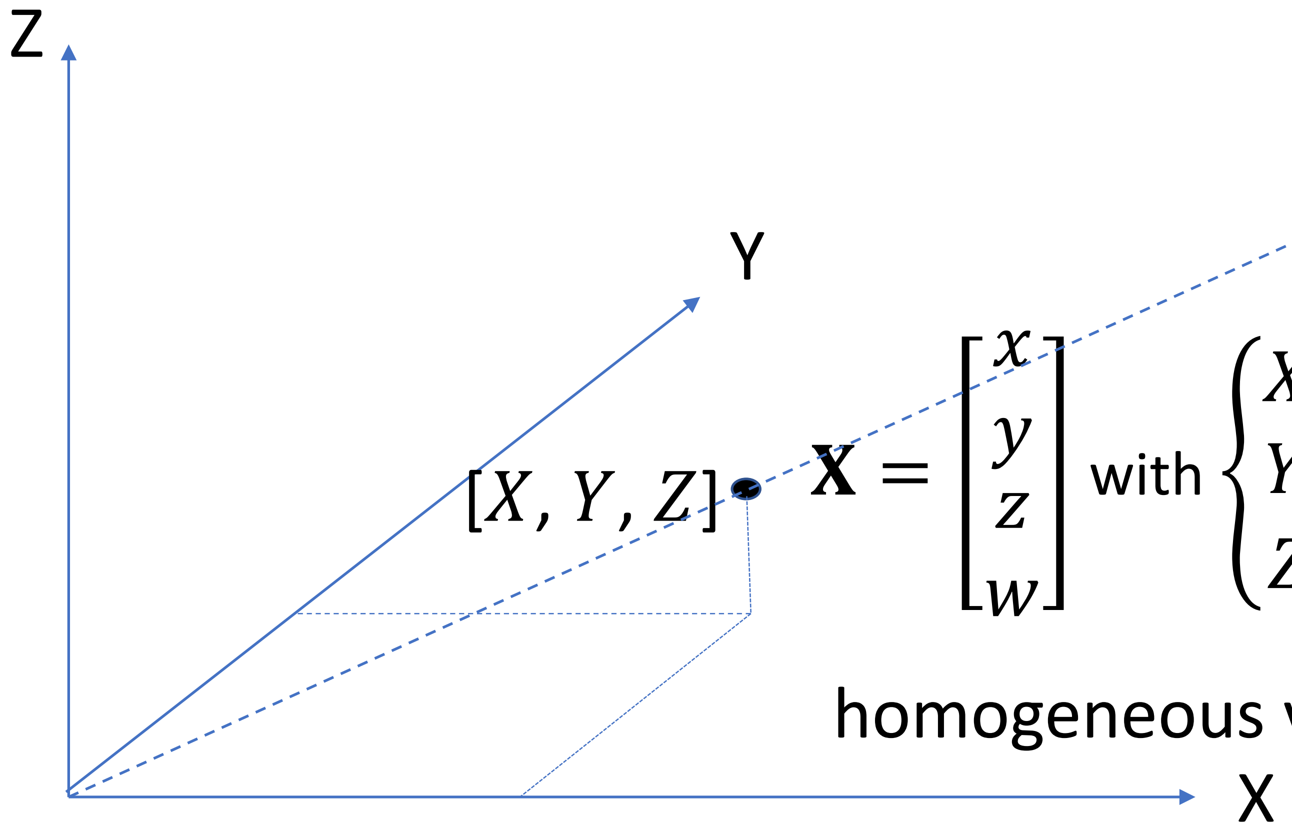


$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

homogeneous vector: $\lambda\mathbf{X} \Leftrightarrow \mathbf{X}$

Projective space $\mathbb{P}^3$: points at the ∞

$$\mathbf{X}_\infty = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$



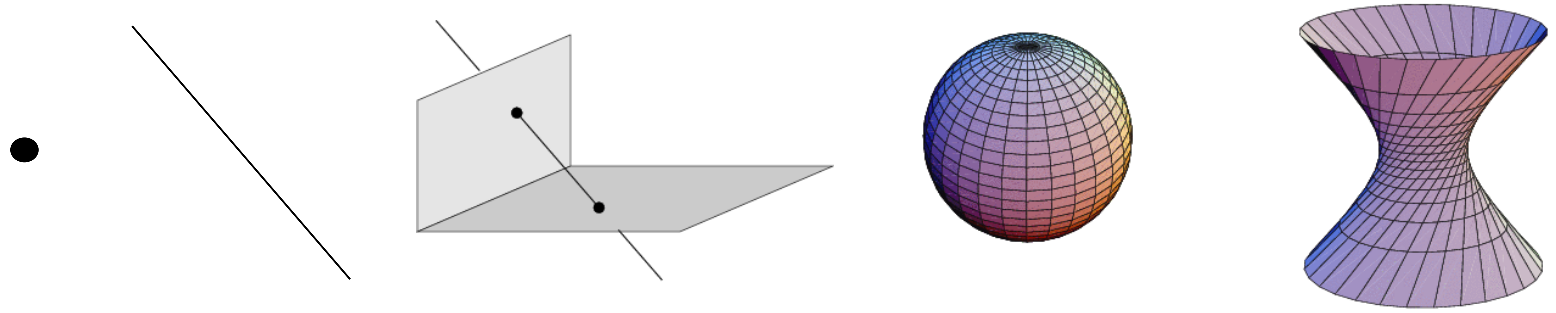
$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

homogeneous vector: $\lambda\mathbf{X} \Leftrightarrow \mathbf{X}$

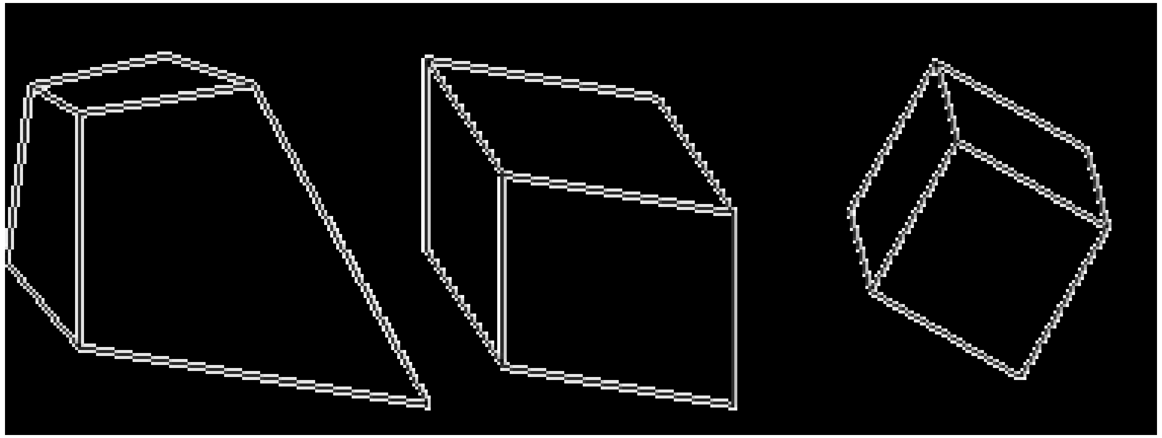
$$\mathbb{P}^3 = \{\mathbf{X} \in \mathbb{R}^4 - \{[0 \quad 0 \quad 0 \quad 0]^T\}\}$$

3D Space Projective Geometry

- **Elements**
- Points
- **Planes**
- Quadrics
- (Dual quadrics)

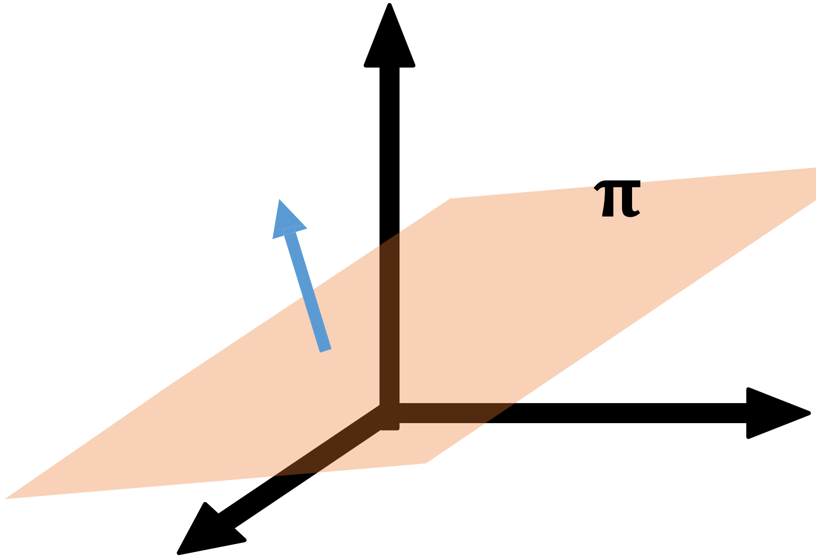


- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



Planes in the projective space

Planes in 3D Projective Geometry

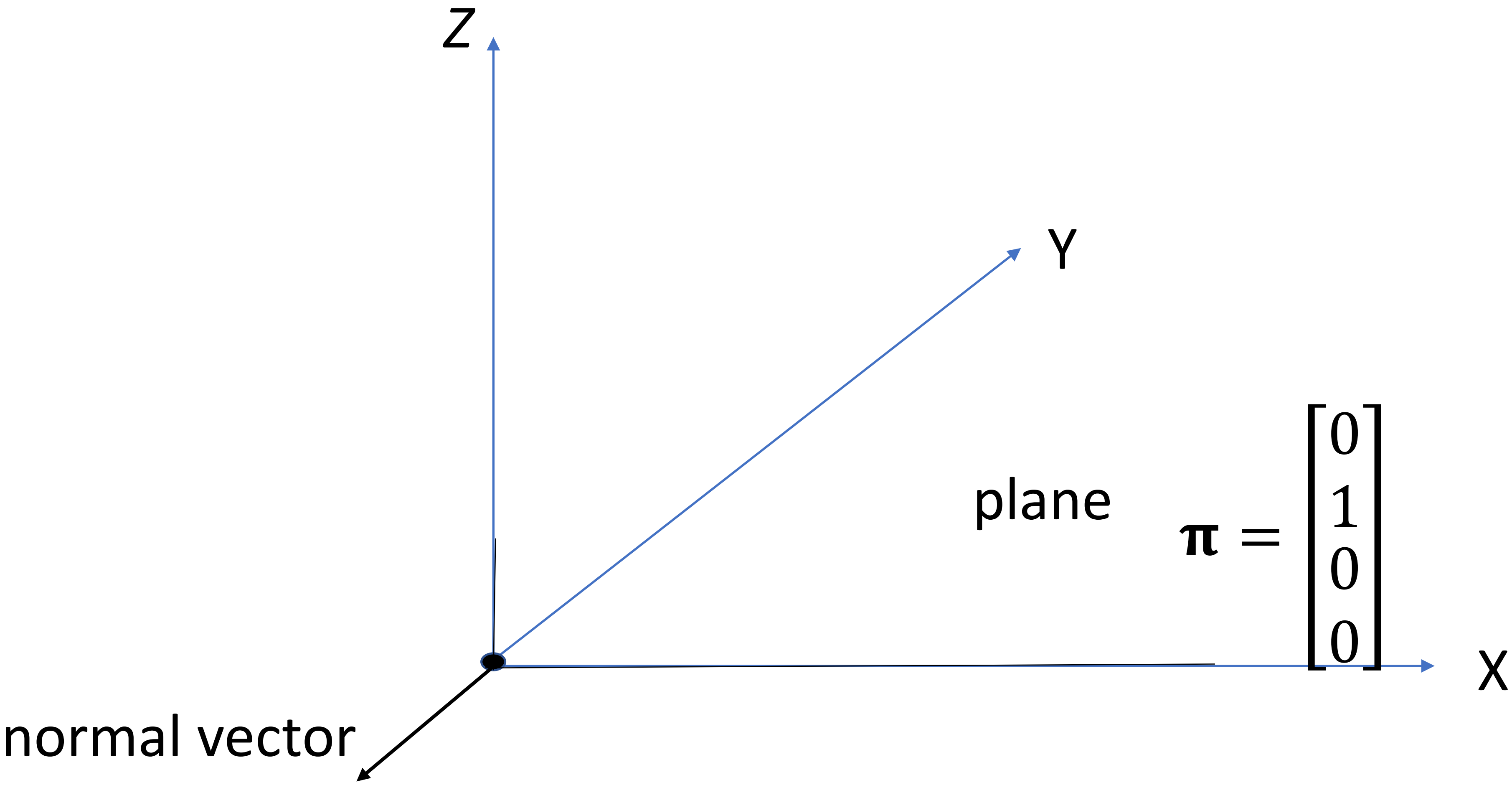


$\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with direction $\begin{pmatrix} a & b & c \end{pmatrix}$ normal to the plane,

and $\dfrac{-d}{\sqrt{a^2+b^2+c^2}}$ is the distance between the origin and the plane

$\boldsymbol{\pi}$ is a homogeneous vector: $\lambda\boldsymbol{\pi} \Leftrightarrow \boldsymbol{\pi}$

Example: the X-Z plane



$$\boldsymbol{\pi} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

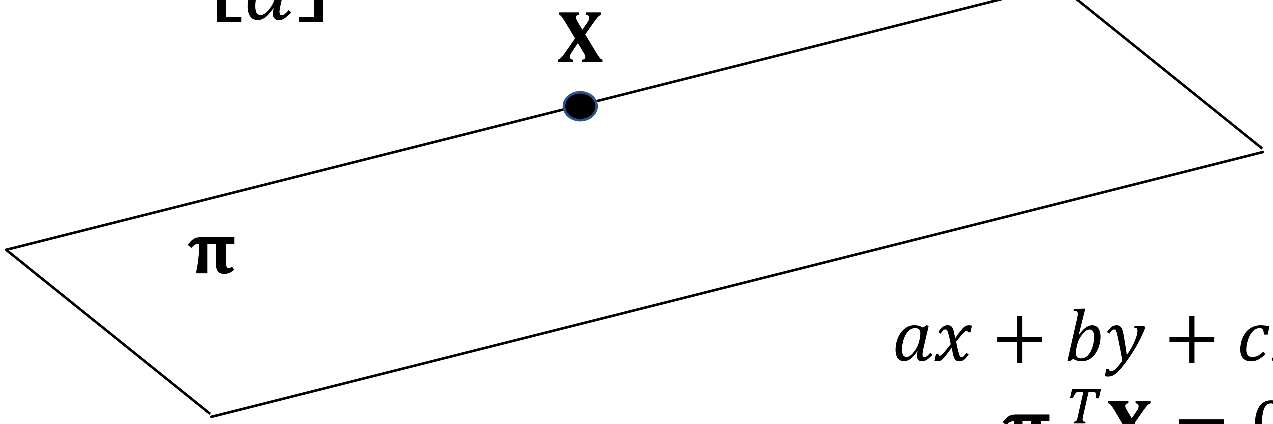
The incidence relation:
a point is on a plane, or a plane goes through a point

Incidence relation

the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is on the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

or

the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ goes through the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$



$$ax + by + cz + dw = 0$$
$$\boldsymbol{\pi}^T\mathbf{X} = 0 = \mathbf{X}^T\boldsymbol{\pi}$$

Dividing by $\boldsymbol{w}$ we find the cartesian coordinates again

The plane at the infinity:
the locus of the points at the infinity

The «locus» of the points at the infinity

As there are infinite points at the infinity (one for each direction), what is the aspect of the set of these points at the infinity?

Simply: $$w = 0$$

This set is a plane: $[\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c} \quad \boldsymbol{d}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\\ \boldsymbol{w}\end{bmatrix} = 0$, actually $[\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\\ \boldsymbol{w}\end{bmatrix} = 0$

namely, **the plane at the infinity** $\boldsymbol{\pi}_\infty = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

NOTE: this plane has undefined normal direction

The duality principle between points and planes

Since dot product is commutative
→ incidence relation is commutative

$$\boldsymbol{\pi}^T\mathbf{X} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix} = 0 = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = 0 = \mathbf{X}^T\boldsymbol{\pi} = \mathbf{0}$$

point $\mathbf{X}$ is on plane $\boldsymbol{\pi}$

$\Leftrightarrow$

point $\boldsymbol{\pi}$ is on **plane** $\mathbf{X}$

point $\mathbf{X}$ is on plane $\boldsymbol{\pi}$ (i.e. plane $\boldsymbol{\pi}$ goes through point $\mathbf{X}$)

$\leftrightarrow$

point $\boldsymbol{\pi}$ is on **line** $\mathbf{X}$ (i.e. line $\mathbf{X}$ goes through point $\boldsymbol{\pi}$)

Principle of **duality** between points and planes
in 3D Projective Geometry

For any true sentence containing the words

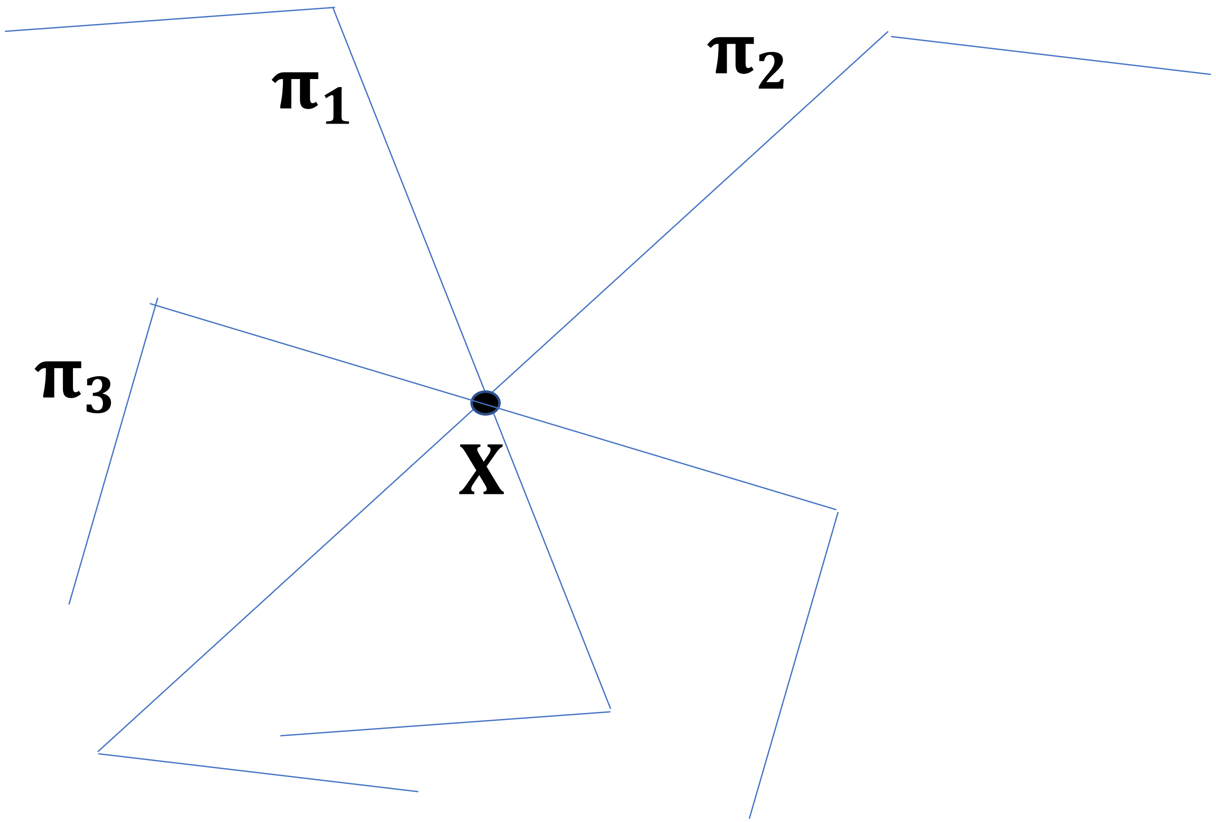
- point
- plane
- is on
- goes through

there is a DUAL sentence -also true- obtained by substituting, in the previous one, each occurence of

- point	by	- plane
- plane	by	- point
- is on	by	- goes through
- goes through	by	- is on

The point on three planes

The point on three planes



$$\begin{cases} \boldsymbol{\pi}_1{}^T\mathbf{X} = 0 \\ \boldsymbol{\pi}_2{}^T\mathbf{X} = 0 \\ \boldsymbol{\pi}_3{}^T\mathbf{X} = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \\ \boldsymbol{\pi}_3{}^T \end{bmatrix}}_{3 \times 4} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

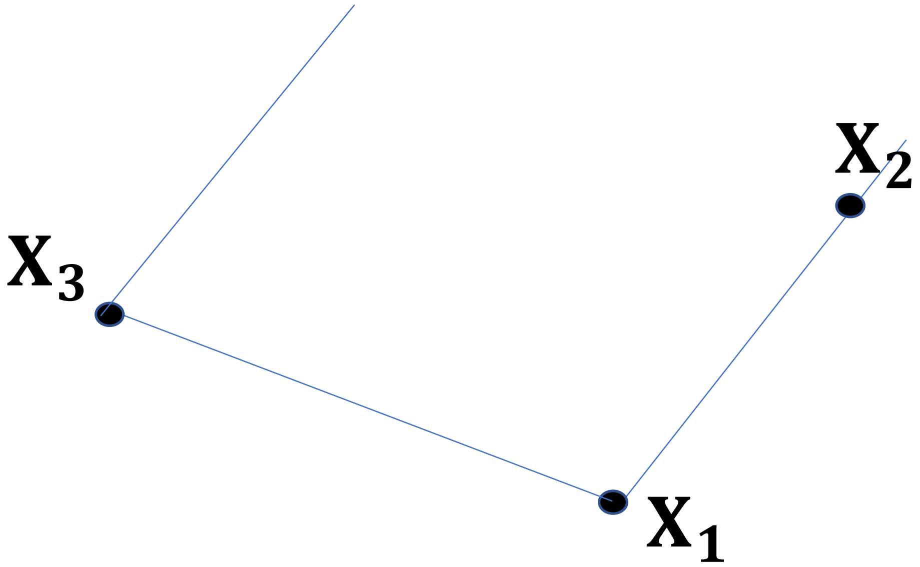
$$\mathbf{X} = \mathbf{RNS}\Bigg(\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \\ \boldsymbol{\pi}_3{}^T \end{bmatrix}_{3 \times 4}\Bigg)$$

a solution vector + all its mutiples

The plane through three points

The plane through three points: DUAL of the point on three planes

The plane through three points



$$\begin{cases} \mathbf{X_1}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_2}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_3}^T \boldsymbol{\pi} = 0 \end{cases}$$

$$\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \\ \mathbf{X_3}^T \end{bmatrix}_{3 \times 4} \boldsymbol{\pi} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\pi} = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \\ \mathbf{X_3}^T \end{bmatrix}_{3 \times 4}\right)$$

a solution vector + all its mutiples

Lines

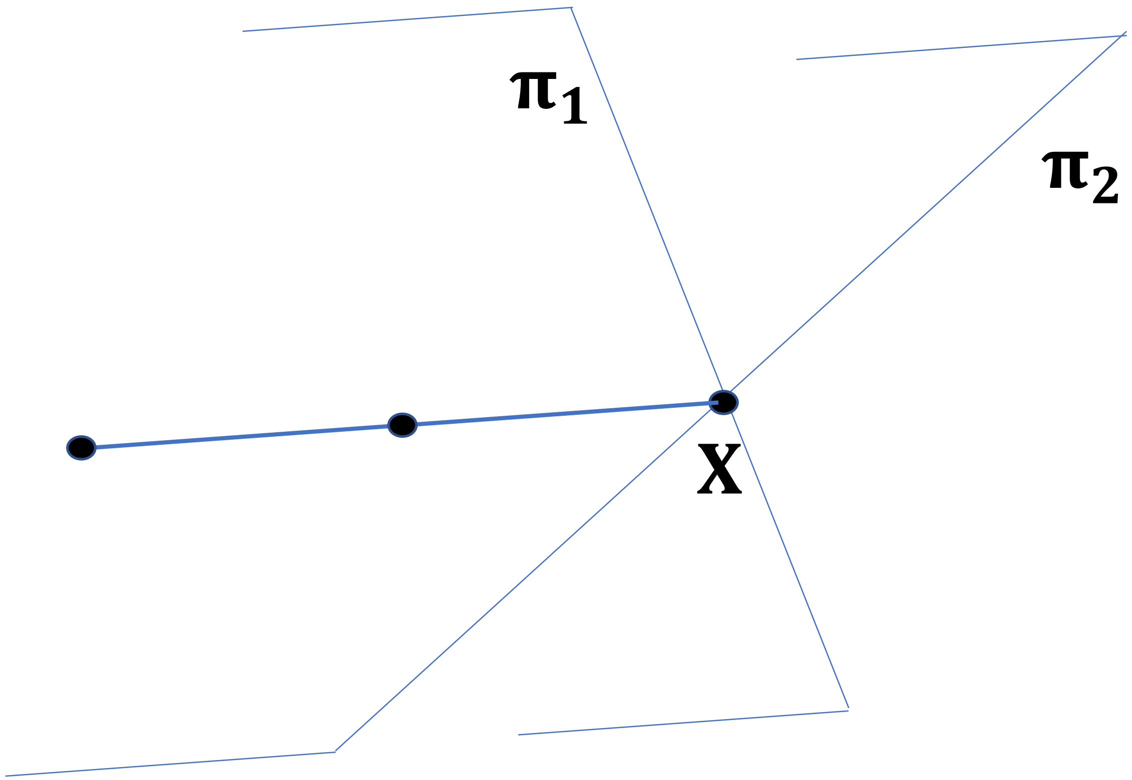
WHAT ABOUT LINES ?

Lines are primitive elements in the planar geometry
but they are **not** primitive elements in the **space** geometry

Lines can be defined through points or through planes.

