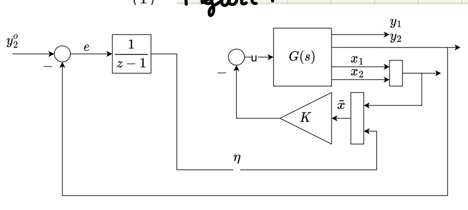
Ex. 1: Given the discrete-time system

$$G(z) = \begin{bmatrix} \dfrac{z-1}{(z-0.5)^2} \\ \dfrac{z}{(z-0.5)^2} \end{bmatrix} \qquad (1)$$

Figure 1



and the state space representation

$$\begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{cases}$$

Find

1. Find the poles and zeros of G(s),
2. Determine how many outputs can be regulated to constant references,
3. Verify there are no invariant zeros in z=1, using the state space model.
4. Assume the states are measurable, show how to design a pole placement scheme guaranteeing zero error robust regulation (refer to Figure 1)
5. In case the state are not measurable, can we use a static observer?

① poles / zeros $\begin{cases} \text{S.S} \\ \text{T.F} \end{cases}$ 2 ways of evaluate it.

(POLES) through minors (of order 1)

$$\begin{cases} M_1 = \dfrac{z-1}{(z-0.5)^2} \\ M_2 = \dfrac{z}{(z-0.5)^2} \end{cases}$$

$\varphi(s) = (z-0.5)^2 \longrightarrow z_{1,2} = 0.5$ Poles

(ZEROS)

$r_m = 1$ (normal rank) $\Rightarrow$ NO ZEROS (taking the $\varphi(s)$ and common...)

② We can use state enlargment to regulate output to constant reference

↳ ENLARGMENT:

$$\begin{cases} p \le m : \begin{Bmatrix} p: \#\text{ outputs} \\ m: \#\text{ inputs} \end{Bmatrix} \rightarrow 2 \le 1 \text{ NO} \\ \text{NO (invariant ZEROS) derivative action} \end{cases}$$

← condition to check for enlargment!

We can control at least one output?

↳ neglecting one output, What happen to the other? ⇒

(Neglecting y_2)

NO

$$G(z) = \left[\frac{z-1}{(z-0.5)^2} \right]$$

ZERO in $z=1$: Derivative action! → y_1 CANNOT be controlled to 0 tracking error

(Neglecting y_1)

OK

$$G(z) = \left[\frac{z}{(z-0.5)^2} \right]$$

YES ✓ it is possible to control y_2 to zero error at regime..

We need to recompute the ZERO of reduced system

↳ we obtain 2 systems with one zero each!

- another way to check IF controllable output → {through static gain matrix}

considering our system subject to a step $K \to \infty$ (acting for $k \to \infty$)

$$y_\infty = G(z=1)\, u_\infty \quad \text{where} \quad G(z=1) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

0 ← no influence of our input on Y_1

4 ← here yes! Y_2 is influenced by input! → so we can control Y_2 to constant ref!

{same result}

③ Verify there are NO invariant zero in one, through s.s. model

↓

in DISCRETE time:

$$\begin{cases} x(k+1) = A x(k) + B u(k) \\ y(k) = C x(k) + D u(k) \end{cases} \rightarrow \begin{cases} z\, x(k) - A x(k) - B u(k) = 0 \\ y(k) - C x(k) - D u(k) = 0 \end{cases}$$

↳ by collecting $x(k)$ and rearrange

$$\begin{cases} (zI - A)\, x(k) - B u(k) = 0 \\ -C x(k) - D u(k) + y(k) = 0 \end{cases}$$

↓ From here, the SYSTEM MATRIX:

$$\boxed{\begin{bmatrix} (zI-A) & -B \\ -C & -D \end{bmatrix}} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} y(k) = 0$$

$P(z)$: used to compute invariant ZEROS of our syst:

invariant zeros, values of z for which $P(z)$ lose rank!

$\text{rank}(P(z)) < r_m$? → by using det of the matrix

We check for $z=1$

$\left\{ \text{rank}(P(z)) \text{ for a general } z \to r_m = 3 \right\}$

$r_m = 3$

$$P(z=1) = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0.25 & 0 & -1 \\ \hline 1 & -1 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$\text{rank}(P(1)) = 3$

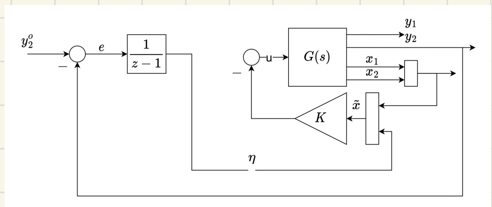
FULL RANK → NO invariant ZERO

(4) IF measurable states, pole placement regulator which guarantee 0 error robust reg

↓

We should design the control on enlarged system

(STEP 1) Enlarge the system using new variables...



