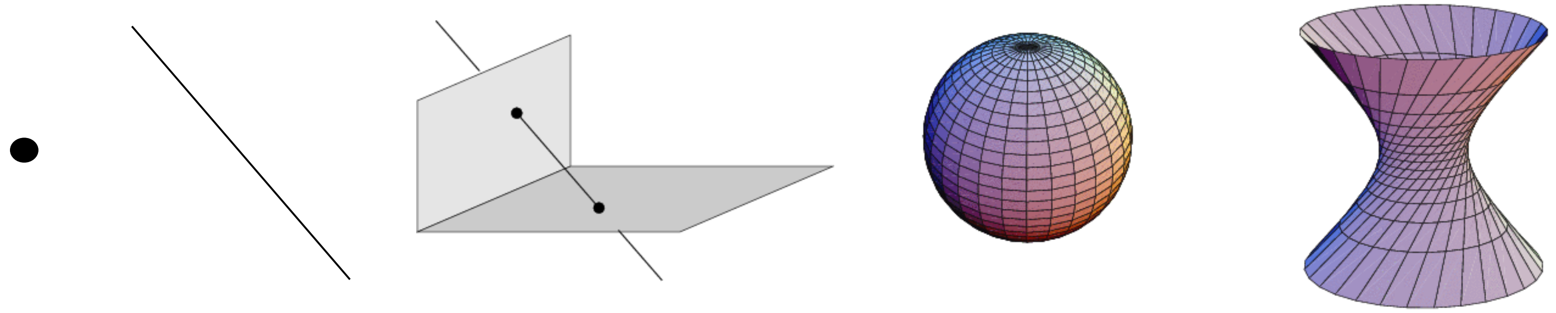
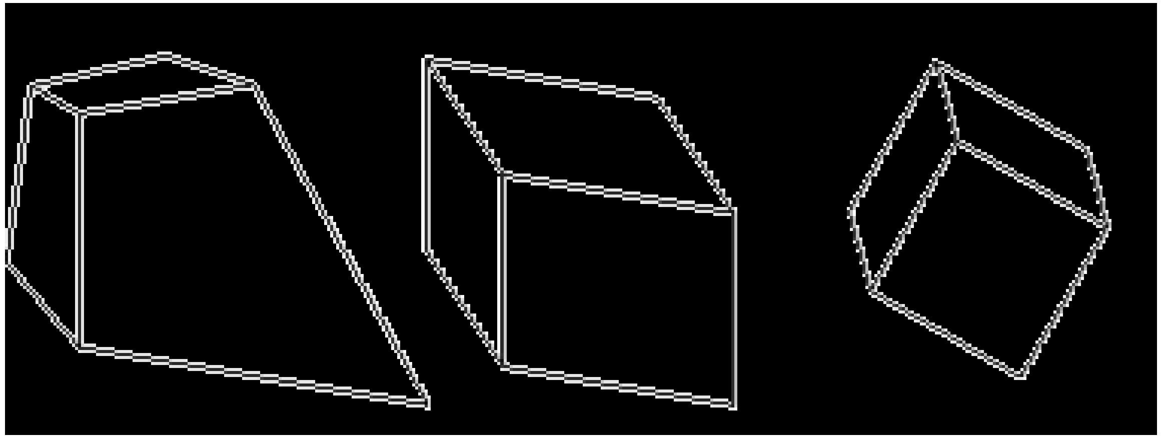
Space (3D) Projective Geometry

3D Space Projective Geometry

- **Elements**
- Points
- Planes
- Quadrics
- Dual quadrics



- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



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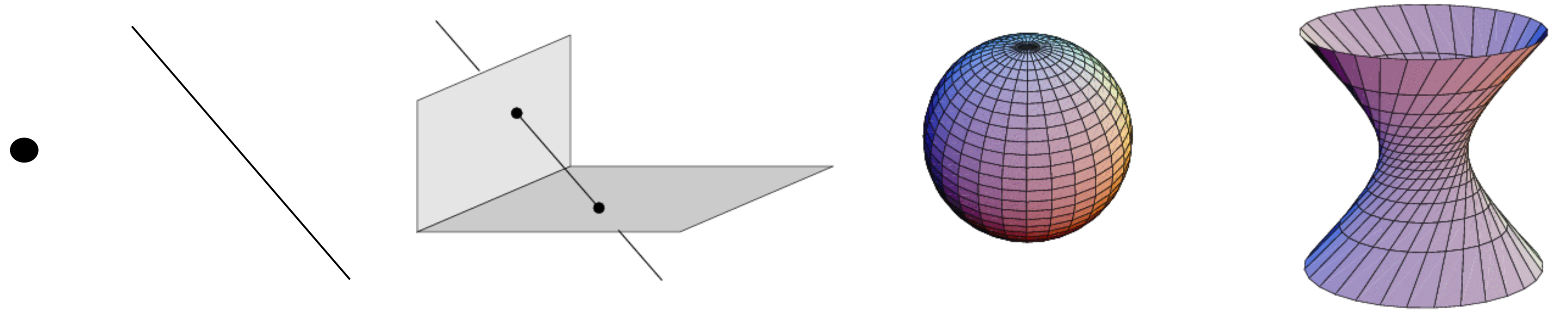

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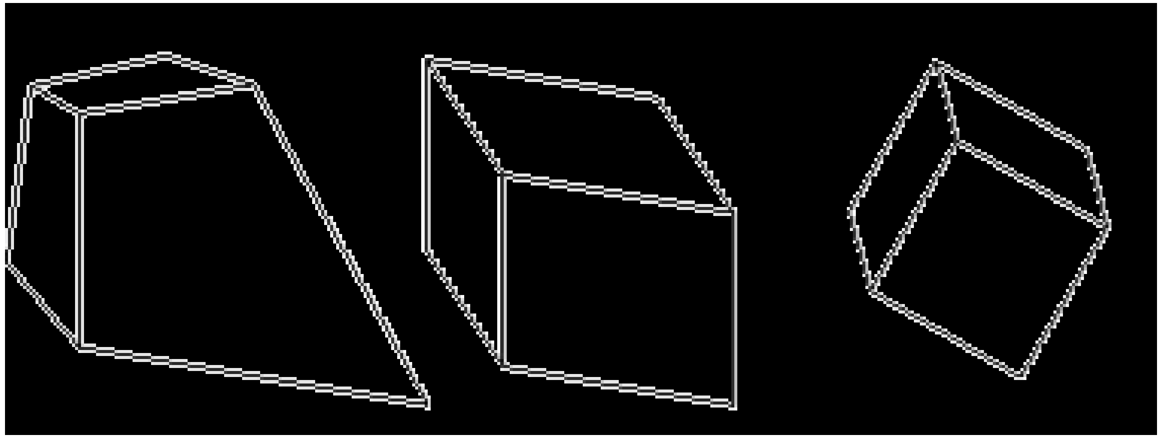
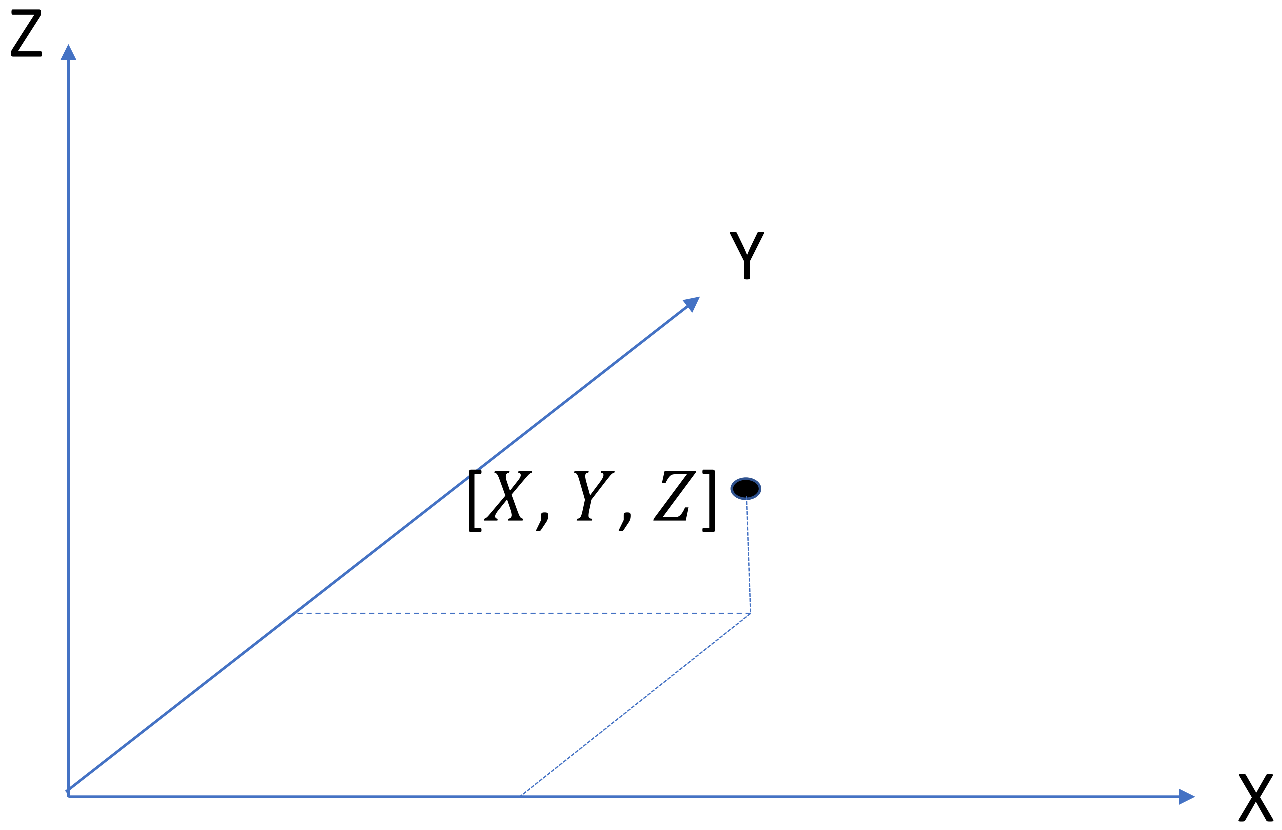
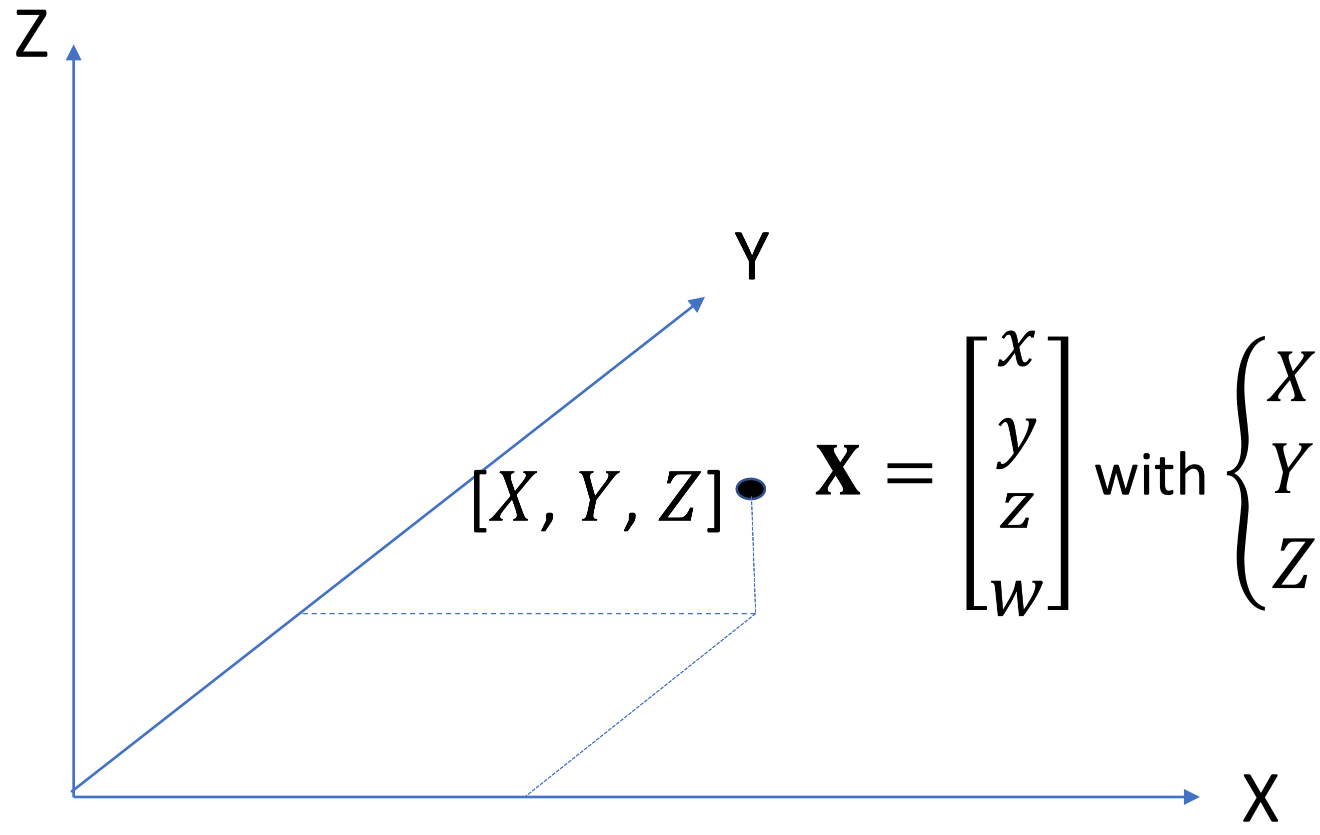


Points in the projective space

Euclidean space (3D) cartesian coordinates

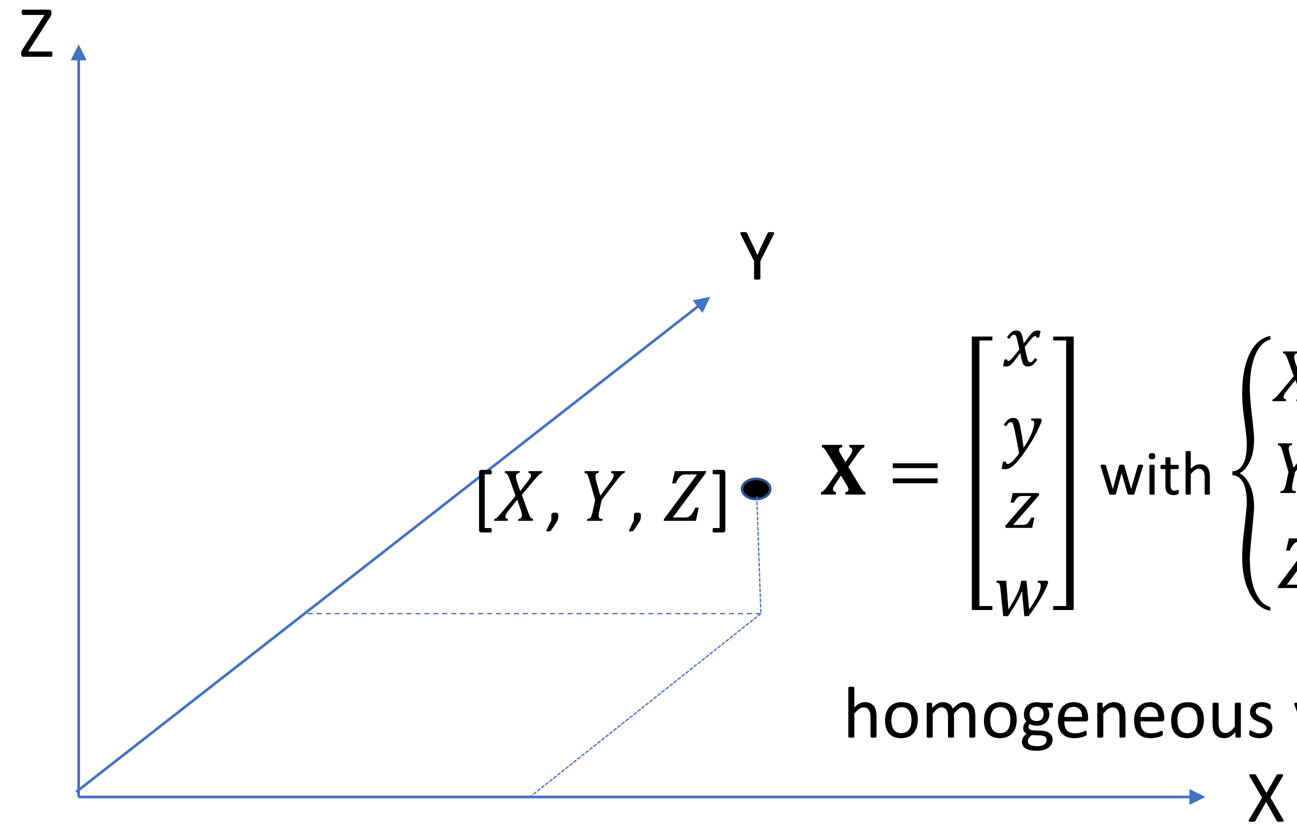


Projective space (3D)
4 homogeneous coordinates



$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

Projective space (3D)
4 homogeneous coordinates

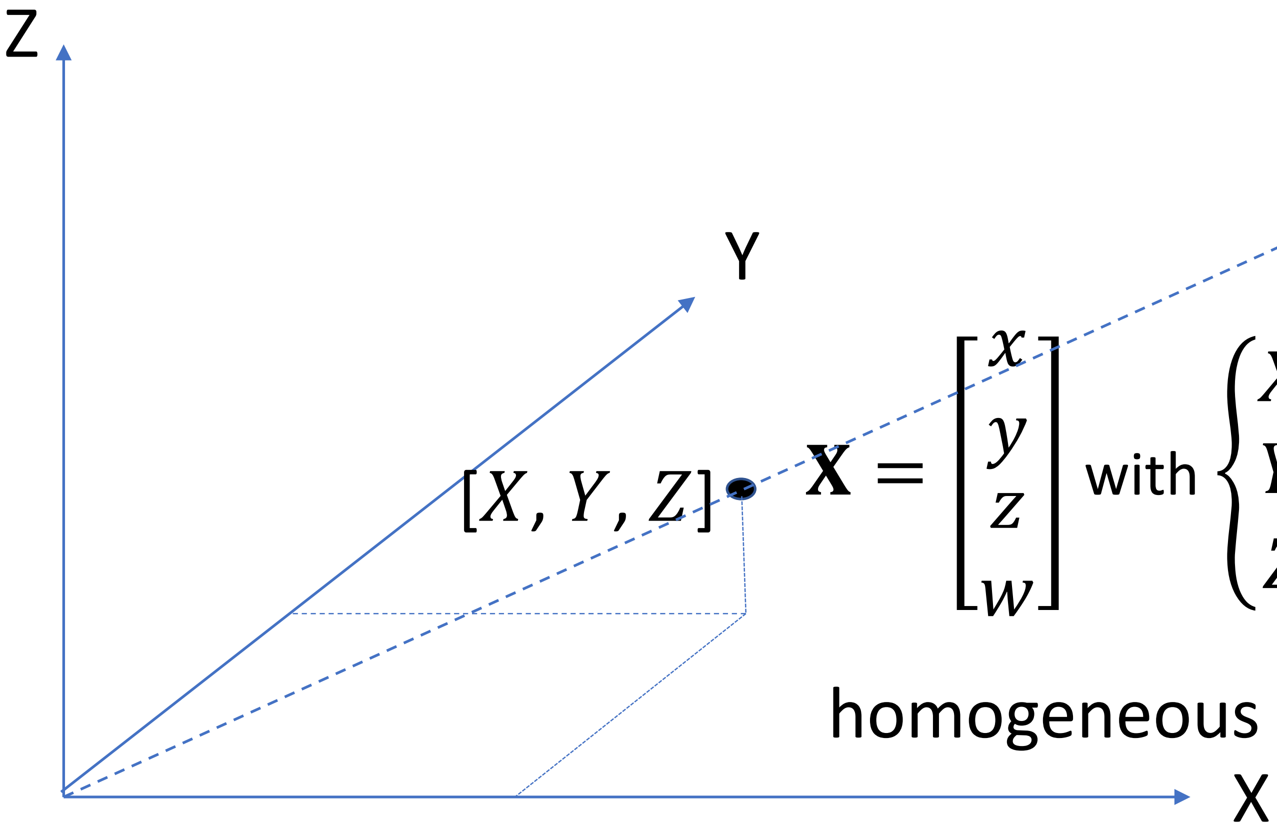


$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

homogeneous vector: $\lambda\mathbf{X} \Leftrightarrow \mathbf{X}$

Projective space $\mathbb{P}^3$: points at the ∞

$$\mathbf{X}_\infty = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$



$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ with } \begin{cases} X = x/w \\ Y = y/w \\ Z = z/w \end{cases}$$

homogeneous vector: $\lambda\mathbf{X} \Leftrightarrow \mathbf{X}$

$$\mathbb{P}^3 = \{\mathbf{X} \in \mathbb{R}^4 - \{[0 \quad 0 \quad 0 \quad 0]^T\}\}$$

redundancy

4 homogeneous coordinates to represent points in the 3D space (3 dof)

an infinite number of equivalent representations for a single point, namely all nonzero multiples of the vector $[X \quad Y \quad Z \quad 1]^T$

the null vector $[0 \quad 0 \quad 0 \quad 0]^T$ **does not** represent any point

→ Projective space $\mathbb{P}^3 = \{[x \quad y \quad z \quad w]^T \in \mathbb{R}^4\} - \{[0 \quad 0 \quad 0 \quad 0]^T\}$

→ its three degrees of freedom are the three independent ratios between the four coordinates $x : y : z : w$

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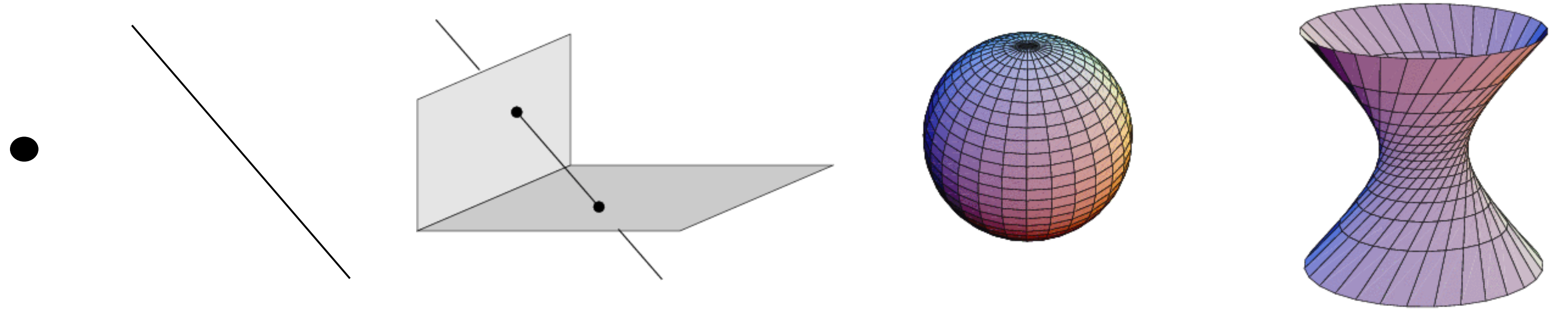

- **Transformations**
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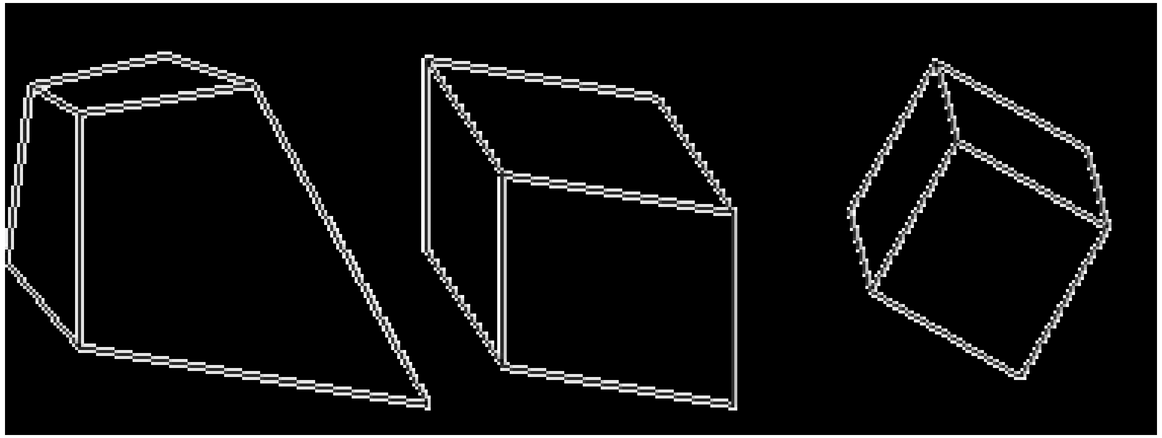
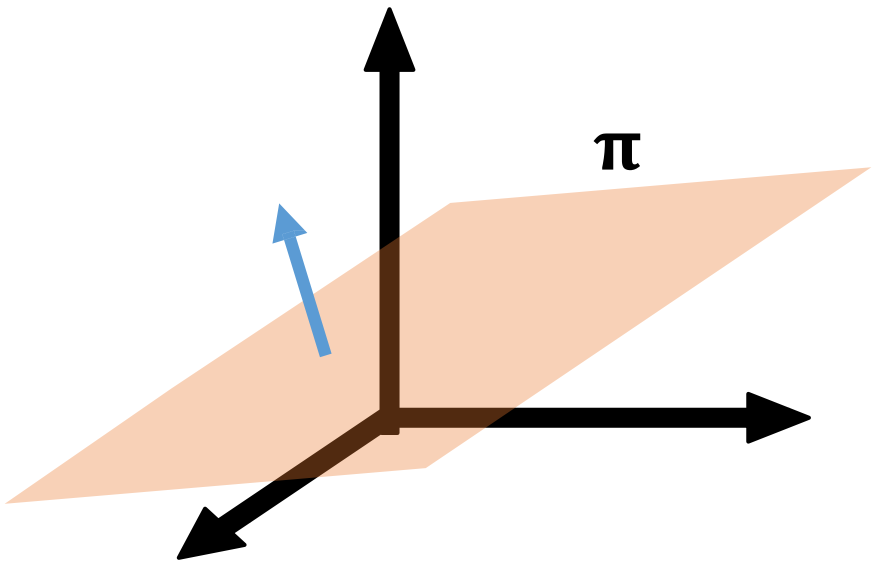


Planes in the projective space

Planes in 3D Projective Geometry



$\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with direction $\begin{pmatrix} a & b & c \end{pmatrix}$ normal to the plane,

and $\frac{-d}{\sqrt{a^2+b^2+c^2}}$ = the distance between the origin and the plane

$\boldsymbol{\pi}$ is a homogeneous vector: $\lambda\boldsymbol{\pi} \Leftrightarrow \boldsymbol{\pi}$

redundancy

4 homogeneous parameters to represent planes in the 3D space (3 dof)

an infinite number of equivalent representations for a single planes, namely all nonzero multiples of the homogeneous vector
$[a \quad b \quad c \quad d]^T$

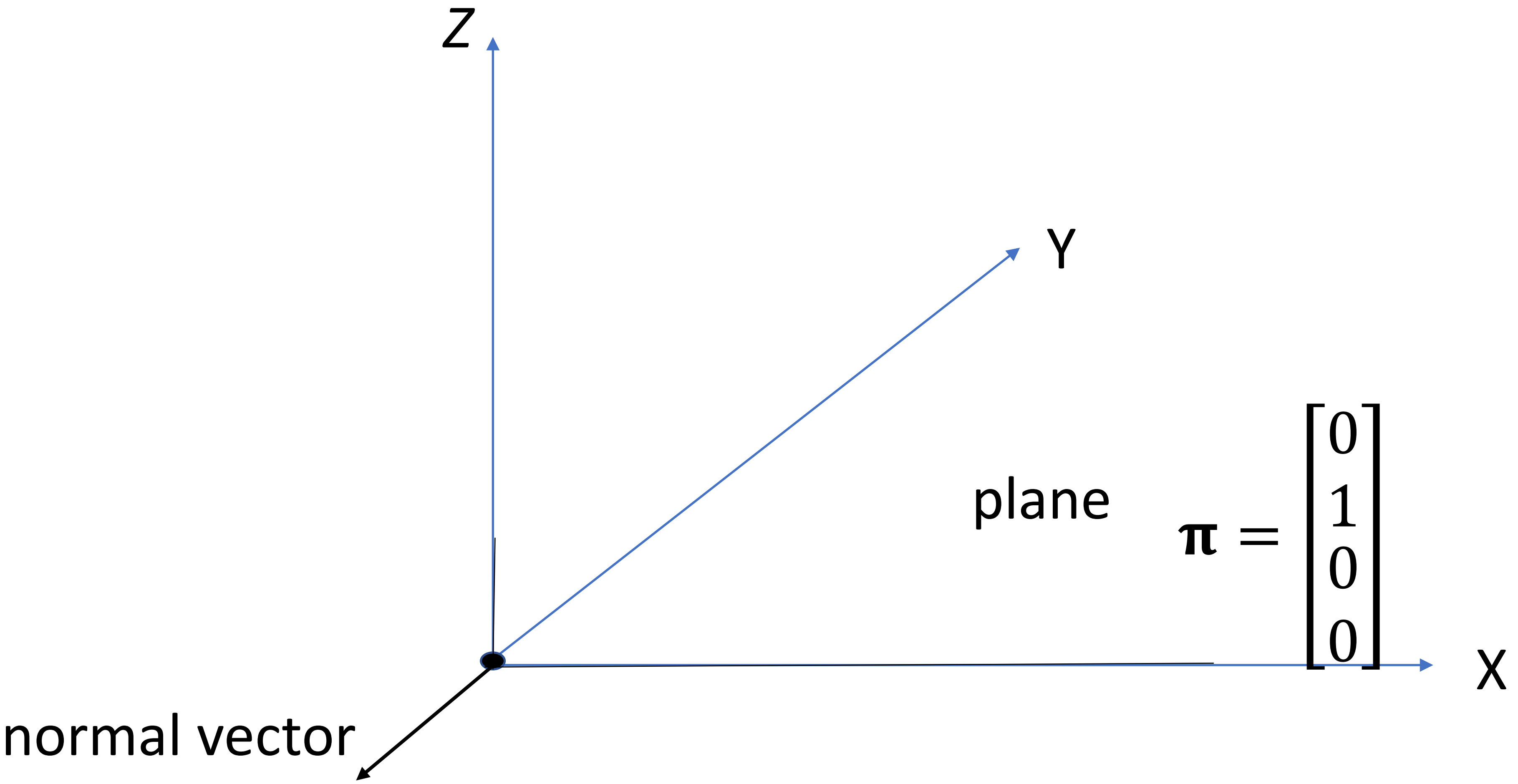
the null vector $[0 \quad 0 \quad 0 \quad 0]^T$ **does not** represent any plane