Lines are intermediate entities between points and planes
they are self-dual

Line: the set of points $\mathbf{X}$ on **two** planes



$$\begin{cases} \boldsymbol{\pi}_1{}^T \mathbf{X} = 0 \\ \boldsymbol{\pi}_2{}^T \mathbf{X} = 0 \end{cases}$$

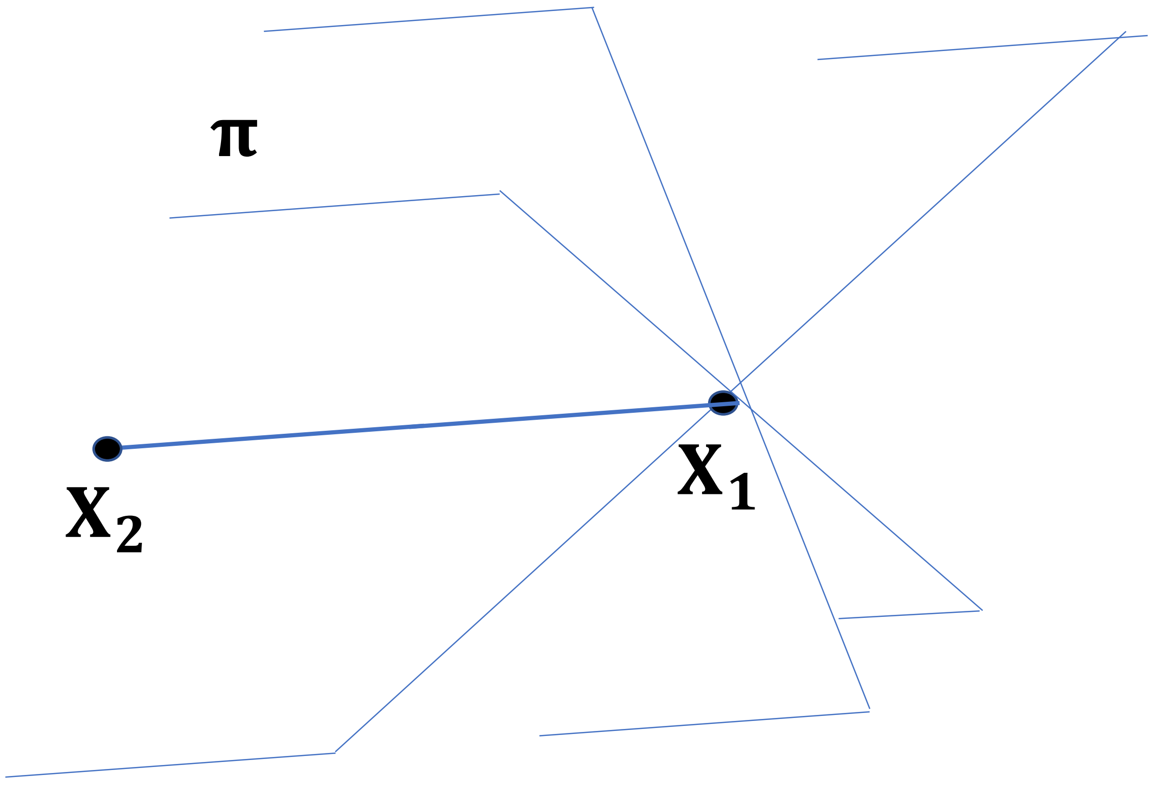
$$\underbrace{\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \end{bmatrix}}_{2 \times 4} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \end{bmatrix}_{2 \times 4}\right)$$

2D set of solution vectors: two points and all their linear combinations

→ due to homogeneity: 1D set of points (parameter abscissa)

Line: the set of planes $\boldsymbol{\pi}$ through **two** points



$$\begin{cases} \mathbf{X_1}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_2}^T \boldsymbol{\pi} = 0 \end{cases}$$

$$\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}_{2 \times 4} \boldsymbol{\pi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

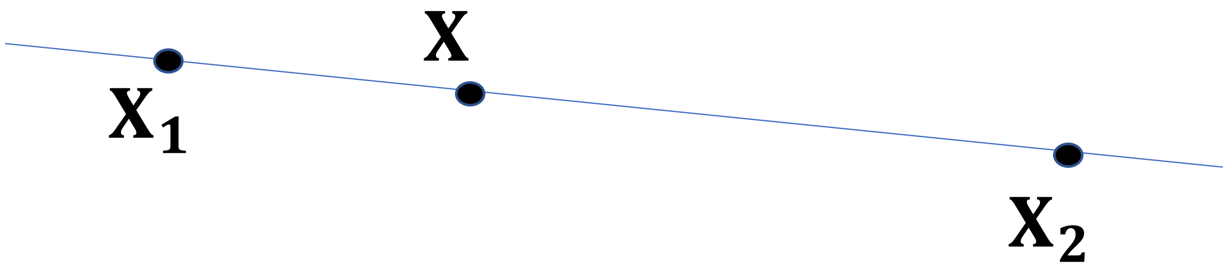
$$\mathbf{L}^* = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}_{2 \times 4}\right)$$

2D set of vector solutions: two planes and all their linear combinations

→ due to homogeneity: 1D set of planes (parameter: rotation angle)

linear combination of two points

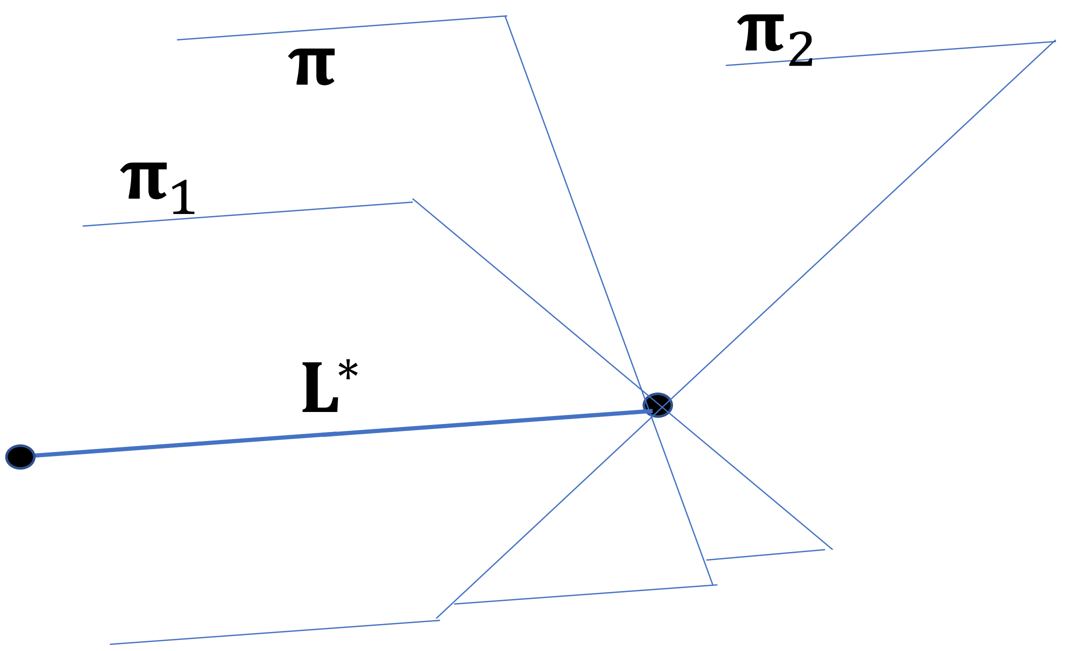
Property: the point $\mathbf{X}$ given by the linear combination $\mathbf{X} = \propto \mathbf{X}_1 + \beta\mathbf{X}_2$ of two points $\mathbf{X}_1$ and $\mathbf{X}_2$ is on the line $\mathbf{L}$ through $\mathbf{X}_1$ and $\mathbf{X}_2$



A **line** $\mathbf{L}$ can also be defined as the set of all points, that are linear combinations of two given points: $\mathbf{X} = \propto \mathbf{X}_1 + \beta\mathbf{X}_2$

DUAL: linear combination of two planes

Dual property: the plane $\boldsymbol{\pi}$**,** given by the linear combination $\boldsymbol{\pi} = \propto \boldsymbol{\pi}_1 + \beta \boldsymbol{\pi}_2$ of two planes $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$, goes through the line $\mathbf{L}^*$ on $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$

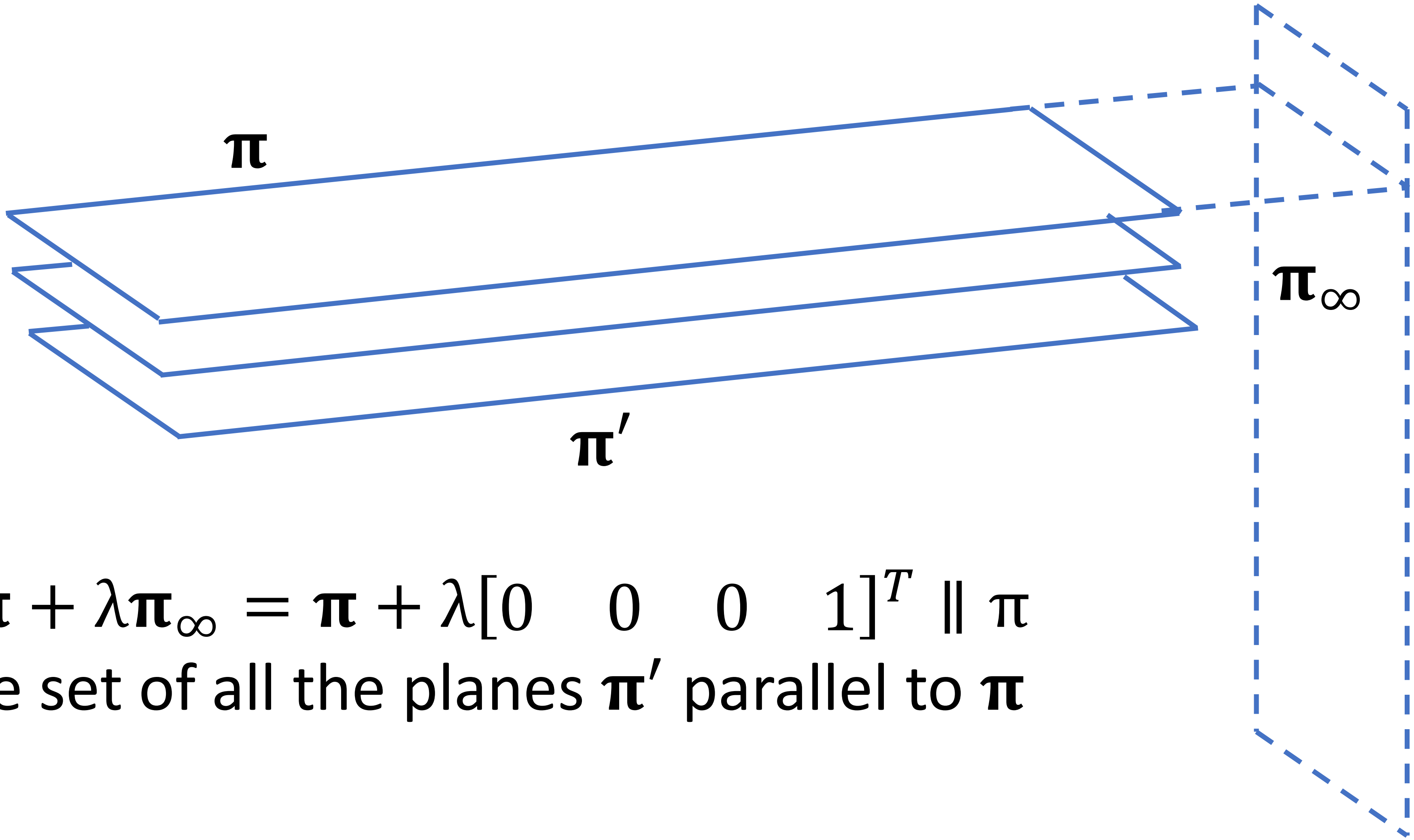


A **line** $\mathbf{L}^*$ can also be defined as the set of all planes, that are linear combinations of two given planes: $\boldsymbol{\pi} = \propto \boldsymbol{\pi}_1 + \beta\ \boldsymbol{\pi}_2$

pairs of DUALLY corresponding words

- point	→	- plane
- line	→	- line
- plane	→	- point
- is on	→	- goes through
- goes through	→	- is on

A special case:
linear combinations of a plane $\boldsymbol{\pi}$ and the plane $\boldsymbol{\pi}_\infty$



$$\boldsymbol{\pi}' = \boldsymbol{\pi} + \lambda\boldsymbol{\pi}_\infty = \boldsymbol{\pi} + \lambda\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T \parallel \pi$$

→ the set of all the planes $\boldsymbol{\pi}'$ parallel to $\boldsymbol{\pi}$

Remark: planes and lines at the infinity

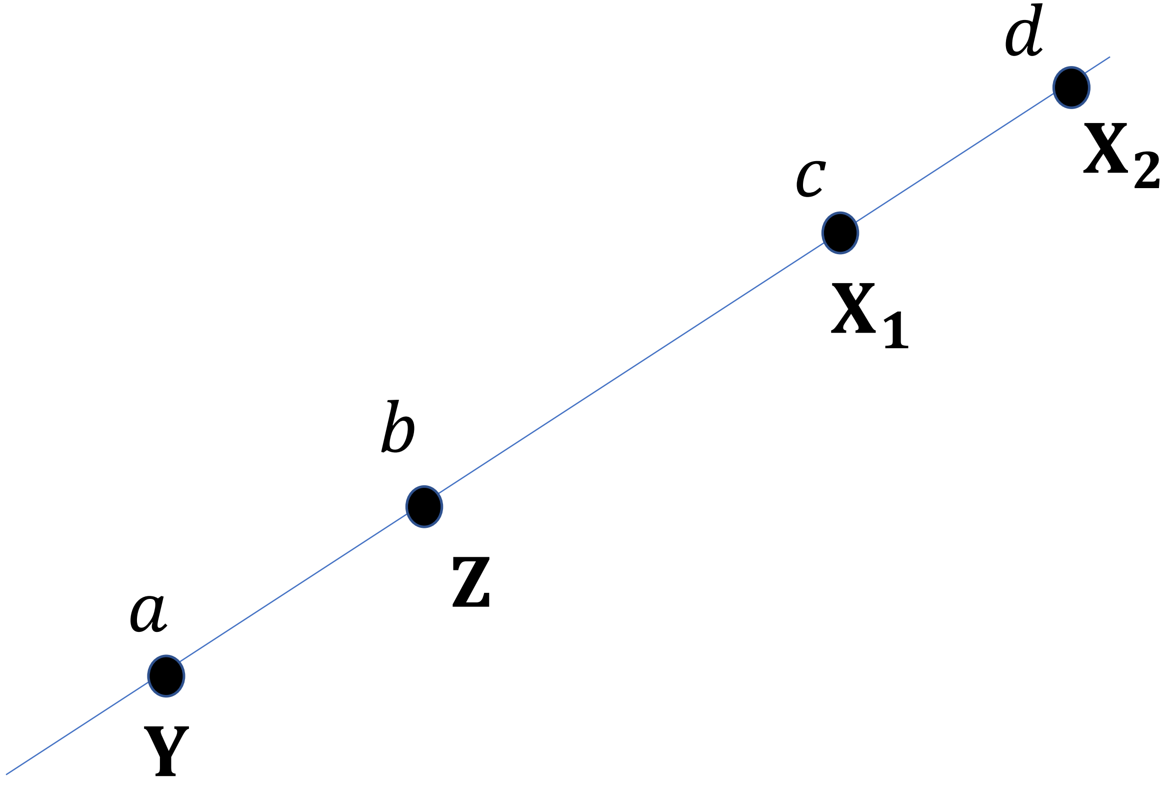
Each plane $\boldsymbol{\pi}$ has its own line at the infinity $\boldsymbol{l}_{\infty}(\boldsymbol{\pi})$
and also its own circular points $\boldsymbol{I}(\boldsymbol{\pi})$ and $\boldsymbol{J}(\boldsymbol{\pi})$

parallel planes share the same $\boldsymbol{l}_{\infty}$
and the same circular points $\boldsymbol{I}$ and $\boldsymbol{J}$

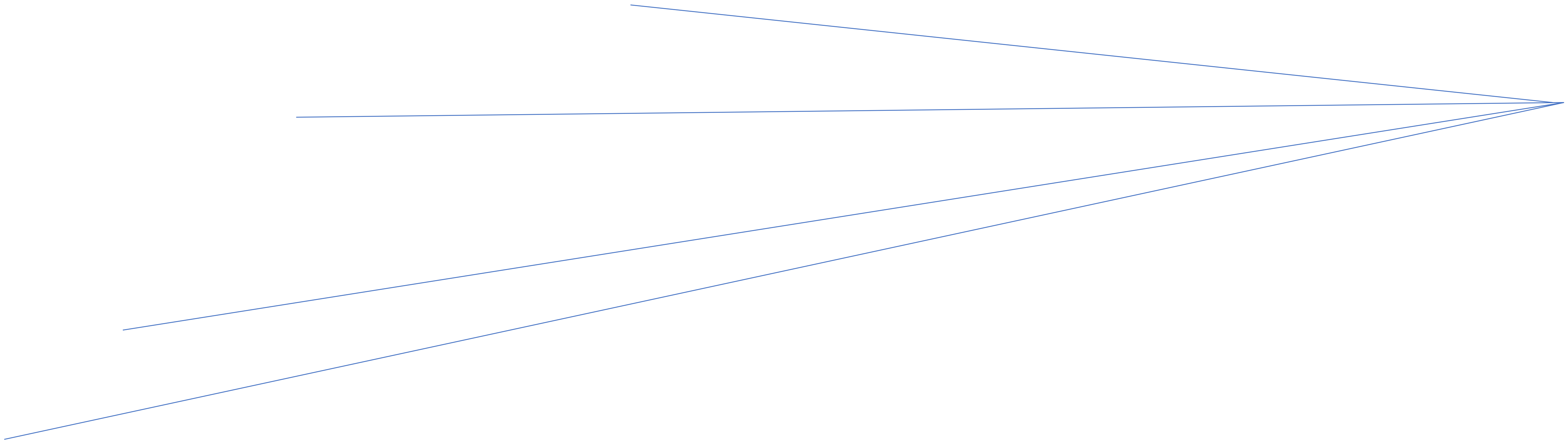
The cross ratio

1D cross ratio of a 4-tuple of colinear points

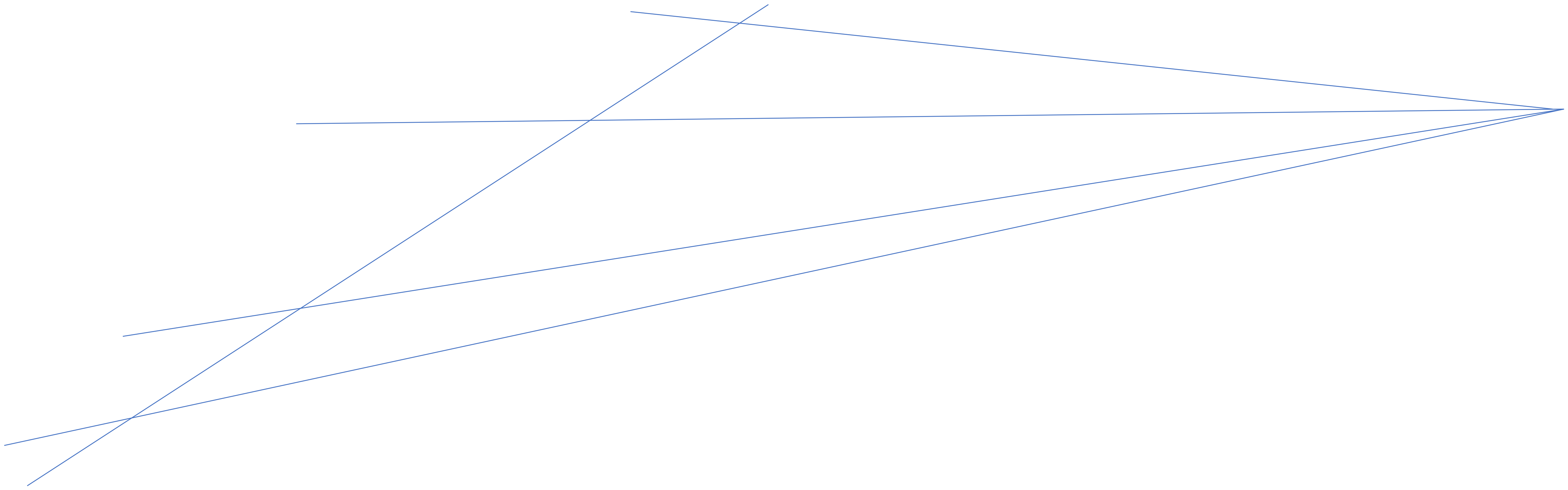
$$CR_{Y,Z,\mathrm{X}_1,\mathrm{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$



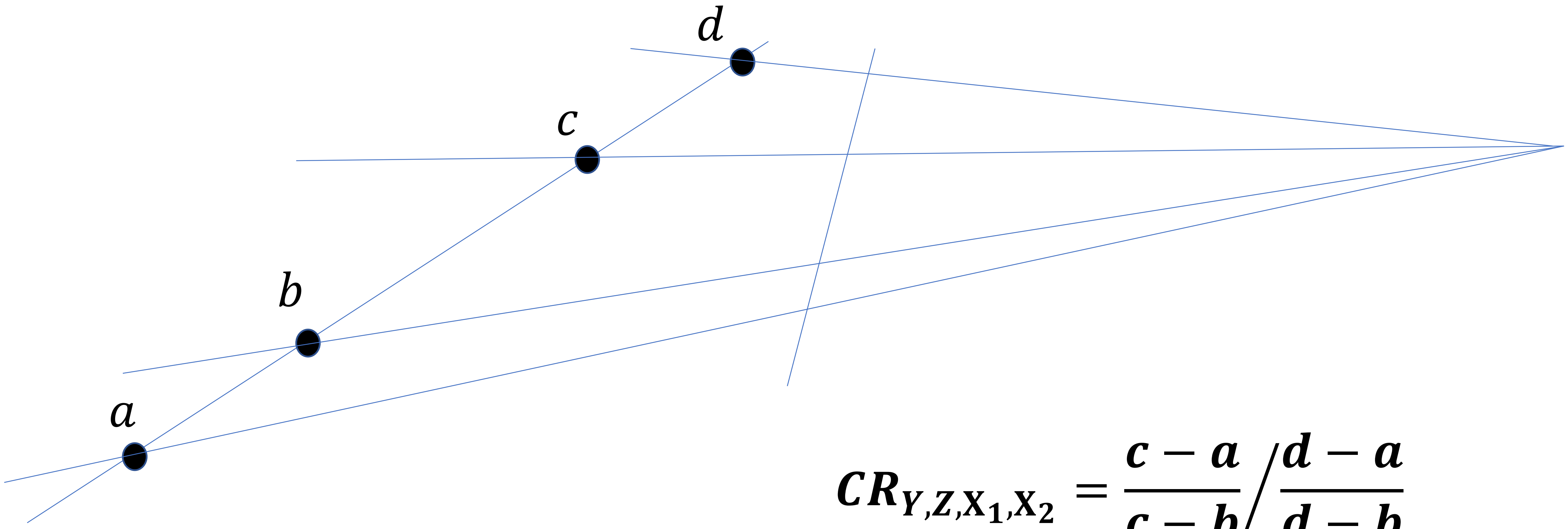
2D cross ratio of a 4-tuple of coplanar, concurrent lines



2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line …



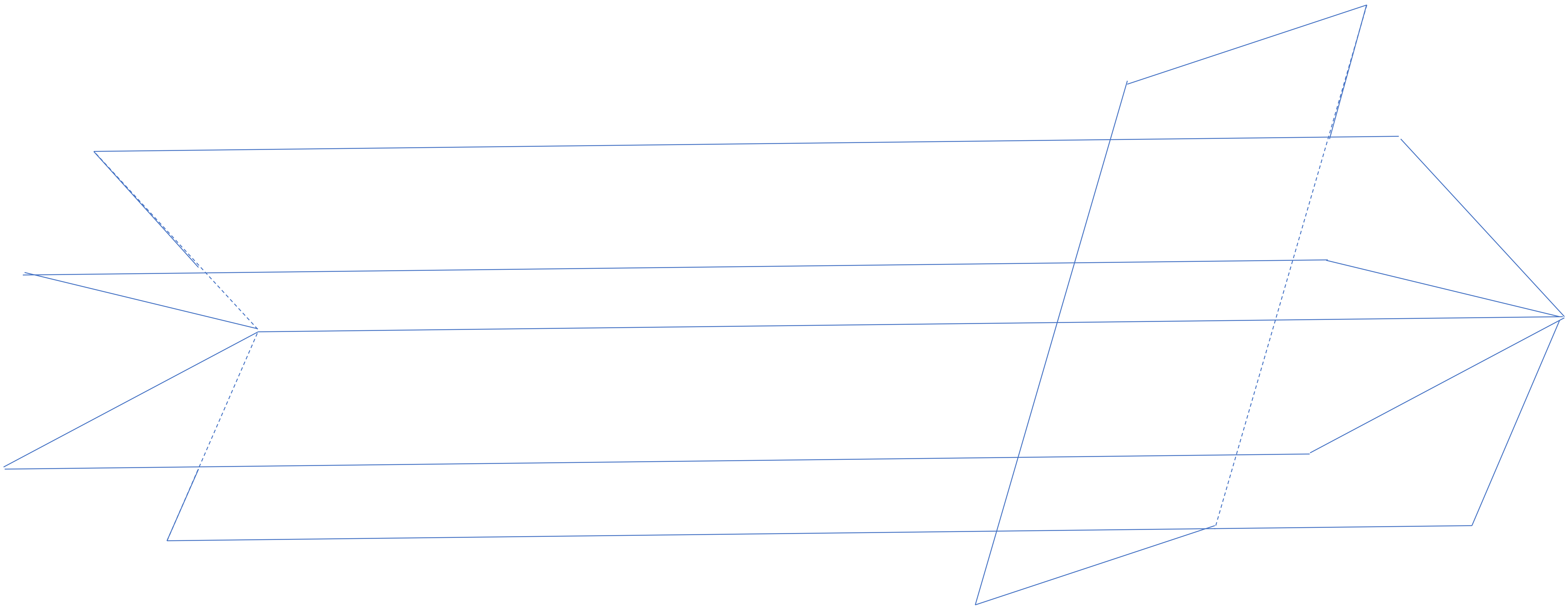
2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line … compute the 1D cross ratio of intersection points



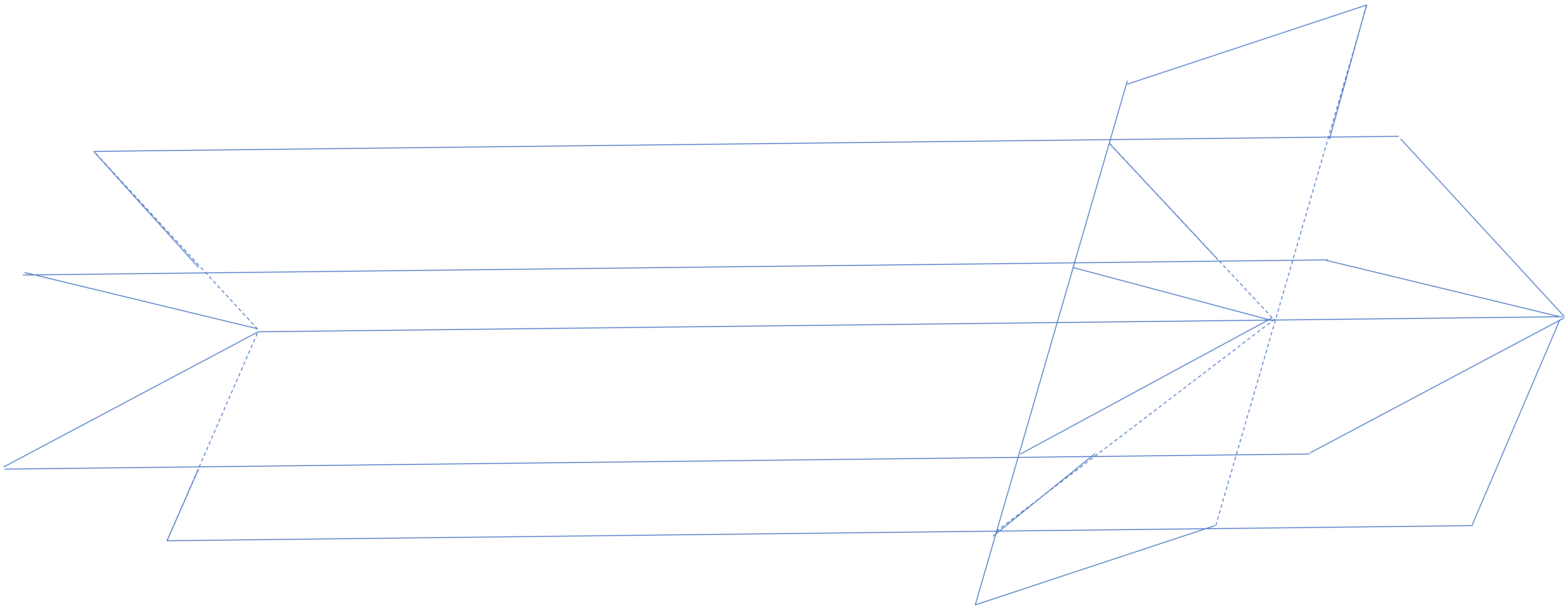
$$CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

3D cross ratio of a 4-tuple of coaxial planes:

3D cross ratio of a 4-tuple of coaxial planes: take any crossing plane …

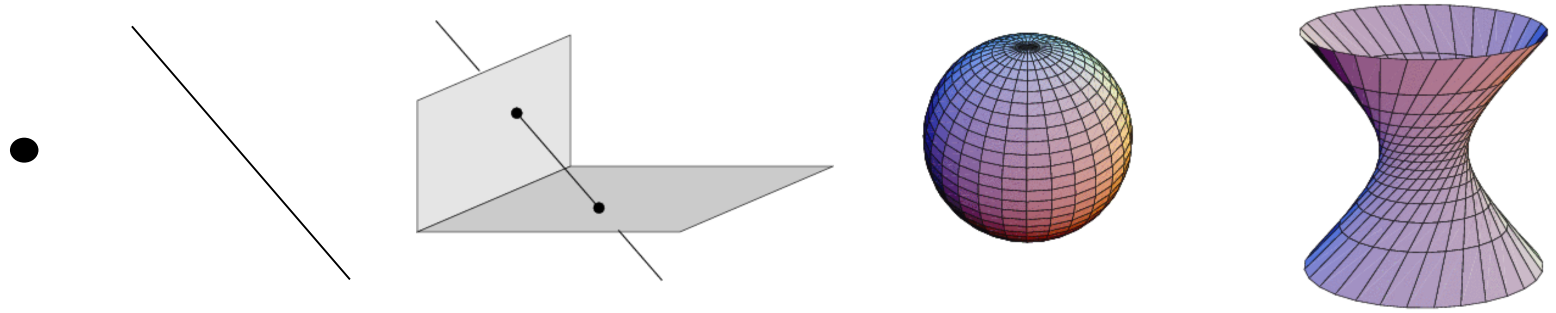


3D cross ratio of a 4-tuple of coaxial planes: take any crossing plane … compute the 2D cross ratio of intersection lines

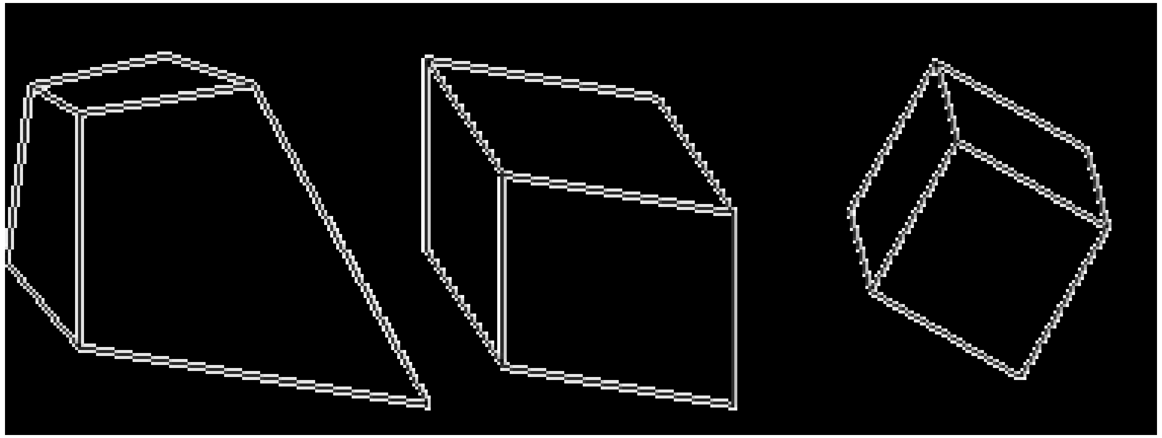


3D Space Projective Geometry

- **Elements**
- Points
- Planes
- **Quadrics**
- (Dual quadrics)



- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



QUADRICS

Quadric: a point $\mathbf{X}$ is on a quadric $\boldsymbol{Q}$ if it satisfies a homogeneous *quadratic* equation, namely

$$\mathbf{X}^{T}\boldsymbol{Q}\,\mathbf{X} = \mathbf{0}$$

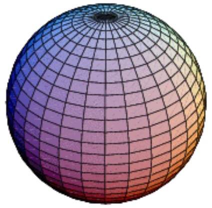
where $\boldsymbol{Q}$ is a 4x4 symmetric matrix.

