Advanced and Multivariable Control

Model Predictive Control – Part 1

Riccardo Scattolini

General characteristics of MPC

By far the ***most popular advanced control method*** in industry (see the following slides) and in embedded applications. All the main automation companies (ABB, Siemens, …) have SW tools for its implementation

Developed in the late ’70 and early ’80 in the process industry to cope with large scale systems, ***constraints*** on the process variables, time varying reference signals (known in advance)

The basic idea is ***to transform the control synthesis problem into an optimization one***. A finite horizon control problem is stated and solved

Empirical models (impulse/step response) can be used to reduce the time required by the design phase

Nonlinear models obtained through physical modeling can be used

With respect to finite horizon optimal control previously studied, a ***time invariant control law*** is obtained by means of ***the Receding Horizon principle***, see next slides,

S.J. Qin, T.A. Badgwell / Control Engineering Practice 11 (2003) 733–764 745

Table 6
Summary of linear MPC applications by areas (estimates based on vendor survey; estimates do not include applications by companies who have licensed vendor technology)[a]

Area	Aspen Technology	Honeywell Hi-Spec	Adersa[b]	Invensys	SGS[c]	Total
Refining	1200	480	280	25		1985
Petrochemicals	450	80	—	20		550
Chemicals	100	20	3	21		144
Pulp and paper	18	50	—	—		68
Air & Gas	—	10	—	—		10
Utility	—	10	—	4		14
Mining/Metallurgy	8	6	7	16		37
Food Processing	—	—	41	10		51
Polymer	17	—	—	—		17
Furnaces	—	—	42	3		45
Aerospace/Defense	—	—	13	—		13
Automotive	—	—	7	—		7
Unclassified	40	40	1045	26	450	1601
Total	1833	696	1438	125	450	4542
First App.	DMC:1985	PCT:1984	IDCOM:1973			
	IDCOM-M:1987	RMPCT:1991	HIECON:1986	1984	1985	
	OPC:1987					
Largest App.	603 × 283	225 × 85	—	31 × 12	—	

[a] The numbers reflect a snapshot survey conducted in mid-1999 and should not be read as static. A recent update by one vendor showed 80% increase in the number of applications.

[b] Adersa applications through January 1, 1996 are reported here. Since there are many embedded Adersa applications, it is difficult to accurately report their number or distribution. Adersa's product literature indicates over 1000 applications of PFC alone by January 1, 1996.

[c] The number of applications of SMOC includes in-house applications by Shell, which are unclassified. Therefore, only a total number is estimated here.

some data from

Economic assessment of advanced process control – a survey and framework

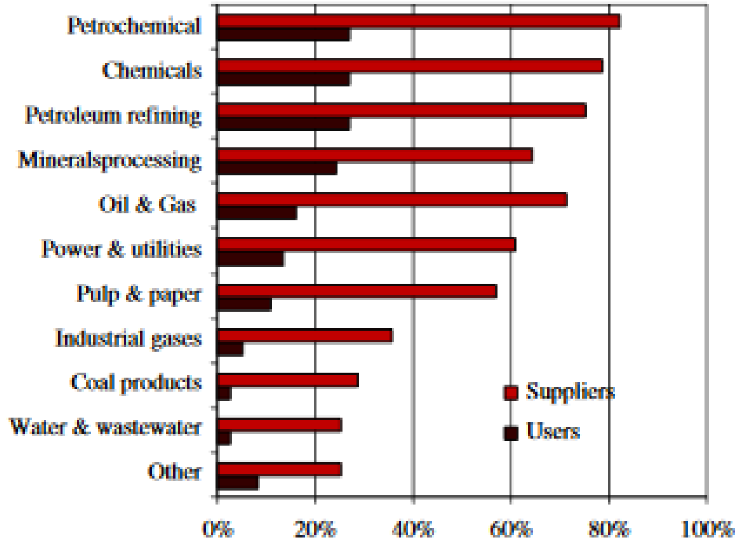
Margret Bauer, Ian K. Craig

Journal of Process Control 18 (2008) pp. 2–18

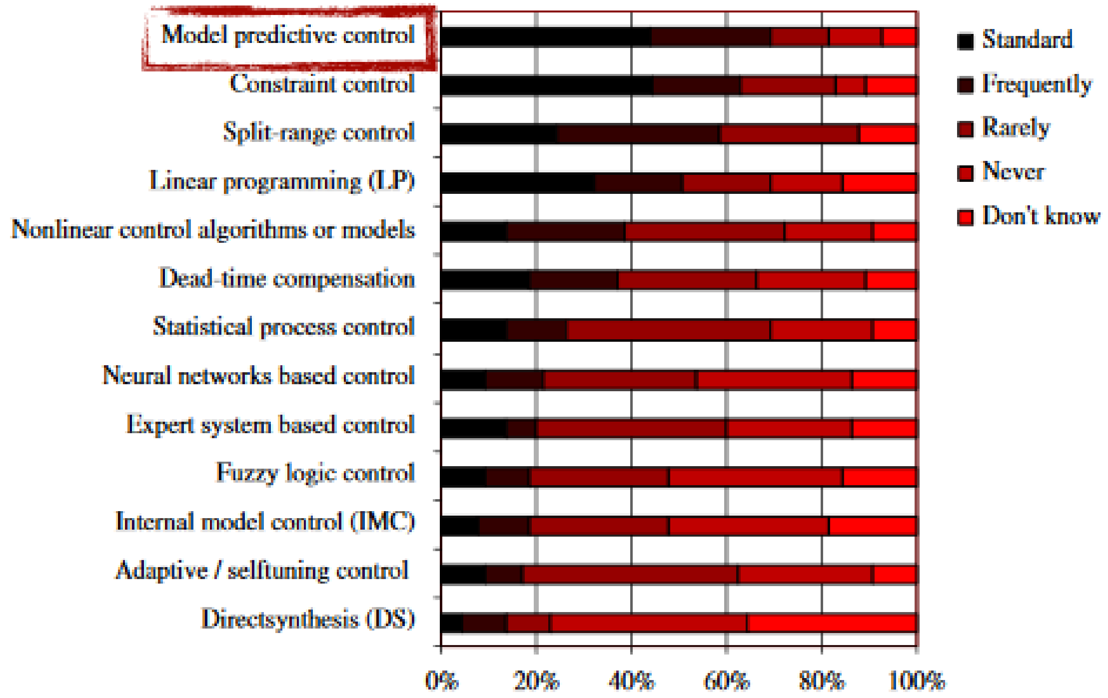
A key objective of industrial advanced process control (APC) projects is to stabilize the process operation. In order to justify the cost associated with the introduction of new APC technologies to a process, ***the benefits have to be quantified in economic terms***. In the past, economic assessment methods have been developed that link the variation of key controlled process variables to economic performance quantities. This paper reviews these methods and incorporates them in a framework for the economic evaluation of APC projects. A web-based survey on the economic assessment of process control has been completed by over 60 industrial APC experts. The results give information about the state-of-the-art assessment of economic benefits of advanced process control.

(Bauer, Craig, 2008)

- Economic assessment of Advanced Process Control (APC)



participants of APC survey by industry (worldwide)



Industrial use of APC methods: survey results

(Samad, IEEE CS Magazine, 2017)