→ its three degrees of freedom are the three independent ratios between the four parameters a $: b : c : d$

remark

If the fourth parameter d is null, $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, then the plane goes through point $[0,0,0]$

Example: the X-Z plane



$$\boldsymbol{\pi} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

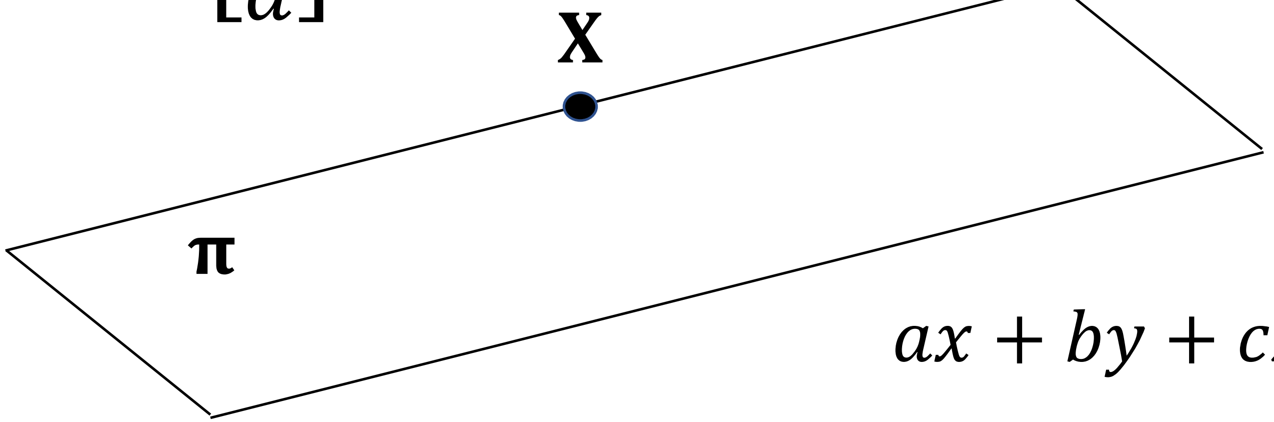
The incidence relation:
a point is on a plane (or a plane goes through a point)

Incidence relation

the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is on the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

or

the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ goes through the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$



$$ax + by + cz + dw = 0$$

Incidence relation

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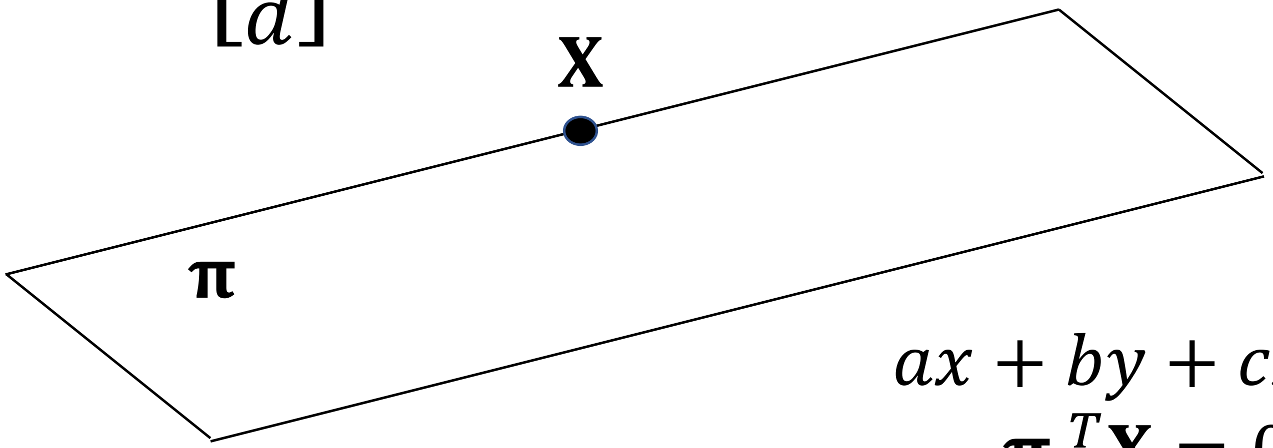

$$ax + by + cz + dw = 0$$
$$\boldsymbol{\pi}^T\mathbf{X} = 0 = \mathbf{X}^T\boldsymbol{\pi}$$

Incidence relation

the point $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is on the plane $\boldsymbol{\pi} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

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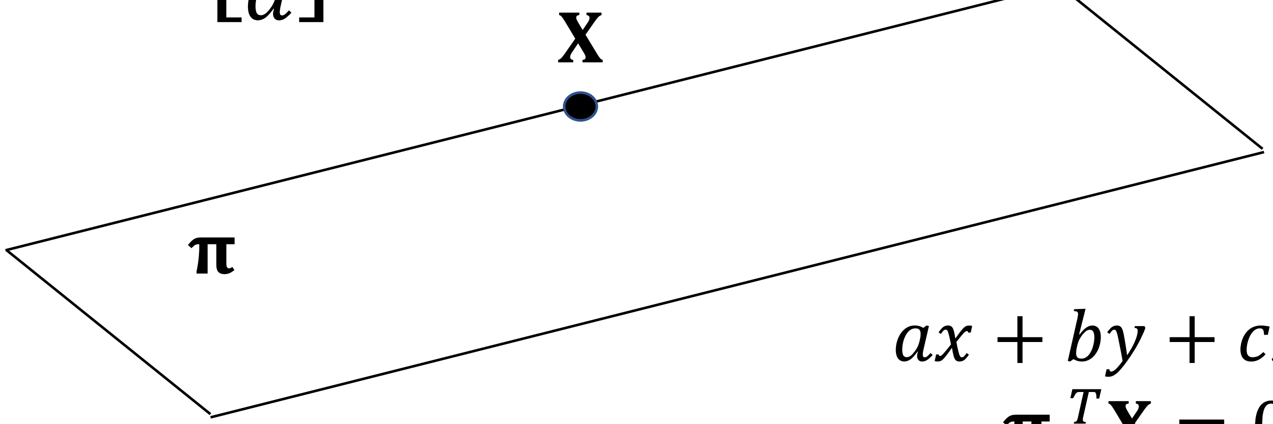


$$ax + by + cz + dw = 0$$

$$\boldsymbol{\pi}^T\mathbf{X} = 0 = \mathbf{X}^T\boldsymbol{\pi}$$

Dividing by $\boldsymbol{w}$ we find the cartesian coordinates again

The plane at the infinity: the locus of the points at the infinity

The «locus» of the points at the infinity

As there are infinite points at the infinity (one for each direction), what is the aspect of the set of these points at the infinity?

Simply: $w = 0$

This set is a plane: $[\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c} \quad \boldsymbol{d}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\\\boldsymbol{w}\end{bmatrix} = 0$, actually $[\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\\\boldsymbol{w}\end{bmatrix} = 0$

namely, **the plane at the infinity** $\boldsymbol{\pi}_\infty = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

NOTE: this plane has undefined normal direction

The duality principle between points and planes

2. Since dot product is commutative → incidence relation is commutative

$$\boldsymbol{\pi}^T\mathbf{X} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix} = 0 = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} & \boldsymbol{w} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = 0 = \mathbf{X}^T\boldsymbol{\pi} = \mathbf{0}$$

point $\mathbf{X}$ is on plane $\boldsymbol{\pi}$

$\Leftrightarrow$

point $\boldsymbol{\pi}$ is on **plane** $\mathbf{X}$

point $\mathbf{X}$ is on plane $\boldsymbol{\pi}$ (i.e. plane $\boldsymbol{\pi}$ goes through point $\mathbf{X}$)

$\leftarrow\rightarrow$

point $\boldsymbol{\pi}$ is on **line** $\mathbf{X}$ (i.e. line $\mathbf{X}$ goes through point $\boldsymbol{\pi}$)

Principle of **duality** between points and planes
in 3D Projective Geometry

For any true sentence containing the words

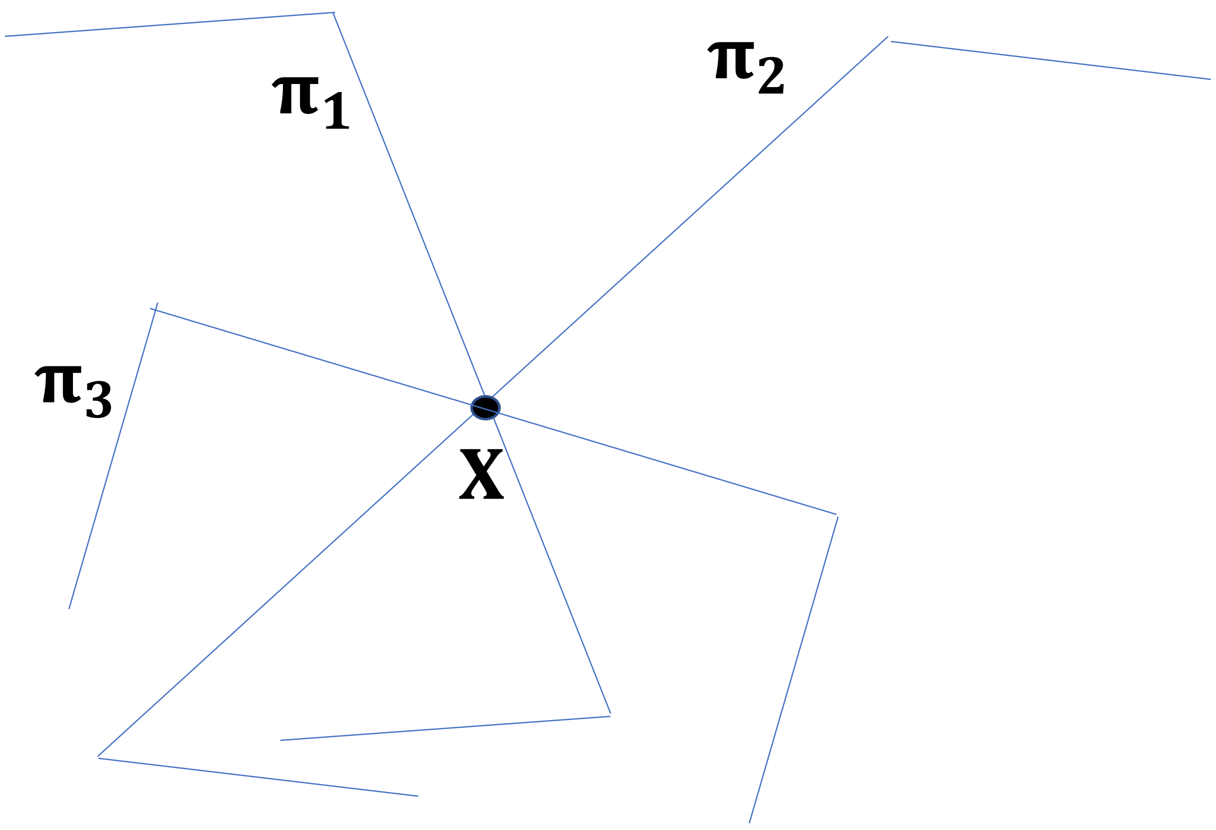
- point
- plane
- is on
- goes through

there is a DUAL sentence -also true- obtained by substituting, in the previous one, each occurence of

- point	by	- plane
- plane	by	- point
- is on	by	- goes through
- goes through	by	- is on

The point on three planes

the point on three planes



$$\begin{cases} \boldsymbol{\pi}_1{}^T\mathbf{X} = 0 \\ \boldsymbol{\pi}_2{}^T\mathbf{X} = 0 \\ \boldsymbol{\pi}_3{}^T\mathbf{X} = 0 \end{cases}$$

$$\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \\ \boldsymbol{\pi}_3{}^T \end{bmatrix}_{3 \times 4} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

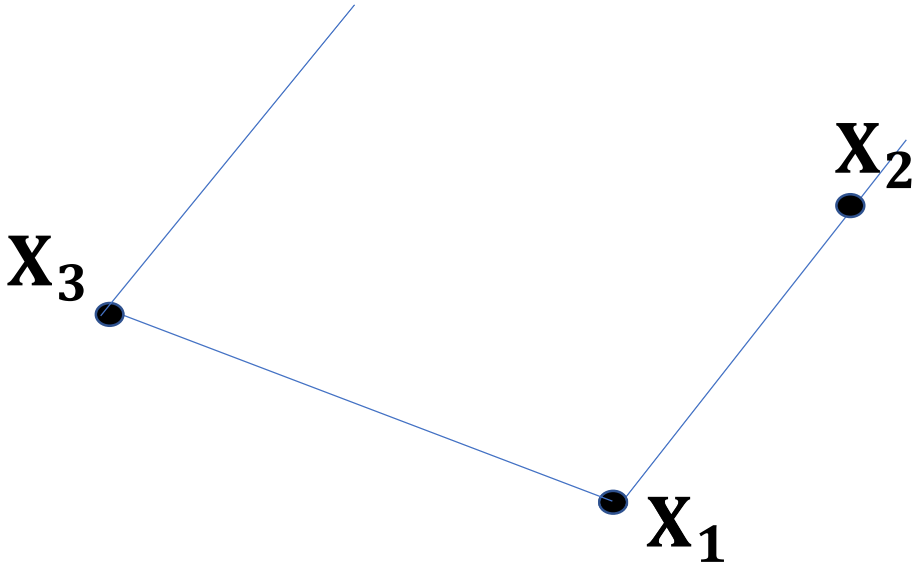
$$\mathbf{X} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \\ \boldsymbol{\pi}_3{}^T \end{bmatrix}_{3 \times 4}\right)$$

a solution vector + all its mutiples

the plane through three points

the plane through three points:
dual of the point through three planes

the plane through three points



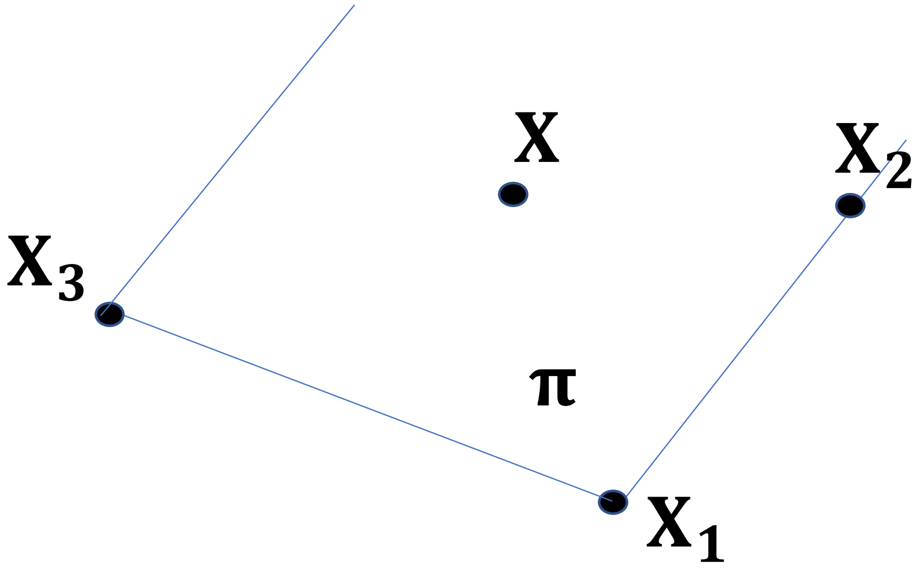
$$\begin{cases} \mathbf{X_1}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_2}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_3}^T \boldsymbol{\pi} = 0 \end{cases}$$

$$\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \\ \mathbf{X_3}^T \end{bmatrix}_{3 \times 4} \boldsymbol{\pi} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\pi} = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \\ \mathbf{X_3}^T \end{bmatrix}_{3 \times 4} \right)$$

a solution vector + all its mutiples

the plane as its span



$\mathbf{X}$ is a linear combination $\propto \mathbf{X_1} + \beta \mathbf{X_2} + \gamma \mathbf{X_3}$

→ $\mathbf{X}$ is coplanar to $\mathbf{X_1}, \mathbf{X_2},$ and $\mathbf{X_3}$

i.e. $\mathbf{X} = [\mathbf{X_1} \quad \mathbf{X_2} \quad \mathbf{X_3}] \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix} = \mathrm{M}\mathbf{x}$ where

$\mathbf{x} = \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix}$ can be regarded as homogeneous coordinates within the 2D geometry of plane $\boldsymbol{\pi}$

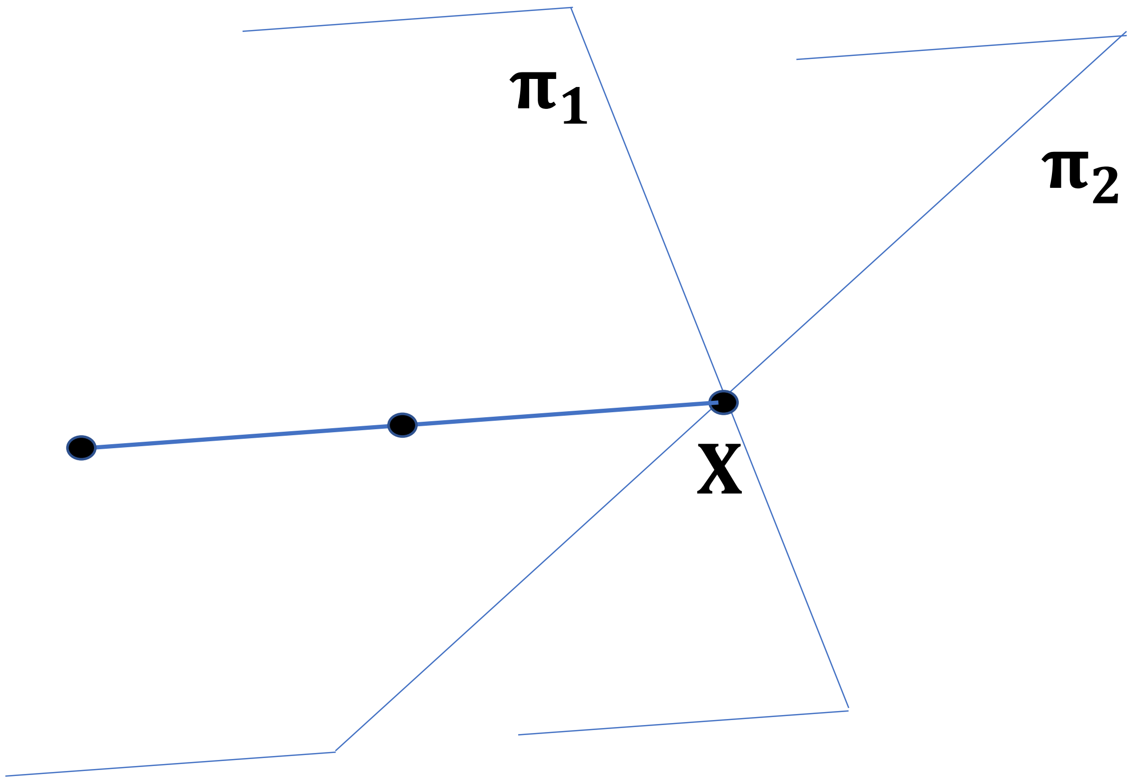
$$\mathbf{X} = \mathrm{M}\mathbf{x}$$

LINES ?

Lines are primitive elements in the planar geometry
but they are **not** primitive elements in the space geometry
There is no minimal parameterization for lines in 3D
However
in the 3D space there are (∞) planes, and planes contain lines

Lines are intermediate entities between points and planes
they are self-dual

Line: the set of points $\mathbf{X}$ on **two** planes



$$\begin{cases} \boldsymbol{\pi}_1{}^T \mathbf{X} = 0 \\ \boldsymbol{\pi}_2{}^T \mathbf{X} = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \end{bmatrix}}_{2\text{ x }4} \mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

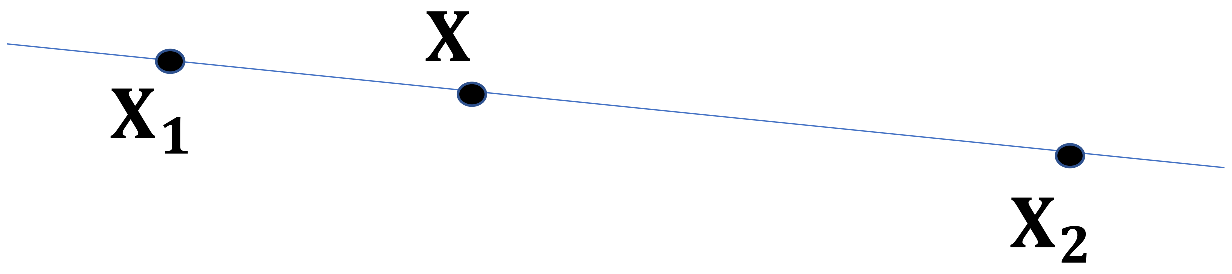
$$\mathbf{L} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{\pi}_1{}^T \\ \boldsymbol{\pi}_2{}^T \end{bmatrix}_{2\text{ x }4}\right)$$

2D set of solution vectors: two points and all their linear combinations

→ due to homogeneity: 1D set of points (parameter abscissa)

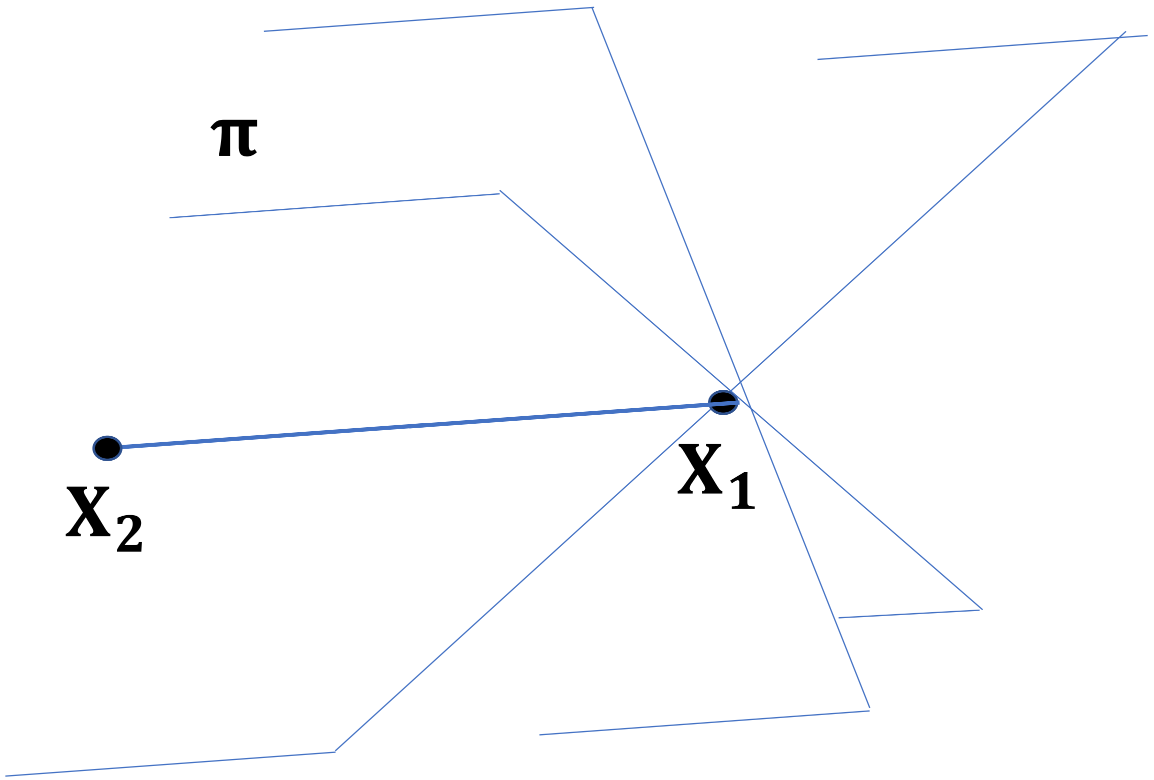
linear combination of two points

Property: the point $\mathbf{X}$ given by the linear combination $\mathbf{X} = \propto \mathbf{X}_1 + \beta\mathbf{X}_2$ of two points $\mathbf{X}_1$ and $\mathbf{X}_2$ is on the line $\mathbf{L}$ through $\mathbf{X}_1$ and $\mathbf{X}_2$



A **line $\mathbf{L}$** can also be defined as the set of all points, that are linear combinations of two given points: $\mathbf{X} = \propto \mathbf{X}_1 + \beta\mathbf{X}_2$

Line: the set of planes **π** through **two** points



$$\begin{cases} \mathbf{X_1}^T \boldsymbol{\pi} = 0 \\ \mathbf{X_2}^T \boldsymbol{\pi} = 0 \end{cases}$$

$$\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}_{2 \times 4} \boldsymbol{\pi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

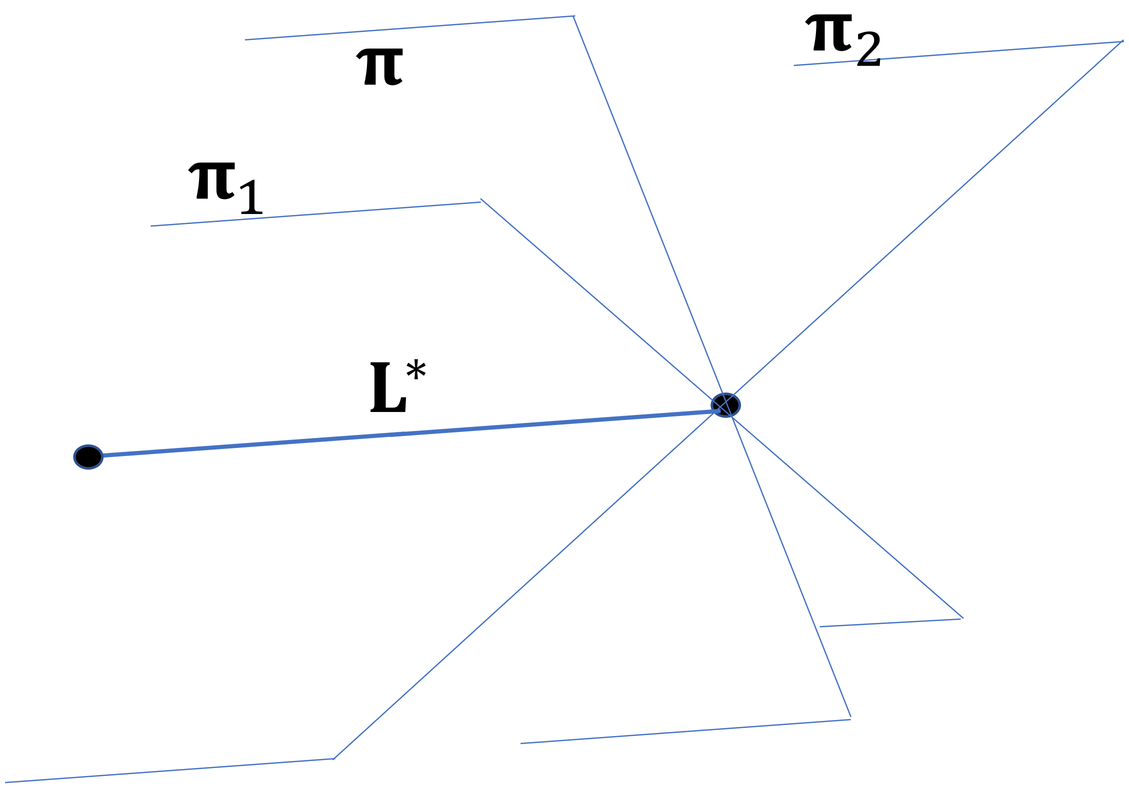
$$\mathbf{L}^* = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}_{2 \times 4}\right)$$

2D set of vector solutions: two planes and all their linear combinations

→ due to homogeneity: 1D set of planes (parameter: rottion angle)

DUAL: linear combination of two planes

Dual property: the plane $\boldsymbol{\pi}$**,** given by the linear combination $\boldsymbol{\pi} = \propto \boldsymbol{\pi}_1 + \beta\boldsymbol{\pi}_2$ of two planes $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$, goes through the line $\mathbf{L}^*$ on $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$



pairs of DUALLY corresponding words

- point	→	- plane
- line	→	- line
- plane	→	- point
- is on	→	- goes through
- goes through	→	- is on

Each plane $\boldsymbol{\pi}$ has its own line at the infinity $\boldsymbol{l}_{\infty}(\boldsymbol{\pi})$
and also its own circular points $\boldsymbol{I}(\boldsymbol{\pi})$ and $\boldsymbol{J}(\boldsymbol{\pi})$

parallel planes share the same $\boldsymbol{l}_{\infty}$
and the same circular points $\boldsymbol{I}$ and $\boldsymbol{J}$

Angle between two 3D directions

E.g.

