Advanced and Multivariable Control

Lyapunov Stability

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Dynamic, finite dimensional, time invariant system:

$$\dot{x}(t) = f\left(x(t), u(t)\right), \quad x(t_0) = x_0 \quad x \in R^n \quad u \in R^m$$

$$\dot{x}(t) = f\left(x(t)\right)$$ autonomous

$$\dot{x}(t) = Ax(t) + Bu(t)$$ linear

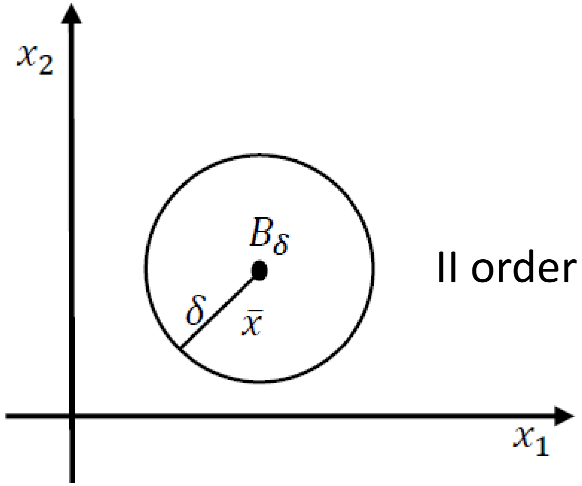
equilibrium pair $(\bar{x}, \bar{u})$ → $f(\bar{x}, \bar{u}) = 0$

$$A\bar{x} = -B\bar{u}$$ linear

- *A* non-singular → unique equilibrium for constant $\bar{u}$
- *A* singular
  - infinite solutions
  - no solutions

The equilibrium $\bar{x}$ is *isolated* if there exists $\delta > 0$ such that there does not exist any other equilibrium in

$$B_\delta\left(\bar{x}, \delta\right) := \{x : \|x - \bar{x}\| \leq \delta\}$$



II order system, norm-2 $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$

For any fixed $\bar{u}$, the system takes the form $\dot{x}(t) = f\left(x(t), \bar{u}\right) = \varphi\left(x(t)\right)$

If the equilibrium is $\bar{x}$, set $x(t) = \bar{x} + \delta x(t)$ and write the system as $\delta\dot{x}(t) = \varphi\left(\bar{x} + \delta x(t)\right)$ with equilibrium at the origin

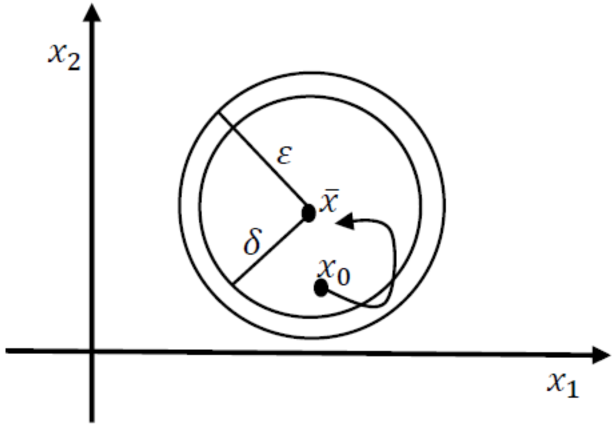
Stability

The equilibrium $\bar{x}$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all the initial states x_0 satisfying

$$\|x_0 - \bar{x}\| \leq \delta$$

it holds that

$$\|x(t) - \bar{x}\| \leq \varepsilon, \quad \forall t \geq 0$$



Asymptotic stability

If $\bar{x}$ is a stable equilibrium and, in addition,

$$\lim_{t \to \infty} \|x(t) - \bar{x}\| = 0$$

then $\bar{x}$ is an *asymptotically stable* equilibrium.

Asymptotic stability is a *local* property, since δ can be very small. The equilibrium $\bar{x}$ is *globally (asymptotically) stable* if it is asymptotically stable for any $x_0 \in R^n$

Linear systems → stability is a property of the system

the system is asymptotically stable **if and only if** all the eigenvalues of A have negative real part

Nonlinear systems → stability is a property of the equilibrium

Consider $\boxed{\dot{x}(t) = f\left(x(t), u(t)\right)}$ $f \in C^1, \quad f(\bar{x}, \bar{u}) = 0$

and let
$$\begin{array}{rcl} x(t) & = & \bar{x} + \delta x(t) \\ u(t) & = & \bar{u} + \delta u(t) \end{array}$$

The linearized model is

$$\boxed{\delta \dot{x}(t) = A\delta x(t) + B\delta u(t)}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}, u=\bar{u}}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x}, u=\bar{u}}$$

Theorem

- if all the eigenvalues of A have negative real part, then the equilibrium $(\bar{x}, \bar{u})$ is asymptotically stable; *only sufficient*
- if at least one eigenvalue of A has positive real part, then the equilibrium $(\bar{x}, \bar{u})$ is unstable;
- if all the eigenvalues of A have negative or null real part, no conclusion can be drawn on the stability of the equilibrium from the analysis of the linearized system.

Problems

- What to do in the case of nondifferentiable functions *f*?
- No information about the stability region (δ in the definition of stability)

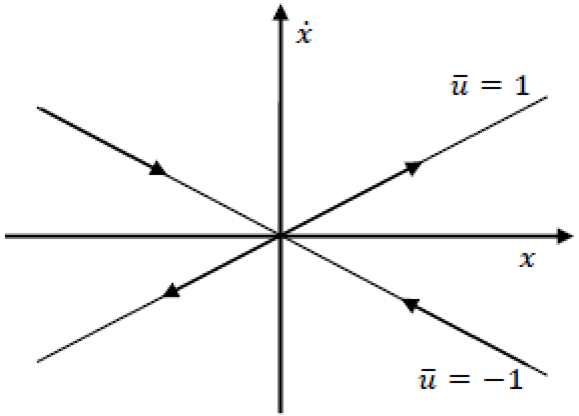
Example 1

$$\dot{x}(t) = x(t)u(t) \quad , \quad u = \bar{u} = \mp 1 \quad ,$$ → in any case $\bar{x} = 0$ is an equilibrium, but …

$\bar{u} = -1$ → $\dot{x}(t) = -x(t)$ → negative eigenvalue, asymptotically stable equilibrium

$\bar{u} = 1$ → $\dot{x}(t) = x(t)$ → positive eigenvalue, unstable equilibrium

Another way to study the system (for first order systems)



Why for $\bar{u} = -1$ the origin is a **globally asymptotically stable** equilibrium?

Example 2

$\dot{x}(t) = -x(t) + x^2(t)$ → $\bar{x} = 0$ is an equilibrium

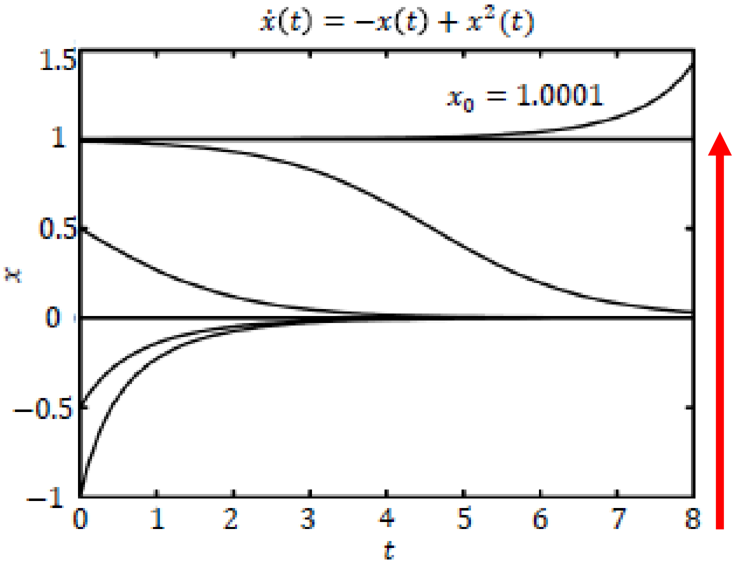
Linearized model

$\delta\dot{x}(t) = -\delta x(t)$ → asymptotically stable equilibrium

*What about the **region of attraction**?*

Analytical solution

$$x(t) = \frac{e^{-t}x_0}{1-x_0+e^{-t}x_0}$$

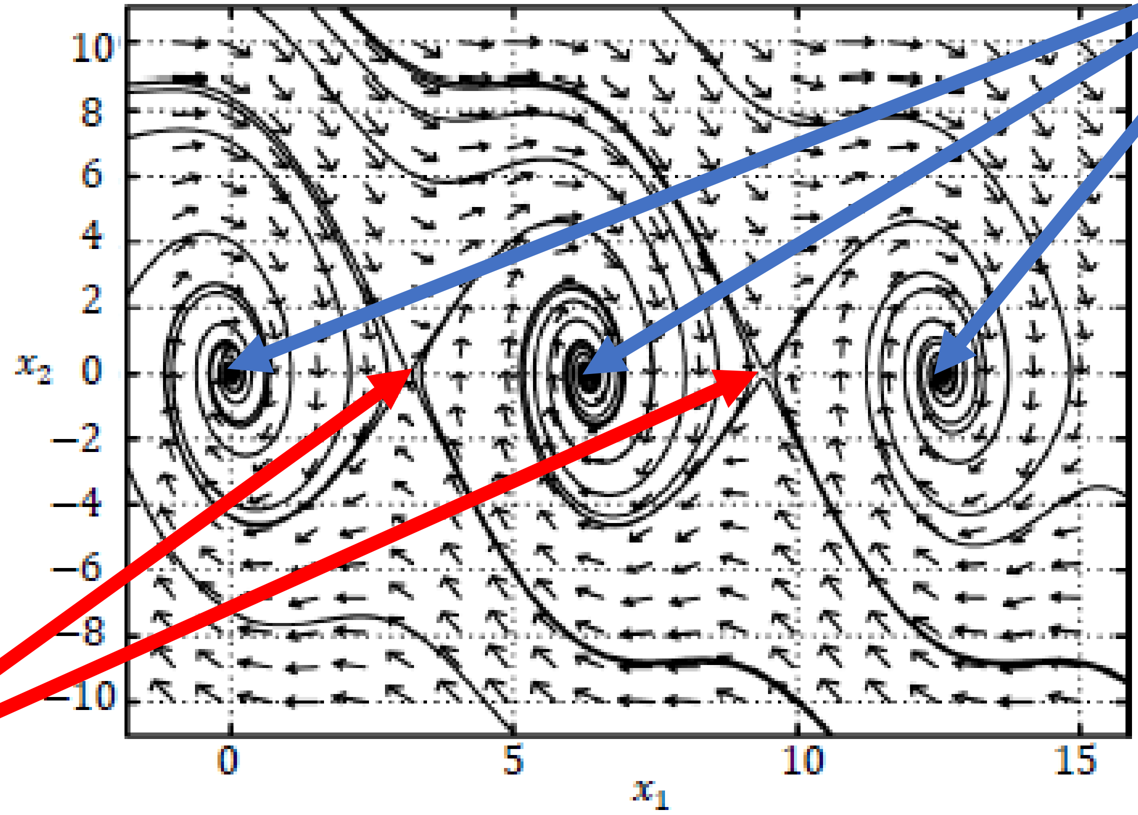


Region of attraction
(-inf,1)

