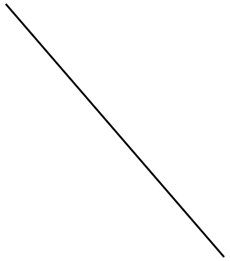
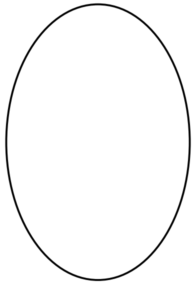
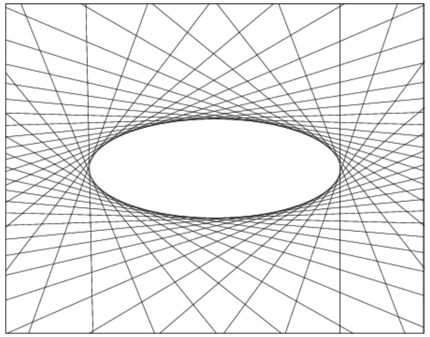
Planar (2D) Projective Geometry

Planar Projective Geometry

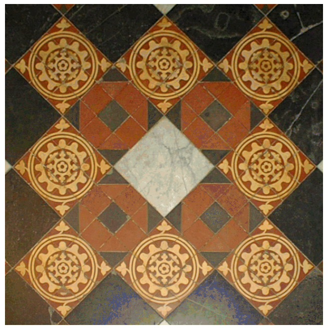
- **Elements**
  - Points
  - Lines
  - Conics
  - Dual conics







- **Transformations**
  - Isometries
  - Similarities
  - Affinities
  - Projectivities



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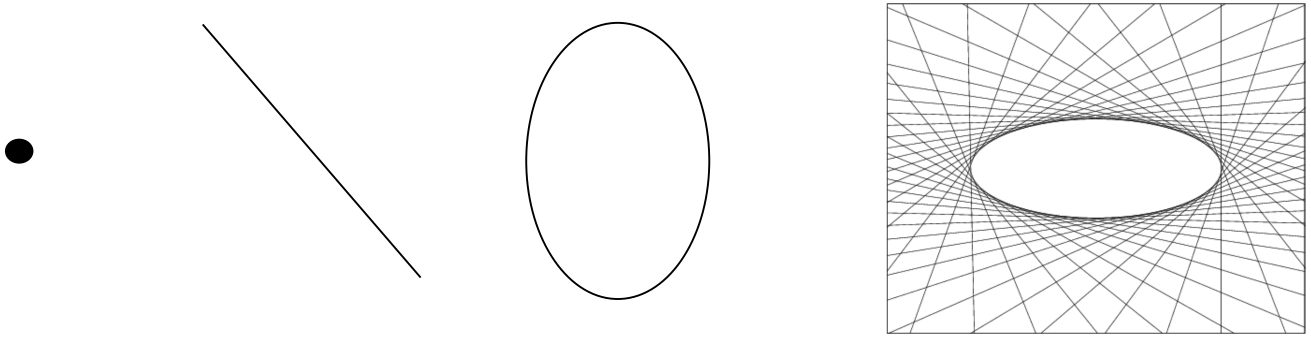

- **Transformations**
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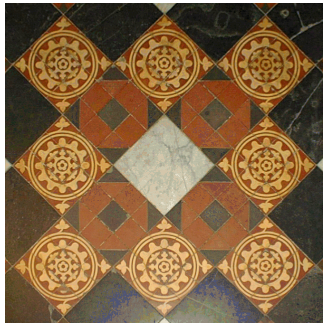
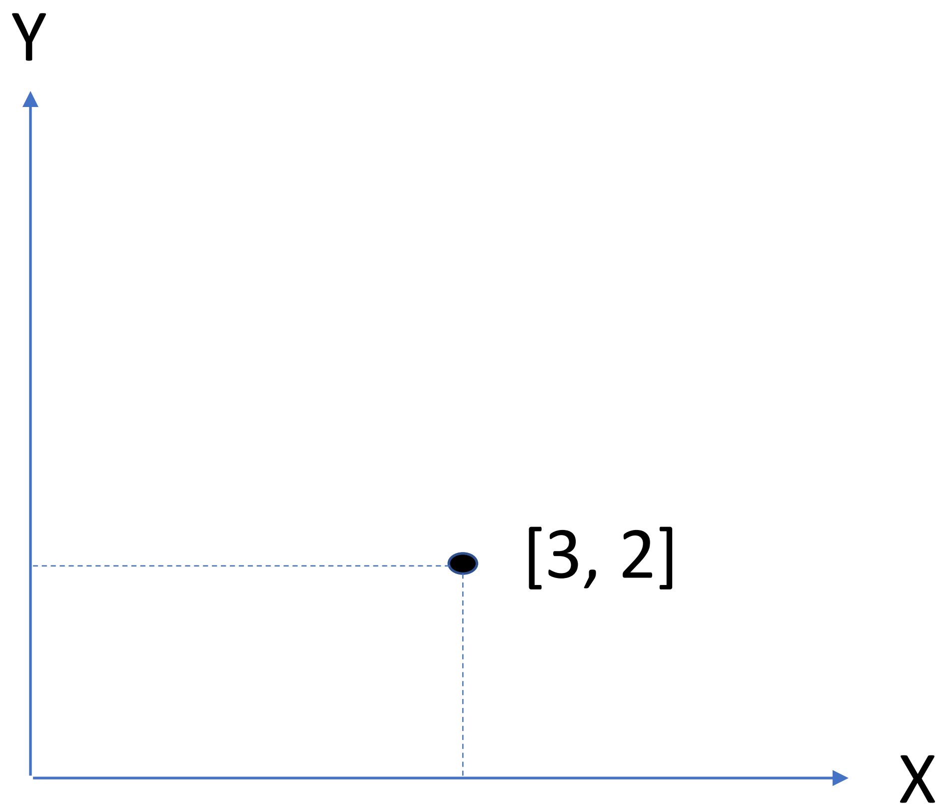


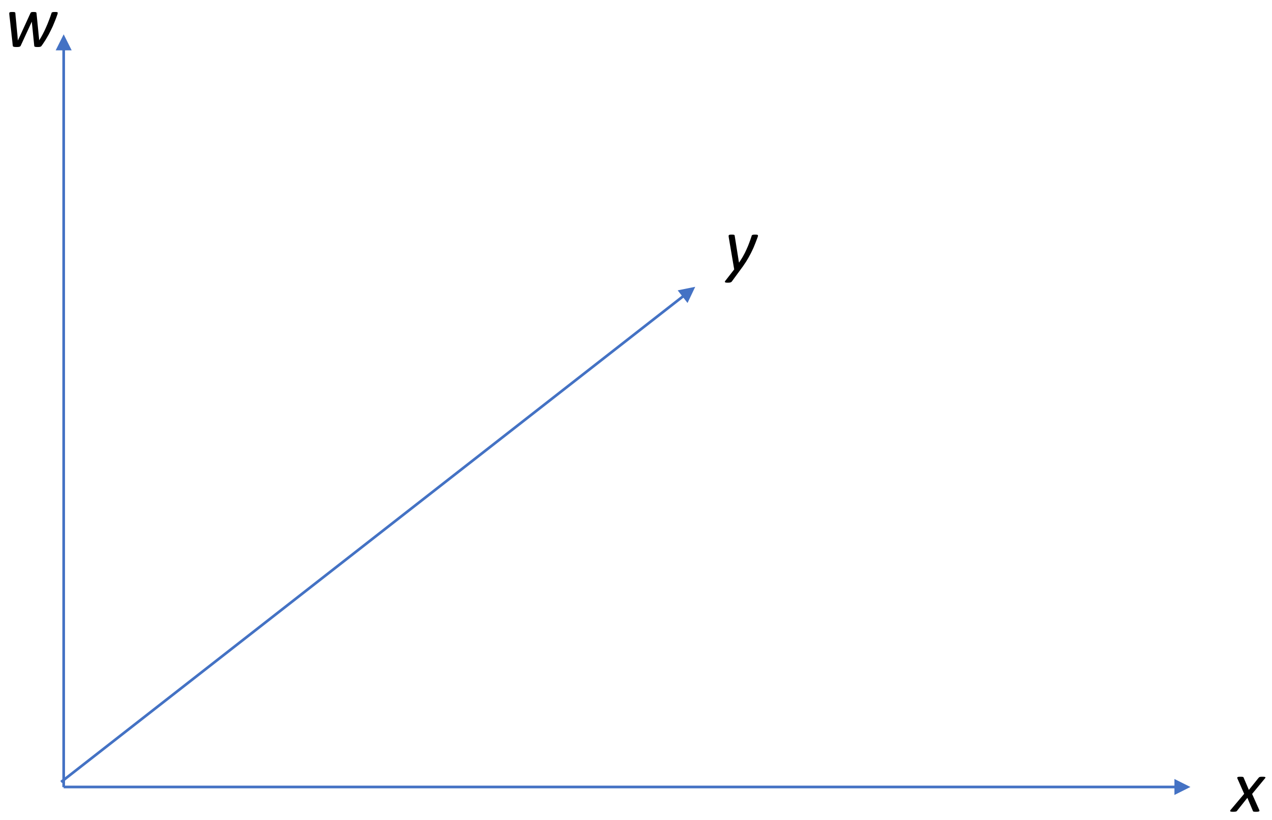
Points in 2D Projective Geometry

Euclidean plane – cartesian coordinates

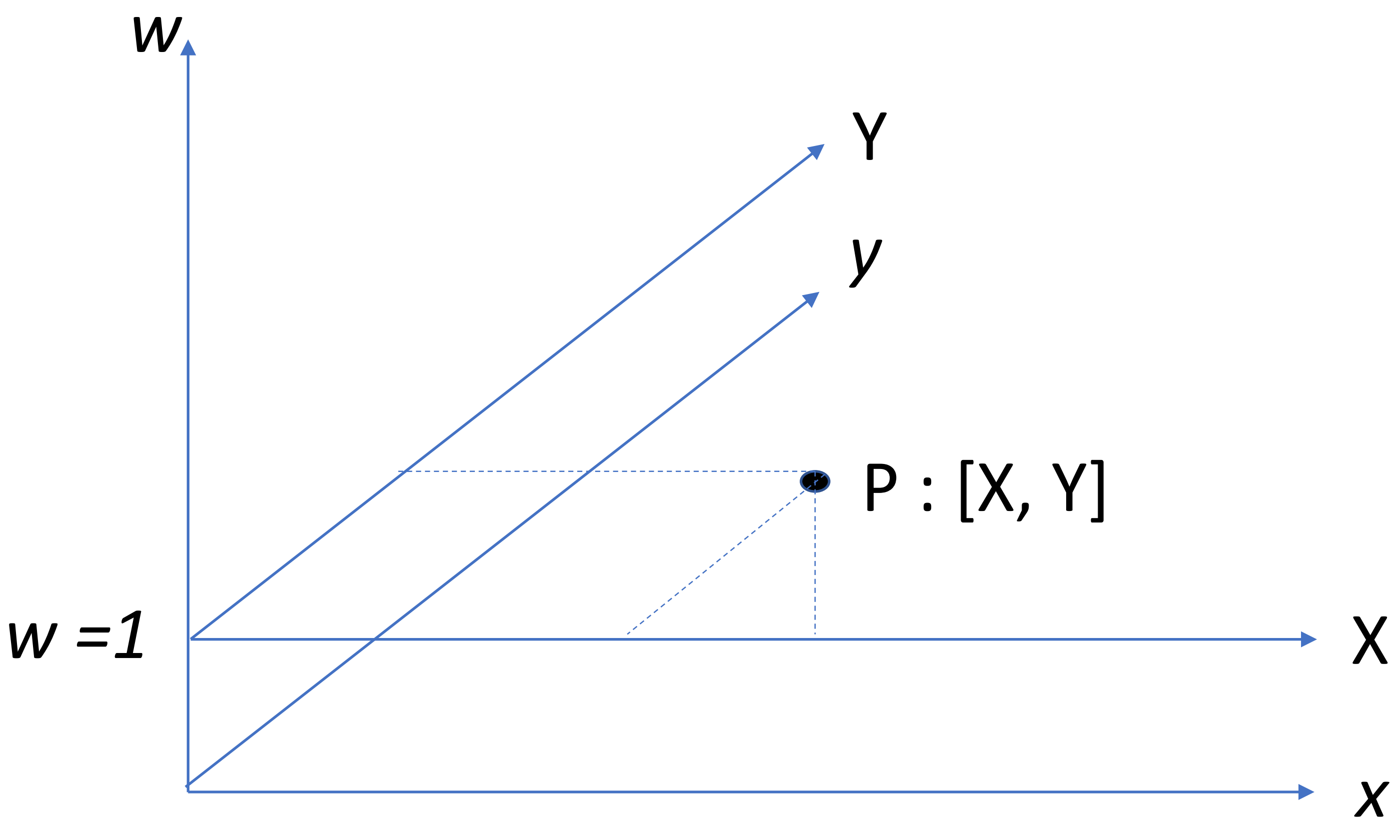


Homogeneous coordinates

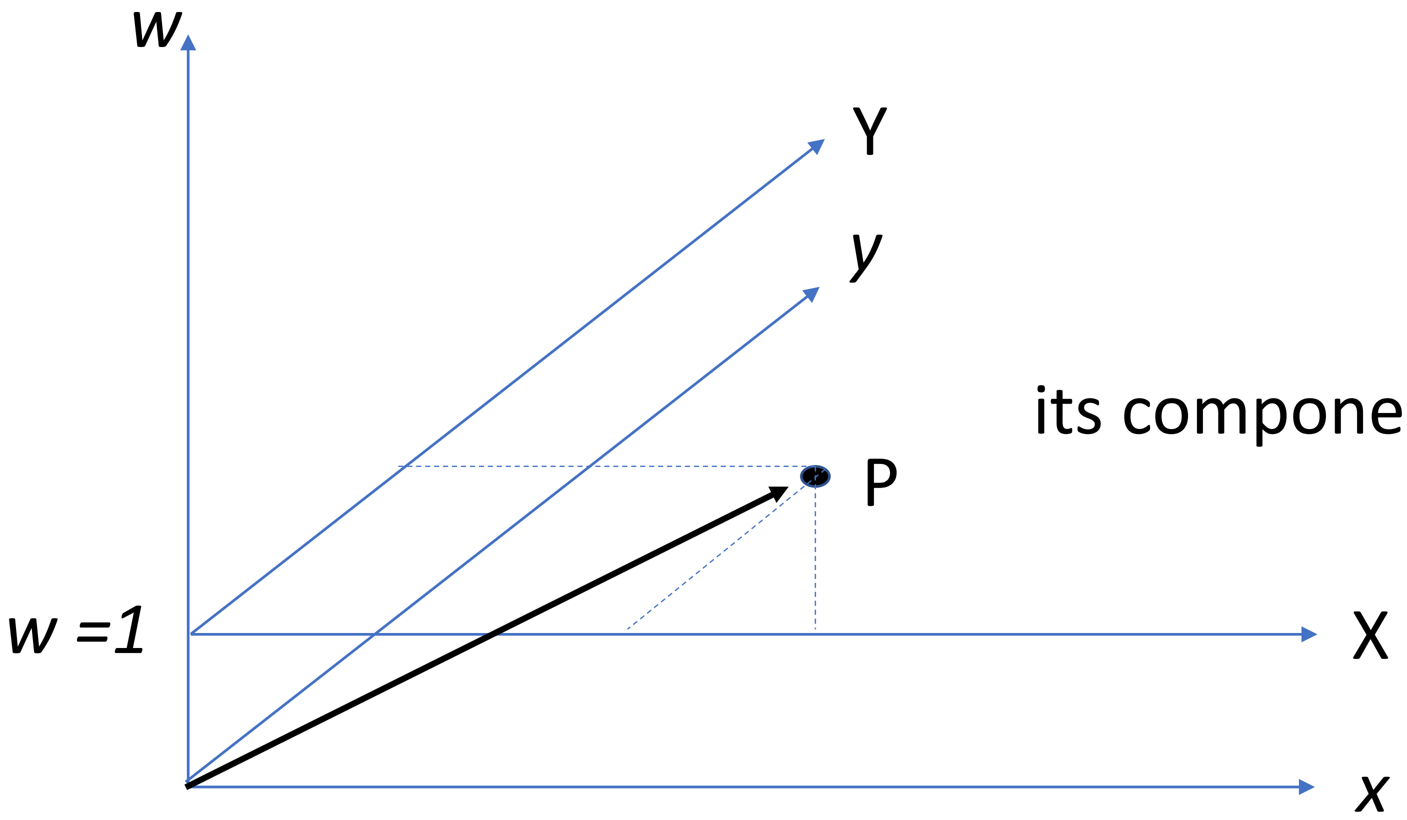
Consider a (3D) space of coordinate vectors



Embed the Euclidean plane into the (3D) space of coordinate vectors as the plane $w = 1$

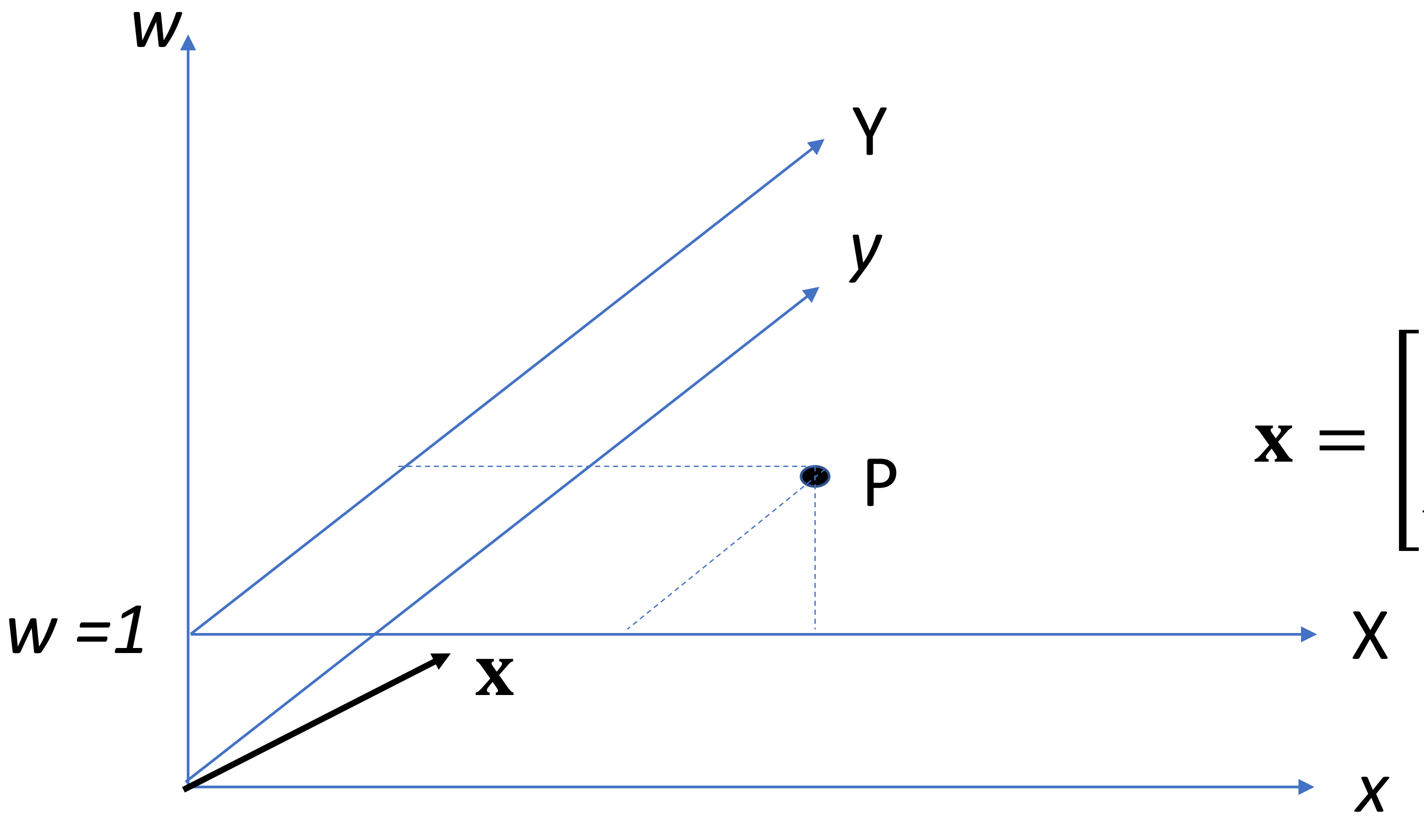


Consider the vector from the origin of the space to the point P



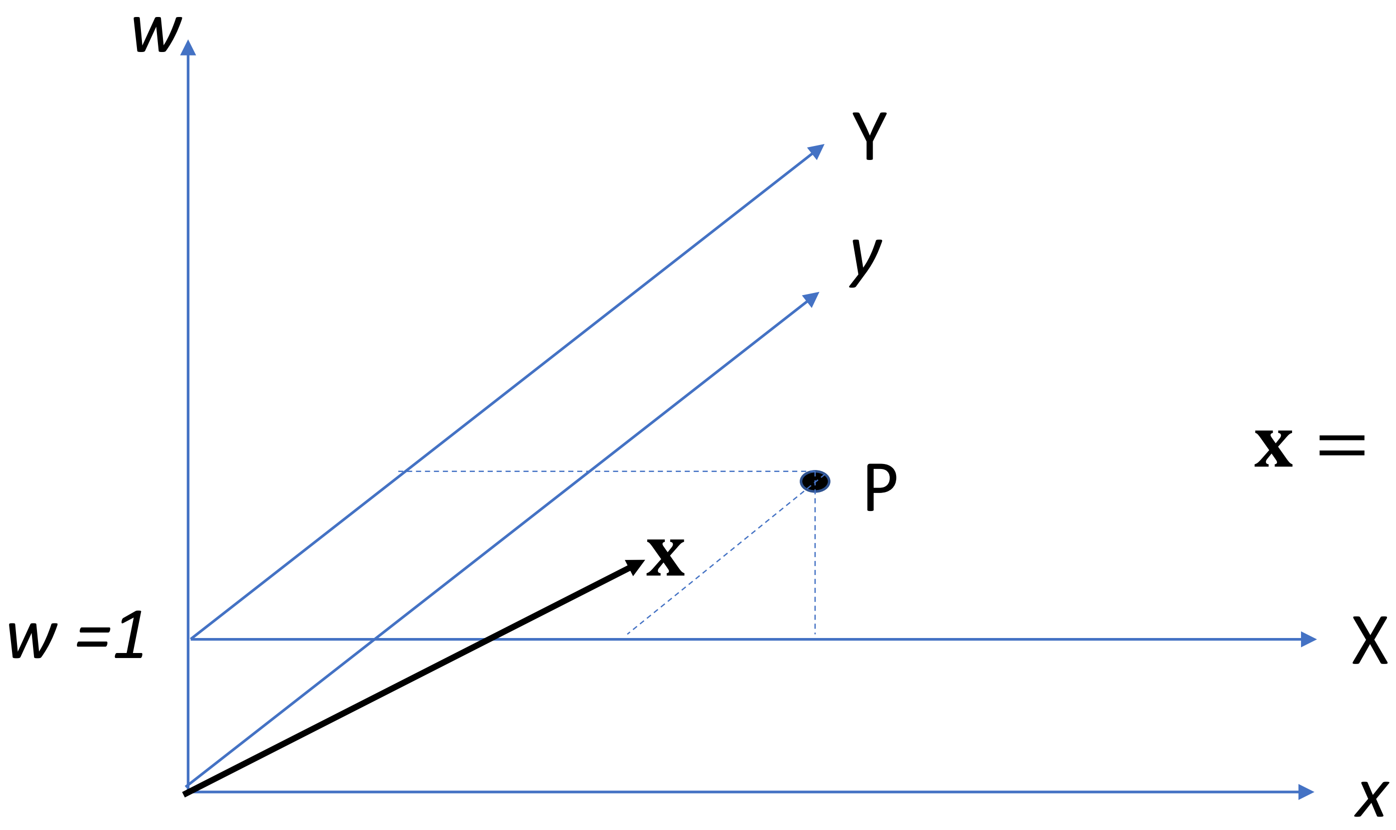
its components are $\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$

the point P is represented by any vector **x** , that is a nonzero multiple of it