$\eta(k) = \frac{1}{z-1} e(k)$ → computed backward from scheme block

$z\eta(k) - \eta(k) = e(k)$

↳ $\eta(k+1) = \eta(k) + e(k)$ → error: $e(k) = y_2^o(k) - y_2(k)$

We are controlling 2^{nd} output

$\eta(k+1) = \eta(k) + y_2^o(k) - y_2(k)$

$$\begin{cases} \eta(k+1) = \eta(k) - \underbrace{C_2}_{\text{from s.s}} x(k) + \overbrace{y_2^o(k)}^{\text{disturbance}} \\ x(k+1) = Ax(k) + Bu(k) \end{cases}$$

{ENLARGED system}

↳ (MATRIX FORM)

$$\begin{bmatrix} x(k+1) \\ \eta(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C_2 & I \end{bmatrix}}_{\tilde{A}\text{ : enlarged state matrix}} \begin{bmatrix} x(k) \\ \eta(k) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}\text{ : enlarged output}} u(k) + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\tilde{M}} y_2^o(k)$$

↓

FROM here we place the poles... $K = \text{place}(\tilde{A}, \tilde{B}, \tilde{z})$ (by Matlab)

POLE PLACEMENT possible if $(\tilde{A}, \tilde{B})$ is Reachable

$\Updownarrow$ IFF

(A, B) is Reachable

to design p.p

5 state observer (IF states non measurable ?!) remove hyp

$\downarrow$

Estimate $x(k)$ using $y(k)$ inverting the model

$$y(k) = \underbrace{C x(k)}_{\text{invert matrix } C} + \cancel{Du}(k) \to 0$$

invert matrix $C \Rightarrow$ so if C squared & NON singular!

$C^{-1} \exists \Leftrightarrow C \begin{cases} \text{squared} \\ \text{NON singular} \end{cases}$

given

$$C = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

YES! we can use a STATIC OBSERVER inverting matrix C

$$C^{-1} = -\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to C^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{\det(C)}$$

Ex. 2: Given the regulated system in Figure 2, the regulator matrix

$$R(s) = \begin{bmatrix} 1 & 1 \\ 0 & -\dfrac{s-0.5}{s+1} \end{bmatrix}$$

and transfer function

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s-0.5} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

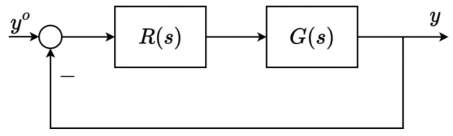


Figure 2

1. Compute the poles and zeros of $G(s)$,
2. Is it enough to check the stability of $S(s) = (I + G(s)R(s))^{-1}$ to check the stability of the closed loop?
3. Specify which transfer functions should be considered to asses the stability of the closed-loop system.

(1) from the minors

POLES

$$M_{22} = \frac{1}{(s+1)(s+2)} \rightarrow \varphi(s) = (s+1)(s+2)(s-0.5) \text{ charact. polynomial}$$

↓

3 POLES $\begin{cases} s=-1 \\ s=-2 \\ s=0.5 \end{cases}$

ZEROS

from normal rank

$r_m = 2$

$$M_{22} = \frac{1}{(s+1)(s+2)} \cdot \frac{(s-0.5)}{(s-0.5)} \; \varphi(s) \rightarrow$$ max common divisors

1 ZERO : $s = 0.5$

(2) enough $S(s)$ to check closed loop stability?

↓

check STABILITY from T.F possible only IF NO hidden cancellation on the closed loop system → to use only one 1 T.F to check Asymp stability!

$$L(s) = G(s)R(s) \neq L_u = R(s)\cdot G(s)$$

(MATRIX PRODUCT!)

$$= \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s-0.5} \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -\dfrac{s-0.5}{s+1} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & -\dfrac{s+0.5}{(s+1)(s+2)} \end{bmatrix}$$

to check for critical cancellation.. compute poles of it:

$$M_{22} = \frac{s-0.5}{(s+1)^2(s+2)}$$

respect minor of order 1 --> this identify characteristic polym.