II order systems – the phase (or state) plane

Interesting to study the *trajectories* (evolution) of the system in the plane (x_1,x_2)

Asymptotically stable equilibria (but NOT globally stable)



Unstable equilibria

This allows one to have an idea of the region of attraction

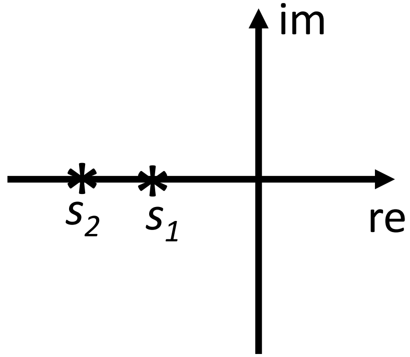
Figure 1.10: Pendulum - phase portrait ($M = L = k = 1$).

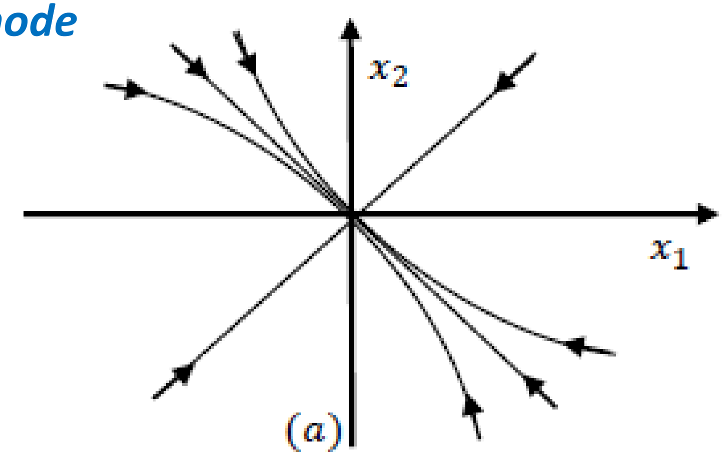
Linear systems

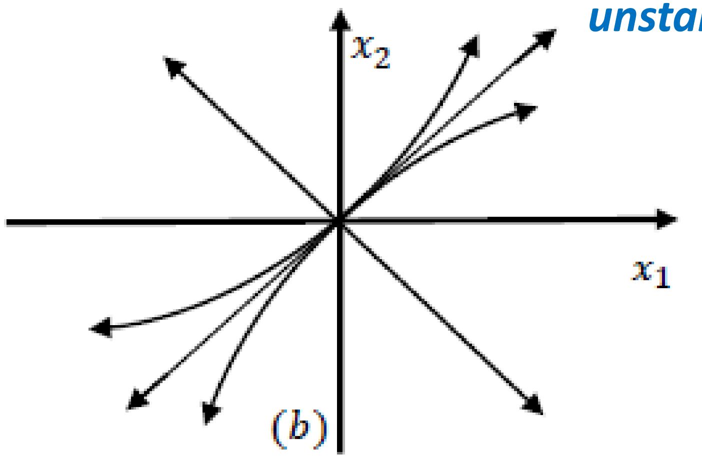
Easy to characterize the form of the state trajectories as functions of the eigenvalues

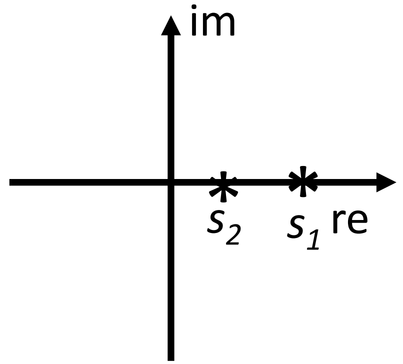
stable node

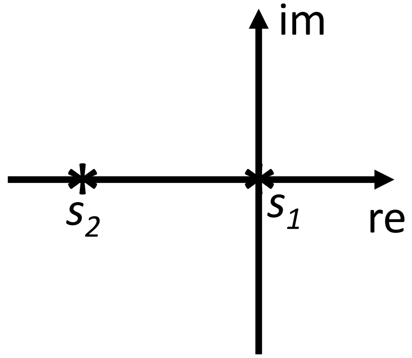
unstable node

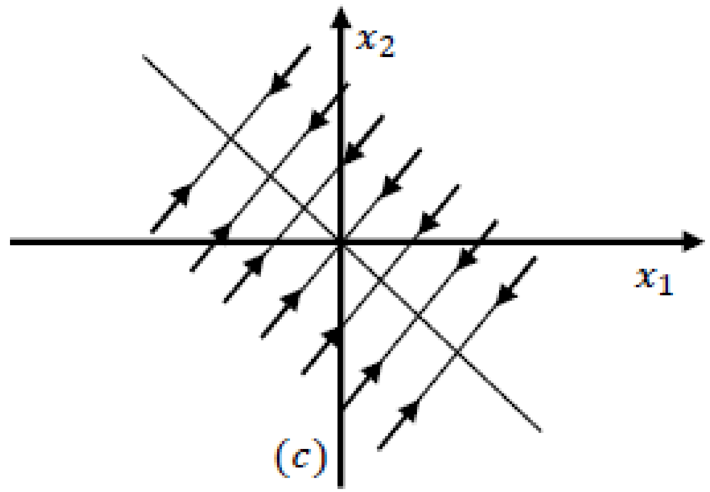


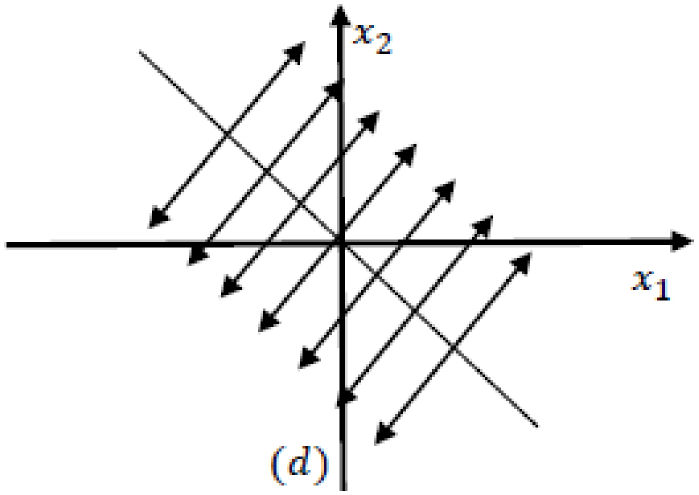


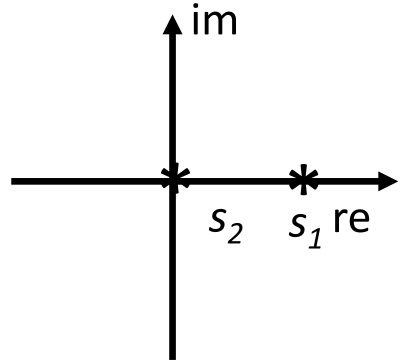


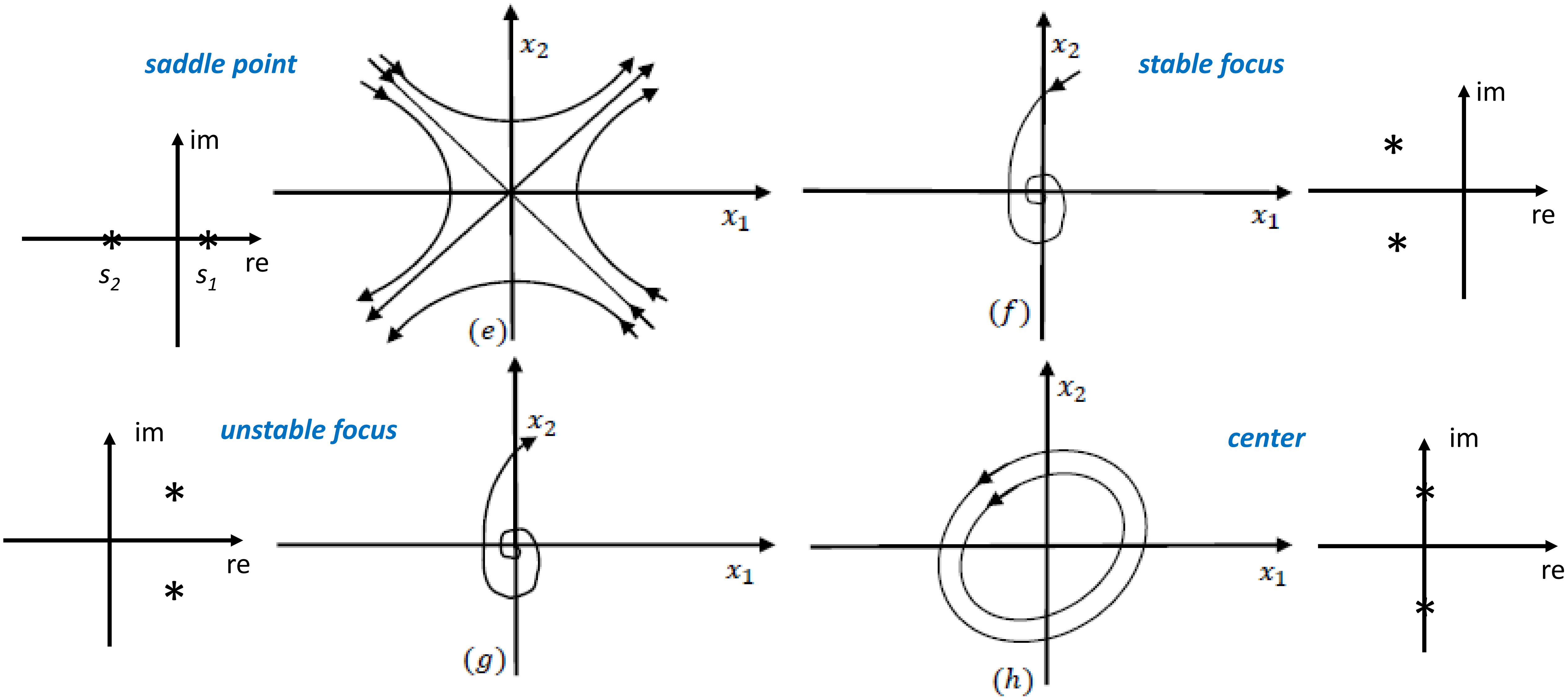












(plus other forms for repeated eigenvalues)