$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

the point P is represented by any vector **x** , that is a nonzero multiple of it



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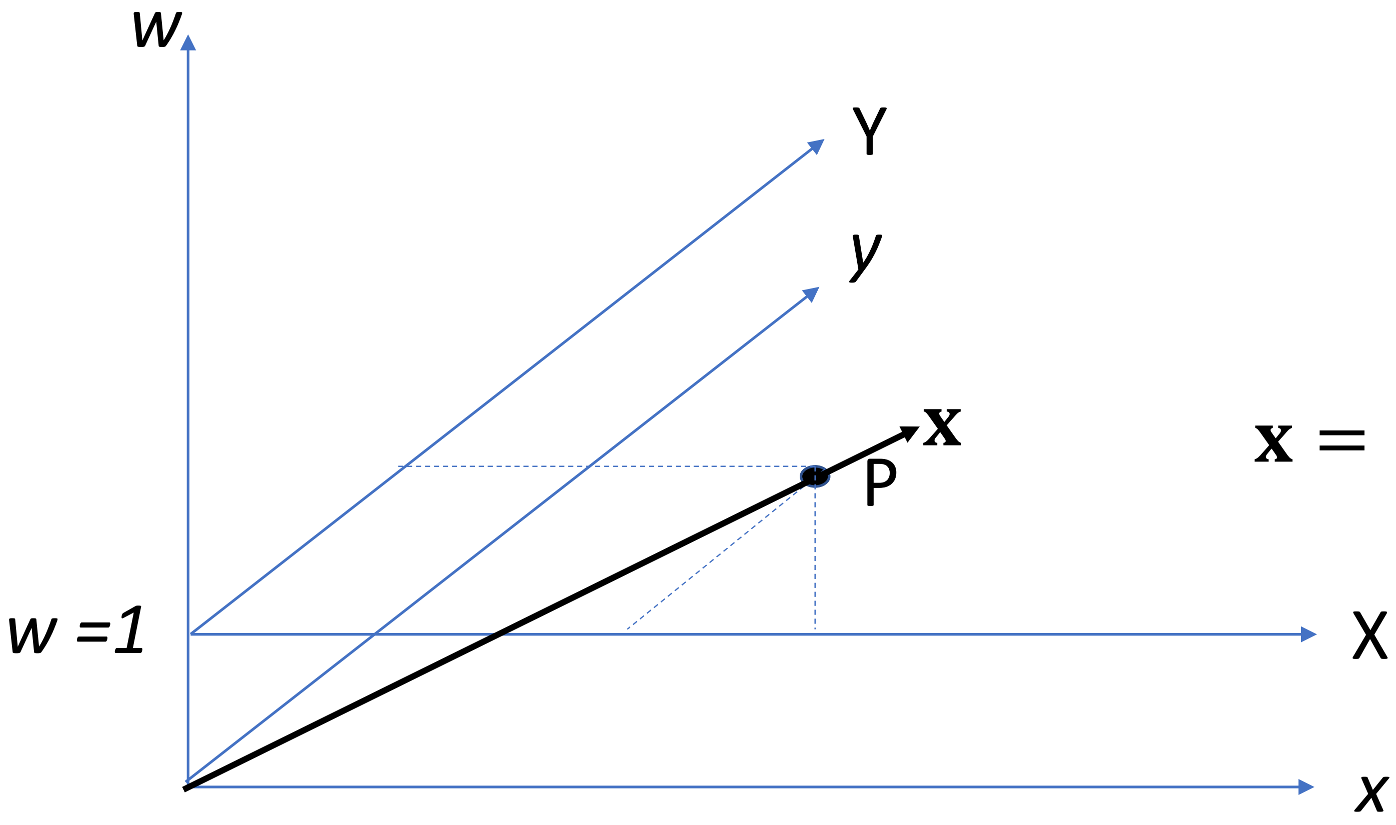

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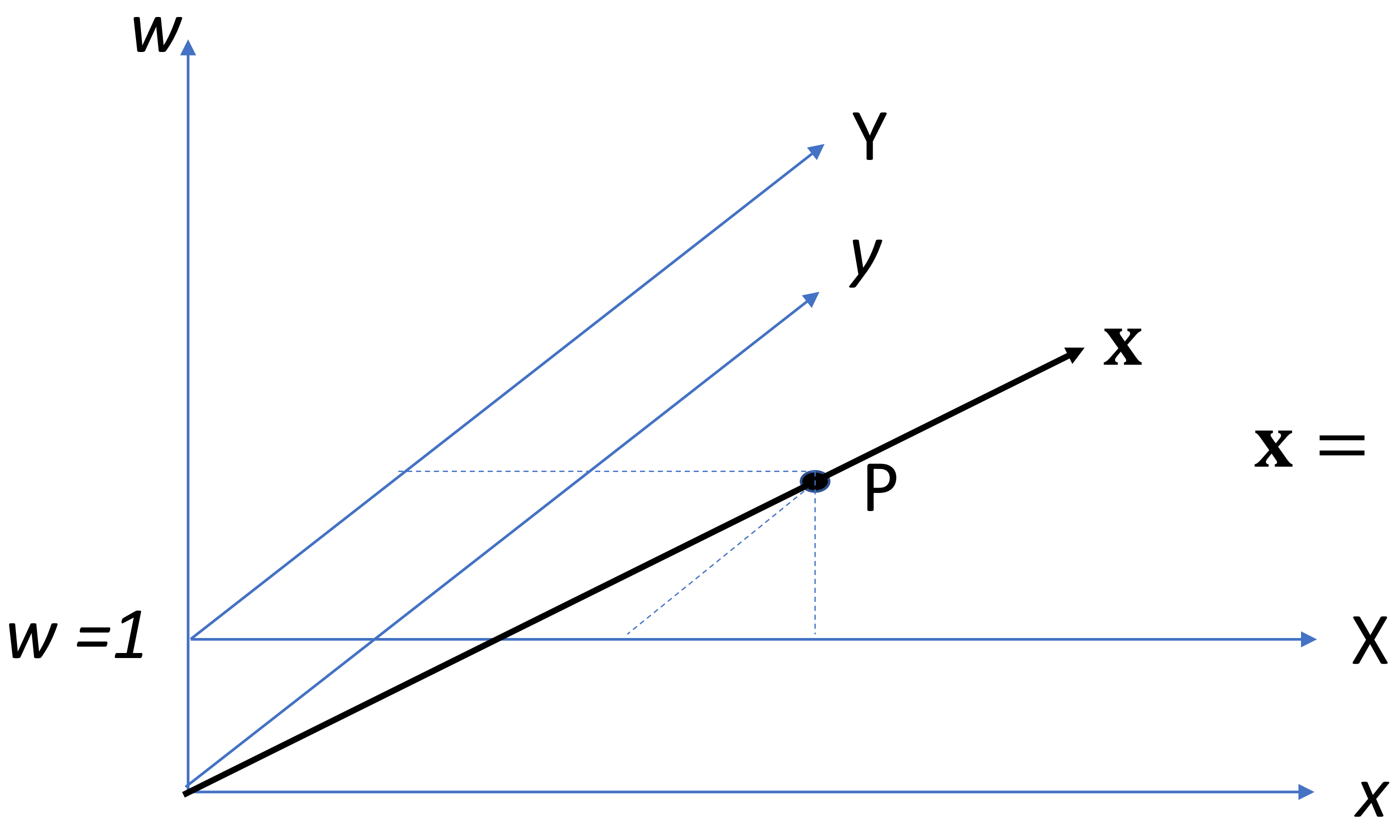

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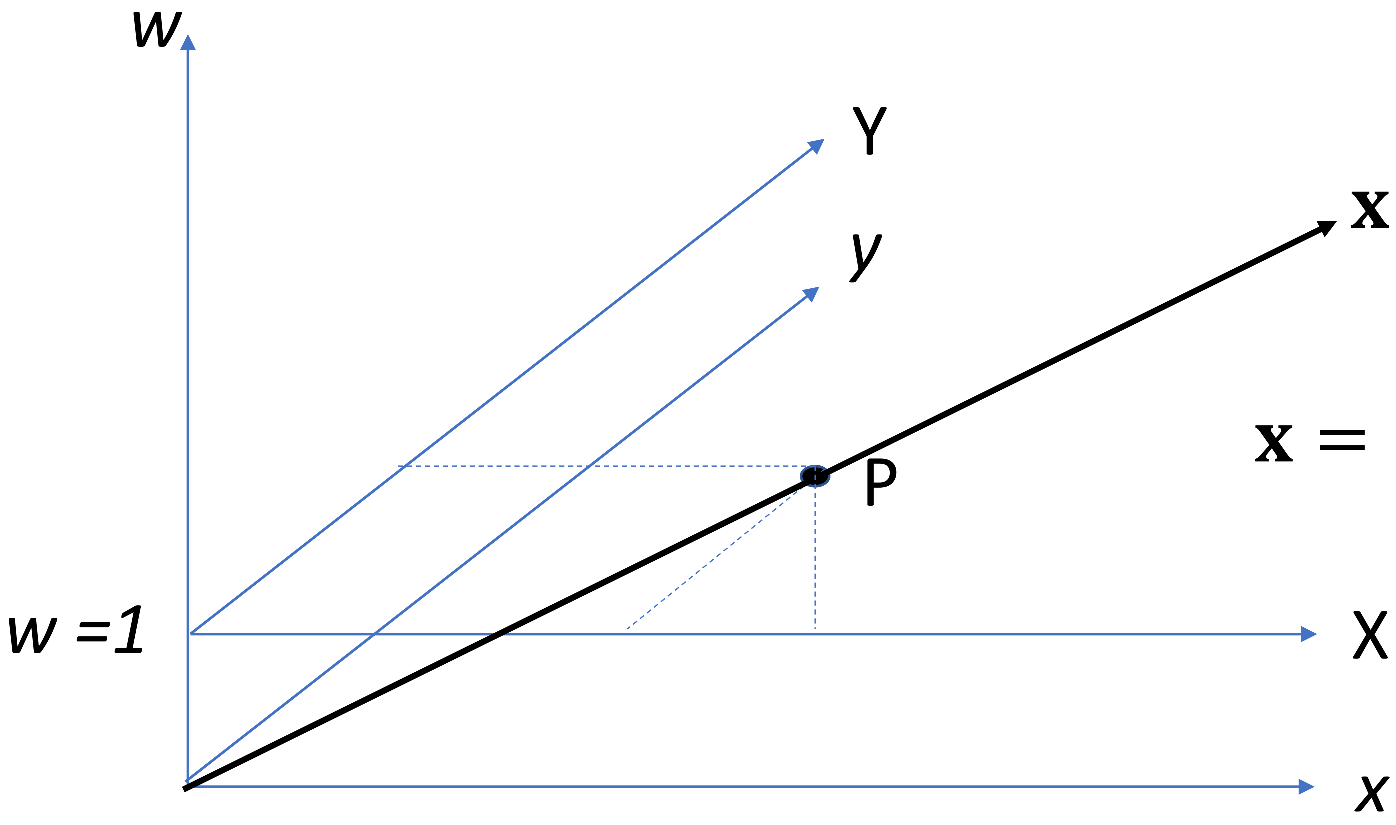


$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

A vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and all its nonzero multiples $\lambda \begin{bmatrix} x \\ y \\ w \end{bmatrix}$, including $\begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$,

represent the point of cartesian coordinates $[X \quad Y] = [x/w \quad y/w]$

on the Euclidean plane

→ homogeneity: any vector $\mathbf{x}$ is equivalent to all its nonzero multiples $\lambda\mathbf{x}$, $\lambda \neq \mathbf{0}$

since they represent the same point

→ $[x \quad y \quad w]$ are **homogeneous** coordinates of the point on the plane

redundancy

3 homogeneous coordinates to represent points in the 2D plane (2 dof)

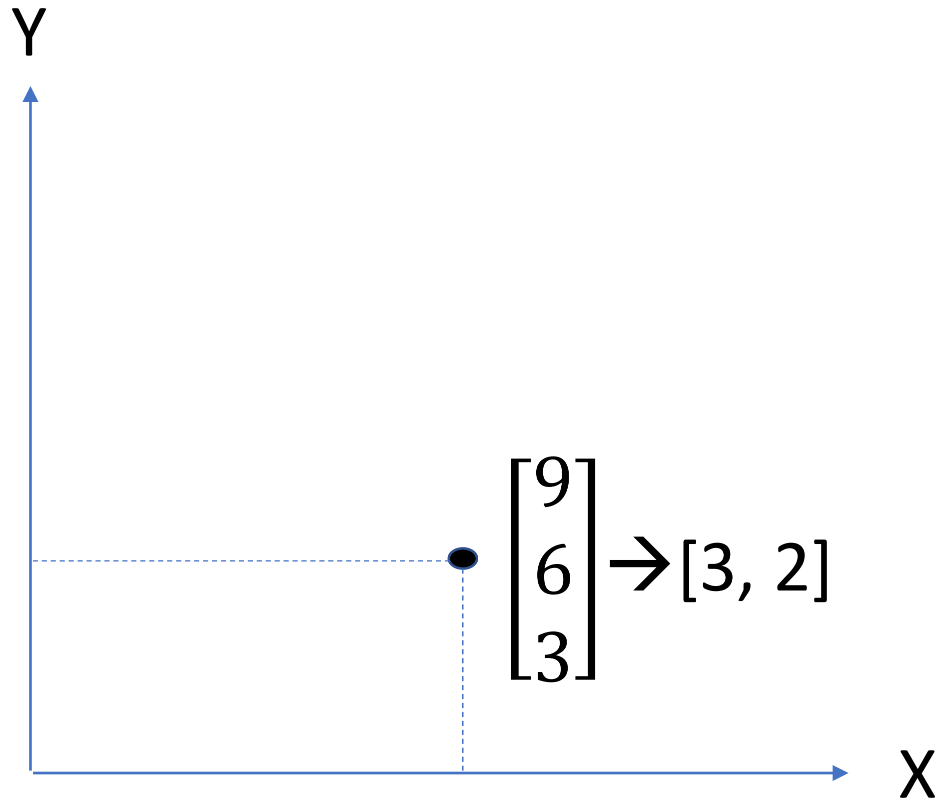
an infinite number of equivalent representations for a single point, namely all nonzero multiples of the vector $[X \quad Y \quad 1]^T$

the null vector $[0 \quad 0 \quad 0]^T$ **does not** represent any point

→ Projective plane $\mathbb{P}^2 = \{[x \quad y \quad w]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

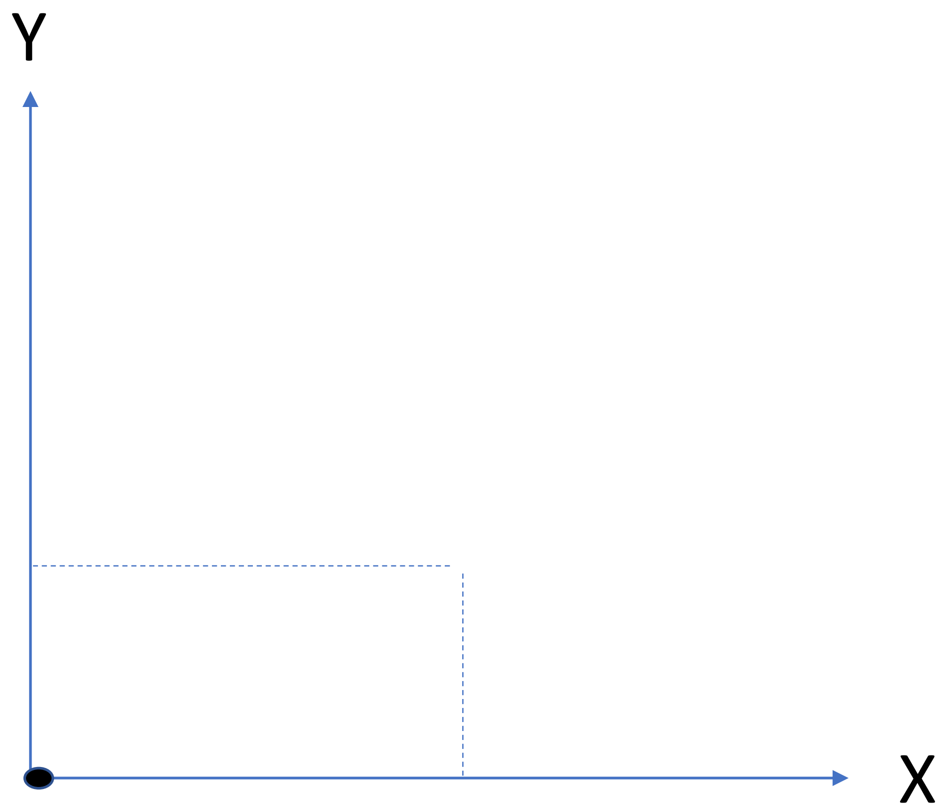
→ its two degrees of freedom are the two independent ratios between the three coordinates $x : y : w$

cartesian coordinates vs homogeneous coordinates



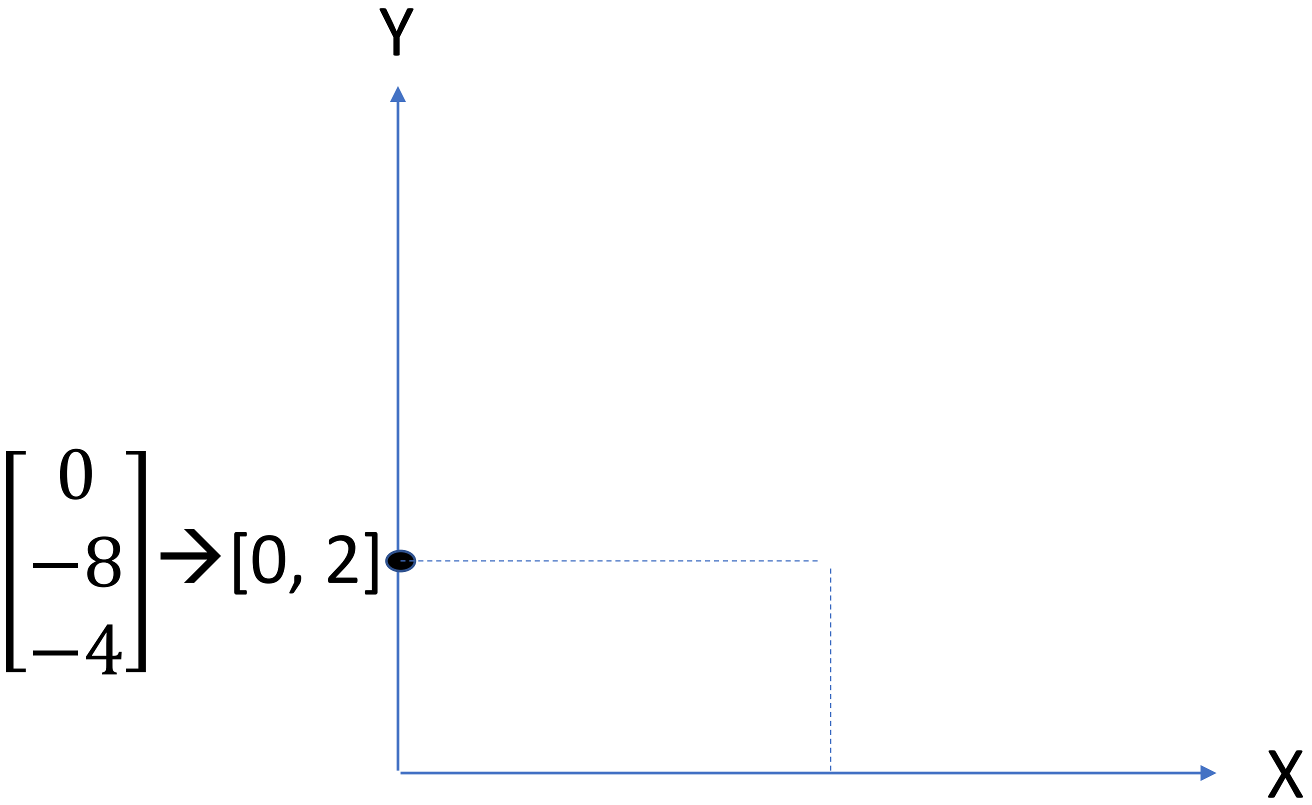
$$\begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \rightarrow [3, 2]$$

cartesian coordinates vs homogeneous coordinates



the origin $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow [0, 0]$

cartesian coordinates vs homogeneous coordinates



caveat

in the sequel, for short, we will use

«the point $\mathbf{x}$»

in place of the correct

«the point represented by the vector $\mathbf{x}$»

points at the ∞

what are points with w=0?

