1

(3 punti)

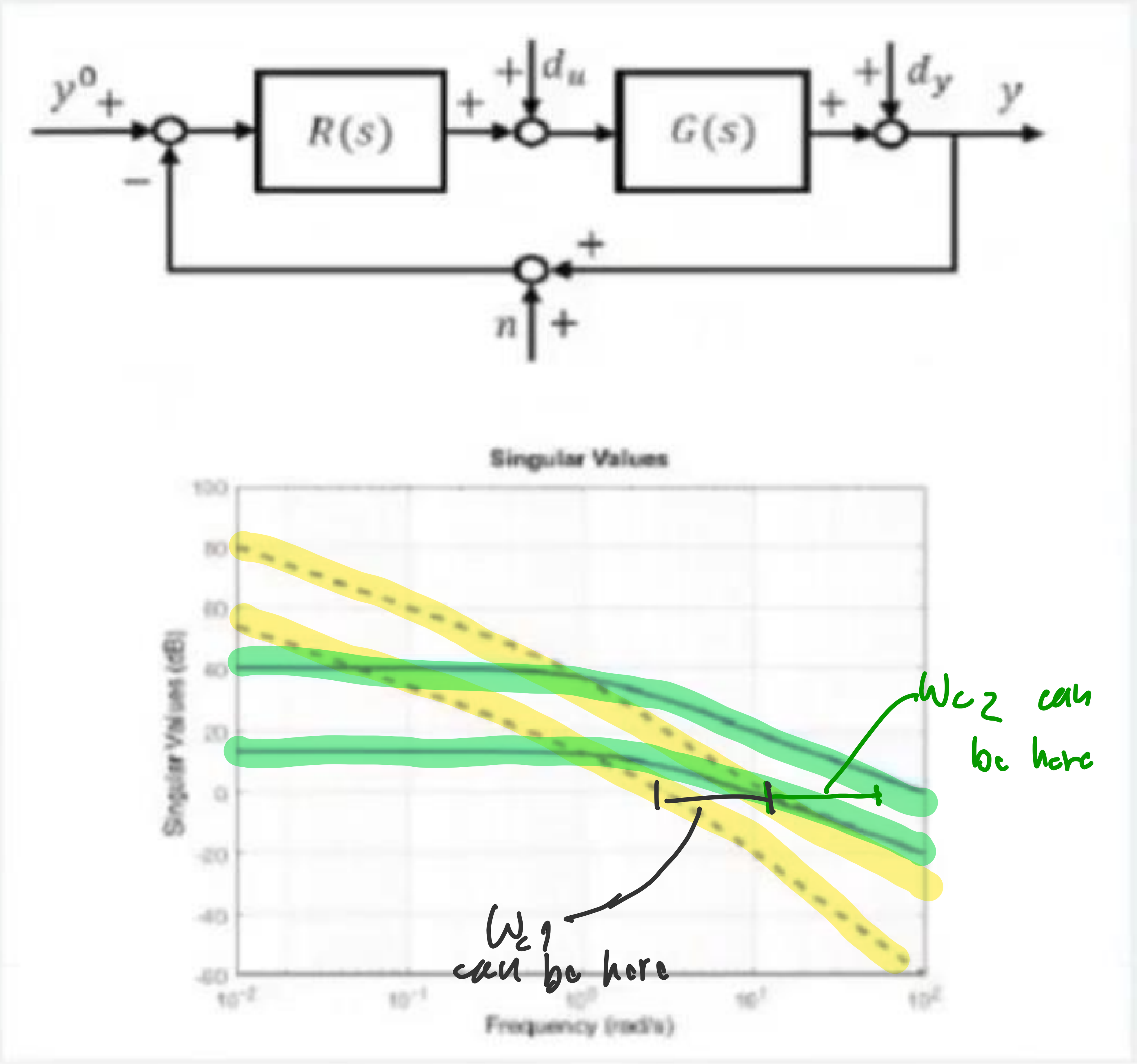
Consider a second order dynamic system with state $x = [x_1\ x_2]'$, described by $\dot{x} = f(x)$ and $f(0) = 0$. For this system, take a Lyapunov function $V(x)$ globally positive definite and radially unbounded. Assume also that $\dot{V}(x) = (x_2^2 - 1)x_1^2$. The the origin is an equilibrium:

- ◯ unstable
- ◯ locally asymptotically stable
- ◯ globally asymptotically stable
- ☒ stable