Note that the eigenvectors depend on the basis you consider

Nonlinear systems

If f is C_1, in a neighbor of an equilibrium, the trajectories are the ones of the corresponding linearized system

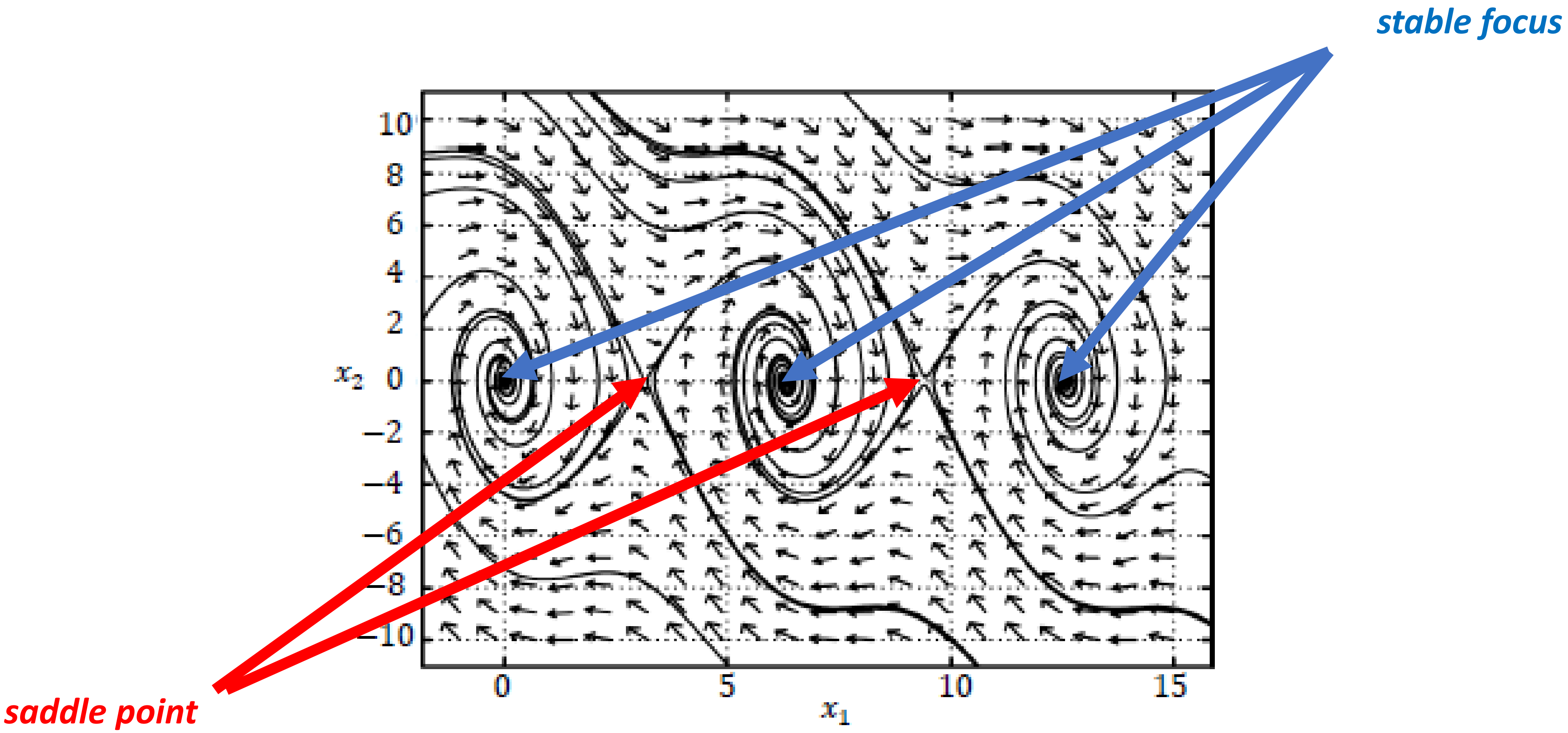
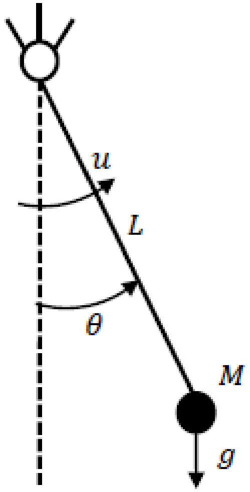


Figure 1.10: Pendulum - phase portrait ($M = L = k = 1$).

Pendulum



$$ML^2\ddot{\vartheta}(t) = -k\dot{\vartheta}(t) - MLg\sin\vartheta(t) + u(t), \quad M,\ L,\ k > 0$$

$$\Big\downarrow \quad x_1 = \vartheta,\ x_2 = \dot{\vartheta}$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{L}\sin\left(x_1(t)\right) - \frac{k}{ML^2}x_2(t) + \frac{1}{ML^2}u(t) \end{aligned}$$

$$\Big\downarrow \quad \bar{u} = 0$$

$$\begin{aligned} \boxed{\bar{x}_2 = 0} \\ \frac{g}{L}\sin\left(\bar{x}_1\right) &= 0 \quad \rightarrow \quad \boxed{\bar{x}_1 = i\pi,\ i = 0, 1, \ldots} \end{aligned}$$

equilibria

$$\Big\downarrow$$

$$\begin{aligned} \delta\dot{x}_1(t) &= \delta x_2(t) \\ \delta\dot{x}_2(t) &= -\frac{g}{L}\cos\left(\bar{x}_1\right)\delta x_1(t) - \frac{k}{ML^2}\delta x_2(t) + \frac{1}{ML^2}\delta u(t) \end{aligned}$$

Pendulum – linearized model

$$
\begin{aligned}
\delta \dot{x}_1(t) &= \delta x_2(t) \\
\delta \dot{x}_2(t) &= -\frac{g}{L}\cos(\bar{x}_1)\,\delta x_1(t) - \frac{k}{ML^2}\delta x_2(t) + \frac{1}{ML^2}\delta u(t)
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(\bar{x}_1) & -\frac{k}{ML^2} \end{bmatrix}
$$

For $M = 1,\ k = 1,\ L = 1,\ g = 9.8$ the eigenvalues are

$$
s = -0.5 \mp j3\ (\bar{x}_1 = 2\pi i,\ \text{stable focus})
$$

$$
s_1 = 2.67,\ s_2 = -3.67\ (\bar{x}_1 = (2i+1)\,\pi,\ \text{saddle point})
$$

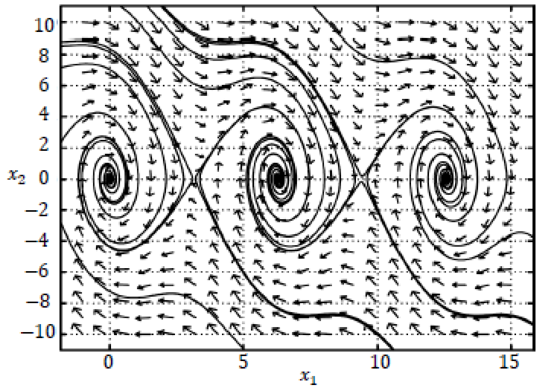


Figure 1.10: Pendulum - phase portrait ($M = L = k = 1$).

How to compute the phase plane?

Use the SW *pplaneM.m* (free download), where *M* is the release of Matlab

Example: van der Pol oscillator

But what to do for non differentiable and/or higher order systems?

The **Lyapunov theory** can provide a satisfactory answer

(and it is the most popular and useful method for the analysis of nonlinear systems and for the nonlinear control synthesis)

A gentle introduction through examples…

Another example – van der Pol oscillator

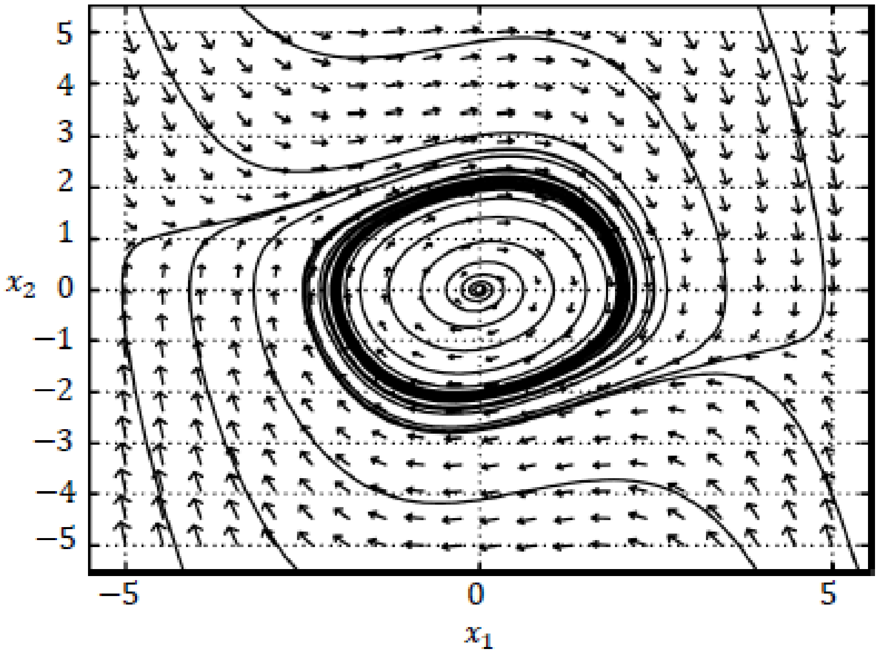
$$m\ddot{x}(t) + 2c\left(x^2(t) - 1\right)\dot{x}(t) + kx(t) = 0, \quad m,\ c,\ k > 0$$

$$x_1 = x,\ x_2 = \dot{x}$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{2c}{m}\left(1 - x_1^2(t)\right)x_2(t) - \frac{k}{m}x_1(t) \end{aligned}$$