- Impact of advanced control technologies in industry

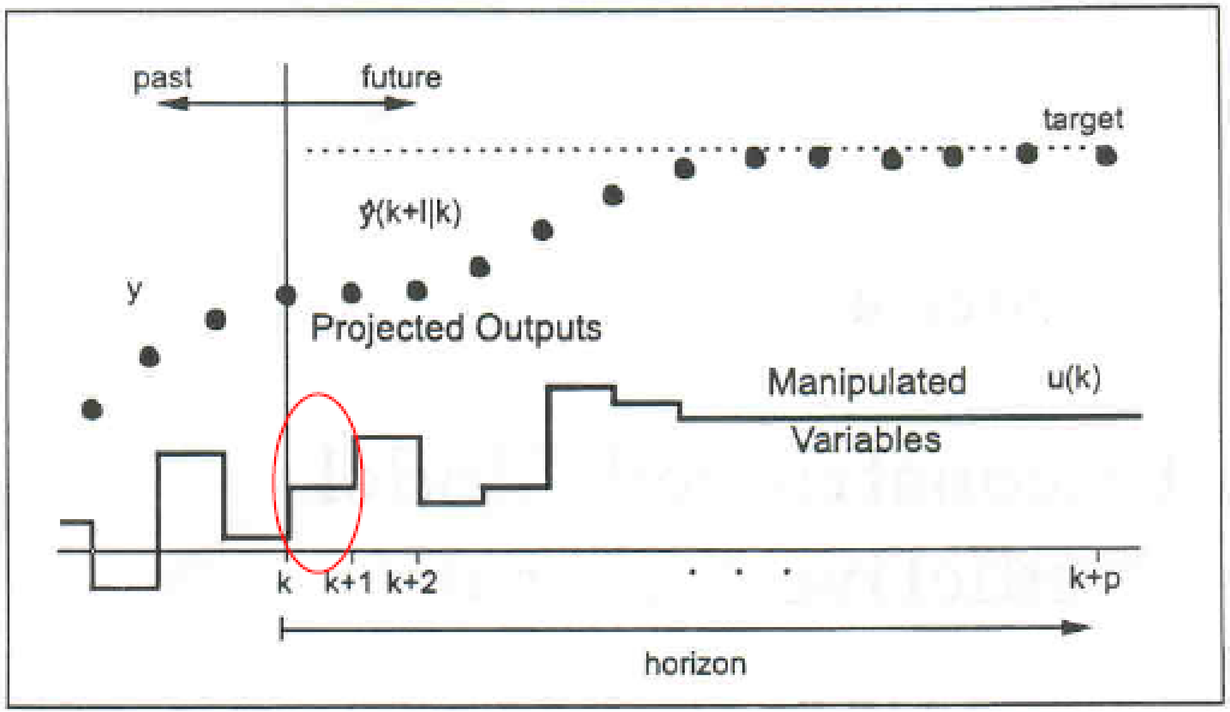
TABLE 1 A list of the survey results in order of industry impact as perceived by the committee members.

Rank and Technology	High-Impact Ratings	Low- or No-Impact Ratings
PID control	100%	0%
Model predictive control	78%	9%
System identification	61%	9%
Process data analytics	61%	17%
Soft sensing	52%	22%
Fault detection and identification	50%	18%
Decentralized and/or coordinated control	48%	30%
Intelligent control	35%	30%
Discrete-event systems	23%	32%
Nonlinear control	22%	35%
Adaptive control	17%	43%
Robust control	13%	43%
Hybrid dynamical systems	13%	43%

How is it explained in industrial courses?

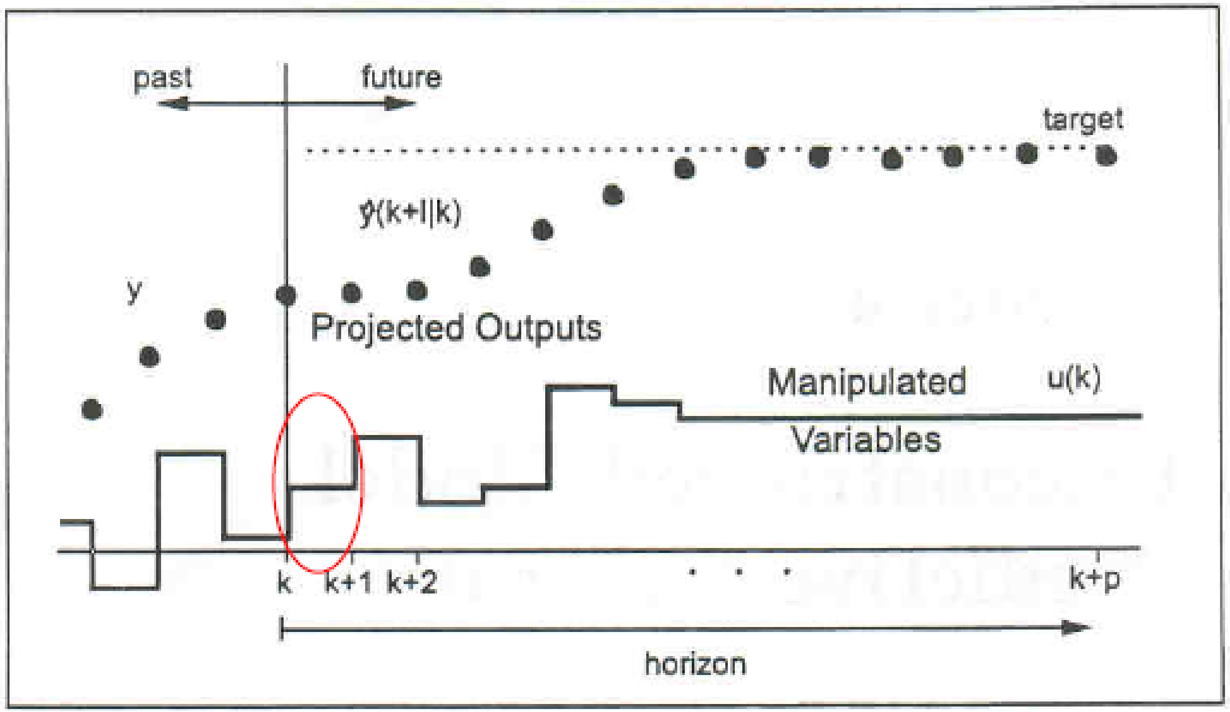
Let's have a look at it, then we will place it in the context of this course

MPC is based on the knowledge of a **dynamic model of the system** in order to compute the **future evolution of the controlled variables** as function of **the future evolution of the control inputs**. The input sequence is computed minimizing a **cost function under state, input, and output constraints**



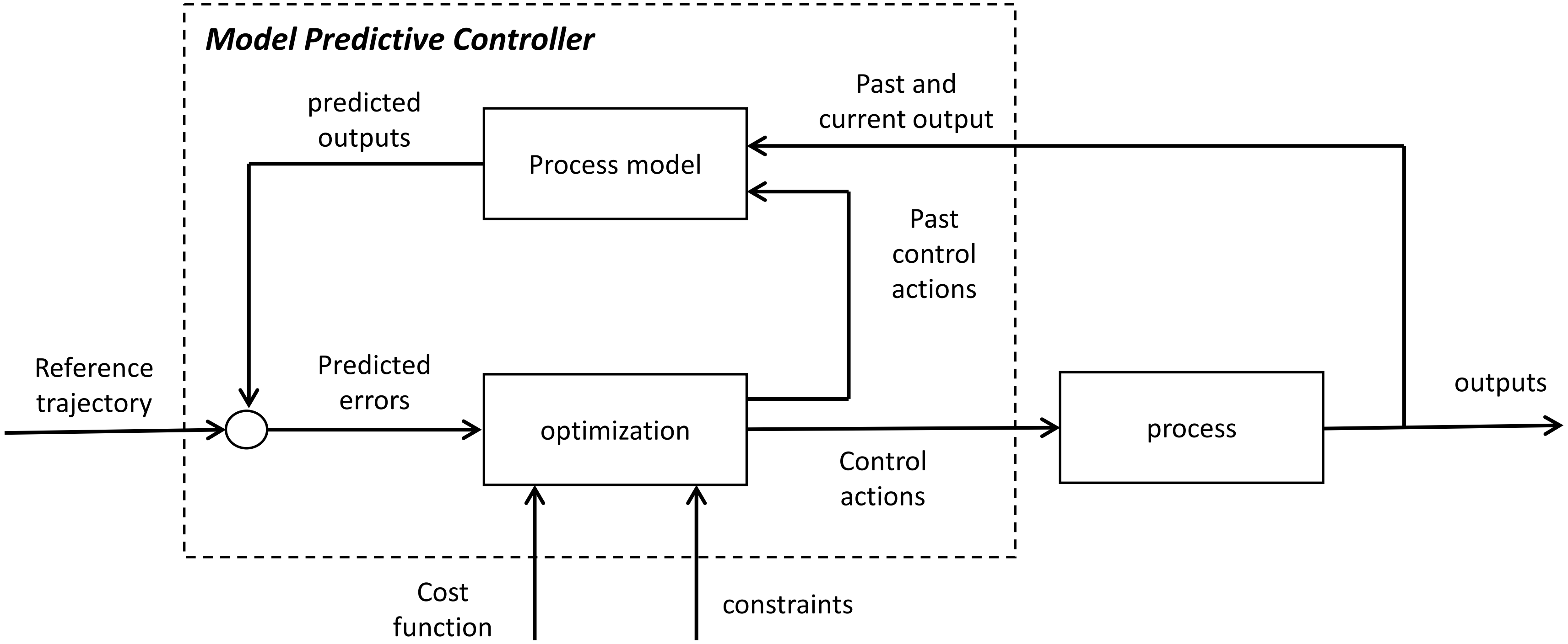
At time *k* the future sequence of control variables is computed, but only its first value is used