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

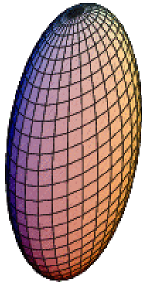
- $\boldsymbol{Q}$ is a homogeneous matrix: $\lambda\boldsymbol{Q} \Leftrightarrow \boldsymbol{Q}$
- 9 degrees of freedom
- 9 points in general positions define a quadric

Quadric classification

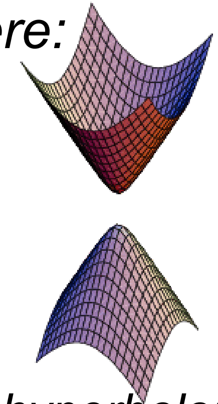
Projectively equivalent to *sphere:*



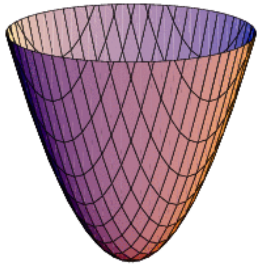
sphere



ellipsoid

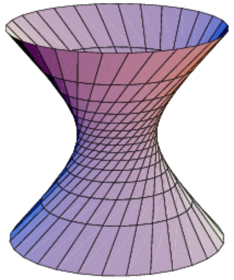


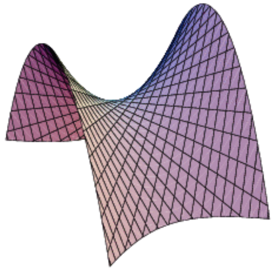
hyperboloid of two sheets



paraboloid

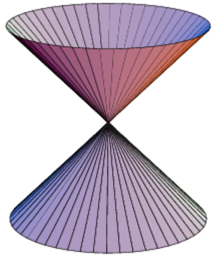
Ruled quadrics:



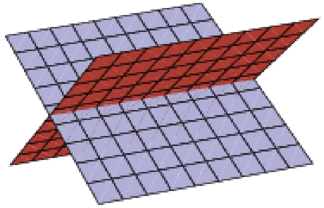


hyperboloids of one sheet

Degenerate ruled quadrics:



cone



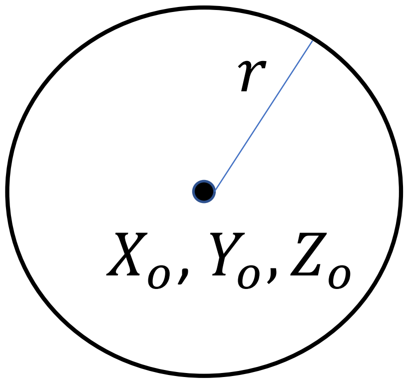
two planes

Example: the sphere

First in cartesian coordinates:

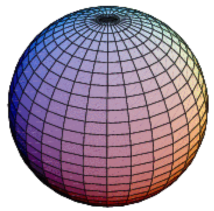
$$(X - X_o)^2 + (Y - Y_o)^2 + (Z - Z_o)^2 - r^2 = 0$$

X_o, Y_o, Z_o are the center coordinates, r is the radius.



… then in homogeneous coordinates:

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & -X_o \\ 0 & 1 & 0 & -Y_o \\ 0 & 0 & 1 & -Z_o \\ -X_o & -Y_o & -Z_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix}}_{} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

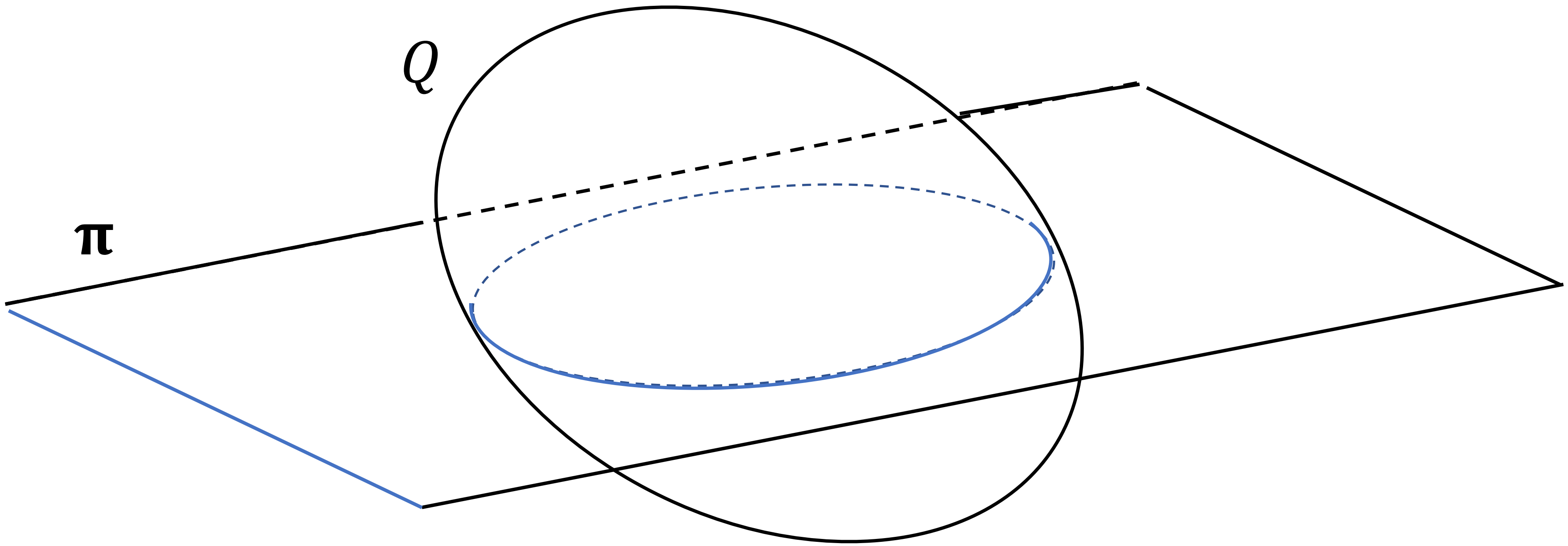


$$\mathbf{X}^{\intercal} \; \boldsymbol{Q} \; \mathbf{X} = 0$$

Intersection of a quadric and a plane

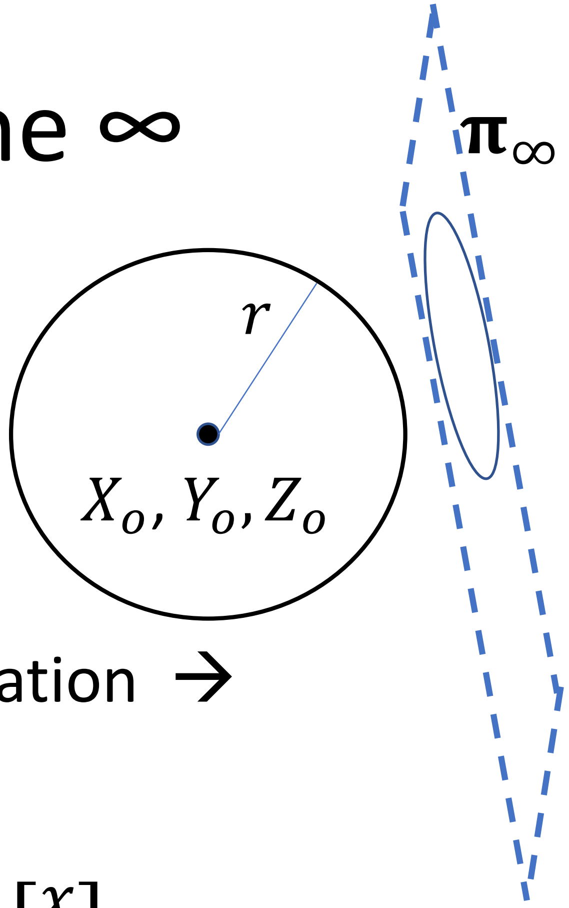
plane – quadric intersection: quadratic equation → a conic

.



The **absolute conic**: an extension of the circular points

A noteworthy example: intersecting a sphere and the plane at the ∞



$$\begin{cases} (x - X_o w)^2 + (y - Y_o w)^2 + (z - Z_o w)^2 - r^2 w^2 = 0 \\ w = 0 \end{cases}$$

$$\rightarrow$$

$$\begin{cases} x^2 + y^2 + z^2 = 0 \\ w = 0 \end{cases}$$

The sphere parameters (center and radius) disappear from the equation → the intersection **conic** is the **same for all** spheres:

$$x^2 + y^2 + z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \Omega_\infty \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

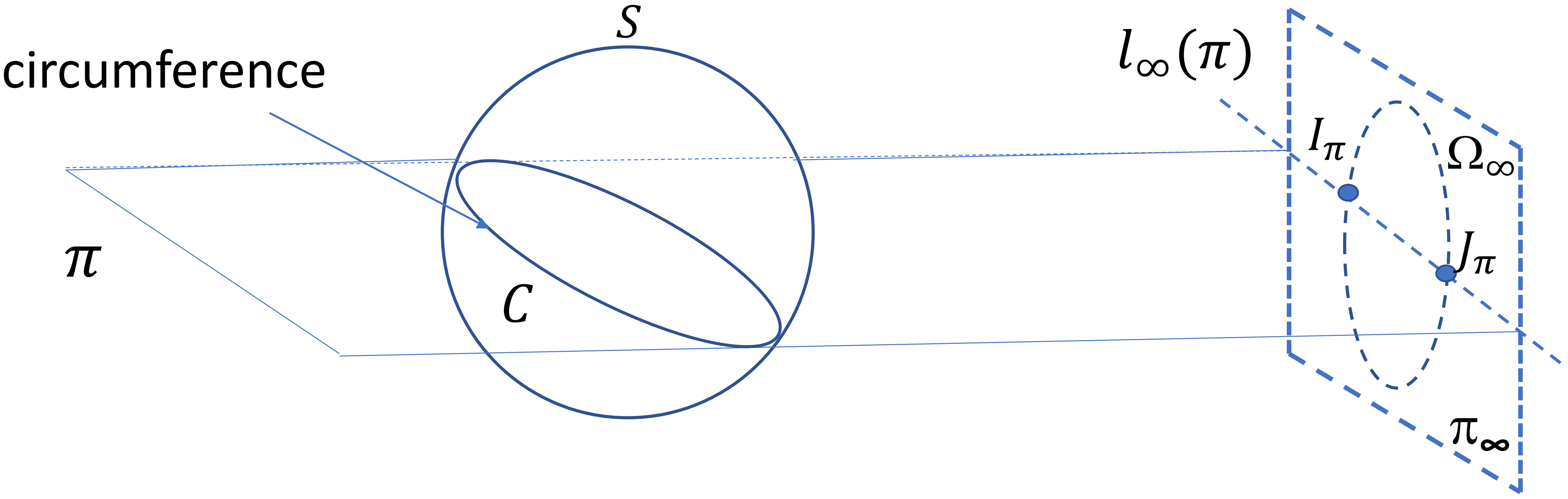
A conic within $\boldsymbol{\pi}_\infty$: $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **ABSOLUTE CONIC**

Property:

The absolute conic Ω_∞ contains the circular points I_π, J_π of any plane π

Proof.

Cutting sphere $\cap\ \boldsymbol{\pi}_\infty = \Omega_\infty$ with plane $\boldsymbol{\pi}$ gives circle $\cap\ \boldsymbol{l}_\infty(\boldsymbol{\pi}) = \{I_\pi, J_\pi\} \subset \Omega_\infty$

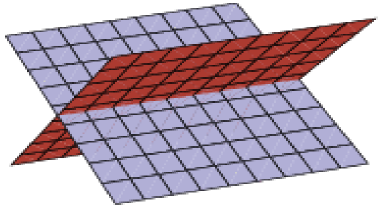


Degenerate Quadrics

DEGENERATE QUADRICS

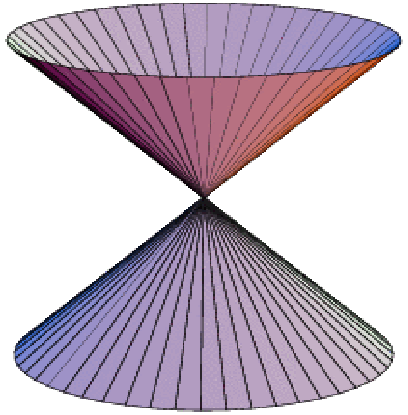
$$\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{1}$ → $\boldsymbol{Q} = \mathbf{A}\mathbf{A}^T$ repeated plane $\mathbf{A}$

- rank $\boldsymbol{Q} = \mathbf{2}$ → $\boldsymbol{Q} = \mathbf{A}\mathbf{B}^T + \mathbf{B}\mathbf{A}^T$  two planes $\mathbf{A}$ and $\mathbf{B}$

- rank $\boldsymbol{Q} = \mathbf{3}$ a **cone**

  vertex = RNS($\boldsymbol{Q}$)

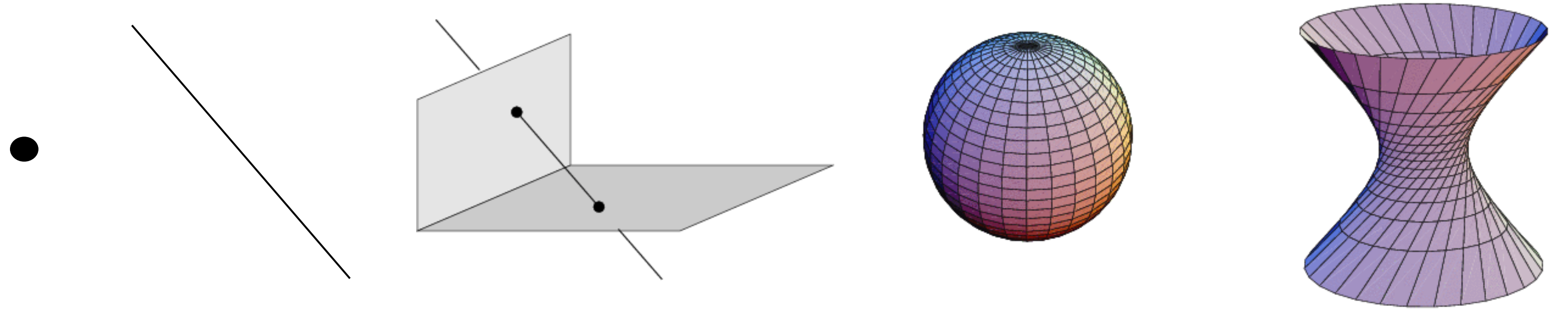


a cylinder is a cone

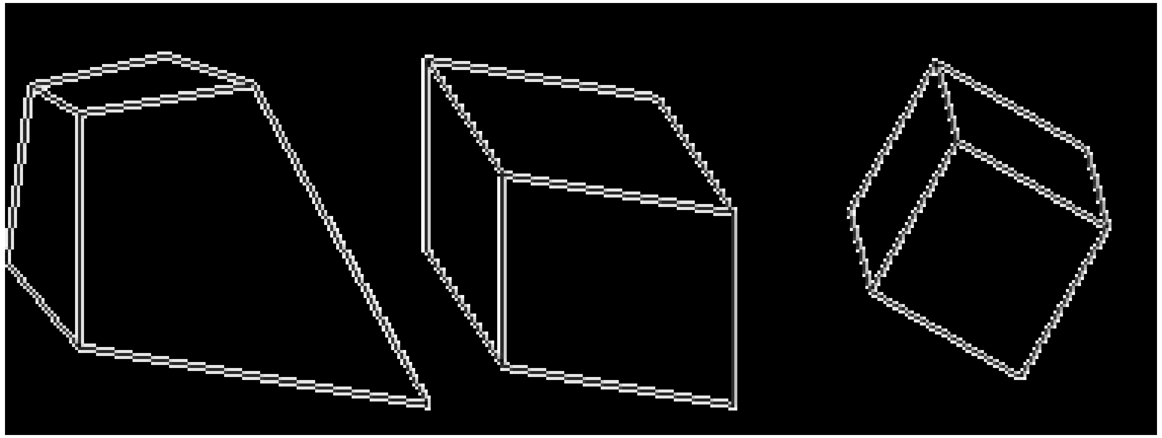
whose vertex is at the infinity

3D Space Projective Geometry

- **Elements**
  - Points
  - Planes
  - **Quadrics**
  - (Dual quadrics)



- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities



Projective 3D Geometry: Projective Transformations

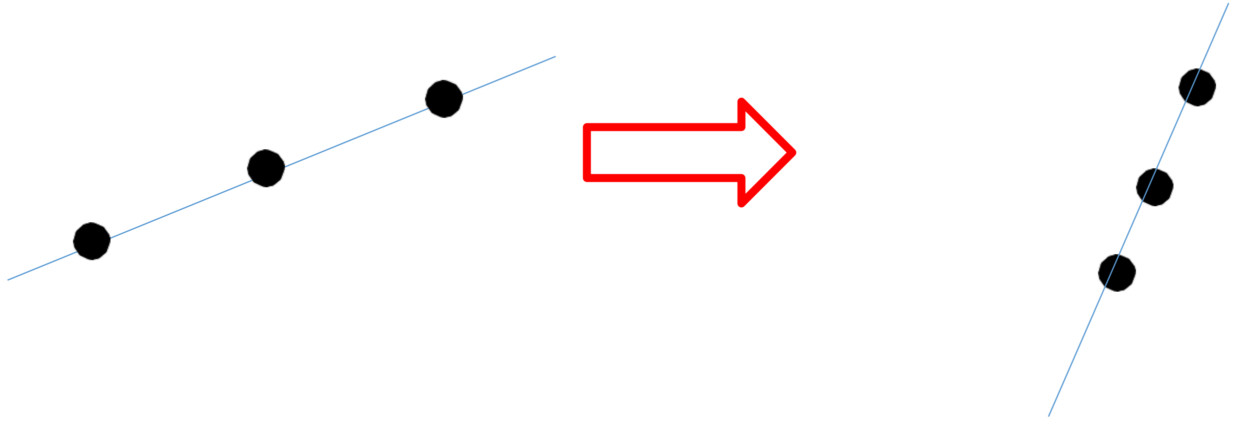
Projective mappings

Def. *A* ***projective mapping*** *between a projective space* $\mathbb{P}^3$ *and an other projective space* $\mathbb{P}'^3$ *is an* ***invertible*** *mapping which preserves colinearity:*

$$h\colon \mathbb{P}^3 \rightarrow \mathbb{P}'^3, \mathbf{X}' = h(\mathbf{X}),\ \mathbf{X}_1,\ \mathbf{X}_2, \mathbf{X}_3 \text{ are colinear}$$

$$\leftrightarrow$$

$$\mathbf{X}'_1 = h(\mathbf{X}_1),\ \mathbf{X}'_2 = h(\mathbf{X}_2), \mathbf{X}'_3 = h(\mathbf{X}_3) \text{ are colinear}$$



Alternative names:

- *Projectivity*

Fundamental Theorem of Projective Geometry

Theorem: *A mapping* h *:* $\mathbb{P}^3 \rightarrow \mathbb{P}'^3$ *is projective* if and only if *there exists an invertible 4x4 matrix* H *such that for any point in* $\mathbb{P}^3$ *represented by the vector* $\mathbf{X}$ *, is* $h(\mathbf{X}) = H\,\boldsymbol{X}$

i.e. projective mappings are LINEAR in the homogeneous coordinates

(they are not linear in cartesian coordinates)

Projectivity: 15 degrees of freedom

From the theorem

$$h(\mathbf{X}) = \mathbf{X}' = H\,\mathbf{X}$$

Therefore, if we multiply the matrix H by any nonzero scalar λ, the relation is satisfied by the same points

$$\mathbf{X}' = \lambda H\,\mathbf{X}$$

Thus any nonzero multiple of the matrix H represents the same projective mapping as H.

Hence H is a homogeneous matrix: in spite of its 16 entries,

H has only 15 degrees of freedom, namely the ratios between its elements.

Transformation of points, planes, quadrics, dual quadrics

Transformation rules for the space elements

A homography transforms **each point** $\boldsymbol{X}$ into a point $\boldsymbol{X}'$ such that:

$$\boldsymbol{X} \to H\boldsymbol{X} = \boldsymbol{X}'$$

A homography transforms **each plane $\boldsymbol{\pi}$** into a plane $\boldsymbol{\pi}'$ such that:

$$\boldsymbol{\pi} \to H^{-T}\boldsymbol{\pi} = \boldsymbol{\pi}'$$

A homography transforms **each quadric** $\boldsymbol{Q}$ into a quadric Q$'$ such that:

$$Q \to H^{-T}QH^{-1} = Q'$$

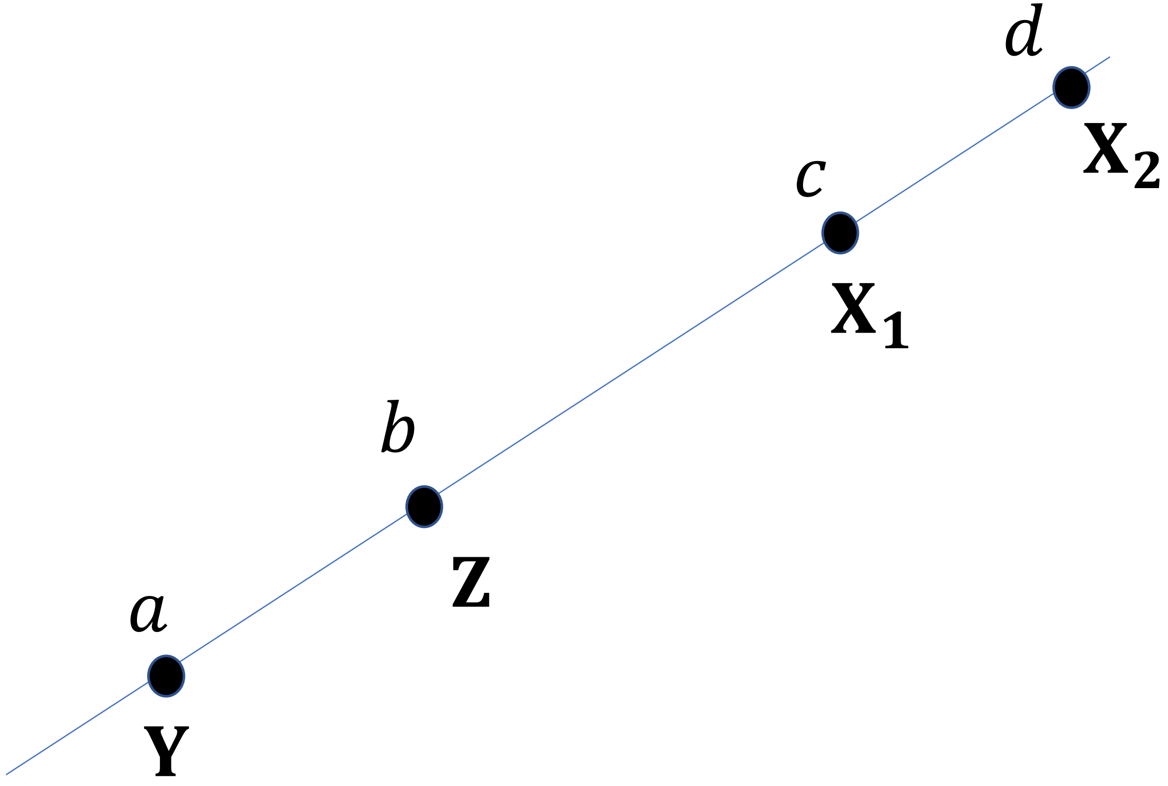
A homography transforms **each dual quadric** Q^* into a dual quadric $Q^{*\prime}$

$$Q^* \to HQ^*H^T = Q^{*\prime}$$

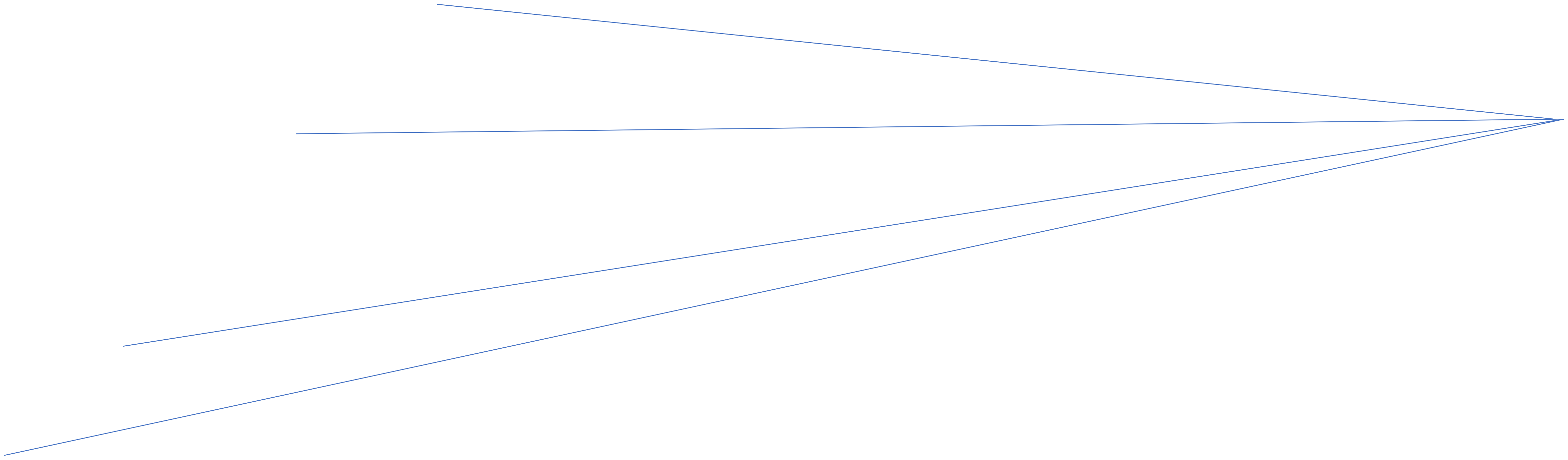
Cross ratios: invariant under projective mappings

1D cross ratio of a 4-tuple of colinear points

$$CR_{Y,Z,\mathrm{X}_1,\mathrm{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$



2D cross ratio of a 4-tuple of coplanar, concurrent lines



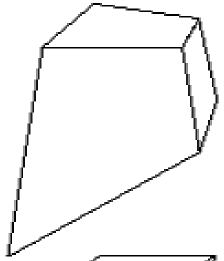
3D cross ratio of a 4-tuple of coaxial planes:

Hierarchy of projective transformations

Hierarchy of transformations

Projective
15dof

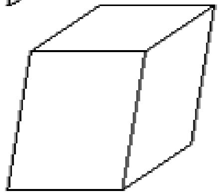
$$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$



Intersection and tangency

Affine
12dof

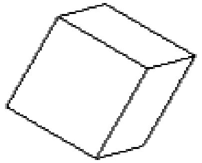
$$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$



Parallellism of planes,
Volume ratios, centroids,
The plane at infinity π_∞

Similarity
7dof

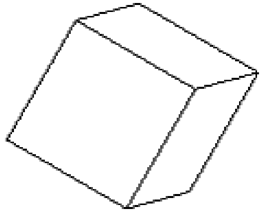
$$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$



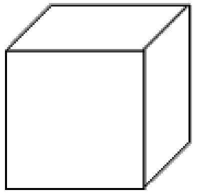
The absolute conic Ω_∞

Euclidean
6dof

$$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$



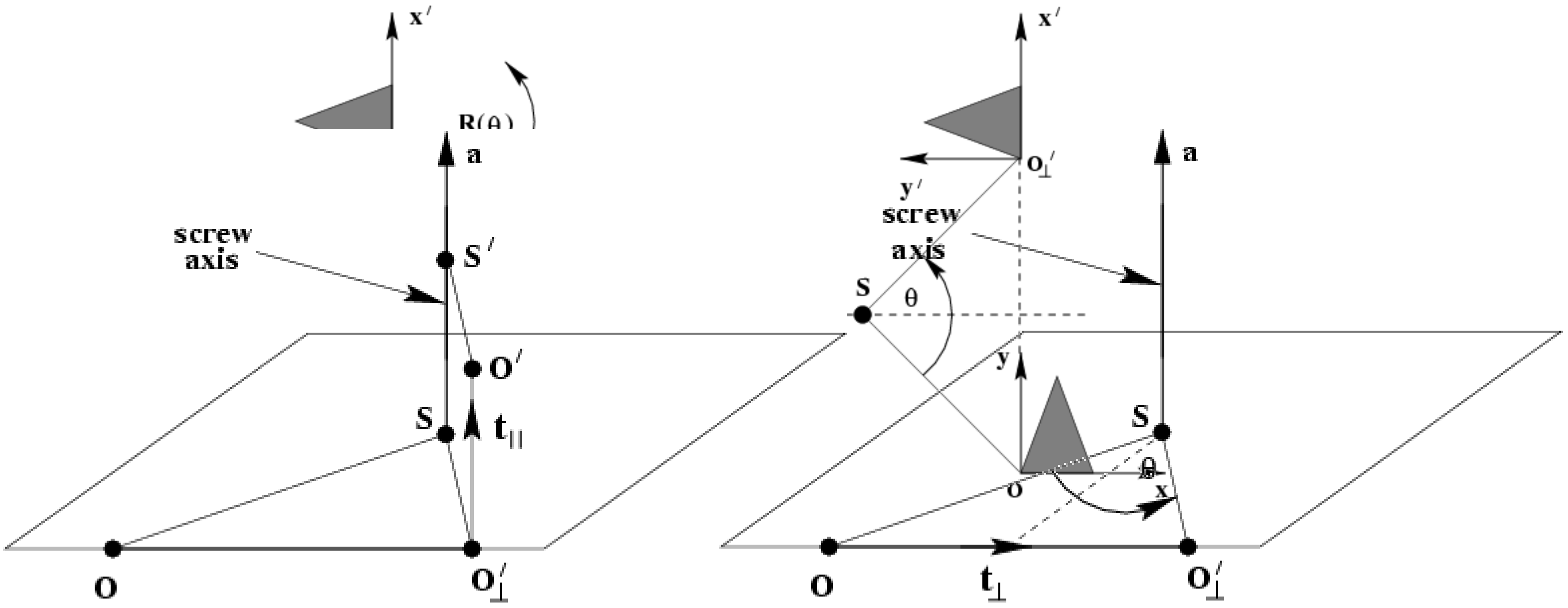
Volume



Caveat

A 3D rototranslation **is not** a pure rotation: screw decomposition

Any rotor-translation is equivalent to a rotation about a screw axis and a translation along the screw axis.



screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

Isometries (or Euclidean mappings)

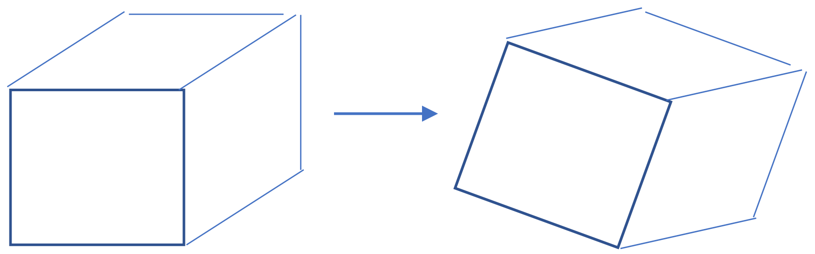
$$H_I = \begin{bmatrix} R_\perp & & & \boldsymbol{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_\perp$ is a 3x3 orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^T$

det ${R_\perp}^{-1} = 1$ planar rigid displacement (-1 for reflection)

6 dofs: translation $\boldsymbol{t}$ + Euler angles $\vartheta, \varphi,$?