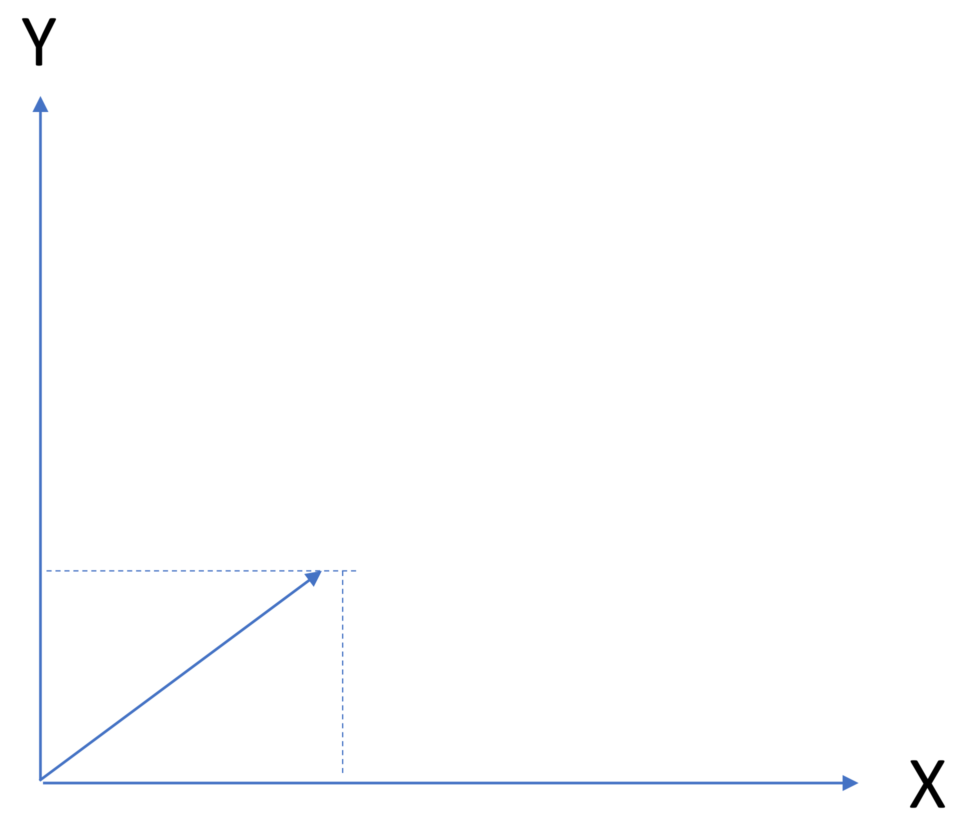
Consider the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and let w (slowly) drop to 0, starting from w =1

The cartesian coordinates start with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/1 \\ y/1 \end{bmatrix}$ and become $\begin{bmatrix} x/w \\ y/w \end{bmatrix}$

As w decreases, the point will move along a constant direction $[x, y]$, with increasing distance from the origin. As w tends to 0, this points tends to the infinity along the direction $[x, y]$.

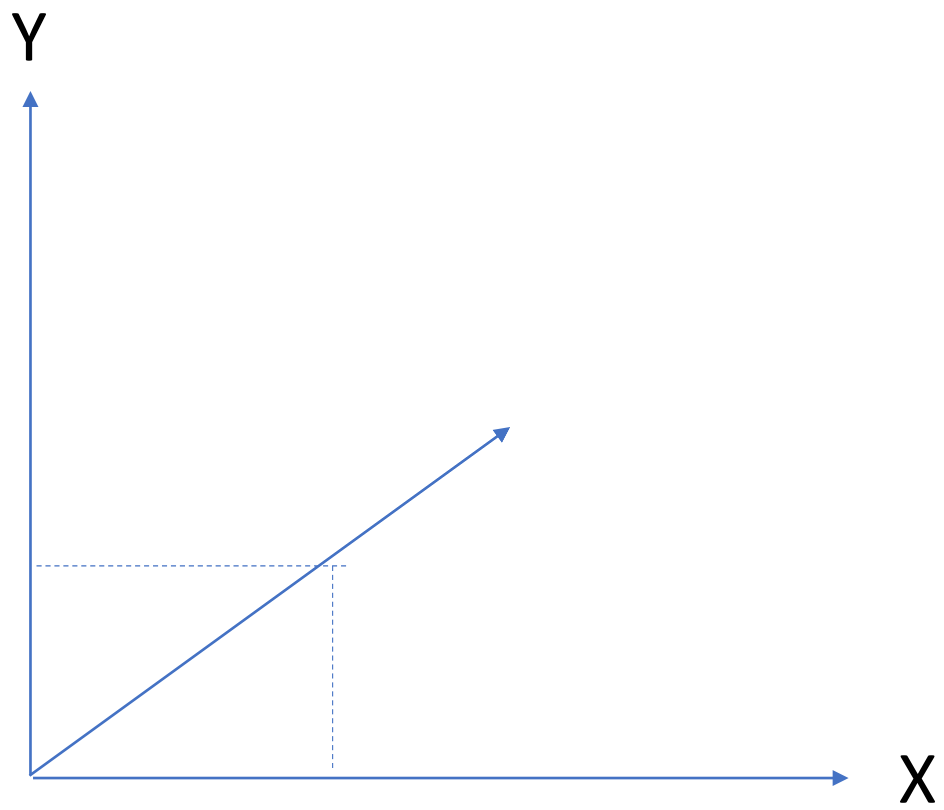
The point $\mathbf{x} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is called the **point at the infinity along the direction** $[x, y]$.

points at the infinity



$[3 \quad 2 \quad 0]^T \rightarrow ?$

points at the infinity



$[3 \quad 2 \quad 0]^T \rightarrow ?$

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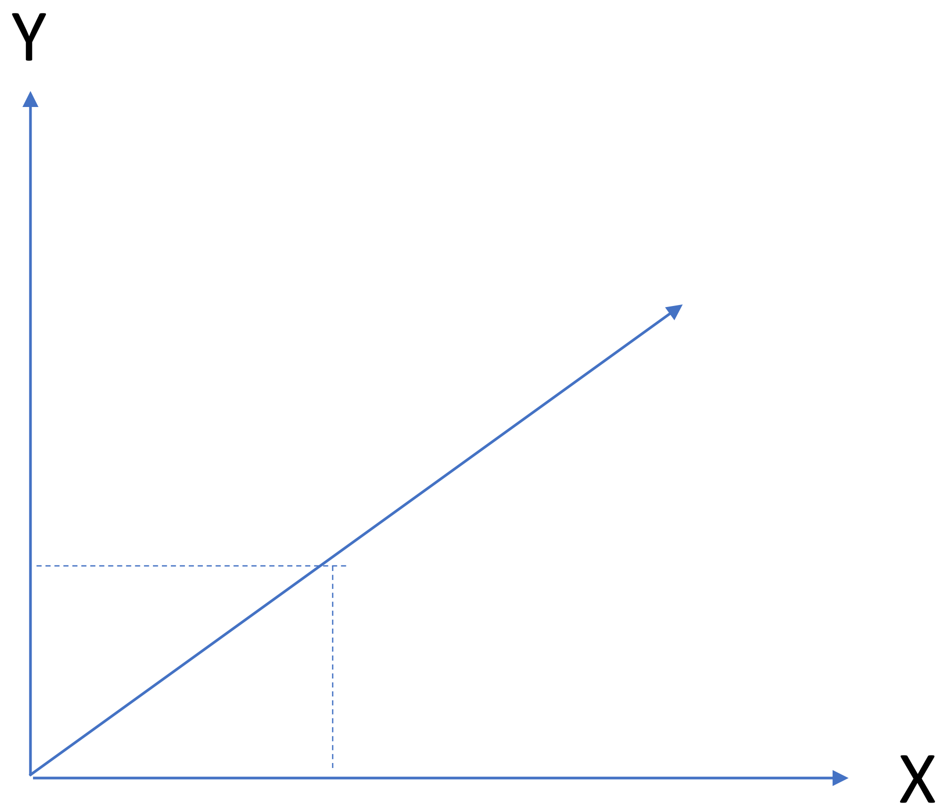
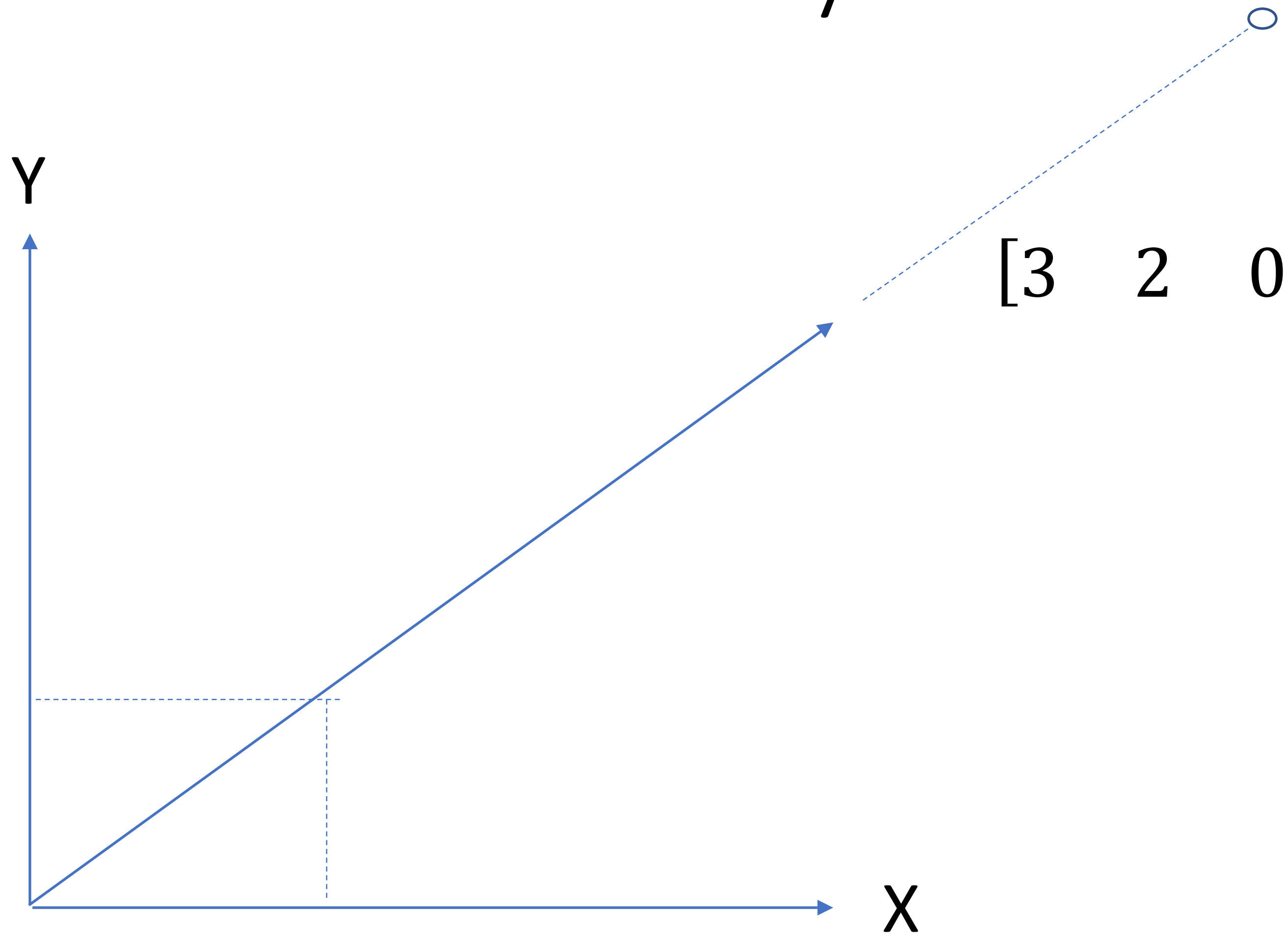


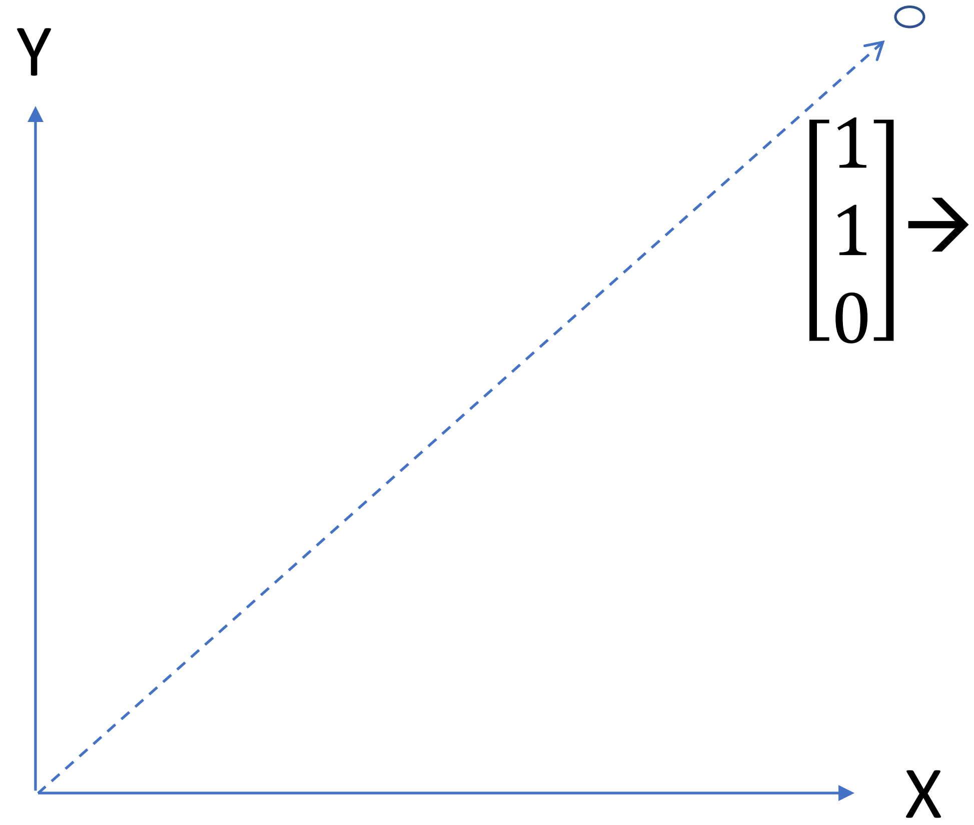
$[3 \quad 2 \quad 0]^T \rightarrow ?$

points at the infinity



$[3 \quad 2 \quad 0]^T \rightarrow ?$

Points at the infinity, who represent directions, are not part of the Euclidean plane: they are extra points, well defined within the Projective plane.



$\begin{bmatrix}1\\1\\0\end{bmatrix} \rightarrow$ point at the infinity

along direction $[x, y]$

Euclidean plane and Projective plane

Projective plane $\mathbb{P}^2 = \{[x \quad y \quad w]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

= Euclidean plane ∪ set of the points at the infinity

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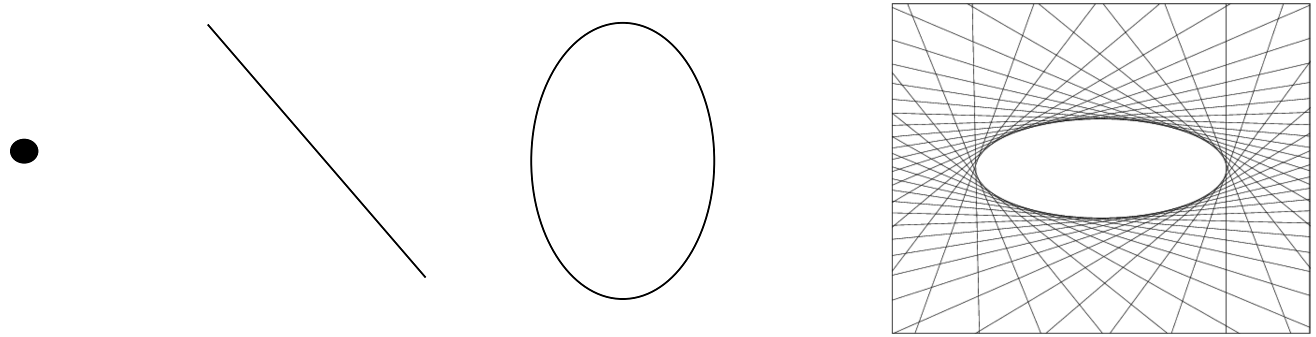

- **Transformations**
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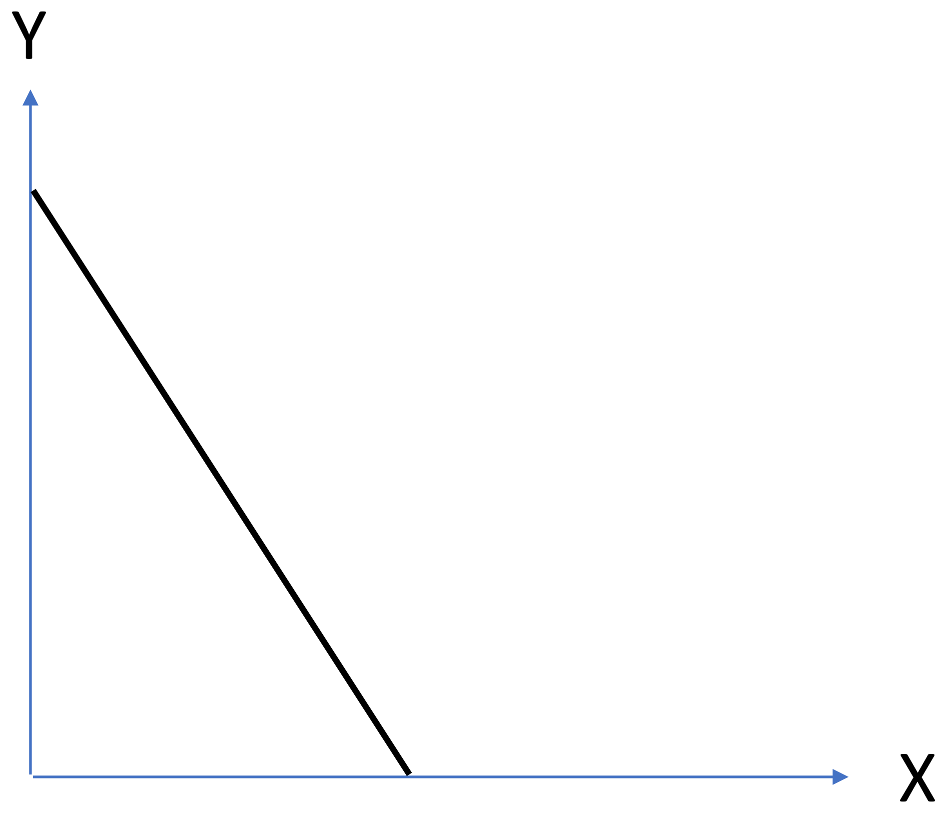
Lines in 2D Projective Geometry

Consider a line on the Euclidean plane



$$aX + bY + c = 0$$

$$a\frac{x}{w} + b\frac{y}{w} + c = 0$$

$$ax + by + cw = 0$$

$$ax + by + cw = 0 \rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

homogeneous, **linear** equation in $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$: $\boldsymbol{l}^T\mathbf{x} = 0$, where the vector $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

and all its nonzero multiples $\lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ represent a line

→ homogeneity: any vector $\boldsymbol{l}$ is equivalent to all its nonzero multiples $\lambda\boldsymbol{l}$, $\lambda \neq \mathbf{0}$ and they represent the same line

→ $\begin{bmatrix} a & b & c \end{bmatrix}$ are **homogeneous** parameters of the line

redundancy

3 homogeneous parameters to represent lines in the 2D plane (2 dof)

an infinite number of equivalent representations for a single line, namely all nonzero multiples of the unit normal vector

the null vector $[0 \quad 0 \quad 0]^T$ **does not** represent any line

→ Projective «dual» plane $\mathbb{P}^2 = \{[a \quad b \quad c]^T \in \mathbb{R}^3\} - \{[0 \quad 0 \quad 0]^T\}$