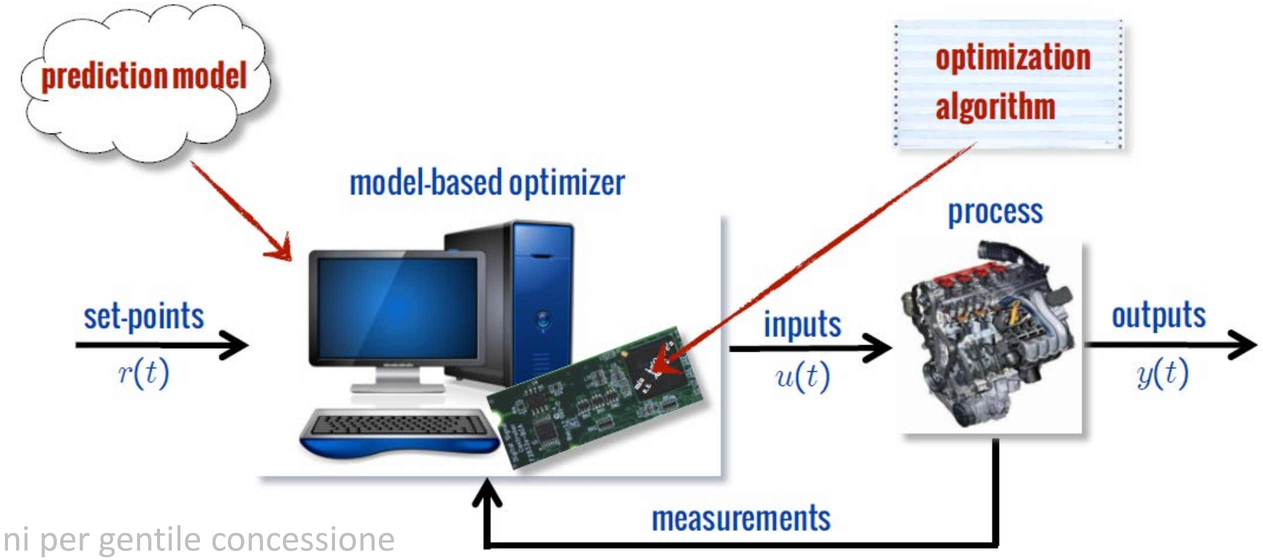
At time *k+1* the optimization procedure is repeated with the same prediction horizon



strategy named
Receding Horizon
Rolling Horizon
Moving Horizon

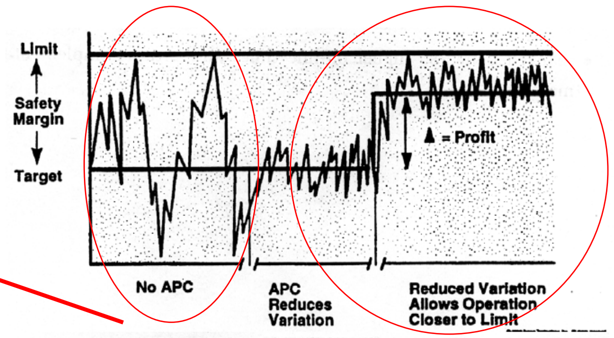
In this way, a time invariant control law is obtained

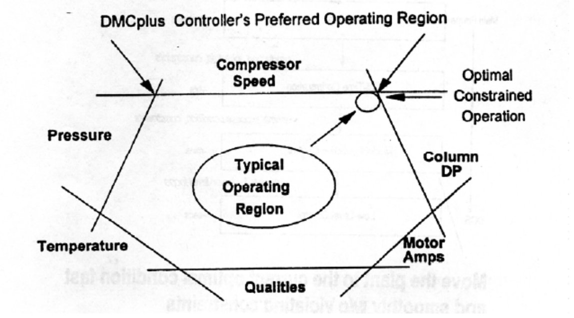


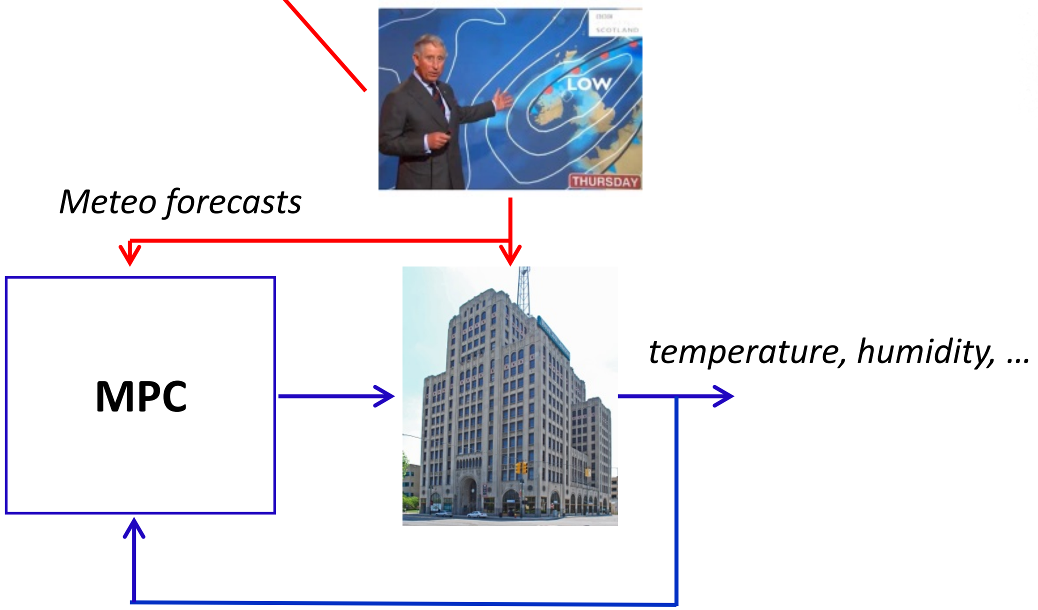


Immagini per gentile concessione del Prof. A. Bemporad

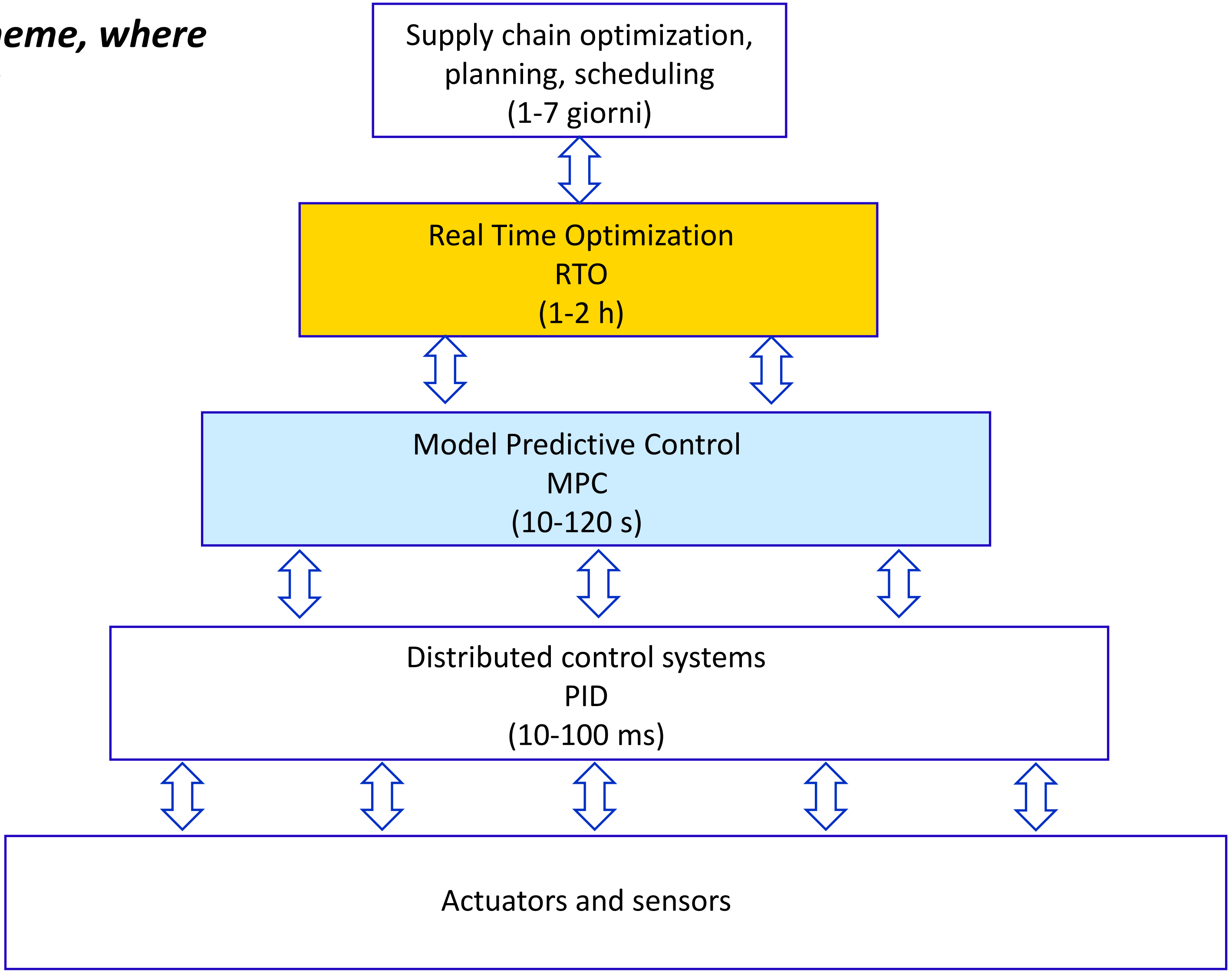
- More efficiency and tighter control with respect to traditional PID control schemes. This means that the reference signals can be set to values near the operating constraints, with economic advantages
- It can easily consider the knowledge of future external disturbances to improve the control action







