June 2013

Ex 1

a) Linearized model

$$\delta \dot{x}_1 = -\alpha \delta x_1 + \vartheta \delta x_2$$

$$\delta \dot{x}_2 = -\vartheta \delta x_1 - \gamma \delta x_2$$

$$\delta \dot{x}_3 = -\xi \delta x_3$$

$$A = \begin{vmatrix} -\alpha & \vartheta & 0 \\ -\vartheta & -\gamma & 0 \\ 0 & 0 & \xi \end{vmatrix} \rightarrow sI - A = \begin{vmatrix} s+\alpha & -\vartheta & 0 \\ \vartheta & s+\gamma & 0 \\ 0 & 0 & s+\xi \end{vmatrix}$$

$$\det(sI - A) = (s+\xi)\left(s^2 + (\alpha+\gamma)s + \alpha\gamma + \vartheta^2\right)$$

Conditions for stability (asymptotic)

$$\xi > 0$$

$$\alpha + \gamma > 0$$

$$\alpha\gamma + \vartheta^2 > 0$$

b) Lyapunov theory

Use the quadratic function $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3$$

$$\dot V = -\alpha x_1^2 - \beta x_1^3 - \gamma x_1^2 x_2 + \vartheta \cancel{x_1 x_2} - \gamma x_1 x_2^2$$

$$- \varepsilon x_2^3 - \vartheta \cancel{x_1 x_2} - \eta x_2^2 + \gamma x_1 x_2 x_3 - \zeta x_3^2$$

$$+ \varepsilon x_2^2 x_3 =$$

$$= -\alpha x_1^2 - \eta x_2^2 - \zeta x_3^2 + \text{higher order terms}$$

$\dot V < 0$ for $\alpha > 0, \ \eta > 0, \ \zeta > 0$

$\boxed{\text{Ex 2}}$

multiplicative uncertainty

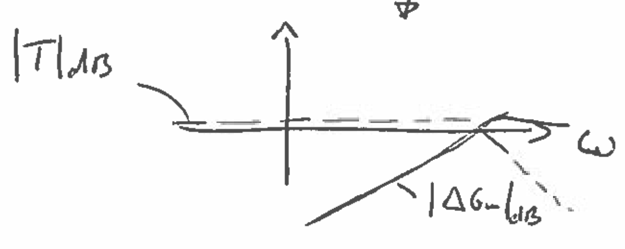
$$G(s) = \bar G(s) \cdot \left[1 + \Delta G_m(s)\right]$$

$$\frac{1}{\cancel{(1+sT)}(1+\alpha s)} \cdot \cancel{(1+sT)} = 1 + \Delta G_m(s)$$

$$\Delta G_m(s) = \frac{1}{1+\alpha s} - 1 = \frac{-\alpha s}{1+\alpha s}$$

Letting $T(s) = \dfrac{R(s)\bar G(s)}{1+R(s)\bar G(s)}$ the condition

for robust stability is $\|T \cdot \Delta G_m\|_\infty < 1$



additive uncertainty

$$G(s) = \bar{G}(s) + \Delta G_a(s) \rightarrow \Delta G_a(s) = \frac{1}{(1+sT)(1+\alpha s)} - \frac{1}{1+sT}$$

$$\Delta G_a(s) = \frac{-\alpha s}{(1+sT)(1+\alpha s)}$$

choice of W_S, W_T → see the notes

Ex 3

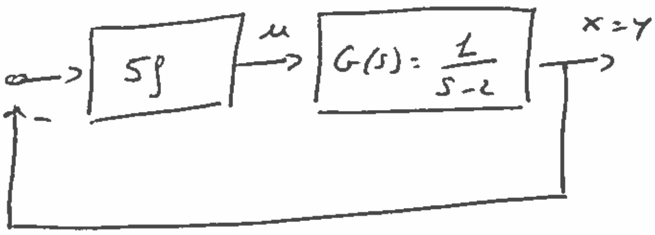
$A = 2,\ B = 1,\ Q = 5,\ R = 1$

eq. for control $\quad A'P + PA + Q - PBR^{-1}B'P = 0$

$$4P + 5 - P^2 = 0 \longrightarrow P = 5 > 0$$

$$K = R^{-1}B'P = 5 \longrightarrow A - BK = 2 - 5 = -3$$

gain margin



nominal case: $\rho = 1$ (ρ: gain variation)

characteristic eq. in perturbed conditions:

$$s - 2 + 5\rho = 0 \rightarrow s = 2 - 5\rho < 0$$

↑ for stability

$$\rho > 2/5 = 0.4 \quad (\text{smaller than } 0.5!)$$

With the observer with gain $L = \rho B$ the overall regulator is

$$\begin{cases} u(t) = -K\hat{x}(t) \\ \dot{\hat{x}}(t) = A\hat{x}(t) - BK\hat{x}(t) + \underbrace{\rho B}_{L}\left[y(t) - C\hat{x}(t)\right] \end{cases}$$

or $R(s) = K\left(sI - A + BK + LC\right)^{-1}L$

$$= \frac{5\rho}{s - 2 + 5 + \rho} = \frac{5\rho/3+\rho}{1 + \frac{1}{3+\rho}s}$$

if $\rho \to \infty$ then $R(s) = \frac{5}{1 + \alpha s}$, $\alpha \to 0$

that is $R(s) \longrightarrow 5$ as in the state feedback case

(LTR procedure)

$\boxed{\text{Ex 4}}$

See the notes

$\boxed{\text{Ex 5}}$

1. The disturbance can have at most $r = p$ components.
The enlarged model is $(v = w = 0)$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Md(t) \\ \dot{d}(t) = 0 \\ y(t) = Cx(t) + Nd(t) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \begin{vmatrix} \dot{x}(t) \\ \dot{d}(t) \end{vmatrix} = \begin{vmatrix} A & M \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x(t) \\ d(t) \end{vmatrix} + \begin{vmatrix} B \\ 0 \end{vmatrix} u(t) \\ y(t) = |C \quad N| \begin{vmatrix} x(t) \\ d(t) \end{vmatrix} \end{cases}$$

and the pair $\left(\begin{vmatrix} A & M \\ 0 & 0 \end{vmatrix}, |C \quad N| \right)$ must be observable.

2. The model to be considered is

$$\begin{cases} \dot{x}(t) = a\,x(t) + b\,u(t) + v(t) \\ \dot{a}(t) = 0 \\ y(t) = c\,x(t) + w(t) \end{cases} \qquad (a = A,\ b = B,\ c = C)$$

The EKF can be used and the matrices of the linearized system

$$A(t) = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial a} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial a} \end{vmatrix} = \begin{vmatrix} \hat{a}(t) & \hat{x}(t) \\ 0 & 0 \end{vmatrix}, \quad C = |1 \quad 0|$$

must be used in the Riccati equation to compute the time-varying gain $L(t)$