→ its two degrees of freedom are the two independent ratios between the three parameters $a : b : c$

caveat

in the sequel, for short, we will use

«the line $\boldsymbol{l}$»

in place of the correct

«the line represented by the vector $\boldsymbol{l}$»

three remarks

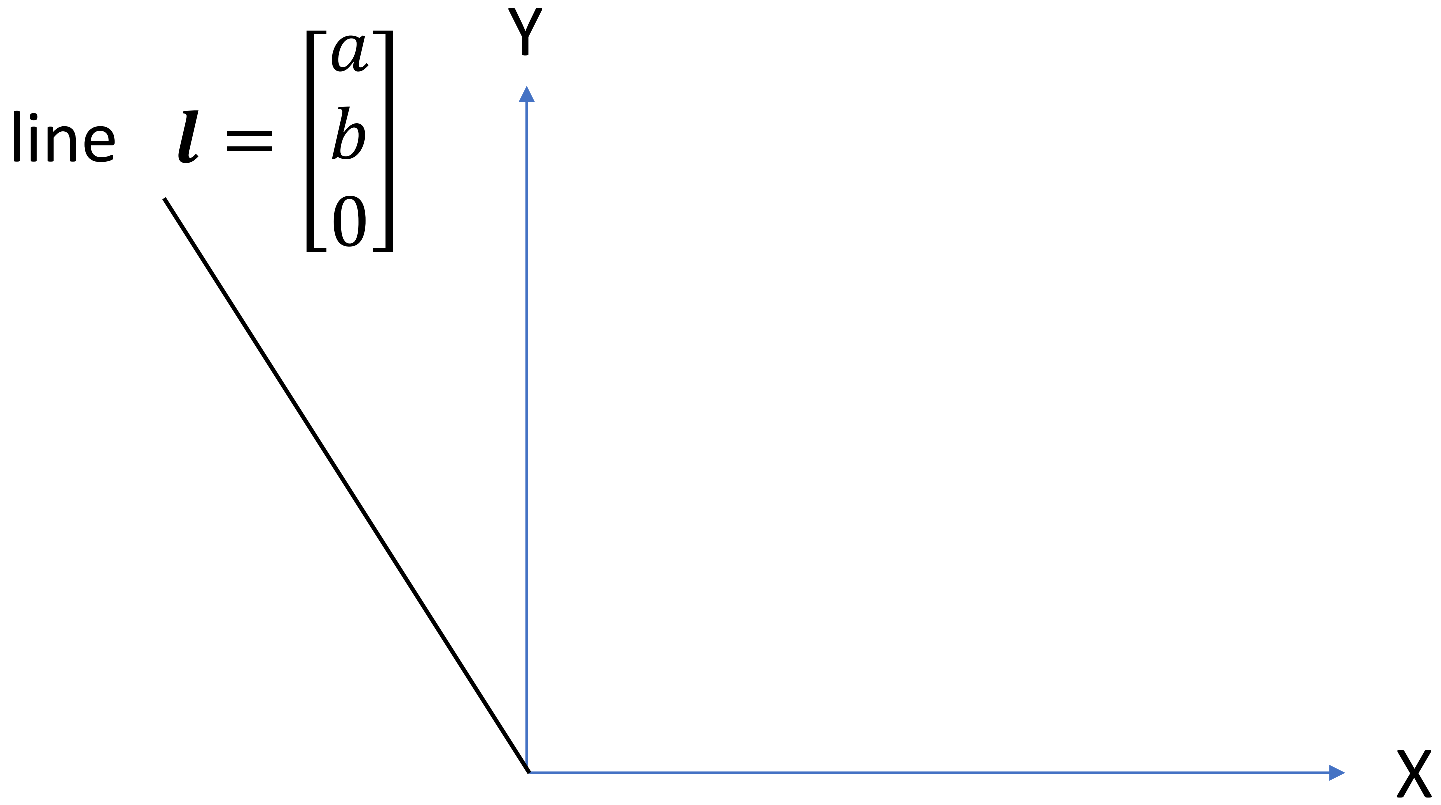
1. If the third parameter is null, $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, then the line goes through point $[0,0]$

2. within the euclidean plane, direction $[a, b]$ is normal to the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

(exercise: why?)

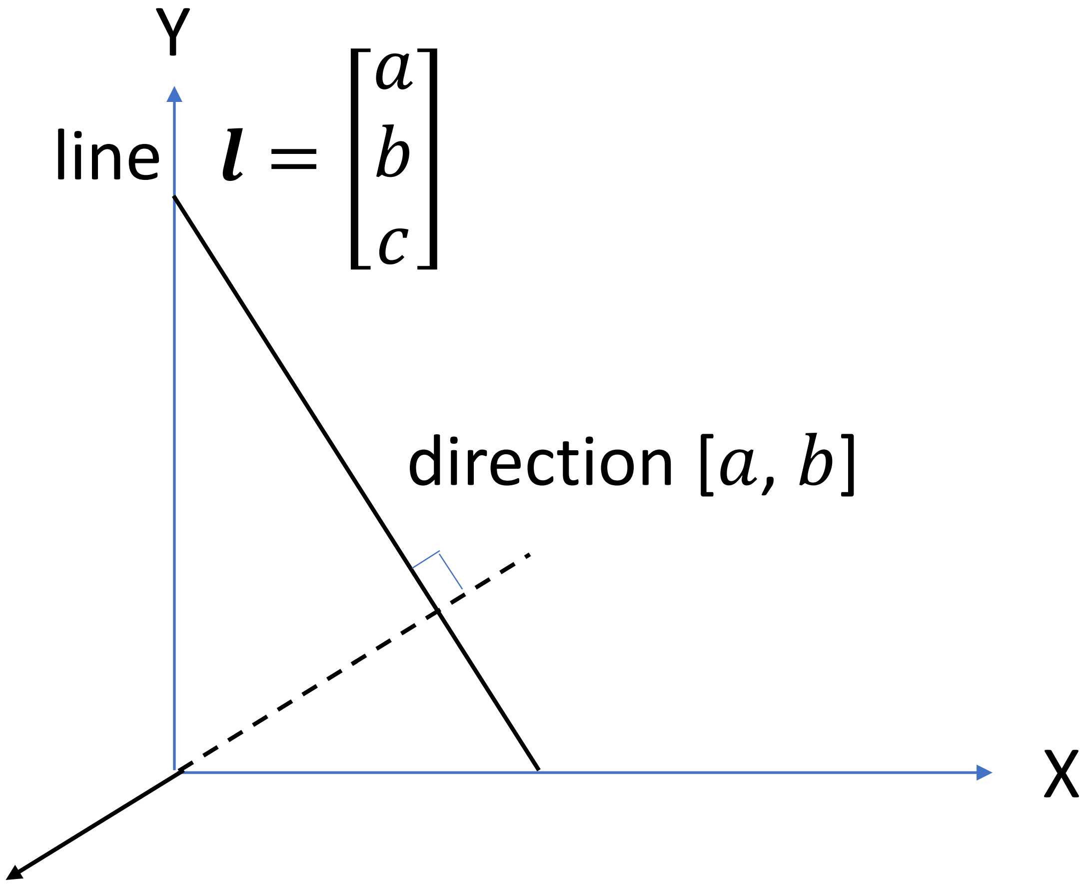
3. two lines $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} a \\ b \\ c' \end{bmatrix}$ are parallel: their common direction is $[b$,-$a]$

1.



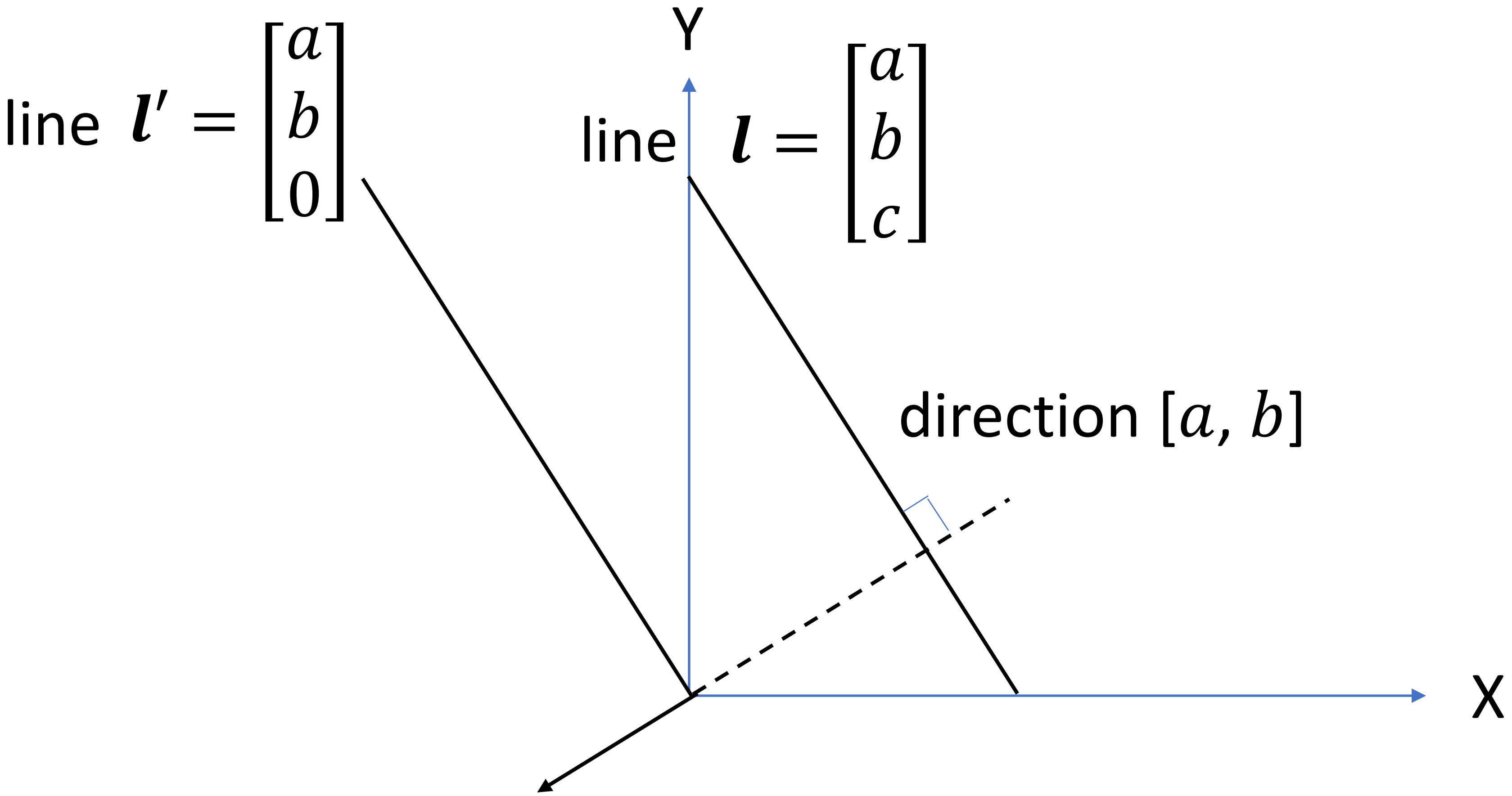
A line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{0}]^T$ whose third parameter is zero, goes through the origin of the plane

2.



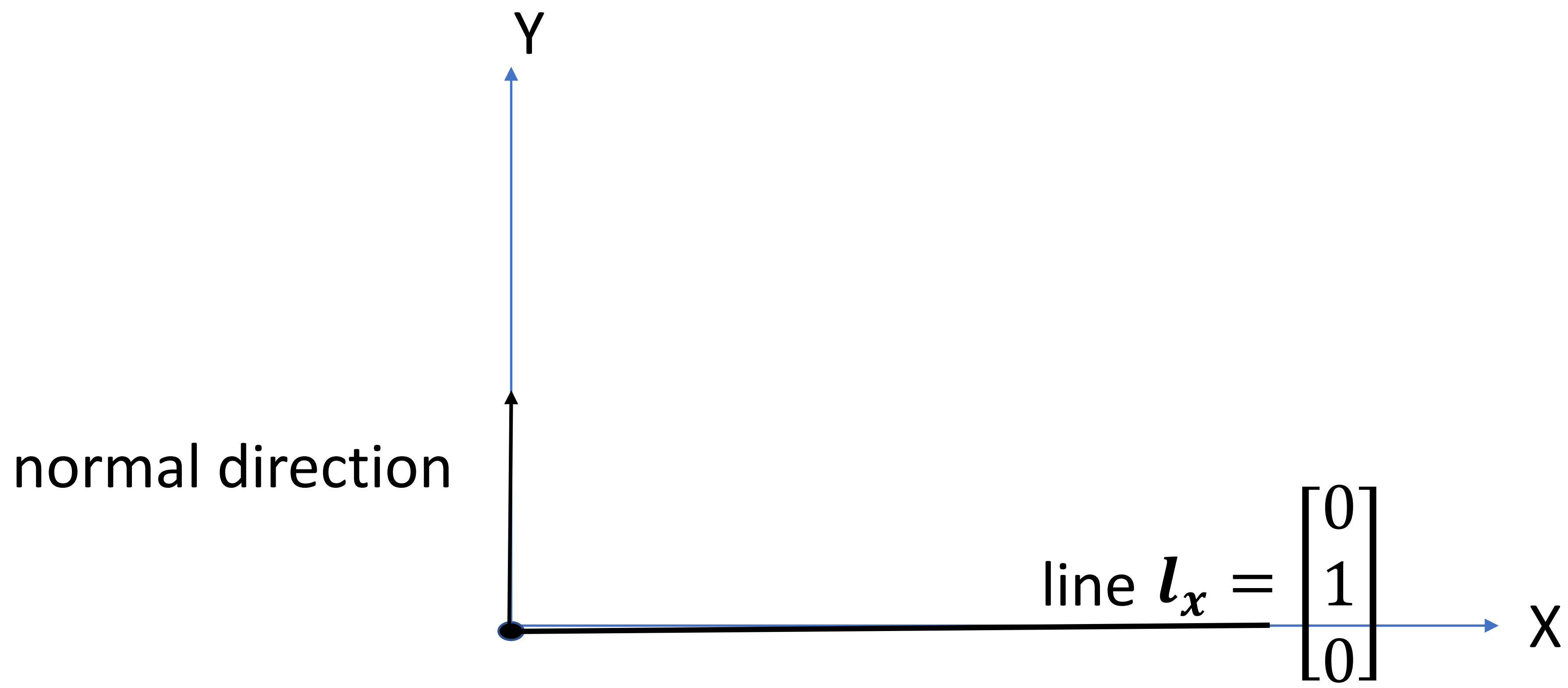
direction $[a, b]$ is normal to the line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]^{\boldsymbol{T}}$

3.

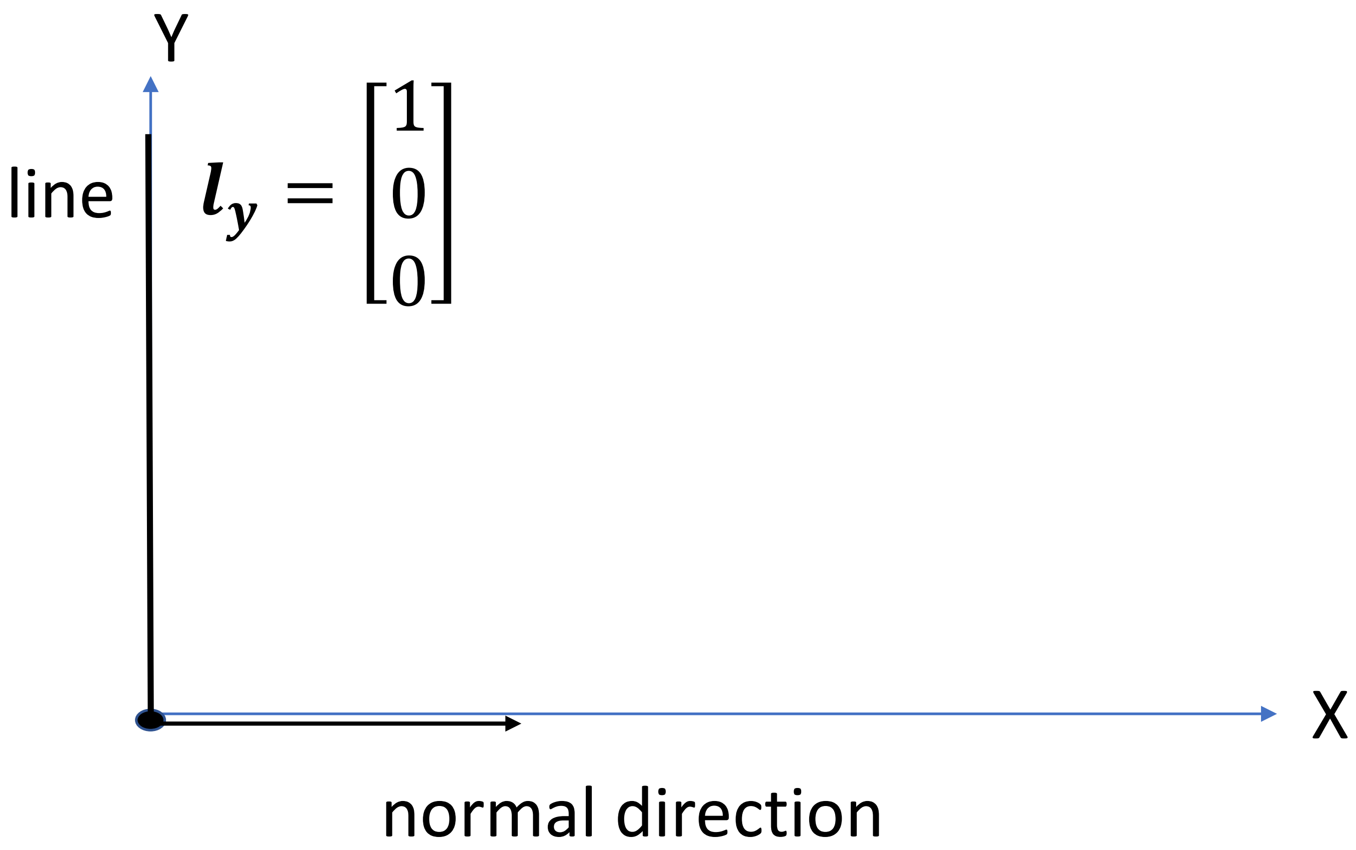


direction $[a, b]$ is normal both to the line $\boldsymbol{l} = [a \quad b \quad c]^{\boldsymbol{T}}$
and to the line $\boldsymbol{l} = [a \quad b \quad 0]^{\boldsymbol{T}}$

Example: the X-axis



Example: the y-axis



$$\boldsymbol{l_y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

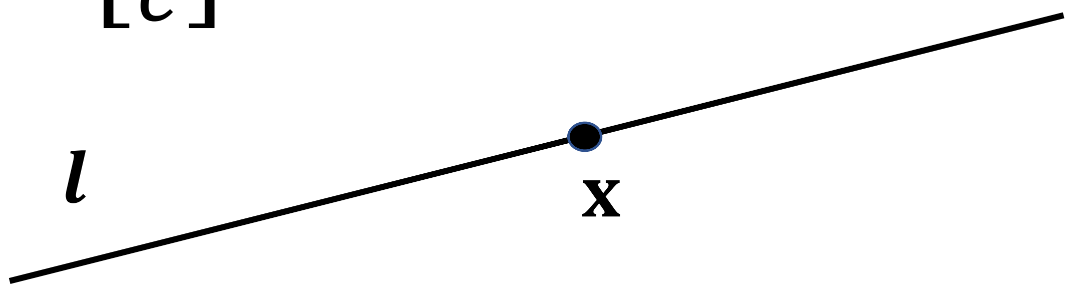
The incidence relation
a point is on a line (or a line goes through a point)

Incidence relation: $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \boldsymbol{l}^T \mathbf{x} = 0$

the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ is on the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

or

the line $\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ goes through the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$



The line at the infinity:
the locus of the points at the infinity

The «locus» of the points at the infinity

As there are infinite points at the infinity (one for each direction), what is the aspect of the set of these points at the infinity?