In the overall control scheme, where to use MPC?



but now, let's proceed smoothly according to the course layout …

… and start from linear models, state feedback, state and control weighting in the cost function

Anyway remember … MPC is a very large family of control algorithms, the goal in this course is to transmit the main ideas behind them. For the whole analysis of all the developments and applications of MPC 10 CFU would not be enough

Finite Horizon LQ control

System

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \qquad x \text{ measurable}$$

Optimization problem

$$\min_{u(k),\, u(k+1),\, \dots,\, u(k+N-1)} \quad J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\|x(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) + \|x(k+N)\|_S^2$$

«classical» Finite Horizon optimal control problem

Closed-loop solution (Riccati equation)

$$u^o(k+i) = -K(i)x(k+i), \quad i = 0, 1, ..., N-1$$

$$K(i) = \left(R + B'P(i+1)B\right)^{-1} B'P(i+1)A$$

$$P(i) = Q + A'P(i+1)A - A'P(i+1)B\left(R + B'P(i+1)B\right)^{-1} B'P(i+1)A$$

$$P(N) = S$$

Defined over a finite horizon and time varying. How to obtain a time invariant control law?

Receding Horizon (RH) principle

$$u^{MPC}(k) = -K(0)x(k)$$

No constraints on state and inputs can be included

Open-loop solution

Lagrange equation $$x(k+i) = A^i x(k) + \sum_{j=0}^{i-1} A^{i-j-1} B u(k+j), \quad i > 0$$

Define

$$X(k) = \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N-1) \\ x(k+N) \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^{N-1} \\ A^N \end{bmatrix}, U(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-2) \\ u(k+N-1) \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} B & 0 & 0 & \cdots & 0 & 0 \\ AB & B & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A^{N-2}B & A^{N-3}B & A^{N-4}B & \cdots & B & 0 \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & AB & B \end{bmatrix}$$

$$X(k) = \mathcal{A}x(k) + \mathcal{B}U(k)$$

state evolution along the prediction horizon as function of the initial state and future inputs

Define the matrices, with N blocks on the diagonal

$$\mathcal{Q} = \begin{bmatrix} Q & 0 & \cdots & 0 & 0 \\ 0 & Q & \cdots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & \cdots & Q & 0 \\ 0 & 0 & \cdots & 0 & S \end{bmatrix} \geq 0, \quad \mathcal{R} = \begin{bmatrix} R & 0 & \cdots & 0 & 0 \\ 0 & R & \cdots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & \cdots & R & 0 \\ 0 & 0 & \cdots & 0 & R \end{bmatrix} > 0$$

Define

$$\bar{J}(x(k), U(k), k) = X'(k)\mathcal{Q}X(k) + U'(k)\mathcal{R}U(k)$$

Note that

$$J(x(k), U(k), k) = \bar{J}(x(k), U(k), k) + x'(k)Qx(k)$$

term independent on ***u(k+i), i≥0***

Therefore

$$arg(\min_{u(\cdot)} J(x(k), u(\cdot), k)) = arg(\min_{U(k)} \bar{J}(x(k), U(k), k))$$

and we can minimize with respect to ***U(k)***

$$\begin{aligned} & \bar{J}(x(k), U(k), k) = (\mathcal{A}x(k) + \mathcal{B}U(k))' \, \mathcal{Q} \, (\mathcal{A}x(k) + \mathcal{B}U(k)) + U'(k)\mathcal{R}U(k) \\ = \quad & x'(k)\mathcal{A}'\mathcal{Q}\mathcal{A}x(k) + 2x'(k)\mathcal{A}'\mathcal{Q}\mathcal{B}U(k) + U'(k)\left(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R}\right)U(k) \end{aligned}$$

*If there are no constraints, one can simply derive with respect to **U(k)** and set the derivative to zero (the quadratic function in **U(k)** is positive definite)*

$$U^o(k) = -\left(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R}\right)^{-1} \mathcal{B}'\mathcal{Q}\mathcal{A}x(k)$$

$$U^o(k) = -\left(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R}\right)^{-1} \mathcal{B}'\mathcal{Q}\mathcal{A}x(k) = \begin{bmatrix} \mathcal{K}(0) \\ \mathcal{K}(1) \\ \vdots \\ \mathcal{K}(N-1) \end{bmatrix} x(k), \quad \mathcal{K}(i) \in R^{m,n}$$

$$u^o(k+i) = -\mathcal{K}(i)x(k), \quad i = 0, 1, ..., N-1$$

The numerical values of the optimal control sequence are the same obtained with the closed-loop strategy

Closed-loop

$$u^o(k+i) = -K(i)x(k+i), \quad i = 0, 1, ..., N-1$$

Open-loop

$$u^o(k+i) = -\mathcal{K}(i)x(k), \quad i = 0, 1, ..., N-1$$

Receding Horizon

$$u^{MPC}(k) = -K(0)x(k) = -\mathcal{K}(0)x(k)$$

If the state is not measurable?

A state observer must be used (pole placement, KF,…)

Is the receding horizon strategy stabilizing?

$x(k+1) = 3x(k) - u(k)$ (unstable)

$$J = \sum_{i=0}^{N-1} \left(x^2(k+i) + 2u^2(k+i) \right)$$

$\boxed{N \to \infty}$ $\rightsquigarrow$ LQ_∞ solution $(A=3, B=-1, Q=1, R=2, S=0)$

$$\bar{P} = 9\bar{P} + 1 - \frac{9\bar{P}^2}{\bar{P}+2} \longrightarrow \bar{P} = 17.12,\ \bar{K} = -2.69,\ A - B\bar{K} = 0.31$$

$\boxed{\text{RH with different horizons}}$

$P_N = 0 \rightarrow K_{N-1} = 0 \;,\; A - BK_{N-1} = 3$ unstable

$P_{N-1} = 1 \rightarrow K_{N-2} = -1 \;,\; A - BK_{N-2} = 2$ "

$P_{N-2} = 7 \rightarrow K_{N-3} = -\frac{21}{9} \;,\; A - BK_{N-3} = \frac{2}{3} < 1$ STABLE

The minimum prediction horizon required to have closed-loop stability is $N = 3$

The issue of the stabilizing properties of MPC will be discussed later on

How to consider constraints?

$$u_m \leq u(k+i) \leq u_M, \quad i = 0, ..., N-1$$

$$x_m \leq x(k+i) \leq x_M, \quad i = 1, ..., N$$

(elementwise)

$$y_m \leq y(k+i) \leq y_M, \quad i = 1, ..., N$$

Define

$$U_m = \begin{bmatrix} u_m \\ u_m \\ \vdots \\ u_m \end{bmatrix}, \quad U_M = \begin{bmatrix} u_M \\ u_M \\ \vdots \\ u_M \end{bmatrix}, \quad X_m = \begin{bmatrix} x_m \\ x_m \\ \vdots \\ x_m \end{bmatrix}, \quad X_M = \begin{bmatrix} x_M \\ x_M \\ \vdots \\ x_M \end{bmatrix}$$

$$Y_m = \begin{bmatrix} y_m \\ y_m \\ \vdots \\ y_m \end{bmatrix}, \quad Y_M = \begin{bmatrix} y_M \\ y_M \\ \vdots \\ y_M \end{bmatrix}, \quad Y(k) = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+N-1) \\ y(k+N) \end{bmatrix}$$

Constrained optimization problem

$$\min_{U(k)} \quad \bar{J}(x(k), U(k), k) = (\mathcal{A}x(k) + \mathcal{B}U(k))' \, \mathcal{Q} \, (\mathcal{A}x(k) + \mathcal{B}U(k)) + U'(k)\mathcal{R}U(k)$$

$$\begin{array}{rcl} X_m & \leq & X(k) = \mathcal{A}x(k) + \mathcal{B}U(k) \leq X_M \\ U_m & \leq & U(k) \leq U_M \\ Y_m & \leq & Y(k) \leq Y_M \end{array}$$

This problem **does not have an explicit solution**, which can be computed with suitable **optimization algorithms**.