Linearization at the origin

$$\begin{aligned} \delta\dot{x}_1(t) &= \delta x_2(t) \\ \delta\dot{x}_2(t) &= -\frac{k}{m}\delta x_1(t) + \frac{2c}{m}\delta x_2(t) \end{aligned}$$



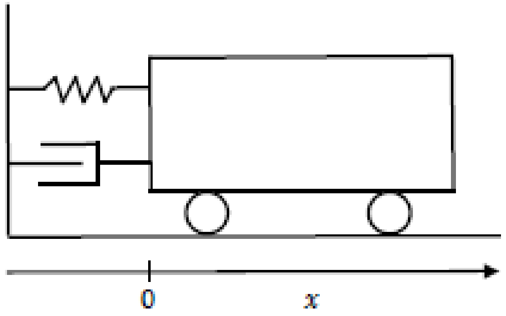
eigenvalues

$$s = \frac{c}{m} \mp \sqrt{\frac{c^2}{m^2} - \frac{k}{m}}$$

c=0.1,m=1,k=1

unstable focus

The cart example



$$m\ddot{x} = -h(\dot{x}) - k(x)$$

$k(x) = k_0 x + k_1 x^3$ spring

$h(\dot{x}) = b\dot{x}\,|\dot{x}|$ damper

$b, k_0, k_1 > 0$

$$m\ddot{x} + b\dot{x}\,|\dot{x}| + k_0 x + k_1 x^3 = 0$$

Skip the dependence on time (if not necessary)

$x_1 = x \quad , x_2 = \dot{x}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\{-bx_2|x_2| - k_0 x_1 - k_1 x_1^3\} \end{aligned}$$

Linerization method not useful

$T = \frac{1}{2} m x_2^2$ kynetic energy

$U = k_o \frac{x_1^2}{2} + k_1 \frac{x_1^4}{4}$ potential energy

$V(x_1, x_2) = T(x_2) + U(x_1)$ total energy

$$V(x_1, x_2) = k_o \frac{x_1^2}{2} + k_1 \frac{x_1^4}{4} + \frac{1}{2} m x_2^2$$

$V(x_1, x_2) = 0 \quad \text{for } x_1 = 0 \text{ and } x_2 = 0 \text{ (the origin)}$

$V(x_1, x_2) > 0 \quad \text{for } x_1 \neq 0 \text{ and/or } x_2 \neq 0$

$$V(x_1, x_2) = k_o \frac{x_1^2}{2} + k_1 \frac{x_1^4}{4} + \frac{1}{2} m x_2^2$$

Derivative of *V(x), x=[*x_1 x_2*]',* with respect to time and along the trajectories

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{dV(x)}{dx}\frac{dx}{dt} = k_0 x_1 \dot{x_1} + k_1 x_1^3 \dot{x_1} + m x_2 \dot{x_2} = -b x_2^2 |x_2| \leq 0$$

This means that we have dissipation of energy until $x_2=0$ (null velocity)

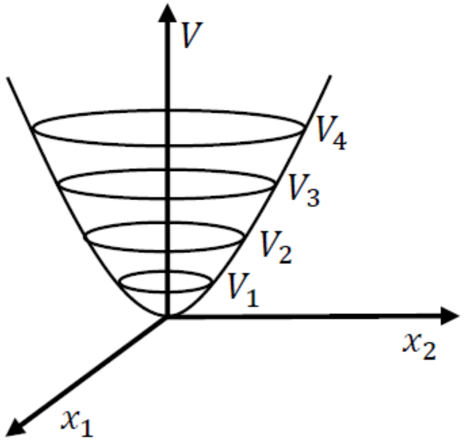
Why $\dot{V}(x) \leq 0$ and not $\dot{V}(x) < 0$? Because it can be null also for $x_1 \neq 0$

In any case, this means that the system will tend to an equilibrium condition with null velocity (but maybe not null position).

General idea: try to study the equilibria of a system by analyzing a suitable «*energy function*»

Positive definite functions

A *scalar* function $V(x)$, *continuous* with its first derivatives, is locally:



positive definite in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) > 0$ for any x belonging to an open neighbor of $\bar{x}$ (see the figure where $\bar{x} = 0$)

semidefinite positive in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) \geq 0$ for any x belonging to an open neighbor of $\bar{x}$

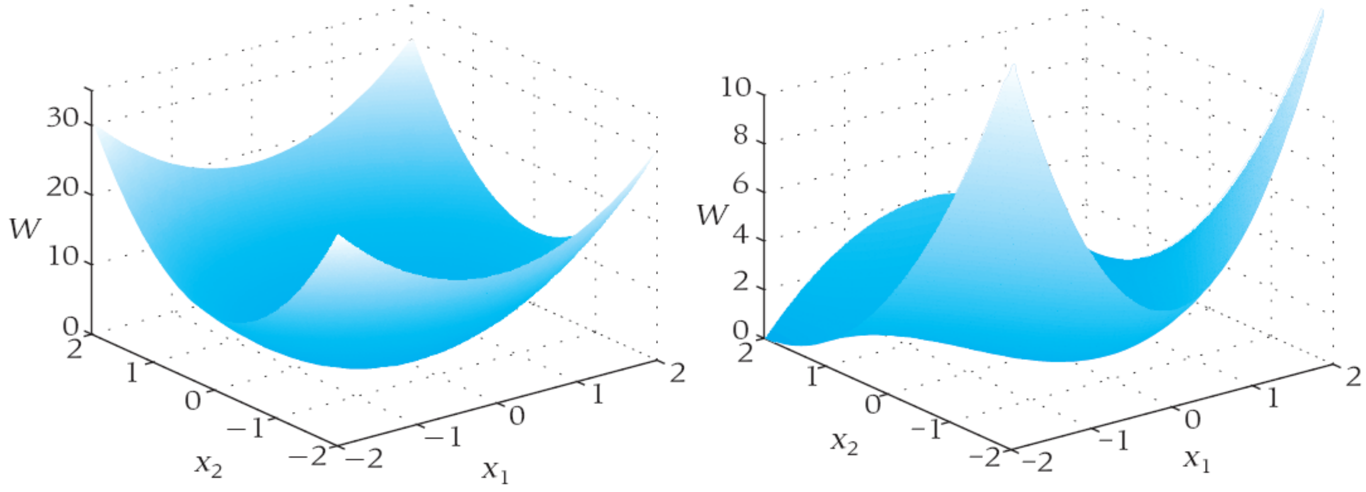
definite negative in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) < 0$ for any x belonging to a neighbor of $\bar{x}$

semidefinite negative in $\bar{x}$ if $V(\bar{x}) = 0$ and $V(x) \leq 0$ for any x belonging to an open neighbor of $\bar{x}$

All these definitions are **global** if the corresponding conditions are fulfilled for any $x \neq \bar{x}$

If $V(x)$ is positive definite in $\bar{x}$ and $\lim_{\|x\|\to\infty} V(x) = +\infty$, then $V(x)$ is *radially unbounded*

$$\boxed{\bar{x} = 0}$$



a) b)

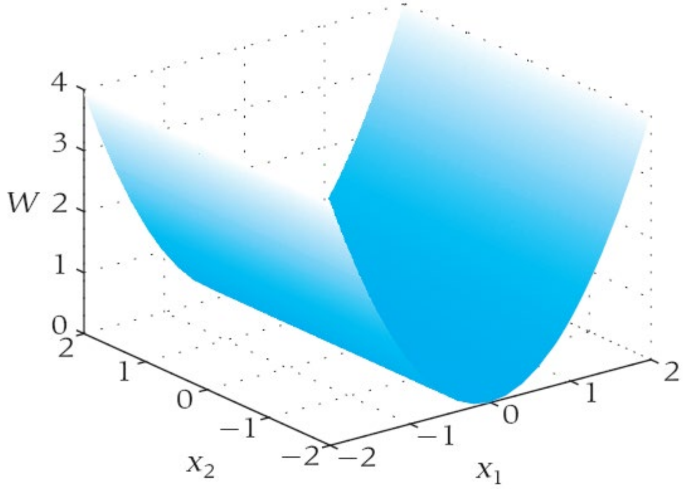
$$V(x) = 3x_1^2 + 5x_2^2$$

Globally positive definite and radially unbounded

$$V(x) = x_1^2 + x_2^2 - x_1^2 x_2$$

(locally) positive definite
In a neighbor of the origin the second order terms dominate

$$\boxed{\bar{x} = 0}$$

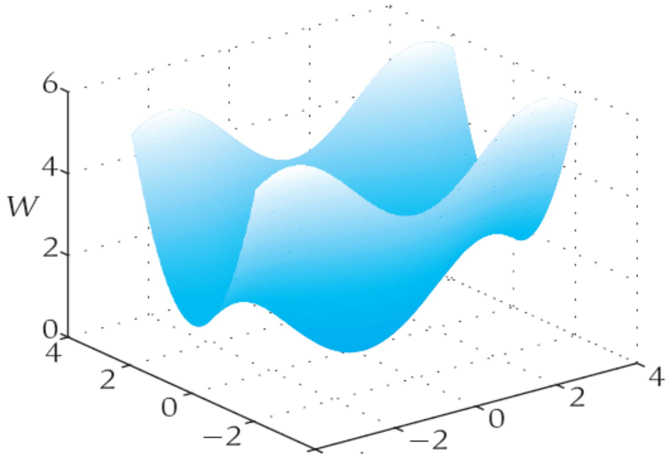


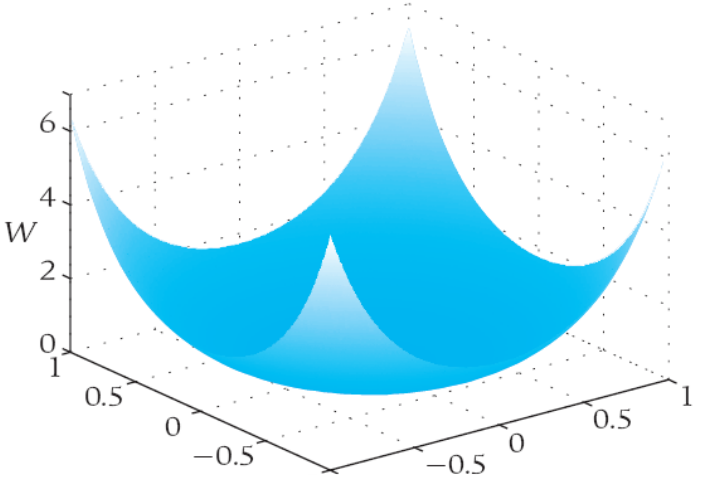
c)

$$V(x) = x_1^2$$

Positive semidefinite

$$\boxed{\bar{x} = 0}$$





$$V(x) = 1 - \cos x_1 + x_2^2$$

Positive definite

$$V(x) > 0$$

$$\text{for } -2\pi < x_1 < 2\pi \textit{ and for any } x_2$$

$$V(x) = e^{x_1^2 + x_2^2} - 1$$

Globally positive definite and radially unbounded

c)

d)