$\dot{V}(x) = x_1^2 x_2^2 - x_1^2 \leq 0$

↳ We can have $\dot{V}(x) = 0$ even if $x_2 \neq 0$ (hola Sallo)

2



Consider the feedback system reported in the block diagram and two possible loop transfer functions L1(s)=R1(s)G1(s) and L2(s)=R2(s)G2(s) with the principal gains reported in the figure. Let L1(s) be associated to the dashed lines and L2(s) to the continuous ones. Select the true answer among the following ones:

(3 punti)

- ◯ Assuming that at low and high frequency all the singular values diagrams have the same slope shown in the figure, it is likely that the two loop transfer functions do not have poles at the origin. FALSE → L1 has $\rho > 0$ (so poles in the origin)
- ◯ L1(s) and L2(s) have roughly the same crossover frequency. → FALSE It doesn't make sense to talk about crossover freq. for MIMO (They should have more than one!)
- ◯ At high frequency (ω > 20) the guaranteed attenuation of the noise disturbance n provided by L2(s) is always greater than the one guaranteed by L1(s). FALSE → $\overline{\sigma}_2 > \overline{\sigma}_1$ in $\omega > 20$
- ☒ At low frequency (ω < 1) the guaranteed attenuation of the disturbance dy provided by L1(s) is always greater than the one guaranteed by L2(s). → TRUE $\underline{\sigma}_1 > \underline{\sigma}_2 \geq L$ (limit)

3

Assume to have a linear system with n states and $\overset{?}{\overbrace{n-1}}$ outputs. Then, it is possible to design with pole placement a regulator of order 1 such that all the eigenvalues of the closed-loop system are in prescribed positions.
(3 punti)

- Regulator order is usually of order $n-p = n-n+1 = 1$ ✓
- The sys. must be in minimal form → $G(s)$ only describe reach. and obs. part of the sys.

No, the regulator must be of order n

yes provided that the system is stabilizable and detectable
↳ we can design a stable regulator but we'll be able to move the eigenvalues only of the reachable and obs. part

yes, provided that an estimator of unknown disturbances is also used

~~Yes, provided that the system is reachable and observable~~

4

(3 punti)

Consider the system described by

Poles: $\begin{cases} -5 \\ -5 \\ -5 \\ -1 \end{cases}$

$\varphi(s) = (s+5)^3(s+1)$

Invariant Zeros: (Normal rank order = 1)

$\begin{cases} M_1 = \dfrac{(s+1)^2(s+\alpha)}{\varphi(s)} \\ M_2 = \dfrac{2(s+5)^2}{\varphi(s)} \end{cases}$
↳ no zeros (for $\alpha \neq 5$)

$$\underset{2\times 1}{G(s)} = \begin{bmatrix} \dfrac{(s+1)(s+\alpha)}{(s+5)^3} \\ \dfrac{2}{(s+1)(s+5)} \end{bmatrix}$$

$p=2$
$m=1$

and select the *wrong* statement

~~the zeros are s=-1, s=-alfa~~

for a generic value of alfa, it is not possible to jointly bring the two outputs to arbitrary constant values must be $p \leq m$ $p = 2 \nleq m = 1$ → TRUE

the poles are s=-1, s=-5 (multiplicity 3) → TRUE

for alfa=0, only the second output can be asymptotically regulated to an arbitrary constant value
DERIVATIVE ACTION (zero in $\emptyset$) → TRUE

$$G(s) = \begin{bmatrix} \dfrac{(s+1)s}{(s+5)^3} \\ \dfrac{2}{(s+1)(s+5)} \end{bmatrix}$$

5

(3 punti)

Consider the system

$$\dot{x}(t) = 2x(t) + u(t)$$

$A=2 \quad B=1$

Sys.: (for $y=x$)

$G(s) = \frac{1}{s-2}$

For this system compute the infinite horizon LQ control law with $Q = 2.25$, $R = 1$.
The Riccati equation for LQ control is