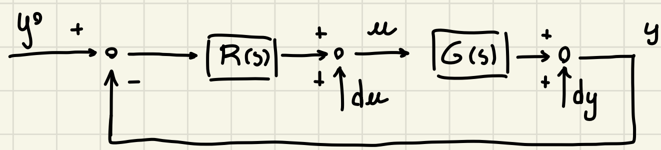
→ Poles: $s_{1,2} = -1$, $s_3 = -2$ vs original syst pole in 1/2, zero in 1/2 there was an hidden cancellation! $\longrightarrow$ so it is NOT enough to check stability from one T.F

$\Rightarrow$ NO it is NOT possible
because there is a forbidden cancellation in $L(s) = G(s) \cdot R(s)$

Remind... Which are the T.F of a MIMO system

↳ overall 5 T.F characterizing MIMO syst. $\rightarrow$

$$\begin{cases} S(s) = (I + L(s))^{-1} & \text{SENSITIVITY} \\ T(s) = (I + L(s))^{-1} L(s) & \text{COMPLEMENTARY SENSITIVITY} \\ \dfrac{Y(s)}{du} = (I + L(s))^{-1} G(s) & \\ \dfrac{U(s)}{du} = (I + Lu)^{-1} & \\ \dfrac{U(s)}{dy} = (I + Lu)^{-1} R(s) & \end{cases}$$

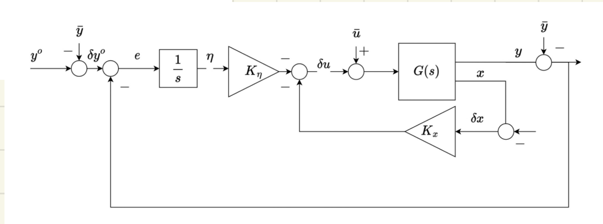


Ex. 3: Consider the continuous time system

$$\begin{cases} \dot{x} = x^3 - x \cdot u^2 \\ y = x \end{cases} \tag{6}$$

and the equilibrium point $\bar{x} = 1$, $\bar{u} = 1$. Design a pole placement controller that allows asymptotically stable equilibrium in two ways:

1. Pole placement to stabilize and a cascade PI for performances,
2. Pole placement on the enlarged system.



Pole placement → for LINEAR SYSTEM!

{ FIRST we have to linearize around equilibrium }

$\bar{x} = 1 \quad \bar{u} = 1$ LINEARIZE : (1)

$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}} \delta u = (3\bar{x}^2 - \bar{u}^2)\delta x - 2\bar{x}\bar{u}\,\delta u =$$

unstable linearized syst

$$\begin{cases} \delta\dot{x} = 2\,\delta x - 2\,\delta u \\ \delta y = \delta x \end{cases}$$

We stabilize it by pole placement

T.F of linearized syst.

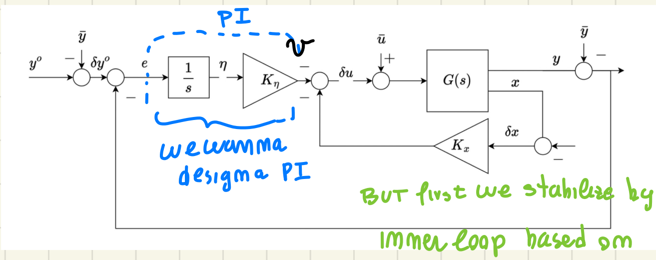
⇓

$$\begin{cases} s\,\delta x - 2\,\delta x = -2\,\delta u \\ \delta y = \delta x \end{cases} \rightarrow \delta y = \boxed{-\frac{2}{s-2}}\,\delta u$$

T.F of linearized syst $= G(s)$

on this T.F as we have an unstable pole, we cannot design a simple PI

↳ We design a POLE PLACEMENT controller



BUT first we stabilize by an inner loop based on pole placement

$v \to y$ stabilized

⇓

than design PI controller for v!

POLE PLACEMENT

↳ IF (A,B) pair is Reachable → simple SISO system

$M_R = B = -2$ — OK REACHABLE! full Rank → we can apply P.place in SISO we can do pole placement computation by hand

↳ compute explicitly the CONTROL LAW

$\Downarrow$

compute $\bar{K}$: $\quad \delta u = \underbrace{-K\delta x}_{\text{POLE PLACEM.}} + v$

exogenous signal cause pole placem. inside inner loop coming from PI

(LIN. SYST.)

$$\delta\dot{x} = 2\delta x - 2(-K\delta x + v)$$

$$\delta\dot{x} = \underbrace{(2+2K)}_{\uparrow}\delta x - 2v$$

pole of our system → compute K ⇒ to place the pole where we want!

for example,

place the pole in $s=-1$

↳ $2+2K=-1 \Rightarrow K = -3/2$

↓

once stabilized the system

$$\begin{cases} \delta\dot{x} = -\delta x - 2v \\ \delta y = \delta x \end{cases}$$

T.F of new inner loop?