- Angle between the directions normal to two planes
- Angle between the directions of two points at the infinity

Angle between two 3D directions

Angle between the normals to planes $\boldsymbol{\pi_1} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$ and $\boldsymbol{\pi_2} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}$:

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{\left(a_1{}^{\mathbf{2}} + b_1{}^{\mathbf{2}} + c_1{}^{\mathbf{2}}\right)\left(a_2{}^{\mathbf{2}} + b_2{}^{\mathbf{2}} + c_2{}^{\mathbf{2}}\right)}}$$

Angle between two 3D directions

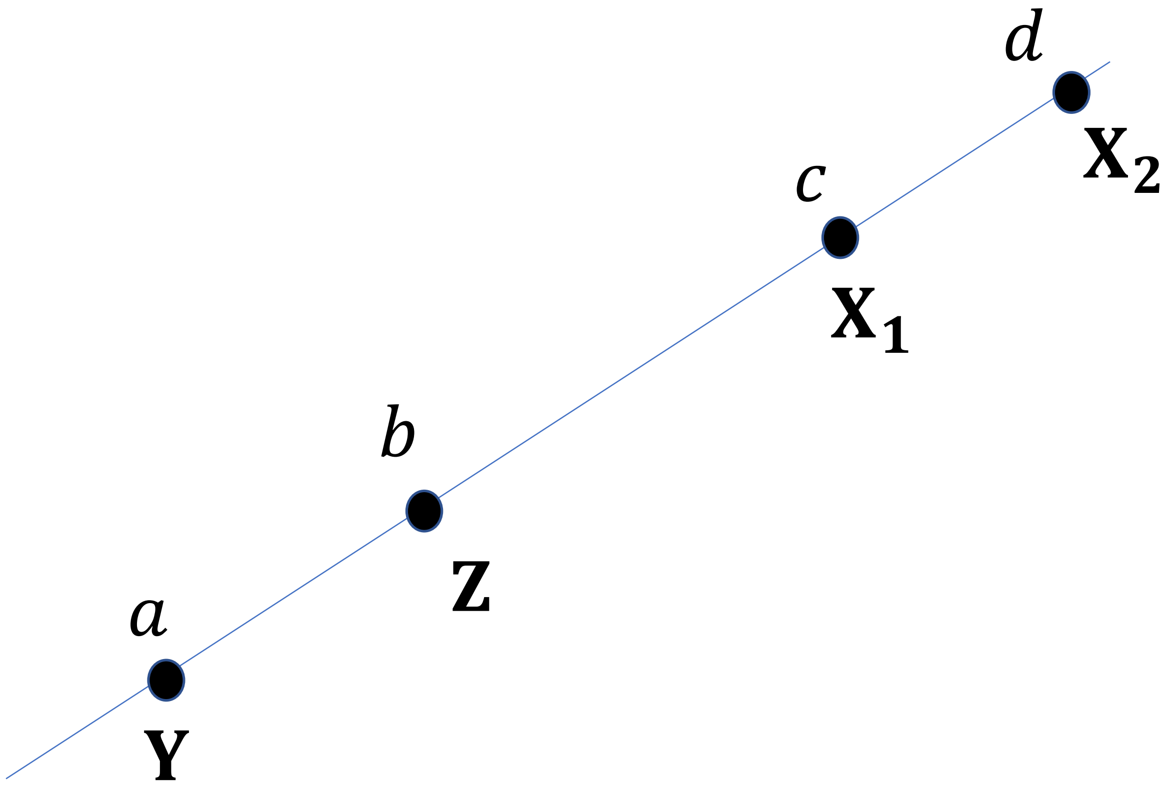
Angle between the directions of points $\mathbf{X}_{\infty\mathbf{1}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{bmatrix}$ and $\mathbf{X}_{\infty\mathbf{2}} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 0 \end{bmatrix}$:

$$\cos\vartheta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{({x_1}^{\mathbf{2}} + {y_1}^{\mathbf{2}} + {z_1}^{\mathbf{2}})({x_2}^{\mathbf{2}} + {y_2}^{\mathbf{2}} + {z_2}^{\mathbf{2}})}}$$

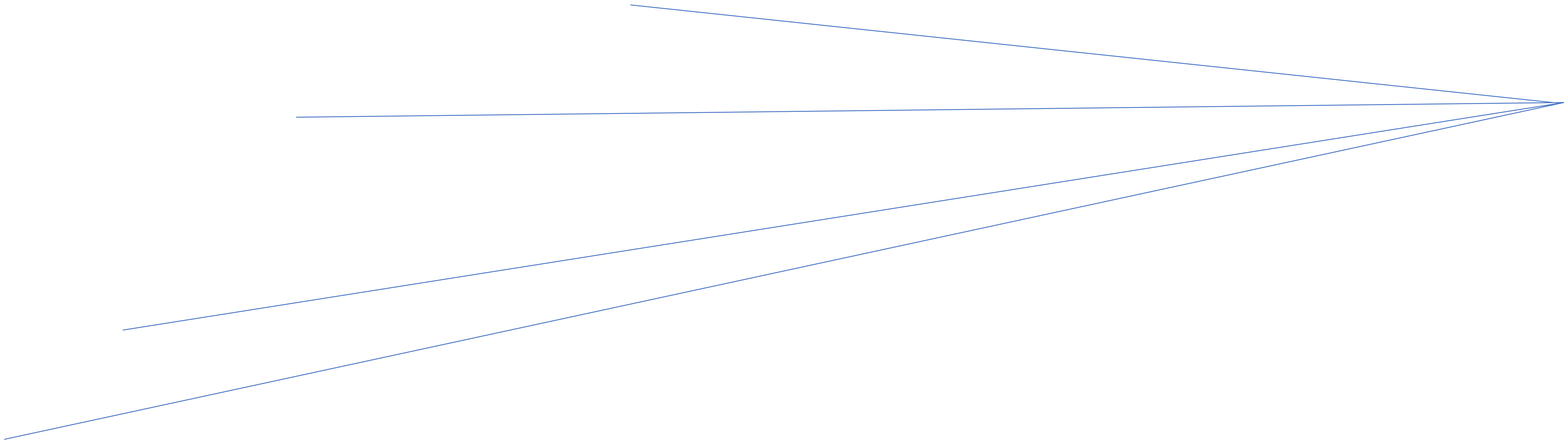
The cross ratio

1D cross ratio of a 4-tuple of colinear points

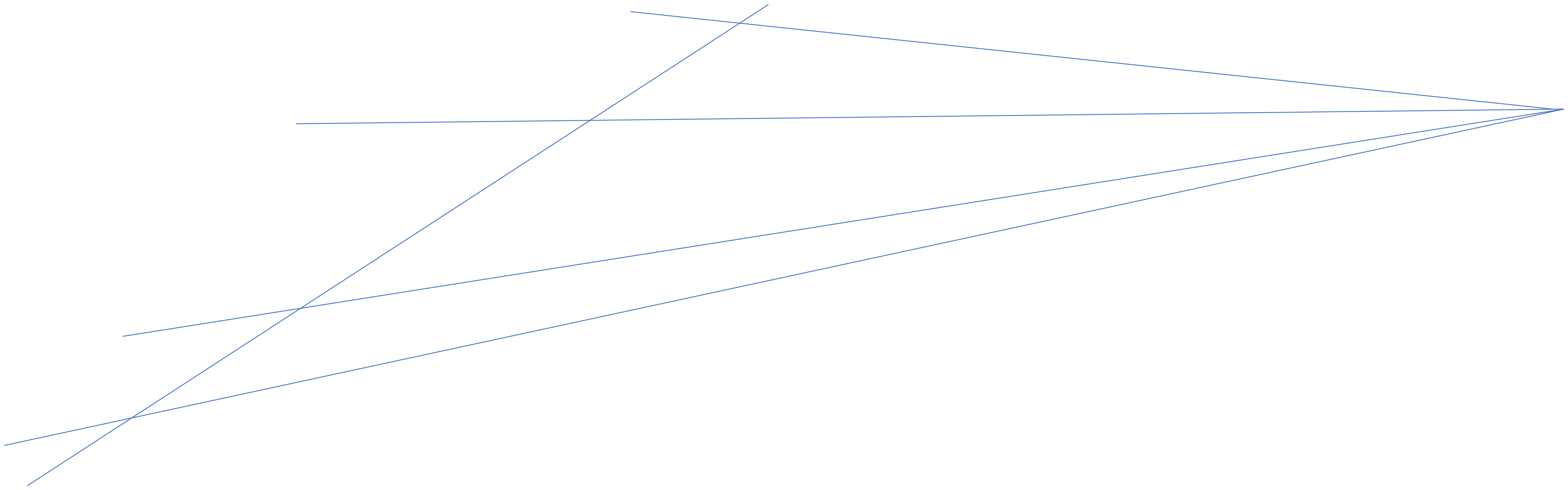
$$CR_{Y,Z,\mathrm{X}_1,\mathrm{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$



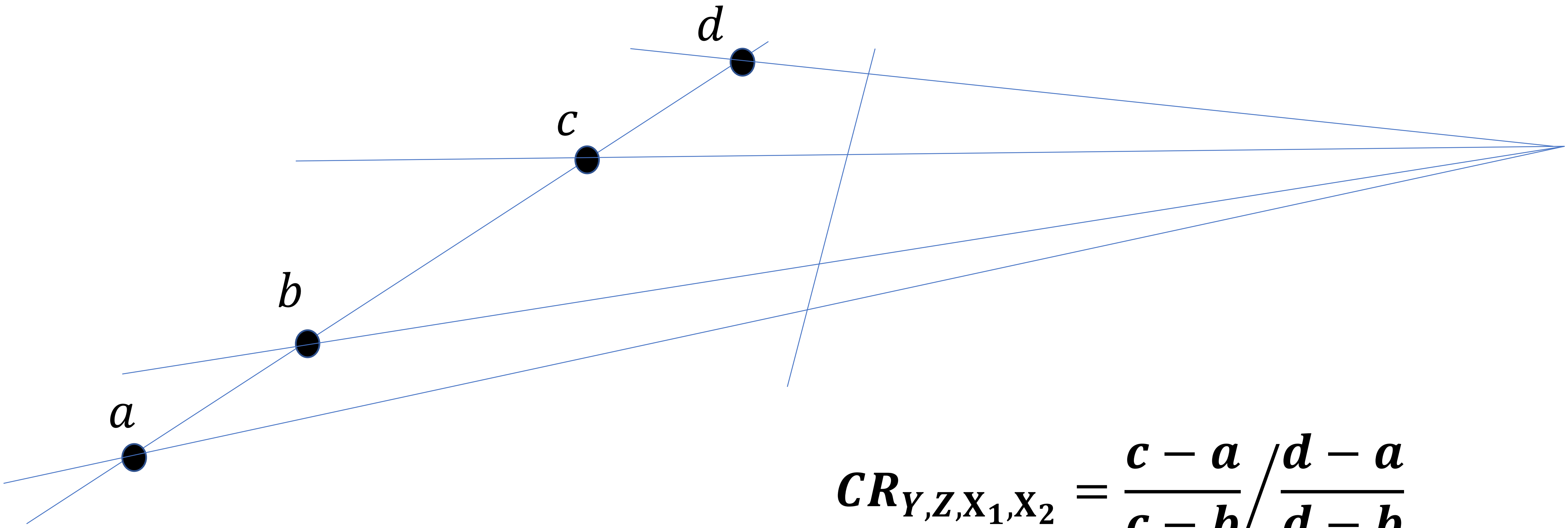
2D cross ratio of a 4-tuple of coplanar, concurrent lines



2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line …



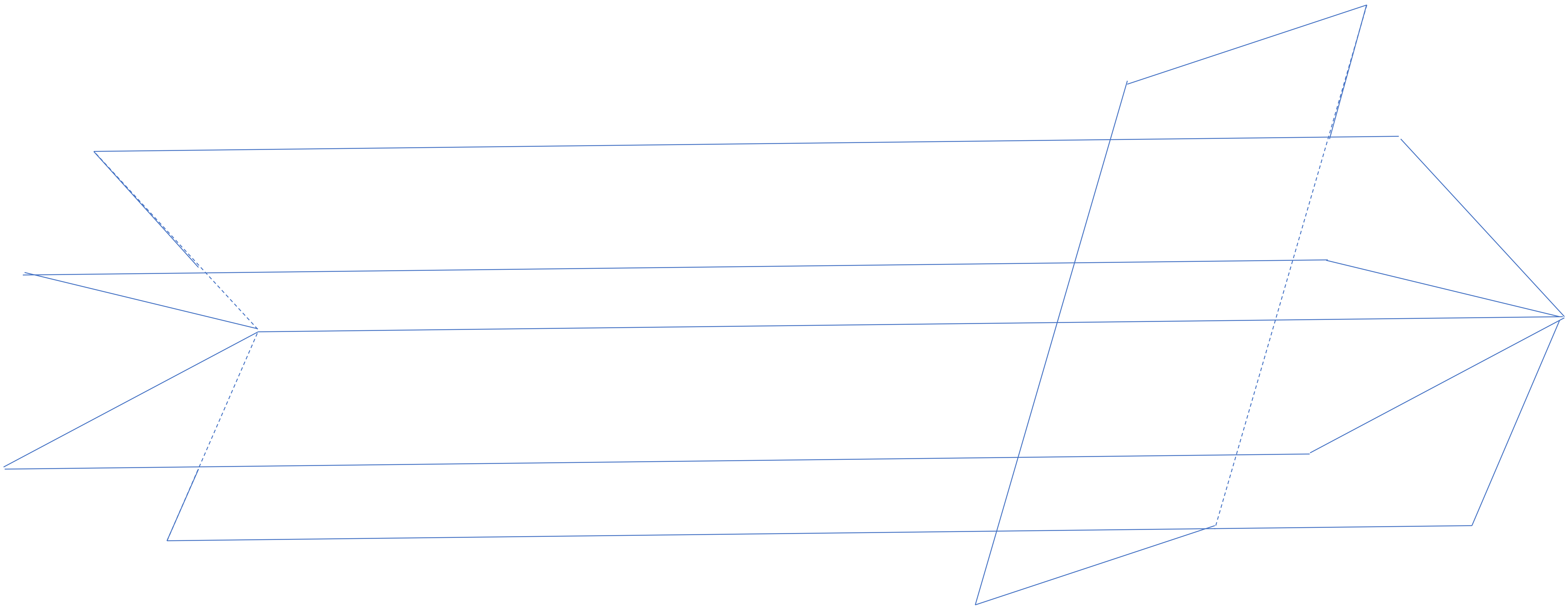
2D cross ratio of a 4-tuple of coplanar, concurrent lines: take any crossing line … compute the 1D cross ratio of intersection points



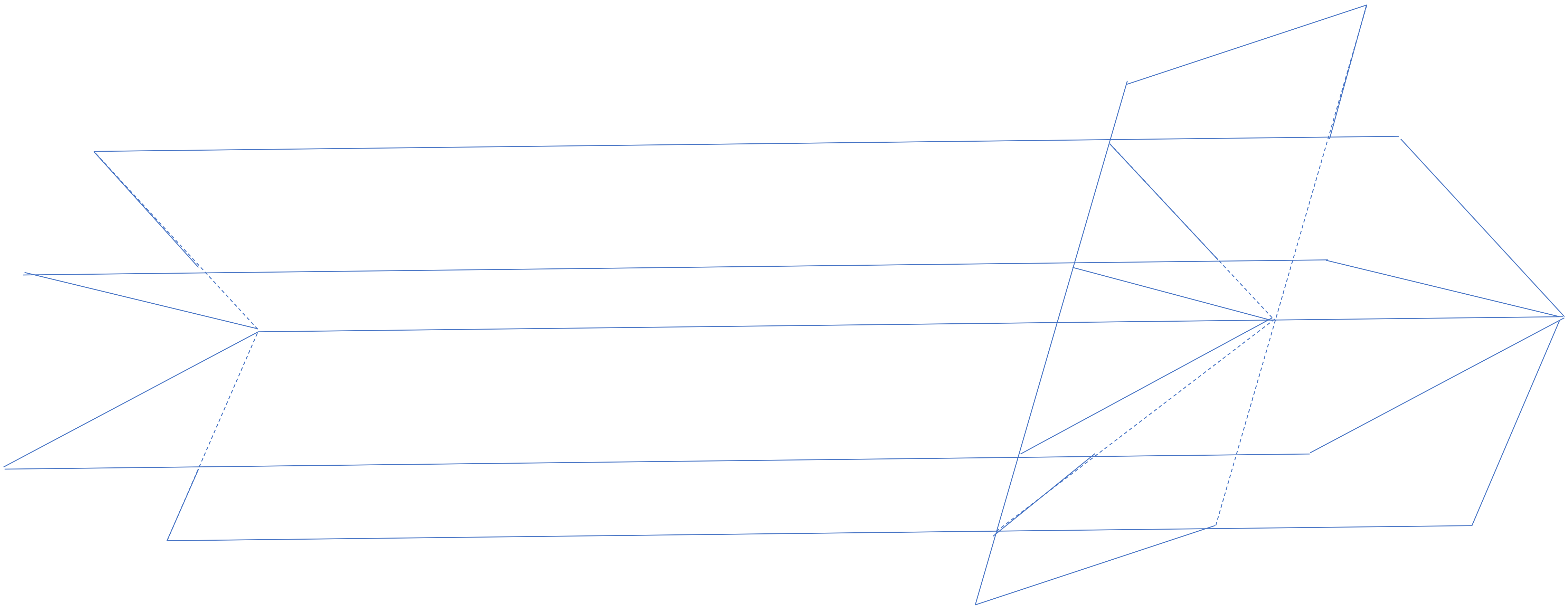
$$CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

3D cross ratio of a 4-tuple of coaxial planes:

3D cross ratio of a 4-tuple of coaxial planes: take any crossing plane …



3D cross ratio of a 4-tuple of coaxial planes: take any crossing plane … compute the 2D cross ratio of intersection lines



QUADRICS

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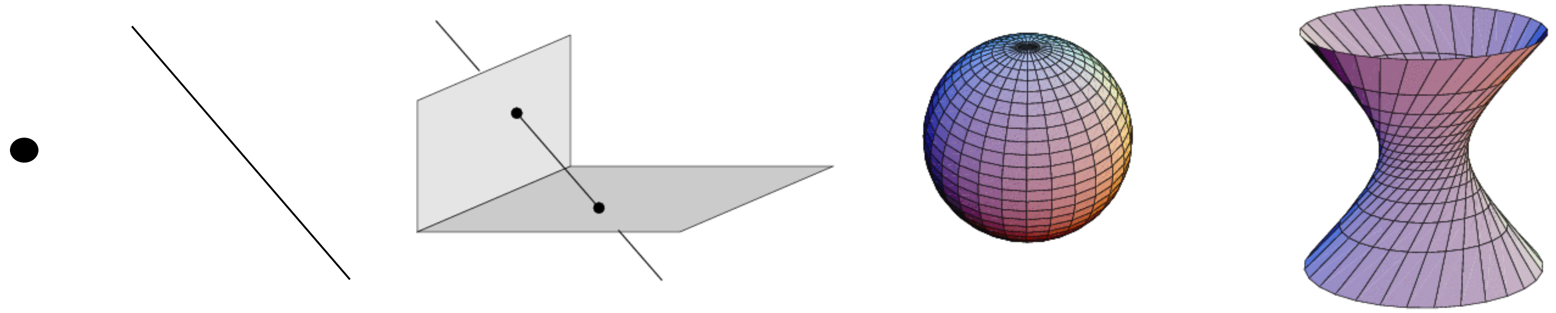
- Dual quadrics



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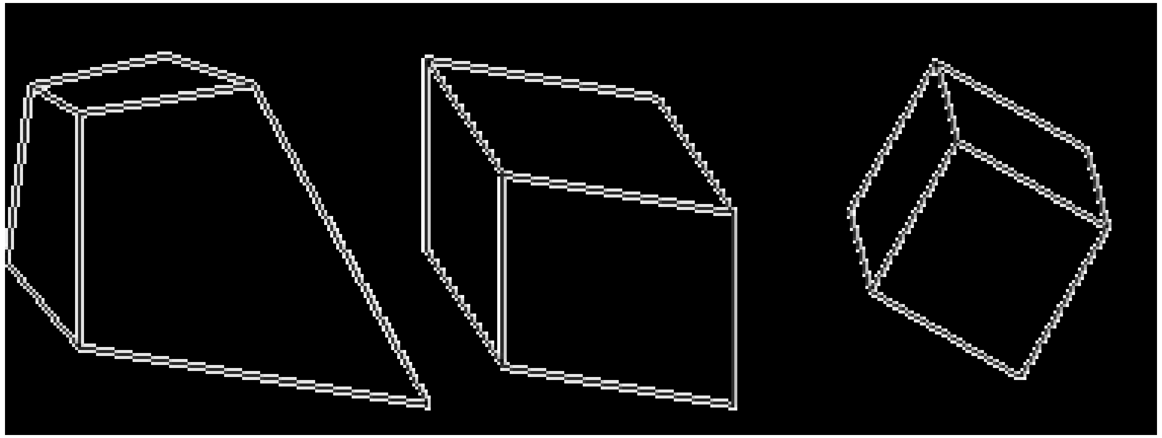


Quadric: a point $\mathbf{X}$ is on a quadric $\boldsymbol{Q}$ if it satisfies a homogeneous *quadratic* equation, namely

$$\mathbf{X}^{T}\boldsymbol{Q}\,\mathbf{X} = \mathbf{0}$$

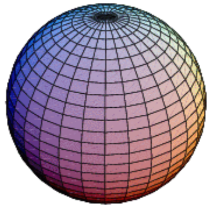
where $\boldsymbol{Q}$ is a 4x4 symmetric matrix.

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

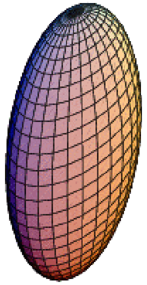
- $\boldsymbol{Q}$ is a homogeneous matrix: $\lambda\boldsymbol{Q} \Leftrightarrow \boldsymbol{Q}$
- 9 degrees of freedom
- 9 points in general positions define a quadric

Quadric classification

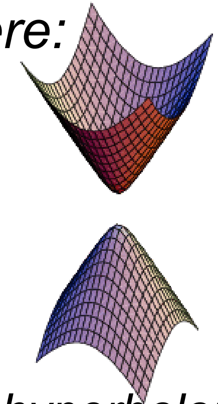
Projectively equivalent to *sphere:*



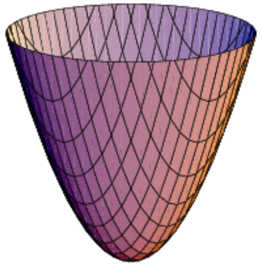
sphere



ellipsoid

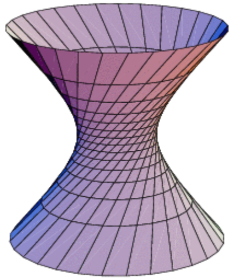


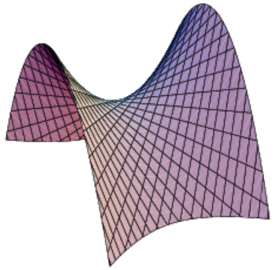
hyperboloid of two sheets



paraboloid

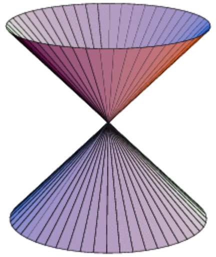
Ruled quadrics:



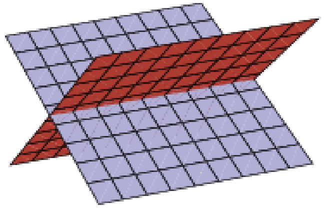


hyperboloids of one sheet

Degenerate ruled quadrics:

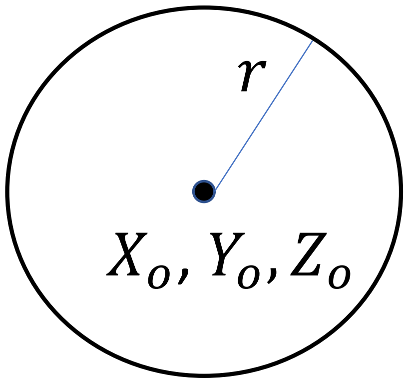


cone



two planes

Example: the sphere



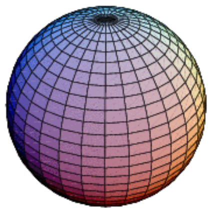
First in cartesian coordinates:

$$(X - X_o)^2 + (Y - Y_o)^2 + (Z - Z_o)^2 - r^2 = 0$$

X_o, Y_o, Z_o are the center coordinates, r is the radius.

… then in homogeneous coordinates:

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -X_o \\ 0 & 1 & 0 & -Y_o \\ 0 & 0 & 1 & -Z_o \\ -X_o & -Y_o & -Z_o & X_o^2 + Y_o^2 - r^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$



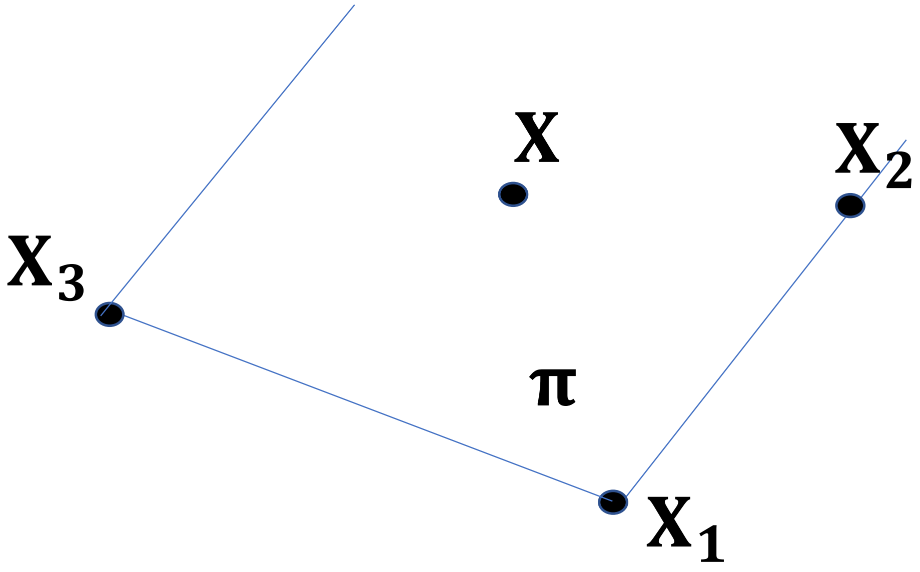
$$\mathbf{X}^\intercal \; \boldsymbol{Q} \; \mathbf{X} = 0$$

Intersection of a plane and a quadric

Plane defined by its span: $\boldsymbol{\pi}$: $\mathbf{X} = \mathbf{Mx}$ ($\mathbf{M}$ is a 4x3 matrix)
where vector $\mathbf{x}$ represent homogeneous coordinates within $\boldsymbol{\pi}$

…

remember: the plane as its span



$\mathbf{X}$ is a linear combination $\propto \mathbf{X_1} + \beta\mathbf{X_2} + \gamma\mathbf{X_3}$
→ $\mathbf{X}$ is coplanar to $\mathbf{X_1}, \mathbf{X_2},$ and $\mathbf{X_3}$

i.e. $\mathbf{X} = [\mathbf{X_1} \quad \mathbf{X_2} \quad \mathbf{X_3}] \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix} = \mathrm{M}\mathbf{x}$ where

$\mathbf{x} = \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix}$ can be regarded as homogeneous coordinates within the 2D geometry of plane $\boldsymbol{\pi}$

$$\mathbf{X} = \mathrm{M}\mathbf{x}$$

Intersection of a plane and a quadric

Plane defined by its span: $\boldsymbol{\pi}$: $\mathbf{X} = \mathbf{M}\mathbf{x}$ ($\mathbf{M}$ is a 4x3 matrix)
where vector $\mathbf{x}$ represent homogeneous coordinates within $\boldsymbol{\pi}$

Quadric $\mathbf{Q}$: $\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$

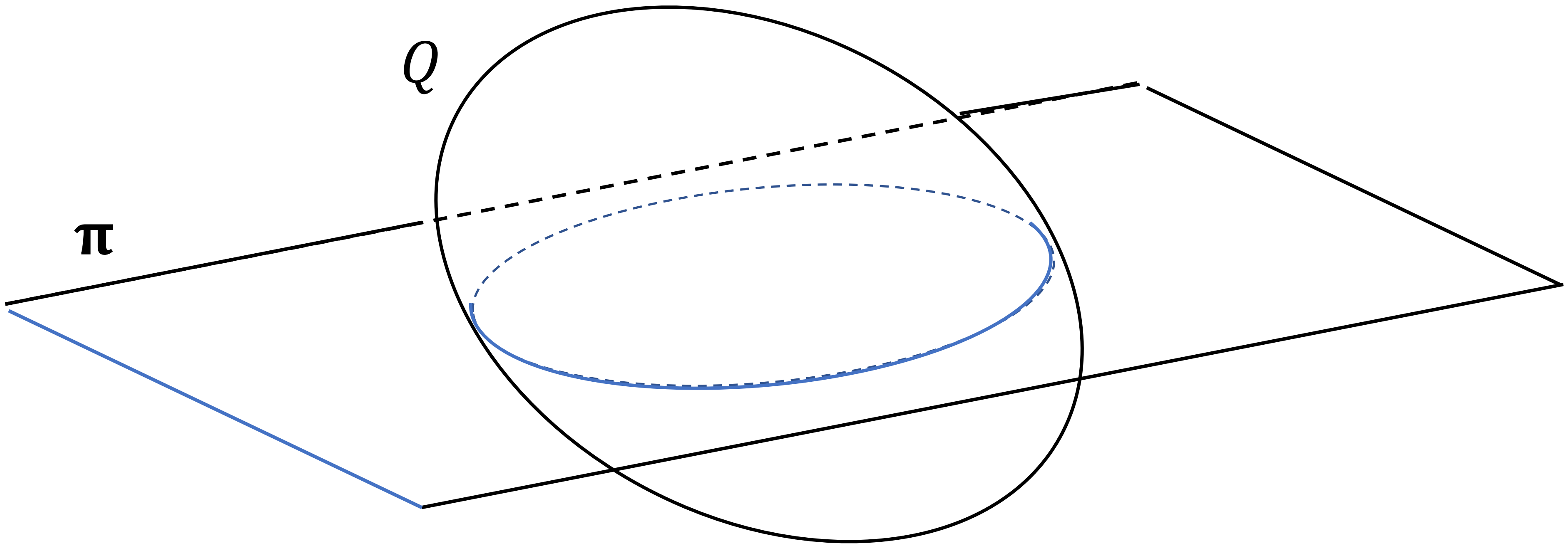
Plane-quadric intersection: $\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{x}^T \mathbf{M}^T \boldsymbol{Q} \mathbf{M}\mathbf{x} = \mathbf{x}^T \boldsymbol{C}\, \mathbf{x} = \mathbf{0}$

… is a **conic** $\boldsymbol{C}$

$$\boldsymbol{C} = \mathbf{M}^T \boldsymbol{Q} \mathbf{M}$$

plane – quadric intersection: a conic

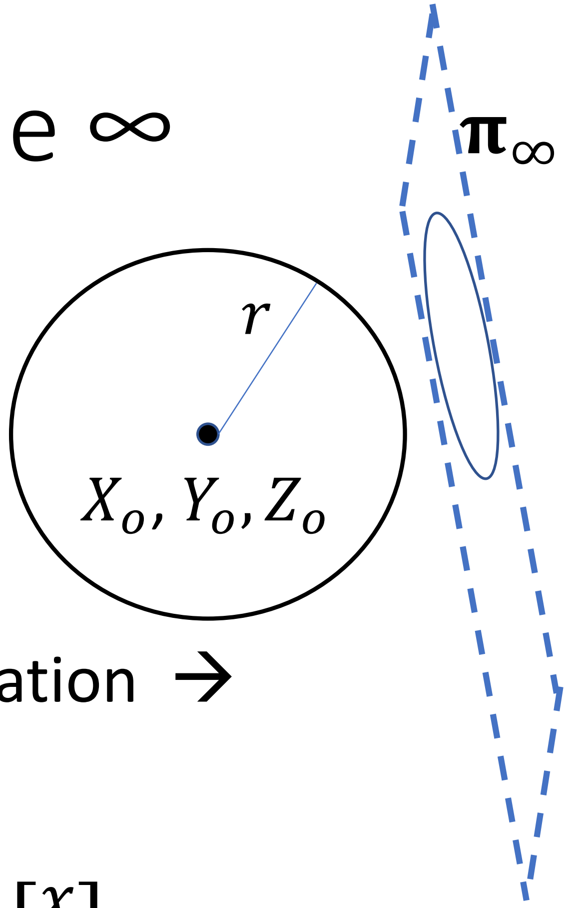
.