Invariants: lenghts, distances, areas → shape and size → relative positions



Similarities

$$H_S = \begin{bmatrix} s\,R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix}$$

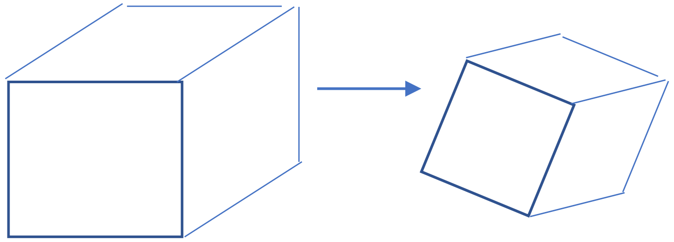
$R_\perp$ is a 3x3 orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^{T}$

7 dofs: rigid displacement + *scale*

Invariants: ratio of lenghts, angles → shape (not size)

the absolute conic Ω_∞

and the absolute dual quadric $\boldsymbol{Q}^*{}_\infty$



Affinities (or affine mappings)

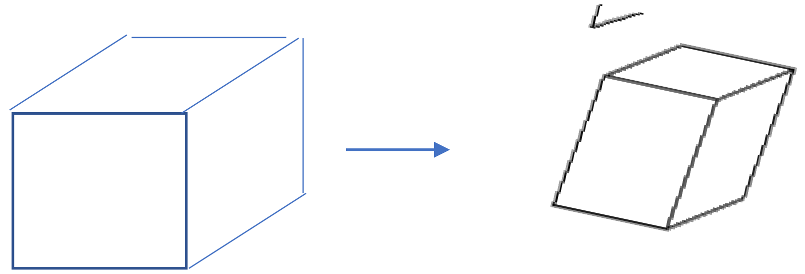
$$H_A = \begin{bmatrix} & A & & \boldsymbol{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A is any 3x3 invertible matrix

12 dofs: $A + \boldsymbol{t}$

Invariants: parallelism, ratio of parallel lenghts, ratio of areas

<u>the plane at the infinity</u> $\boldsymbol{\pi}_\infty$



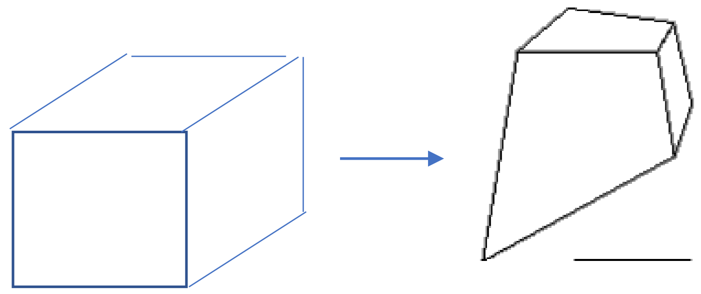
Projectivities (or projective mappings)

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix}$$

A is any 3x3 invertible matrix

15 dofs: $A + \boldsymbol{v} + \boldsymbol{t}$

Invariants: colinearity, incidence, order of contact (crossing, tangency, inflections), <u>the 1D cross ratio, the 2D cross ratio, the 3D cross ratio</u>



3D reconstruction problem formulation

3D reconstruction problem formulation

Unknown original scene = set of points in the 3D space

→

An unknown 3D projective mapping is applied to them

→

Suppose that the transformed **3D points** can be observed

→

From the observed points (different from the original)

recover a model of the original scene

HOW? Images are 2D, not 3D: **several views** (see later)

3D Shape reconstruction:

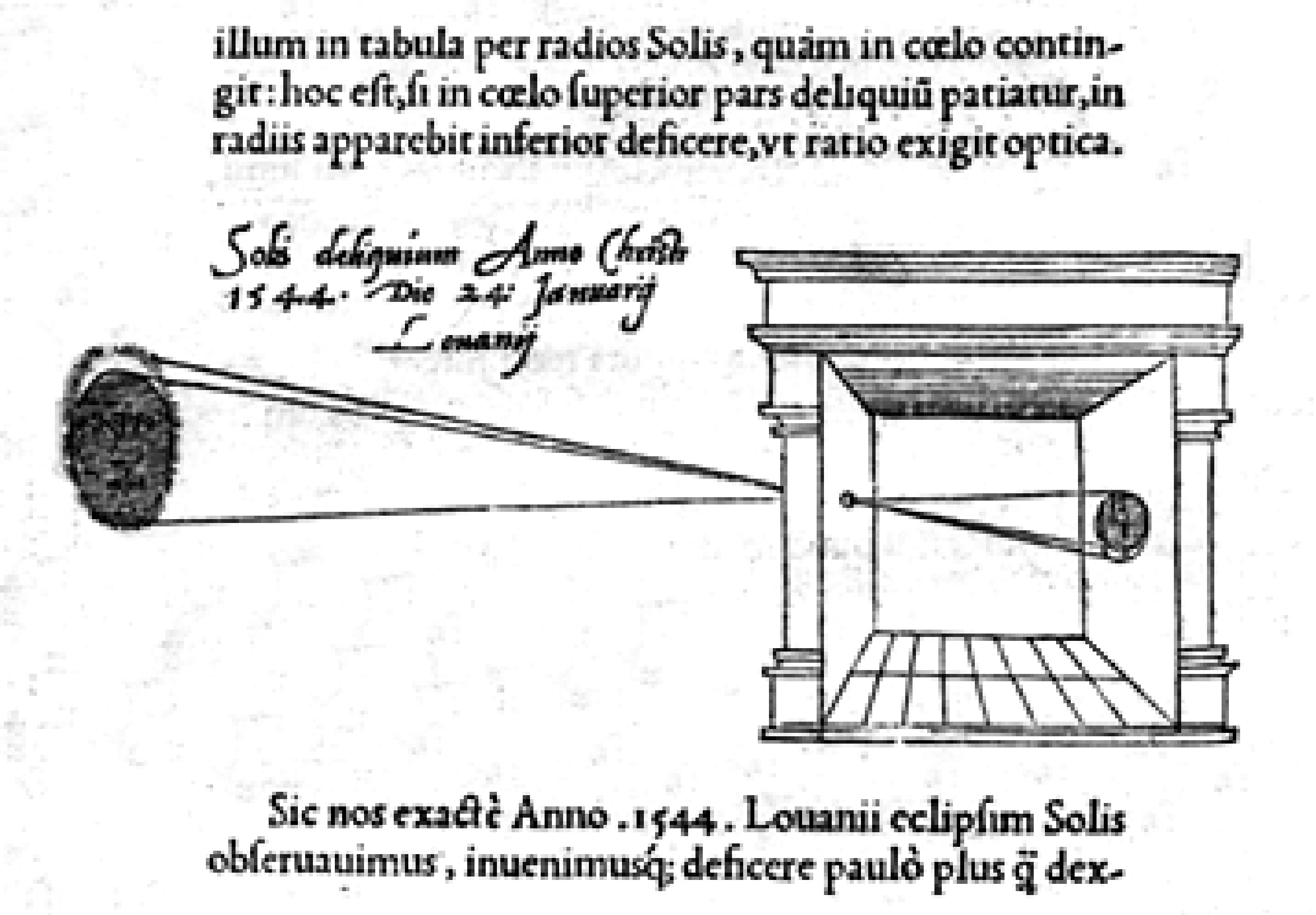
will be studied in Multi-view Geometry

- Introduction and the Camera Optical System
- Planar (2D) Projective Geometry
- Spatial (3D) Projective Geometry
- **Camera Geometry (3D → 2D Projection) and single-view Geometry**

Camera Geometry

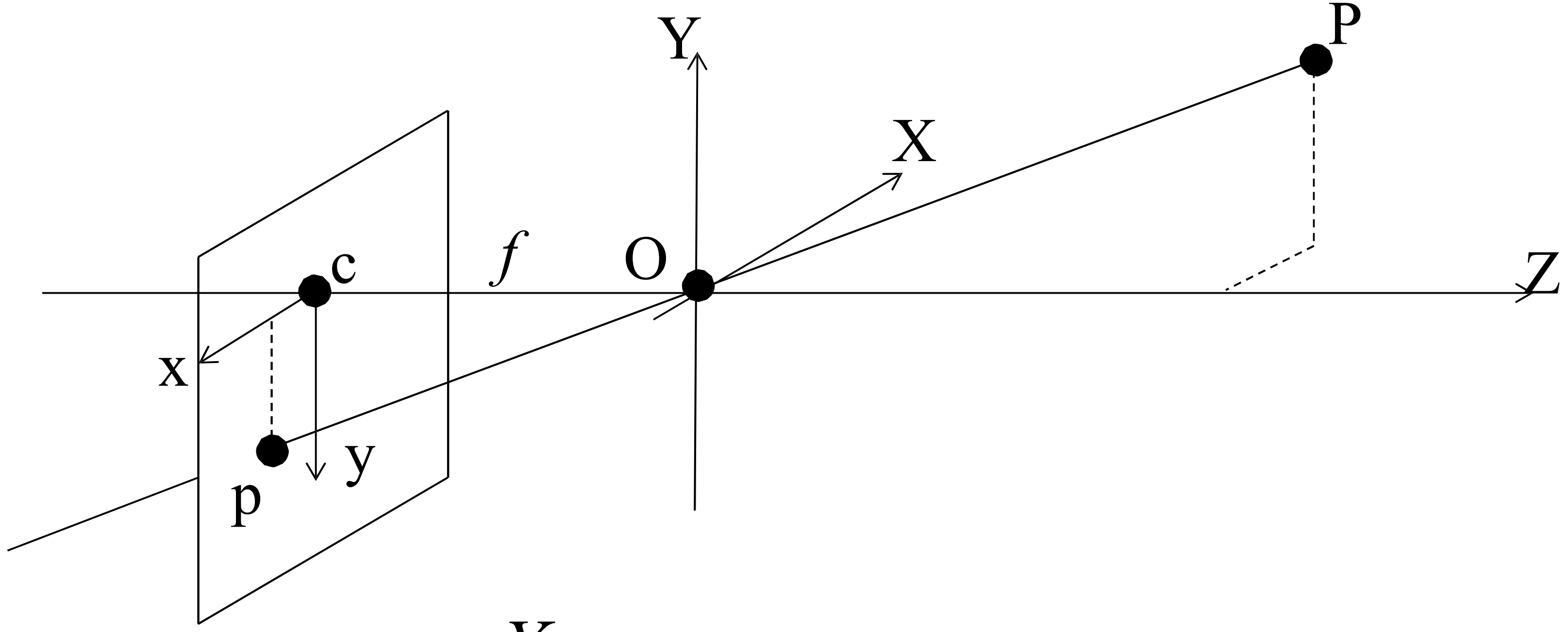
camera projective model

Pinhole camera



illum in tabula per radios Solis, quàm in cœlo contingit: hoc eſt, ſi in cœlo ſuperior pars deliquiū patiatur, in radiis apparebit inferior deficere, vt ratio exigit optica.

Sic nos exactè Anno .1544. Louanii eclipſim Solis obſeruauimus, inuenimusq; deficere paulò plus q̃ dex-



$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

perspective projection

-nonlinear

-not shape-preserving

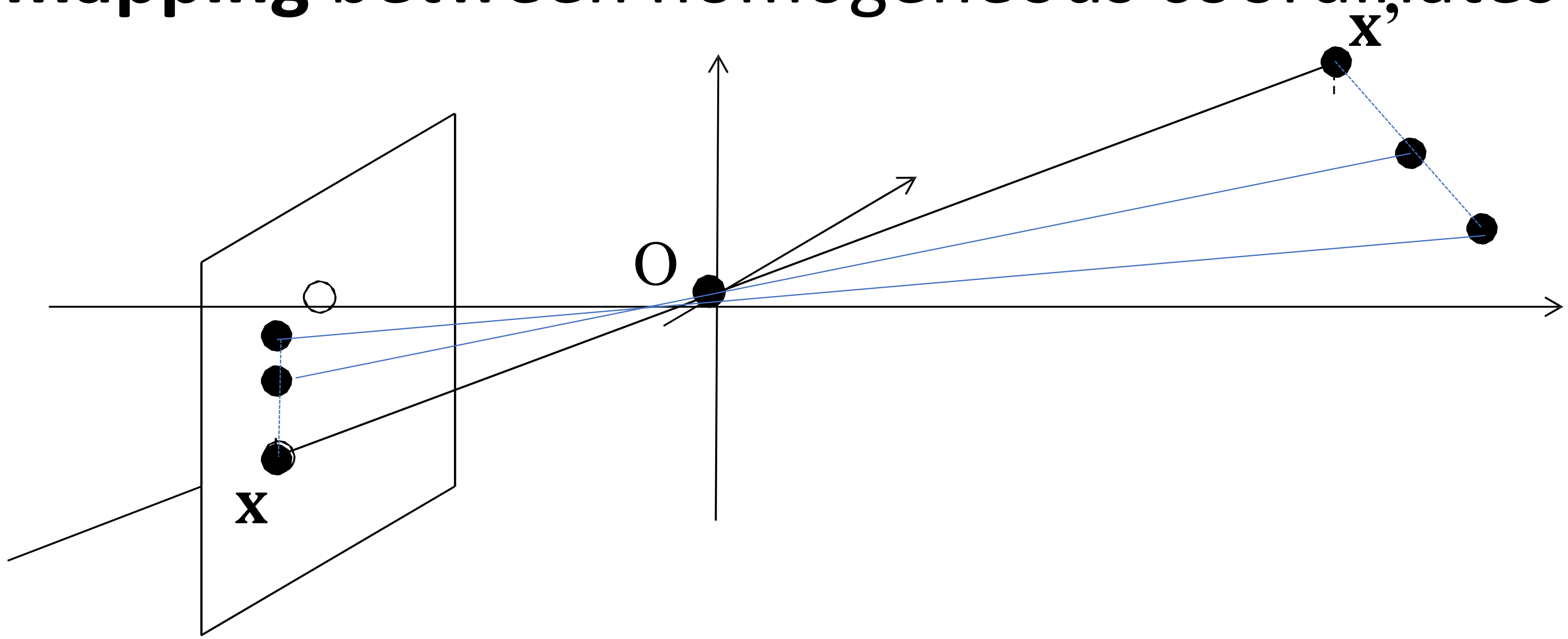
-not length-ratio preserving

Scene-to-image projection

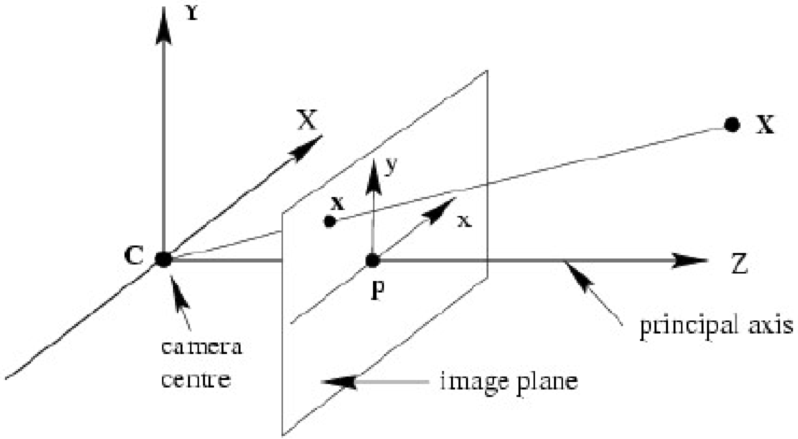
Colinear points are projected onto colinear image points

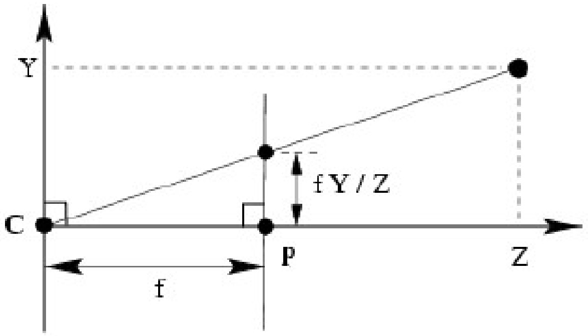
→ colinearity is preserved

→ **linear mapping** between homogeneous coordinates



CAMERA GEOMETRY





colinearity is preserved → linear relation among homogeneous coords

camera projection matrix

$$\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}_{3x4} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \mathbf{P}_{3x4}\mathbf{X} = \left| \; \mathbf{M}_{3x3} \quad \mathbf{m}_{3x1} \; \right| \mathbf{X}$$

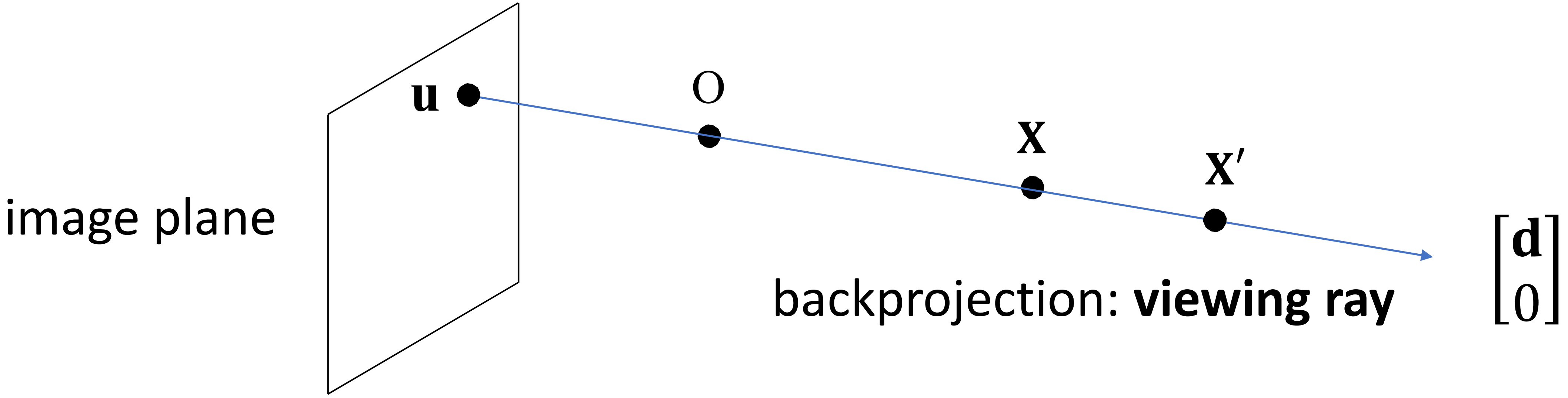
invertible

3D space

image

- **SCENE**
- CAMERA

The viewing ray associated to an image point $\mathbf{u}$



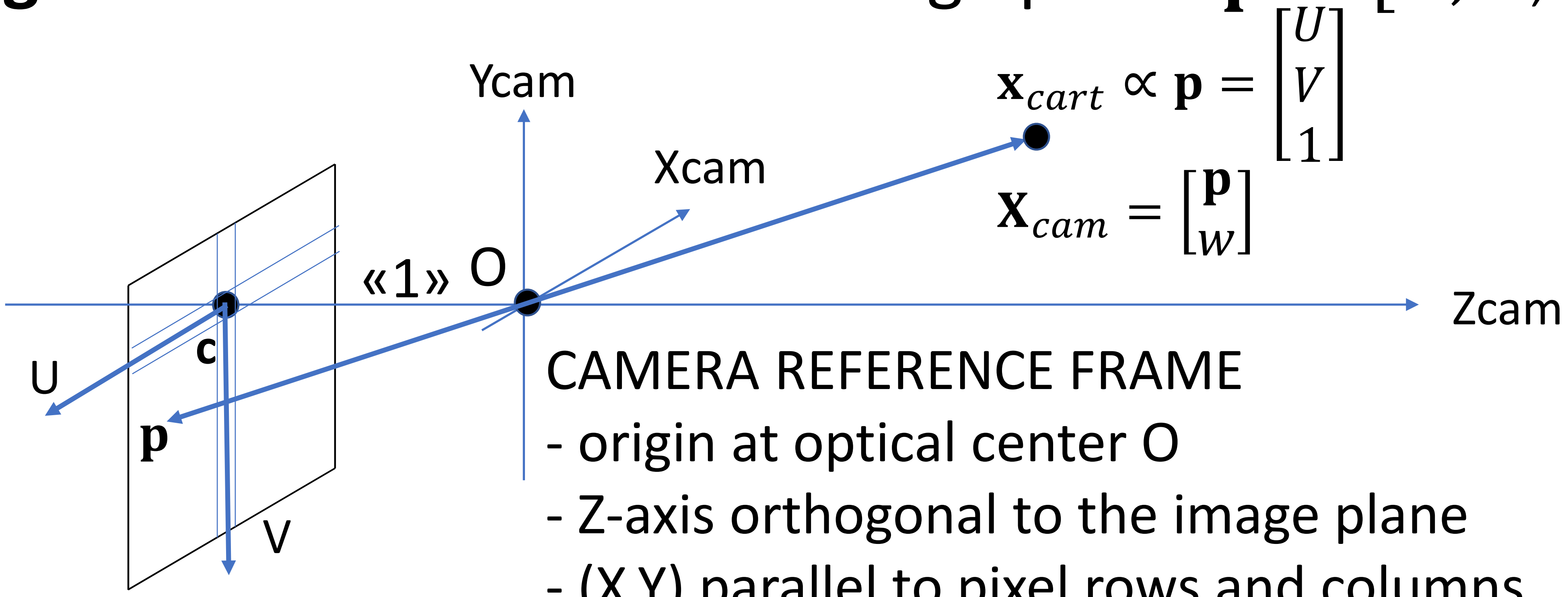
The backprojection of image point $\mathbf{u}$ for the camera $\mathrm{P} = [\mathbf{M} \quad \mathbf{m}]$, is a straight line

- through $\mathrm{O} = \ RNS(\mathrm{P})$ *Proof*: any $\mathbf{X}' = \mathrm{O} + \lambda\mathbf{X}$ projects to $\mathrm{P}\mathbf{X}' = \overset{0}{\mathrm{PO}} + \mathrm{P}\mathbf{X} = \mathrm{P}\mathbf{X}$
- whose direction is $\mathbf{d} = \mathbf{M}^{-1}\mathbf{u}$ *Proof*: $\mathbf{u} = [\mathbf{M} \quad \mathbf{m}]\begin{bmatrix}\mathbf{d}\\0\end{bmatrix} = \mathbf{M}\mathbf{d} \rightarrow \mathbf{d} = \mathbf{M}^{-1}\mathbf{u}$

Where is O ? from $\mathrm{PO} = [\mathbf{M} \quad \mathbf{m}]\begin{bmatrix}\mathbf{o}\\1\end{bmatrix} = 0 \ \rightarrow \ \mathbf{o} = \ -\mathbf{M}^{-1}\mathbf{m}$ cartesian coordinates

- SCENE
- **CAMERA**

homog. **geometric** coordinates of image point: $\mathbf{p} = [U, V, 1]^T$



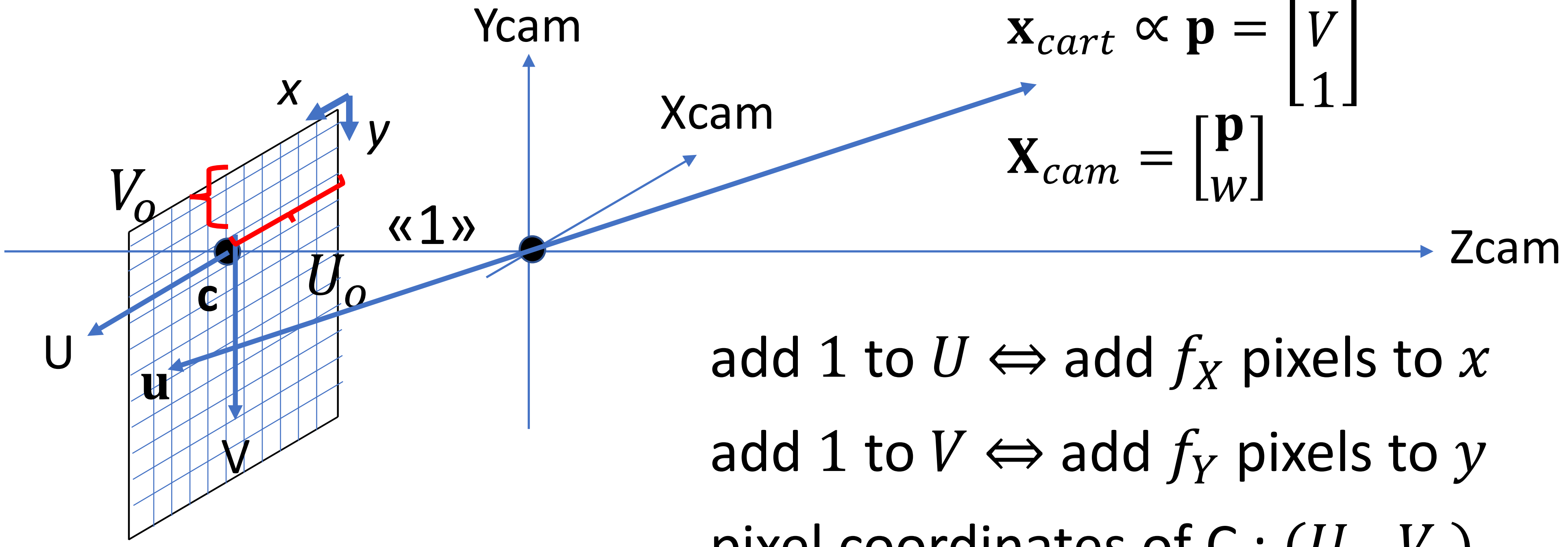
CAMERA REFERENCE FRAME

- origin at optical center O
- Z-axis orthogonal to the image plane
- (X,Y) parallel to pixel rows and columns
- measure unit along X,Y and Z = focal distance f

IMAGE **GEOMETRIC** COORDINATE SYSTEM

- origin at the principal point C
- (U, V) parallel to pixel rows and columns, but opposite to (X, Y)
- measure unit along U and V = focal distance f

homogeneous **pixel** coordinates of image point: $\mathbf{u} = [x, y, 1]^T$



$\mathbf{x}_{cart} \propto \mathbf{p} = \begin{bmatrix} U \\ V \\ 1 \end{bmatrix}$