In this case however, the problem is a ***Quadratic Programming QP*** one, which can be solved with very efficient algorithms

Excursus: Quadratic Programming problems in Matlab

$$min_{\varphi} 0.5\varphi' H\varphi + f'\varphi$$

$$\varphi = quadprog(H, f, A, b, A_{eq}, b_{eq}, low_b, up_b)$$

$$\begin{array}{rcl} A\varphi & \leq & b \\ A_{eq}\varphi & = & b_{eq} \\ low_b & \leq & \varphi \\ \varphi & \leq & up_b \end{array}$$

$$\min_{U(k)} \quad \bar{J}(x(k), U(k), k) = x'(k)\cancel{\mathcal{A}'\mathcal{Q}}\mathcal{A}x(k) + \underbrace{2x'(k)\mathcal{A}'\mathcal{Q}\mathcal{B}}_{f'}U(k) + U'(k)\underbrace{(\mathcal{B}'\mathcal{Q}\mathcal{B}+\mathcal{R})}_{0.5H}\underbrace{U}_{\varphi}(k))$$

Try by yourself to write the state, input, and output constraints in the form required by *quadprog*

Tracking of reference signals and disturbances

System

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + Md(k) \\
y(k) &= Cx(k) + d(k)
\end{aligned}
$$

Cost function

$$
\begin{aligned}
J(x(k), u(\cdot), k) &= \sum_{i=0}^{N-1} \left(\|y^o(k+i) - y(k+i)\|_Q^2 + \|u(k+i)\|_R^2 \right) \\
&+ \|y^o(k+N) - y(k+N)\|_S^2
\end{aligned}
$$

The future evolution of the output along the prediction horizon is

$$Y(k) = \mathcal{A}_c x(k) + \mathcal{B}_c U(k) + \mathcal{M}_c D(k)$$

where

$$\mathcal{A}_c = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-1} \\ CA^N \end{bmatrix}, \quad D(k) = \begin{bmatrix} d(k) \\ d(k+1) \\ \vdots \\ d(k+N-2) \\ d(k+N-1) \\ d(k+N) \end{bmatrix}$$

$$\mathcal{M}_c = \begin{bmatrix} CM & I & 0 & \cdots & 0 & 0 & 0 \\ CAM & CM & I & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^{N-2}M & CA^{N-3}M & CA^{N-4}M & \cdots & CM & I & 0 \\ CA^{N-1}M & CA^{N-2}M & CA^{N-3}M & \cdots & CAM & CM & I \end{bmatrix}$$

$$\mathcal{B}_c = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 & 0 \\ CAB & CB & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^{N-2}B & CA^{N-3}B & CA^{N-4}B & \cdots & CB & 0 \\ CA^{N-1}B & CA^{N-2}B & CA^{N-3}B & \cdots & CAB & CB \end{bmatrix}$$

Letting

$$Y^o(k) = \begin{bmatrix} y^o(k+1) \\ y^o(k+2) \\ \vdots \\ y^o(k+N-1) \\ y^o(k+N) \end{bmatrix}$$

by properly defining the matrices $\mathcal{Q}$ e $\mathcal{R}$ one obtains

$$arg(\min_{U(k)} J(x(k), u(\cdot), k)) = arg(\min_{U(k)} \bar{J}(x(k), U(k), k))$$

and the cost function to be minimized becomes

$$\bar{J}(x(k), U(k), k) = (Y^o(k) - Y(k))' \, \mathcal{Q} \, (Y^o(k) - Y(k)) + U'(k)\mathcal{R}U(k)$$

possibly ***under state, control, and output constraints***

Note the cost function (and the solution of the optimization problem)

$$\bar{J}(x(k), U(k), k) = (Y^o(k) - Y(k))' \, \mathcal{Q}(Y^o(k) - Y(k)) + U'(k)\mathcal{R}U(k)$$

depend on future reference signals and disturbances

$$Y(k) = \mathcal{A}_c x(k) + \mathcal{B}_c U(k) + \mathcal{M}_c D(k)$$

This allows one to consider the problem of time varying reference signals

If these quantities are known in advance, MPC produces an anticipative response

If *d* and y^o are unknown in advance, it is customary to set

$$y^o(k+i) = y^o(k) \quad , \quad d(k+i) = d(k)$$

Soft and hard constraints

An optimization problem is solved on-line. It must be guaranteed that a solution is always computed by the optimization algorithm

*These constraints can always be satisfied because the control variables are our design choices (the results of the optimization procedure). They are implemented as **hard constraints***

$$
\begin{aligned}
U_m &\leq U(k) \leq U_M \\
X_m &\leq X(k) \leq X_M \\
Y_m &\leq Y(k) \leq Y_M
\end{aligned}
$$

*These constraints can not (temporarilly or permanently) be satisfied, due for example to the effects of disturbances. They are implemented as **soft constraints** by means of suitable slack variables*

MPC with soft constraints

$$\min_{U(k),\varepsilon} \left(Y^o(k) - Y(k)\right)' \mathcal{Q} \left(Y^o(k) - Y(k)\right) + U'(k)\mathcal{R}U(k) + \rho\varepsilon$$

subject to

$$\begin{aligned} U_m &\leq U(k) \leq U_M \\ Y_m - \varepsilon I &\leq Y(k) \leq Y_M + \varepsilon I \\ \varepsilon &\geq 0 \end{aligned}$$

where $I = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}'$ and ρ must be selected sufficiently high to guarantee that the optimum is $\varepsilon = 0$ when a feasible solution exists

Constant reference signals and integral action

Assume that $y^o(k) = y^o$ and $d = 0$ (for simplicity) . If $p \leq m$ and the system does not have transmission zeros in 1, it is possible to compute the equilibrium $(\bar{x}, \bar{u})$ such that

$$\begin{aligned} \bar{x} &= A\bar{x} + B\bar{u} \\ y^o &= C\bar{x} \end{aligned}$$

and the cost function to be minimized can be written as (plus additional weight on the states)

$$J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\|y^o - y(k+i)\|_Q^2 + \|u(k+i) - \bar{u}\|_R^2 \right) + \|y^o - y(k+N)\|_S^2$$

Pros: $J = 0 \text{ for } y(k+i) = y^o \text{ and } u(k+i) = \bar{u}$

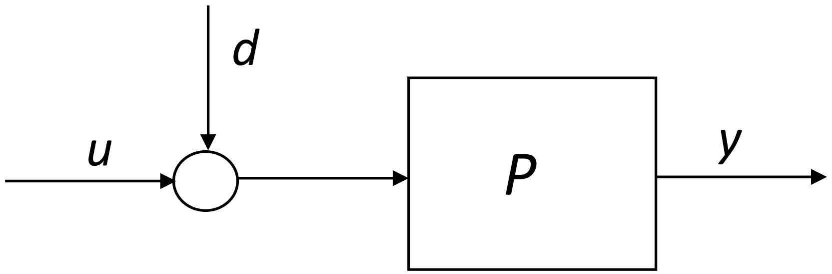
Cons: in case of unknown disturbances and/or modeling errors the computation of the steady state is not correct, and steady state errors occur