The absolute conic:
an extension of the circular points

A noteworthy example: intersecting a sphere and the plane at the ∞

$$\begin{cases}(x - X_o w)^2 + (y - Y_o w)^2 + (z - Z_o w)^2 - r^2 w^2 = 0 \\ w = 0\end{cases}$$



$$\rightarrow$$

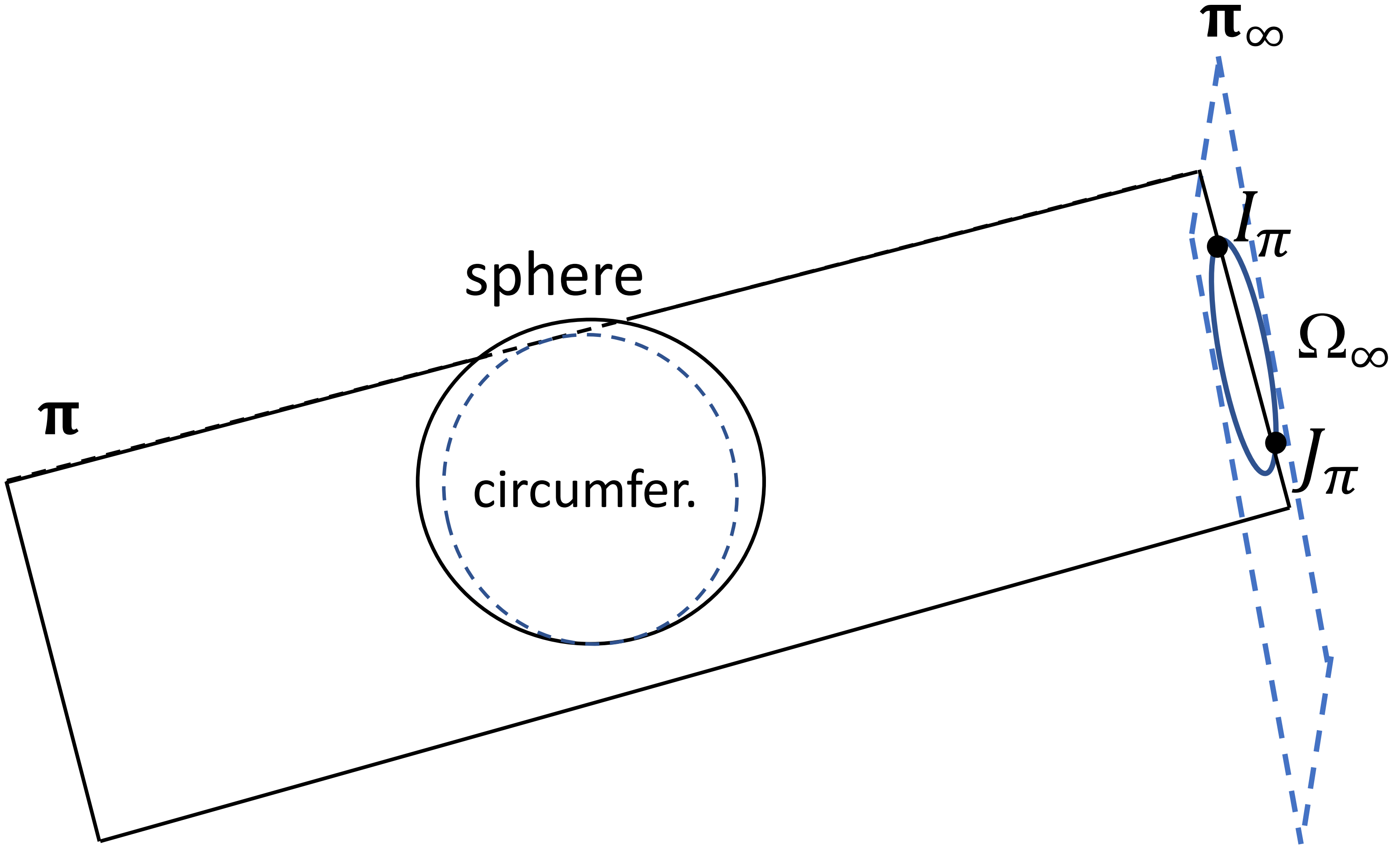
$$\begin{cases}x^2 + y^2 + z^2 = 0 \\ w = 0\end{cases}$$

The sphere parameters (center and radius) disappear from the equation $\rightarrow$ the intersection **conic** is the **same for all** spheres:

$$x^2 + y^2 + z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \Omega_\infty \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

A conic within $\boldsymbol{\pi}_\infty$: $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **ABSOLUTE CONIC**

absolute conic: made of (∪) circular points of all planes



THE POLAR PLANE

Polar plane of a point wrt a quadric

Given a point $\mathbf{Y}$ and a quadric $\boldsymbol{Q}$, the plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ is called the *polar plane* of point $\mathbf{Y}$ with respect to the quadric $\boldsymbol{Q}$.

Polar plane of a point wrt a quadric

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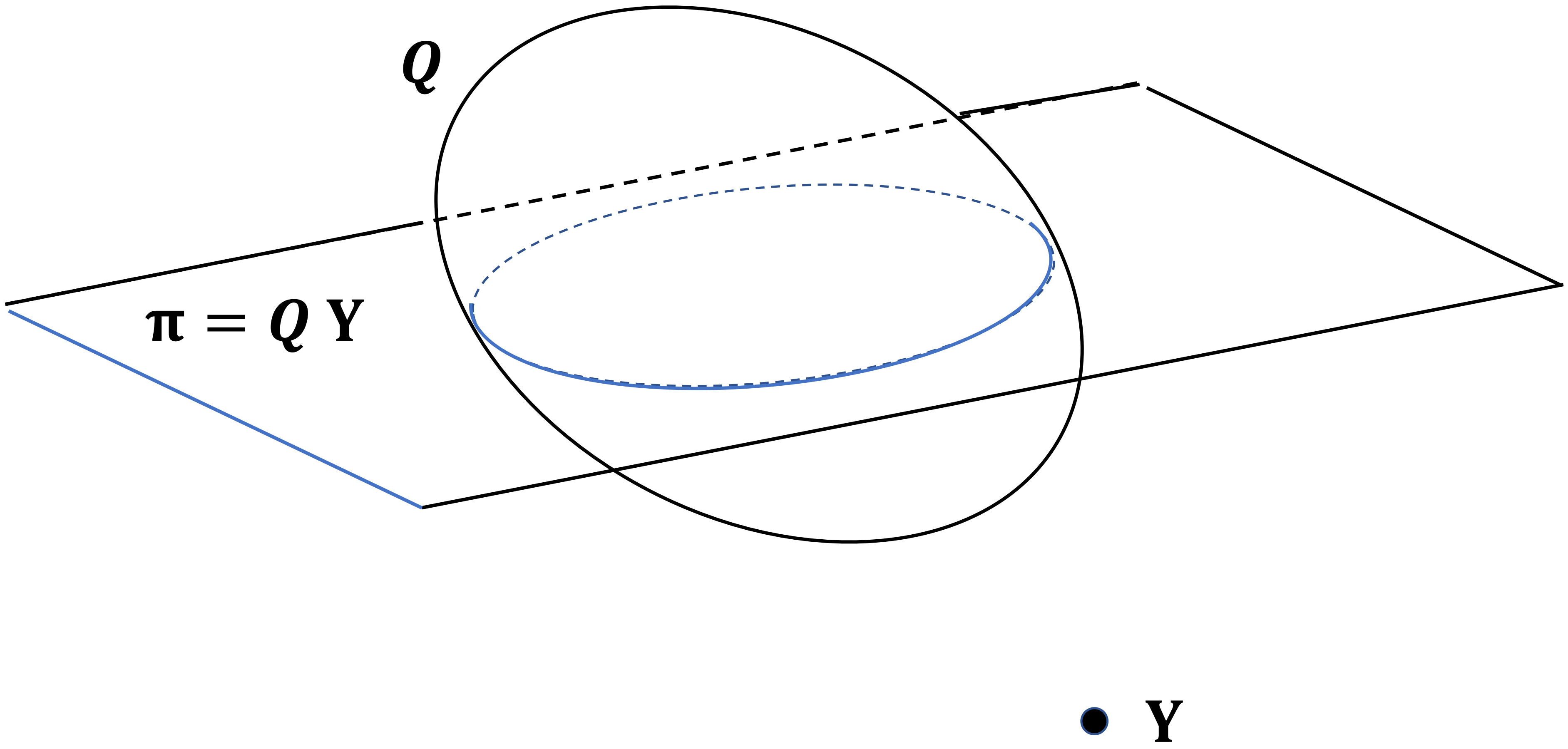
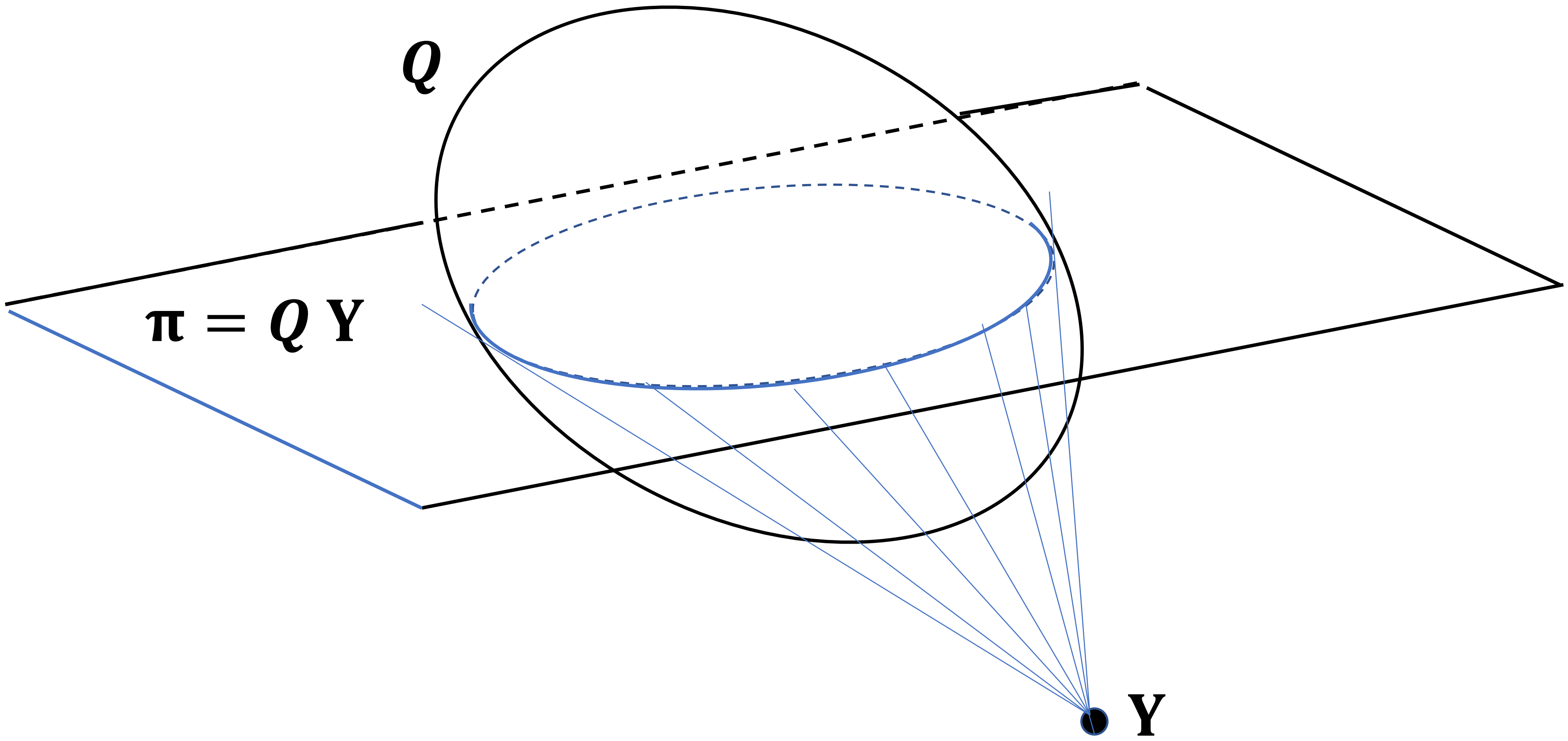


Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

The intersection conic $\boldsymbol{\pi} \cap \boldsymbol{Q}$ is the set of tangency points of the planes through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$ (or the lines through $\mathbf{Y}$ that are tangent to $\boldsymbol{Q}$)



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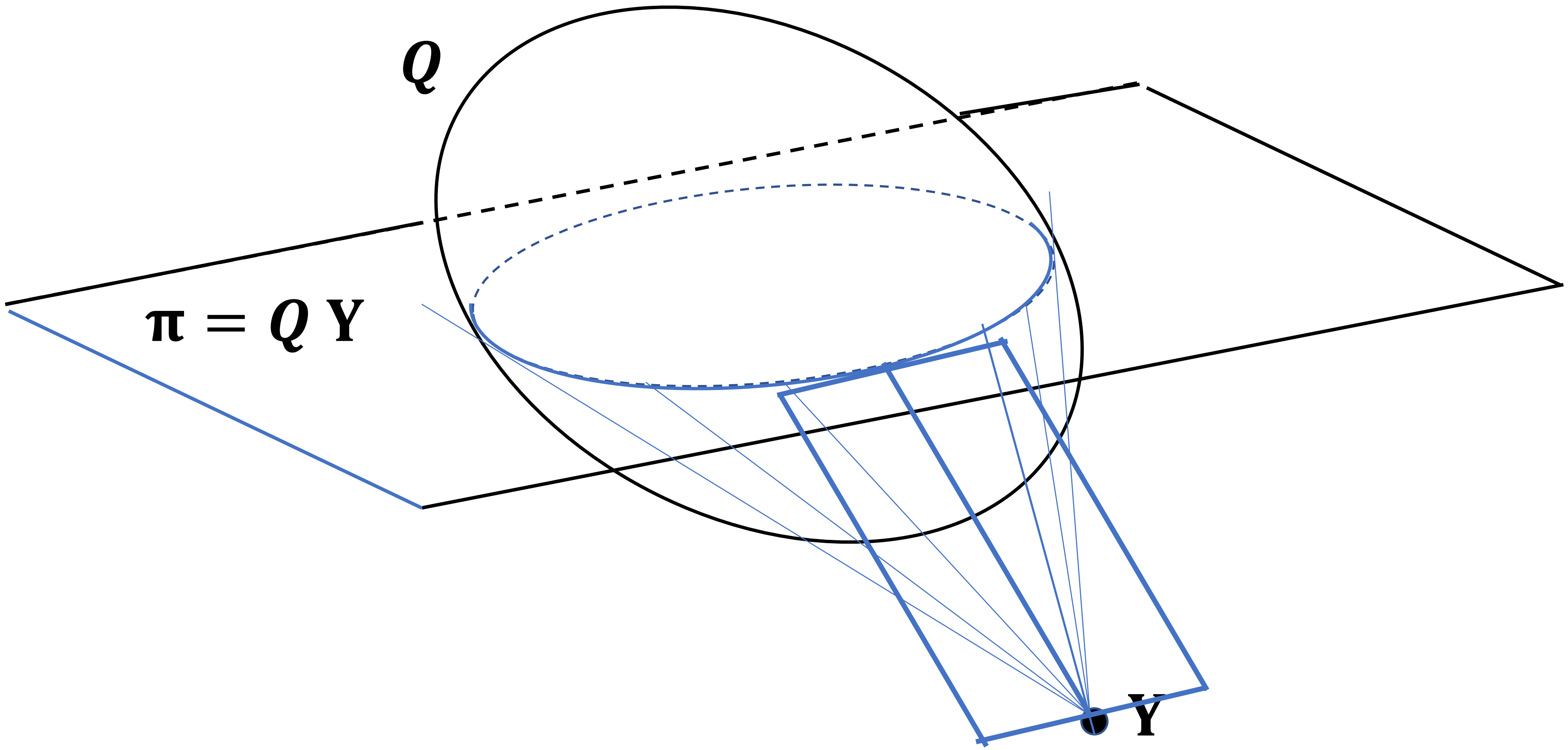

Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

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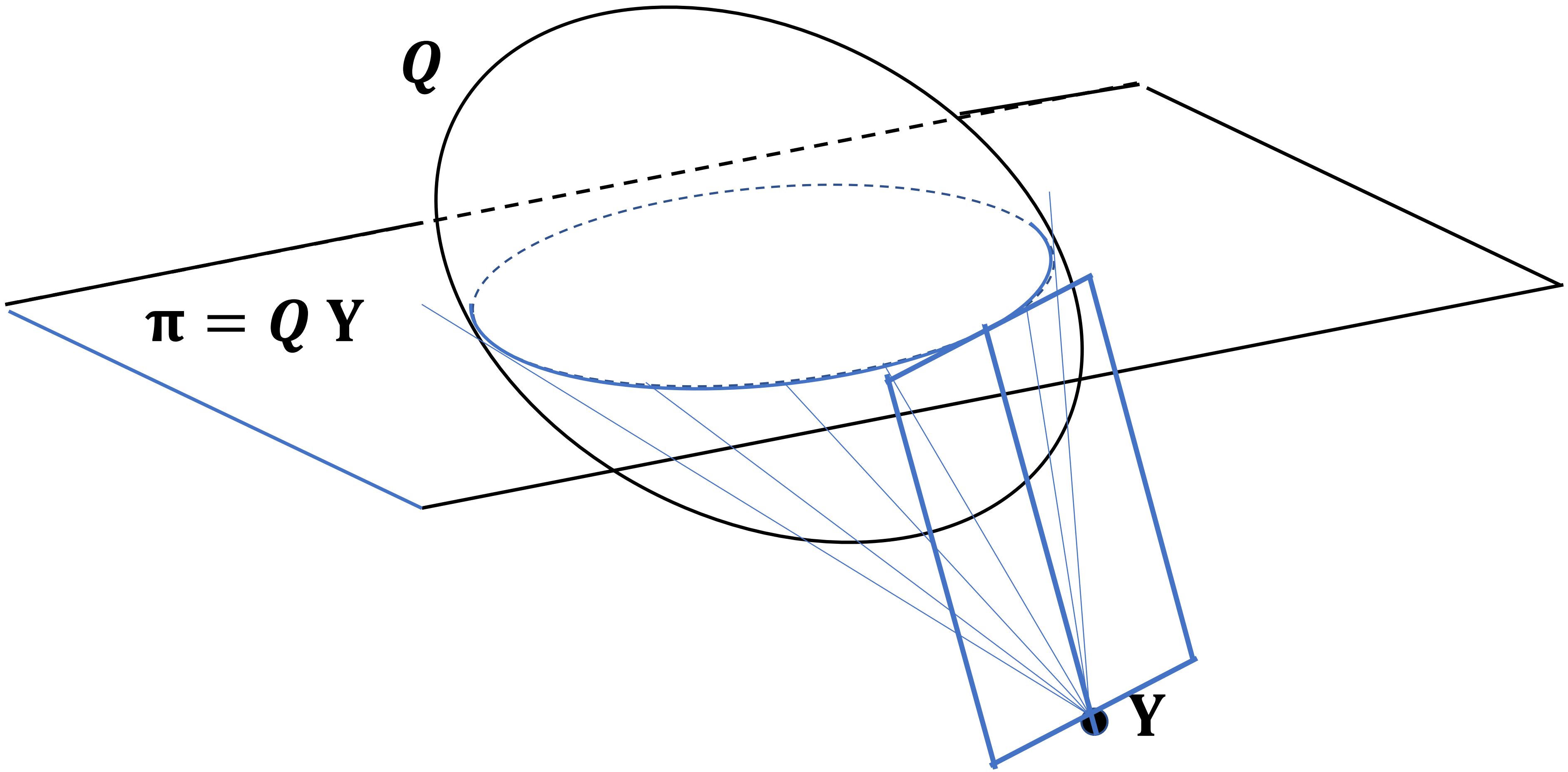

Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

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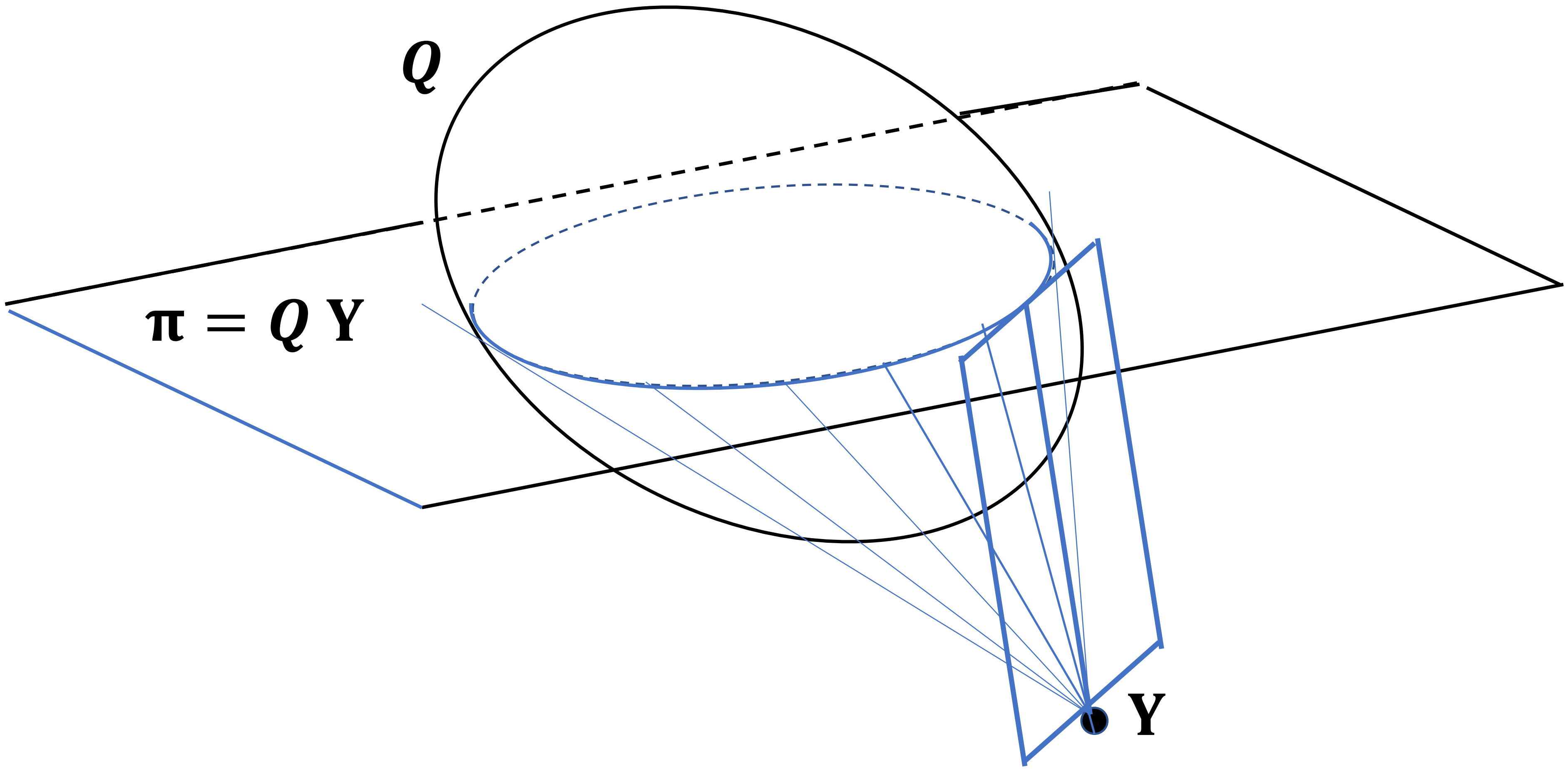
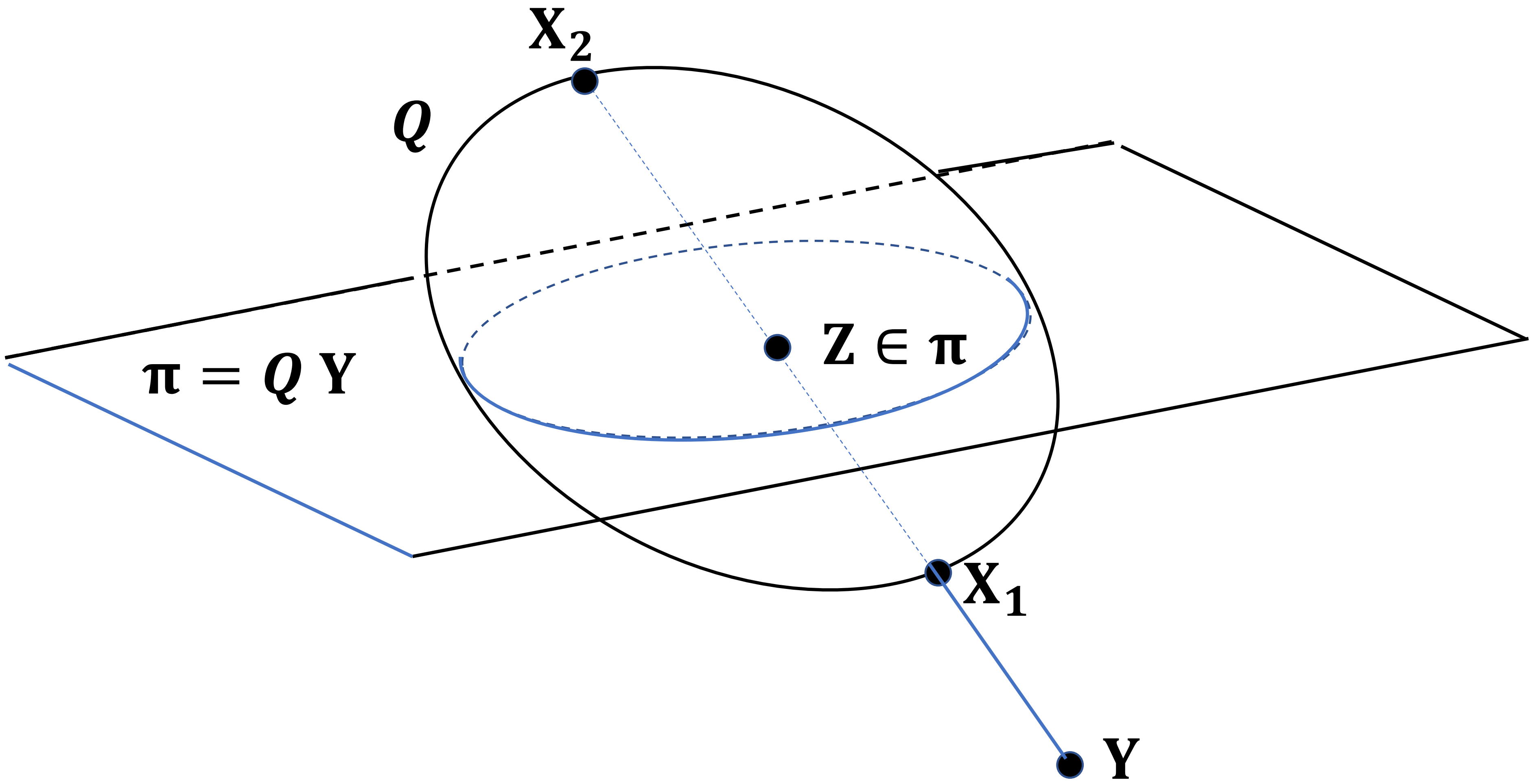
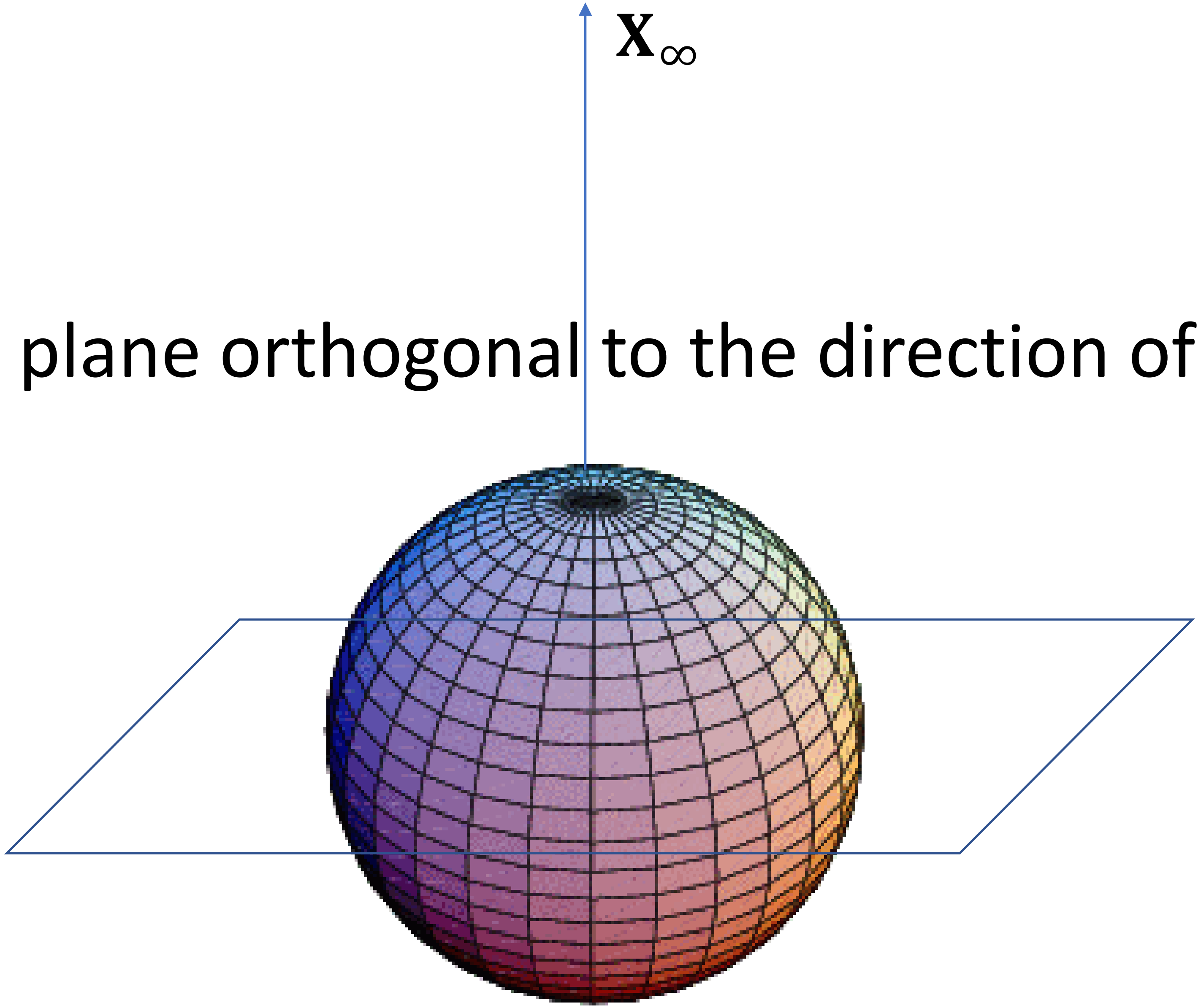


Polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{Y}$ of a point $\mathbf{Y}$ wrt a quadric $\boldsymbol{Q}$

Any 4-tuple $(\mathbf{Y}, \mathbf{Z}, \mathbf{X_1}, \mathbf{X_2})$ is harmonic (CR = -1)



Example: polar of a point at the infinity wrt a sphere

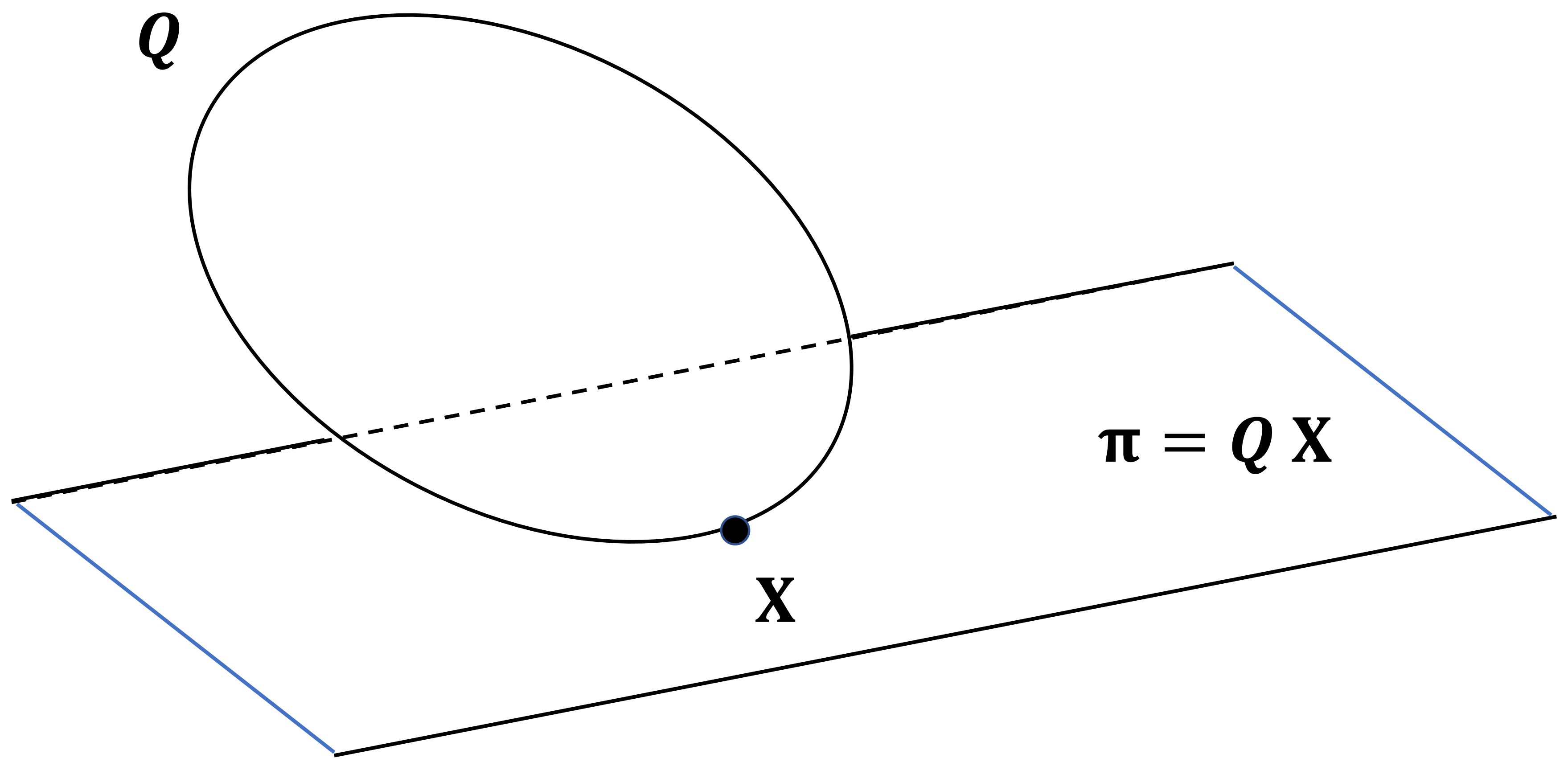


equatorial plane orthogonal to the direction of $\mathbf{X}_\infty$

the polar plane of a point **ON** the quadric $\boldsymbol{Q}$

the polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{X}$ of a point $\mathbf{X}$ ON the quadric $\boldsymbol{Q}$

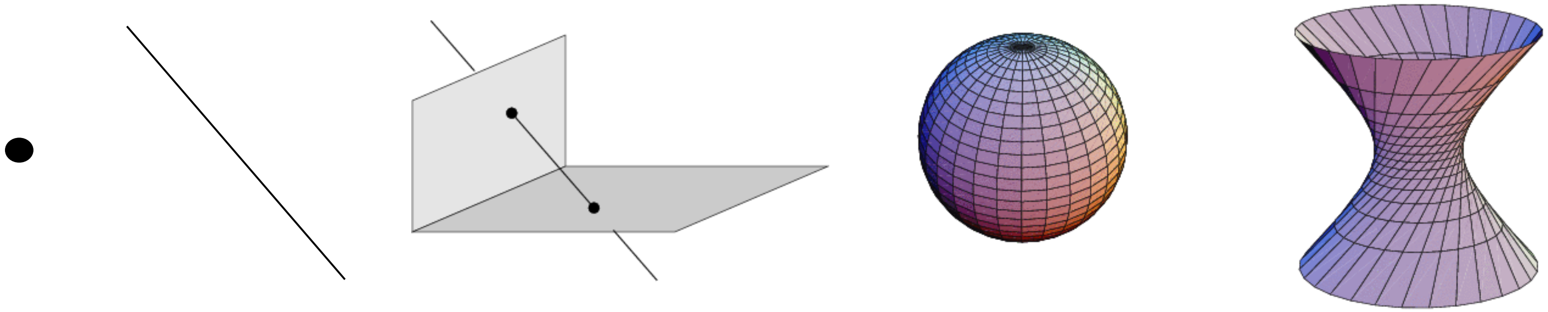
… the plane tangent to $\boldsymbol{Q}$ through $\mathbf{X}$



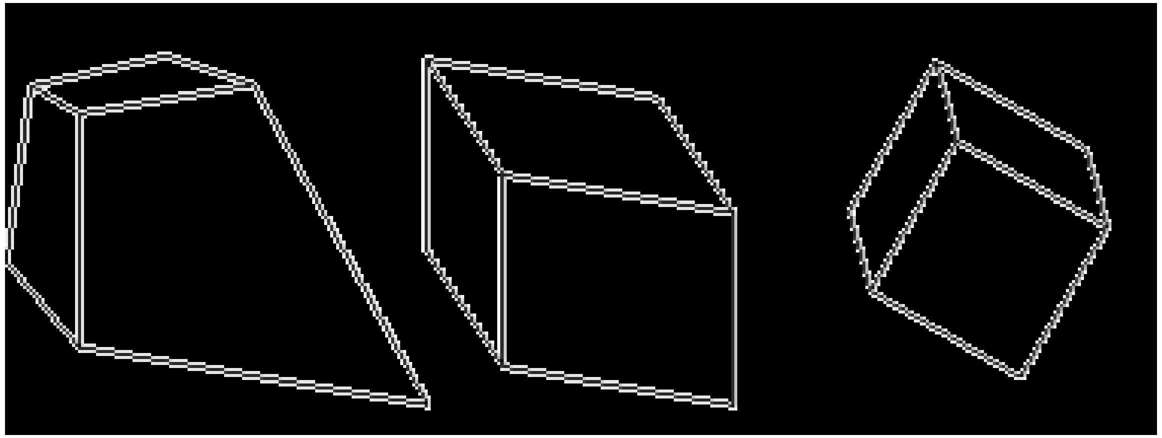
(NONDEGENERATE) DUAL QUADRICS

3D Space Projective Geometry

- **Elements**
- Points
- Planes
- Quadrics
- **Dual quadrics**



- **Transformations**
- Isometries
- Similarities
- Affinities
- Projectivities



Dual quadric: a plane $\boldsymbol{\pi}$ is on a dual quadric $\boldsymbol{Q}^*$ if it satisfies a homogeneous *quadratic* equation, namely

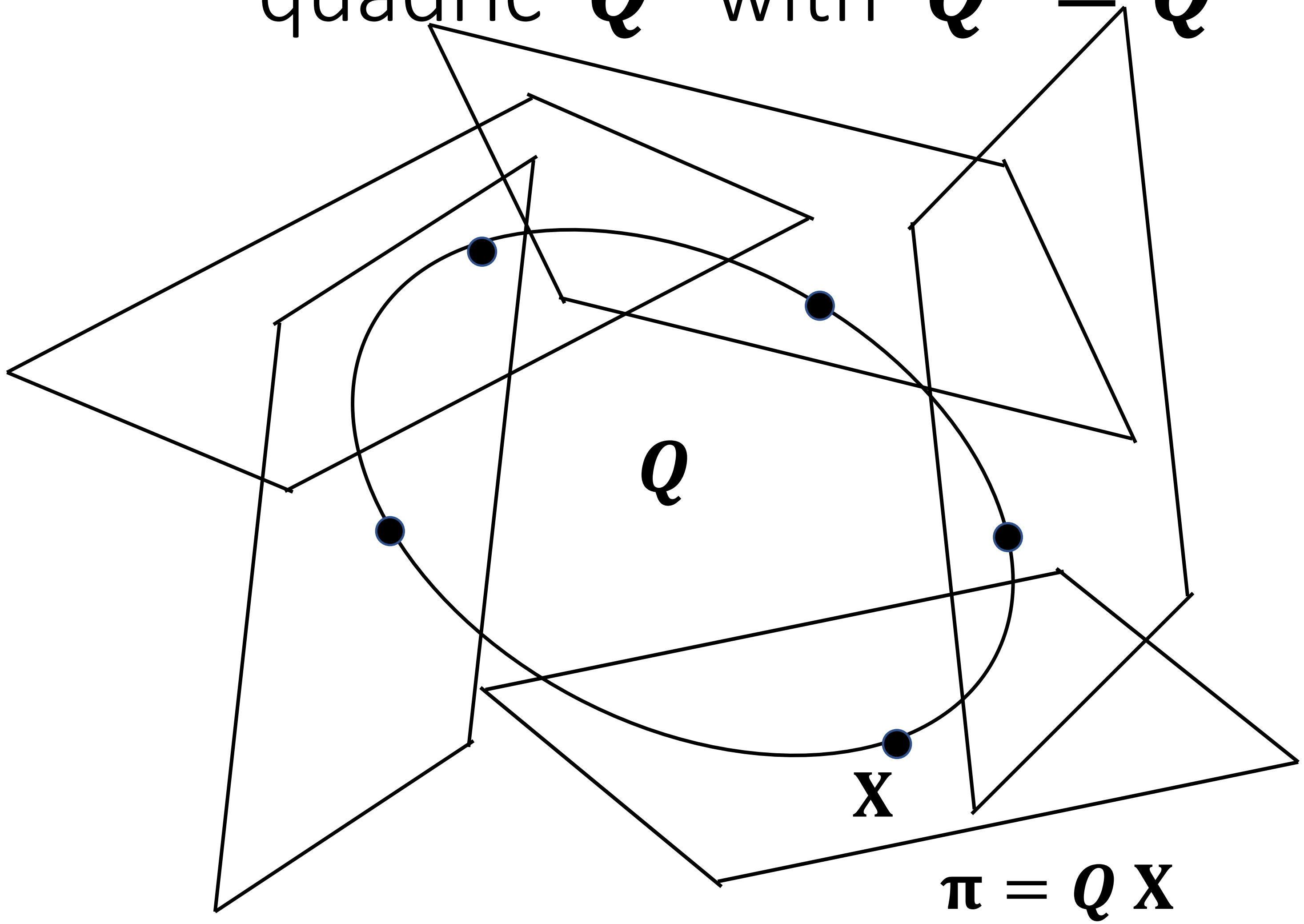
$$\boldsymbol{\pi}^{T} \boldsymbol{Q}^* \, \boldsymbol{\pi} = \mathbf{0}$$

where $\boldsymbol{Q}^*$ is a 4x4 symmetric matrix.

$\boldsymbol{Q}^*$ is a set of **planes**

- $\boldsymbol{Q}^*$is a homogeneous matrix: $\lambda \, \boldsymbol{Q}^* \Leftrightarrow \boldsymbol{Q}^*$
- 9 degrees of freedom

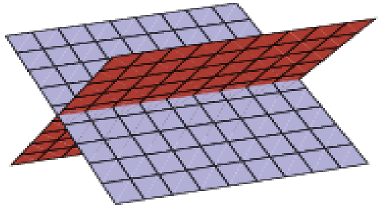
the set of planes tangent to a quadric $\boldsymbol{Q}$ is the dual quadric $\boldsymbol{Q}^*$ with $\boldsymbol{Q}^* = \boldsymbol{Q}^{-1}$



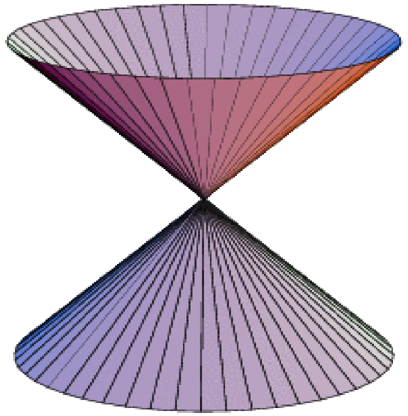
DEGENERATE QUADRICS

$$\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{1}$ → $\boldsymbol{Q} = \mathbf{A}\mathbf{A}^T$ repeated plane $\mathbf{A}$

- rank $\boldsymbol{Q} = \mathbf{2}$ → $\boldsymbol{Q} = \mathbf{A}\mathbf{B}^T + \mathbf{B}\mathbf{A}^T$  two planes $\mathbf{A}$ and $\mathbf{B}$

- rank $\boldsymbol{Q} = \mathbf{3}$ a **cone**



DEGENERATE QUADRICS

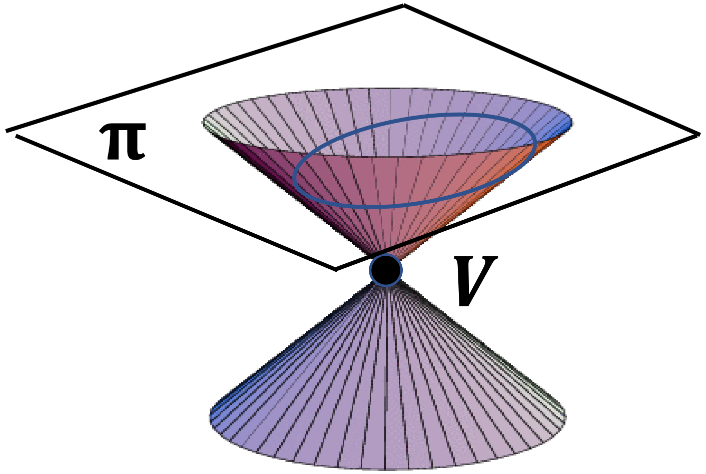
$$\mathbf{X}^T \boldsymbol{Q}\, \mathbf{X} = \mathbf{0}$$

$$\text{rank } \boldsymbol{Q} = \mathbf{3}$$

Consider $\boldsymbol{V} = \text{RNS}(\boldsymbol{Q}) : \boldsymbol{Q}\, \boldsymbol{V} = \mathbf{0}$

- $\boldsymbol{V} \in \boldsymbol{Q}$ $\quad [\mathbf{V}^T \boldsymbol{Q}\, \mathbf{V} = \mathbf{V}^T\ \mathbf{0} = \mathbf{0}]$
- $\forall \boldsymbol{X} \in \boldsymbol{Q}$ **,** any point $\boldsymbol{Y}$ colinear with $\mathbf{X}$ and $\boldsymbol{V}$, namely $\boldsymbol{Y} = \alpha \boldsymbol{X} + \boldsymbol{V}$, is also $\in \boldsymbol{Q}$

  in fact $\boldsymbol{Y}^T \boldsymbol{Q}\, \boldsymbol{Y} = (\boldsymbol{\alpha X} + \boldsymbol{V})^T \boldsymbol{Q}\, (\alpha \boldsymbol{X} + \boldsymbol{V}) = \boldsymbol{\alpha}^2 \boldsymbol{X}^T \boldsymbol{Q} \boldsymbol{X} = \mathbf{0}$
- A generic plane $\boldsymbol{\pi}$ crosses $\boldsymbol{Q}$ in a conic



DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q}^{*} = \mathbf{1}$ → $\boldsymbol{Q}^{*} = \boldsymbol{X}\boldsymbol{X}^{\boldsymbol{T}}$ repeated point: planes through point $\boldsymbol{X}$

- rank $\boldsymbol{Q}^{*} = \mathbf{2}$ → $\boldsymbol{Q}^{*} = \boldsymbol{X}\boldsymbol{Y}^{\boldsymbol{T}} + \boldsymbol{Y}\boldsymbol{X}^{\boldsymbol{T}}$ two points: planes through $\boldsymbol{X}$ or $\boldsymbol{Y}$

- rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\, \boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} = \mathrm{RNS}(\boldsymbol{Q})$ and any point on a conic

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T} \boldsymbol{Q}^{*} \boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} = \text{RNS}(\boldsymbol{Q})$ and any point on a conic

NOT a primitive element in 3D geometry

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T} \boldsymbol{Q}^{*} \boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} = \mathrm{RNS}(\boldsymbol{Q})$ and any point on a conic (conic = an intersection $\boldsymbol{\pi} \cap \boldsymbol{Q_o}$ of a plane and a quadric)

NOT a primitive element in 3D geometry

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\, \boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q} = \mathbf{3}$ the **dual** of a cone
- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} = \mathrm{RNS}(\boldsymbol{Q})$ and any point that (i) belongs to a quadric $\boldsymbol{Q_o}$ and (ii) is on a plane $\boldsymbol{\pi}$.

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

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- **Cone**: set of points that are linear combinations of the point $\boldsymbol{V} =$ RNS($\boldsymbol{Q}$) and any point that (i) belongs to a quadric $\boldsymbol{Q_o}$ and (ii) is on a plane $\boldsymbol{\pi}$.
- **Dual cone**: set of planes that are linear combinations of the plane $\boldsymbol{R} =$RNS($\boldsymbol{Q}^{*}$) and any plane that (i) belongs to a dual quadric $\boldsymbol{Q_o}^{*}$and (ii) goes through a point $\boldsymbol{X}$.

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{\,T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone
- **Dual cone**: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) **belongs to a dual quadric** $\boldsymbol{Q_o}^{*}$and (ii) goes through a point $\boldsymbol{X}$.

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Dual cone: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) **is tangent to a quadric** $\boldsymbol{Q}_{\boldsymbol{o}} = \boldsymbol{Q}_{\boldsymbol{o}}{}^{*-\mathbf{1}}$ and (ii) goes through a point $\boldsymbol{X}$.

DEGENERATE DUAL QUADRICS

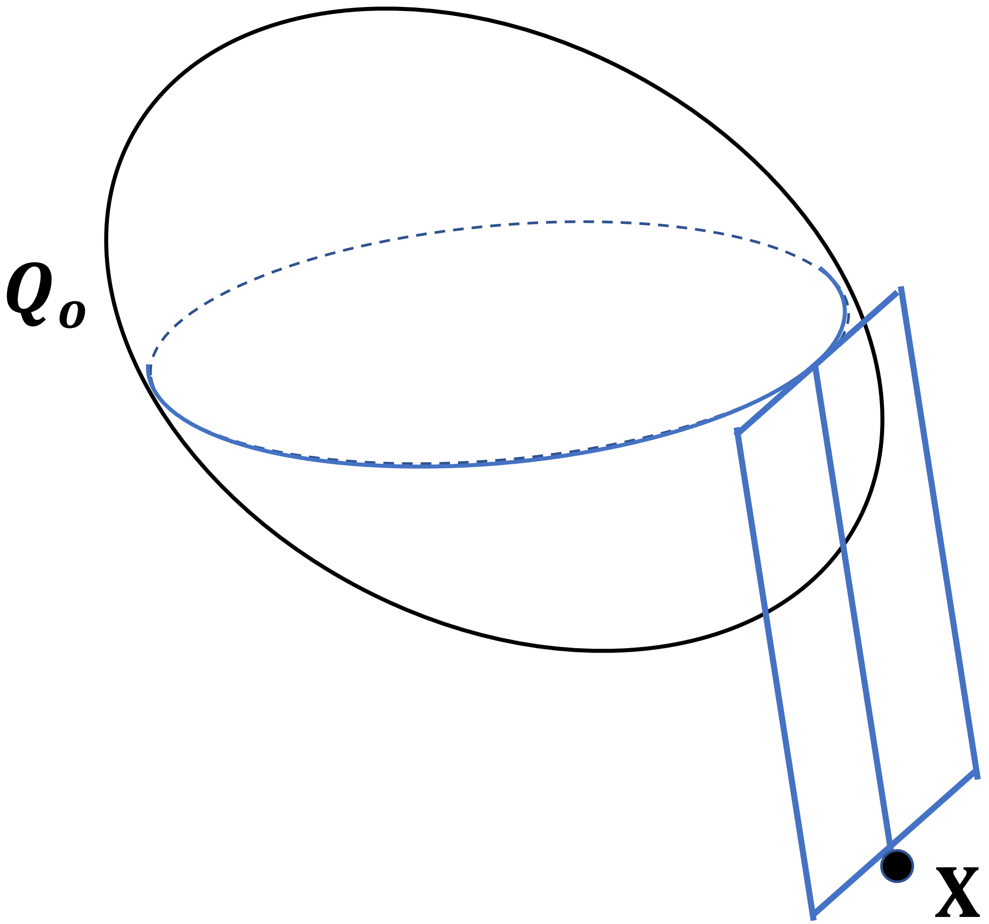
$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone

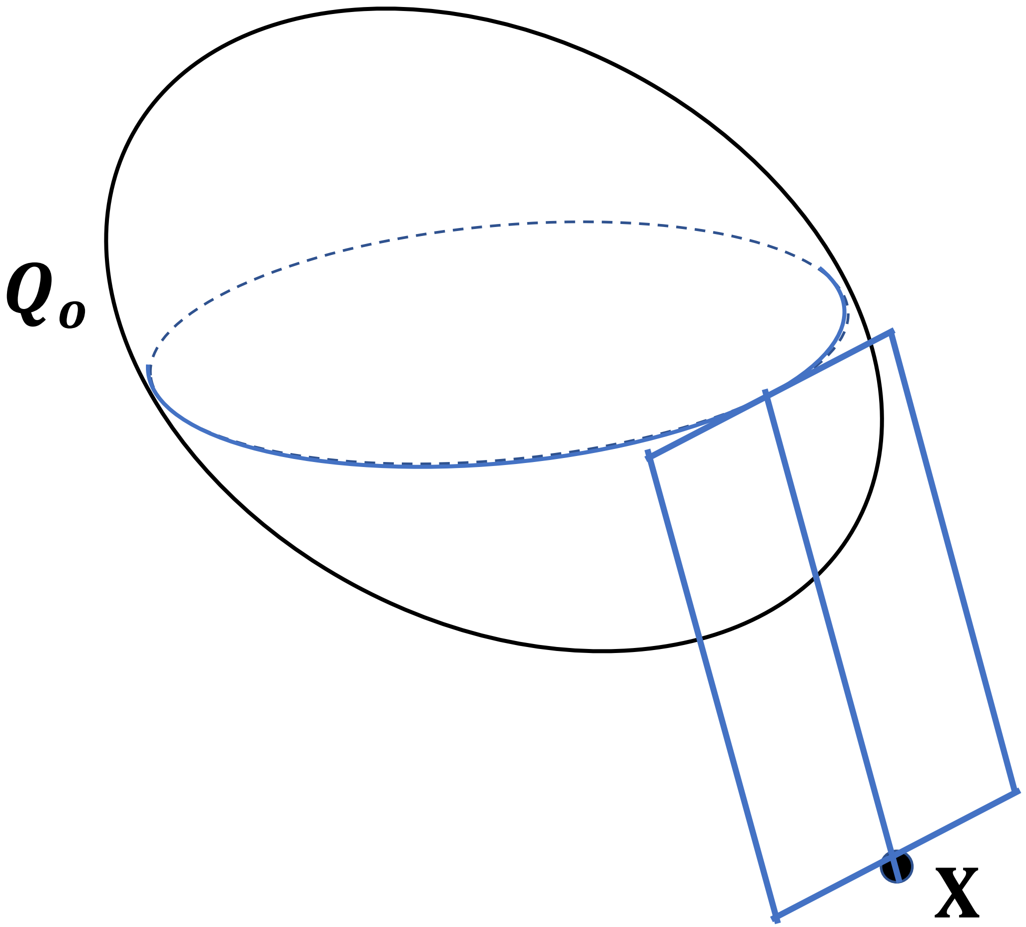
Dual cone: set of planes that are linear combinations of the plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$) and any plane that (i) **is tangent to a quadric** $\boldsymbol{Q_o} = \boldsymbol{Q_o}^{*-\mathbf{1}}$ and (ii) goes through a point $\boldsymbol{X}$.

what is the locus of such planes?

