ADVANCED AND MULTIVARIABLE CONTROL

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Surname and Name …….

University Id……………………………………………………………..Signature…………………………………………………………………..

Exercise 1

Consider the **discrete time** system

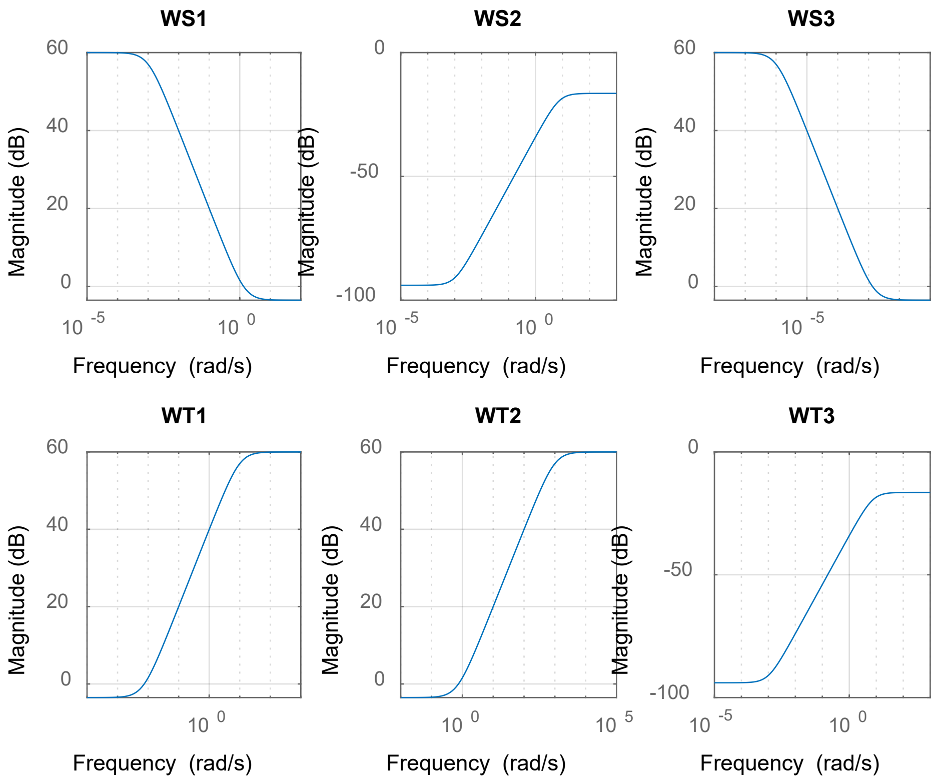
$$x_1(k+1) = \frac{a x_2(k)}{1 + x_1^2(k)}$$
$$x_2(k+1) = \frac{b x_1(k)}{1 + x_2^2(k)}$$

with $0 \le a < 1$, $0 \le b < 1$.

Prove that the origin is an asymptotically stable equilibrium using a quadratic Lyapunov function.

Exercise 2

Consider the design of a *H2/Hinf* controller with shaping functions. Which pair (*Ws, WT*) of the functions shown in the following figure is reasonable and coherent with the goals of the control design procedure (in terms of roughly the same cutoff frequency)?



Exercise 3

Consider the following **discrete-time** system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u(k)}$$
$$\mathrm{y(k)} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

1. which output variable can be forced to be asymptotically equal to a given arbitrary constant reference value?

2. Assuming that the state of the system is measurable, show the structure of a control scheme guaranteeing the zero error regulation property for the selected output, and write the model to be considered in the design of the stabilizing state-feedback control law.

3. In this problem, is it necessary to use a dynamic state observer?

Exercise 4

Given the system

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + d(t)$$

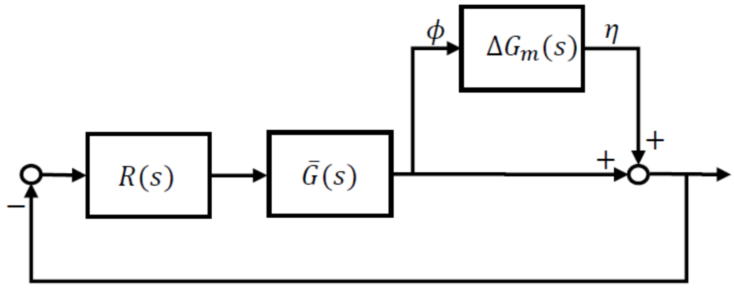
1. Assume that *d(t)=0* and compute the infinite horizon LQ regulator with weighting matrices *R=5I*, *Q=4*. Compute the resulting closed-loop eigenvalue.
2. Show how to estimate the disturbance *d*, assumed to be constant but unknown, starting from the measurement of the state (NOTE: it is sufficient to write the enlarged system to be used for the estimation of the extended state with an observer).