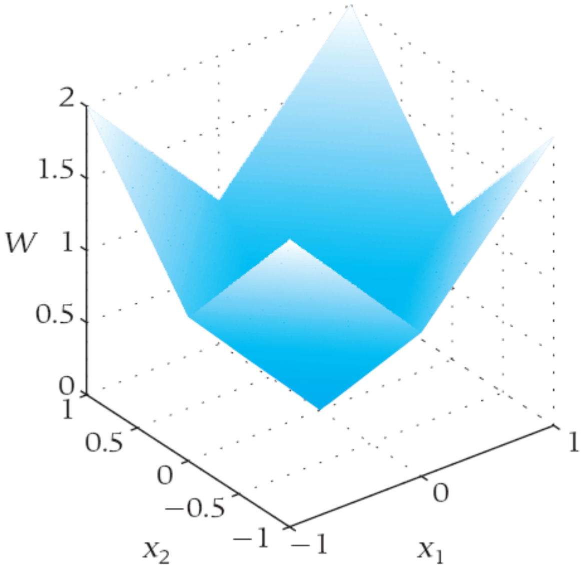
$$\boxed{\bar{x} = 0}$$

$$V(x) = \sum_{i=1}^{n} |x_i|$$

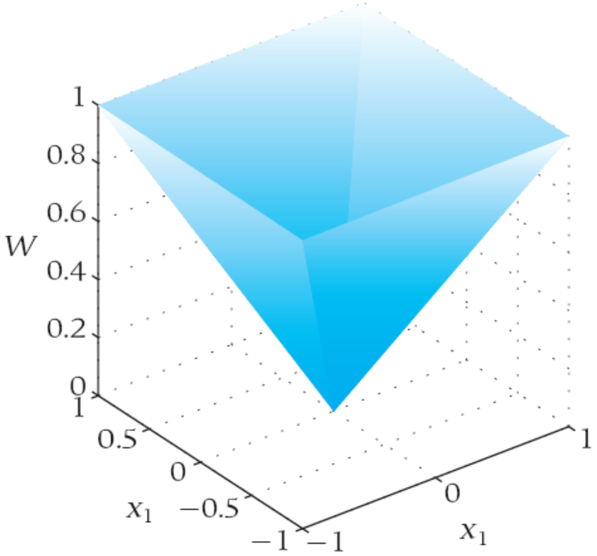
Globally positive definite and radially unbounded

$$V(x) = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

Globally positive definite and radially unbounded



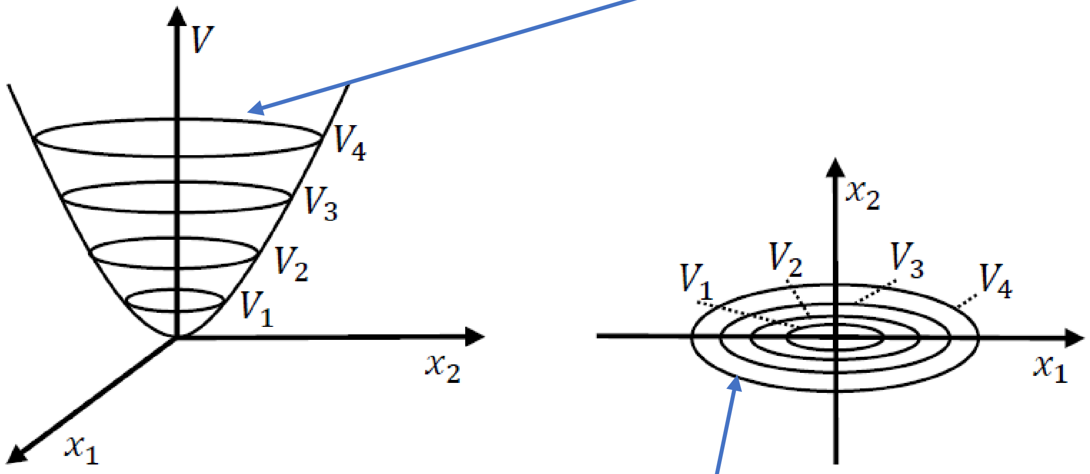
c)



d)

Given a function $V(x)$ positive definite in $\bar{x}$, the values of x such that $V(x) = \bar{V}$, where $\bar{V}$ is a positive value, define a *level surface*, or a *level line* in the case of second order systems

$\boxed{\bar{x} = 0}$



Given $\bar{V} > 0$ sufficiently small, the corresponding level line is a closed curve that includes $\bar{x}$

Given $V_2 > V_1 > 0$ the set of values of x such that $V(x) \leq V_1$ is contained into the set of values of x such that $V(x) \leq V_2$

Stability - Lyapunov theorem

Consider

$$\dot{x}(t) = \varphi\left(x(t)\right) \quad , \quad \varphi \in C^1 \quad , \quad \varphi(\bar{x}) = 0$$

If there exists a function $V(x)$, continuous with its derivative, positive definite in $\bar{x}$ and such that its derivative, along the state trajectories, $\dot{V}(x)$ is semidefinite negative in $\bar{x}$, i.e. such that for any x belonging to a neighbor of $\bar{x}$

$$\dot{V}(x) = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}\varphi\left(x\right) \leq 0$$

then $\bar{x}$ is a stable equilibrium

V(x) is called a **Lyapunov function**

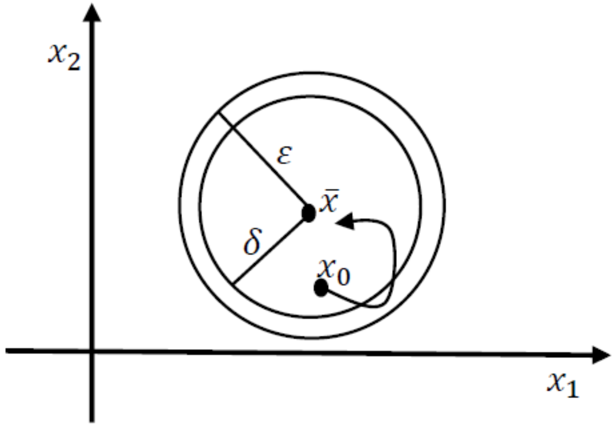
Recall the definition of stability

The equilibrium $\bar{x}$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all the initial states x_0 satisfying

$$\|x_0 - \bar{x}\| \leq \delta$$

it holds that

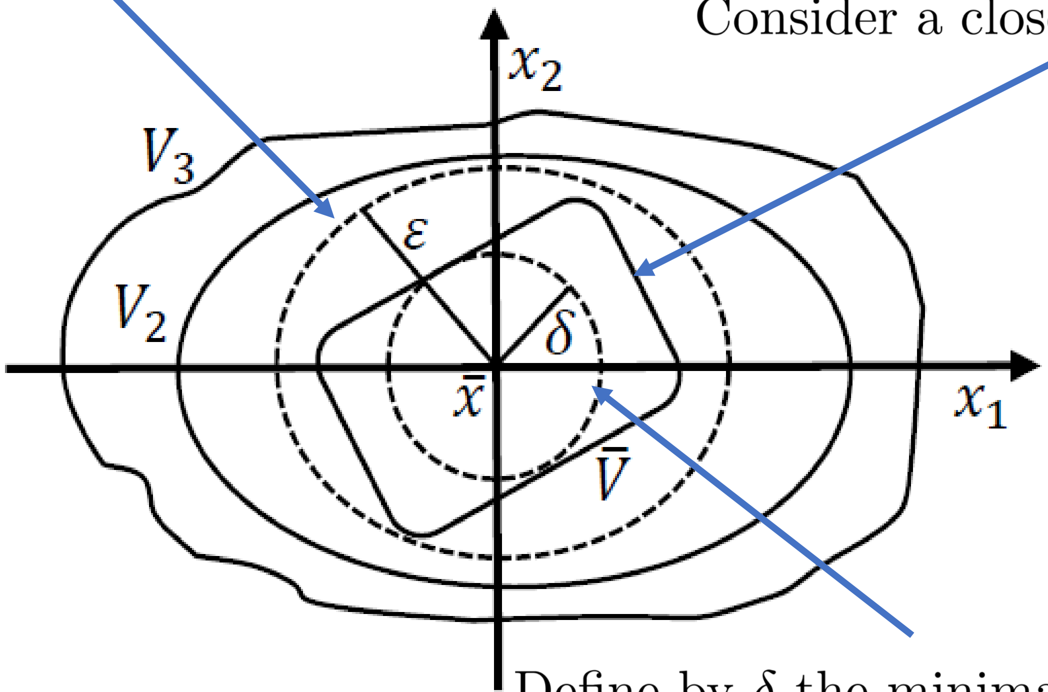
$$\|x(t) - \bar{x}\| \leq \varepsilon, \quad \forall t \geq 0$$



Sketch of the proof

Given $\varepsilon > 0$, consider the set $B_\varepsilon := \{x : \|x - \bar{x}\| \ < \varepsilon\}$

Consider a closed level line $\bar{V}$ contained in B_ε



Define by δ the minimal distance of this curve from $\bar{x}$

Since by assumption $\dot{V} \leq 0$, for any initial state $x_0 \in B_\delta$, the state trajectory is fully contained in $\bar{V}$, i.e. it is inside B_ε

Asymptotic stability

If the assumptions of the previous Theorem hold and $\dot{V}$ is negative definite in $\bar{x}$, then $\bar{x}$ is an asymptotically stable equilibrium

Global asymptotic stability

If there exists a function $V(x)$, continuous with its derivative, globally positive definite in $\bar{x}$, radially unbounded, and such that its derivative, along the state trajectories, $\dot{V}(x)$ is globally definite negative in $\bar{x}$, then $\bar{x}$ is the unique globally asymptotically stable equilibrium of the system