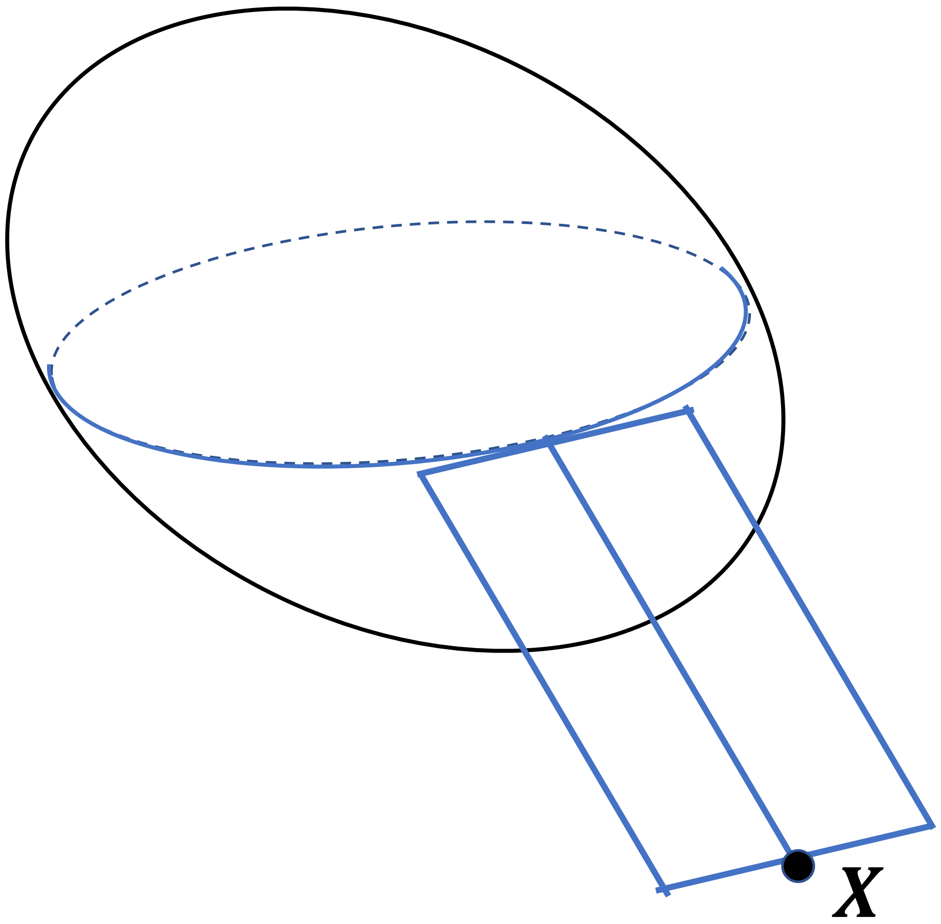
Planes through point $\boldsymbol{X}$ tangent to quadric $\boldsymbol{Q_o}$



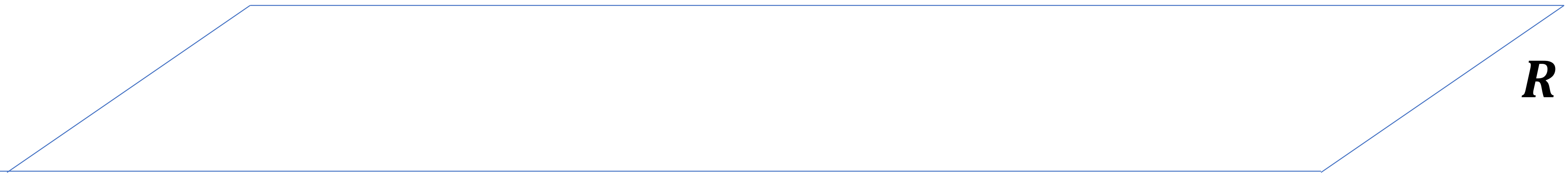
Planes through point $\boldsymbol{X}$ tangent to quadric $\boldsymbol{Q_o}$



Planes through point $\boldsymbol{X}$ tangent to quadric $\boldsymbol{Q_o}$



Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



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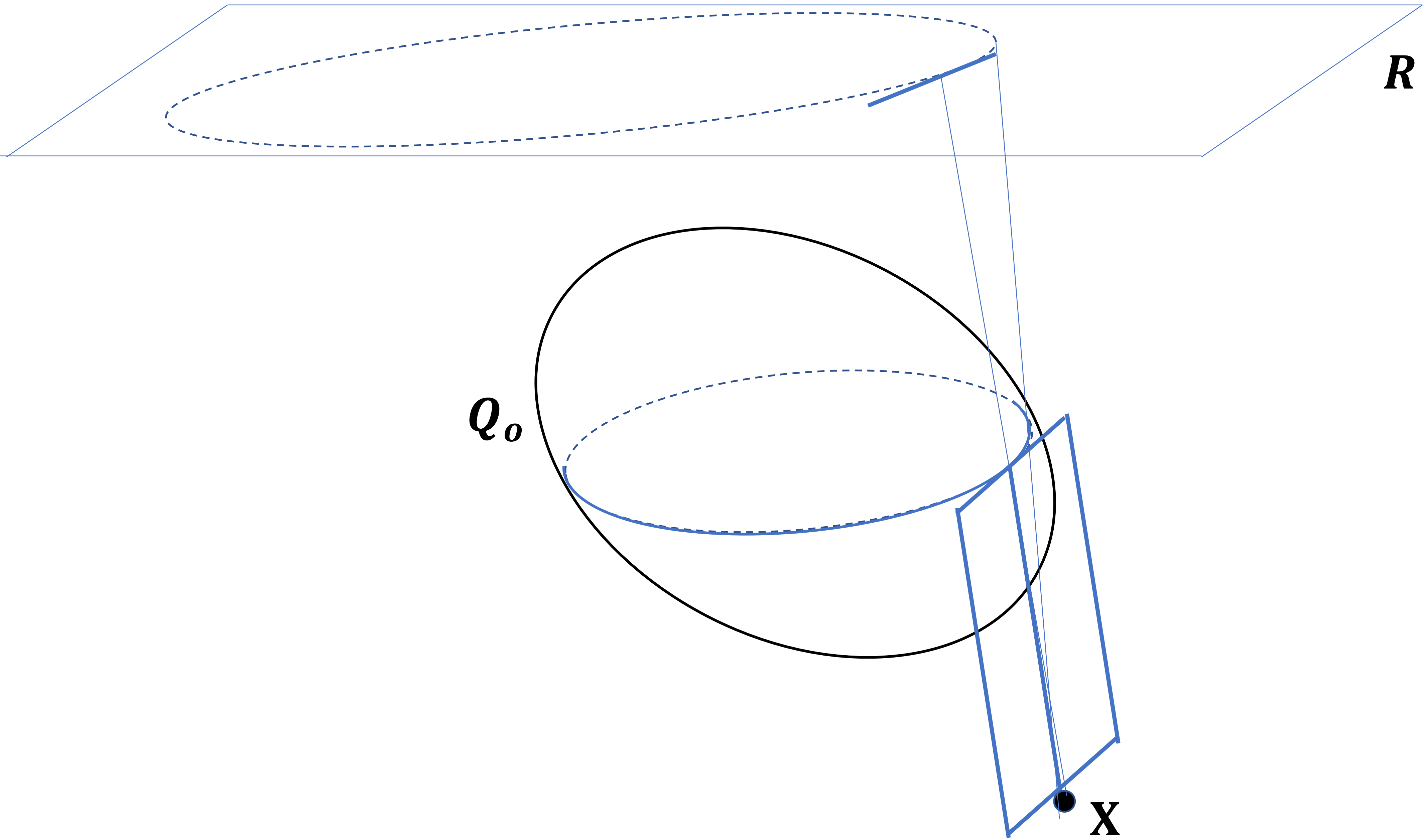

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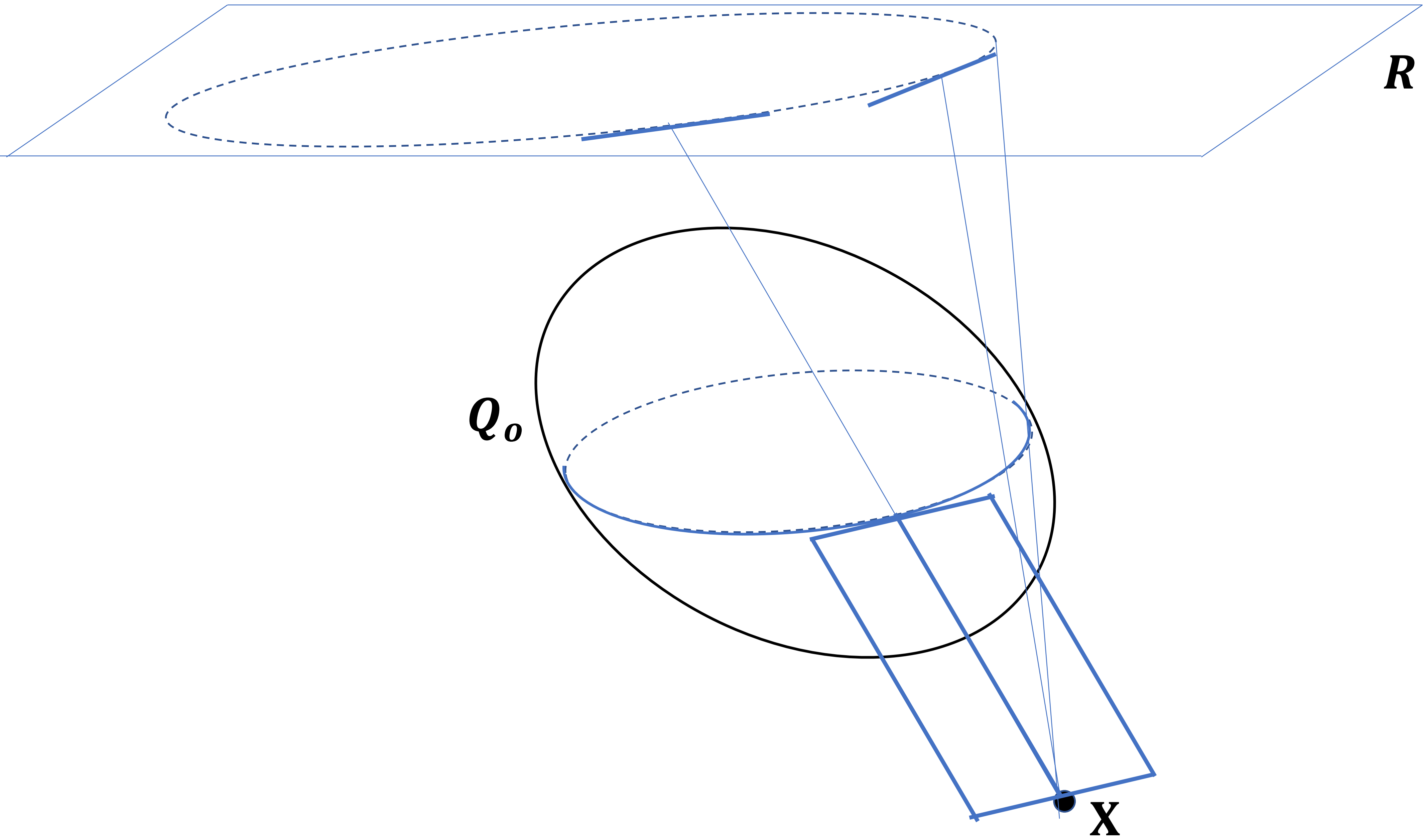
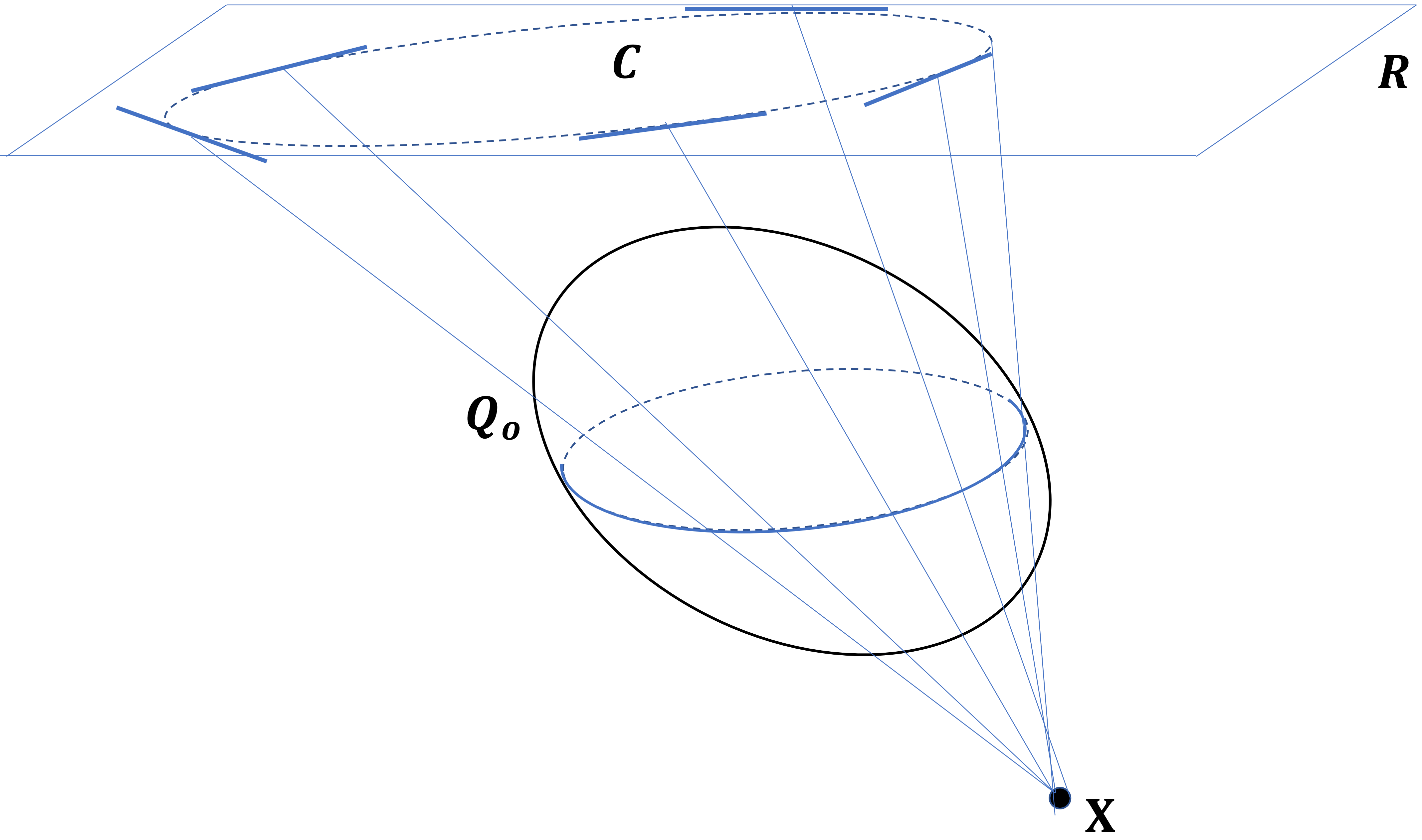
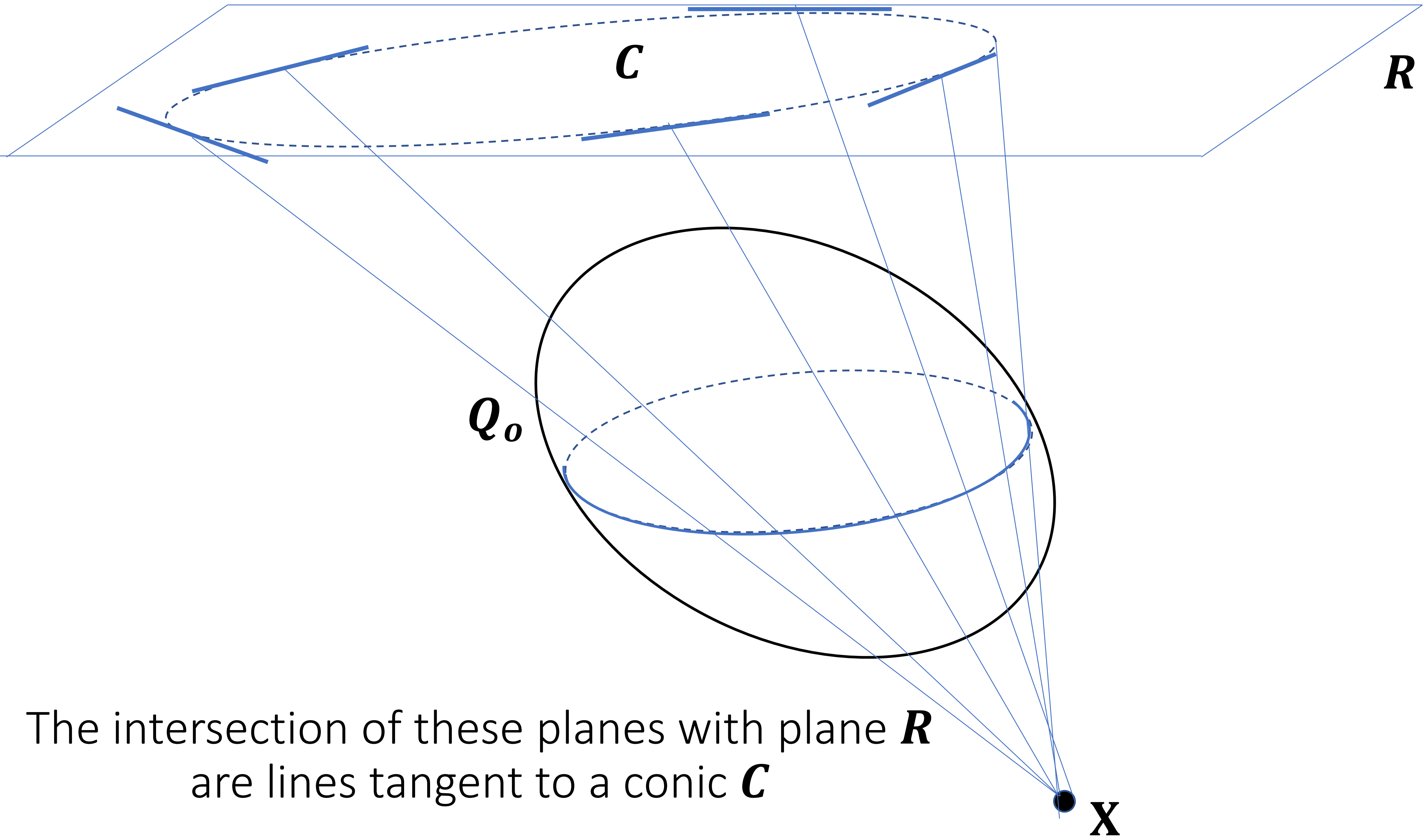


Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^{*}$)

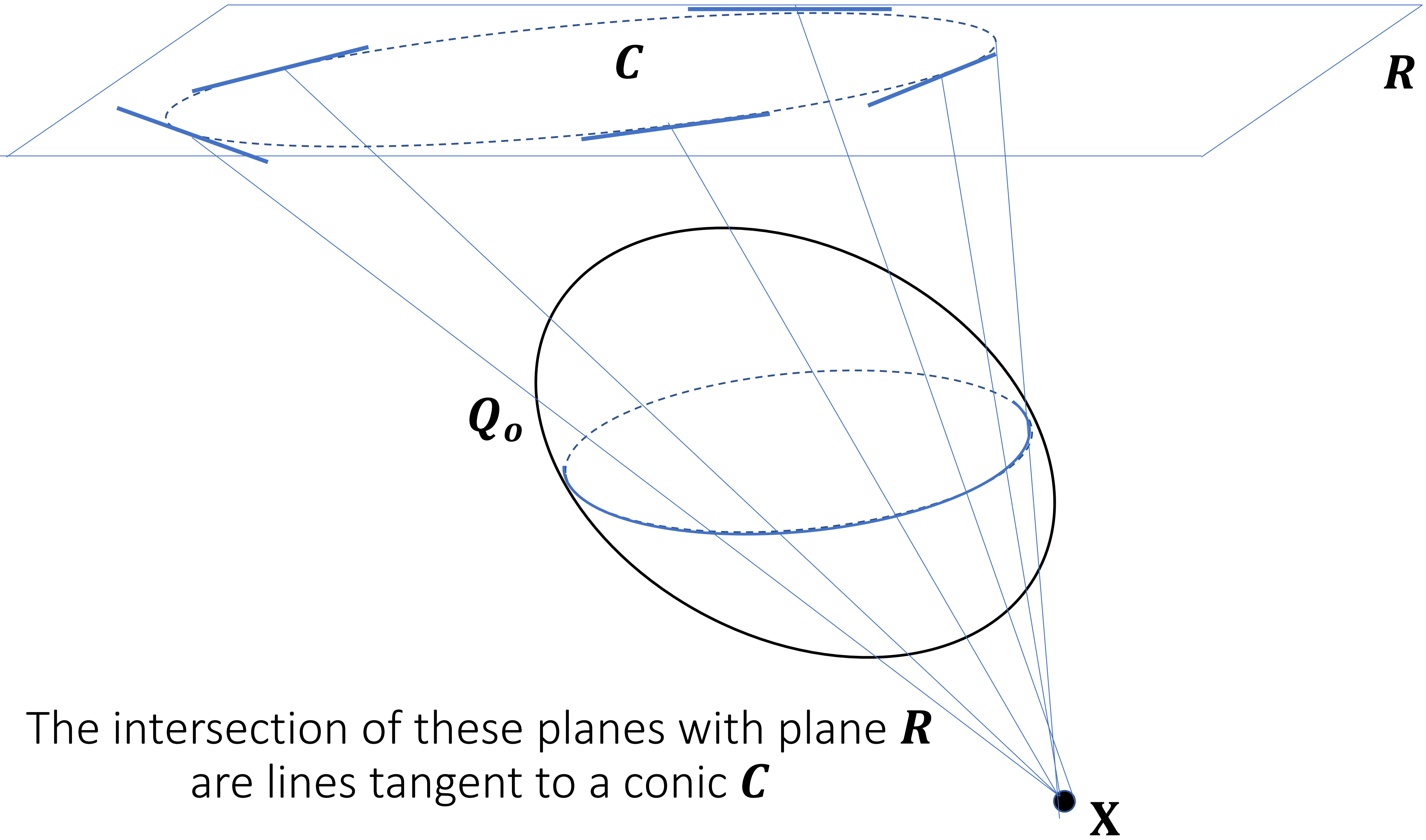


Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



The intersection of these planes with plane $\boldsymbol{R}$ are lines tangent to a conic $\boldsymbol{C}$

Their intersections with plane $\boldsymbol{R}$ =RNS($\boldsymbol{Q}^*$)



The intersection of these planes with plane $\boldsymbol{R}$ are lines tangent to a conic $\boldsymbol{C}$

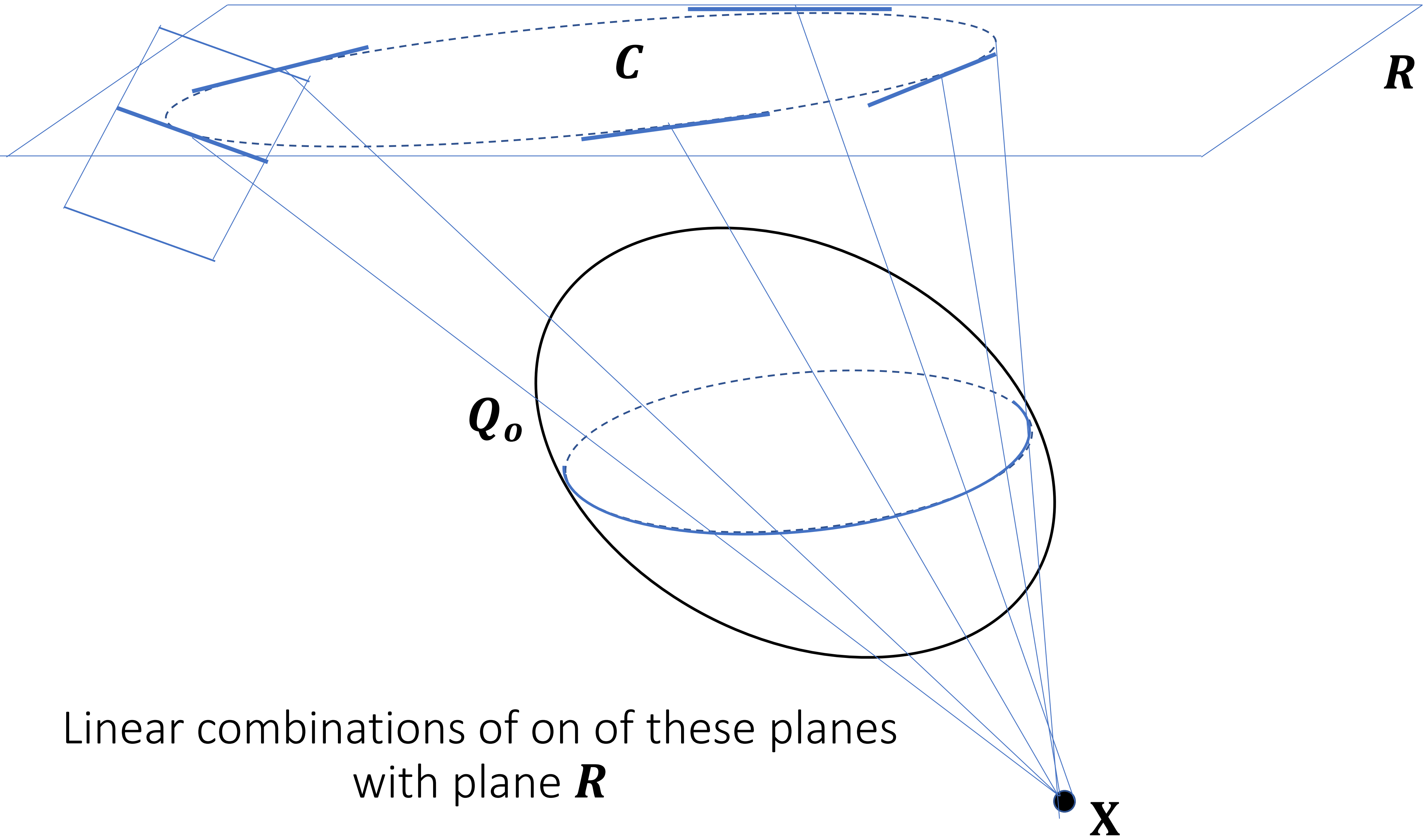
DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\, \boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

Dual cone: set of **planes that are linear combinations of the plane $\boldsymbol{R}$** $=$RNS($\boldsymbol{Q}^{*}$) **and any plane** that (i) is tangent to a quadric $\boldsymbol{Q_o} = \boldsymbol{Q_o}^{*-\mathbf{1}}$and (ii) goes through a point $\boldsymbol{X}$.

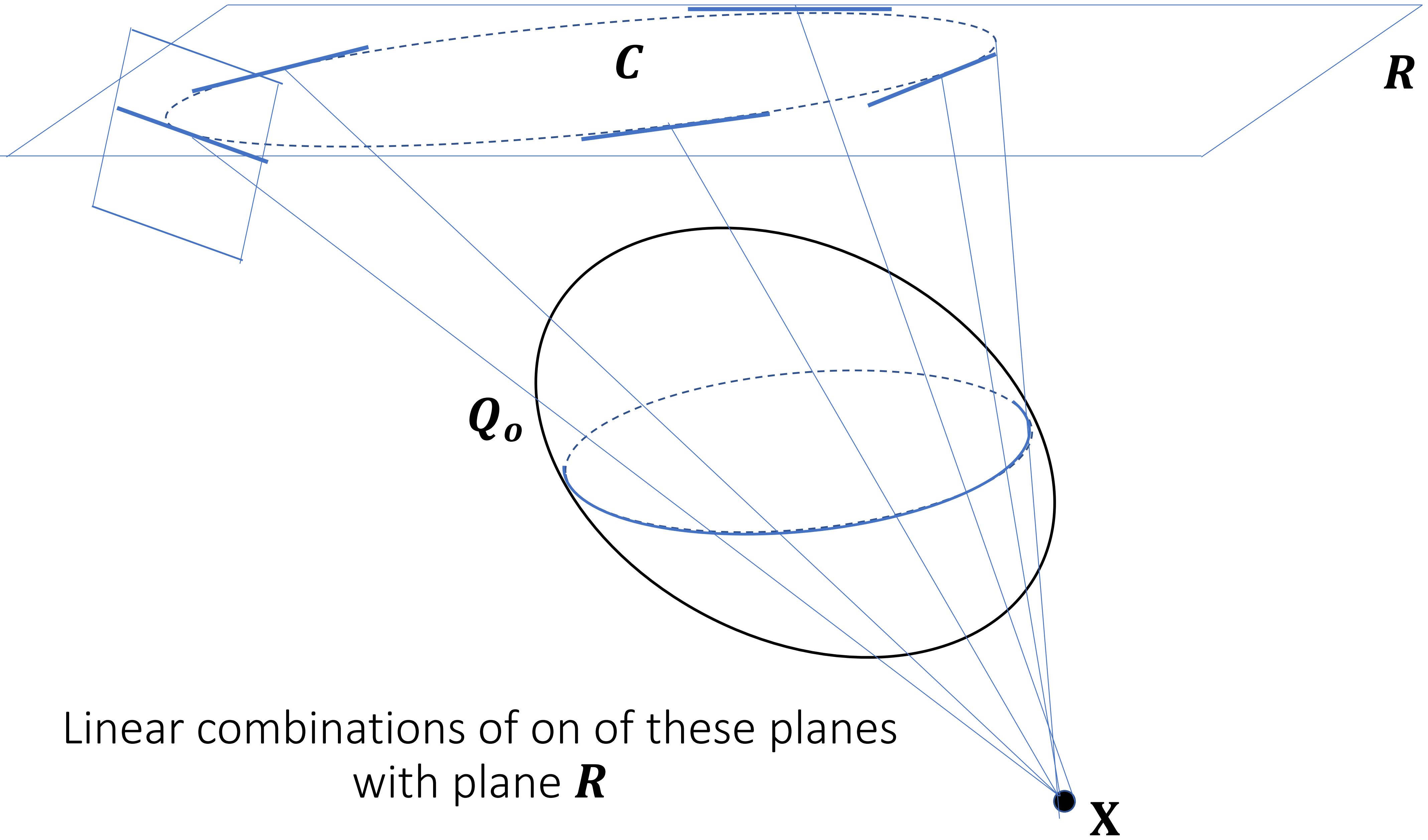
Now that we know these planes, we have to construct –for each of them- the set of linear combinations with plane $\boldsymbol{R}$. This set of linear combinations is a pencil of planes: the set of coaxial planes, whose axis is the intersection line of the plane and plane $\boldsymbol{R}$. This axis is a tangent to conic $\boldsymbol{C}$.