Simply:

$$w = 0$$

This set is a line: $[\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{w} \end{bmatrix} = 0,\ \text{actually}\ [\mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{w} \end{bmatrix} = \boldsymbol{w} = 0$

namely, **the line at the infinity** $\boldsymbol{l}_{\infty} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The duality principle between points and lines

2. Since dot product is commutative → incidence relation is commutative

$$\boldsymbol{l}^T\mathbf{x} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{w}\end{bmatrix} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\\ \boldsymbol{c}\end{bmatrix} = \mathbf{x}^T\boldsymbol{l} = \mathbf{0}$$

point $\mathbf{x}$ is on line $\boldsymbol{l}$

$\Leftrightarrow$

point $\boldsymbol{l}$ is on **line** $\mathbf{x}$

point $\mathbf{x}$ is on line $\boldsymbol{l}$ (i.e. line $\boldsymbol{l}$ goes through point $\mathbf{x}$)

$\leftrightarrow$

point $\boldsymbol{l}$ is on **line** $\mathbf{x}$ (i.e. line $\mathbf{x}$ goes through point $\boldsymbol{l}$)

Principle of **duality** between points and lines

(50% discount principle)

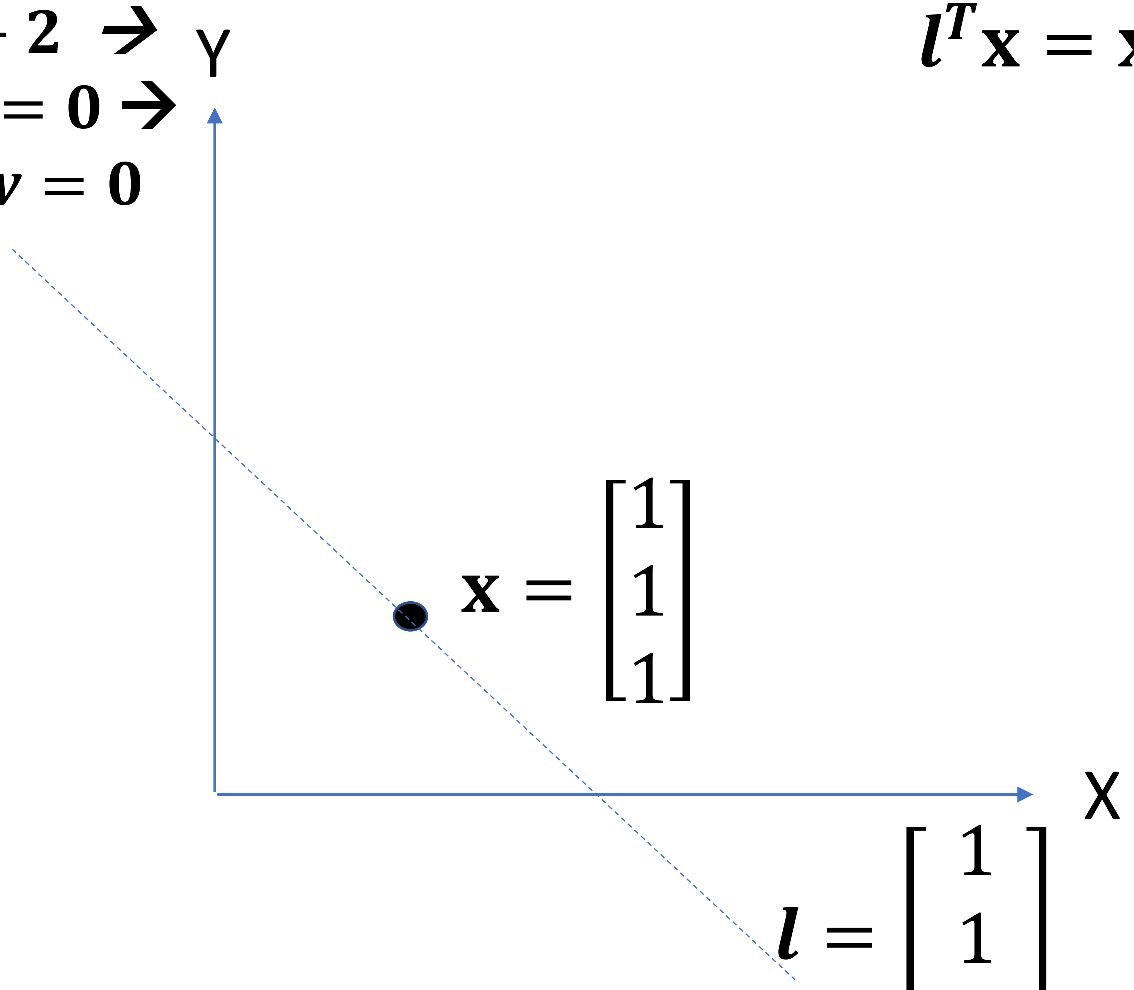
point $\mathbf{x}$ is on line $\boldsymbol{l}$ (i.e. line $\boldsymbol{l}$ goes through point $\mathbf{x}$)

$\boldsymbol{Y = -X + 2} \rightarrow$

$\boldsymbol{X + Y - 2 = 0} \rightarrow$

$\mathbf{x} + \boldsymbol{y} - \mathbf{2}\boldsymbol{w} = \mathbf{0}$

$$\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$$



$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

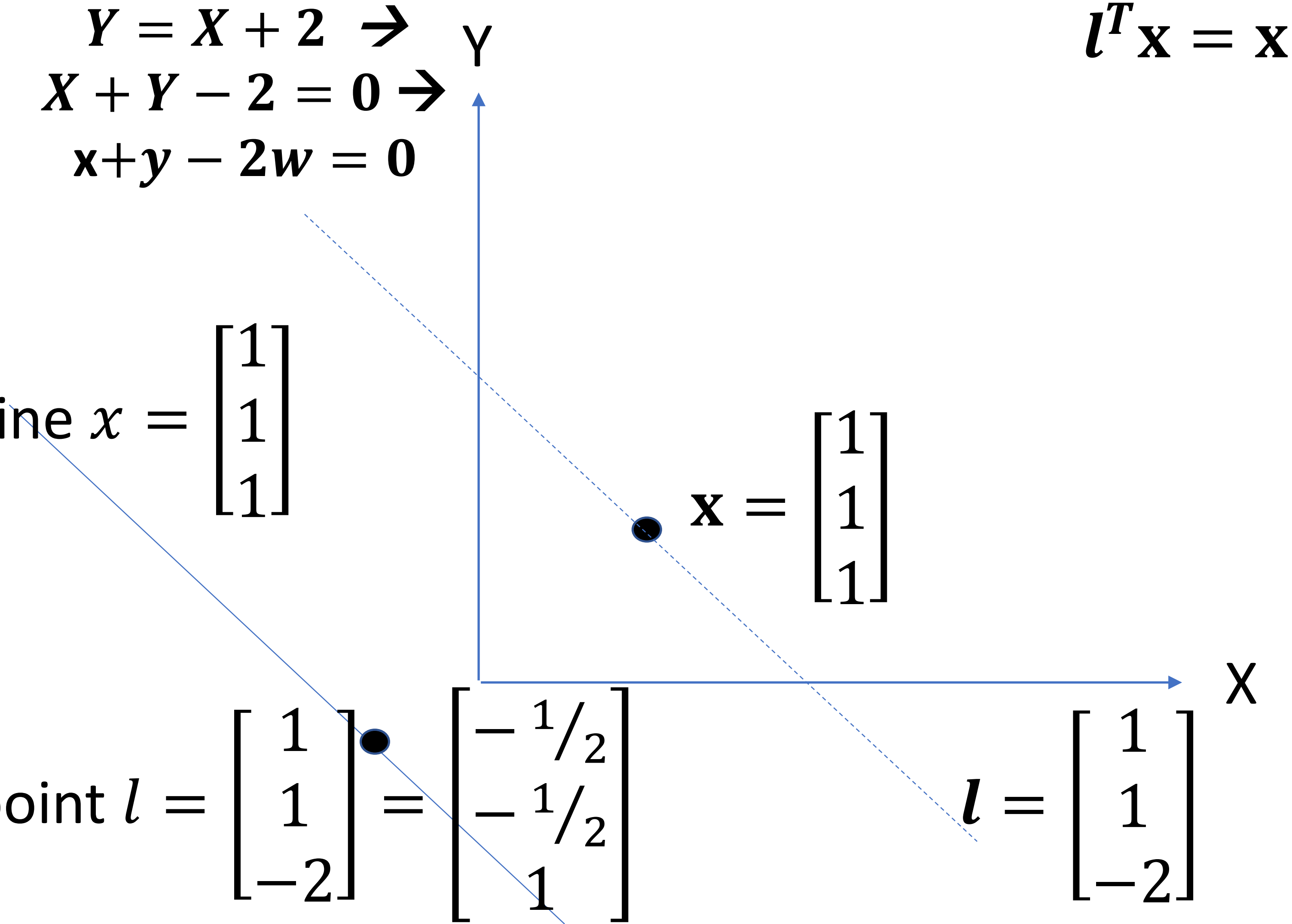
point $\boldsymbol{l}$ is on line $\boldsymbol{x}$ (i.e. line $\boldsymbol{x}$ goes through point $\boldsymbol{l}$)

$\boldsymbol{Y = X + 2} \rightarrow$

$\boldsymbol{X + Y - 2 = 0} \rightarrow$

$\mathbf{x} + \boldsymbol{y} - \mathbf{2}\boldsymbol{w} = \mathbf{0}$

$\boldsymbol{l}^{\boldsymbol{T}}\mathbf{x} = \mathbf{x}^{\boldsymbol{T}}\boldsymbol{l} = \mathbf{0}$



line $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

point $l = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -{}^{1}/_{2} \\ -{}^{1}/_{2} \\ 1 \end{bmatrix}$

$\boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

For any true sentence containing the words

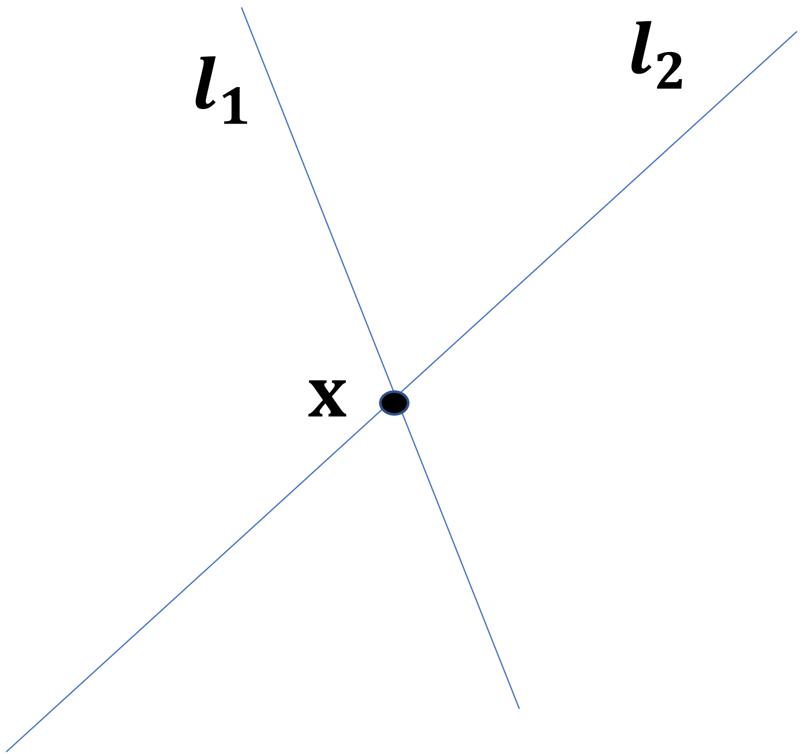
- point
- line
- is on
- goes through

there is a DUAL sentence -also true- obtained by substituting, in the previous one, each occurence of

- point	by	- line
- line	by	- point
- is on	by	- goes through
- goes through	by	- is on

The point on two lines

the point on two lines

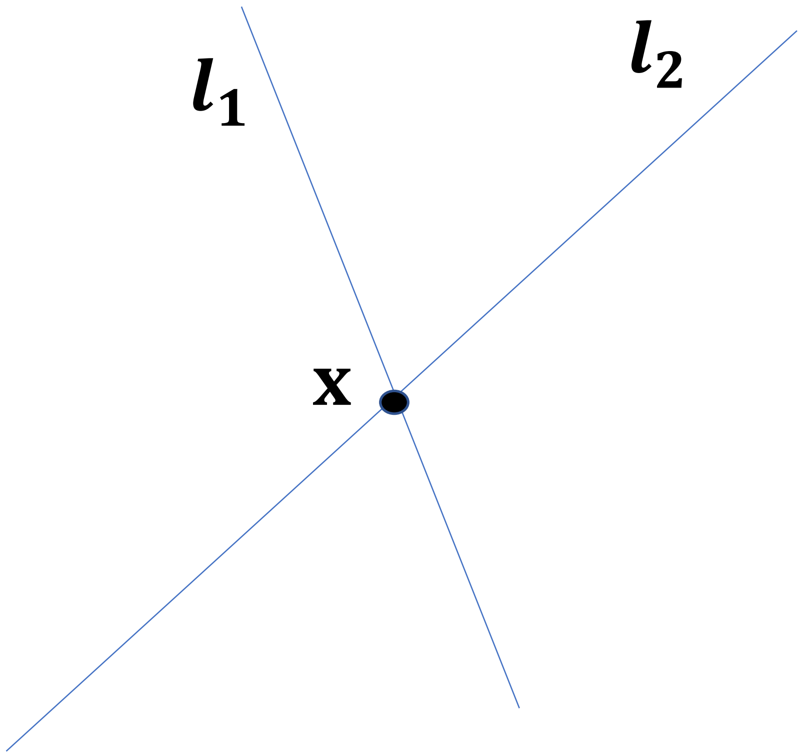


$$\begin{cases} \boldsymbol{l_1}^T \mathbf{x} = 0 \\ \boldsymbol{l_2}^T \mathbf{x} = 0 \end{cases}$$

$$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\right)$$

the point on two lines



$$\begin{cases} \boldsymbol{l_1}^T\mathbf{x} = 0 \\ \boldsymbol{l_2}^T\mathbf{x} = 0 \end{cases}$$

$$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{RNS}\left(\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}\right)$$

$\begin{bmatrix} \boldsymbol{l_1}^T \\ \boldsymbol{l_2}^T \end{bmatrix}$ is a 2x3 matrix → RNS is a 1D vector space: how many points?

Just ONE: in fact the ∞ solutions for x are all multiples of a common x

→ but they represent just ONE point !!

Observation:

in

$$\begin{cases} \boldsymbol{l_1}^T\mathbf{x} = 0 \\ \boldsymbol{l_2}^T\mathbf{x} = 0 \end{cases}$$

$\mathbf{x}$ is a vector orthogonal to both $\boldsymbol{l_1}$and $\boldsymbol{l_2}$ vectors

→

in the 2D Projective Geometry

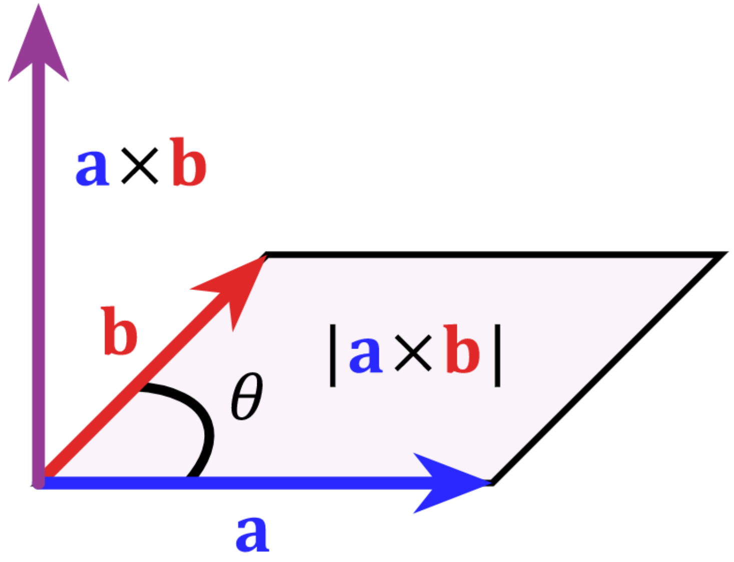
$\mathbf{x}$ is (a multiple of) the cross product of $\boldsymbol{l_1}$and $\boldsymbol{l_2}$

$$\mathbf{x} = \boldsymbol{l_1} \times \boldsymbol{l_2}$$

Cross Product

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ be two vectors, their cross product is a vector $\boldsymbol{a} \times \boldsymbol{b} \in \mathbb{R}^3$

- That is perpendicular to the plane $< \boldsymbol{a}, \boldsymbol{b} >$
- Has orientation of the right-hand rule
- Has lenght proportional to the area of the parallelogram spanned by the vectors, $||\boldsymbol{a} \times \boldsymbol{b}|| = ||\boldsymbol{a}|| ||\boldsymbol{b}|| \sin \boldsymbol{\theta}$



Cross Product

Rmk: the cross product can be also computed as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

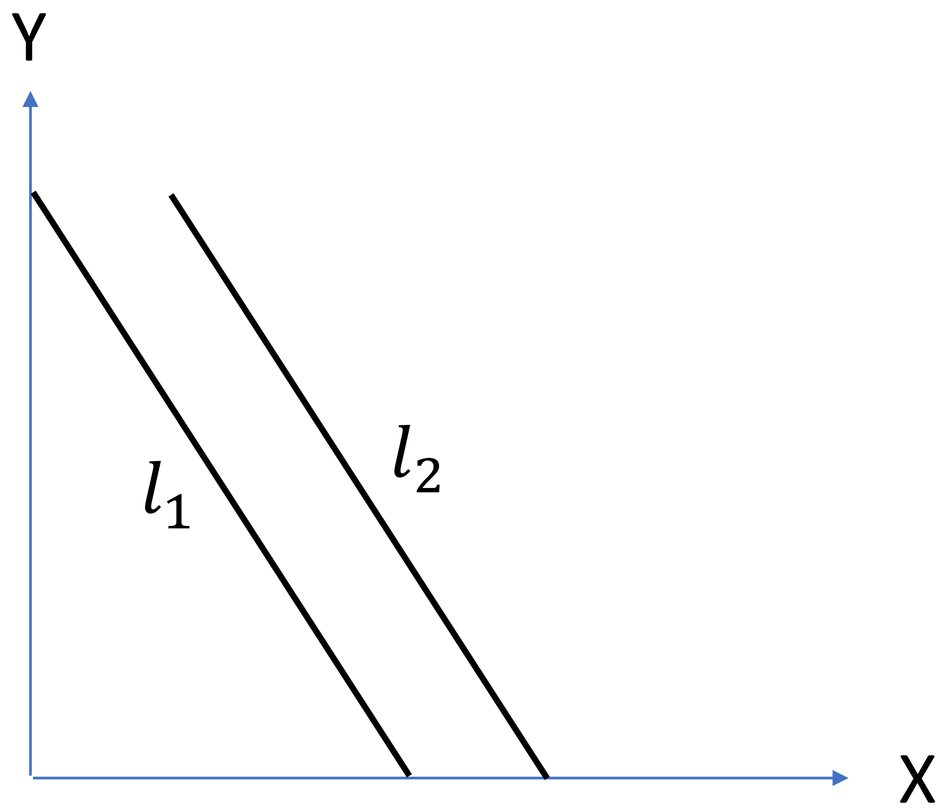
being $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ the versors of $\mathbb{R}^3$ and $|\cdot|$ the determinant

Rmk: the cross product is anti-commutative

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$$

But this is not an issue when we want to intersect two lines, since the result in the same point of $\mathbb{P}^2$ (equivalence up to a multiplication by -1)

Example: intersection of two parallel lines



Example: intersection of two parallel lines

Suppose that lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ are parallel: this means that

$l_1 = [a \quad b \quad c_1]^T$ and
$l_2 = [a \quad b \quad c_2]^T$

The point $\mathbf{x} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{w}]^{\boldsymbol{T}}$ common two these two lines satisfies both

$$ax + by + c_1 w = 0$$

and

$$ax + by + c_2 w = 0$$

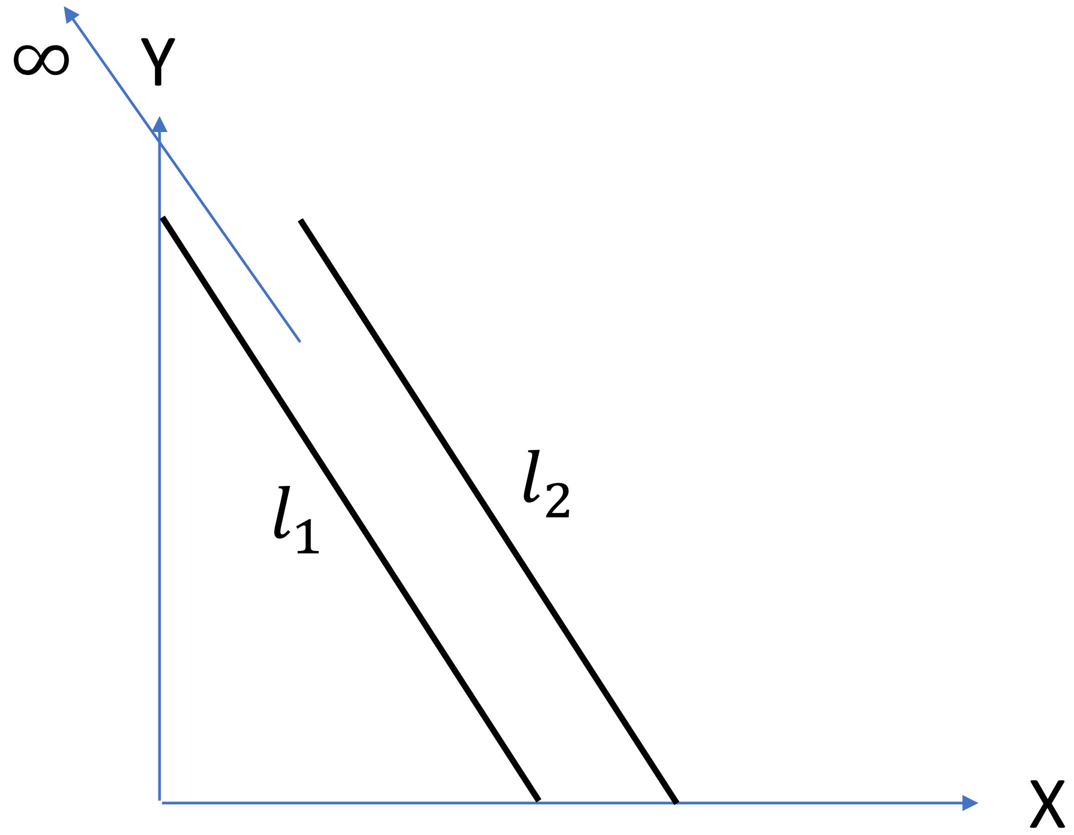
$$\downarrow$$

$$\mathbf{x} = [b \quad -a \quad 0]^T$$

Namely, the point at the infinty along the direction of both lines

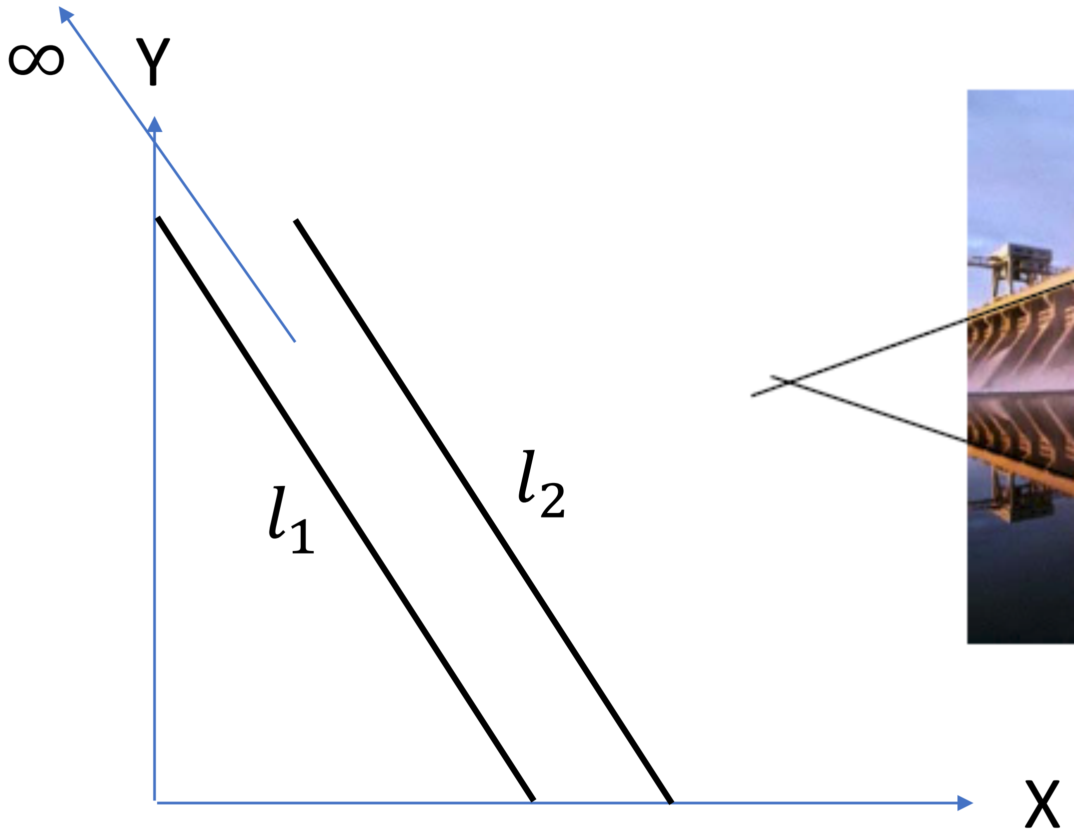
(remember: [a, b] is the direction **normal** to both lines)

