

ADVANCED AND MULTIVARIABLE CONTROL

June 28, 2018

Surname and Name

University Id.....Signature.....

Exercise 1

Consider the system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)(x_2(t) - 4) \\ \dot{x}_2(t) &= -x_2(t)\end{aligned}$$

- Compute the equilibrium.
- Show that it is asymptotically stable by using a quadratic Lyapunov function.
- Compute a region of attraction of the equilibrium.

SOLUTION

a) $0 = \bar{x}_1(\bar{x}_2 - 4) = 0 \rightarrow \bar{x}_1 = 0$
 $0 = \bar{x}_2$

b) $V(x) = 0.5(x_1^2 + x_2^2) \rightarrow \dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2(x_2 - 4) - x_2^2$ so the origin is locally asymptotically stable.
 $\dot{V}(x) < 0 \leftrightarrow x_2 < 4$

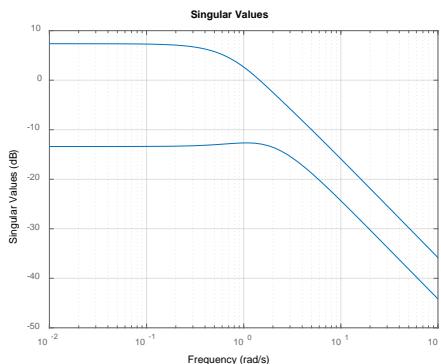
c) $\forall x_1, x_2 < 4$

Exercise 2

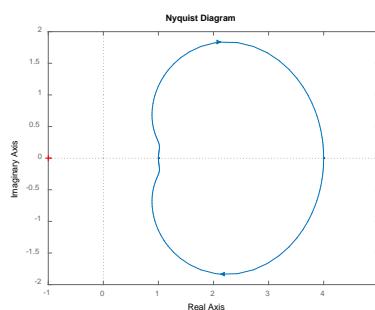
Consider the system described by the transfer function

$$G(s) = \begin{bmatrix} \frac{2}{(s+1)^2} & \frac{1}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix}$$

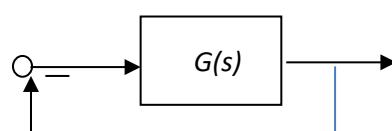
- a. Compute poles and zeros.
- b. Specify if the computed poles and/or zeros can limit the performance of a control system designed for $G(s)$.
- c. The singular values of $G(s)$ are reported in the following figure. Specify if there is a frequency band where the system can guarantee:
 - c1. amplification for any input signal,
 - c2. attenuation for any input signal



- d. The Nyquist diagram of $\det(I+G(s))$ is reported in the following figure.



Is it possible to conclude something about the stability of the following closed-loop system (motivate)?



SOLUTION

Poles: $s=-1, s=-1, s=-2$

Zeros: $s=1$

Yes the presence of a real positive zero limits the dynamic performance of a control system for $G(s)$.

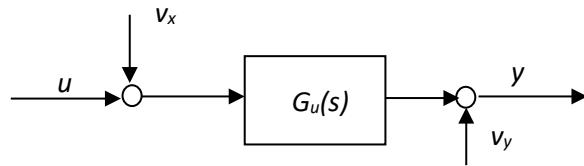
There is no any frequency band where the system can guarantee amplification for any input signal

Attenuation is guaranteed for any input signal in the frequency band $[1, \infty]$.

Loop transfer function asymptotically stable + no encirclements around the origin \rightarrow closed-loop system asymptotically stable (Nyquist criterion).

Exercise 3

Given the system



where $G_u(s) = \frac{1}{s+a}$, $a > 0$

- Assuming that v_x and v_y are null, compute the LQ control law with $R=1$, $Q=3a^2$ and the corresponding closed-loop eigenvalue.
- Assuming that y (not x) is measurable and $v_x = WN(0, 3a^2)$, $v_y = WN(0, 1)$
design the Kalman predictor and write its formula.
- Compute the overall regulator transfer function with LQ + KP and the closed-loop poles.
- Describe, in general, the Loop Transfer Recovery procedure: motivations, system assumptions, procedure.

Stationary Riccati equation of LQ control

$$0 = A'P + PA + Q - PBR^{-1}B'P$$

SOLUTION

Riccati eq.

$$p^2 + 2ap - q = 0 \rightarrow p = -a + 2a = a \quad (\text{recall that } a > 0)$$

$$K = a$$

$$A - BK = -2a$$

Concerning the Kalman predictor, for duality

$$\tilde{p} = p, L = K$$

$$\dot{\hat{x}}(t) = -2a\hat{x}(t) + bu(t) + ay(t)$$

$$\text{Regulator transfer function } R(s) = \frac{a^2}{s+3a}$$

Closed-loop poles: $A-BK=-2a$, $A-LC=-2a$

Question d: see the notes (recall that the system assumptions had to be specified)

Exercise 4

Consider a SISO system with impulse response coefficients $g_1, g_2, g_3, g_4, g_5=g_6=\dots=g_\infty=0$

Show how to design a MPC algorithm, and in particular:

- How to predict the future output as a function also of the current output;
- How to define a cost function penalizing: (a) the difference over the prediction horizon between the future outputs and a constant reference signal, (b) the future control values.
- How to include constraints on both the future outputs and the future control values, specify the constraints which can be considered as hard and the ones which must be defined as soft, and discuss how to use slack variables for soft constraints.

SOLUTIONS

$$y(k+i) = \sum_{j=1}^4 g_j u(k+i-j) + y(k) - \sum_{j=1}^4 g_j u(k-j)$$

$$\Downarrow$$

$$\begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \\ \vdots \end{bmatrix} = \begin{bmatrix} g_1 & 0 & 0 & \dots \\ g_2 & g_1 & 0 & \dots \\ g_3 & g_2 & g_1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \\ \vdots \end{bmatrix} + \begin{bmatrix} g_2 & g_3 & g_4 & 0 \\ g_3 & g_4 & 0 & 0 \\ g_4 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \\ \vdots \end{bmatrix} +$$

$$+ \begin{bmatrix} y(k) \\ y(k) \\ y(k) \\ \vdots \end{bmatrix} - \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ g_1 & g_2 & g_3 & g_4 \\ g_1 & g_2 & g_3 & g_4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \\ u(k-4) \end{bmatrix}$$

In matrix form

$$Y_{future}(k) = BU_{future}(k) + Y(k) - B_{old}U_{old}(k)$$

where

$$Y_{future}(k) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \\ \vdots \end{bmatrix}, U_{future}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \\ \vdots \end{bmatrix}, Y(k) = \begin{bmatrix} y(k) \\ y(k) \\ y(k) \\ \vdots \end{bmatrix}, U_{old}(k) = \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \\ u(k-4) \end{bmatrix}$$

Questions B), C), see the notes