Linear combinations with plane $\boldsymbol{R}$



Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



Linear combinations of on of these planes with plane $\boldsymbol{R}$

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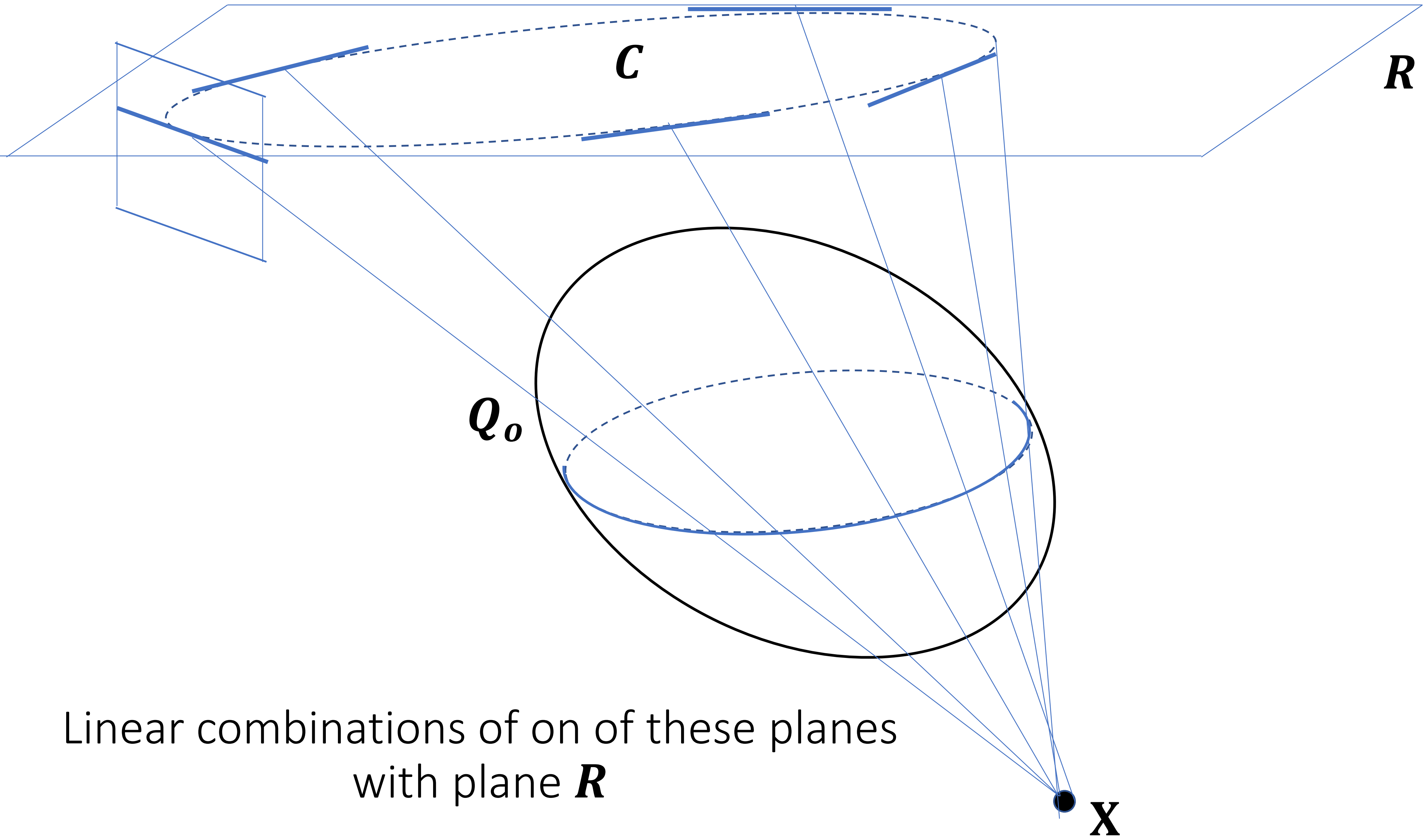
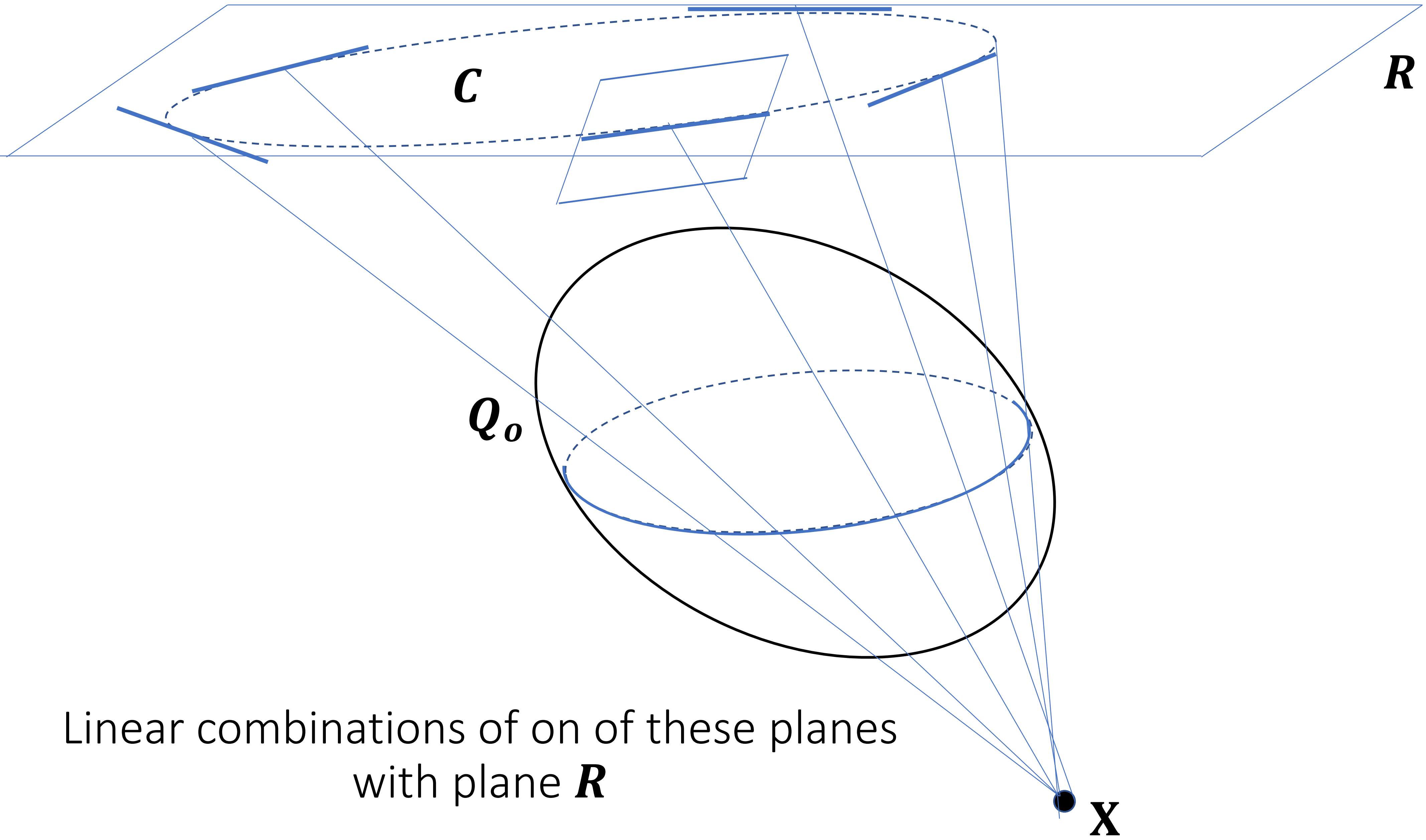


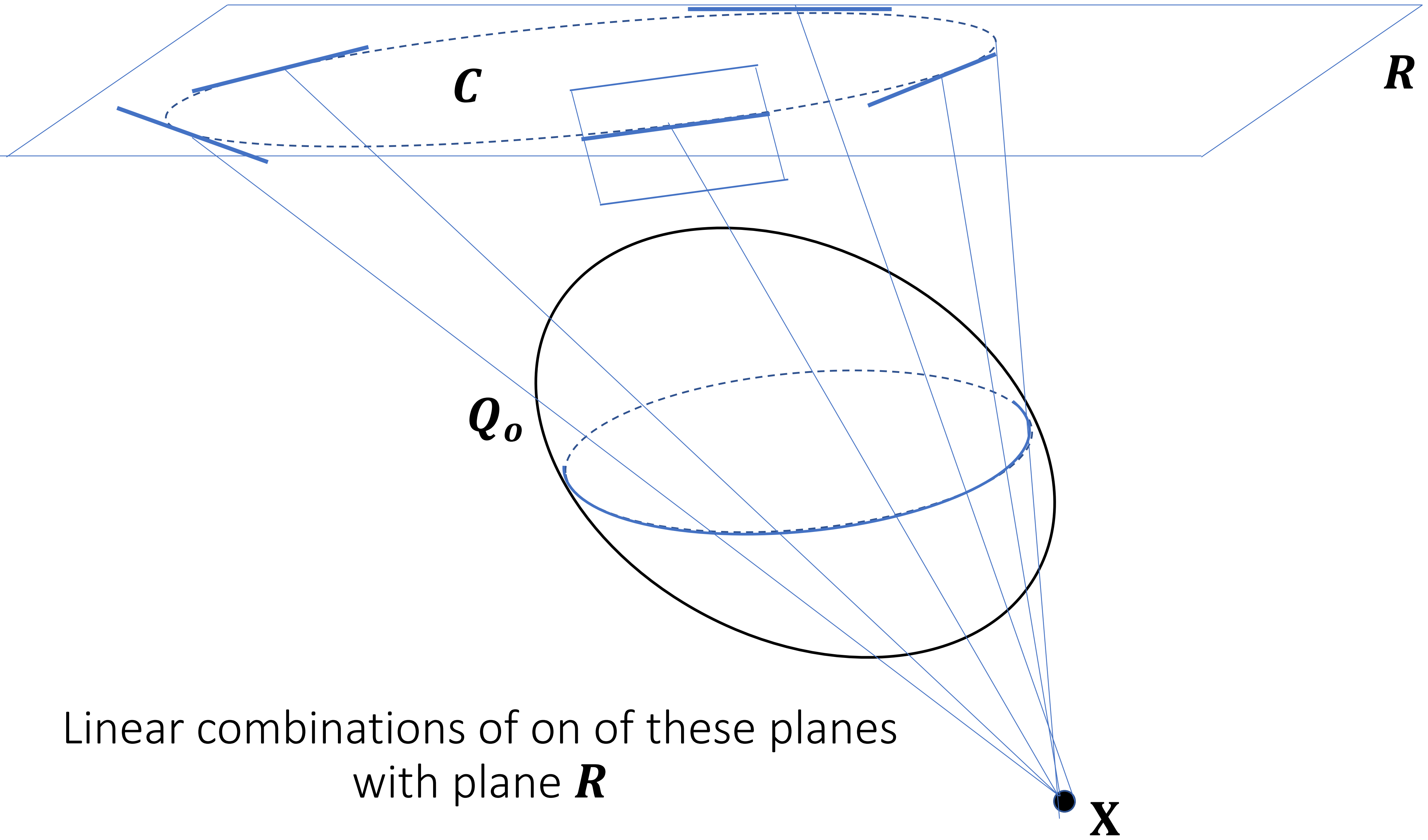
Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



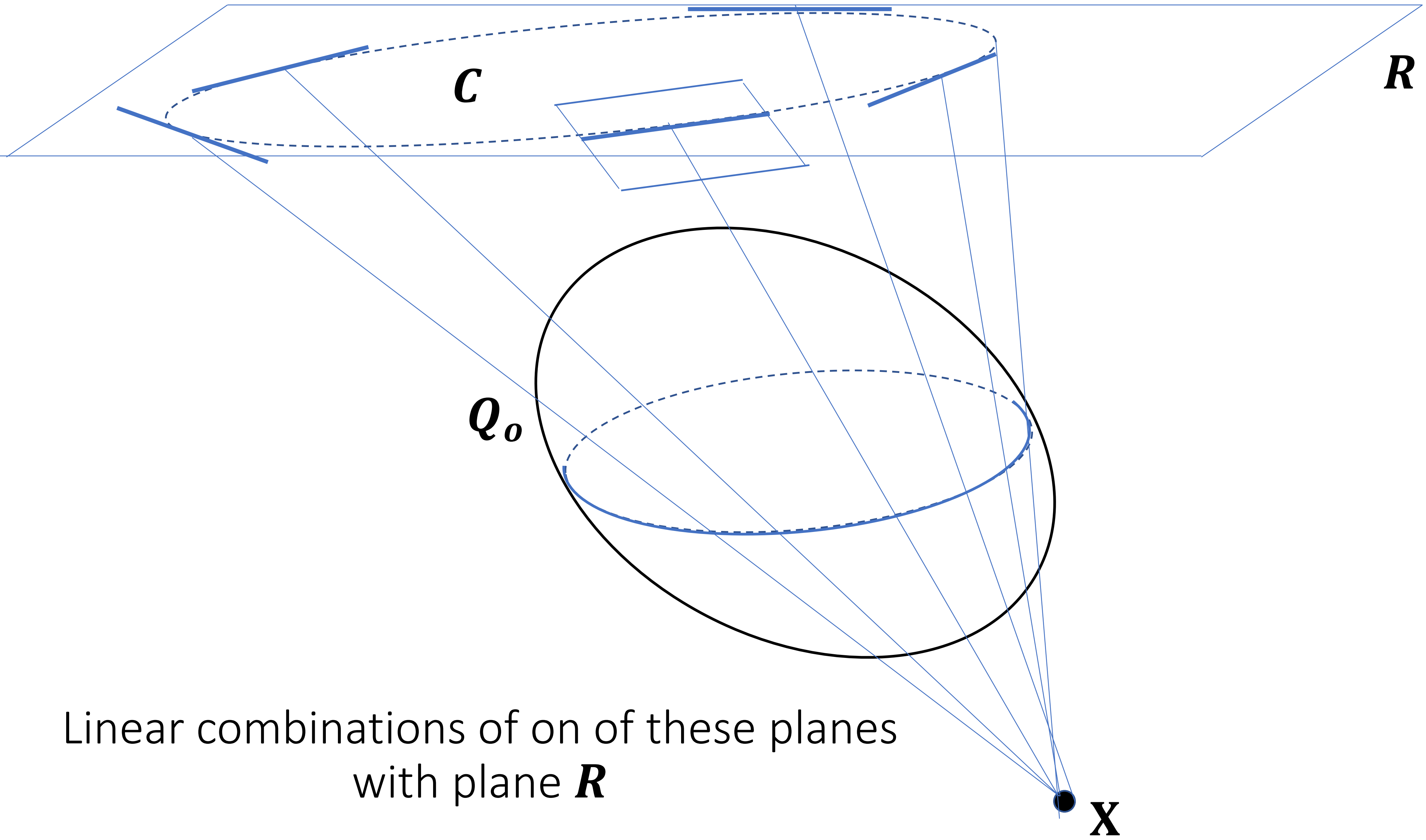
Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$



Linear combinations of on of these planes with plane $\boldsymbol{R}$

Linear combinations with plane $\boldsymbol{R}$

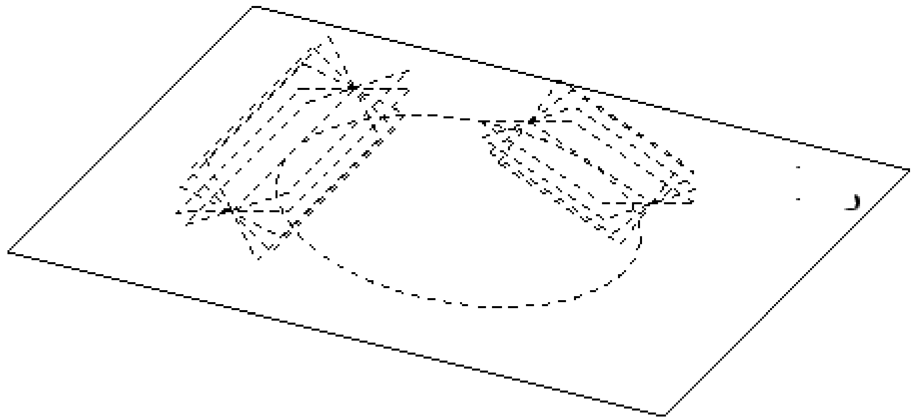


Linear combinations of on of these planes with plane $\boldsymbol{R}$

DEGENERATE DUAL QUADRICS

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

- rank $\boldsymbol{Q}^{*} = \mathbf{3}$ the **dual** of a cone
- Dual of a cone: → The set of planes that are tangent to a conic

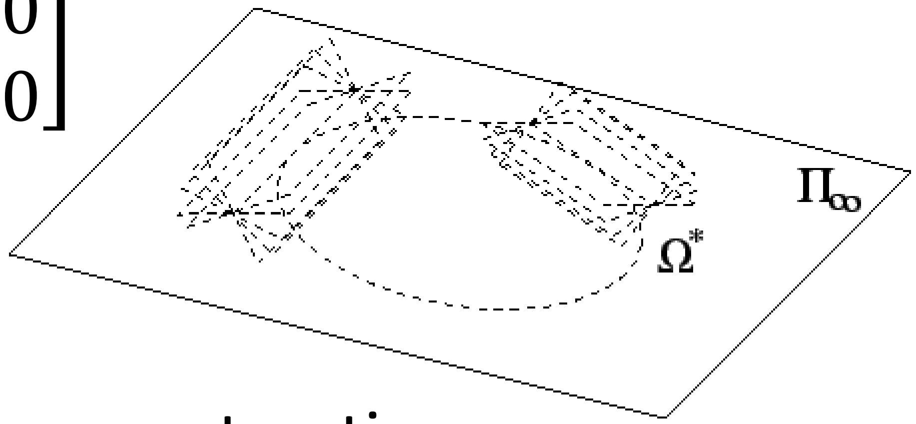


A noteworthy example:
THE ABSOLUTE (dual) QUADRIC

$$\boldsymbol{\pi}^{T}\boldsymbol{Q}^{*}_{\infty}\boldsymbol{\pi} = \mathbf{0}$$

The set of planes that are tangent to the absolute conic is the quadric $\boldsymbol{Q}^{*}_{\infty}$ with

$$\boldsymbol{Q}^{*}_{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



The absolute dual quadric $\boldsymbol{Q}^{*}_{\infty}$ is useful in the 3D reconstruction

Proof:

Dual absolute conic $\Omega_\infty^* = \Omega_\infty^{-1}$ in $\boldsymbol{\pi}_\infty$: set of lines tangent to $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A plane through one of such lines: $\boldsymbol{\pi} = [a \quad b \quad c \quad d]^T$, which intersects π_∞ at a line l: $\begin{cases} ax + by + cz + dw = 0 \\ w = 0 \end{cases}$ → $ax + by + cz = 0 = l^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

with $l = [a \quad b \quad c]^T$, where $l \in \Omega_\infty^*$, i.e. $l^T \Omega_\infty^* l = 0$. But

$$l^T \Omega_\infty^* l = [a \quad b \quad c]^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a \quad b \quad c \quad d]^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Hence

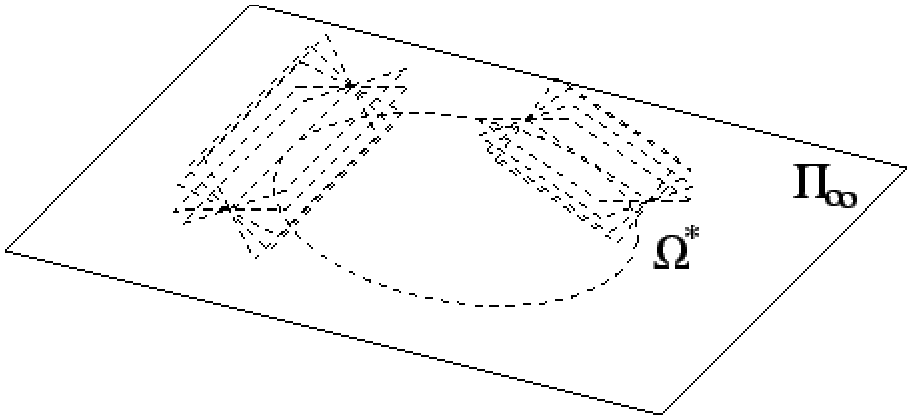
$$\boldsymbol{\pi}^T \boldsymbol{Q}^*{}_\infty \boldsymbol{\pi} = \mathbf{0}$$

A property of the THE ABSOLUTE (dual) QUADRIC

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

→The set of planes that are tangent to the absolute conic:

$$\boldsymbol{Q}^{*}{}_{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Property: the RNS of absolute dual quadric $\boldsymbol{Q}^{*}{}_{\infty}$ is the plane at the infinity $\boldsymbol{\pi}_\infty$

$$\boldsymbol{Q}^{*}{}_{\infty}\boldsymbol{\pi}_\infty = 0 \;\rightarrow\; \text{RNS}(\,\boldsymbol{Q}^{*}{}_{\infty}\,) = \boldsymbol{\pi}_\infty$$

Projective 3D Geometry: Projective Transformations

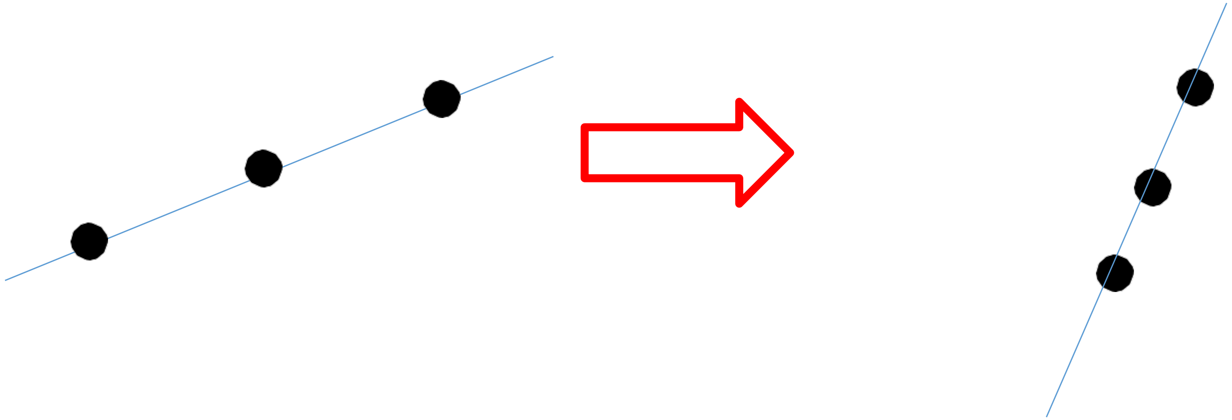
Projective mappings

Def. *A **projective mapping** between a projective space* $\mathbb{P}^3$ *and an other projective space* $\mathbb{P}'^3$ *is an **invertible** mapping which preserves colinearity:*

$$h\colon \mathbb{P}^3 \rightarrow \mathbb{P}'^3, \mathbf{X}' = h(\mathbf{X}),\ \mathbf{X}_1,\ \mathbf{X}_2, \mathbf{X}_3 \textit{ are colinear}$$

$$\leftrightarrow$$

$$\mathbf{X}'_1 = h(\mathbf{X}_1),\ \mathbf{X}'_2 = h(\mathbf{X}_2), \mathbf{X}'_3 = h(\mathbf{X}_3) \textit{ are colinear}$$



Alternative names:

- *Projectivity*

Fundamental Theorem of Projective Geometry

Theorem: *A mapping* h *:* $\mathbb{P}^3 \rightarrow \mathbb{P}'^3$ *is projective* if and only if *there exists an invertible 4x4 matrix* H *such that for any point in* $\mathbb{P}^3$ *represented by the vector* $\mathbf{X}$ *, is* $h(\mathbf{X}) = H\,\boldsymbol{X}$

i.e. projective mappings are LINEAR in the homogeneous coordinates

(they are not linear in cartesian coordinates)

Projectivity: 15 degrees of freedom

From the theorem

$$h(\mathbf{X}) = \mathbf{X}' = H\,\mathbf{X}$$

Therefore, if we multiply the matrix H by any nonzero scalar λ, the relation is satisfied by the same points

$$\mathbf{X}' = \lambda H\,\mathbf{X}$$

Thus any nonzero multiple of the matrix H represents the same projective mapping as H.

Hence H is a homogeneous matrix: in spite of its 16 entries,

H has only 15 degrees of freedom, namely the ratios between its elements.

Projectivity estimation

H has only 15 degrees of freedom, namely the ratios between its elements.
E.g.

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix}$$

Therefore, it can be estimated by just FIVE point correspondences,

since each point correspondence $\mathbf{X}' = H\,\mathbf{X}$ yelds **three** independent equations

Transformation of points, planes, quadrics, dual quadrics

Transformation rules for the space elements

A homography transforms **each point $\boldsymbol{X}$** into a point $\boldsymbol{X}'$ such that:

$$\boldsymbol{X} \to H\boldsymbol{X} = \boldsymbol{X}'$$

A homography transforms **each plane $\boldsymbol{\pi}$** into a line$\boldsymbol{\pi}'$ such that:

$$\boldsymbol{\pi} \to H^{-T}\boldsymbol{\pi} = \boldsymbol{\pi}'$$

A homography transforms **each quadric $\boldsymbol{Q}$** into a quadric Q$'$ such that:

$$Q \to H^{-T}QH^{-1} = Q'$$

A homography transforms **each dual quadric** Q^* into a dual quadri**c** $Q^{*\prime}$

$$Q^* \to HQ^*H^T = Q^{*\prime}$$

Vanishing points and vanishing line

remember: intersection of two parallel lines on a plane

Suppose that lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ are parallel: this means that

$$l_1 = [a \quad b \quad c_1]^T \text{ and}$$
$$l_2 = [a \quad b \quad c_2]^T$$

The point $\mathbf{x} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]^{\boldsymbol{T}}$ common two these two lines satisfies both

$$ax + by + c_1 w = 0$$

and

$$ax + by + c_2 w = 0$$

$$\rightarrow$$

$$\mathbf{x} = [b \quad -a \quad 0]^T$$

Namely, the point at the infinty along the direction of both lines

(remember: [a, b] is the direction **normal** to both lines)

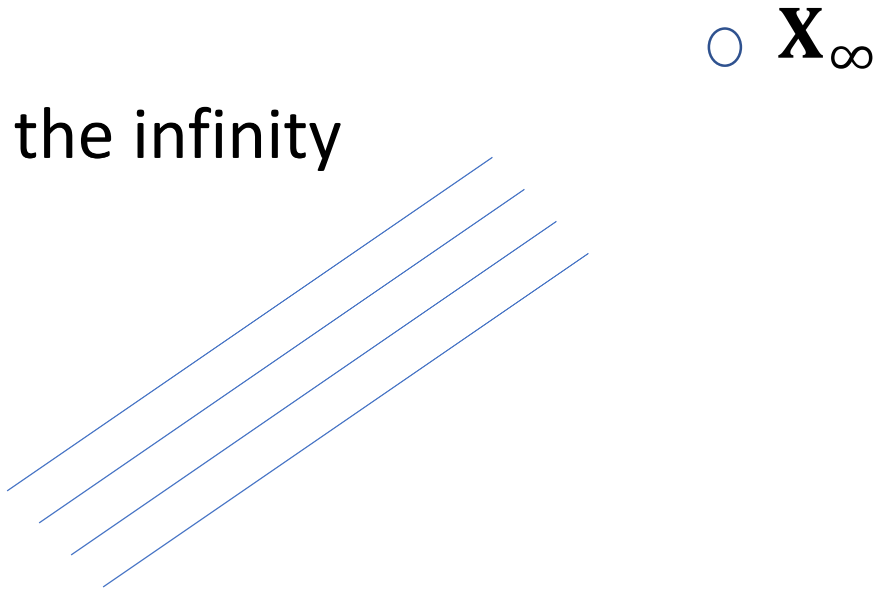
intersection of **many** parallel lines in the 3D space

Suppose that lines are parallel to direction $[a \quad b \quad c]^T$:

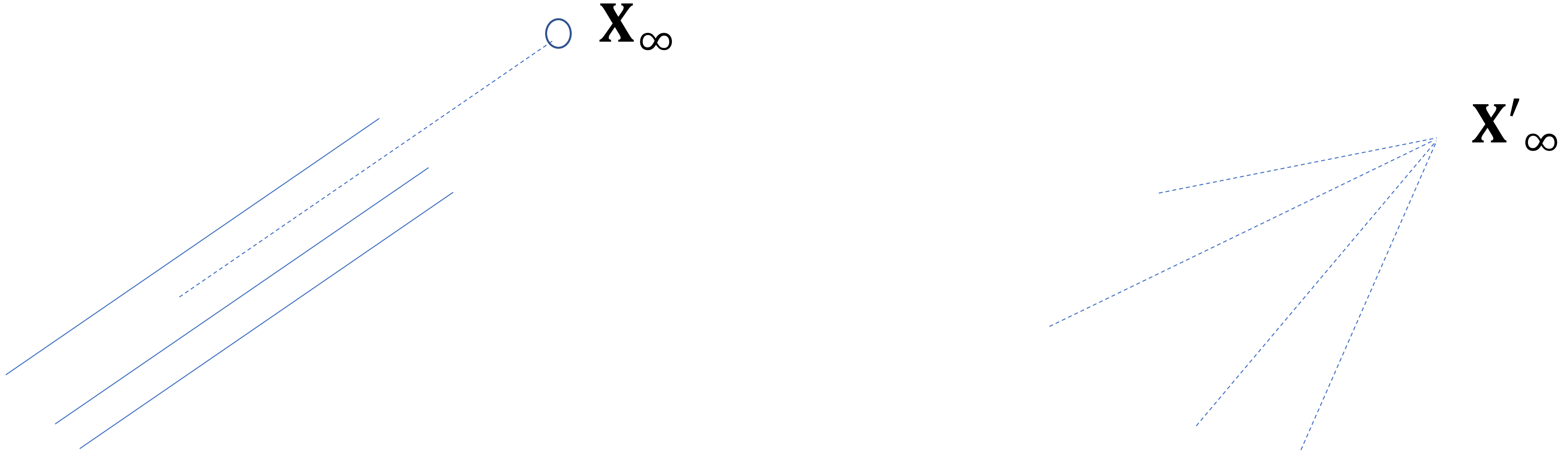
All these lines cross at the common point

$$\mathbf{X}_\infty = [a \quad b \quad c \quad 0]^T$$

This point is at the infinity



Applying a projective transformation (e.g. an image) to all the above parallel lines we obtain concurrent lines. The common point (at the infinity) $\mathbf{X}_\infty$ to all parallel lines is mapped onto a point $\mathbf{X}'_\infty = H\mathbf{X}_\infty$ where all mapped lines concur.



intersection of **many** parallel planes in the 3D space

All parallel planes $[a \quad b \quad c \quad d_i]^T$ contain a common line

$$\rightarrow$$

$$\mathbf{L}^* = \mathbf{RNS}(\begin{bmatrix} \mathbf{X_1}^T \\ \mathbf{X_2}^T \end{bmatrix}) = \mathbf{RNS}(\begin{bmatrix} b & -a & 0 & 0 \\ -c & 0 & a & 0 \end{bmatrix})$$

This line is a linear combination of two points at the infinity, e.g.,

$$\begin{bmatrix} b \\ -a \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -c \\ 0 \\ a \\ 0 \end{bmatrix}$$

intersection of **many** parallel planes in the 3D space

dually: all parallel planes $[a \quad b \quad c \quad d_i]^T$ contain a common line

$$\rightarrow$$

$$\mathbf{L} = \mathbf{RNS}(\begin{bmatrix} \boldsymbol{\pi_1}^T \\ \boldsymbol{\pi_2}^T \end{bmatrix}) = \mathbf{RNS}(\begin{bmatrix} a & b & c & d_1 \\ a & b & c & d_2 \end{bmatrix})$$

This line is a linear combination of two parallel planes, e.g.,

$$\begin{bmatrix} a \\ b \\ c \\ d_1 \end{bmatrix} \text{ and } \begin{bmatrix} a \\ b \\ c \\ d_2 \end{bmatrix}$$