$\mathbf{X}_{cam} = \begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}$

add 1 to $U \Leftrightarrow$ add f_X pixels to x

add 1 to $V \Leftrightarrow$ add f_Y pixels to y

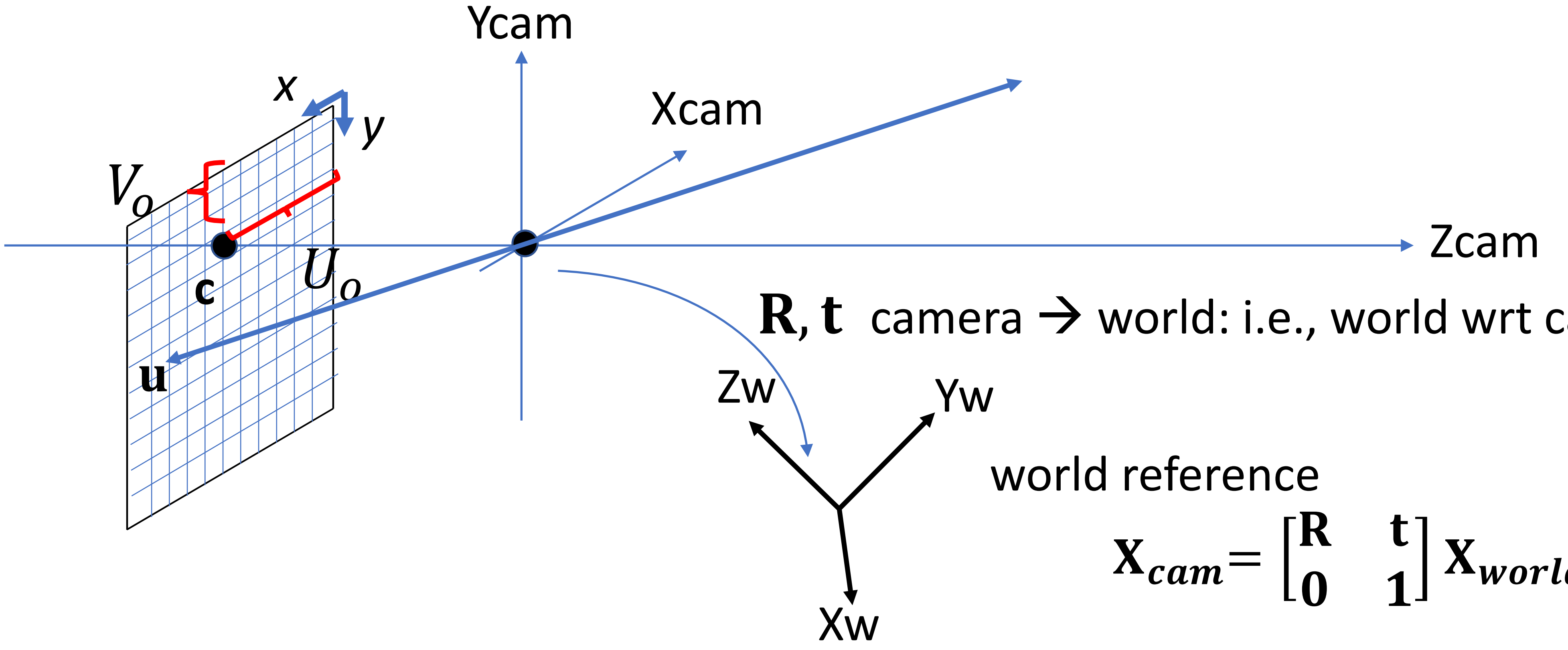
pixel coordinates of C : (U_o, V_o)

pixels: not (always) square

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & U_o \\ 0 & f_y & V_o \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U \\ V \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & 0 & U_o \\ 0 & f_y & V_o \\ 0 & 0 & 1 \end{bmatrix}}_{} \mathbf{p} = [\mathbf{K} \quad \mathbf{0}]\mathbf{X}_{cam} = \mathbf{K}[\mathbf{I} \quad \mathbf{0}]\mathbf{X}_{cam}$$

$\mathbf{K}$: called **calibration matrix**

In general, world reference is ≠ camera reference



$\mathbf{R}, \mathbf{t}$ camera → world: i.e., world wrt camera

$$\mathbf{X}_{cam} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \mathbf{X}_{world}$$

$$\mathbf{u} = \mathbf{K}[\mathbf{I} \quad \mathbf{0}]\mathbf{X}_{cam} = \mathbf{K}[\mathbf{R} \quad \mathbf{t}]\ \mathbf{X}_{world} = [\mathbf{KR} \quad \mathbf{Kt}]\ \mathbf{X}_{world}$$

remember $\mathbf{u} = \quad = \mathbf{P} \quad \mathbf{X}_{world} = [\mathbf{M} \quad \mathbf{m}]\ \mathbf{X}_{world}$

→ $\boxed{\mathbf{M} = \mathbf{KR}}$ and $\boxed{\mathbf{m} = \mathbf{Kt}}$

Calibration matrix

Calibration matrix

$$\mathbf{K} = \begin{bmatrix} f_x & 0 & U_o \\ 0 & f_y & V_o \\ 0 & 0 & 1 \end{bmatrix}$$

Intrinsic parameters: don't vary under camera displacement

- relative position of camera center and image plane
- pixel aspect ratio $a = \frac{f_x}{f_y}$ (for ***natural*** *camera*, square pixels, is $a = 1$)

Camera projection matrix

$$\boldsymbol{P} = [\mathbf{KR} \quad \mathbf{Kt}] = \mathbf{K}[\mathbf{R} \quad \mathbf{t}]$$

Intrinsic parameters + **Extrinsic parameters:**

- relative pose (= position & orientation) of camera reference and world reference

Camera calibration

Intrinsic camera **calibration**:
estimation of matrix $\mathbf{K}$
- focal distance f_X
- focal distance f_Y
- principal point (U_o, V_o)
- skew factor (in old cameras)

aspect ratio $a = f_X/f_Y$:
ratio between pixel width and height

intrinsic camera parameters don't vary
under camera displacement

Extrinsic camera **calibration**:
estimation of matrix $\mathbf{R}$ and vector $\boldsymbol{t}$
- camera→world rotation $\mathbf{R}$
- camera→world translation $\boldsymbol{t}$

extrinsic camera parameters vary
under camera displacement

vanishing points

V image of the point at the ∞ along direction **d**

$$\mathbf{u}_V = \left[\, \mathbf{M} \;\vdots\; \mathbf{m} \,\right] \cdot \begin{bmatrix} \mathbf{d} \\ \cdots \\ 0 \end{bmatrix} = \mathbf{M} \cdot \mathbf{d}$$

Remember: the direction of the backprojection of image point $\mathbf{V}$ (viewing ray associated to $\mathbf{V}$) is $\mathbf{M}^{-1}\mathbf{u}_\mathbf{V} = \mathbf{M}^{-1}\ \mathbf{M}\mathbf{d} = \mathbf{d}$

→

Vanishing Point Theorem:

The viewing ray associated to the vanishing point $\mathbf{V}$ of a direction $\mathbf{d}$ is parallel to $\mathbf{d}$

Navigation based on vanishing point of desired direction

mobile robot navigation

vehicle driving

Find $\mathbf{V}$ and, knowing $\mathbf{M}$, follow the direction $\mathbf{d} = \mathbf{M}^{-1}\mathbf{V}$ of the corridor

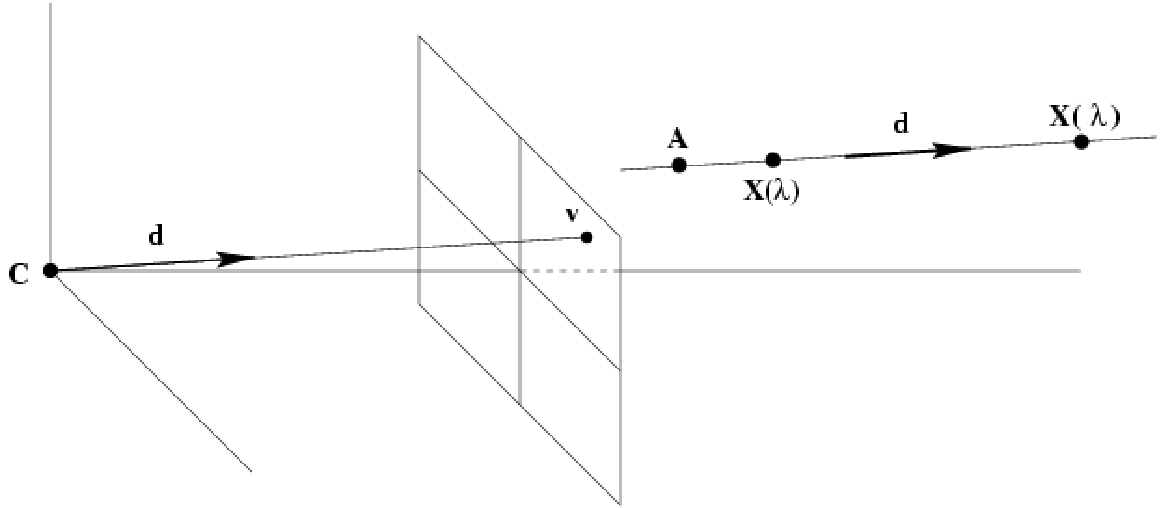
$\mathbf{M} = \mathbf{KR}$ can be estimated through extrinsic + intrinsic calibration

Vanishing points (d measured wrt camera reference)

$$\mathrm{x}(\lambda) = \mathrm{PX}(\lambda) = \mathrm{PA} + \lambda\mathrm{PD} = \mathrm{a} + \lambda\mathrm{Kd}$$

$$\mathrm{v} = \lim_{\lambda\to\infty} \mathrm{x}(\lambda) = \lim_{\lambda\to\infty}(\mathrm{a} + \lambda\mathrm{Kd}) = \mathrm{Kd}$$

$$\mathrm{v} = \mathrm{PX}_\infty = \mathrm{Kd}$$



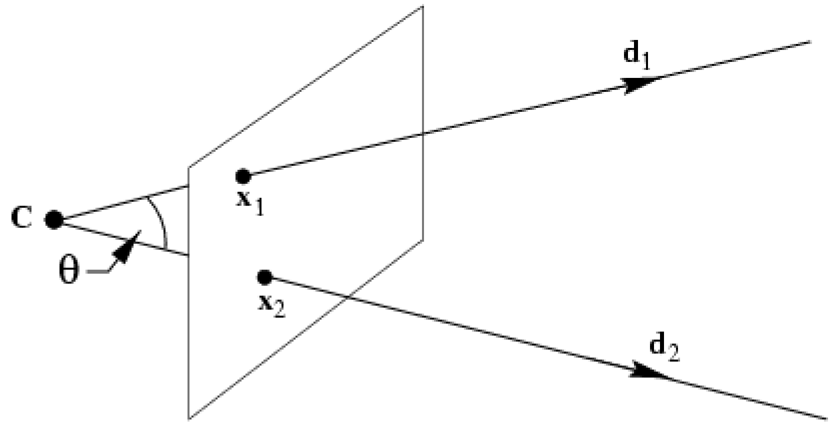
Vanishing points (d measured wrt world reference)

$$\mathrm{v} = \mathrm{PX}_\infty = \mathrm{KRd}_w$$

What does INTRINSIC calibration give?

$$u = K[I|0]\begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$d = K^{-1}u$$



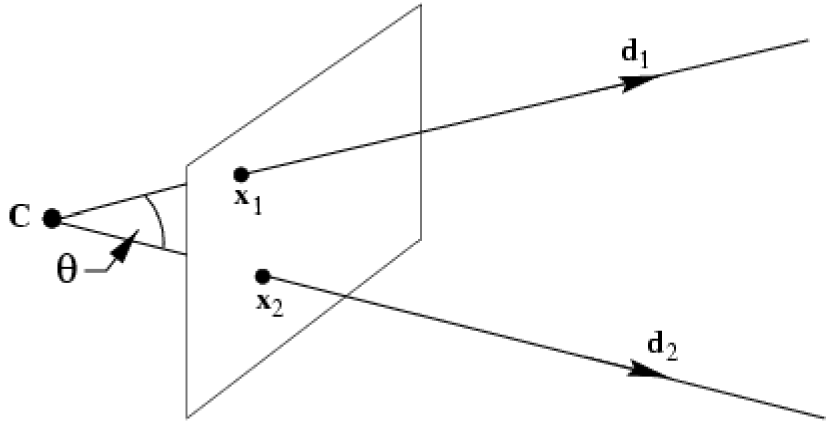
$$\cos\theta = \frac{{d_1}^T d_2}{\sqrt{({d_1}^T d_1)({d_2}^T d_2)}} = \frac{{u_1}^T (K^{-T}K^{-1}) u_2}{\sqrt{({u_1}^T (K^{-T}K^{-1}) u_1)({u_2}^T (K^{-T}K^{-1}) u_2)}}$$

These angles don't depend on the absolute position of the camera

→ Relative position of viewing rays associated to different image points

The ω matrix (sometimes called the IAC)

Define $\omega \triangleq (\mathbf{KK}^T)^{-1}$ (ω is a symmetric matrix)



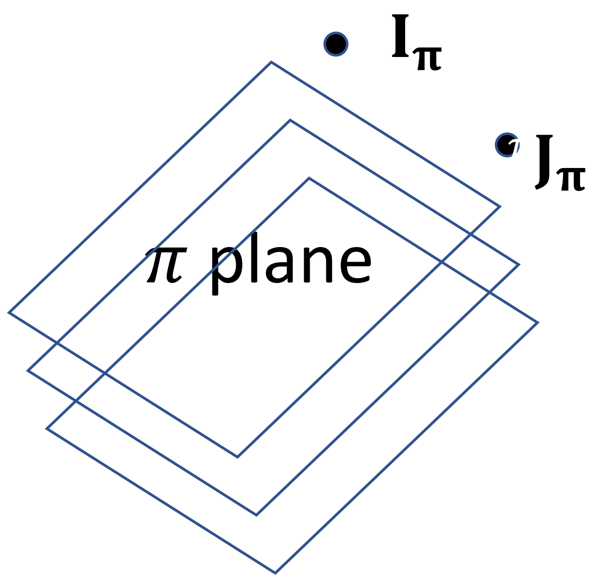
Property: The directions d_1 and d_2 of the viewing rays of two image points u_1 and u_2 form an angle given by

$$\cos\theta = \frac{\mathrm{u}_1{}^{\mathrm{T}}\omega\,\mathrm{u}_2}{\sqrt{(\mathrm{u}_1{}^{\mathrm{T}}\omega\,\mathrm{u}_1)(\mathrm{u}_2{}^{\mathrm{T}}\omega\mathrm{u}_2)}}$$

Useful properties

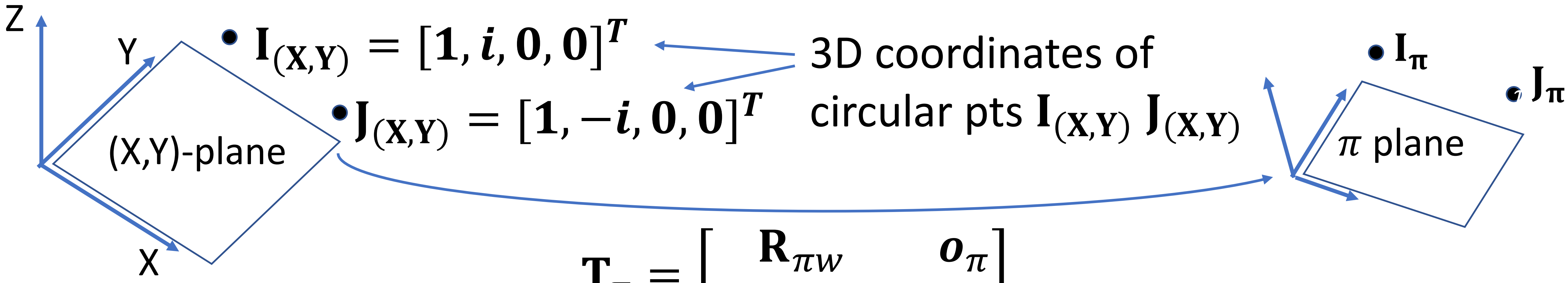
The circular points in the space

The circular points in the space



- Each plane π has its own pair of circular points $\mathbf{I_\pi}, \mathbf{J_\pi}$
- Parallel planes share the same circular points

The circular points $I = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$ and $J = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$ in the space



$\mathbf{I}_{(\mathbf{X},\mathbf{Y})} = [\mathbf{1}, \boldsymbol{i}, \mathbf{0}, \mathbf{0}]^T$

$\mathbf{J}_{(\mathbf{X},\mathbf{Y})} = [\mathbf{1}, -\boldsymbol{i}, \mathbf{0}, \mathbf{0}]^T$

3D coordinates of circular pts $\mathbf{I}_{(\mathbf{X},\mathbf{Y})}$ $\mathbf{J}_{(\mathbf{X},\mathbf{Y})}$

$$\mathbf{T}_{\pi} = \begin{bmatrix} & \mathbf{R}_{\pi w} & & \boldsymbol{o}_{\pi} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Generic plane π obtained by rototranslating the (X,Y)-plane with isometry $\mathbf{T}_{\pi}$

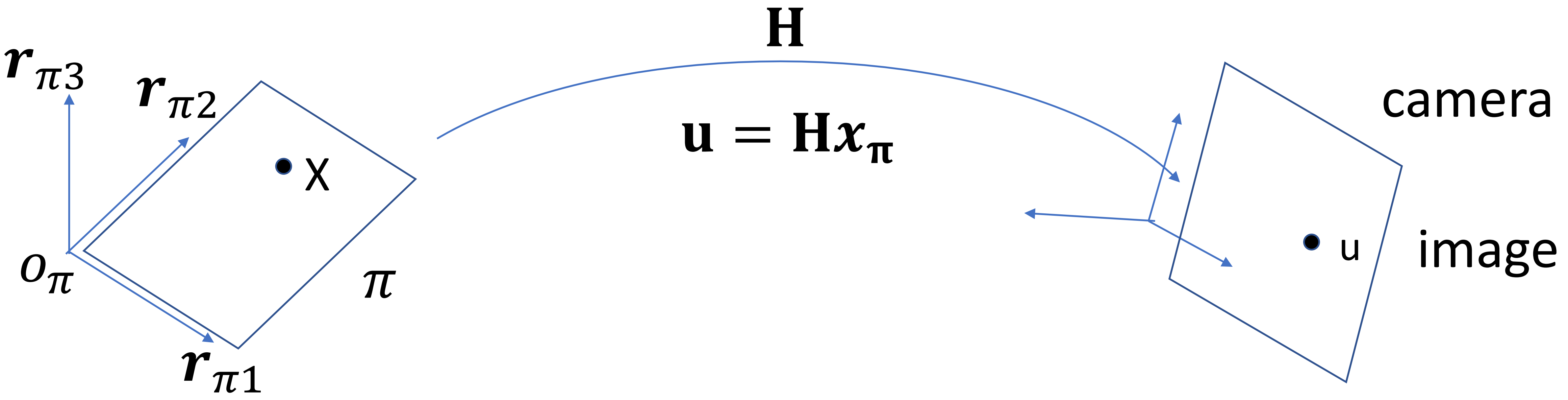
→ 3D coordinates of $\mathbf{I}_{\boldsymbol{\pi}}, \mathbf{J}_{\boldsymbol{\pi}}$: $\mathbf{I}_{\boldsymbol{\pi}} = \mathbf{T}_{\pi}\, \mathbf{I}_{(\mathbf{X},\mathbf{Y})} = \begin{bmatrix} \mathbf{R}_{\pi w} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}$ $\quad \mathbf{J}_{\boldsymbol{\pi}} = \mathbf{T}_{\pi}\, \mathbf{J}_{(\mathbf{X},\mathbf{Y})} = \begin{bmatrix} \mathbf{R}_{\pi w} \boldsymbol{J} \\ \mathbf{0} \end{bmatrix}$

→ 2D coordinates of circular points of π within π_{∞}: $\boldsymbol{I}_{\boldsymbol{\pi}} = \mathbf{R}_{\pi w} \boldsymbol{I},\ \boldsymbol{J}_{\boldsymbol{\pi}} = \mathbf{R}_{\pi w} \boldsymbol{J}$

Useful properties

The image of a planar scene:
Homography between a plane π and its image

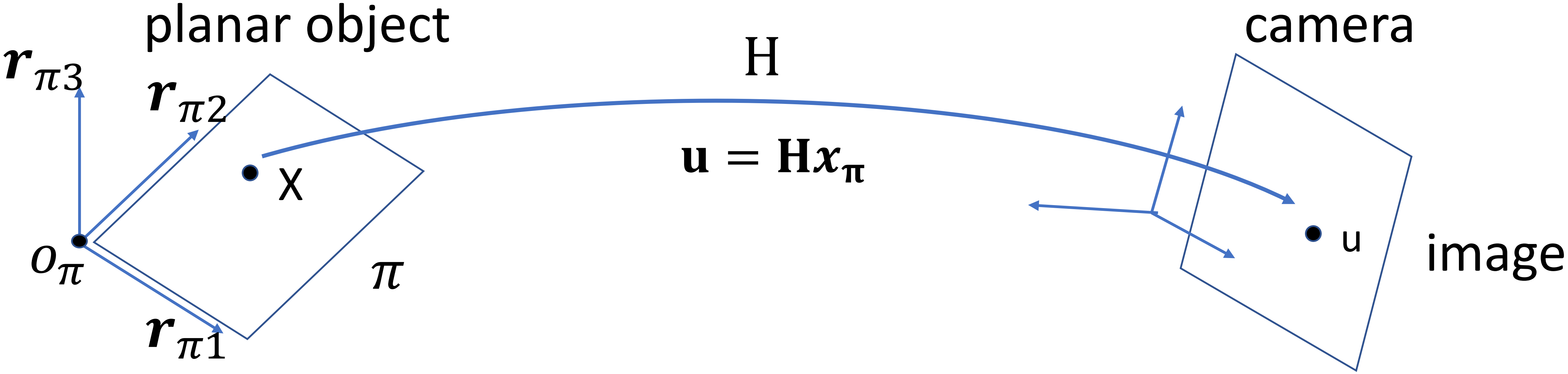
Homography between a plane π and its image



plane π reference: relative pose plane wrt camera → rototranslation $\mathbf{R}_\pi$, $\boldsymbol{o}_\pi$

world reference ≡ camera reference

$$\mathrm{P} = [\mathbf{K} \quad \mathbf{0}]$$



plane reference

world reference ≡ camera reference → $\mathrm{P} = [\mathbf{K} \quad \mathbf{0}]$

$$\mathbf{X}_{\boldsymbol{\pi}} = \begin{bmatrix} x \\ y \\ 0 \\ w \end{bmatrix} \qquad \mathbf{X}_{\mathbf{w}} = \begin{bmatrix} \boldsymbol{r}_{\boldsymbol{\pi}\mathbf{1}} & \boldsymbol{r}_{\boldsymbol{\pi}\mathbf{2}} & \boldsymbol{r}_{\boldsymbol{\pi}\mathbf{3}} & \mathbf{o}_{\boldsymbol{\pi}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_{\pi 1} & \boldsymbol{r}_{\pi 2} & \mathbf{o}_{\pi} \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ w \end{bmatrix}}_{\boldsymbol{x}_{\boldsymbol{\pi}}}$$

$$\mathbf{u} = \mathbf{P}\mathbf{X}_{\mathbf{w}} = [\mathbf{K} \quad \mathbf{0}] \begin{bmatrix} \boldsymbol{r}_{\pi 1} & \boldsymbol{r}_{\pi 2} & \mathbf{o}_{\pi} \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{\boldsymbol{\pi}} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}]\boldsymbol{x}_{\boldsymbol{\pi}}$$

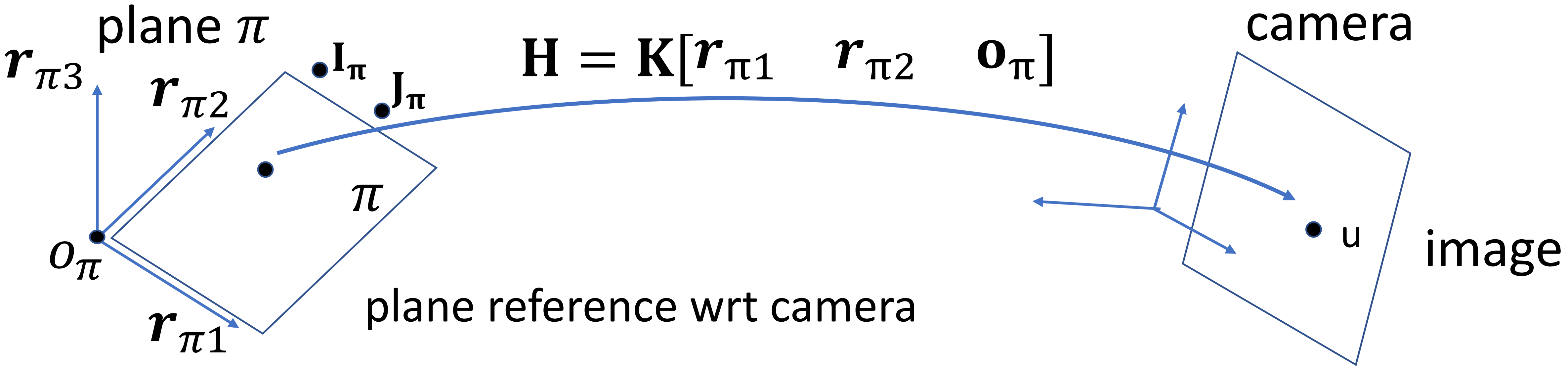
plane-to-image homography $\mathbf{H} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}]$

The image of a planar scene: Homography between a plane π and its image

Example:

- Image of the circular points of plane π

$$\mathbf{R}_\pi = [\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \boldsymbol{r}_{\pi 3}] : \text{rotation of plane } \pi \text{ wrt camera}$$



$$(\boldsymbol{I'}_\pi, \boldsymbol{J'}_\pi) = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_\pi]\left(\begin{bmatrix}\mathbf{1}\\ \boldsymbol{i}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\ -\boldsymbol{i}\\ \mathbf{0}\end{bmatrix}\right) = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \boldsymbol{r}_{\pi 3}]\left(\begin{bmatrix}\mathbf{1}\\ \boldsymbol{i}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{1}\\ -\boldsymbol{i}\\ \mathbf{0}\end{bmatrix}\right)$$

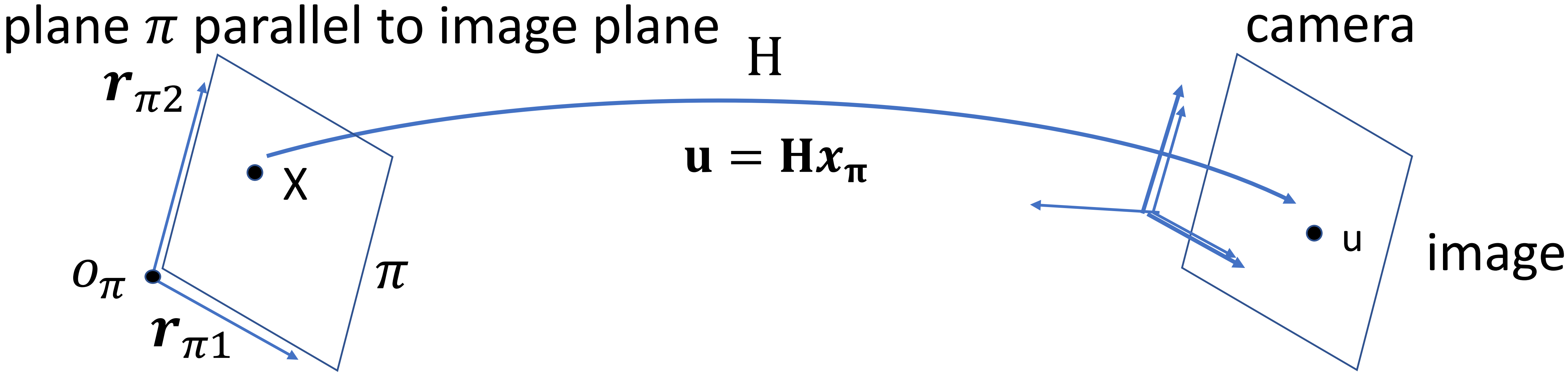
$$\rightarrow$$

$$\boxed{\boldsymbol{I'}_\pi = \mathbf{K}\mathbf{R}_\pi \boldsymbol{I}, \boldsymbol{J'}_\pi = \mathbf{K}\mathbf{R}_\pi \boldsymbol{J}}$$

The image of a planar scene:
Homography between a plane π and its image

Particular case:

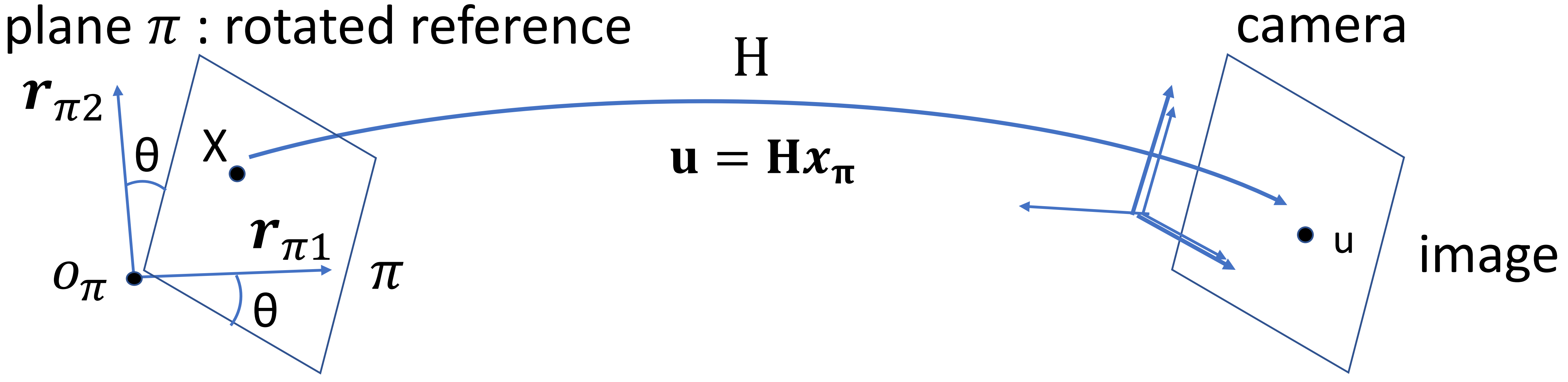
- plane π parallel to the image plane



$$\boldsymbol{r}_{\boldsymbol{\pi}\mathbf{1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \boldsymbol{r}_{\boldsymbol{\pi}\mathbf{2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow [\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad o_\pi] = \begin{bmatrix} 1 & 0 & X_o \\ 0 & 1 & Y_o \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_\pi]\boldsymbol{x}_{\boldsymbol{\pi}} = \mathbf{K}\begin{bmatrix} 1 & 0 & X_o \\ 0 & 1 & Y_o \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{x}_{\boldsymbol{\pi}} = \begin{bmatrix} f_x & 0 & U_o \\ 0 & f_y & V_o \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & X_o \\ 0 & 1 & Y_o \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{x}_{\boldsymbol{\pi}}$$

plane-to-image homography $\mathbf{H}$ = 2D-affine * 2D-isometry = 2-D affinity



$$\boldsymbol{r}_{\boldsymbol{\pi}\mathbf{1}} = \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}, \ \boldsymbol{r}_{\boldsymbol{\pi}\mathbf{2}} = \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix} \rightarrow [\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad o_\pi] = \begin{bmatrix} & R & X_o\\ & & Y_o\\ 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{u} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_\pi]\boldsymbol{x}_{\boldsymbol{\pi}} = \mathbf{K}\begin{bmatrix} & R & X_o\\ & & Y_o\\ 0 & 0 & 1\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\pi}} = \begin{bmatrix} f_x & 0 & U_o\\ 0 & f_y & V_o\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} & R & X_o\\ & & Y_o\\ 0 & 0 & 1\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\pi}}$$

plane-to-image homography $\mathbf{H}$ = 2D-affine * 2D-isometry = 2-D affinity

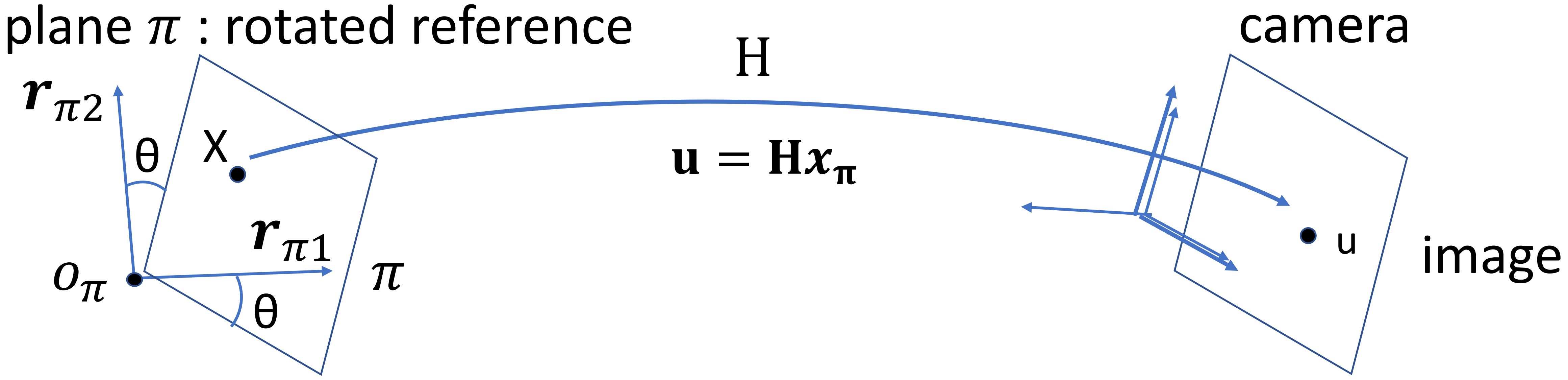
The image of a planar scene:
Homography between a plane π and its image

