Ex. 1: Given the scheme block in Figure 1

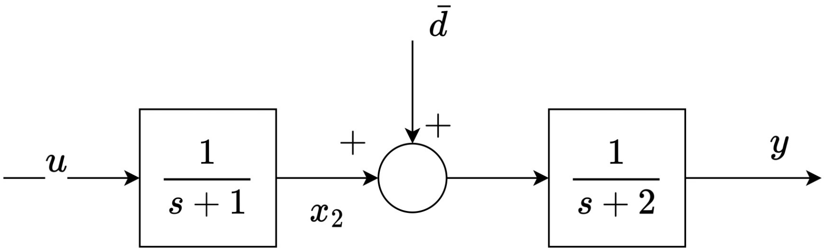


Figure 1

1. Design a reduced order observer in case $\bar{d}$ is a constant unknown disturbance.

OBSERVER: to get state estimate when the state cannot be measured

$\downarrow$

we start from S.S representation of the system

$$\begin{cases} X_2(s) = \frac{1}{s+1} u(s) \\ y(s) = \frac{1}{s+2}(X_2(s) + d(s)) \end{cases} \xrightarrow{t} \begin{cases} \dot{x}_2(t) = -X_2(t) + u(t) \\ \dot{y}(t) = -2y(t) + X_2(t) + d(t) \end{cases}$$

↳ NOT yet S.S representation!

$$y = x_1: \begin{cases} \dot{x}_1(t) = -2x_1(t) + x_2(t) + d(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ y(t) = x_1(t) \end{cases}$$

S.S representation

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0] \quad D = 0$$

$\downarrow$ to design the OBSERVER we have to check:

1) (A, C) OBSERVABLE $\quad \mathcal{M}_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rightarrow \text{rank}(\mathcal{M}_0) = 2$ OK ✓

OBS syst.

• condition to estimate $\bar{d}$

2) $p \geq n_d$ where $\begin{cases} p := \text{number of outputs} \\ n_d := \text{number of disturbances} \end{cases}$

Whe have $p=1=md$ OK✓ We can estimate $\bar{d}$

↓ to do this in practice:

we enlarge the state $d = x_3$ $\Rightarrow$ Enlarged S.S system

↓

dynamic of disturbance $\dot{x}_3 = \frac{d}{dt}(\bar{d}) = 0$

in MATRIX form:

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + x_2(t) + x_3(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ \dot{x}_3(t) = 0 \\ y(t) = x_1(t) \end{cases}$$

$$\tilde{A} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = [0]$$

3^{rd} ORD system

↓ Now we can estimate $\bar{d}$ by design OBS on enlarged system so 3^{rd} state obs. through all state estimation

We need to transform the system through $\Leftarrow$

T (non sing, square matrix) $\rightarrow$ $T = \begin{bmatrix} C \\ T_1 \end{bmatrix}$ apply the transformation!

↓

any matrix which makes T square non sing

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$ $\Leftarrow$ simple solution

easy computation!

$y = x_1$: our output is x_1 → so $x_1, \dot{x}_1$ are known since $y = x_1$

⇙ (measurable)

this allows us to properly collect $\eta = \dot{x}_1 + 2x_1$ ⤳ rewrite S.S system $\Rightarrow$

$$\begin{cases} \dot{x}_2 = -x_2 + u \\ \dot{x}_3 = 0 \\ \eta = x_2 + x_3 \end{cases}$$

defining the s.s system in this way. we have one order less

→ new s.s, on which we wanna design the state estimator:

$$A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_d = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D_d = 0$$

state observer construct from this matrix:

Dynamic of the observer $(\hat{y} = C_d \hat{x})$

$$\dot{\hat{x}} = \underbrace{A_d \hat{x} + B_d u}_{\text{prediction}} + \underbrace{L(y - \hat{y})}_{\text{error on the output}}$$

⇓ $y = \eta$ on our case

$$\dot{\hat{x}}_r = (A_d - LC_d)\hat{x}_r + B_d u + L\eta = (A_d - LC_d)\hat{x}_r + B_d u + L(\dot{y} - 2y)$$

$\hat{x}_r$: reduced syst. state

$$\begin{cases} \eta = \dot{x}_1 - 2x_1 \\ y = x_1 \end{cases} \longrightarrow \dot{y} - 2y = \eta$$

But we measure the state... ⇒ $\dot{\hat{x}}_r - L\dot{y} = (A_d - LC_d)\hat{x}_r + B_d u + 2Ly$

