



Exercises session 6: LQ control, KF, LQG, loop transfer recovery

Ex. 1: Given the continuous time system

$$\begin{cases} \dot{x}(t) = -x(t) + u(t) \\ y(t) = x(t) \end{cases} \quad (1)$$

and the Riccati Differential Equation

$$\dot{P}(t) + A^T P(t) + Q - P(t) B R^{-1} B^T P(t) + P(t) A = 0 \quad (2)$$

1. Find the LQ_∞ control law with $Q = 1$, $R = 1$.
2. Find the corresponding closed-loop poles, the closed loop T.F., the maximum gain variation and evaluate the phase margin.
3. Design a steady-state Kalman Filter with $\tilde{Q} = \rho^2$, $\tilde{R} = 1$.
4. Compute the overall LQG regulator T.F.
5. Show how to apply the loop transfer recovery procedure (LTR).

Ex. 2: Given the system

$$\dot{x}(t) = 0.5x(t) + u(t) \quad (3)$$

1. Find the LQ_∞ control law with $Q = 1$, $R = 1$.
2. Find the corresponding closed-loop poles.
3. Given $u(t) = -\rho K_{LQ} x(t)$, find the set of ρ for which the closed loop system is A.S.
4. Find the phase margin.
5. Which is the maximum time-delay that allows to maintain the asymptotic stability?
6. Enforce a closed loop pole faster than $s = -2$

Ex. 3: Given discrete time system

$$\begin{cases} x(k+1) = -x(k) + u(k) + \nu_x(k) \\ y(k) = x(k) + v_y(k) \end{cases} \quad (4)$$

where $\nu_x \sim WGN(0, \tilde{Q})$, $\nu_y \sim WGN(0, \tilde{R})$

1. Find the LQ_∞ control law with $Q = 1$, $R = 2$.
2. Compute the maximum gain variation allowed by the LQ_∞ .
3. Design a Kalman Filter using $\tilde{Q} = 2$, $\tilde{R} = 1.5$.
4. Compute the regulator T.F.
5. Compute the closed loop poles.

The steady-state Riccati equation for discrete time systems is

$$\bar{P} = Q + A^T \bar{P} A - A^T \bar{P} B (R + B^T \bar{P} B)^{-1} B^T \bar{P} A \quad (5)$$

Ex. 4: Given the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ y = x_2 \end{cases} \quad (6)$$

1. Design a LQ_∞ control law with

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (7)$$

and $R = 1$.

2. Compute the closed loop poles.
3. Design a Kalman Filter and apply the LTR procedure.

Ex. 4: Given the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ y = x_2 \end{cases}$$

1. Design a LQ_∞ control law with

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and $R = 1$.

2. Compute the closed loop poles.
3. Design a Kalman Filter and apply the LTR procedure.

... complete

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \quad u = -\sqrt{2} x_1 - x_2 = -K_{LQ} x$$

2

$$\dot{x} = Ax + Bu = Ax - BK_{LQ} x = \underbrace{(A - BK_{LQ})} x$$

$$A - BK_{LQ} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix}$$

$$\varphi(s) = \det(sI - (A - BK_{LQ})) = \det \begin{bmatrix} s + \sqrt{2} & 1 \\ -1 & s \end{bmatrix} = s^2 + \sqrt{2}s + 1$$

$$s_{1,2} = -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} j$$

Re part neg! stable

↓ solving this polynomial we find the poles...

③ K.F + LTR

$\tilde{R} = 1$ $\tilde{Q} = \rho B \tilde{R} B^T$ helps to recovery stability of LQ control

$$= \rho \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \text{ diagonal} \Rightarrow B_q = \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & 0 \end{bmatrix}$$

to design a K.F we should check the conditions...

CONDITIONS FOR K.F

1. (A, C) OBSERVABLE ✓
2. (A, B_q) REACHABLE ✓

I can take B_q from \tilde{Q}

(checking OBS, REACH. matrix)

↑

$$\mathcal{M}_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ } \det(\mathcal{M}_0) \neq 0 \text{ clearly Full Rank } \checkmark \text{ } (A, C) \text{ OBS}$$

$$\mathcal{M}_R = \begin{bmatrix} \tilde{B}_q & A \tilde{B}_q \end{bmatrix} = \begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\rho} & 0 \end{bmatrix} \text{ } \text{rank}(\mathcal{M}_R) = 2 = \text{sys ORD} \text{ } \underline{\text{OK}} \text{ REACHABLE}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & 0 \end{bmatrix}$$

↓

OK → we can apply K.F

$$\dot{\hat{x}} = A \hat{x} + B u - \tilde{L}_{KF} (y - \hat{y})$$

solved by K.F Riccati eq.

$$\text{R.E. (KF): } A \tilde{P} + \tilde{Q} - \tilde{P} C^T \tilde{R}^{-1} C \tilde{P} + \tilde{P} A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

↓

we need to define a parametric

$$\tilde{P} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \text{ } \left. \begin{array}{l} \text{symm.} \\ \text{SILVESTER} \\ \text{CRITERION} \end{array} \right\} \text{ to ensure pos. def}$$

... computation ...

$$\tilde{P} = \begin{bmatrix} \sqrt{2} \sqrt[4]{\rho^3} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2} \sqrt[4]{\rho} \end{bmatrix}$$

$$L_{KF} = \tilde{P} C^T \tilde{R}^{-1} = \begin{bmatrix} \curvearrowright \\ // \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{r}^{-1} = \begin{bmatrix} \sqrt{\rho} \\ \sqrt{2} \sqrt[4]{\rho} \end{bmatrix}$$