Integral action in MP – method 1 : DISTURBANCE ESTIMATION

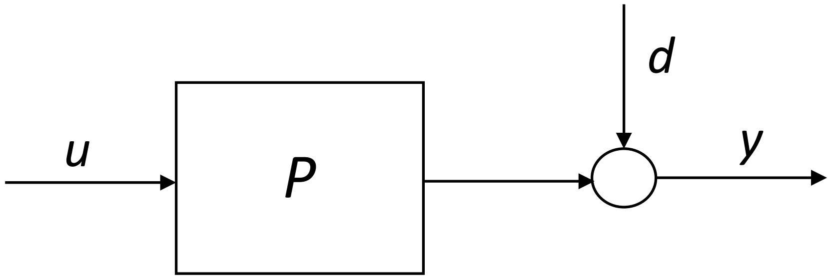
Assume that the system is affected by an *unknown*, *constant* disturbance

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Md(k) \\ d(k+1) &= d(k) \\ y(k) &= Cx(k) + Fd(k) \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} &\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} = \begin{bmatrix} A & M \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \\ &y(k) = \begin{bmatrix} C & F \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \end{aligned}$$

Possible choices



M=B, F=0



M=0, F=I

For the enlarged system
$$\begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} = \begin{bmatrix} A & M \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} C & F \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}$$

The following result holds (proof by yourselves)

The pair

$$\bar{A} = \begin{bmatrix} A & M \\ 0 & I \end{bmatrix} \quad , \quad \bar{C} = \begin{bmatrix} C & F \end{bmatrix}$$

is observable if and only if the pair (A, C) is observable and

$$rank \begin{bmatrix} A - I & M \\ C & F \end{bmatrix} = n + r \quad \longleftrightarrow \quad r \leq p, d \in R^r$$

Under the previous conditions, it is possible to build an ovbserver to estimate, at any time instant $\begin{bmatrix} \hat{x}(k) \\ \hat{d}(k) \end{bmatrix}$

Then, assuming that the future disturbance is constant, $(d(k+i) = \hat{d}(k),\ i \geq 0)$, one can compute the equilibrium pair $(\bar{x}, \bar{u})$ such that

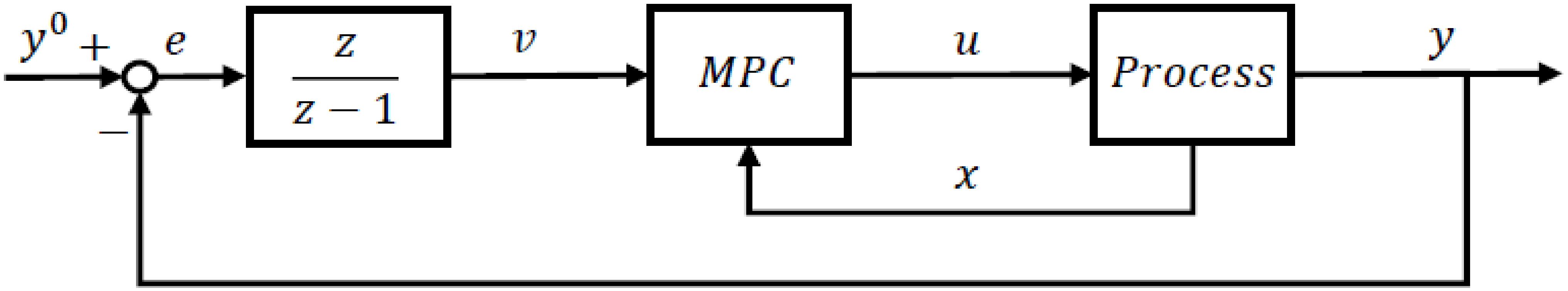
$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} M & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} \hat{d}(k) \\ y^o \end{bmatrix}$$

and use these values in the cost function (possibly with a weight on the states)

$$J(x(k), u(\cdot), k) = \sum_{i=0}^{N-1} \left(\|y^o - y(k+i)\|_Q^2 + \|u(k+i) - \bar{u}\|_R^2 \right) + \|y^o - y(k+N)\|_S^2$$

Integral action in MP – method 2 : EFFECTIVE INTEGRAL ACTION

The idea is to resort to the scheme



analogous to the schemes previously studied

Plant + integrators

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ v(k+1) &= v(k)+e(k+1) \end{aligned}$$

$$e(k)=y^o-y(k)$$

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ v(k+1) &= v(k)+y^o-Cx(k+1)=v(k)+y^o-CAx(k)-CBu(k) \end{aligned}$$

define $\begin{cases} \delta x(k)=x(k)-x(k-1) \\ \delta u(k)=u(k)-u(k-1) \end{cases}$

subtract the same equations at *k-1*

recall that $v(k+1)-v(k)=e(k+1)$

$$\begin{bmatrix} \delta x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -CA & I \end{bmatrix} \begin{bmatrix} \delta x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} B \\ -CB \end{bmatrix} \delta u(k)$$

system in velocity form

$$\begin{bmatrix} \delta x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -CA & I \end{bmatrix} \begin{bmatrix} \delta x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} B \\ -CB \end{bmatrix} \delta u(k)$$

reachable if (A,B) reachable and the system under control does not have transmission zeros at z=1

An interesting cost function

$$\min_{\delta u(k+i), i=0,\ldots,N-1} J(\delta x(k), \delta u(\cdot), k) = \sum_{i=0}^{N-1} \left(\|e(k+i)\|_Q^2 + \|\delta u(k+i)\|_R^2 \right) + \|e(k+N)\|_S^2$$

J=0 for ***e=0***, ***δu=0***, no computation of the steady state values of states and inputs is required

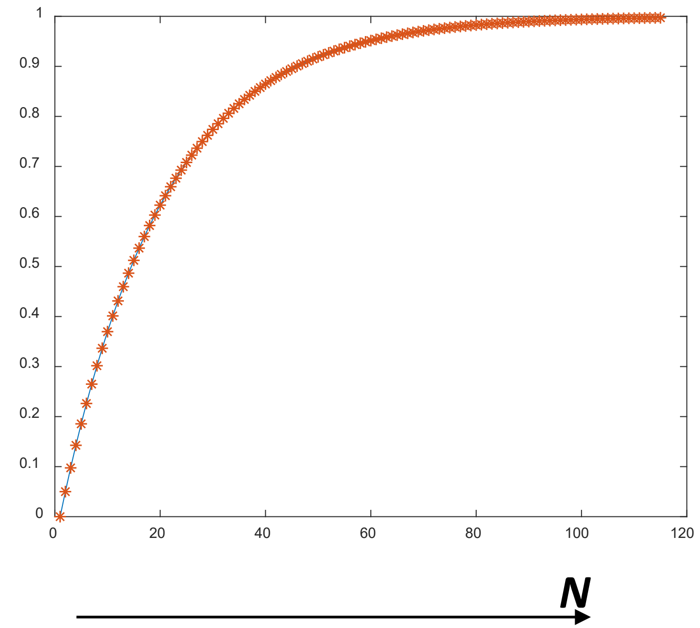
weight and constraints on ***δu*** have physical meaning, and are related to the «speed of variation» of the inputs (think to a valve). In addition, they can be used to speed up or slow down the control action by means of a proper selection of the bounds on ***δu***

Nce the optimal sequence of future control variations $\delta u^o(k+i),\ i=0,\ldots,N-1$ is computed, one sets

$$u(k) = u(k-1) + \delta u^o(k)$$

Choice of the prediction horizon *N*

In general, it is suggested to set ***N*** to cover the settling time of the process (in case of asymptotically stable systems) . It obviously depends on the adopted sampling period

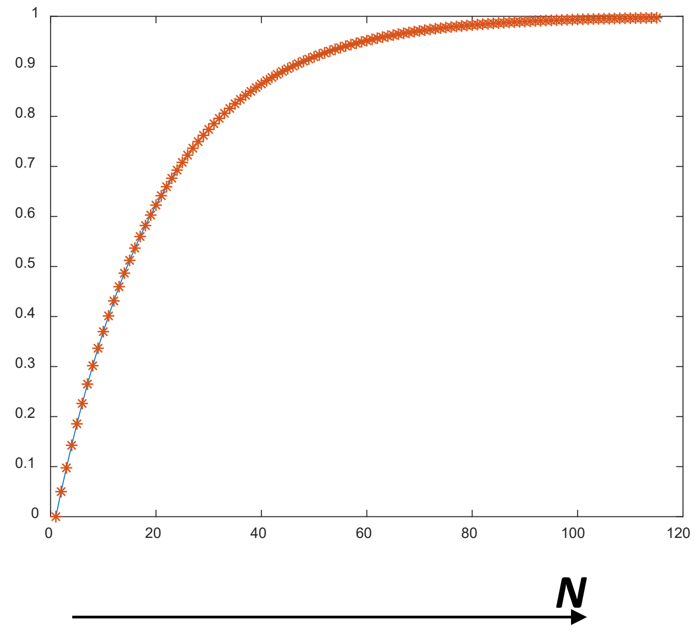


Empirical motivation: in the case of unconstrained systems (closed-loop solution), the Riccati equation converges and the RH-MPC control law is stabilizing