(OBSERVER)

add/subtract

$$\begin{cases} \dot{\hat{x}}_r - L\dot{y} = (A_d - LC_d)\hat{x}_r + B_d u + 2Ly \pm (A_d - LC_d)Ly \\ \xi = \hat{x} - Ly \\ \dot{\xi} = \dot{\hat{x}} - L\dot{y} \end{cases}$$

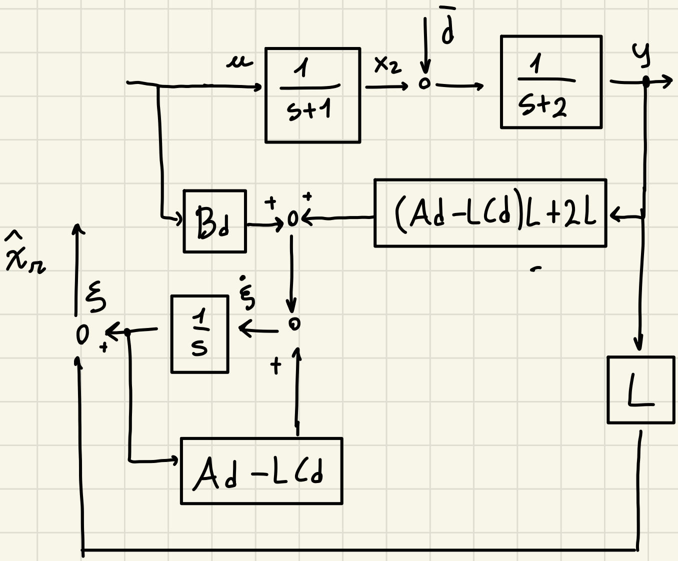
← we obtain a new dynamic

Known

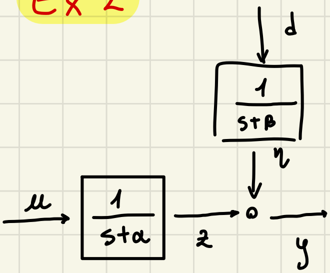
$$\dot{\xi} = (A_d - LC_d)\xi + (A_d - LC_d)Ly + B_d u + 2Ly$$

(observer dynamic) without $\dot{y}$

to implement that state obs on the syst:



Ex 2



$\alpha, \beta > 0$

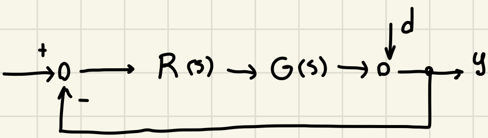
a) Show how to design a pole placement controller with integral action (placing all the poles in $s=-1$), directly from tf representation

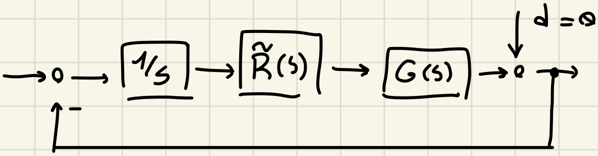
b) Show how to estimate the disturbance d in case it is constant and verify which conditions should be satisfied by α, β to do so

c) Use the estimated disturbance $\hat{d}$ to compensate d

$$G(s) = \frac{1}{s+\alpha}$$

a





$$\frac{1}{s}\tilde{R}(s)\,G(s) = \tilde{R}\,\boxed{1/s\;G(s)}$$

$\tilde{G}(s) :=$ (Enlarged system)

pole placement from T.F

$$\tilde{R}(s) = \frac{\delta_\nu s^\nu + \delta_{\nu-1} s^{\nu-1} + \ldots + \delta_1 s + \delta_0}{s^\nu + \gamma_{\nu-1} s^{\nu-1} + \ldots + \gamma_1 s + \gamma_0}$$

General formula of regulator T.F pole placement

1) ν: order of $\tilde{R}$

2) $\delta_\nu \ldots \delta_0$
$\gamma_{\nu-1} \ldots \gamma_0$ $\quad 2\nu - 1$ parameter

to design!

STEP 1: fix order of Regulator → we deduce the equations to find it:

$$L(s) = \tilde{R}(s)\cdot\tilde{G}(s) = \underbrace{\frac{\delta_\nu s^\nu + \dots + \delta_0}{s^\nu + \gamma_{\nu-1}s^{\nu-1} + \dots + \gamma_0}}_{\tilde{R}} \underbrace{\frac{1}{s(s+\alpha)}}_{\tilde{G}}$$

⇓

charact. polynomial $\varphi_L = N_L + D_L$ → degree of $\varphi_L = \underbrace{\nu}_{\#\text{ poles of } \tilde{R}} + \underbrace{m}_{\#\text{ poles of } G} + 1$

enlarge by integrator

⇓

$$\underbrace{2\nu + 1}_{\#\text{ param}} = \underbrace{\nu + m + 1}_{\#\text{ equation}}$$