Particular case:

- plane π parallel to the image plane

AND

- natural camera, i.e., square pixels $f_x = f_y = f$



$$\boldsymbol{r}_{\boldsymbol{\pi}\mathbf{1}} = \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix},\ \boldsymbol{r}_{\boldsymbol{\pi}\mathbf{2}} = \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix} \rightarrow [\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad o_{\pi}] = \begin{bmatrix} & R & & X_o\\ & & & Y_o\\ 0 & & 0 & 1\end{bmatrix}$$

$$\mathbf{u} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}]\boldsymbol{x}_{\boldsymbol{\pi}} = \mathbf{K}\begin{bmatrix} & R & & X_o\\ & & & Y_o\\ 0 & & 0 & 1\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\pi}} = \begin{bmatrix} f & 0 & U_o\\ 0 & f & V_o\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} & R & & X_o\\ & & & Y_o\\ 0 & & 0 & 1\end{bmatrix}\boldsymbol{x}_{\boldsymbol{\pi}}$$

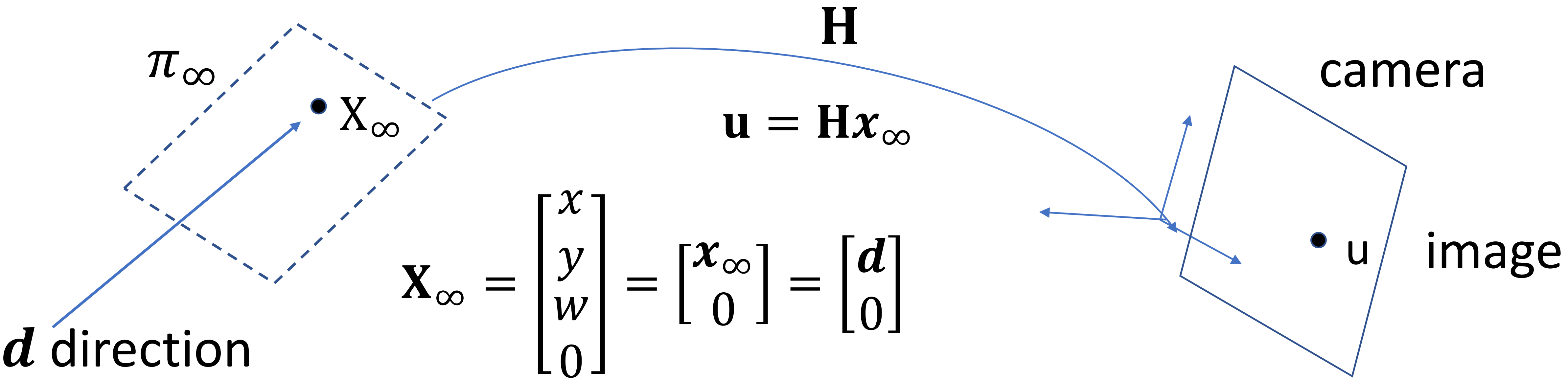
plane-to-image homography $\mathbf{H}$ = 2D-affine * 2D-isometry = 2D-isometry

The image of a planar scene:
Homography between a plane π and its image

Particular case:

- plane π at the infinity: $\pi = \pi_\infty$

Homography $\mathbf{H}$ between the **plane at the infinity $\boldsymbol{\pi}_\infty$** and its image



$$\mathbf{u} = \mathbf{H}\boldsymbol{x}_\infty$$

$$\mathbf{X}_\infty = \begin{bmatrix} x \\ y \\ w \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_\infty \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{d} \\ 0 \end{bmatrix}$$

$$\mathbf{P} = [\mathbf{M} \quad \boldsymbol{m}] = [\mathbf{KR} \quad \mathbf{K}\boldsymbol{t}]$$

camera → world

$$\mathrm{u} = \mathbf{P}\mathbf{X}_\infty = [\mathbf{M} \quad \boldsymbol{m}] \begin{bmatrix} \boldsymbol{x}_\infty \\ 0 \end{bmatrix} = \mathbf{M}\boldsymbol{x}_\infty = \mathbf{KR}\,\boldsymbol{x}_\infty$$

$\boldsymbol{\pi}_\infty$ **to image homography :** $\mathbf{H} = \mathbf{M} = \mathbf{KR} = \mathbf{K}[\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \boldsymbol{r}_3]$

Example: The image of the circular points of a plane π parallel to the image plane

Since π is parallel to the image plane, the relative rotation $\mathbf{R}_\pi$ is the Identity matrix →

$$\boldsymbol{I}'_{\boldsymbol{\pi}} = \mathbf{K}\mathbf{R}_\pi \boldsymbol{I} = \mathbf{K}\boldsymbol{I} = \begin{bmatrix} f_x & 0 & U_o \\ 0 & f_y & V_o \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \boldsymbol{i} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} f_x \\ \boldsymbol{i} f_y \\ \mathbf{0} \end{bmatrix}$$

similarly, $\boldsymbol{J}'_{\boldsymbol{\pi}} = \begin{bmatrix} f_x \\ -\boldsymbol{i} f_y \\ \mathbf{0} \end{bmatrix}$

remember $\omega \triangleq (\mathbf{KK}^T)^{-1}$ (ω is a symmetric matrix)

From $\boldsymbol{\omega}$ to calibration matrix $\mathbf{K}$: Cholesky factorisation of inverse $\boldsymbol{\omega}^{-1} = \mathbf{KK}^{\mathrm{T}}$

remember **Property 1**:

for any plane π, $\boldsymbol{I'}_{\boldsymbol{\pi}} = \mathbf{KR}_{\pi}\, \boldsymbol{I}$ and $\boldsymbol{J'}_{\boldsymbol{\pi}} = \mathbf{KR}_{\pi}\, \boldsymbol{J}$

Property 2:

for any plane π, $\boldsymbol{I'}_{\boldsymbol{\pi}}{}^T \omega\, \boldsymbol{I'}_{\boldsymbol{\pi}} = \mathbf{0}$ and $\boldsymbol{J'}_{\boldsymbol{\pi}}{}^T \omega\, \boldsymbol{J'}_{\boldsymbol{\pi}} = \mathbf{0}$

Proof sketch:

for any plane π, $\boldsymbol{I'}_{\boldsymbol{\pi}} = \mathbf{KR}_{\pi}\boldsymbol{I}$ $\rightarrow \boldsymbol{I} = (\mathbf{KR}_{\pi}{}^{-\mathbf{1}}\boldsymbol{I'}_{\boldsymbol{\pi}}$

from self-orthogonality $\boldsymbol{I}^T\boldsymbol{I} = [\mathbf{1} \quad \boldsymbol{i} \quad \mathbf{0}][\mathbf{1} \quad \boldsymbol{i} \quad \mathbf{0}]^{\boldsymbol{T}} = \mathbf{0}$ and thus

$\mathbf{0} = \boldsymbol{I}^T\boldsymbol{I} = \boldsymbol{I'}_{\boldsymbol{\pi}}{}^T(\mathbf{KR}_{\pi})^{-\boldsymbol{T}}(\mathbf{KR}_{\pi})^{-\mathbf{1}}\boldsymbol{I'}_{\boldsymbol{\pi}} = \boldsymbol{I'}_{\boldsymbol{\pi}}{}^T(\mathbf{KK}^{\boldsymbol{T}})^{-\mathbf{1}}\boldsymbol{I'}_{\boldsymbol{\pi}} = \boldsymbol{I'}_{\boldsymbol{\pi}}{}^T\omega\, \boldsymbol{I'}_{\boldsymbol{\pi}} = \mathbf{0}$

From projection matrix P to calibration matrix $\mathbf{K}$

remember

$$\mathrm{P} = [\mathbf{M} \quad \boldsymbol{m}] = [\mathbf{KR} \quad \mathbf{K}\boldsymbol{t}]$$

from $\omega \triangleq \left(\mathbf{K}\mathbf{K}^{T}\right)^{-1}$

$$\mathbf{M}\mathbf{M}^{\mathrm{T}} = \mathbf{KR}(\mathbf{KR})^{\mathrm{T}} = \mathbf{K}\mathbf{K}^{\mathrm{T}} = \boldsymbol{\omega}^{-1}$$

From $\mathrm{P} = [\mathbf{M} \quad \boldsymbol{m}]$:

- take $\omega^{-1} = \mathbf{M}\mathbf{M}^{\mathrm{T}}$
- $\mathbf{K}$ = Choleski factorization of $\omega^{-1} = \mathbf{M}\mathbf{M}^{\mathrm{T}} = \mathbf{K}\mathbf{K}^{\mathrm{T}}$

Camera calibration through the ω matrix

USEFUL FACTS

vanishing points = image of points at the ∞: $\mathbf{v_d} = \mathrm{PX}_\infty = \mathbf{K}\,[\mathbf{R} \quad \mathbf{t}] \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = \mathbf{KR}\,\mathbf{d}$

→ Homography from $\boldsymbol{\pi}_\infty$ to image plane: $\mathbf{H} = \mathbf{KR}$ → inverse: $\mathbf{d} = (\mathbf{KR})^{-1}\mathbf{v_d}$

From $\boldsymbol{\omega}$ to calibration matrix $\mathbf{K}$: Cholesky factorisation of inverse $\boldsymbol{\omega}^{-1} = \mathbf{KK}^{\mathrm{T}}$

Constraints on $\boldsymbol{\omega}$ from:

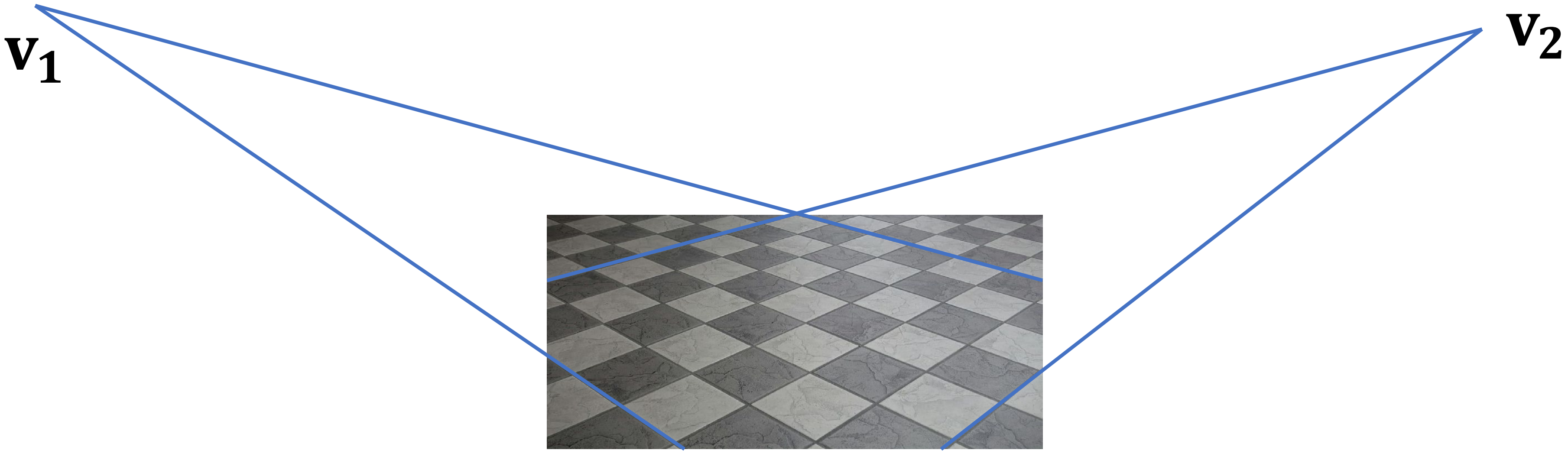
• known angles btw directions,	observing:	their vanishing points
• (includes) self-orthogonality,	observing:	images of circular points
• known planar shapes,	observing:	their images

Constraints on the ω matrix

- **known angles btw directions** $\mathbf{d}_i$, and **their vanishing points** $\mathbf{v}_i = \mathbf{KRd}_i$

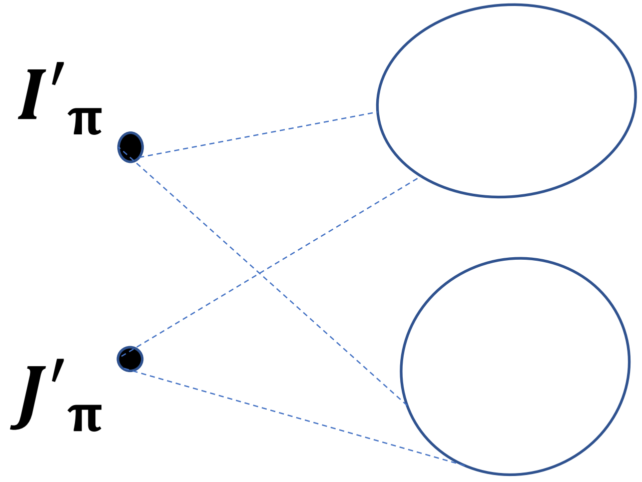
$$\cos\theta = \frac{\mathbf{d_1}^{\mathrm{T}}\mathbf{d_2}}{\sqrt{(\mathbf{d_1}^{\mathrm{T}}\mathbf{d_1})(\mathbf{d_2}^{\mathrm{T}}\mathbf{d_2})}} = \frac{\mathbf{v_1}^{\mathrm{T}}(\mathbf{KR})^{-\mathrm{T}}(\mathbf{KR})^{-1}\mathbf{v_2}}{\sqrt{(\mathbf{v_1}^{\mathrm{T}}..\mathbf{v_1})(\mathbf{v_2}^{\mathrm{T}}..\mathbf{v_2})}} = \frac{\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_2}}{\sqrt{(\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_1})(\mathbf{v_2}^{\mathrm{T}}\omega\mathbf{v_2})}}$$

if directions are orthogonal $\theta = 90°$ → **linear** constraint: $\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_2} = \mathbf{0}$



Constraints on the ω matrix

- **images of circular points**
  (e.g., intersection of imaged circumferences)

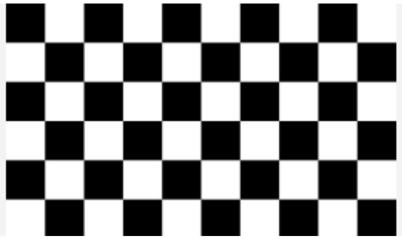
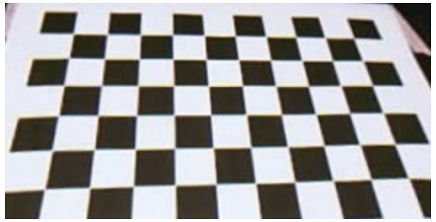


from Property 2: $\boldsymbol{I'_\pi}^T \omega\, \boldsymbol{I'_\pi} = \mathbf{0}$

Amounts to 2 independent eqns:
- Real part = 0
and
- Imaginary part = 0

($\boldsymbol{J'_\pi}^T \omega\, \boldsymbol{J'_\pi} = \mathbf{0}$ leads to equivalent equations)

Constraints on the ω matrix

- **known planar shapes and**  $\xrightarrow{\mathbf{H}}$  **their images**

plane-to-image homography $\mathbf{H} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}]$

$\rightarrow [\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}] = \mathbf{K}^{-1}\mathbf{H} = \left[\mathbf{K}^{-1}\boldsymbol{h}_1 \quad \mathbf{K}^{-1}\boldsymbol{h}_2 \quad \mathbf{K}^{-1}\boldsymbol{h}_3\right]$

vectors $\mathbf{K}^{-1}\boldsymbol{h}_1, \mathbf{K}^{-1}\boldsymbol{h}_2$ (i.e. $\boldsymbol{r}_{\pi 1}, \boldsymbol{r}_{\pi 2}$) are orthogonal and have the same module

$\rightarrow$

$$\mathbf{0} = \boldsymbol{h}_1^T \mathbf{K}^{-T}\mathbf{K}^{-1}\boldsymbol{h}_2 = \boldsymbol{h}_1^T\, \omega\, \boldsymbol{h}_2$$

and

$$\mathbf{0} = \boldsymbol{h}_1^T \mathbf{K}^{-T}\mathbf{K}^{-1}\boldsymbol{h}_1 - \boldsymbol{h}_2^T \mathbf{K}^{-T}\mathbf{K}^{-1}\boldsymbol{h}_2 = \boldsymbol{h}_1^T\, \omega\, \boldsymbol{h}_1 - \boldsymbol{h}_2^T\, \omega\, \boldsymbol{h}_2$$

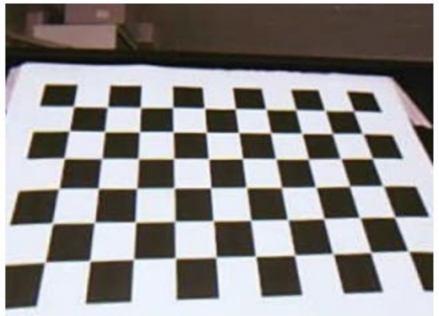
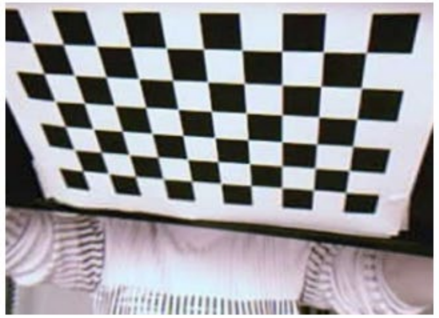
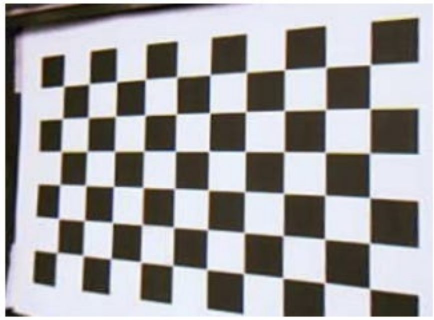
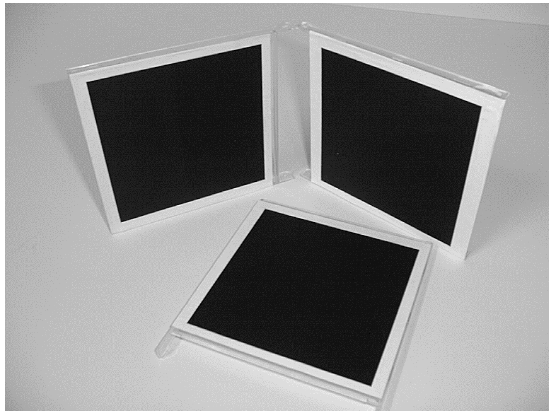
Camera calibration from images of known planar shapes (Zhang method)

For each homography $\mathbf{H} = [\boldsymbol{h}_1 \quad \boldsymbol{h}_2 \quad \boldsymbol{h}_3]$

$$\boldsymbol{h_1}^T \omega \boldsymbol{h_2} = 0$$

$$\boldsymbol{h_1}^T \omega \boldsymbol{h_1} - \boldsymbol{h_2}^T \omega \boldsymbol{h_2} = 0$$

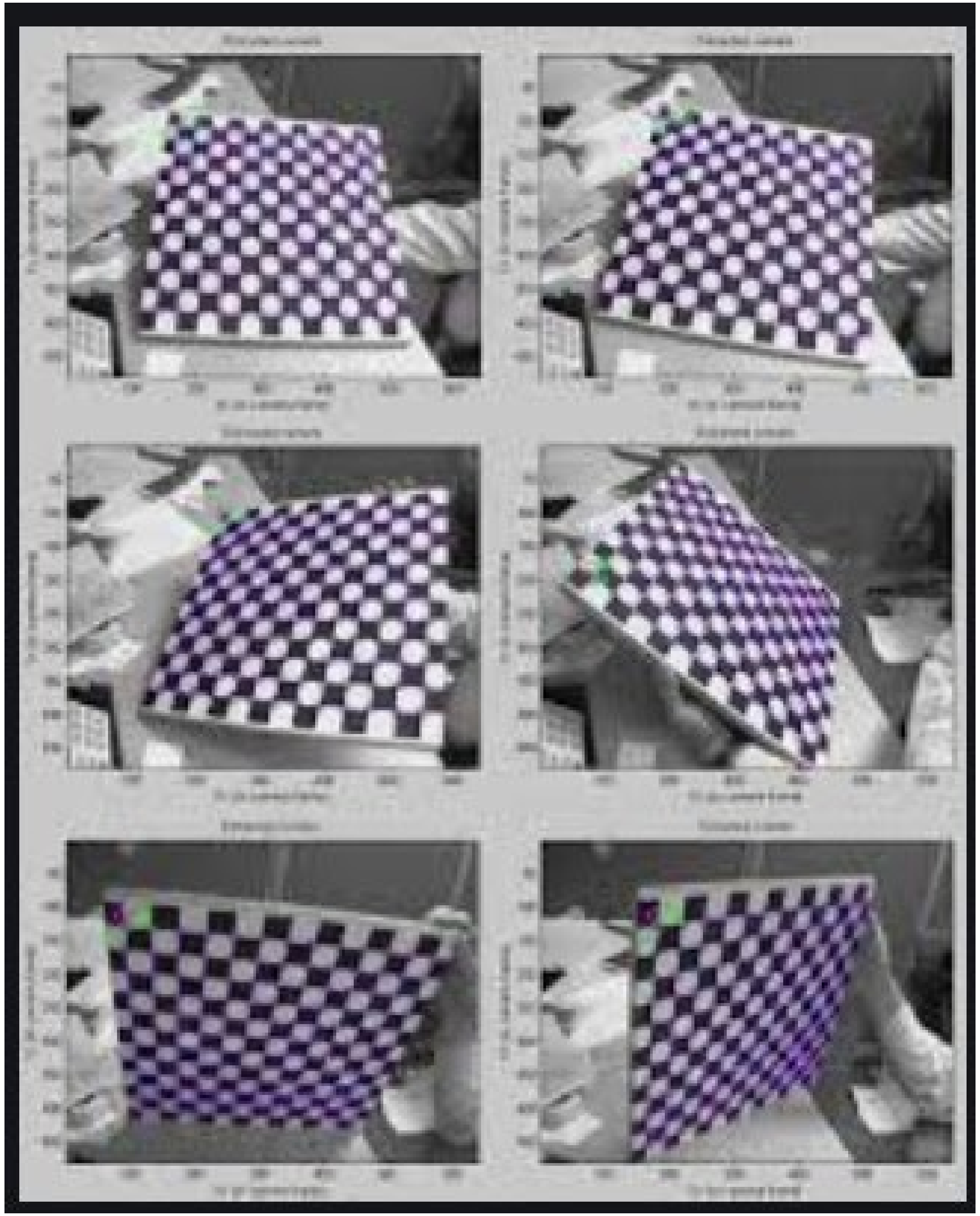
2 homogeneous equations in ω → at least 3 homographies needed

e.g.    or 

from Cholesky factorisation of ω^{-1} → $\mathbf{K}$

Matlab camera calibration toolbox

- implements Zhang method
- planar target (easily printable)
- several images ($\sim$ 20) to cope with noise
- also estimates distortion param.
- provides accurate calibration



Matlab Calibration Toolbox also contains estimation of **distortion** parameters

$$x = x_o + (x_o - c_x)(K_1 r^2 + K_2 r^4 + \ldots)$$
$$y = y_o + (y_o - c_y)(K_1 r^2 + K_2 r^4 + \ldots)$$

$$r = (x_o - c_x)^2 + (y_o - c_y)^2 \quad .$$

Scenario n. 1

Known: vanishing points and angle θ between directions

$$\cos\theta = \frac{\mathbf{d_1}^{\mathrm{T}}\mathbf{d_2}}{\sqrt{\left(\mathbf{d_1}^{\mathrm{T}}\mathbf{d_1}\right)\left(\mathbf{d_2}^{\mathrm{T}}\mathbf{d_2}\right)}} = \frac{\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_2}}{\sqrt{(\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_1})(\mathbf{v_2}^{\mathrm{T}}\omega\mathbf{v_2})}}$$

$\rightarrow$

constraint on the ω matrix

(linear if directions are orthogonal)

Scenario n. 2

Known: vanishing points and ω matrix

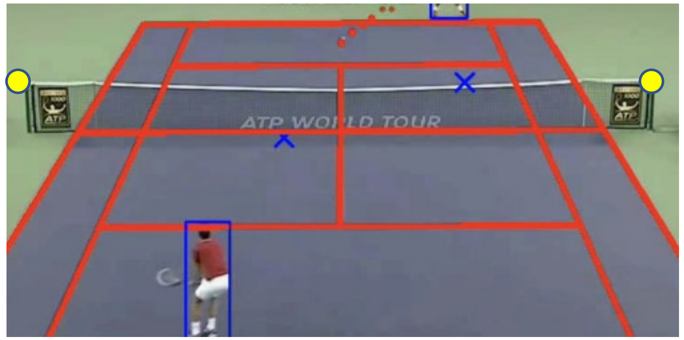
$$\cos\,\theta = \frac{\mathbf{d_1}^{\mathrm{T}}\mathbf{d_2}}{\sqrt{\left(\mathbf{d_1}^{\mathrm{T}}\mathbf{d_1}\right)\left(\mathbf{d_2}^{\mathrm{T}}\mathbf{d_2}\right)}} = \frac{\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_2}}{\sqrt{(\mathbf{v_1}^{\mathrm{T}}\omega\mathbf{v_1})(\mathbf{v_2}^{\mathrm{T}}\omega\mathbf{v_2})}}$$

→

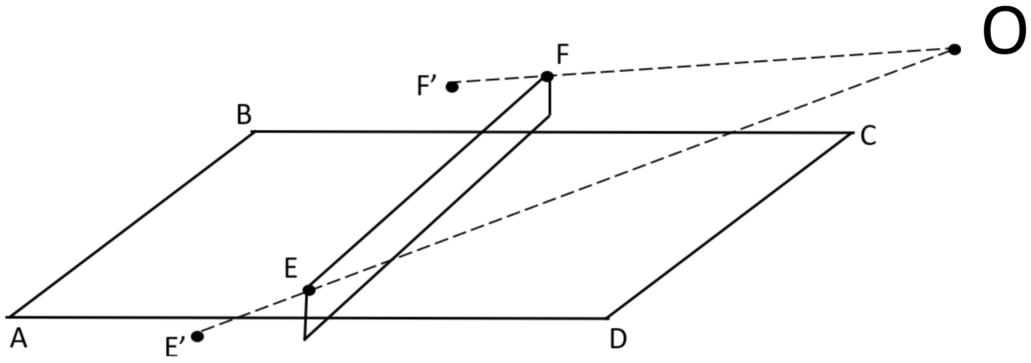
compute angle θ between directions

→ reconstruction of the shape

Extrinsic + intrinsic camera calibration from a single image of a known planar shape and known camera center position