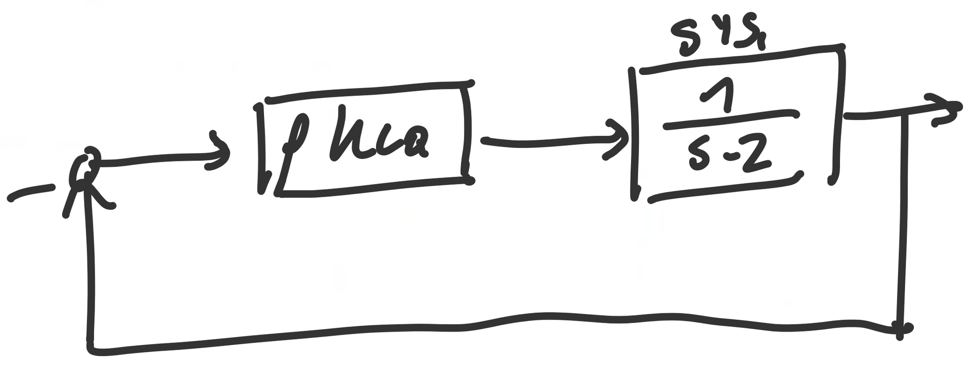
$$-\dot{P}(t) = A'P(t) + P(t)A + Q - P(t)BR^{-1}B'P(t)$$

The corresponding closed-loop system is characterized by:

- [x] pole in -2.5, gain margin (0.445,inf)
- [ ] pole in -2.5, gain margin (4.5,inf)
- [ ] pole in -4.5, gain margin (2,inf)
- [ ] pole in -2.5, gain margin undefined since the open loop system is unstable

LQ

- Conditions: (A,B) reach. $M_R = B = 1$ ✓
  (A, C_Q) obs. $M_O = C_Q = \sqrt{2.25}$ ✓
- Solve the steady-state RE:
  $0 = 4\bar{P} + 2.25 - \bar{P}^2 \rightarrow \bar{P} = \frac{9}{2} = 4.5$ or $-\frac{1}{2}$ ✗
- Controller gain
  $K_{LQ} = R^{-1}B'\bar{P} = \frac{9}{2}$
- Closed-loop
  $\text{eig}(A - BK_{LQ}) = 2 - \frac{9}{2} = -2.5 < 0$ (A.S.)



- Gain margin
  $T(s) = \dfrac{\frac{\rho K_{LQ}}{s-2}}{\frac{s-2+\rho K_{LQ}}{s-2}} \rightarrow$ Pole: $s - 2 + \rho K_{LQ}$
  To be A.S. $2 - \rho K_{LQ} < 0$
  $\rightarrow \rho > \frac{2}{9} \cdot 2 = \frac{4}{9}$
  0.444

6

In model order reduction methods:
(3 punti)

- [x] It is possible to guarantee a prescribed approximation error expressed in terms of the Hinf norm of the difference between the real and the approximate transfer function
  $\| G(s) - G_a^k(s) \|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_{k+n})$
- [ ] with balanced residualization the approximation error is smaller than in balanced truncation (for a fixed order of the approximant)
- [ ] the procedure can be applied only to transfer functions of a regulator computed with H2 or Hinf control
- [ ] both balanced truncation and balanced reasidualization guarantee that the steady state gain is preserved

just this guarantees it

7

In Model Predictive Control stability of the origin can be obtained by including in the problem formulation
(3 punti)

- [x] a zero terminal constraint and no weight on the final state
- [x] a quadratic terminal constraint and a quadratic weight on the final state
- [ ] a zero terminal constraint and a control horizon smaller than the prediction horizon (no sense)
- [ ] a quadratic terminal constraint and a linear weight on the final state

Control horizon is usually introduced to simplify the computations

8

Domanda
(5 punti)

• Enlarged the state: $\tilde{x} = \begin{bmatrix} x \\ a \end{bmatrix}$

• a fictious dynamics: $a(k+1) = a(k)$

• sys
$$\begin{cases} x(k+1) = x(k) + ax(k)u(k) + v_x(k) \\ a(k+1) = a(k) \\ y(k) = x(k) + v_y(k) \end{cases}$$

Consider the system

non linear

$$x(k+1) = x(k) + \underbrace{ax(k)u(k)}_{\text{non linear}} + v_x(k)$$
$$y(k) = x(k) + v_y(k)$$

where a is an unknown, but constant parameter, and v_x, v_y satisfy the standard assumptions of Kalman filtering. Show how to implement an Extended Kalman Predictor to estimate x and a, including the computation of the matrices $A(k)$, $C(k)$ (Riccati equation not required).