← must be equal to have enough info → $\nu = m = 1$

$$\tilde{R} = \frac{\delta_1 s + \delta_0}{s + \gamma_0}$$ 1st order regulator

↓

to write char polynomial step 2

$$L(s) = \frac{s\delta_1 + \delta_0}{s + \gamma_0}\,\frac{1}{s(s+\alpha)}$$

$$\hookrightarrow \varphi_L(s) = \delta_1 s + \delta_0 + (s + \gamma_0)(s^2 + \alpha s) =$$

$$= 1s^3 + s^2(\alpha + \gamma_0) + (\delta_1 + \gamma_0\alpha)s + \gamma_0$$

(=)

⇒ All poles in (−1) ⇒ $\varphi^*(s) = (s+1)^3 = s^3 + 3s^2 + 3 + 1$

⇓

make equal the coefficents:

$$\begin{cases} \delta_0 = 1 \\ \delta_1 + \alpha\gamma_0 = 3 \\ \gamma_0 + \alpha = 3 \\ 1 = 1 \end{cases} \quad \begin{cases} \delta_0 = 1 \\ \delta_1 = 3 - \alpha\gamma_0 = 3 - \alpha(3-\alpha) \\ \gamma_0 = 3 - \alpha \end{cases}$$

⇓

$$\boxed{\tilde{R} = \frac{(\alpha^2 - 3\alpha + 3)s + 1}{s + (3-\alpha)}}$$

(pole placement T.F)

(for higher order system → HARD by hand)

⇓

so there is another formulation → when reg order increases we use **MATRIX FORMULATION**

$$\tilde{G}(s) = \frac{\tilde{B}}{\tilde{A}} = \frac{1}{s(s+\alpha)} = \frac{1}{s^2+s\alpha+0}$$

we consider a matrix... where we place the coeff in a column vector $\begin{bmatrix}1\\ \alpha\\ 0\end{bmatrix}$

coefficients of $\tilde{A}$ — coeff of $\tilde{B}$ — Denom. of $\tilde{R}$ — num. of $\tilde{R}$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ \alpha & 1 & 0 & 0\\ 0 & \alpha & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ \gamma_0\\ \delta_1\\ \delta_0\end{bmatrix} = \begin{bmatrix}1\\ 3\\ 3\\ 1\end{bmatrix}$$

desired coefficents, desired characteristic polynomial $\varphi^*(s) = (s+1)^3$

(fill the rest with 0)

build with num/den of $\tilde{R}$

⇐ (matrix form)

↓

$Fx = b \Rightarrow x = F^{-1}b$

(F invertible)! ↑ we must be sure F^{-1} exist!

ⓑ estimate disturbance

↓ done as before... ⇒ $\bar{d}$ estimation

$$\begin{cases}\dot{z} = -\alpha z + u\\ \dot{\eta} = -\beta\eta + d\\ y = z+\eta\end{cases} \qquad \begin{cases}\dot{x} = Ax + Bu + Md\\ y = Cx + Du\end{cases}$$

↓

conditions to build an observer on the system

1) (A, C) observable

2) $p \geq md \rightarrow p = md = 1$ ok

$$C = \begin{bmatrix}1 & 1\end{bmatrix}$$

$$A = \begin{bmatrix}-\alpha & 0\\ 0 & -\beta\end{bmatrix}$$

to check observability

$$M_o = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 1\\ -\alpha & -\beta\end{bmatrix} \quad \forall \alpha, \beta \mid \alpha \neq \beta \quad \text{OK}$$

enlarge the s.s

$$\begin{cases} \dot{z} = -\alpha z + u \\ \dot{\eta} = -\beta \eta + d \\ \dot{d} = 0 \\ y = z + \eta \end{cases} \longrightarrow A_e = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\beta & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B_e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_e = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

↓

• Build OBS dynamic:

$$\dot{\hat{x}}_e = A_e \hat{x}_e + B_e u + L(y - C\hat{x}_e)$$

$$\dot{\hat{x}}_e = (A_e - LC_e)\hat{x}_e + B_e u + Ly$$

in analogy with pole-placement...

⇓

equivalence..

$$(A - LC)^T \leftrightarrow (A - BK)$$

same...