E


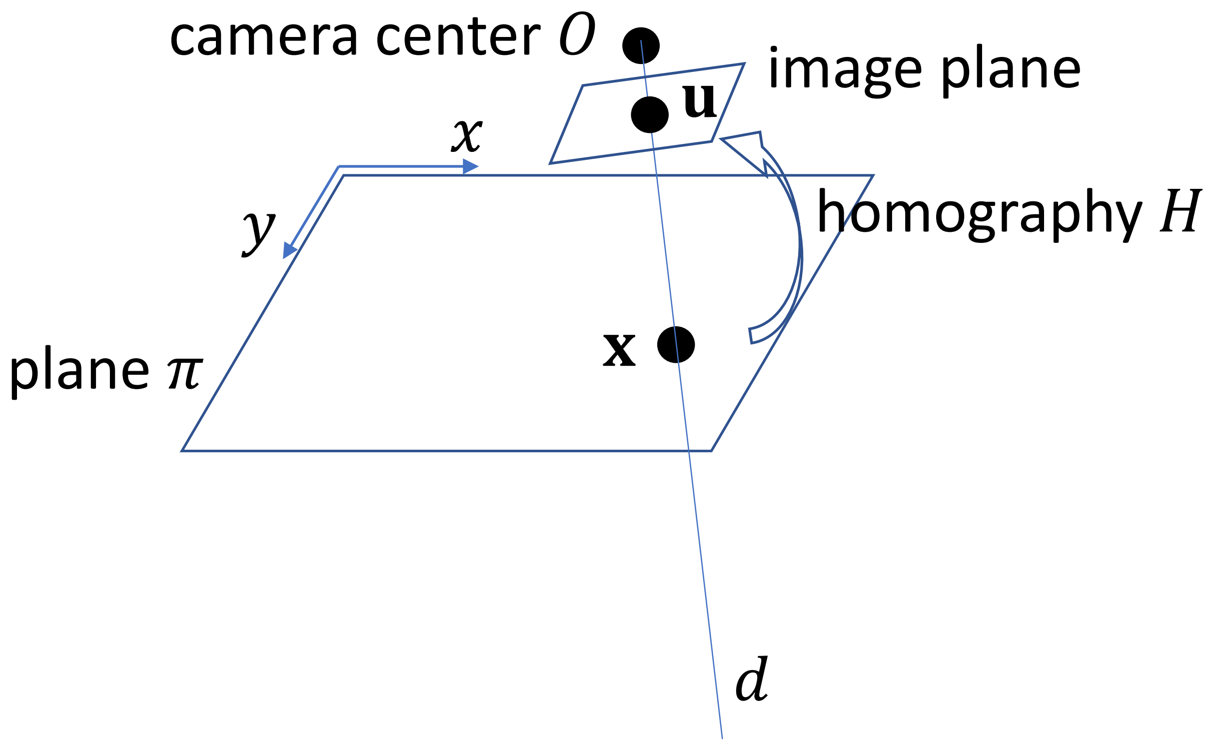
F



Calibration from known plane π and camera center O

Let us refer the coordinates to a reference attached to a known plane π : a generic point on this plane has homogeneous coordinates

$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, while $O = \begin{bmatrix} \mathrm{x}_O \\ \mathrm{y}_O \\ \mathrm{z}_O \end{bmatrix}$ are the known cartesian coordinates of the camera center,



Calibration from known plane π and camera center O

1) Estimate homography H from known points on the plane π and their images

2) Call M_o the matrix relating any point $\mathbf{x}$ on π to the direction $\mathbf{d}$ of a ray from O to $\mathbf{x}$:

$$\mathbf{d} = \begin{bmatrix} x - \mathrm{x}_o \\ y - \mathrm{y}_o \\ -\mathrm{z}_o \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\mathrm{x}_o \\ 0 & 1 & -\mathrm{y}_o \\ 0 & 0 & -\mathrm{z}_o \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \rightarrow {M_o}^{-1} = \begin{bmatrix} 1 & 0 & -\mathrm{x}_o \\ 0 & 1 & -\mathrm{y}_o \\ 0 & 0 & -\mathrm{z}_o \end{bmatrix}$$

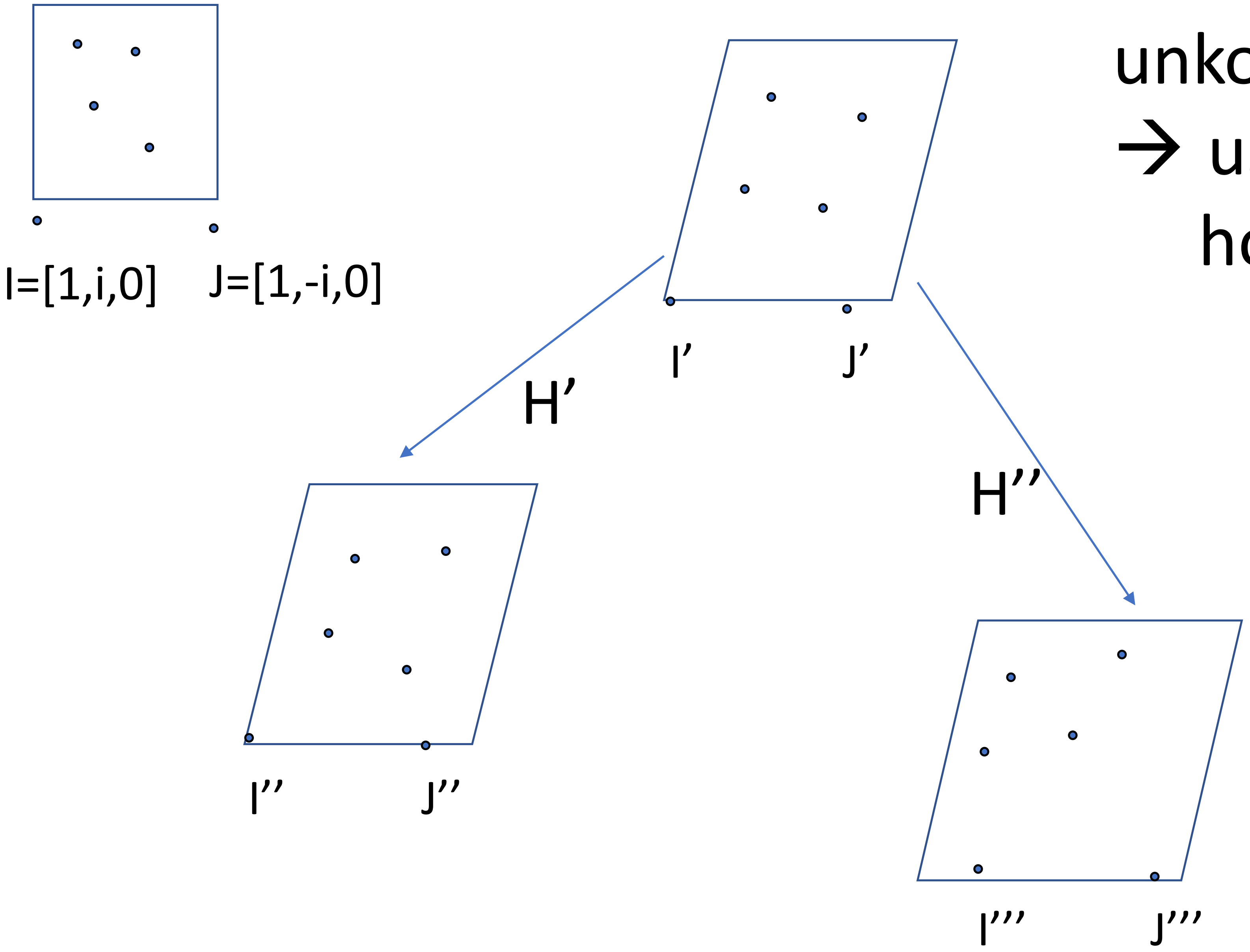
3) Compute the matrix $\mathbf{M}$ relating any image point $\mathbf{u}$ to the direction $\mathbf{d}$ of its viewing ray from $\mathbf{u} = H\,\mathbf{x}$, is $\mathbf{d} = {M_o}^{-1}\mathbf{x} = {M_o}^{-1}H^{-1}\mathbf{u} \rightarrow \mathbf{M}^{-1} = {M_o}^{-1}H^{-1}$

4) Q-R decompose matrix $\mathbf{M}^{-1}$ as $\mathbf{M}^{-1} = \mathbf{R}^{-1}\mathbf{K}^{-1}$, where $\mathbf{K}$ is the camera intrinsic calibration matrix and $\mathbf{R}^{-1}$ is the rotation matrix from the world (reference attached to π) to the camera

Camera calibration from images of an **unknown** planar scene

unkown planar scene

→ use image-to-image homographies



$$I'' = H'I'$$
$$I'^T \omega I' = 0$$
$$I''^T \omega I'' = 0$$

$$I'^T H'^T \omega H' I' = 0$$

Unknowns: I' and J' and ω → at least 5 images
(each nonlinear eqn leads to 2 constraints: Re and Im part)

Camera calibration from images of an **unknown** planar scene

- take images of a planar scene with camera with constant $\mathbf{K}$
- estimate image-image homographies
- formulate equations

$$I'^{T} \omega I' = 0$$

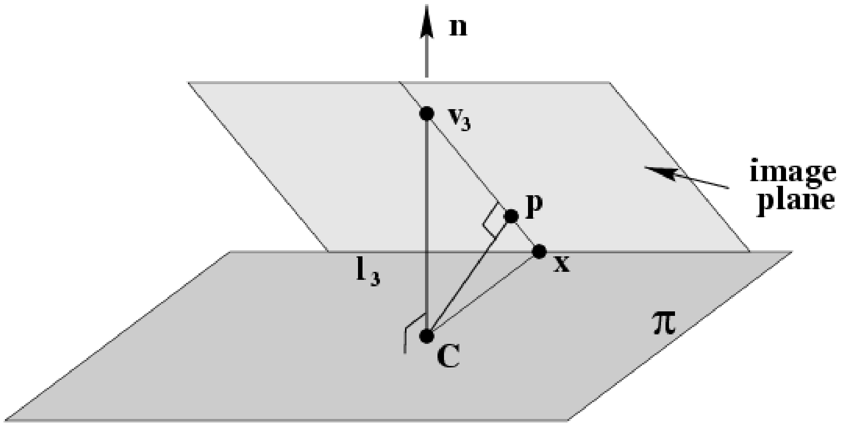
$$I'^{T} H'^{T} \omega H' I' = 0$$

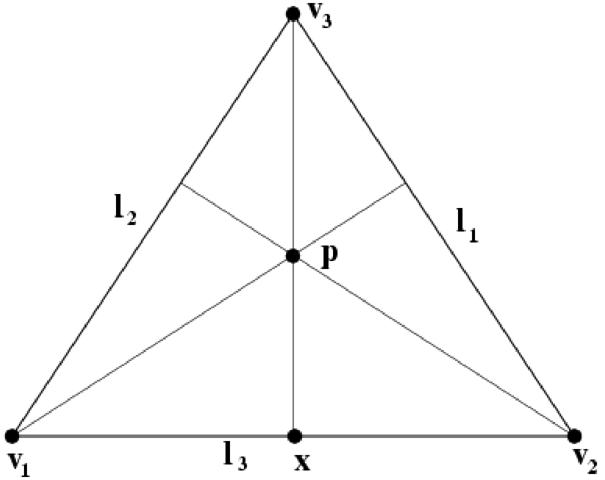
- solve them for ω and I'
- then take the inverse matrix ω^{-1}
- find $\mathbf{K}$ by Cholesky factorisation of $\omega^{-1} = \mathbf{K}\mathbf{K}^{\mathrm{T}}$

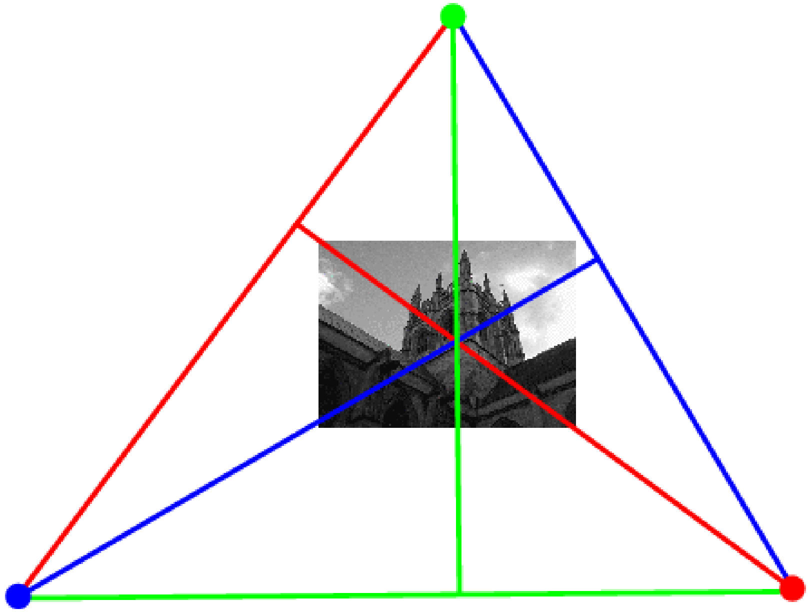
USUALLY LESS ACCURATE THAN ZHANG METHOD

Calibration of natural cameras ($f_X = f_Y = f$) from vanishing points of mutually orthogonal directions

Calibration of NATURAL CAMERAS from vanishing points of orthogonal directions







Orthogonality relation

$$\cos\theta = \frac{\mathrm{v_1}^{\mathrm{T}}\omega\mathrm{v_2}}{\sqrt{(\mathrm{v_1}^{\mathrm{T}}\omega\mathrm{v_1})(\mathrm{v_2}^{\mathrm{T}}\omega\mathrm{v_2})}} = 0$$

$$\mathbf{K} = \begin{bmatrix} f & 0 & U_o \\ 0 & f & V_o \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \omega = (\mathbf{K}\mathbf{K}^T)^{-1} = \begin{bmatrix} 1 & 0 & -U_0 \\ * & 1 & -V_0 \\ * & * & f^2 + U_0^2 + V_0^2 \end{bmatrix}$$ 3 unknowns

$$\begin{aligned} \mathrm{v_1}^{\mathrm{T}}\omega\mathrm{v_2} &= 0 \\ \mathrm{v_2}^{\mathrm{T}}\omega\mathrm{v_3} &= 0 \\ \mathrm{v_3}^{\mathrm{T}}\omega\mathrm{v_1} &= 0 \end{aligned}$$ 3 equations

Camera calibration from

- a planar face of known (reconstructed) shape and
- a vanishing point of the normal to the plane

calibration after reconstruction of a planar face

Shape of the face:
known after metric rectificaton

How much additional information is needed to calibrate the camera?

- Vanishing point of the direction normal to the reconstructed face

Vanishing point of vertical direction

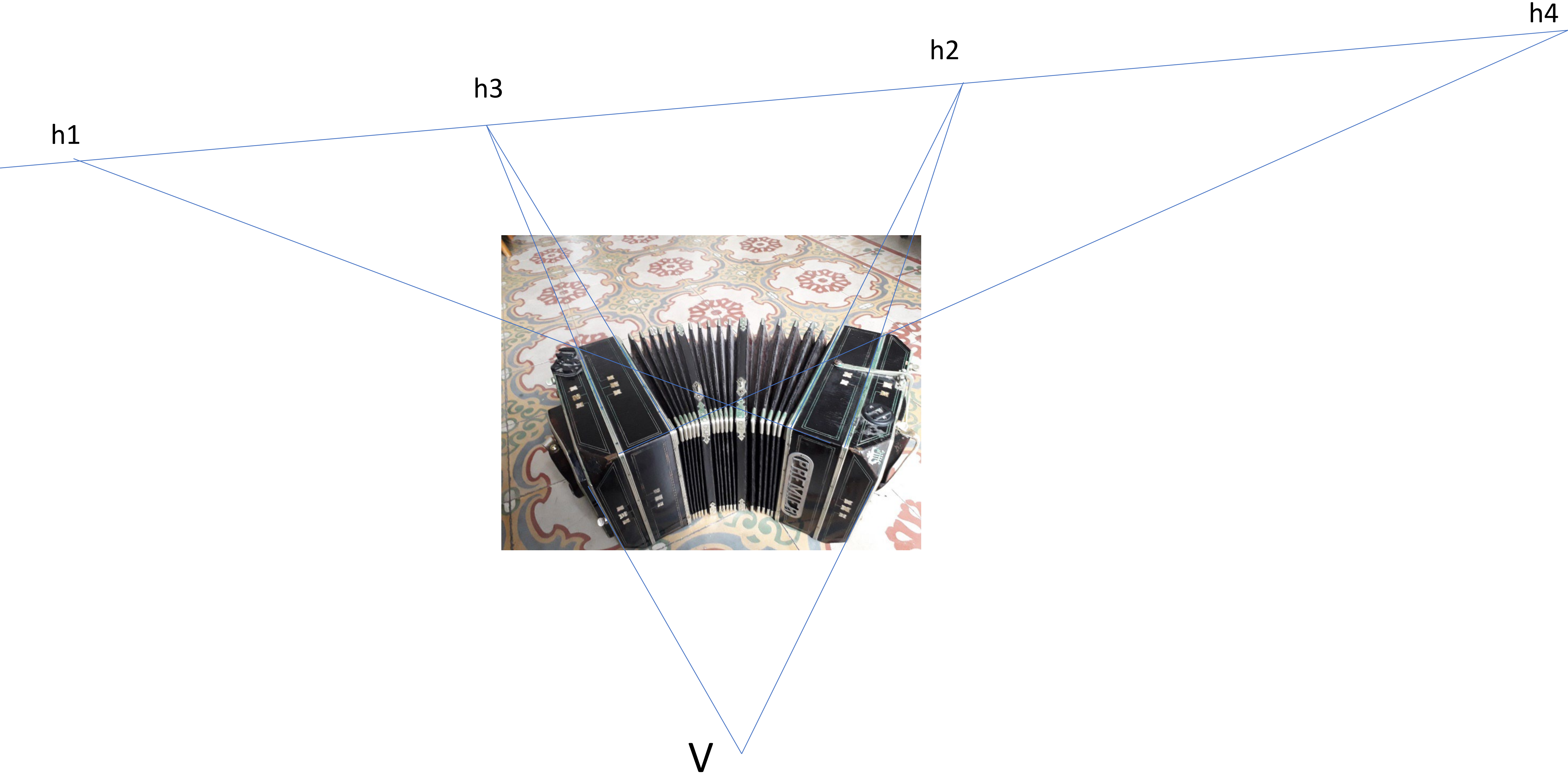
V

Camera calibration (assume skew = 0)

- Only four unknowns:

$$\omega = (KK^T)^{-1} = \begin{vmatrix} a^2 & 0 & -u_0 a^2 \\ * & 1 & -v_0 \\ * & * & f_Y^2 + a^2 u_0^2 + v_0^2 \end{vmatrix}$$

→ only four equations are needed


h4
h2
h3
h1
V

Calibration: rectified planar face plus vanishing point of the direction normal to the face

$$h_1^T \omega h_2 = 0$$
$$h_3^T \omega h_4 = 0$$
$$v^T \omega h_1 = 0$$
$$v^T \omega h_2 = 0$$

3° and 4° equations are linearly independent,

but there are no further ones (why?)

- → solve for ω
- → find $\mathbf{K}$ (by Cholesky factorisation of $\omega^{-1} = \mathbf{K}\mathbf{K}^{\mathrm{T}}$)

Calibration from rectified face plus orthogonal vanishing point

direct method:
independent of the chosen pairs of mutually orthogonal vanishing points

from

- the reconstructing homography H_R from given img to rectified img
- the image of the line at the infinity $\boldsymbol{l'}_{\infty}$
- the vanishing point $\mathbf{v}$ along the direction orthogonal to the face

Calibration from rectified face plus orthogonal vanishing point

- From plane-to-image homography $\mathbf{H} = [h_1 \quad h_2 \quad h_3]$ inverse of the rectifying homography $\mathbf{H}_R$: $\mathbf{H} = \mathbf{H}_R^{-1}$

$\rightarrow$

$$\boxed{\begin{array}{c} {h_1}^{\boldsymbol{T}} \omega h_2 = 0 \\ {h_1}^{\boldsymbol{T}} \omega h_1 - {h_2}^{\boldsymbol{T}} \omega h_2 = 0 \end{array}}$$

(same as Zhang method)

- from $\mathbf{v}_1^T \omega \mathbf{v} = 0$ and $\mathbf{v}_2^T \omega \mathbf{v} = 0$, and $\boldsymbol{l}'_\infty = \mathbf{v}_1 \times \mathbf{v}_2$ $\rightarrow$

$$\boxed{\boldsymbol{l}'_\infty = \omega\mathbf{v} \ \text{(2 eqns)}}$$

Calibration from rectified face plus orthogonal vanishing point: alternative to the last 2 eqns

- from $I' = H_R^{-1} I = H_R^{-1} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = [h_1 \quad h_2 \quad h_3] \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = h_1 + i h_2,$

since $I'^T \omega\, I' = (h_1 + i h_2)^T \omega (h_1 + i h_2) = 0$ →

$$\boxed{\begin{gathered} h_1{}^T \omega h_2 = 0 \\ h_1{}^T \omega h_1 - h_2{}^T \omega h_2 = 0 \end{gathered}}$$

- for any $\mathbf{x} = [\alpha \quad \beta \quad 0]^{\mathbf{T}} \in \boldsymbol{l}_\infty,\ \mathbf{x}' = H_R^{-1}\mathbf{x} = \alpha h_1 + \beta h_2 \in \boldsymbol{l}'_\infty$ and since $\mathbf{v}$ is a vanishing point of direction ⊥ to $\mathbf{x}$, is $\mathbf{v}^T \omega\, \mathbf{x} = 0\ \forall(\alpha, \beta)$

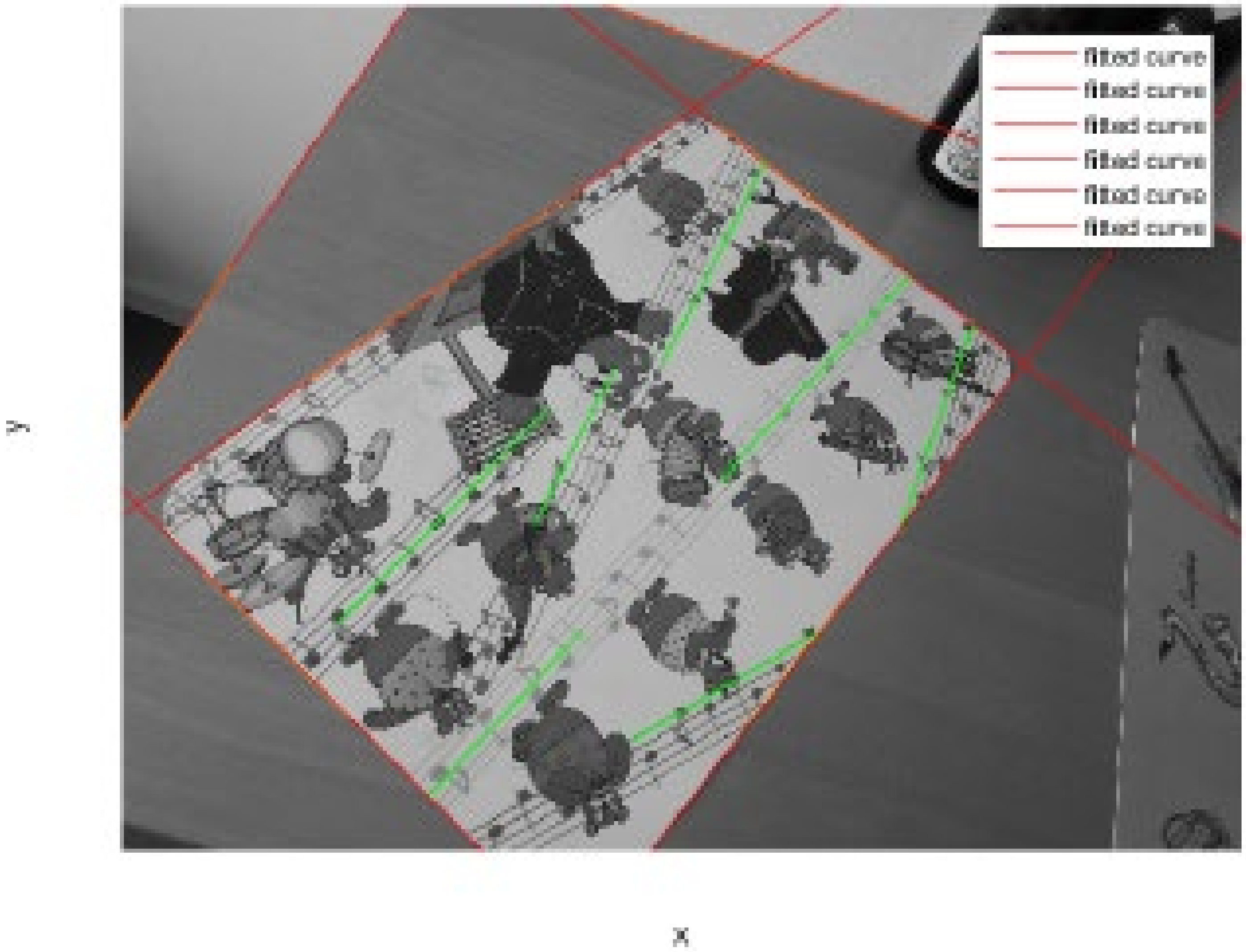
→

$$\boxed{\mathbf{v}^T \omega h_1 = 0, \text{ and } \mathbf{v}^T \omega h_2 = 0}$$

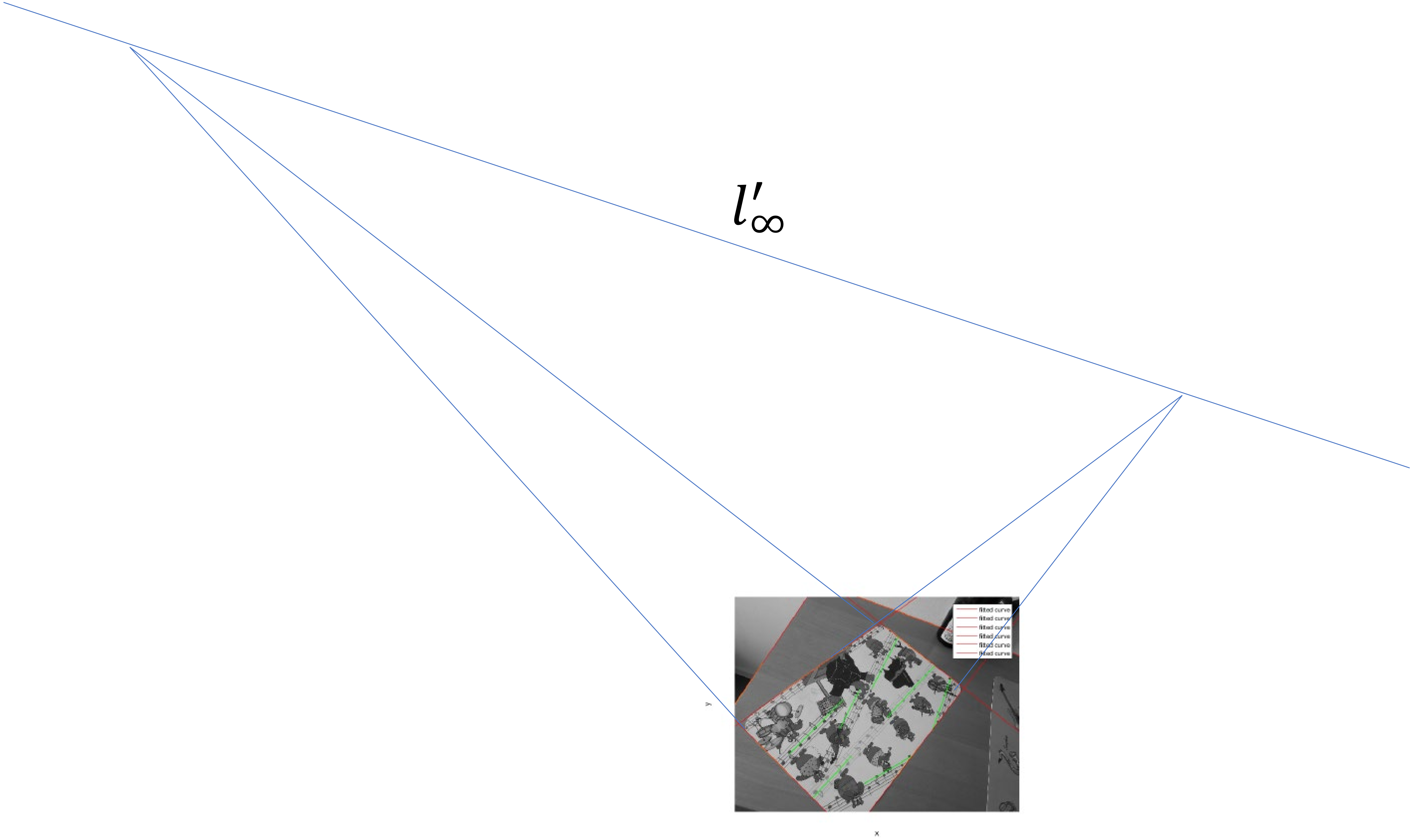
Rectification of a single calibrated image from vanishing points

DONNAS
2003
COLLIO
Saxophone

Images of pairs of parallel lines



Vanishing line l'_∞ from vanishing points



Unknown shape, known image: → plane-to-image homography $\mathbf{H}$ provides position of points on the plane

$\mathbf{H}$?