Polarity is preserved under projective mappings

The polar plane $\boldsymbol{\pi} = \boldsymbol{Q}\mathbf{X}$ of a point $\mathbf{X}$ wrt a quadric $\boldsymbol{Q}$ is mapped onto the polar plane $\boldsymbol{\pi}' = \boldsymbol{Q}'\mathbf{X}'$ of the transformed point $\mathbf{X}' = H\mathbf{X}$ wrt the transformed quadric $\boldsymbol{Q}' = H^{-T}\mathbf{Q}\,H^{-1}$

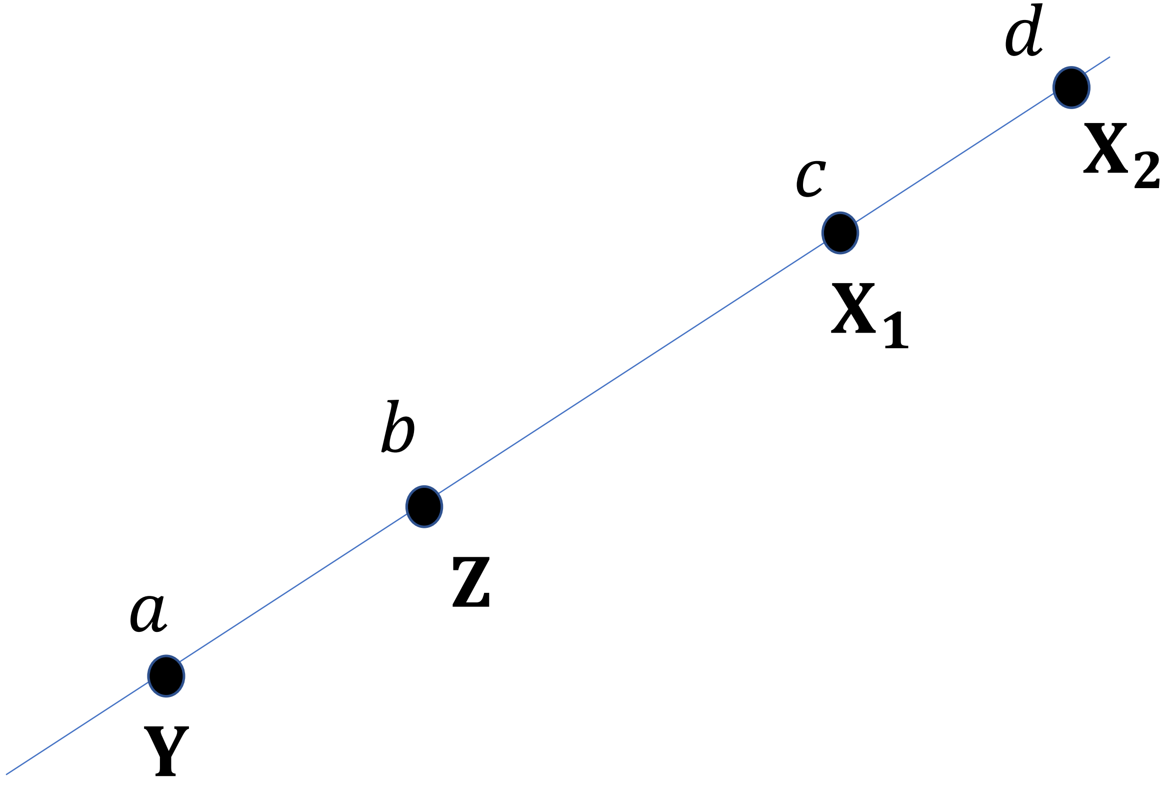
In fact,

$$\boldsymbol{\pi}' = H^{-T}\boldsymbol{\pi} = H^{-T}\mathbf{Q}\mathbf{X} = H^{-T}\mathbf{Q}\,H^{-1}H\mathbf{X} = \boldsymbol{Q}'\mathbf{X}'$$

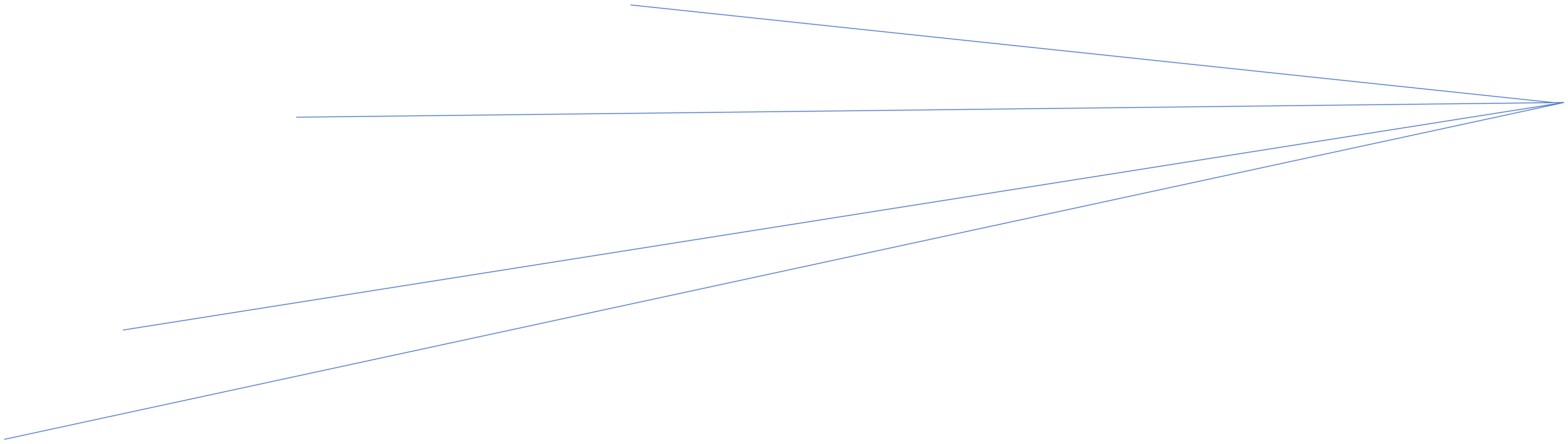
Cross ratios: invariant under projective mappings

1D cross ratio of a 4-tuple of colinear points

$$CR_{Y,Z,\mathrm{X}_1,\mathrm{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$



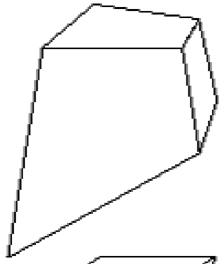
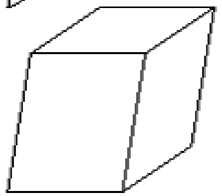
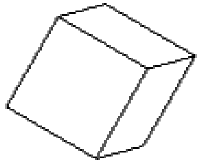
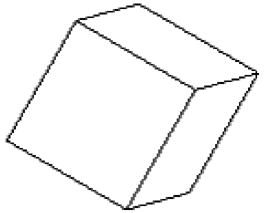
2D cross ratio of a 4-tuple of coplanar, concurrent lines

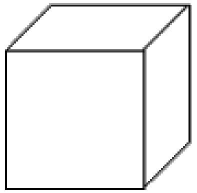


3D cross ratio of a 4-tuple of coaxial planes:

Hierarchy of projective transformations

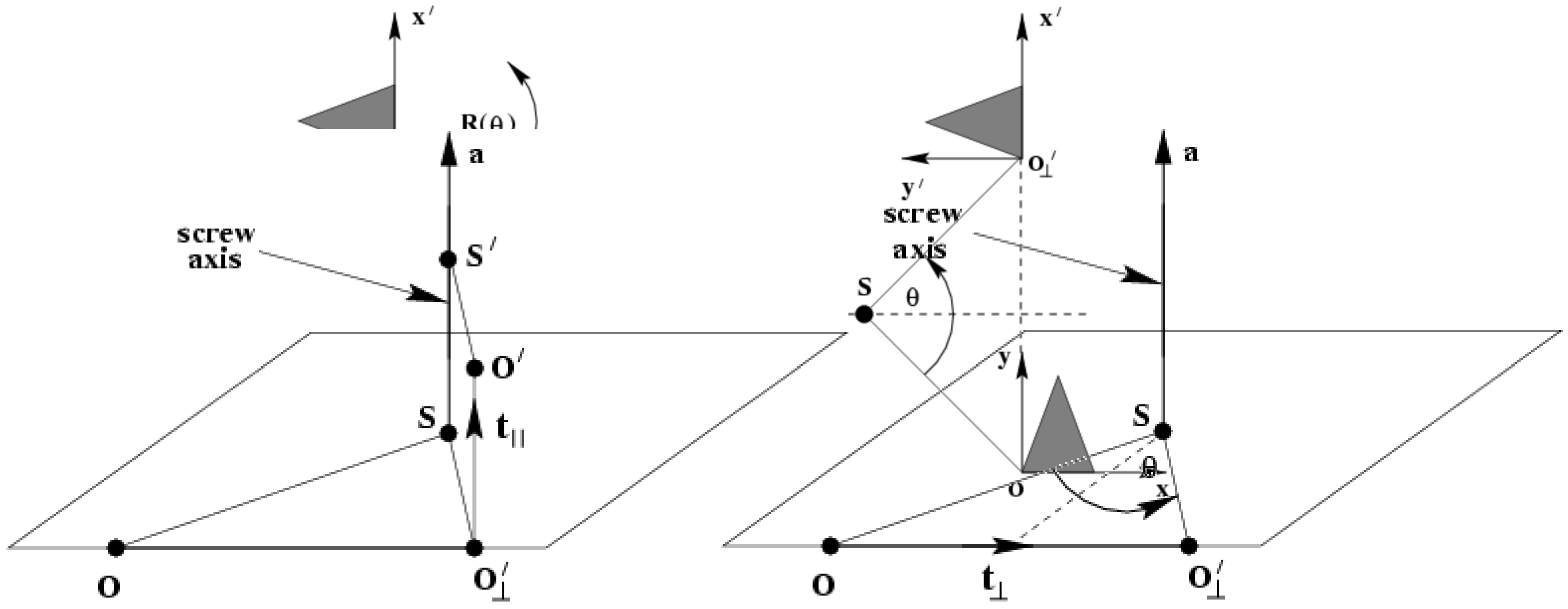
Hierarchy of transformations

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		**The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Volume



A 3D rototranslation **is not** a pure rotation: screw decomposition

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.



screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

Isometries (or Euclidean mappings)

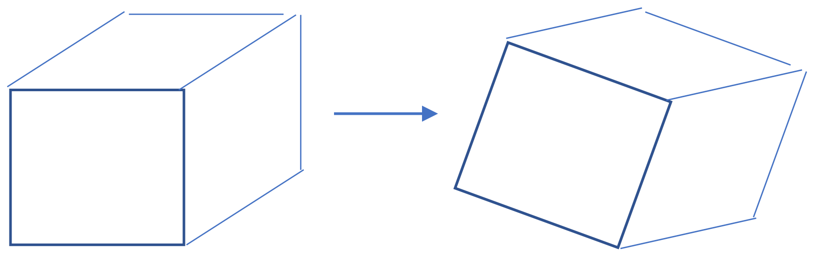
$$H_I = \begin{bmatrix} R_\perp & & \boldsymbol{t} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_\perp$ is a 3x3 orthogonal matrix: $R_\perp{}^{-1} = R_\perp{}^{T}$

det $R_\perp{}^{-1} = 1$ planar rigid displacement (-1 for reflection)

6 dofs: translation $\boldsymbol{t}$ + Euler angles $\vartheta, \varphi,$?

Invariants: lenghts, distances, areas → shape and size → relative positions



Similarities

$$H_S = \begin{bmatrix} s\,R_\perp & \boldsymbol{t} \\ 0 \quad 0 & 1 \end{bmatrix}$$

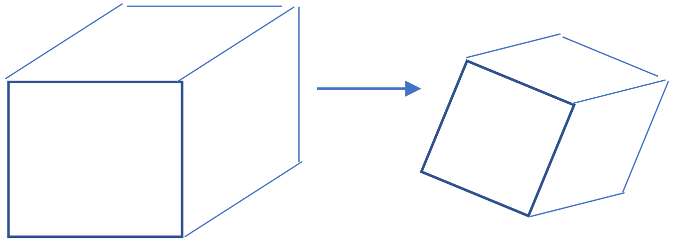
$R_\perp$ is a 3x3 orthogonal matrix: ${R_\perp}^{-1} = {R_\perp}^T$

7 dofs: rigid displacement + *scale*

Invariants: ratio of lenghts, angles → shape (not size)

the absolute conic Ω_∞

and the absolute dual quadric $\boldsymbol{Q}^*{}_\infty$



Affinities (or affine mappings)

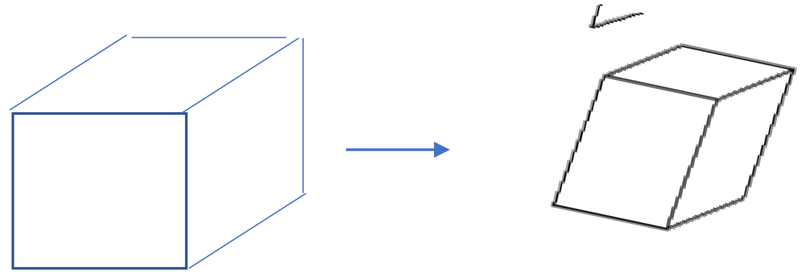
$$H_A = \begin{bmatrix} & A & & \boldsymbol{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A is any 3x3 invertible matrix

12 dofs: $A + \boldsymbol{t}$

Invariants: parallelism, ratio of parallel lenghts, ratio of areas

<u>the plane at the infinity</u> $\boldsymbol{\pi}_\infty$



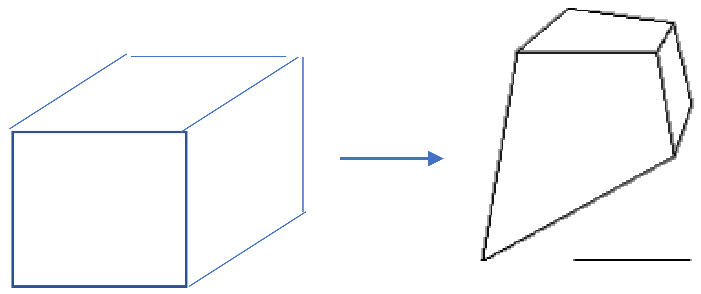
Projectivities (or projective mappings)

$$H = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{v}^T & 1 \end{bmatrix}$$

A is any 3x3 invertible matrix

15 dofs: $A + \boldsymbol{v} + \boldsymbol{t}$

Invariants: colinearity, incidence, order of contact (crossing, tangency, inflections), <u>the 1D cross ratio, the 2D cross ratio, the 3D cross ratio</u>



The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}}\pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{A}\,\mathrm{t} & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canical position $\pi_\infty = (0,0,0,1)^{\mathsf{T}}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^{\mathsf{T}}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

A theorem on an affine invariant

Theorem. *A projective transformation H maps the plane at the infinity $\boldsymbol{\pi}_\infty$ onto itself (i.e., $\boldsymbol{\pi}_\infty$ is invariant under a projective transformation)*

$$\Updownarrow$$

H *is* ***affine***

3D reconstruction problem

Unknown original scene = set of points in the 3D space

→

An unknown 3D projective mapping is applied to them

→

Suppose that the transformed 3D scene can be observed

→

From the observed scene (different from the original)

recover a model of the original scene

3D reconstruction problem

Unknown original scene = set of points in the 3D space

→

An unknown 3D projective mapping is applied to them

→

Suppose that the transformed 3D scene can be observed

→

From the observed scene (different from the original)

recover a model of the original scene

HOW? Images are 2D, not 3D: **issues to be addressed later**

Application to affine reconstruction

Given 3D points obtained by an uknown projective mapping of an unknown original scene (set of points in 3D space)

$\rightarrow$

the plane $\boldsymbol{\pi}'_\infty$ (i.e. the transformed $\boldsymbol{\pi}_\infty$) is in general $\neq \boldsymbol{\pi}_\infty$!!

Use $\boldsymbol{\pi}'_\infty$ as additional information: if we apply to the transfomed set a second mapping $H_{\boldsymbol{AR}}$ which sends $\boldsymbol{\pi}'_\infty$ back to $\boldsymbol{\pi}_\infty$, we obtain a new, reconstructed model

The composed mapping of $\boldsymbol{\pi}_\infty$ is again $\boldsymbol{\pi}_\infty$ $\rightarrow$

From the theorem, the obtained model is an affine mapping of the original scene

$\rightarrow$

The obtained model is an **affine reconstruction** of the scene

1. Use of $\boldsymbol{\pi}'_\infty$ in affine reconstruction

....

apply to the transformed point set a second projective mapping $\boldsymbol{H_{AR}}$ that maps $\boldsymbol{\pi}'_\infty$ back to $\boldsymbol{\pi}_\infty$,

how can we find such a projective mapping H_{AR} ?

$$H_{AR} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & \boldsymbol{\pi}'^{\,T}_{\infty} & & \end{bmatrix},$$

such that H_{AR} it is invertible

To sum up: affine rectification from $\boldsymbol{\pi}'_\infty$

- Find three points $\mathbf{X'_1}, \mathbf{X'_2}, \mathbf{X'_3}$ that result from mapping three points at the infinity

- Fit the transformed plane $\boldsymbol{\pi}'_\infty$ to them: $\boldsymbol{\pi}'_\infty = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{X'_1}^T \\ \mathbf{X'_2}^T \\ \mathbf{X'_3}^T \end{bmatrix}\right)$

- Affine rectification matrix

$$H_{AR} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & \boldsymbol{\pi}'^{\,T}_\infty & & \end{bmatrix}$$

- Affine reconstructed model $M_A = H_{AR}$ given_points

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $(X_1, X_2, X_3)\mathrm{I}(X_1, X_2, X_3)^\mathsf{T}$

(with no real points)

The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

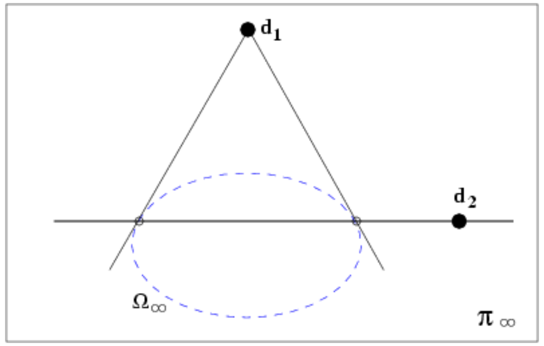
1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two points
3. Spheres intersect π_∞ in Ω_∞

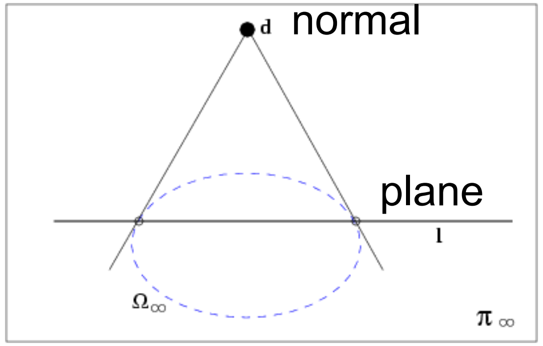
The absolute conic

Euclidean: $$\cos\theta = \frac{\left(d_1^T d_2\right)}{\sqrt{\left(d_1^T d_1\right)\left(d_2^T d_2\right)}}$$

Projective: $$\cos\theta = \frac{\left(d_1^T \Omega_\infty d_2\right)}{\sqrt{\left(d_1^T \Omega_\infty d_1\right)\left(d_2^T \Omega_\infty d_2\right)}}$$

$d_1^T \Omega_\infty d_2 = 0$ (orthogonality=conjugacy)





A theorem on an invariant under similarities

Theorem. *A projective transformation H maps the **absolute conic** Ω_∞ onto itself (i.e., Ω_∞ is invariant under a projective transformation)*

$$\Updownarrow$$

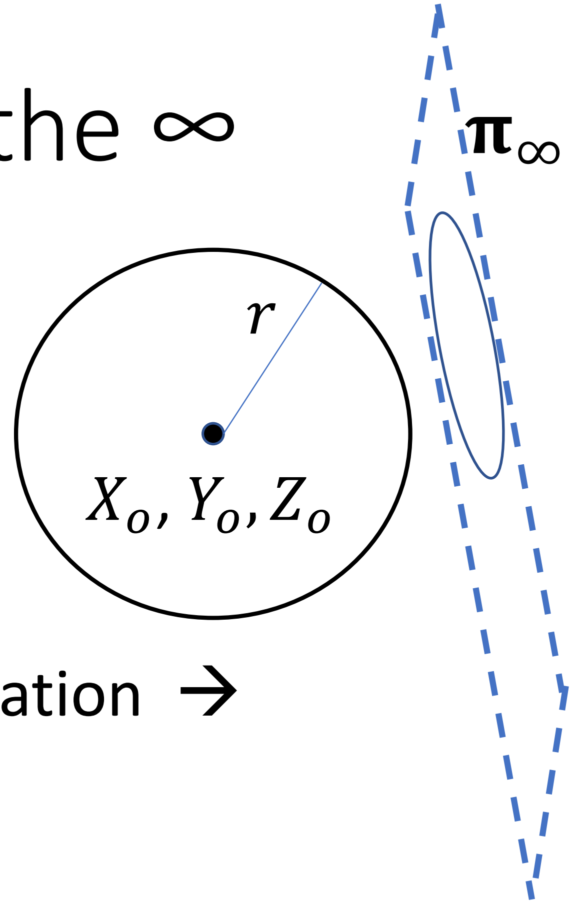
H *is **a similarity***

The absolute conic: intersection of a sphere and the plane at the ∞

$$\begin{cases}(x - X_o w)^2 + (y - Y_o w)^2 + (z - Z_o w)^2 - r^2 w^2 = 0 \\ w = 0\end{cases}$$

$$\rightarrow$$

$$\begin{cases}x^2 + y^2 + z^2 = 0 \\ w = 0\end{cases}$$



The sphere parameters (center and radius) disappear from the equation $\rightarrow$

the intersection **conic** is the **same for all** spheres:

$$x^2 + y^2 + z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

A conic within $\boldsymbol{\pi}_\infty$: $\Omega_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **ABSOLUTE CONIC**

«proof» without computation

- $\Omega_\infty = \boldsymbol{\pi}_\infty \cap$ any sphere
- Similarity maps sphere onto sphere’
- Similarity maps $\boldsymbol{\pi}_\infty$ onto $\boldsymbol{\pi}_\infty$
- Similarity maps sphere $\cap\ \boldsymbol{\pi}_\infty$ onto sphere’ $\cap\ \boldsymbol{\pi}_\infty$
- Similarity maps Ω_∞ onto Ω_∞
- → absolute conic Ω_∞ is **invariant** under similarity

Application to 3D shape reconstruction

Given 3D points obtained by an uknown projective mapping of an unknown original scene (set of points in 3D space)

→

The absolute conic Ω_∞ is mapped onto a conic $\Omega'_\infty \neq \Omega_\infty$!!

Use Ω'_∞ as additional information: if we apply to the transfomed set a second mapping $H_{\boldsymbol{SR}}$ which sends Ω'_∞ back to Ω_∞, we obtain a new, reconstructed model

The composed mapping of Ω_∞ is again Ω_∞→

From the theorem, the obtained model is a similarity of the original scene

→

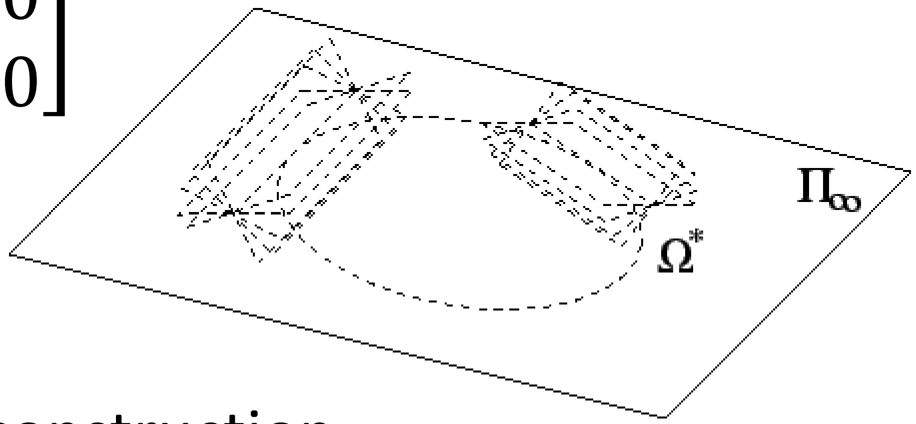
The obtained model is a **shape reconstruction** of the original scene

A noteworthy example:
THE ABSOLUTE (dual) QUADRIC

$$\boldsymbol{\pi}^{T}\,\boldsymbol{Q}^{*}\boldsymbol{\pi} = \mathbf{0}$$

→The set of planes that are tangent to a the absolute conic:

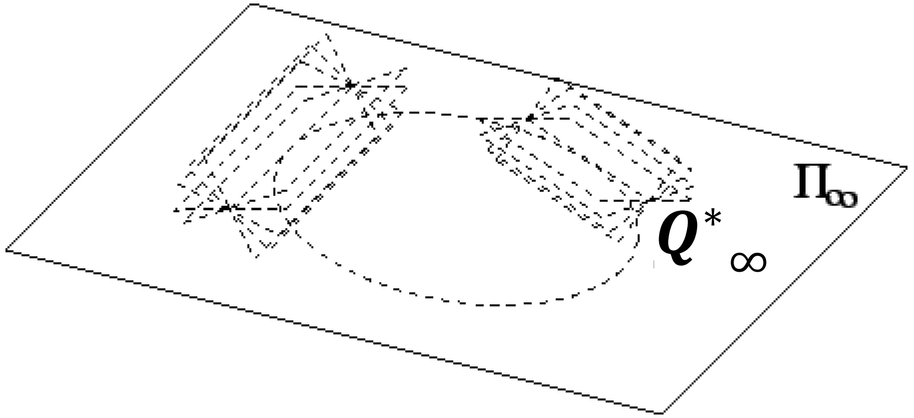
$$\boldsymbol{Q}^{*}{}_{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



The absolute dual quadric $\boldsymbol{Q}^{*}{}_{\infty}$ is useful in the 3D reconstruction

The absolute dual quadric

$$Q^*_\infty = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix}$$



The absolute conic Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the null vector of Q^*_∞
3. Angles:

A theorem on an invariant under similarities

Theorem. *A projective transformation H maps the **absolute dual quadric** $\boldsymbol{Q}^*_\infty$ onto itself (i.e., $\boldsymbol{Q}^*_\infty$ is invariant under a projective transformation)*

$$\Updownarrow$$

H *is **a similarity***

1. Use of $\boldsymbol{Q'}^{*}{}_{\infty}$ in shape reconstruction

finding a projectivity H_{SR} which maps $\boldsymbol{Q'}^{*}{}_{\infty}$ back to $\boldsymbol{Q}^{*}{}_{\infty}$

reduces to finding a projectivity H_{SR} that maps Ω'_{∞} back to Ω_{∞}

$$\boldsymbol{Q}^{*}{}_{\infty} = H_{SR}\, \boldsymbol{Q'}^{*}{}_{\infty}\, {H_{SR}}^{T} \rightarrow$$

$$\boldsymbol{Q'}^{*}{}_{\infty} = {H_{SR}}^{-1}\, \boldsymbol{Q}^{*}{}_{\infty}\, {H_{SR}}^{-T} = {H_{SR}}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ({H_{SR}}^{-1})^{T}$$

$$\mathrm{SVD}(\boldsymbol{Q'}^{*}{}_{\infty}) = U_{\perp} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {U_{\perp}}^{T}$$

$\rightarrow$ one of the ∞^{8} solutions is $H_{SR} = {U_{\perp}}^{-1} = {U_{\perp}}^{T}$

To sum up: shape reconstruction from $\boldsymbol{Q}'^*_\infty$ the transformed absolute dual quadric $\boldsymbol{Q}^*_\infty$

- Find the transformed absolute dual quadric $\boldsymbol{Q}'^*_\infty$

- Singular value decomposition

$$\mathrm{SVD}(\boldsymbol{Q}'^*_\infty) = U_\perp \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} U_\perp{}^T \ = H_{SR}^{-1}\, \boldsymbol{Q}^*_\infty H_{SR}^{-T}$$

- Reconstructing transformation (from svd output U)

$H_{SR} = U^T$ ← notice: $H_{SR} = U^T$ is orthogonal 4x4, not a $\mathbb{P}^3$ isometry

- Euclidean reconstructed model $M_S = H_{SR}$ given_points

2. How to find Ω'_∞(or $\boldsymbol{Q'^*}_\infty$) in practical cases?

In 3D reconstruction we use Ω'_∞, or equivalently $\boldsymbol{Q'^*}_\infty$, as additional information

how can we find Ω'_∞ or equivalently $\boldsymbol{Q'^*}_\infty$**?**

from information on the observed scene
additional constraints are derived:

a. known angles between plane normals
b. known shape of objects, e.g., spheres
c. combinations of a. and b.

Angle ϑ between two planes in the original scene in terms of the mapped elements: $\boldsymbol{\pi}_1', \boldsymbol{\pi}_2'$, $\boldsymbol{Q'^*}_\infty$

$$\cos\vartheta = \frac{\boldsymbol{\pi}_1'^T\ \boldsymbol{Q'^*}_\infty \boldsymbol{\pi}_2'}{\sqrt{(\boldsymbol{\pi}_1'^T\ \boldsymbol{Q'^*}_\infty \boldsymbol{\pi}_1')(\boldsymbol{\pi}_2'^T\ \boldsymbol{Q'^*}_\infty \boldsymbol{\pi}_2')}}$$

Here, $\boldsymbol{\pi}_1', \boldsymbol{\pi}_2'$ are extracted from the mapped scene (see later), whereas $\boldsymbol{Q'^*}_\infty$ is the required information we want to find

Known angle ϑ between two **scene planes** → nonlinear eqn on $\boldsymbol{Q'^*}$

if the scene planes are perpendicular, $\cos\vartheta = 0$ → $\boldsymbol{\pi}_1'^T\ \boldsymbol{Q'^*}_\infty \boldsymbol{\pi}_2' = 0$ **linear**