The intersection of two parallel line is the point at the infinity along their common direction



The vanishing point is the image of point at the infinity (which is where parallel lines intersect)

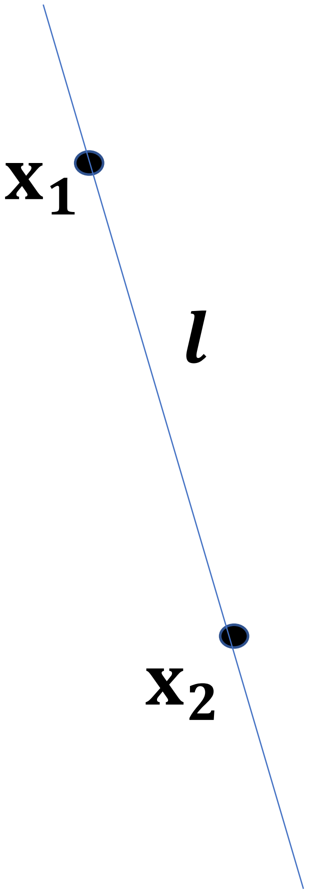
The intersection of two parallel line is the point at the infinity along their common direction



The vanishing point is the image of point at the infinity (which is where parallel lines intersect)

the line through two points

Previous: point on two lines
DUAL: line through two points



$$\boldsymbol{l} = \mathbf{RNS}\left(\begin{bmatrix} \mathbf{x_1}^T \\ \mathbf{x_2}^T \end{bmatrix}\right)$$

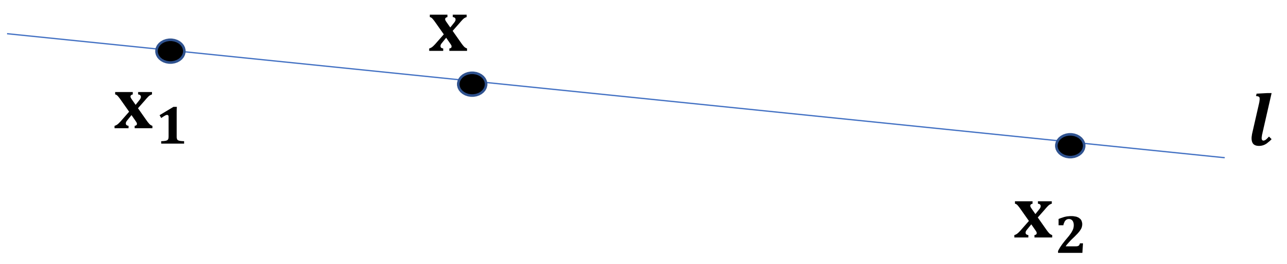
or, just in 2D Proj. Geo.

$$\boldsymbol{l} = \mathbf{x_1} \times \mathbf{x_2}$$

A useful property and its dual

Example: linear combination of two points

Property: the point $\mathbf{x}$ given by the linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$ (i.e. on the line joining $\mathbf{x}_1$ and $\mathbf{x}_2$)



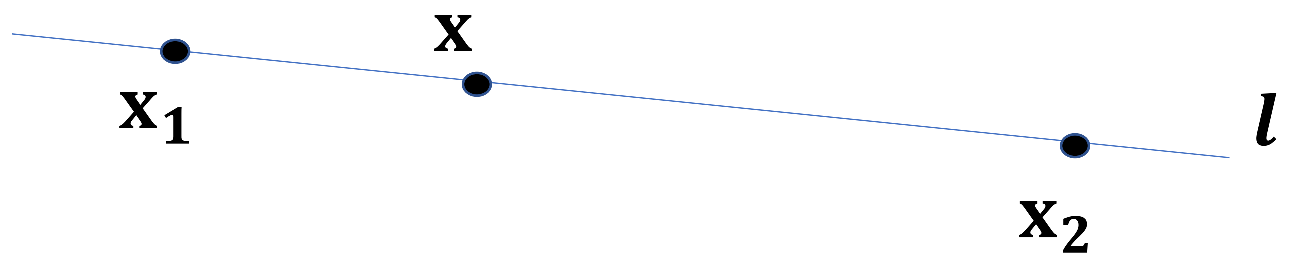
Proof: the line $\boldsymbol{l}$ through both points satisfies $\boldsymbol{l}^T\mathbf{x}_1 = 0$ and $\boldsymbol{l}^T\mathbf{x}_2 = 0$.
By adding α times the first eqn to β times the second one, we obtain

$$0 = \boldsymbol{l}^T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \boldsymbol{l}^T\mathbf{x} = 0$$

i.e. $\mathbf{x}$ is on the same line joining $\mathbf{x}_1$ and $\mathbf{x}_2$.

Example: linear combination of two points

Property: the point $\mathbf{x}$ given by the linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$ (i.e. on the line joining $\mathbf{x}_1$ and $\mathbf{x}_2$)

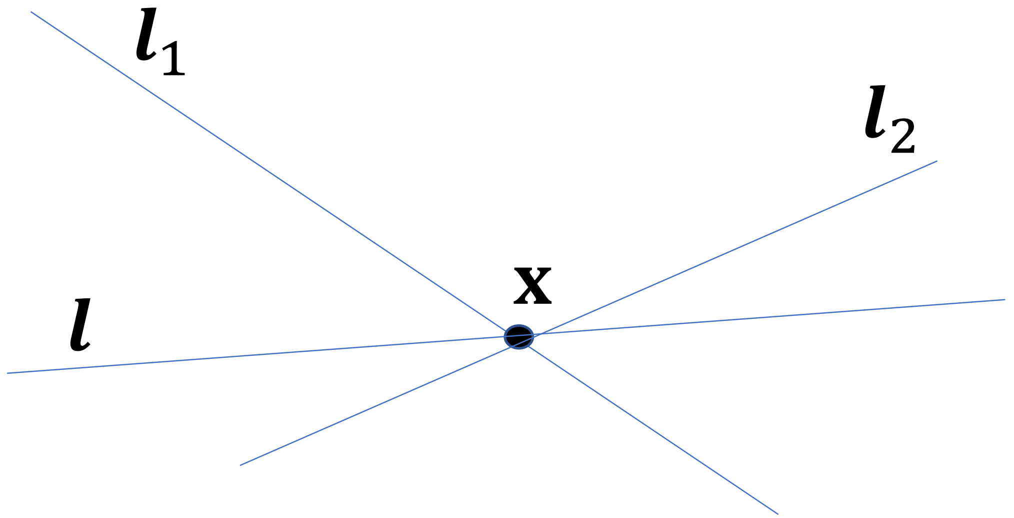


new expression: **«colinear»**

the point $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is **colinear** to points $\mathbf{x}_1$ and $\mathbf{x}_2$

DUAL: linear combination of two lines

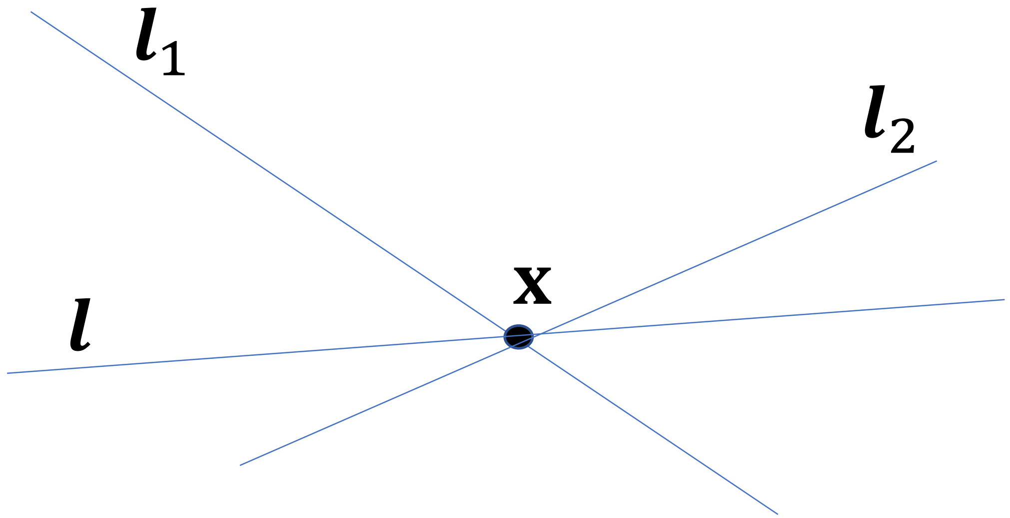
Property: the point $\mathbf{x}$, given by the linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$, is on the line $\boldsymbol{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$



Dual: the line $\boldsymbol{l}$, given by the linear combination $\boldsymbol{l} = \alpha\boldsymbol{l}_1 + \beta\boldsymbol{l}_2$ of two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, goes through the point $\mathbf{x}$ on $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

DUAL: linear combination of two lines

Dual property: the line $\boldsymbol{l}$, given by the linear combination $\boldsymbol{l} = \alpha \boldsymbol{l}_1 + \beta \boldsymbol{l}_2$ of two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, goes through the point $\mathbf{x}$ on $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$



new expression: **«concurrent»**

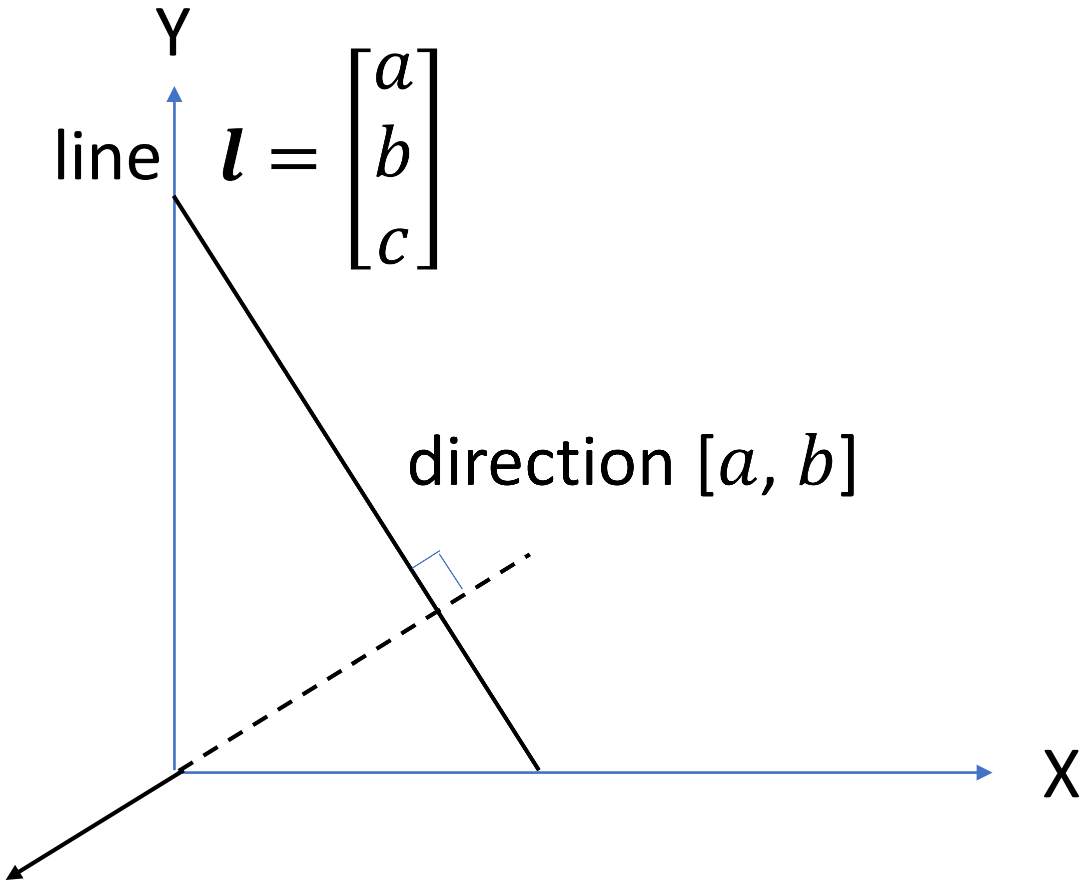
the line $\boldsymbol{l} = \alpha \boldsymbol{l}_1 + \beta \boldsymbol{l}_2$ is **concurrent** to lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

there is a new pair of DUAL corresponding words

- point	→	- line
- line	→	- point
- is on	→	- goes through
- goes through	→	- is on
- colinear	→	- concurrent
- concurrent	→	- colinear

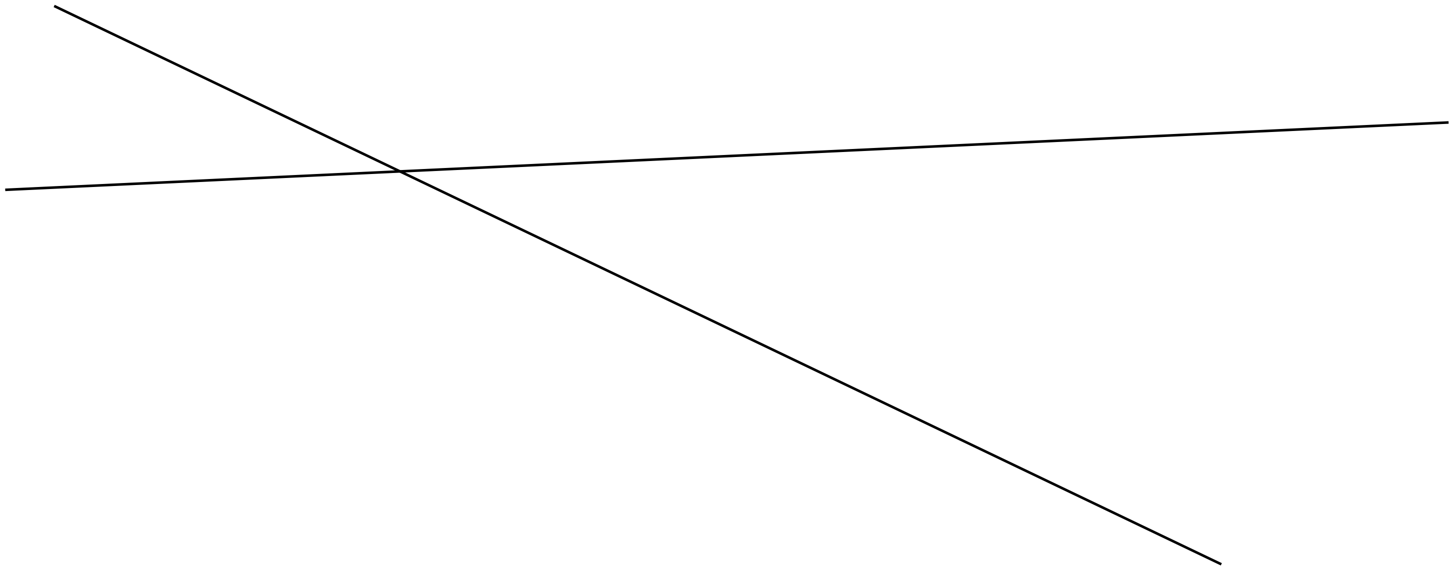
Angle between two lines

Angle between two lines: remember

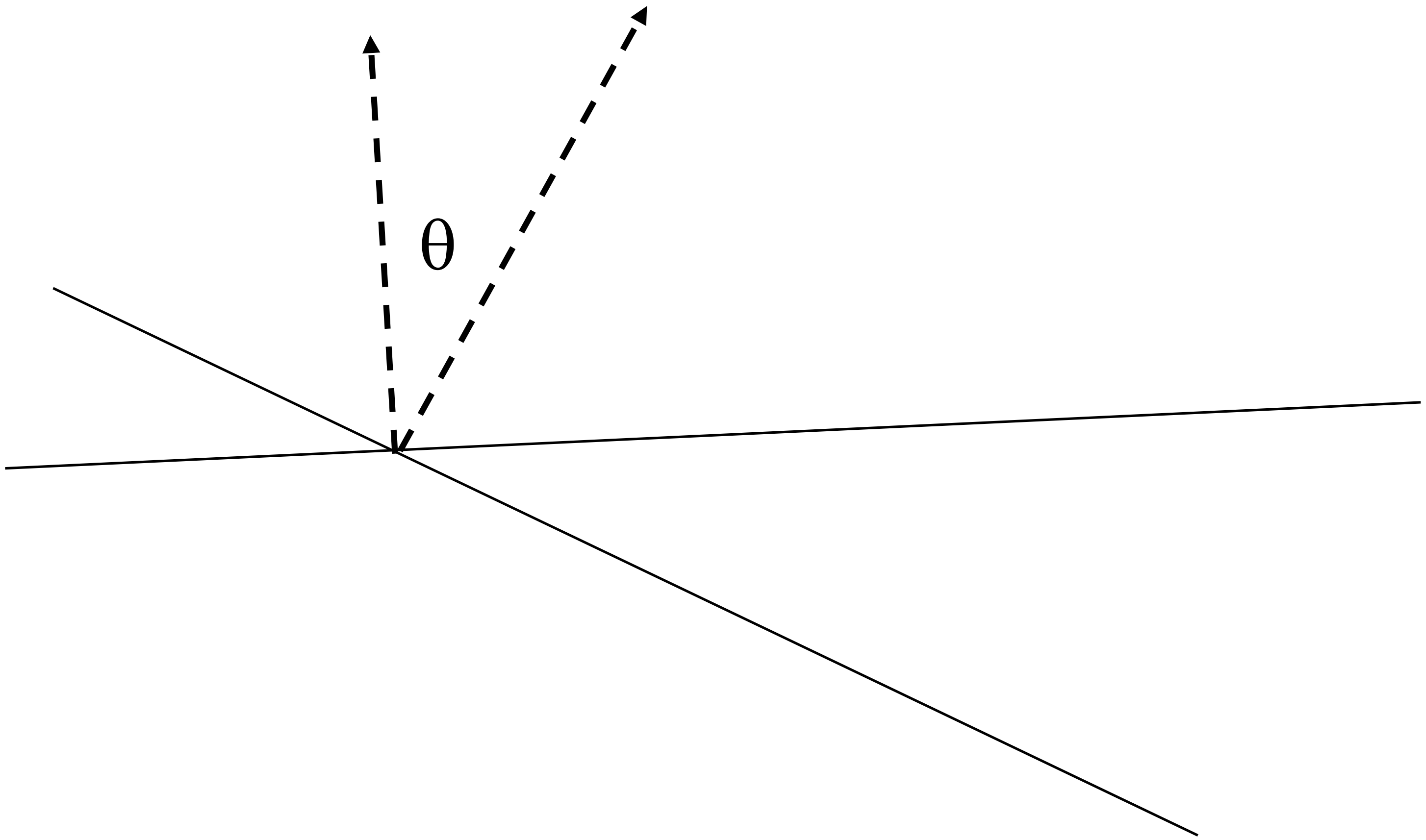


within the euclidean plane, $[a, b]$ is the direction normal to the line $\boldsymbol{l} = [\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}]^{\boldsymbol{T}}$

The angle between two lines …



The angle between two lines is the angle between their normals



Angle between two vectors: $\cos\vartheta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

The angle between two lines $l_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $l_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is