EKF: $\hat{\tilde{x}}(k+1|k) = f(\hat{\tilde{x}}(k|k-1), u(k)) + \bar{L}(k)\left(y(k) - g(\hat{\tilde{x}}(k|k-1))\right)$

with $f(\hat{\tilde{x}}(k|k-1), u(k)) = \begin{bmatrix} \hat{x} + \hat{a}\hat{x}u(k) \\ \hat{a} \end{bmatrix}$ $\quad g(\hat{\tilde{x}}(k|k-1)) = \hat{x}(k|k-1)$

$(\hat{a} = \hat{a}(k|k-1) \quad \hat{x} = \hat{x}(k|k-1))$

with $A(k) = \left.\dfrac{\partial f(\tilde{x}, u)}{\partial \tilde{x}}\right|_{\hat{\tilde{x}}}$ $\quad C(k) = \left.\dfrac{\partial g(\tilde{x}, u)}{\partial \tilde{x}}\right|_{\hat{\tilde{x}}}$

$A(k) = \begin{bmatrix} 1 + \hat{a}u & \hat{x}u \\ 0 & 1 \end{bmatrix}$

$C(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}$

9

(7 punti)

Consider the discrete time system

$A = 1 \quad B = -2$

$$x(k+1) = x(k) - 2u(k) + d(k)$$
$$y(k) = x(k)$$

for this system

(1) design with pole placement a stabilizing regulator guaranteeing closed-loop poles in $z = 0.5$;
(2) design with LQ_∞ a stabilizing controller. Consider a generic Q and $R = 8Q$.
(3) Show how to estimate the constant, but unknown disturbance d.

steady state Riccati equation:
$$P = A'PA + Q - A'PB(R + B'PB)^{-1}B'PA$$

(1) • conditions (A,B) reach. $M_R = B = -2$ ✓

• We'll have that $\text{eig}(A - BK) = 0.5$ so here (scalar) $A - BK = 0.5$

$2K = -0.5 \rightarrow K = -0.25$ ✓

We could have also use Ackerman Formula:

$P(z) = z - 0.5 \rightarrow P(A) = A - 0.5 = 1 - 0.5 = 0.5$

$K = [1] M_R^{-1} P(A) = \dfrac{1}{-2} \cdot 0.5 = -0.25$ ✓

(2) Q and $R = 8Q$

- Conditions

(A,B) reach ✓

(A,C_q) reach ✓ $(Q>0 \rightarrow C_q = \sqrt{Q} > 0)$

- Soln steady-state RE

$$\cancel{\bar{P}} = \cancel{\bar{P}} + Q - \frac{4\bar{P}^2}{8Q + 4\bar{P}} \longrightarrow \frac{\bar{P}^2}{2Q + \bar{P}} - Q = 0$$

$$\bar{P}^2 - \bar{P}Q - 2Q^2 = 0 \qquad P = \begin{cases} \frac{Q + \sqrt{Q^2 + 8Q^2}}{2} = 2Q \\ < 0 \end{cases}$$

- controller gain

$$K_{LQ} = \frac{-4Q}{16Q} = -\frac{1}{4} = -0.25$$

(3) - Fictitious dist. dynamics

$d(k+1) = d(k)$

- Enlarged the sys.

$$\begin{cases} x(k+1) = x(k) + d(k) - 2u(k) \\ d(k+1) = d(k) \\ y(k) = x(k) \end{cases} \qquad \tilde{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Condition

$\rightarrow ((\tilde{A}, \tilde{C})$ obs since (A,C) obs.)

Then we can use a reduced order obs. since $y(k) = x(k)$

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} = \begin{bmatrix} y(k) \\ x_r(k) \end{bmatrix} \longrightarrow \begin{cases} y(k+1) = y(k) + d(k) - 2u(k) \\ d(k+1) = d(k) \end{cases}$$

$\hookrightarrow \eta(k) = y(k+1) - y(k) + 2u(k)$ (known)

- Reduced order sys. $\begin{cases} d(k+1) = d(k) \\ \eta(k) = d(k) \end{cases}$ $\quad \bar{A} = 1 \quad \bar{C} = 1$

- Reduced order obs.: $\hat{d}(k+1) = \hat{d}(k) + L(\eta(k) - \hat{d}(k)) = (1-L)\hat{d}(k) + \eta(k)$

with $L = \text{place}(\overset{A'}{\overset{\shortparallel}{1}}, \overset{C'}{\overset{\shortparallel}{1}}, [\text{poles}])$

Obs. We could have also use the normal obs. which is better than the reduced order one that has a slighty filter action.

$\hookrightarrow$ In that case the obs. was $\hat{x}(k+1) = \tilde{A}\hat{x}(k) + \tilde{B}u(k) + L(y(k) - \tilde{C}\hat{x}(k))$

$L = \text{place}(\tilde{A}', \tilde{C}', [\text{poles}])$