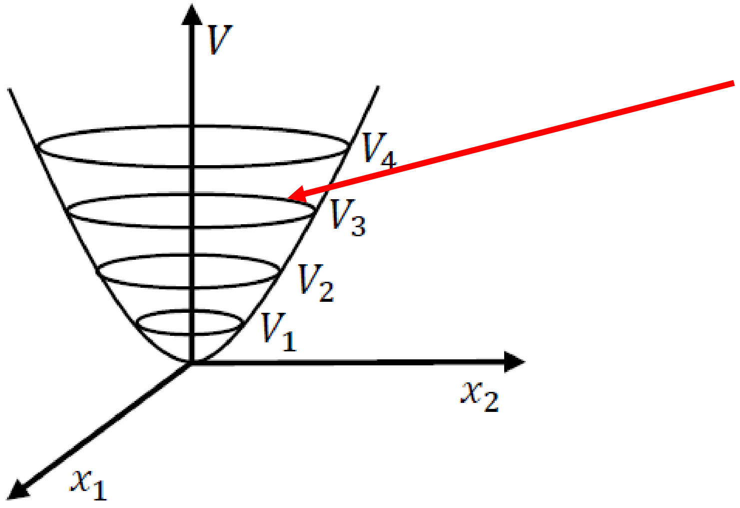
Instability

If there exists a function $V(x)$ continuous with its derivative, positive definite in $\bar{x}$ and such that $\dot{V}(x)$ is positive definite in $\bar{x}$, then $\bar{x}$ is an unstable equilibrium

What if the Lyapunov function is only *semidefinite negative*?

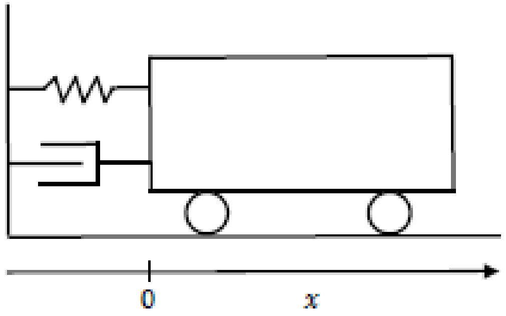
Krasowski – La Salle theorem

If there exists a function $V(x)$ continuous with its derivative, positive definite in $\bar{x}$, such that $\dot{V}(x)$ is semidefinite negative in $\bar{x}$, and the set $S := \left\{ x : \dot{V}(x) = 0 \right\}$ does not contain perturbed (with respect to $\bar{x}$) trajectories compatible with the system, then $\bar{x}$ is an asymptotically stable equilibrium point



In principle, since *Vdot* is only *semidefinite negative*, the state could remain along this level line. However, this behavior should be allowed by the system's dynamics (state equations). If it is not possible, this means that the «*active*» condition is *Vdot<0* and the state converges to the origin

The cart example



$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{1}{m}\{-bx_2|x_2| - k_0x_1 - k_1x_1^3\}\end{aligned}$$

$$V(x_1,x_2) = T(x_2) + U(x_1) = \tfrac{1}{2}mx_2^2 + k_o\tfrac{x_1^2}{2} + k_1\tfrac{x_1^4}{4} > 0$$

$$\dot{V}(x) = -bx_2^2|x_2| \le 0$$

only (simple) stability can be concluded with the Lyapunov theorem, but is it possible to have a trajectory of the system with *Vdot=0*?

$$\dot{V}(x) = 0 \rightarrow x_2 = 0 \rightarrow \dot{x}_1 = 0 \rightarrow x_1 \text{ constant } \bar{x}_1$$

$$-(k_0 + k_1\bar{x}_1^2)\bar{x}_1 = 0 \longrightarrow \bar{x}_1 = 0$$

So, the only possible trajectory compatible with *Vdot=0* is the origin

Comments

The previous results are ***only sufficient conditions***, if you are unable to find a suitable Lyapunov function satisfying one of the above theorems *you cannot conclude anything* about the stability of the equilibrium

The really difficult task is to find a suitable Lyapunov function. If possible, one can resort to the idea of «energy», like in the cart example. Otherwise, it is possible to try with a quadratic function of the form

$$V(x) = (x - \bar{x})' \, P \, (x - \bar{x})$$

where *P* is a positive definite matrix *(P>0),* i.e. $v' P v > 0 \text{ for any } v \neq 0$. More details on the choice of *P* will be discussed next .

Exercise

Consider the system $\dot{x} = -x^3$ and study the stability of the origin.

First of all, note that the origin is an equilibrium (the only one). Moreover, the linearized system is

$$\delta\dot{x}(t) = 0$$

with zero eigenvalue. Therefore, nothing can be concluded from the analysis of the linearized system.

Let's try with a quadratic Lyapunov function

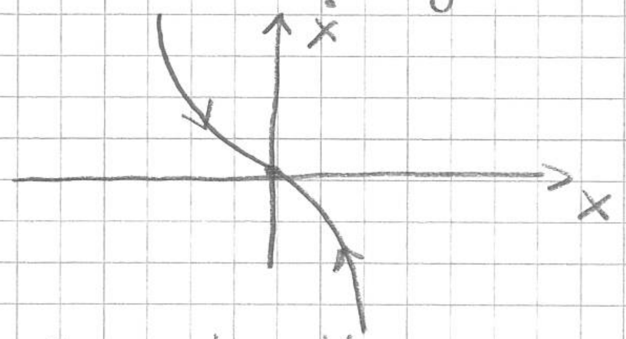
$$V(x) = x^2 > 0$$

Its derivative is

$$\dot{V}(x) = 2x\dot{x} = -2x^4 < 0$$

So the stability is an asymptotically stable equilibrium. Moreover, since $V(x)$ is radially unbounded and $\dot{V}(x) < 0$, $\forall x \neq 0$, the equilibrium is globally asymptotically stable.

Note that, since the system is first order, one can look at the function $x - \dot{x}$

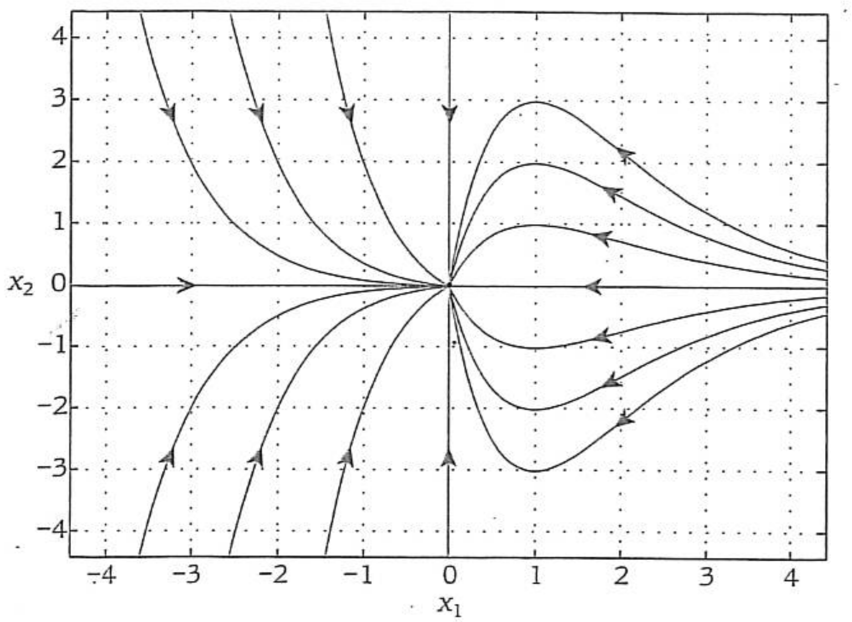


which leads to the same conclusions.

Exercise

Consider the system

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t)\\ \dot{x}_2(t) &= x_2(t)(x_1(t)-1)\end{aligned}$$



It is apparent that the origin is an equilibrium

Moreover, looking at the state trajectories (computed with pplane), One can easily conclude that the equilibrium is unique and globally asymptotically stable

Can we obtain the same result with the Lyapunov theory? Consider

$$V(x) = 0.5(a_1x_1^2 + a_2x_2^2) \quad , \quad a_1 > 0 \quad , \quad a_2 > 0$$

Correspondingly

$$\dot{V}(x) = a_1x_1\dot{x}_1 + a_2x_2\dot{x}_2 = -a_1x_1^2 - a_2x_2^2 + a_2x_1x_2^2$$

Therefore

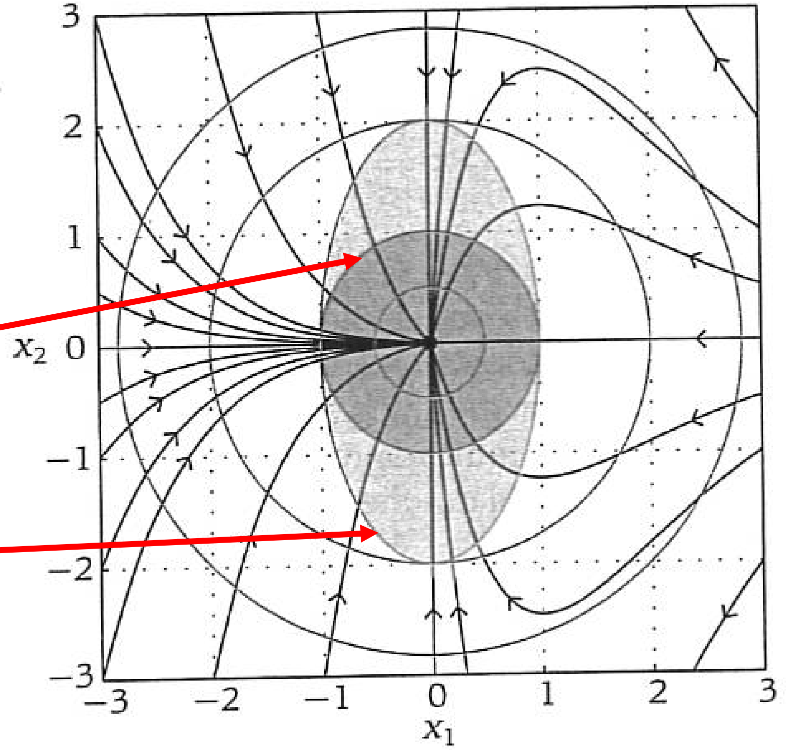
$$\dot{V}(x) = -a_1 x_1^2 - a_2 x_2^2 (1 - x_1) < 0 \text{ for } x_1 < 1$$

and we can conclude that the origin is a *locally* (*not globally!*) asymptotically stable equilibrium