↓

so we can define L^T through matlab place function

P.P	OBS.
A	A^T
B	C^T
K	L^T

OBS dyn to get disturbance extimation

Ex. 3: Given the continuous time system

$$\begin{cases} \dot{x}(t) = -x(t) + u(t) \\ y(t) = x(t) \end{cases} \tag{1}$$

and the Riccati Differential Equation

Given during exam!

$$\dot{P}(t) + A^T P(t) + Q - P(t)BR^{-1}B^T P(t) + P(t)A = 0 \tag{2}$$

1. Find the LQ_∞ control law with $Q = 1$, $R = 1$.
2. Find the corresponding closed-loop poles, the closed loop T.F., the maximum gain variation and evaluate the phase margin.
3. Design a steady-state Kalman Filter with $\tilde{Q} = \rho^2$, $\tilde{R} = 1$.
4. Compute the overall LQG regulator T.F.
5. Show how to apply the loop transfer recovery procedure (LTR).

1) LQ_∞ solved through Riccati equation (given)

Gain found through Riccati eq

state neg feedback control law $\rightarrow u = -K_{LQ}\, x$

(as pole placement control law) $\Rightarrow$ to apply LQ_∞ control

$\downarrow$ check conditions:

1) (A,B) Reachable $\rightarrow$ from s.s compute $M_R \Rightarrow M_R = [B] = 1$: Full Rank! OK ✓ reachable

2) (A, C_q) Observable

where C_q defined such that

$\{C_q^T \cdot C_q = Q\} \rightarrow$ to compute it: Q is a design choice we can easily define it as diagonal

$\Downarrow$

$$Q = \begin{bmatrix} q_1 & & 0 \\ & q_2 & \\ & & \ddots & \\ 0 & & & q_m \end{bmatrix} \rightarrow C_q = \begin{bmatrix} \sqrt{q_1} & & 0 \\ & \sqrt{q_2} & \\ & & \ddots & \\ 0 & & & \sqrt{q_m} \end{bmatrix}$$

since Q (s.p.d) $\leftarrow$

$\Downarrow$

$\forall i = 1, \dots, m \quad q_i \geq 0$

(spd := all eig values of matrix ≥ 0) $\rightarrow$ eig lays on matrix diag values

$Q = 1 \rightarrow C_q = 1 \quad \rightarrow M_o = [C_q] = 1$ OK (A, C_q) obs ✓

conditions satisfyed → we can compute the gain of our control law

K_{LQ} can be found by Riccati eq → Gain K_{LQ} ?

Solve the steady state Riccati eq.

$\Rightarrow \dot{\bar{P}}=0 \qquad A^T\bar{P}+Q-\bar{P}B\underbrace{R^{-1}B\bar{P}}_{K_{LQ}}+\bar{P}A=0$

In our case
$A=[-1]$
$B=[+1]$
$R=Q=[+1]$

$\rightarrow -\bar{P}+1-\bar{P}^2-\bar{P}=0 \Rightarrow -\bar{P}^2-2\bar{P}+1=0$

$\hookrightarrow \begin{cases} \bar{P}=-1+\sqrt{2} \\ \cancel{\bar{P}=-1-\sqrt{2}} \end{cases}$ { $\bar{P}$ must be pos. def. }

We find LQ gain

$K_{LQ}=-1+\sqrt{2}$

control law } $u=-(-1+\sqrt{2})x$

(on MATLAB)
$K_{LQ}=$ lqr (A,B,Q,R)
to compute it after checking required condition!

② closed loop poles, T.F, $\underbrace{\text{max gain var, } \varphi m}$
↳ ROBUSTNESS theory guarantee

on pole placement, poles are given as: $(A-BK)$

CLOSED LOOP poles

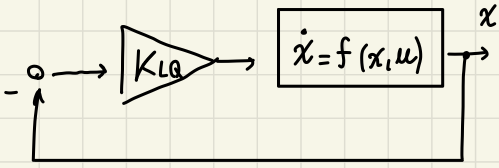
$\rightarrow \dot{x}=-x+u \rightarrow \dot{x}=-x-K_{LQ}x$

$\dot{x}=(-1-K_{LQ})x$

$\dot{x}=(-1+1-\sqrt{2})x=-\sqrt{2}x$

with LQ we don't define pole position...

CLOSE LOOP T.F



$\dot{x}=-x+u$
↓
$sx=-x+u$
$x=\boxed{\frac{1}{s+1}}u$ system T.F

$y=x$
$\rightarrow y/u=\frac{1}{s+1}$

While the loop T.F $L_{LQ}(s)=K_{LQ}\cdot G=\frac{-1+\sqrt{2}}{s+1}$

↳ so the closed loop T.F ⇒

$$T_{LQ} = \frac{L_{LQ}}{1+L_{LQ}} = \frac{\frac{-1+\sqrt{2}}{s+1}}{1+\frac{-1+\sqrt{2}}{s+1}} = \frac{-1+\sqrt{2}}{s+\sqrt{2}}$$

pole closed loop in $(-\sqrt{2})$ as expected!