Problem: ***N*** is also the number of control inputs (multiplied by *m*) to be computed as the solution of the optimization problem, which can become very large and time consuming

Choice of the sampling period

It should be based on the Shannon theorem and on the (required) crossover bandwith in closed-loop



Rule of thumb

Define by *Ts* the settling time at 99% of the open loop step response

Select the sampling time ΔT so as to have 15-50 samples in the settling time

For first order systems, this roughly corresponds to consider a Nyquist frequency guaranteeing an attenuation of 20-30dB with respect to the frequency of the open-loop time constant

Control horizon N_u

Only the first $\boldsymbol{N_u}$ control variables are optimized, with

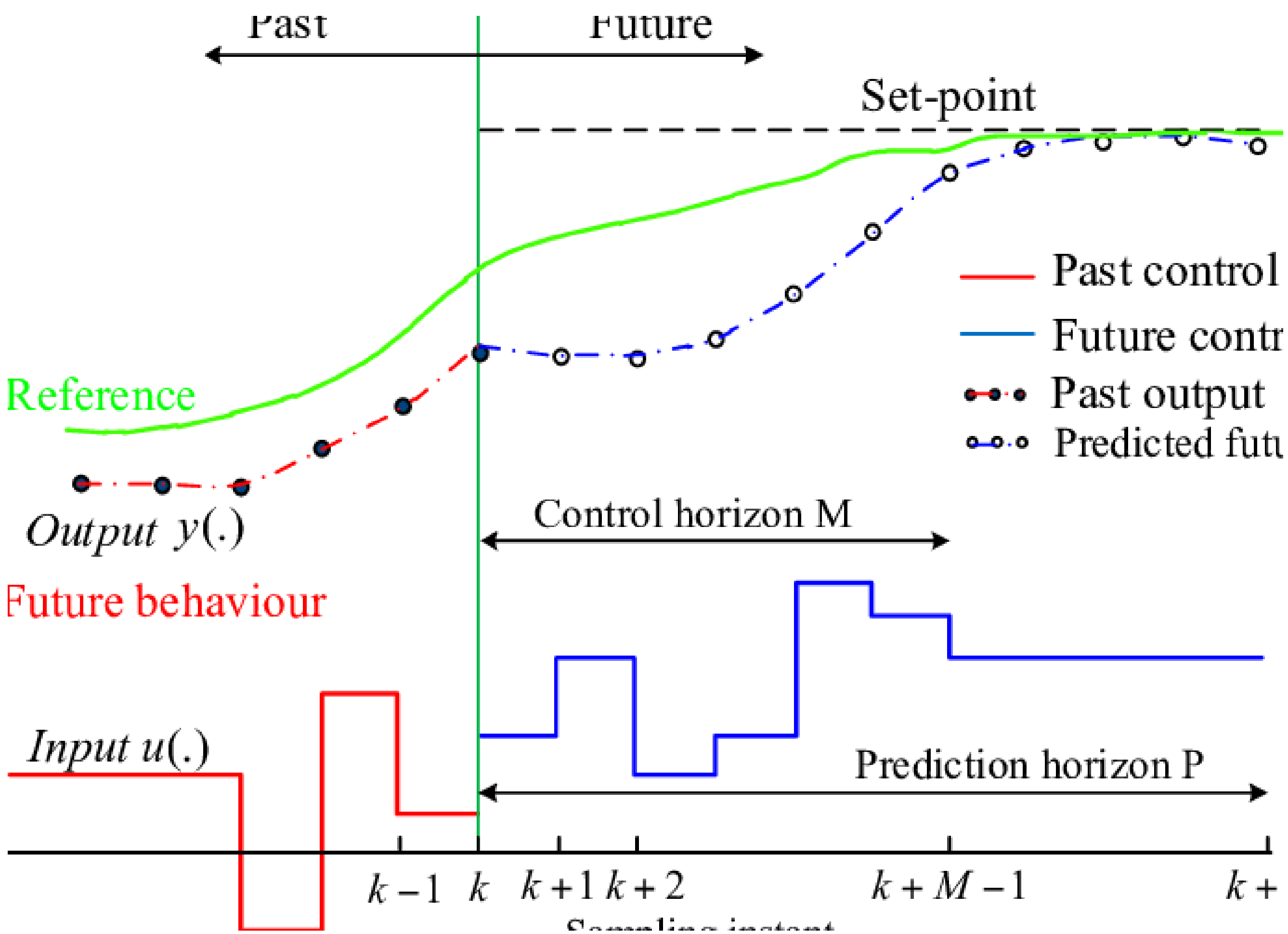
$$0<N_u<N$$

then it is assumed that the control is constant over the remaining part of the prediction horizon, i.e.

$$u(k+i)=u(k+i-1), \quad i=N_u,\ldots,N-1$$

or

$$\delta u(k+i)=0, \quad i=N_u,\ldots,N-1$$



Here $M = N_u$

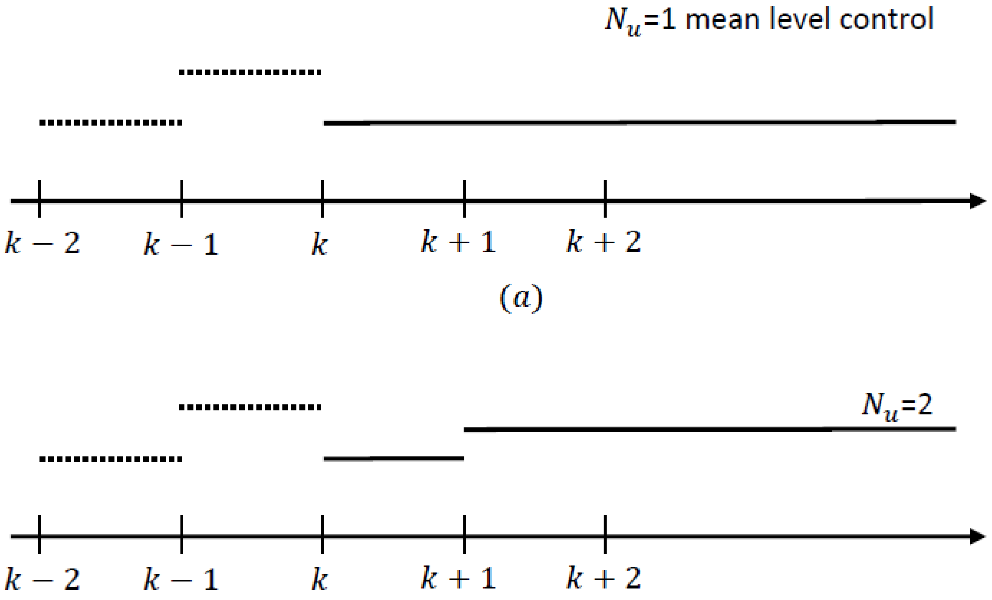
Cost function with control horizon N_u

Using a control horizon different from the prediction horizon implies to slightly modify the cost function (here we use the incremental one for systems in velocity form)

$$J(x(k),u(\cdot),k) = \sum_{i=0}^{N-1} \|y^o(k+i)-y(k+i)\|_Q^2 + \sum_{i=0}^{N_u-1} \|\delta u(k+i)\|_R^2 + \|y^o(k+N)-y(k+N)\|_S^2$$

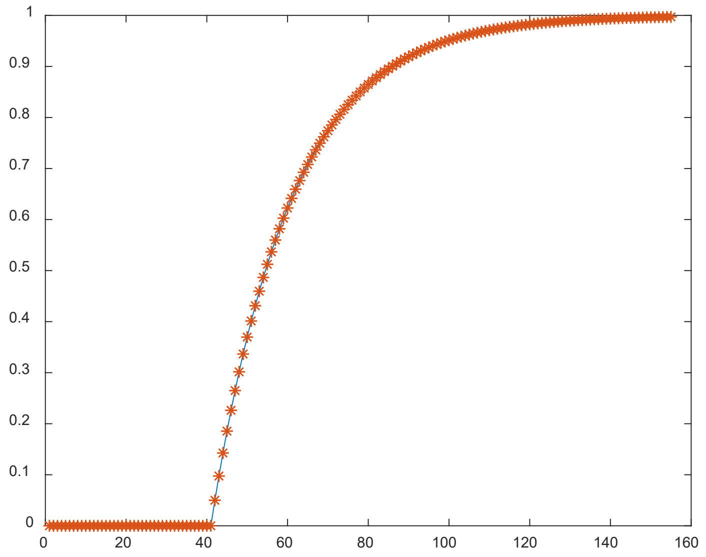
A particular case, usually named ***Mean Level Control***, is the one with ***N_u=1.*** Only one control move is allowed, but at every *k* the control value is changed due to the RH principle, see the figure