Can we estimate the region of attraction? (where *Vdot<0*)

If we take $a_1=a_2$ → the estimated maximum region of attraction is a circle tangent to $x_1=1$

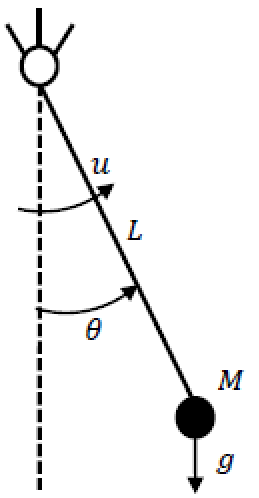
If we take $a_1 \neq a_2$ → the estimated maximum region of attraction is an ellipsoid tangent to $x_1=1$, which can be made larger



In any case, we are unable to conclude the global stability, unless we find a different Lyapunov function

Pendulum

$$x_1 = \vartheta,\ x_2 = \dot{\vartheta}$$



$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{g}{L}\sin\left(x_1(t)\right) - \frac{k}{ML^2}x_2(t) + \frac{1}{ML^2}u(t)\end{aligned}$$

$$\Big\downarrow\ u = 0$$

$$V(x) = U(x) + T(x) = \tfrac{g}{L}(1 - cos(x_1)) + \tfrac{1}{2}x_2^2$$

$$\dot{V}(x) = \tfrac{g}{L}\dot{x}_1 sin(x_1) + x_2\dot{x}_2 = -\tfrac{k}{ML^2}x_2^2 \leq 0 \longrightarrow$$ Simple stability

Krasowski - La Salle asymptotic stability of the origin

$$\dot{V}(x) = 0 \rightarrow \boxed{x_2 = 0} \rightarrow \dot{x}_1 = 0 \rightarrow x_1 \text{ constant } \bar{x}_1$$

$$\swarrow$$

$$\tfrac{g}{L}sin(\bar{x}_1) = 0 \longrightarrow \boxed{\bar{x}_1 = 0}$$

Lyapunov stability for linear systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Theorem

The necessary and sufficient condition for the asymptotic stability of the system is that, given any matrix $Q = Q' > 0$, there exists a matrix $P = P' > 0$ verifying the following Lyapunov equation

$$A'P + PA = -Q$$

Lyapunov equation

What does it mean? How to use it? Take any *Q>0* and solve the Lyapunov eq. with respect to *P*. If (and only if) *P>0*, then the system is asymptotically stable

Is it easier than computing the eigenvalues of *A*? No, but it is very important in many control design methods

Proof – sufficiency (a sort of exercise)

In the case of linear systems, in order to study the stability we can set ***u***=0. Then the system becomes $\dot{x}(t) = Ax(t)$

Given any matrix $Q > 0$, assume that there exists P satisfying the Lyapunov equation and set $V(x) = x'Px$. It follows that

$$\begin{aligned} \dot{V} &= \dot{x}'Px + x'P\dot{x} = x'A'Px + x'PAx \\ &= x'(A'P + PA)x = -x'Qx < 0 \end{aligned}$$

Hence, V is a Lyapunov function for the system

Proof – necessity: ***see the textbook***

How to find a Lyapunov function for a system linearizable at the equilibrium?

Assume you want to consider the origin as equilibrium point

- Linearize the system
- Solve the Lyapunov equation for the linearized system with a $Q>0$ $\rightarrow$ compute the matrix $P>0$
- Use the Lyapunov function $V(x)=x'Px$ for the analysis of the equilibrium of the original nonlinear system. A least locally it must work

This can be useful to estimate the region of convergence

Example

Analyse the stability of the origin of the system

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

Solution 1

Consider the quadratic Lyapunov function $V(x) = 0.5(x_1^2 + x_2^2) > 0$

$$\dot{V}(x) = \cancel{-x_1 x_2} + \cancel{x_1 x_2} + (x_1^2 - 1)x_2^2 \leq 0 \text{ locally } (x_1^2 < 1)$$

Why only semidefinite negative? Because in a neighbor of the origin $x_2=0$ and x_1 «small» leads to *Vdot(x)=0*

Try to use the Krasowski La Salle theorem to prove that the origin is an asymptotically stable equilibrium

Solution 2

Compute the linearized model

$$\begin{aligned}\delta\dot{x}_1 &= -\delta x_2 \\ \delta\dot{x}_2 &= \delta x_1 - \delta x_2\end{aligned} \longrightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Solve the Lyapunov equation with *Q=I* → $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} > 0$

$$V(x) = x'Px = 1.5x_1^2 - x_1x_2 + x_2^2$$

$$\dot{V}(x) = -x_1^2 - x_2^2 \underbrace{- x_1^3x_2 + 2x_1^2x_2^2}_{\text{higher order terms}} < 0 \text{ locally}$$

Impossible to conclude something about global stability
With *pplane* it can be verified that the origin **is not** a globally stable equilibrium

Exercise

Given the system

$$
\begin{aligned}
\dot{x}_1(t) &= (x_1(t) - x_2(t))(x_2^2(t) - 1) \\
\dot{x}_2(t) &= (x_1^2(t) + x_2(t))(x_1^2(t) - 1)
\end{aligned}
$$

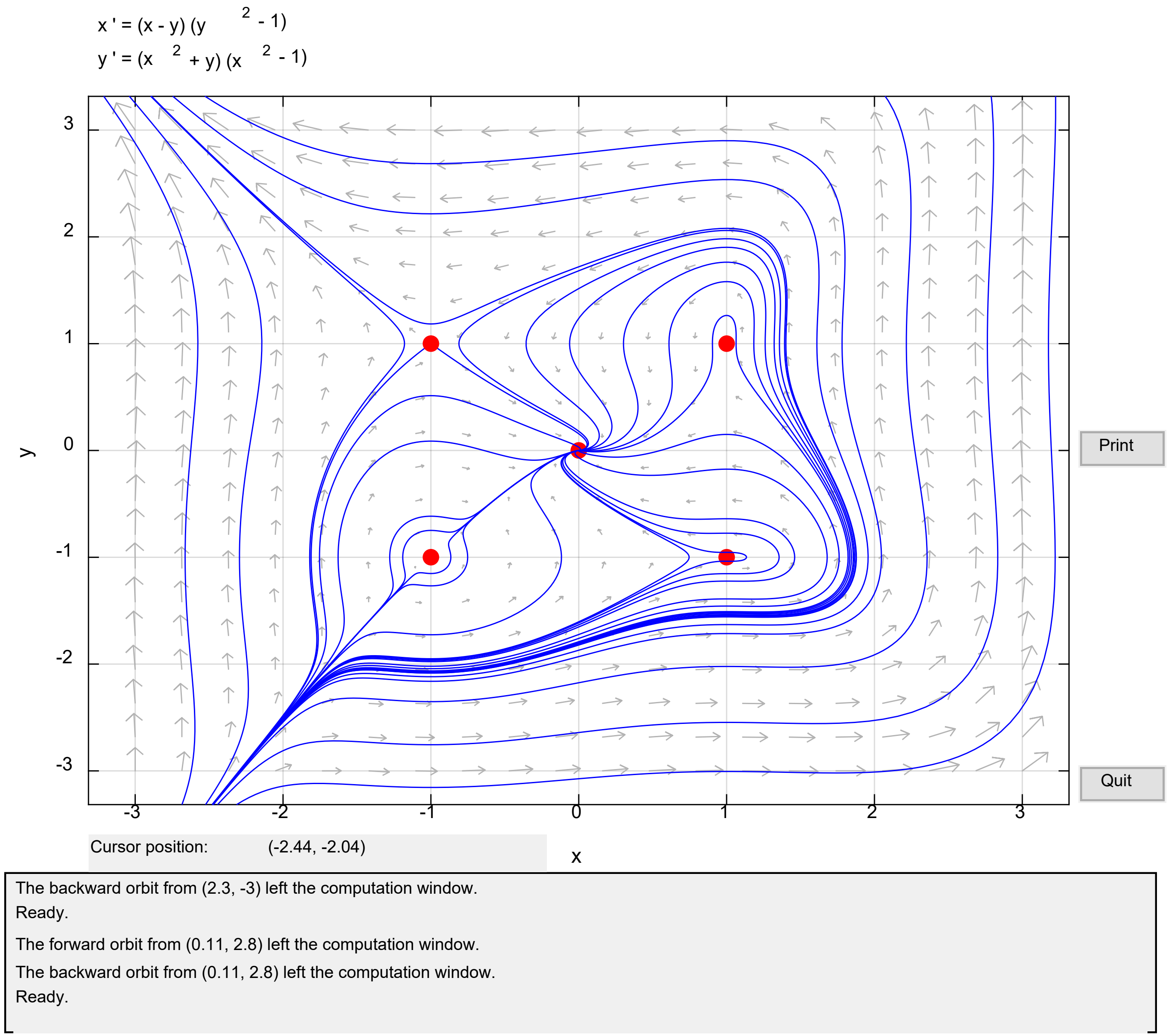
Question 1

Looking at the phase plane, find the equilibria and judge their stability

Answer question 1

$$
\bar{x}_a = (-1, 1), \bar{x}_b = (1, 1), \bar{x}_c = (1, -1),
$$
$$
\bar{x}_d = (-1, -1), \bar{x}_e = (0, 0)
$$

By looking at the trajectories, it seems that the only asymptotically stable equibrium point is (0,0)



Question 2

Check if it is possible to conclude something about the stability of $\bar{x}_a, \bar{x}_b, \bar{x}_e$ by looking at the linearized system

Question 3

Concerning the equilibrium at the origin, check again the result by using the Lyapunov function (in a slightly inproper way) considering

$$P = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

(The proper way is: given a *Q>0*, check if the solution *P* of the Lyapunov equation *A'P+PA=-Q)*