$\Downarrow$

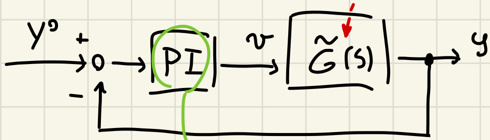
$$\begin{cases} s\delta x + \delta x = -2v \\ \delta y = \boxed{-\dfrac{2}{s+1}}\, v \end{cases}$$

linearized T.F of new syst.

$\| \quad \tilde{G}(s)$

STABLE INNER LOOP (pole in -1) ←

{ our syst. becomes }



PI track the ERROR to 0

⇝ NOW simply design PI for performance ..

② to ENLARGE the system: We check 2 conditions:

ok conditions, we can enlarge ← $\begin{cases} p \leq m & 1 \leq 1 \text{ OK} \\ \text{NO DERIVATIVE ACTION} & \text{OK} \end{cases}$

$$G(s) = -\frac{2}{s-2}$$

↓

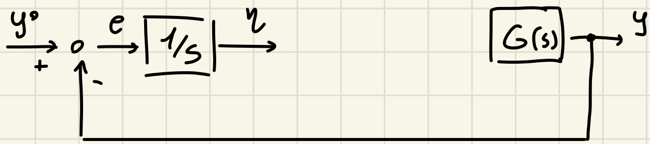
{ We wanna enlarge with integrator because usefull on MIMO cases where the PI tuning is HARD! }

ENLARGMENT is safe option to track reference

↓

step 1) compute new η dynamic

$$\eta = \frac{1}{s} e$$



↓

$$s\eta = e = y^\circ - y \quad \searrow$$

$$\dot{\eta} = y^\circ - y = y^\circ - Cx \qquad \text{(New dynamic)}$$

$$\underbrace{\begin{bmatrix} \delta\dot{x} \\ \dot{\eta} \end{bmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} \delta x \\ \eta \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\tilde{M}} y^\circ$$

[Enlarged Representation]

↓ check Reach of enlarged matrix!

$$(\tilde{A}, \tilde{B}) \text{ Reach} \Leftrightarrow \underbrace{(A, B) \text{ Reach}}$$

YES! as we prove before!

So we can apply pole placement and compute Gains:

$$\dot{\tilde{x}} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y^\circ$$

→ in this case, pole placem is the MATRIX of GAINS K

$$\delta u = \begin{bmatrix} -K_1 & -K_2 \end{bmatrix} \begin{bmatrix} \delta x \\ \eta \end{bmatrix} = -K_1 \delta x - K_2 \eta$$

substituting into MATRIX we can compute K_1, K_2 explicitly:

$$\dot{\tilde{x}} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} (-K_1 \delta x - K_2 \eta) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y^o =$$

$\downarrow$

on S.S representation

$$\begin{cases} \delta \dot{x} = 2\delta x + 2K_1 \delta x + 2K_2 \eta = (2+2K_1)\delta x + 2K_2 \eta \\ \dot{\eta} = -\delta x + y^o \end{cases}$$

from here we compute K_1, K_2 such that the poles are where we desire!

we get from here:

$$\begin{bmatrix} \delta \dot{x} \\ \dot{\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} 2+2K_1 & 2K_2 \\ -1 & 0 \end{bmatrix}}_{\bar{A} = [\tilde{A} - \tilde{B}K]} \begin{bmatrix} \delta x \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y^o$$

this MATRIX is syst. matrix of enlarged syst with pole placem.

$\bar{A}$ is a 2x2 MATRIX... we can place its pole as looking on its minors

characteristic polynomial of $\bar{A}$, placing $\{S_{1,2} = -1\}$ TARGET VALUES!

$$\varphi^*(s) = (s+1)^2$$

$\downarrow$

$$\varphi(s) = \det[(sI - \bar{A})] = \det\begin{bmatrix} s-2-2K_1 & -2K_2 \\ 1 & s \end{bmatrix} =$$

$$= s^2 - (2+2K_1)s + 2K_2 = \varphi^*(s) \text{ TARGET}$$

same coefficients

$$s^2 - (2+2K_1)s + 2K_2 = s^2 + 2s + 1 \rightarrow \begin{cases} -2-2K_1 = 2 \\ 2K_2 = 1 \end{cases} \rightarrow \begin{cases} K_1 = -2 \\ K_2 = 0.5 \end{cases}$$

pole placem Gains!