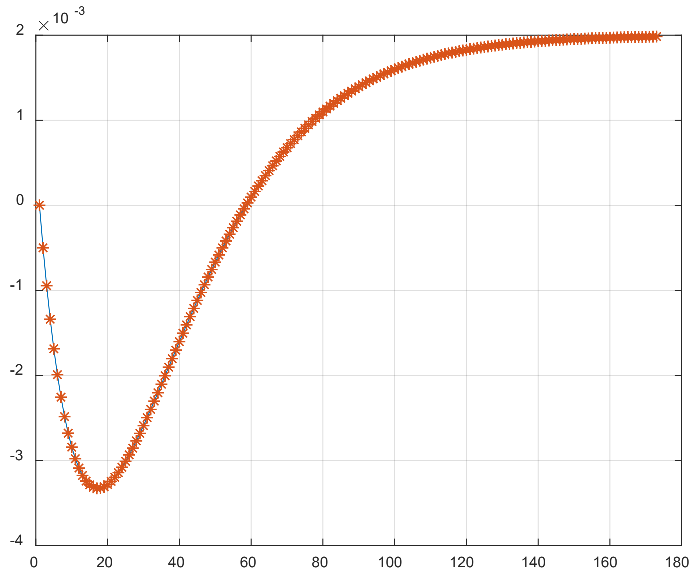
In general, short control horizons slow down the controlled system



Minimum prediction horizon

For nonminimum phase systems, with **delay** or **inverse response** due to unstable zeros, it is convenient not to penalize the future values of the output (or of the error), which would force the system to move in a wrong direction (the standard problem of non minimum phase systems which require a slow control action)



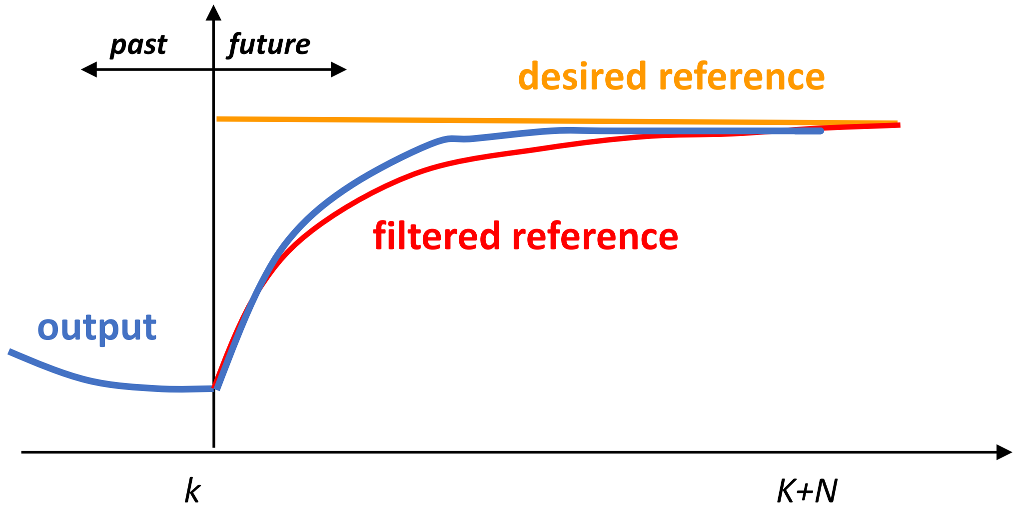


N_1 ***N_1***

$$J(x(k),u(\cdot),k)=\sum_{i=N_1}^{N-1}\|y^o(k+i)-y(k+i)\|_Q^2+\sum_{i=0}^{N_u-1}\|\delta u(k+i)\|_R^2+\|y^o(k+N)-y(k+N)\|_S^2$$

Filtering the reference signal

To reduce the control effort, step variations of the reference signal are avoided, and a filtered reference is computed at any time starting from the current output



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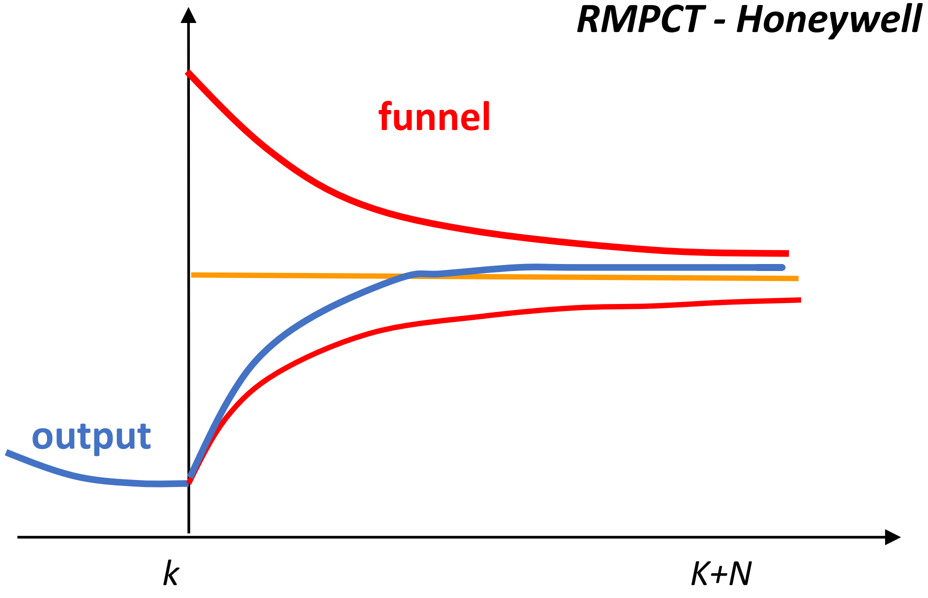
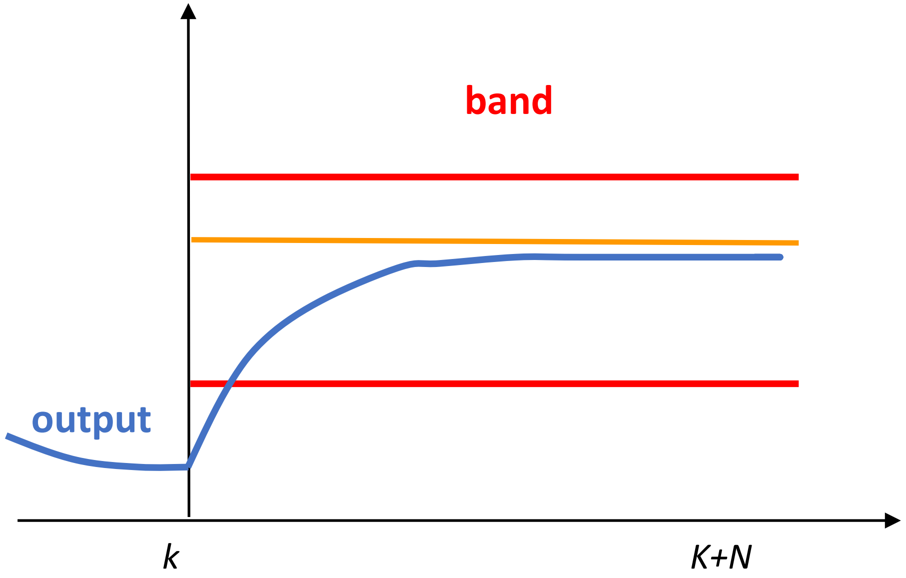
A possible signal generator is

$$y^o(k+i) = \left(1-\alpha^i\right)\bar{y}^o + \alpha^i y(k) \quad , 0 \leq \alpha < 1$$

Note that this causes an additional (external) loop which should be considered in the stability analysis

RMPCT - Honeywell





In many cases it is not mandatory to reach a steady state value, but to remain into prescribed limits. Then, instead of penalizing the error, it is possible to include in the optimization problem some hard (or soft) constraints. Recall that hard constraints on the output are difficult to be satisfied in view of the possible presence of disturbances.