$\downarrow$

ROBUSTNESS (closed loop)

- Gain variation $\rightarrow$ we can parametrize control law

$$u = \underbrace{-\rho K_{LQ}}_{\tilde{K}_{LQ}} x \rightarrow \text{new loop T.F} \quad \tilde{L}_{LQ} = \tilde{K}_{LQ} G =$$

$$\tilde{T}_{LQ} = \frac{\frac{\rho(-1+\sqrt{2})}{\cancel{s+1}}}{\frac{\rho(-1+\sqrt{2})+s+1}{\cancel{s+1}}} = \frac{\rho(-1+\sqrt{2})}{s+(1-\rho+\rho\sqrt{2})} =$$

$$= \frac{\rho(-1+\sqrt{2})}{s+(1+\rho(\sqrt{2}-1))}$$

ROBUSTNESS against gain variation... taking pole of our system such that pole remain neg!

$$(1+\rho(\sqrt{2}-1)) > 0 \qquad \boxed{\rho > -\frac{1}{\sqrt{2}-1}} \simeq -2.4$$

max gain variation to maintain A.S

LQ control guaranteed for max gain variation for $\rho > 0.5$ (apriori) $\leftarrow$ conservative bound

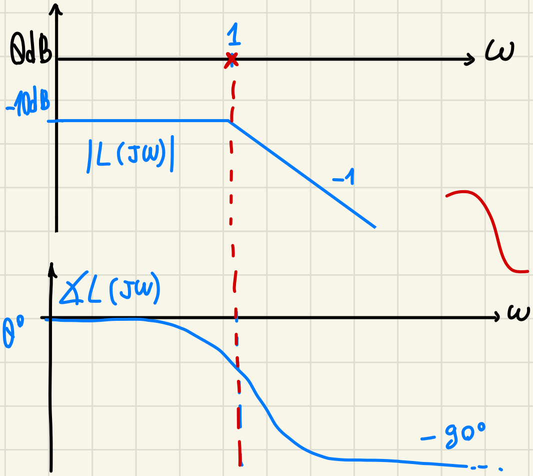
we have proven as valid for $(\rho > -2.4)$

We can do better than theory

- **phase margin** $\quad L_{LQ} = \frac{-1+\sqrt{2}}{s+1} \longrightarrow$ static gain $-1+\sqrt{2} \cong 0.4$

$$\frac{1}{s+\alpha} = \left[\frac{1}{1+s/\alpha}\right]\frac{1}{\alpha}$$

$\hookrightarrow$ Bode diagram of $|L(j\omega)| \Rightarrow$



to find phase margin..

$$\varphi_m = -180 - \measuredangle L(j\omega_c) = -180° - \varphi_{\text{cutting freq}}$$

on cut off frequency!

→ nyquist does not exit unitary circle → $\varphi_m = +\infty$

for theory we are guaranteed $\varphi_m > 60°$ ↗ OK as theory told us! conservative property

Ex. 3: Given the system

$$\dot{x}(t) = 0.5x(t) + u(t)$$

1. Find the LQ_∞ control law with $Q = 1,\ R = 1$.
2. Find the corresponding closed-loop poles.
3. Given $u(t) = -\rho K_{LQ} x(t)$, find the set of ρ for which the closed loop system is A.S.
4. Find the phase margin.
5. Which is the maximum time-delay that allows to maintain the asymptotic stability?
6. Enforce a closed loop pole faster than $s = -2$

Exercises session 5: Reduced order observer, LQ control, LQG and loop transfer recovery

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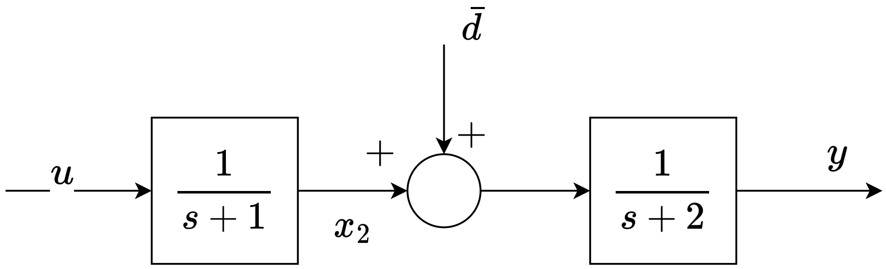

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