Ex. 4: Consider the following open loop transfer functions

$$L(s) = \begin{bmatrix} \frac{1}{\alpha s + 1} & \frac{\beta}{s+2} \\ \frac{\gamma}{\alpha s + 1} & \frac{\delta}{s+1} \end{bmatrix} \tag{7}$$

1. Compute the poles and zeros of the system
2. Which conditions must be imposed on α, β , γ and δ to study the stability of the closed loop system just analyzing $S(s) = (I + L(s))^{-1}$?
3. Which conditions must be imposed on α, β, γ and δ to design a closed loop controller with integral action?

Exercises session 4: MIMO Analysis, Pole Placement and Zero Error regulation

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$$G(z) = \begin{bmatrix} \dfrac{z-1}{(z-0.5)^2} \\ \dfrac{z}{(z-0.5)^2} \end{bmatrix} \tag{1}$$

and the state space representation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.251 & \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{2}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{3}$$

Find

1. Find the poles and zeros of G(s),
2. Determine how many outputs can be regulated to constant references,
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4. Assume the states are measurable, show how to design a pole placement scheme guaranteeing zero error robust regulation (refer to Figure 1)
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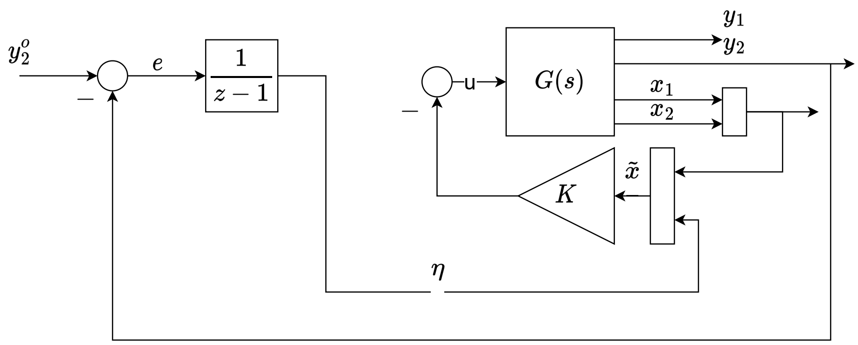

Figure 1

Ex. 2: Given the regulated system in Figure 2, the regulator matrix

$$R(s) = \begin{bmatrix} 1 & \frac{1}{s-0.5} \\ 0 & -\frac{s-0.5}{s+1} \end{bmatrix} \tag{4}$$

and transfer function

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-0.5} \\ 0 & \frac{1}{s+2} \end{bmatrix} \tag{5}$$

1. Compute the poles and zeros of $G(s)$,
2. Is it enough to check the stability of $S(s) = (I + G(s)R(s))^{-1}$ to check the stability of the closed loop?
3. Specify which transfer functions should be considered to asses the stability of the closed-loop system.

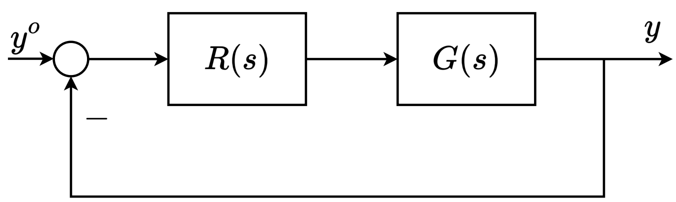

Figure 2

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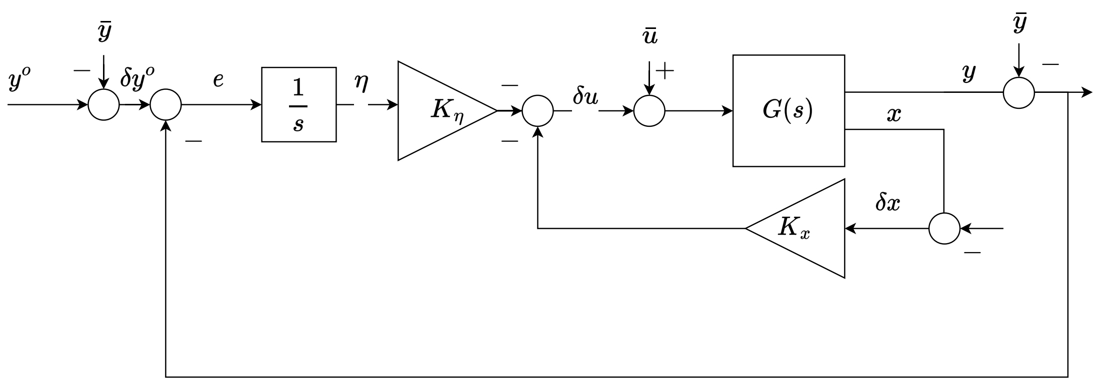

Figure 3

Ex. 4: Consider the following open loop transfer functions

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