Riccati equation:

$$\dot{P}(t) + Q - P(t)BR^{-1}B'P'(t) + P(t)A + A'P'(t) = 0$$

Exercise 5

Given the following feedback system, assume that in the nominal case (*$\Delta G_m(s)=0$*) the system is asymptotically stable and that *$\Delta G_m(s)$* is asymptotically stable, write the sufficient condition guaranteeing that in the perturbed case the asymptotic stability of the closed-loop system is maintained.



SOLUTION EXERCISE 1

Consider the Lyapunov function $V(x) = x_1^2+x_2^2$. It follows that

$$\Delta V(x) = \left(\frac{ax_2}{1+x_1^2}\right)^2 + \left(\frac{bx_1}{1+x_2^2}\right)^2 - x_1^2 - x_2^2 = \left(\left(\frac{a}{1+x_1^2}\right)^2 - 1\right)x_2^2 + \left(\left(\frac{b}{1+x_2^2}\right)^2 - 1\right) \quad x_1^2 < 0$$

SOLUTION EXERCISE 2

WS1 - WT2

SOLUTION EXERCISE 3

There are many ways to answer, for example one can compute the invariant zeros considering the two outputs one at a time. The simplest way however is to directly compute the transfer functions from the input to the two outputs.

$$G(z) = \begin{bmatrix} z-1/(z-0.5)^2 \\ z/(z-0.5)^2 \end{bmatrix}$$

It is apparent that we have one input and two outputs, and there is a derivative action from the input to the first output, so only the second output can be brought to an arbitrary constant value with a proper control action.

Another possibility is to compute the invariant zeros. For the first output one must compute

$$det\begin{bmatrix} zI-A & -B \\ C_1 & 0 \end{bmatrix} = z-1 = 0$$

Therefore there is a derivative term and the first output cannot be brought to an arbitrary constant value.

For the second output

$$det\begin{bmatrix} zI-A & -B \\ C_2 & 0 \end{bmatrix} = z = 0$$

and the output can be asymptotically regulated.

As a third possibility, one could compute the static gain in the two cases

$$det\begin{bmatrix} I-A & -B \\ C_{1/2} & 0 \end{bmatrix}$$

Leading to the same conclusion.

Including an integrator acting on the error related to the second output we have the enlarged system

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C_2 & 1 \end{bmatrix}\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}\mathrm{u(k)}$$

Where

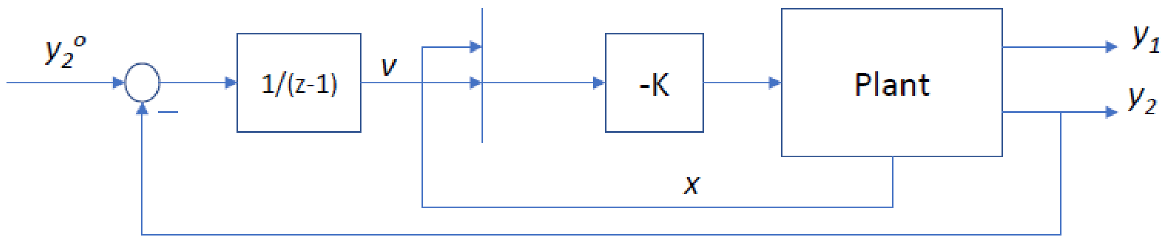
$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -0.25 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathrm{C} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

The control law is

$$\mathrm{u(k)} = -\mathrm{K} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

and the control scheme is



Finally, a dynamic observer is not needed, since one can set

$$\mathrm{x(k)} = \mathrm{C}^{-1} y(k)$$

SOLUTION EXERCISE 4

A=0, B=[1 2], Q=4, R=5I₂

Solution Riccati equation $P=2$, $K=R^{-1}B'P=\begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$, $A-BK=-2$

Disturbance estimation: we consider the enlarged system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$$

Then, any observer can be used to estimate the state and the disturbance, or just the disturbance if a reduced order estimator is considered.

SOLUTION EXERCISE 5

Denoting by *T(s)* the complementary sensitivity function of the nominal system, the condition is

$$|T(j\omega)\Delta G_m(j\omega)| < 1, \quad \forall\omega$$

or

$$\|T\Delta G_m\|_\infty < 1$$