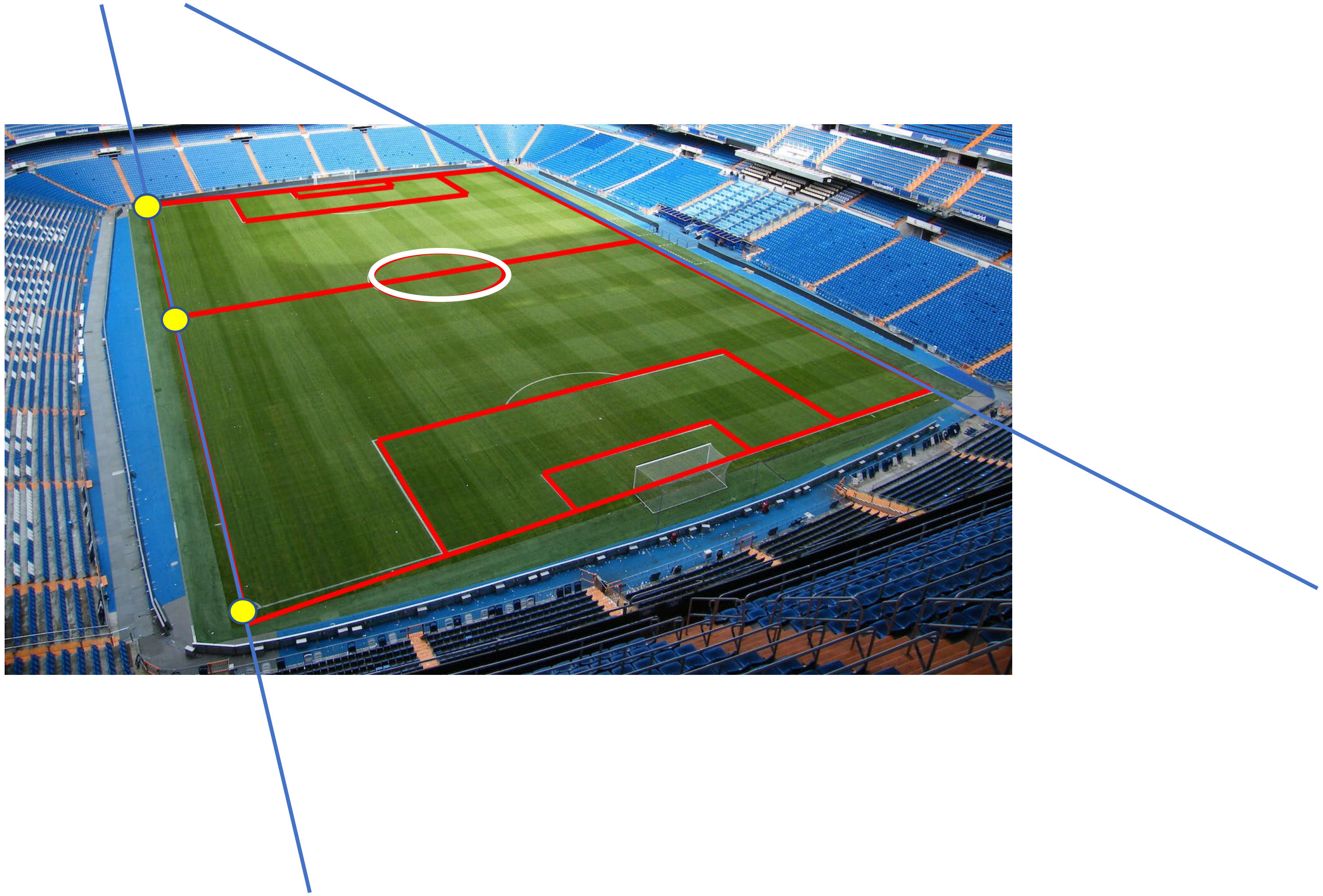
the angle between their normal directions

$[a_1 \quad b_1]$ and $[a_2 \quad b_2]$, namely

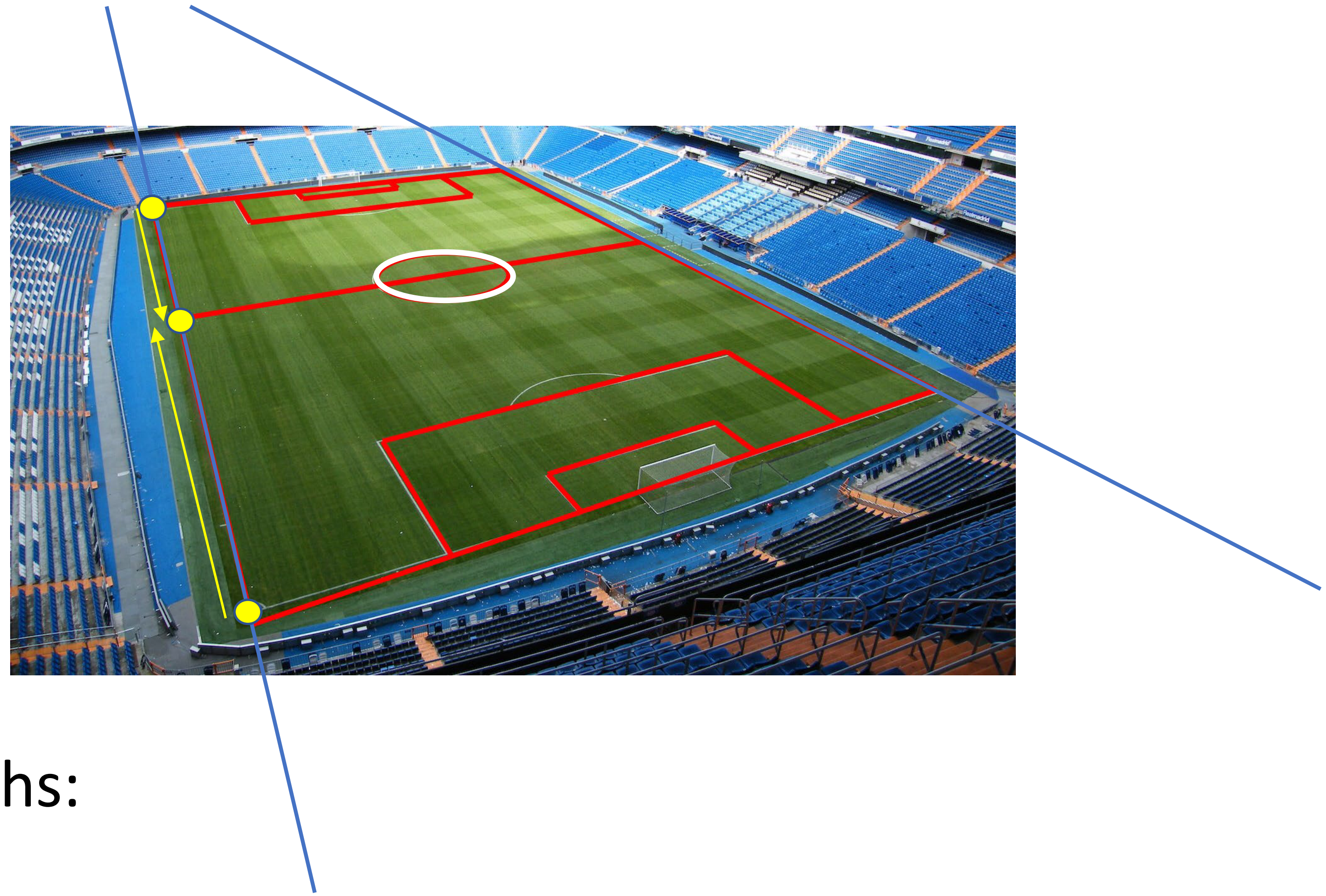
$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{({a_1}^2 + {b_1}^2)({a_2}^2 + {b_2}^2)}}$$

the cross ratio

midpoints **DO NOT PROJECT ONTO** midpoints

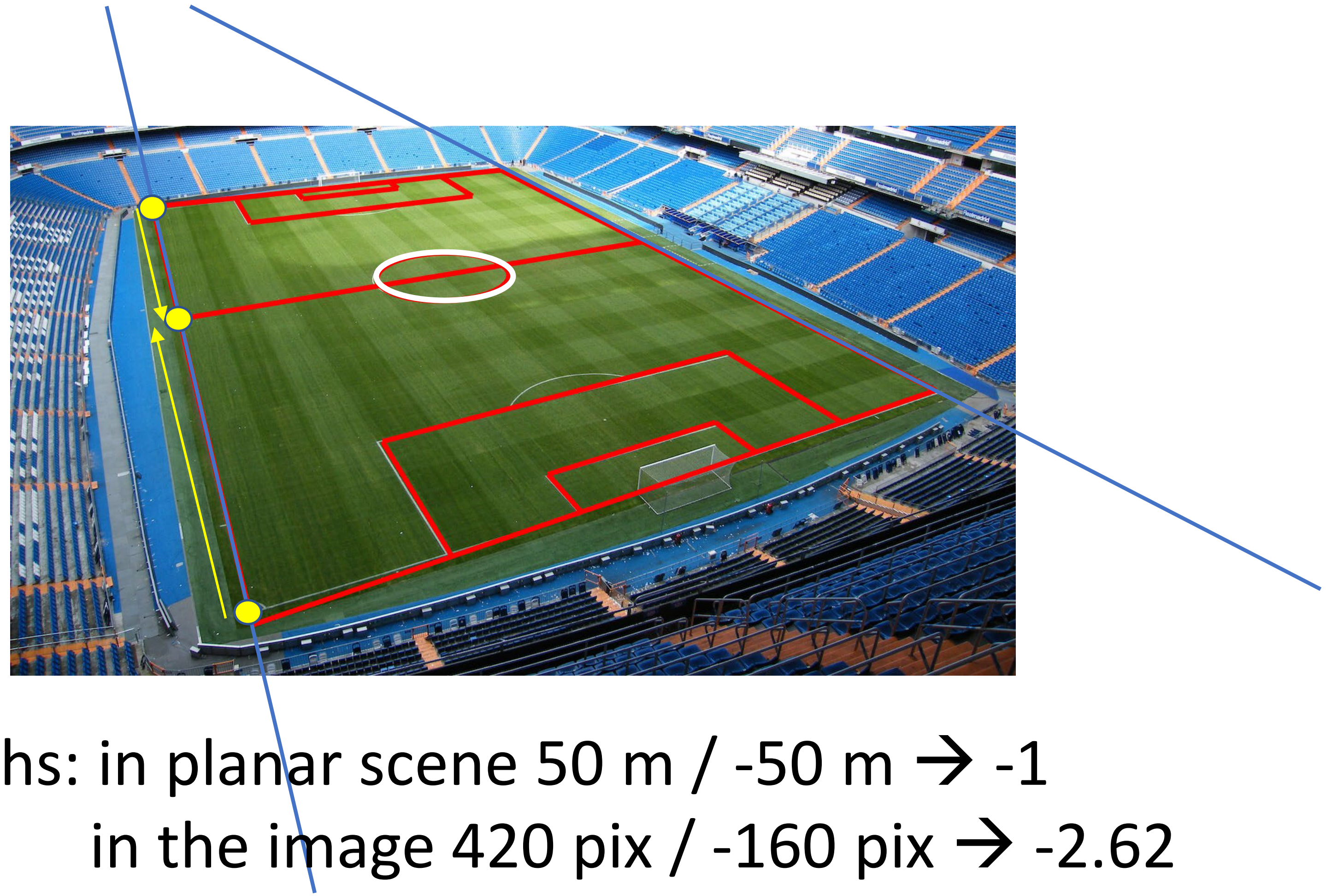


midpoints **DO NOT PROJECT ONTO** midpoints



ratio of lengths:

ratio of lengths: NOT INVARIANT

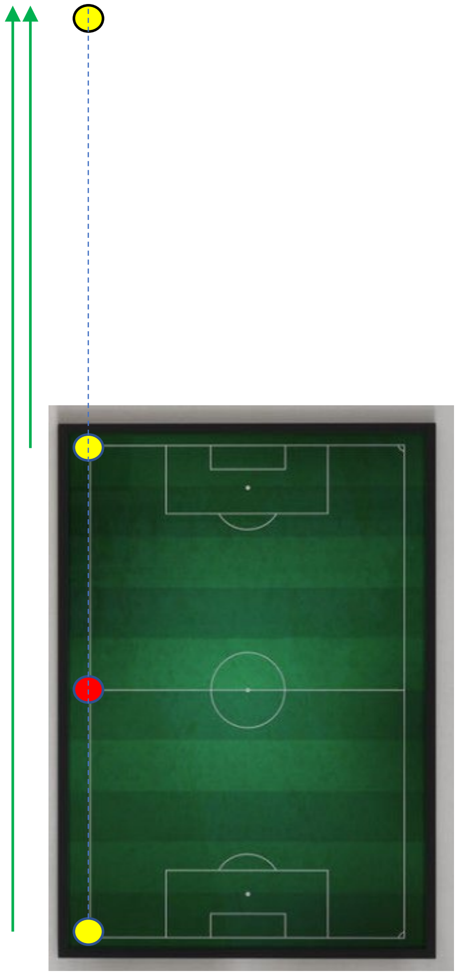


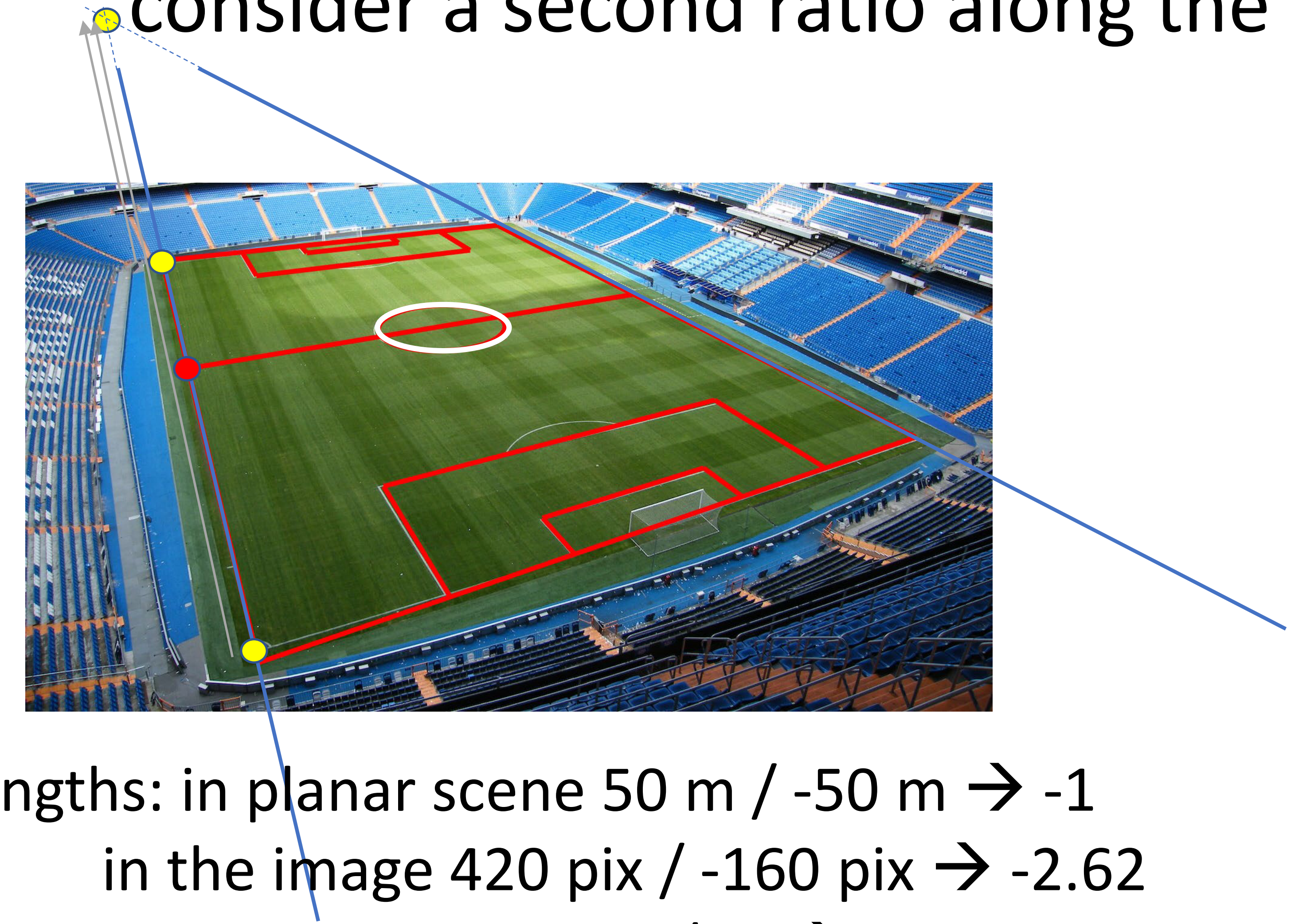
ratio of lengths: in planar scene 50 m / -50 m → -1

in the image 420 pix / -160 pix → -2.62

ratio of lengths: NOT INVARIANT

consider a second ratio along the same line





1° ratio of lengths: in planar scene 50 m / -50 m → -1

in the image 420 pix / -160 pix → -2.62

2° ratio of lengths: in planar scene ∞ / ∞ → 1

in the image 970 pix / 370 pix → +2.62

the ratio between the two ratios seems to be the same for the scene (-1/1= - 1) …

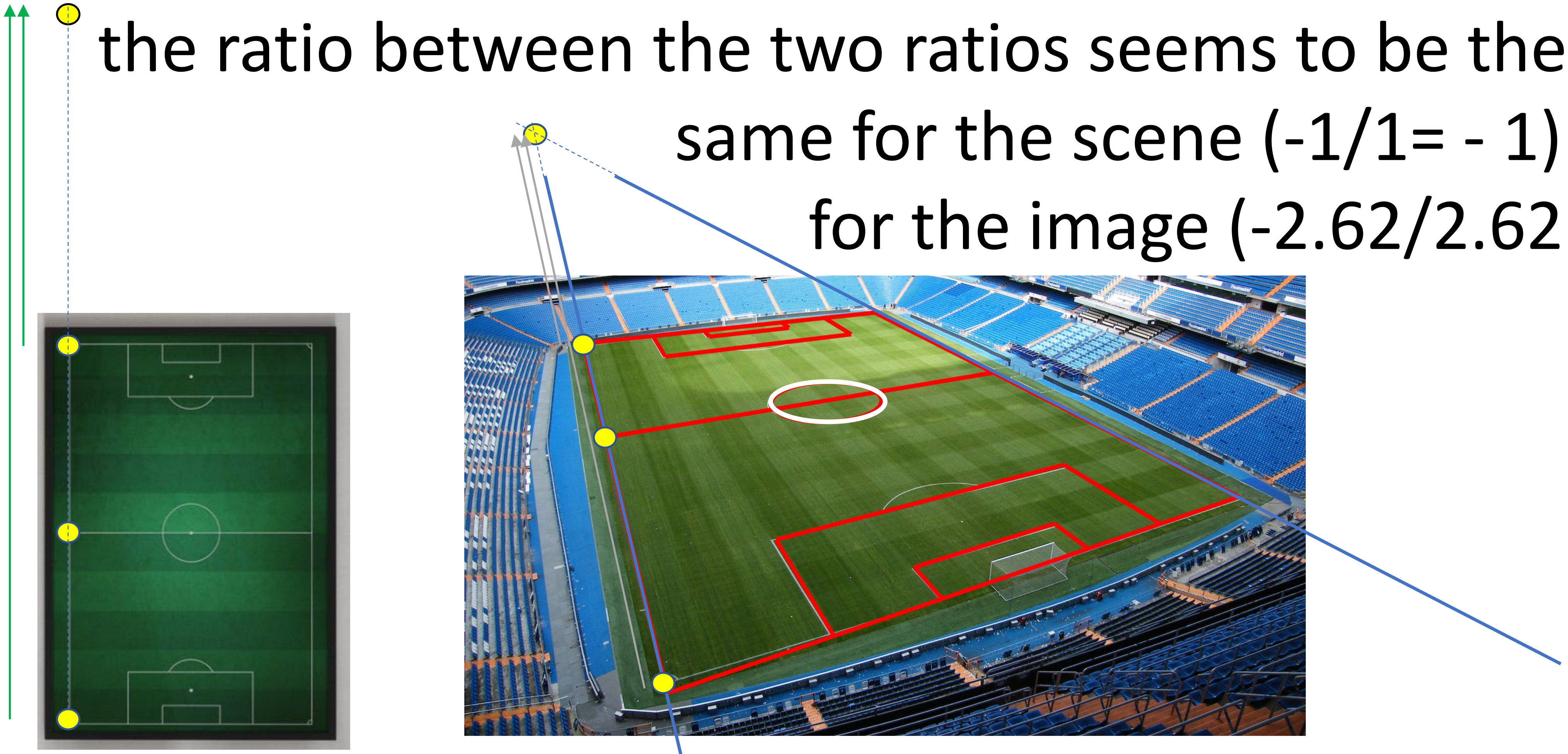
1° ratio of lengths: in planar scene 50 m / -50 m → -1

in the image 420 pix / -160 pix → -2.62

2° ratio of lengths: in planar scene ∞ / ∞ → 1

in the image 970 pix / 370 pix → +2.62

the ratio between the two ratios seems to be the same for the scene (-1/1= - 1) and for the image (-2.62/2.62 = - 1)