→

Solution (shape): image-to-plane homography $\mathbf{H}_R = \mathbf{H}^{-1}$ modulo reference frame on the plane and scale factor

Reconstructing homography $\mathbf{H}_R = \mathbf{H}^{-1}$

plane-to-image homography $\mathbf{H} = \mathbf{K}[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}]$:

$\mathbf{H}$ is related to $\mathbf{K}$ and to the relative pose of plane π wrt camera

→

free to choose a «comfortable» reference frame on the plane π:

the only constrained element is the direction $\mathbf{n}_{\pi}$ normal to plane π

$\mathbf{n}_{\pi} = \mathbf{K}^T l'_{\infty}$ (*for proof see next slide, with* $\mathbf{M} = \mathbf{K}$)

→

Choose any $\boldsymbol{r}_{\pi 1} \perp \boldsymbol{r}_{\pi 2}$ both orthogonal to $\mathbf{n}_{\pi}$ and normalize all of them;

then, take $\mathbf{o}_{\pi} = \mathbf{n}_{\pi}$ → in this way $[\boldsymbol{r}_{\pi 1} \quad \boldsymbol{r}_{\pi 2} \quad \mathbf{o}_{\pi}] = \mathbf{R}_{\pi}$

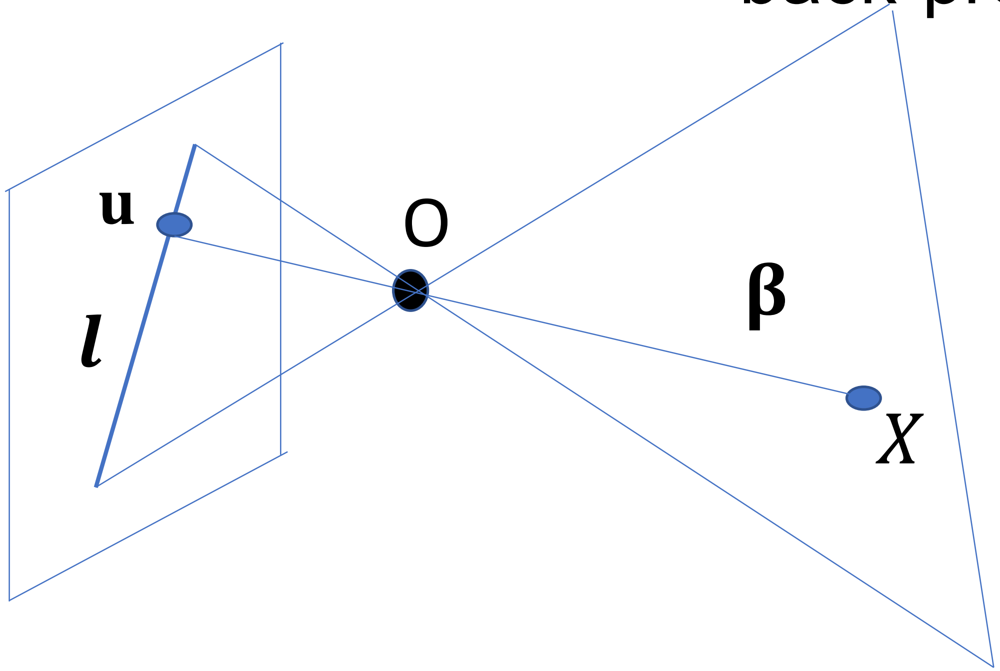
→

$$\mathbf{H}_R = \mathbf{H}^{-1} = (\mathbf{K}\mathbf{R}_{\pi})^{-1} = \mathbf{R}_{\pi}{}^T \mathbf{K}^{-1}$$

DIGRESSION: Back-projection of an image line

set of space points X, whose image projection $\mathbf{u} = \mathbf{P}X$ is on image line $\boldsymbol{l}$

back-projection



$$l^{\mathrm{T}}\mathbf{u} = \underline{l^{\mathrm{T}}\mathbf{P}}X = 0$$

$$\boldsymbol{\beta}^T \mathrm{X} = 0$$

back-projection of $\boldsymbol{l}$: plane $\boldsymbol{\beta} = \mathbf{P}^{\mathrm{T}}\boldsymbol{l}$

through O, in fact, since O $= RNS(\mathbf{P})$

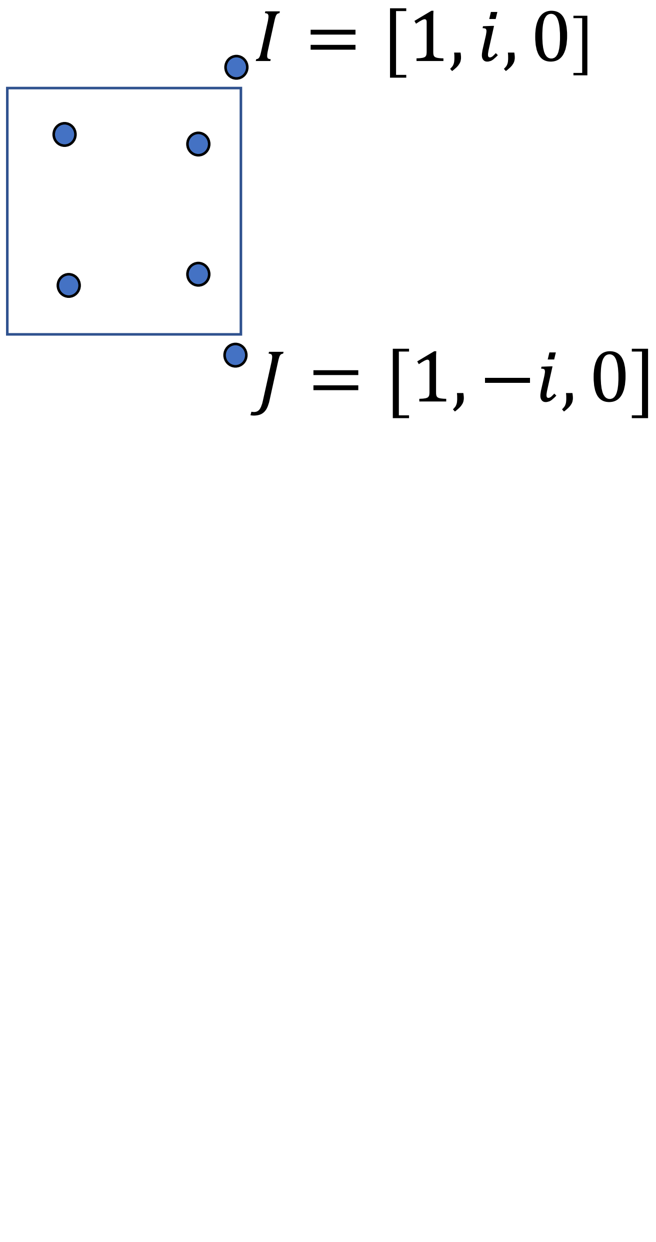
$$\boldsymbol{\beta}^T \mathrm{O} = \boldsymbol{l}^{\mathrm{T}}\mathbf{P}\,\mathrm{O} = \boldsymbol{l}^{\mathrm{T}}0 = 0$$

back-projection of $\boldsymbol{l}$: plane $\boldsymbol{\pi} = \mathbf{P}^{\mathrm{T}}\boldsymbol{l} = \begin{bmatrix} \mathbf{M}^T \\ \boldsymbol{m}^T \end{bmatrix}\boldsymbol{l} \rightarrow$ normal: $\mathbf{n}_{\beta} = \mathbf{M}^T\boldsymbol{l}$

2D shape reconstruction = image rectification

result

Reconstruction of an **unknown** planar scene from calibrated images (i.e. images taken by calibrated cameras)



reconstruction of an **unkown** planar **scene**, from two **calibrated** images

→ use image-to-image homographies

REMEMBER: $I'^T \omega I' = 0$ for any imaged circular point I'

$$I'^T \omega I' = 0$$

$$I''^T \omega I'' = 0$$

$$I'' = H I'$$

$$I'^T H^T \omega H I' = 0$$

Same camera hypothesis can easily be relaxed

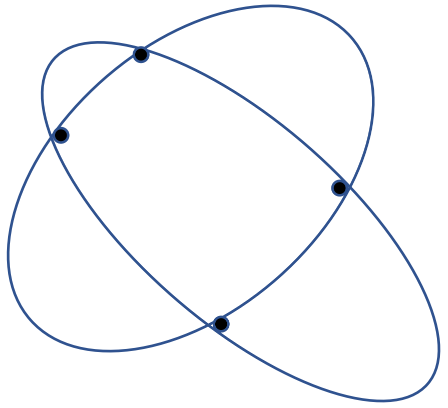
$\omega = \left(\mathbf{K}\mathbf{K}^{\mathrm{T}}\right)^{-1} = \mathbf{K}^{-\mathrm{T}}\mathbf{K}^{-1}$ is known after calibration

just 4 unknowns I' complex coordinates → at least 2 images
(each eqn leads to 2 constraints: Re and Im part)

$$I'^{T}\omega' I' = 0$$
$$I'^{T} H'^{T} \omega'' H' I' = 0$$

two different cameras
ω' and ω''

Geometrically: intersection of two conics:
ω' and $H'^{T}\omega'' H'$
→
two resulting pairs of
imaged circular points
→
selection based on reprojection
or on an additional (third) image



As usual: image rectification from the image $(\boldsymbol{I}', \boldsymbol{J}')$ of the circular points $(\boldsymbol{I}, \boldsymbol{J})$

- Image of the circular points → image of the conic dual to the circular points

$$C_\infty^{*\prime} = \boldsymbol{I}'\boldsymbol{J}'^{\boldsymbol{T}} + \boldsymbol{J}'\boldsymbol{I}'^{\boldsymbol{T}}$$

- Singular value decomposition

$$\text{svd}(C_\infty^{*\prime}) = U \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = H_{SR}^{-1} C_\infty^* H_{SR}^{-T}$$

- Rectifying transformation (from svd output U)

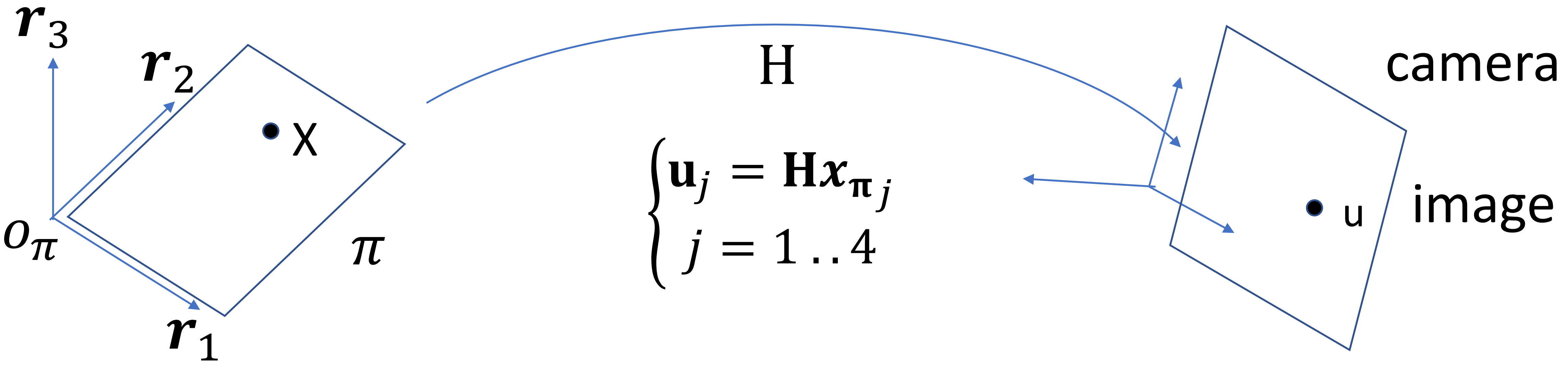
$$H_{SR} = \begin{bmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Euclidean reconstructed model $M_S = H_{SR} * \text{Image}$

Localization of a known planar shape from a calibrated image

Localization of a known planar shape from a calibrated image

Fit homography H between the plane π and its image

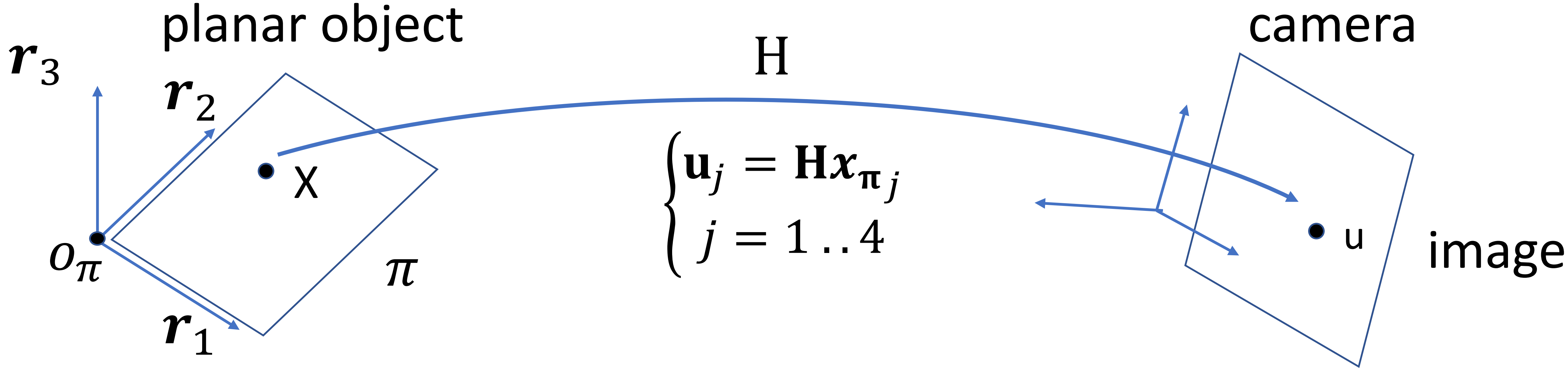


$$\begin{cases} \mathbf{u}_j = \mathbf{H}\boldsymbol{x}_{\boldsymbol{\pi}\,j} \\ j = 1..4 \end{cases}$$

plane π reference: relative pose plane wrt camera → rototranslation $\mathbf{R}_\pi$, $\boldsymbol{o}_\pi$

world reference ≡ camera reference

$$\mathrm{P} = [\mathbf{K} \quad \mathbf{0}]$$



$$\begin{cases} \mathbf{u}_j = \mathbf{H}\boldsymbol{x}_{\boldsymbol{\pi}\,j} \\ j = 1\,..\,4 \end{cases}$$

plane reference

world reference ≡ camera reference → $\mathrm{P} = [\mathbf{K} \quad \mathbf{0}]$

$$\mathbf{X}_{\boldsymbol{\pi}} = \begin{bmatrix} x \\ y \\ 0 \\ w \end{bmatrix} \qquad \mathbf{X}_{\mathbf{w}} = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 & \mathbf{o}_{\boldsymbol{\pi}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \mathbf{o}_\pi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$\boldsymbol{x}_\pi$

$$\mathbf{u} = \mathrm{P}\mathbf{X}_{\mathbf{w}} = [\mathbf{K} \quad \mathbf{0}] \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \mathbf{o}_\pi \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{\boldsymbol{\pi}} = \mathbf{K}[\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \mathbf{o}_\pi]\boldsymbol{x}_{\boldsymbol{\pi}}$$

homography $\mathbf{H} = \mathbf{K}[\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \mathbf{o}_\pi]$ → obj. pose wrt camera $[\boldsymbol{r}_1 \quad \boldsymbol{r}_2 \quad \mathbf{o}_\pi] = \mathbf{K}^{-1}\mathrm{H}$

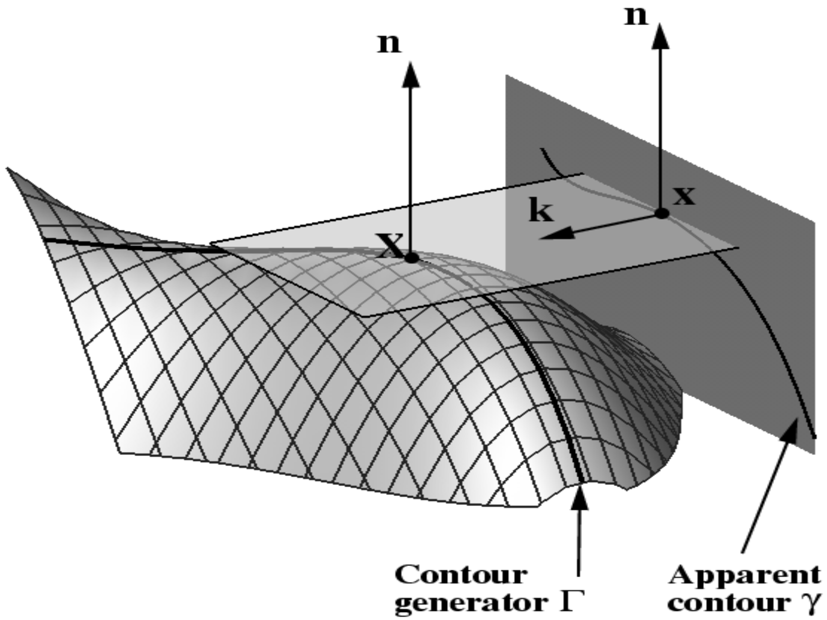
Apparent contour of a smooth surface

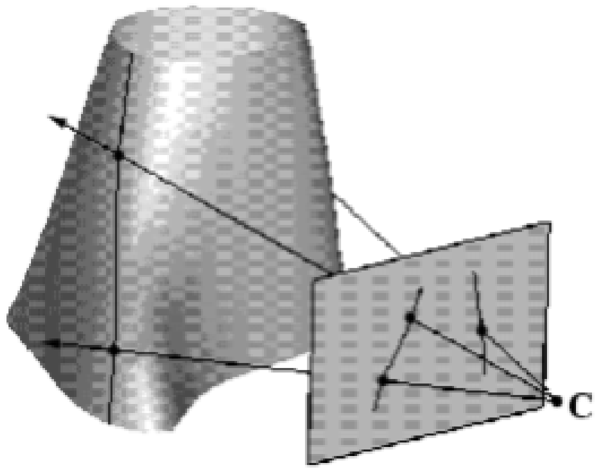
Image projection of smooth surfaces

The **contour generator** Γ is the set of points X on a surface S, whose viewing rays are tangent to S.

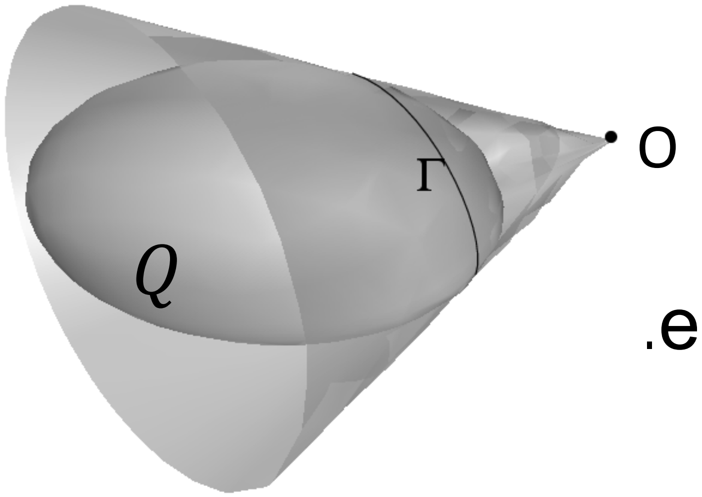
The corresponding **apparent contour** is the image of Γ

Γ depends only on position of projection center O,
Γ depends also on rest of the projection matrix P





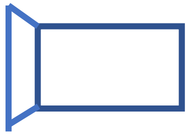
The apparent contour of a quadric



select those planes, tangent to the quadric Q,
i.e. belonging to dual quadric $Q^* = Q^{-1}$
which go through camera center O
.e., that are backprojections $\mathrm{P}^{\mathrm{T}}\boldsymbol{l}$ of some image lines $\boldsymbol{l}$

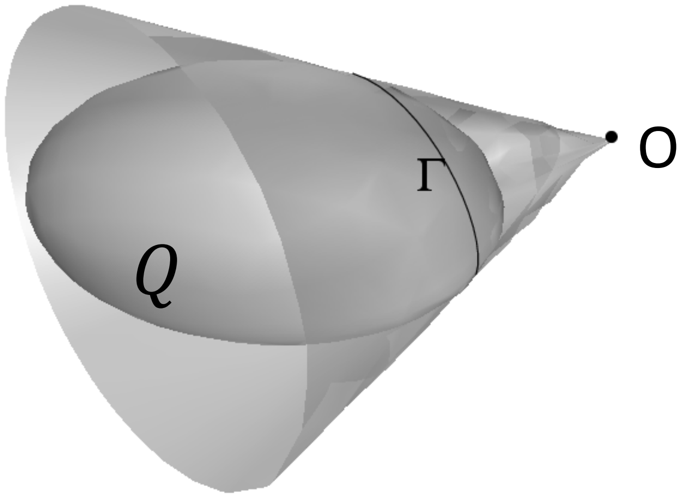
$$\Pi^{\mathrm{T}}\mathrm{Q}^*\Pi = l^{\mathrm{T}}\mathrm{P}\mathrm{Q}^*\mathrm{P}^{\mathrm{T}}l = 0$$

these planes Π are tangent
to the contour generator Γ



these lines $\boldsymbol{l}$ are image of planes Π
→ $\boldsymbol{l}$ are tangent to image of Γ
→ $\boldsymbol{l}$ are tangent to apparent contour γ

The apparent contour of a quadric



select those planes, tangent to the quadric Q,
i.e. belonging to dual quadric $Q^* = Q^{-1}$
which go through camera center O
i.e., that are backprojections of some image lines $\boldsymbol{l}$

$$\Pi^{\mathrm{T}} \mathrm{Q}^* \Pi = l^{\mathrm{T}} \mathrm{P} \mathrm{Q}^* \mathrm{P}^{\mathrm{T}} l = 0$$

these image lines $\boldsymbol{l}$ satisfy a quadratic equation
→ they belong to a dual conic $\mathrm{C}^* = \mathrm{P}\mathrm{Q}^*\mathrm{P}^T$
→ they are tangent to a conic $\mathrm{C} = {\mathrm{C}^*}^{-1} = (\mathrm{P}\mathrm{Q}^*\mathrm{P}^T)^{-1}$

But $\boldsymbol{l}$ are tangent to conic $\mathrm{C} = {\mathrm{C}^*}^{-1} = (\mathrm{P}\mathrm{Q}^*\mathrm{P}^T)^{-1}$ → $\boxed{\text{a.c } \gamma \text{ is the conic C}}$

Exercise: apparent contour of a cone?

- contour generator Γ: set of tangency points from O

→ two straight lines through the vertex V

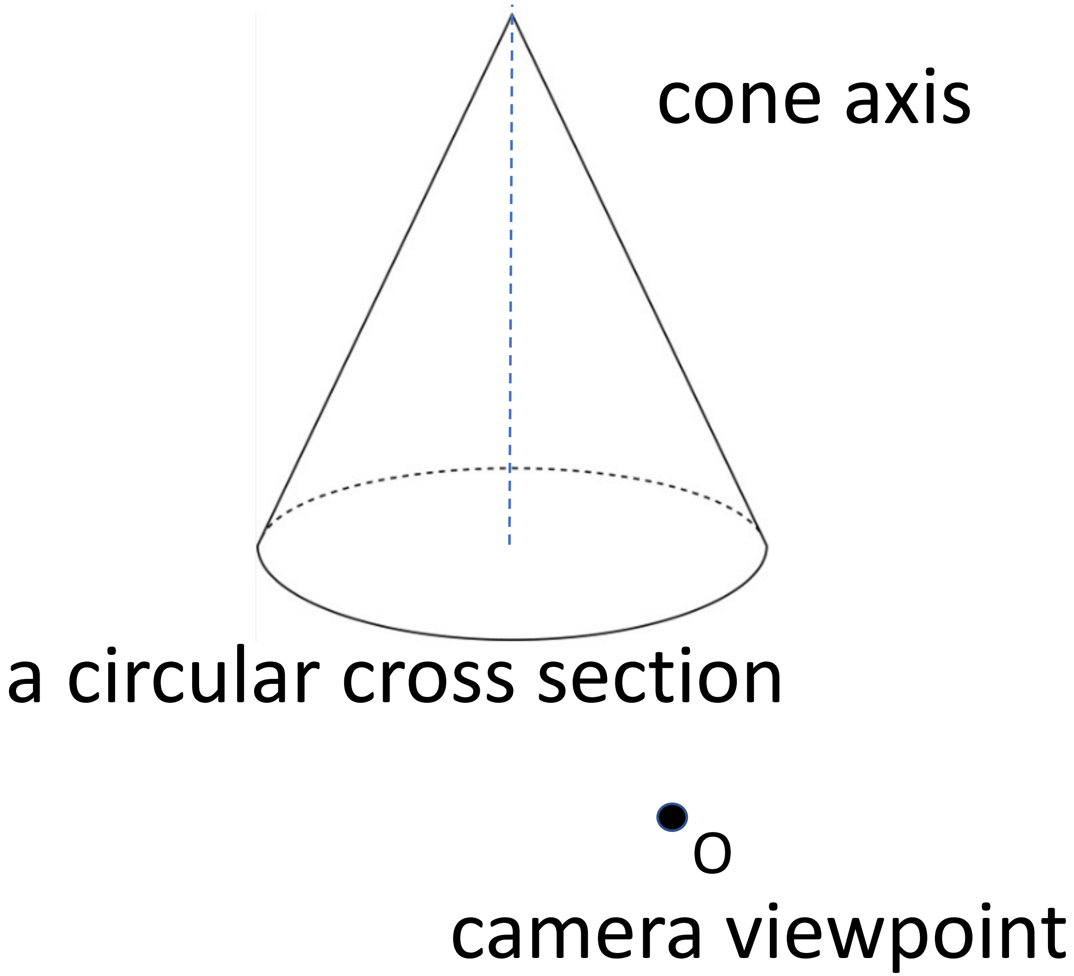
- apparent contour γ : image of Γ

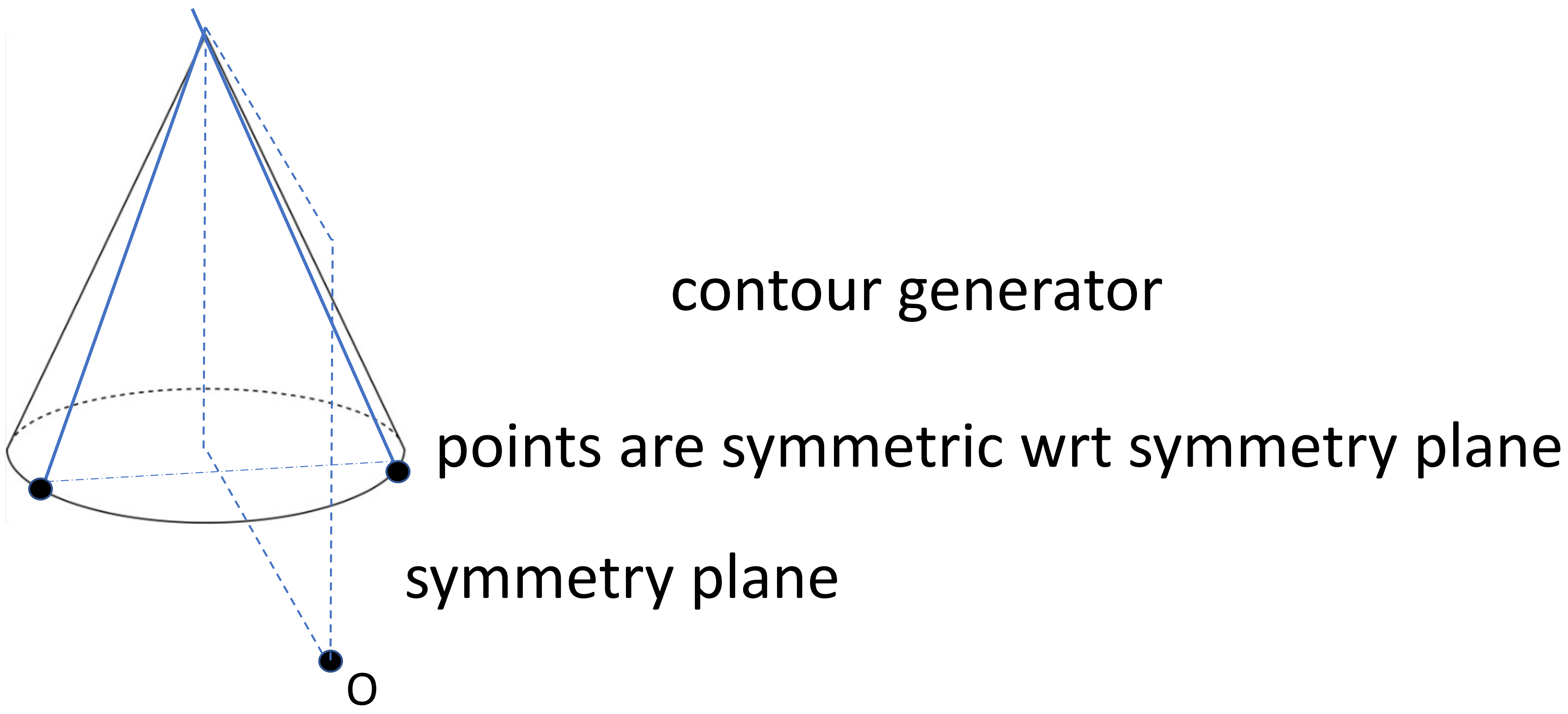
→ two image lines: i.e, a degenerate conic

Example: contour generator of a **right** cone?

- contour generator Γ: set of tangency points from O
→ two straight lines through the vertex V

- **right** cone alone is **symmetric** wrt to its axis
→ right cone + viewpoint O : «less» symmetry,
which symmetry?


cone axis
a circular cross section
o
camera viewpoint


contour generator
points are symmetric wrt symmetry plane
symmetry plane
O

Example: contour generator of a **right** cone?

- contour generator Γ: set of tangency points from O

→ two straight lines through the vertex V

- **right** cone alone is **symmetric** wrt to its axis

→ right cone + viewpoint O : «less» symmetry,

which symmetry?

PLANAR SYMMETRY wrt plane through axis and O

namely, wrt backprojection plane of the imaged axis