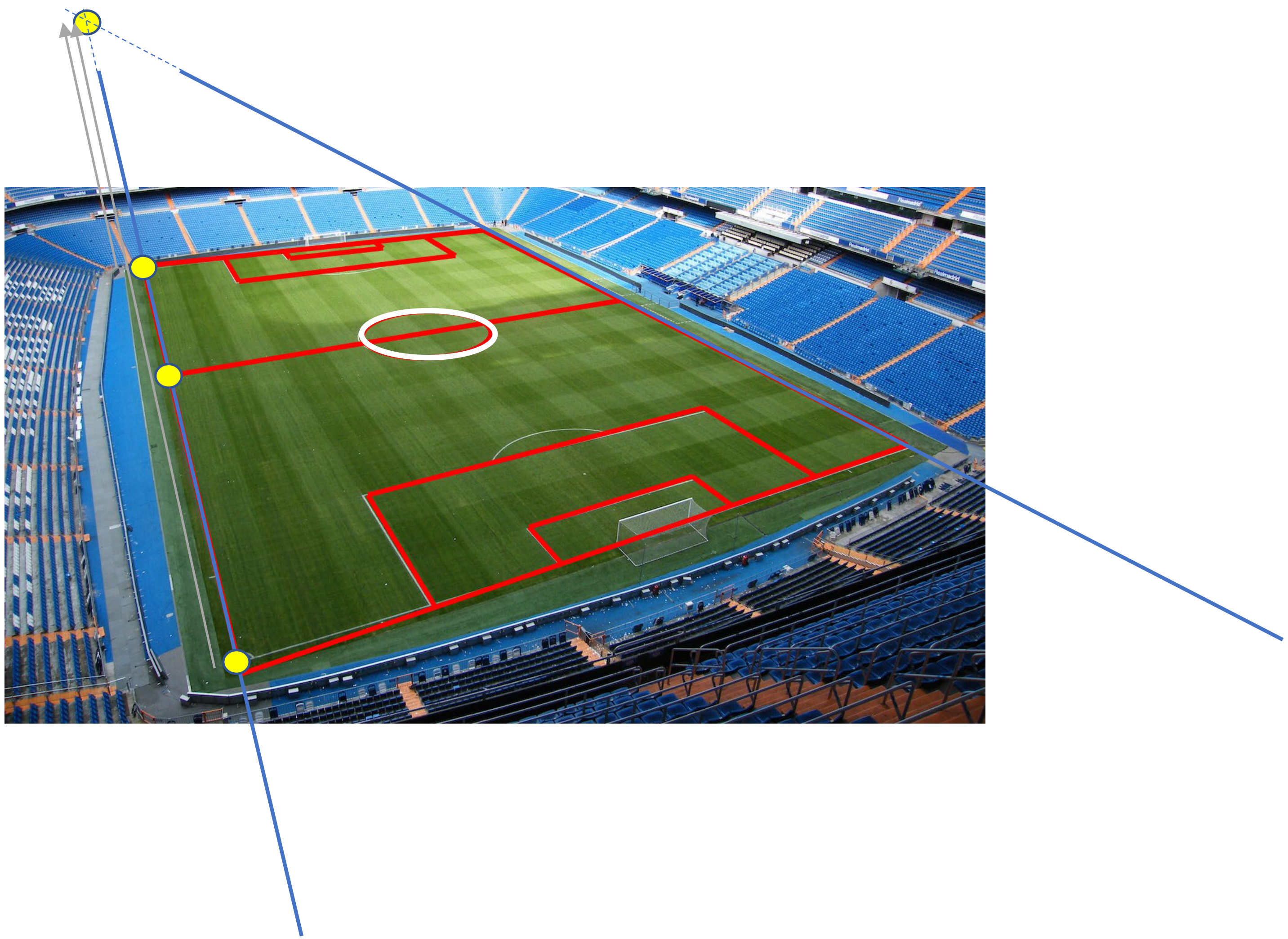
1° ratio of lengths: in planar scene 50 m / -50 m → -1

in the image 420 pix / -160 pix → -2.62

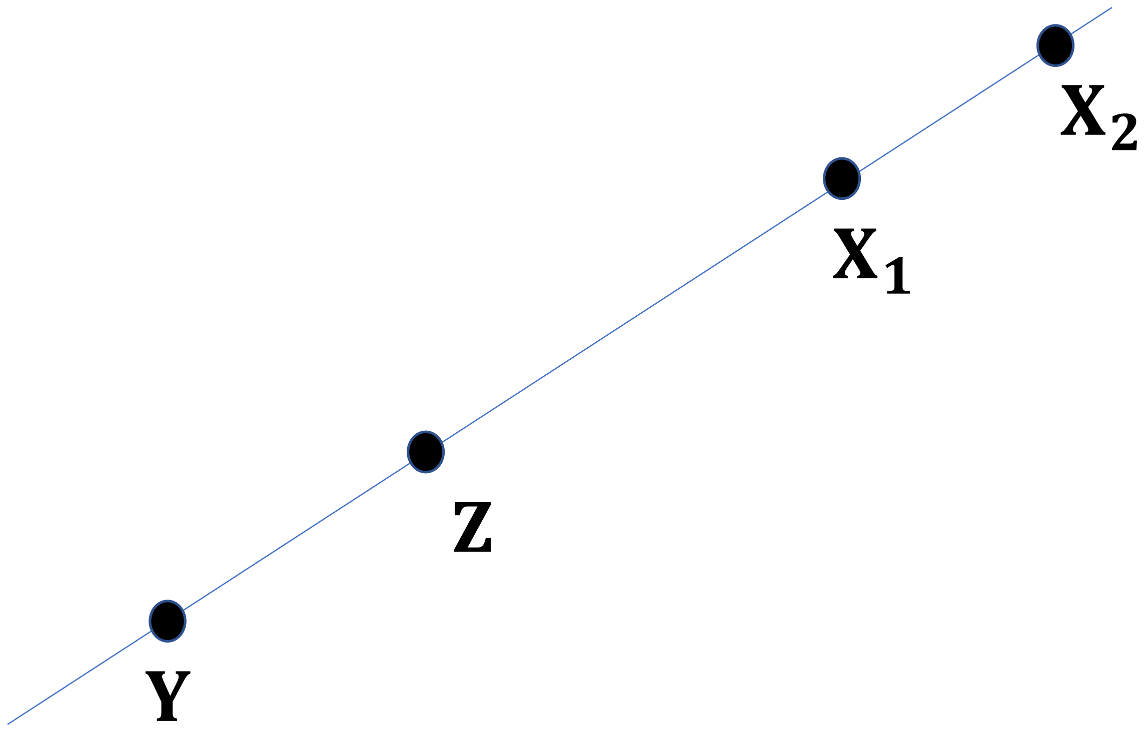
2° ratio of lengths: in planar scene ∞ / ∞ → 1

in the image 970 pix / 370 pix → +2.62

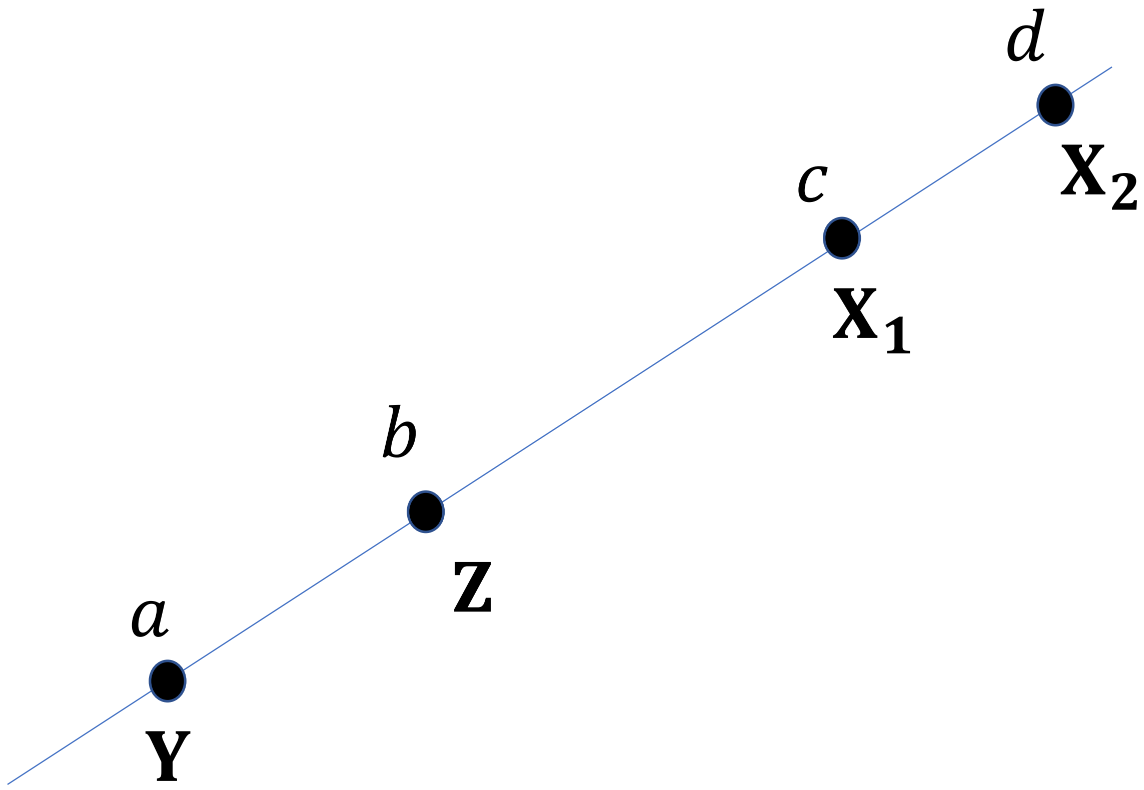
Let's study the ratio of ratios (for 4 colinear points) → **cross ratio**



Cross ratio
of 4 colinear points

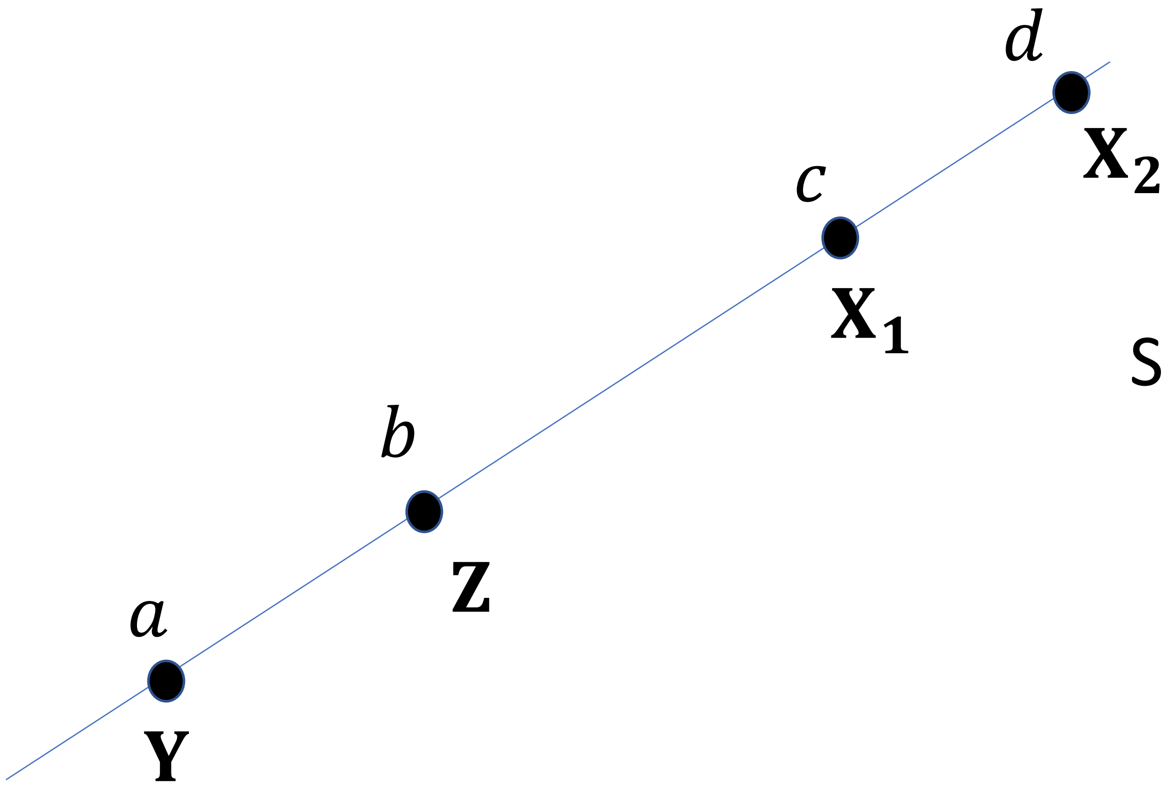


Define a coordinate (abscissa) along the line



Cross ratio of the 4-tuple of colinear points

$$CR_{Y,Z,\mathbf{X}_1,\mathbf{X}_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$



Since $\mathbf{X}_1$ and $\mathbf{X}_2$ are colinear with $\mathbf{Y}, \mathbf{Z}$

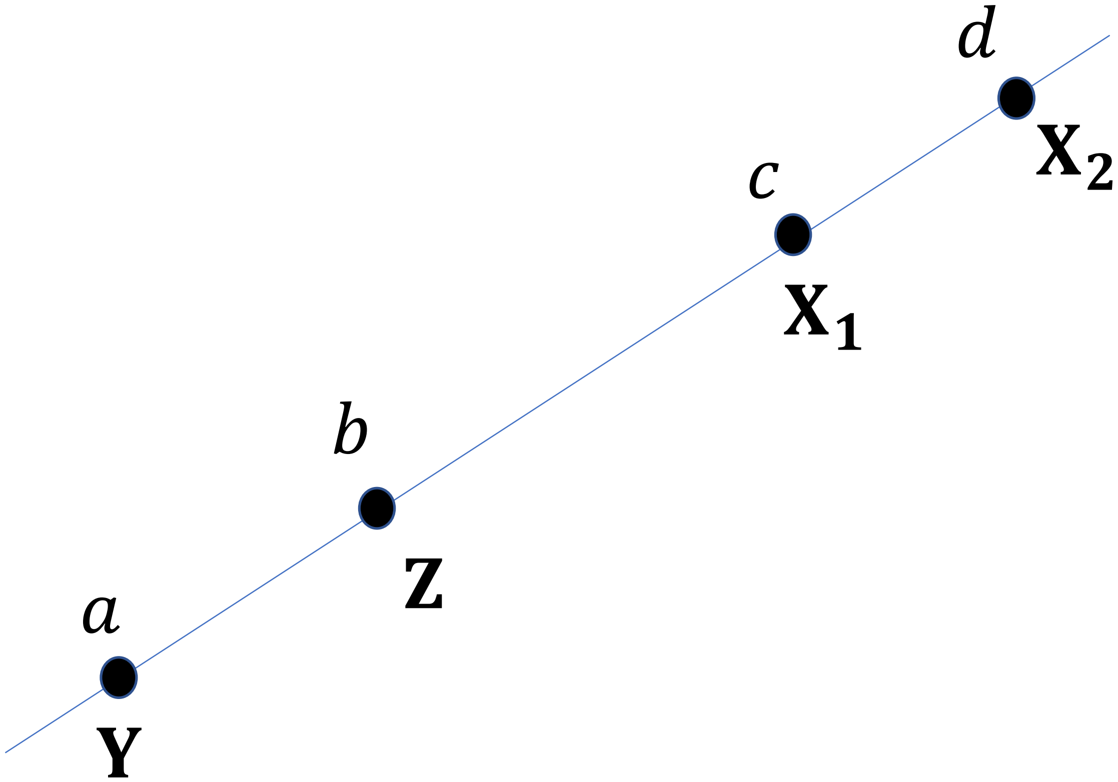
$$\mathbf{X}_1 = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

and

$$\mathbf{X}_2 = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

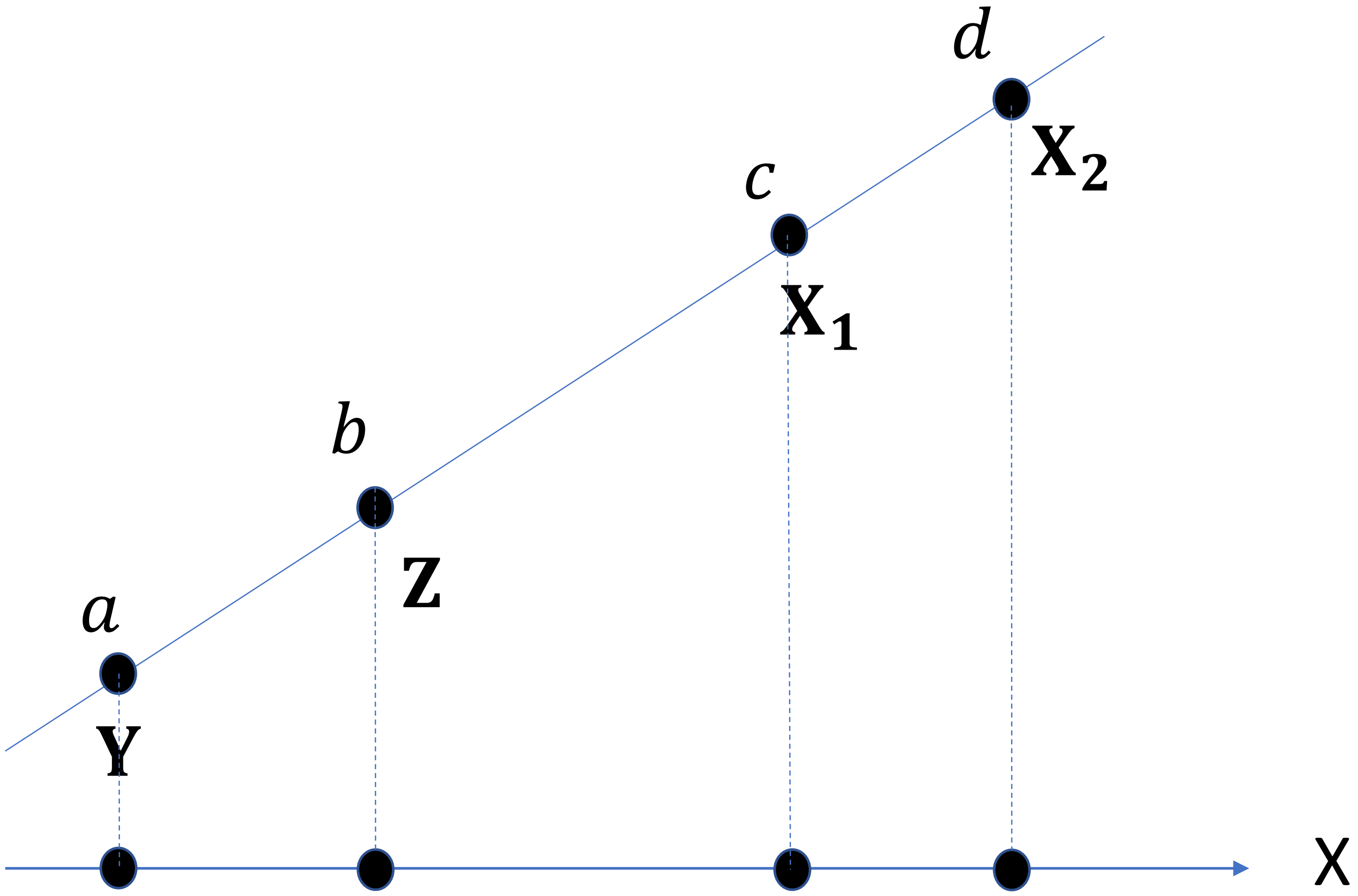
Cross ratio: A result

$$\boldsymbol{CR}_{\boldsymbol{Y},\boldsymbol{Z},\mathbf{X_1},\mathbf{X_2}} = \frac{\boldsymbol{c-a}}{\boldsymbol{c-b}} \Big/ \frac{\boldsymbol{d-a}}{\boldsymbol{d-b}} = \frac{\boldsymbol{\beta_1/\alpha_1}}{\boldsymbol{\beta_2/\alpha_2}}$$



Proof: since the abscissae are proportional to, e.g., the X cartesian coordinates, we can replace the abscissae by the X coordinates (let us reuse the same names)

$$\boldsymbol{CR_{Y,Z,X_1,X_2} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}}$$



Proof: since the abscissae are proportional to, e.g., the X cartesian coordinates, we can replace the abscissae by the X coordinates (let us the same names …)

Let $\mathbf{Y} = \begin{bmatrix} y \\ * \\ v \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} z \\ * \\ w \end{bmatrix}$: then $\boldsymbol{X_1} = \begin{bmatrix} \alpha_1 y + \beta_1 z \\ * \\ \alpha_1 v + \beta_1 w \end{bmatrix}$ and $\boldsymbol{X_2} = \begin{bmatrix} \alpha_2 y + \beta_2 z \\ * \\ \alpha_2 v + \beta_2 w \end{bmatrix}$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Y}$ is

$$c - a = \frac{(\alpha_1 y+\beta_1 z)v-(\alpha_1 v+\beta_1 w)y}{(\alpha_1 y+\beta_1 z)v} = \frac{\beta_1(zv-yw)}{(\alpha_1 y+\beta_1 z)v}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Z}$ is

$$c - b = \frac{(\alpha_1 y+\beta_1 z)w-(\alpha_1 v+\beta_1 w)z}{(\alpha_1 y+\beta_1 z)w} = \frac{-\alpha_1(zv-yw)}{(\alpha_1 y+\beta_1 z)w}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Y}$ is

$$c - a = \frac{(\alpha_1 y+\beta_1 z)v-(\alpha_1 v+\beta_1 w)y}{(\alpha_1 y+\beta_1 z)v} = \frac{\beta_1(zv-yw)}{(\alpha_1 y+\beta_1 z)v}$$

The difference between the X coordinates of $\boldsymbol{X_1}$ and $\mathbf{Z}$ is

$$c - b = \frac{(\alpha_1 y+\beta_1 z)w-(\alpha_1 v+\beta_1 w)z}{(\alpha_1 y+\beta_1 z)w} = \frac{-\alpha_1(zv-yw)}{(\alpha_1 y+\beta_1 z)w}$$

The ratio between these differences is $\frac{c-a}{c-b} = -\frac{\beta_1}{\alpha_1}\frac{w}{v}$ and, similarly,

$$\frac{d-a}{d-b} = -\frac{\beta_2}{\alpha_2}\frac{w}{v}$$

Thus the cross ratio of the four colinear points $\boldsymbol{Y}, \boldsymbol{Z}, \mathbf{X_1}, \mathbf{X_2}$ is

$$\boldsymbol{CR_{Y,Z,X_1,X_2}} = \frac{\boldsymbol{c-a}}{\boldsymbol{c-b}} \Big/ \frac{\boldsymbol{d-a}}{\boldsymbol{d-b}} = \frac{\boldsymbol{\beta_1}}{\boldsymbol{\alpha_1}} \Big/ \frac{\boldsymbol{\beta_2}}{\boldsymbol{\alpha_2}}$$

Q.E.D.