Answer question 2

linearized model

$$
\begin{aligned}
\delta\dot{x}_1 &= (\bar{x}_2^2 - 1)\delta x_1 + (2\bar{x}_1\bar{x}_2 + 1 - 3\bar{x}_2^2)\delta x_2 \\
\delta\dot{x}_2 &= (4\bar{x}_1^3 + 2\bar{x}_1\bar{x}_2 - 2\bar{x}_1)\delta x_1 + (\bar{x}_1^2 - 1)\delta x_2
\end{aligned}
$$

$$\bar{x}_a \to A = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \to \text{eigenvalues } s = \mp 4 \text{ unstable (saddle point)}$$

$$\bar{x}_b \to A = \begin{bmatrix} 0 & 0 \\ -4 & 0 \end{bmatrix} \to \text{eigenvalues } s = 0 \text{ no conclusion}$$

$$\bar{x}_c \to A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \to \text{eigenvalues } s = 1, -1 \text{ unstable}$$

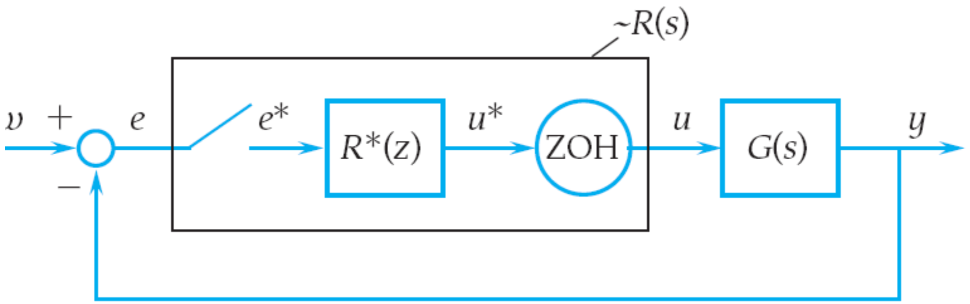
Answer question 3

The solution to the Lyapunov equation is

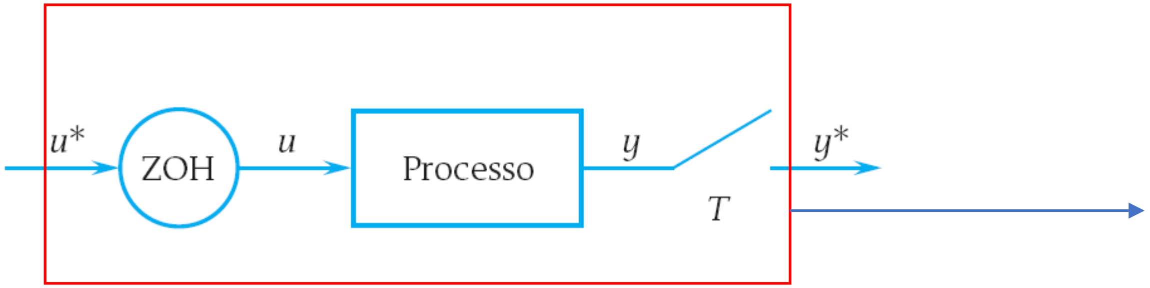
$$Q = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} > 0$$

Design of digital regulators

Strategy 1 – discretise a continuous-time regulator



Strategy 2 – design a discrete time regulator for a discrete time system



$$G(z) = C(zI - A)^{-1}B + D$$

$$\begin{bmatrix} x(k+1) & = Ax(k) + Bu(k) \\ y(k) & = Cx(k) + Du(k) \end{bmatrix}$$

Discrete time systems

$$x(k+1) = f\left(x(k), u(k)\right)$$

$$x(k+1) = f(x(k)) \qquad \text{autonomous systems}$$

$$x(k+1) = Ax(k) + Bu(k) \quad \text{linear systems}$$

Equilibrium

$$\bar{x} = f(\bar{x}, \bar{u})$$

For linear systems

$$\bar{x} = A\bar{x} + B\bar{u}$$

If the eigenvalues of A are different from 1

$$\bar{x} = (I - A)^{-1} B\bar{u}$$

Stability (nothing changes)

The equilibrium $\bar{x}$ is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all the initial states x_0 satisfying

$$\|x_0 - \bar{x}\| \leq \delta$$

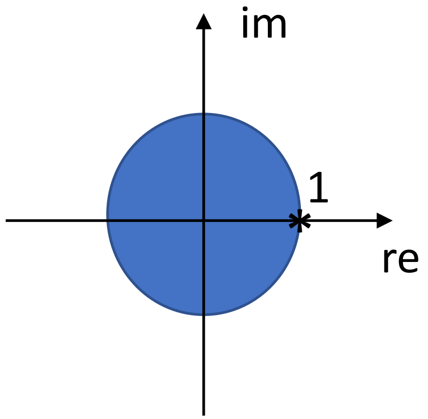
one has

$$\|x(k) - \bar{x}\| \leq \varepsilon \quad , \quad \forall k \geq 0$$

The equilibrium is asymptotically stable if, in addition,

$$\lim_{k \to \infty} \|x(k) - \bar{x}\| = 0$$

Stability region



Linear systems

Necessary and sufficient condition for the asymptotic stability is that all the eigenvalues of *A* have modulus smaller than 1.

Nonlinear systems

$$x(k+1) = f\left(x(k), u(k)\right) \xrightarrow{\text{Linearize (if possible)}} \delta x(k+1) = A\delta x(k) + B\delta u(k)$$

- if all the eigenvalues of A have modulus smaller than 1, then the equilibrium $(\bar{x}, \bar{u})$ is asymptotically stable;
- if at least one eigenvalue of A has modulus greater than 1, then the equilibrium $(\bar{x}, \bar{u})$ is unstable;
- if all the eigenvalues of A have modulus smaller or equal to 1, nothing can be concluded on the stability properties of the equilibrium from the analysis of the linearized system.

Lyapunov method

$$x(k+1) = \varphi(x(k)) \xrightarrow{\textit{equilibrium}} \bar{x} = \varphi(\bar{x})$$

If there exists a function $V(x)$ continuous and *positive definite* in $\bar{x}$ and such that

$$\Delta V(x) = V(\varphi(x)) - V(x) \leq 0$$

in a neighbor of $\bar{x}$, that is if $\Delta V(x)$ is semidefinite negative in $\bar{x}$, then the equilibrium is stable. Moreover, if

$$\Delta V(x) < 0$$

in a neighbor of $\bar{x}$, then $\bar{x}$ is an asymptotically stable equilibrium

Krasowski - La Salle

If there exists a function $V(x)$ *positive definite* in $\bar{x}$, with $\Delta V(x)$ negative semidefinite in $\bar{x}$ and the set

$$S := \{x : \Delta V(x) = 0\}$$

does not contain perturbed (with respect to $\bar{x}$) trajectories compatible with the system, then $\bar{x}$ is an asymptotically stable equilibrium

Linear systems

Necessary and sufficient condition for the asymptotic stability of the linear system

$$x(k+1) = Ax(k)$$

is that for any matrix $Q = Q' > 0$ there exists a matrix $P = P' > 0$ solving the Lyapunov equation

$$A'PA - P = -Q$$

Proof of sufficiency

Given any $Q > 0$, assume that there exists $P > 0$ solving the Lyapunov equation. Now consider the Lyapunov function $V(x) = x'Px$. It follows that

$$\begin{aligned}
\Delta V(x) &= x'(k+1)Px(k+1) - x'(k)Px(k) \\
&= x'(k)A'PAx(k) - x'(k)Px(k) \\
&= x'(k)(A'PA - P)x(k) \\
&= -x'(k)Qx(k) < 0
\end{aligned}$$

Proof of necessity → see the textbook

Example

$$
\begin{cases}
x_1(k+1) = x_2(k)\left(\alpha x_1(k) + \gamma\right), & \gamma \neq -1 \\
x_2(k+1) = x_1(k)\left(\alpha x_2(k) - 1\right)
\end{cases}
$$

Equilibrium

$$
\begin{cases}
\bar{x}_1 = \alpha \bar{x}_1 \bar{x}_2 + \gamma \bar{x}_2 \\
\bar{x}_2 = \alpha \bar{x}_1 \bar{x}_2 - \bar{x}_1
\end{cases}
\longrightarrow (\gamma+1)\bar{x}_2 = 0
$$

$$\downarrow$$

$$\bar{x}_2 = 0, \quad \bar{x}_1 = 0$$

Question 1) Compute the values of α, γ such that the origin is asymptotically stable

Linearized model

$$
\begin{cases}
\delta x_1(k+1) = \gamma\, \delta x_2(k) \\
\delta x_2(k+1) = -\delta x_1(k)
\end{cases}
\longrightarrow A = \begin{vmatrix} 0 & \gamma \\ -1 & 0 \end{vmatrix}
$$

$$\det(zI - A) = z^2 + \gamma = 0 \longrightarrow z = \mp\sqrt{-\gamma}$$

Condition $|\gamma| < 1$, $\forall \alpha$

Question 2) For $\gamma = 0.5$ check the asymptotic stability of the linearized system with the Lyapunov equation

Take $Q = \begin{vmatrix} 0.5 & 0 \\ 0 & 1 \end{vmatrix}$, then

$$A'PA - P = -Q \longrightarrow P = \begin{vmatrix} 2 & 0 \\ 0 & 1.5 \end{vmatrix} > 0 \ !$$

Question 3) Use the previous result to prove the asymptotic stability of the origin of the original system

$$V(x) = x'Px = 2x_1^2 + 1.5x_2^2$$

$$\Delta V(x) = 2\Big[\underbrace{x_2 \cdot (\alpha x_1 + 0.5)}_{x_1(k+1)}\Big]^2 + 1.5\Big[x_1 (\alpha x_2 - 1)\Big]^2$$

$$-2x_1^2 - 1.5x_2^2$$

$$= 2\Big[x_2^2 (\alpha^2 x_1^2 + \alpha x_1 + 0.25\Big] + \Big[x_1^2(\alpha^2 x_2^2 - 2\alpha x_2 + 1\Big]1.5$$

$$-2x_1^2 - 1.5x_2^2$$

$$= -0.5x_1^2 - x_2^2 + \text{higher order terms} < 0$$

LOCALLY

Control synthesis with the Lyapunov theory (some simple ideas)

The Lyapunov theory is the basis for the development of many control synthesis methods for nonlinear systems. Let'a have a look at a couple of examples. Then, we'll introduce a systematic approach for a very specific class of nonlinear systems

A simple example

Given the system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 &= -x_2^3 - x_2\end{aligned}$$

Problem: design a control law *u=k(x) (note: **nonlinear** and **state-feedback**)* such that the origin is an asymptotically stable equilibrium point of the corresponding closed-loop system

One could use the linearized model, but here we would like to obtain global asymptotic stability

System

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 &= -x_2^3 - x_2\end{aligned}$$

Lyapunov function and its derivative

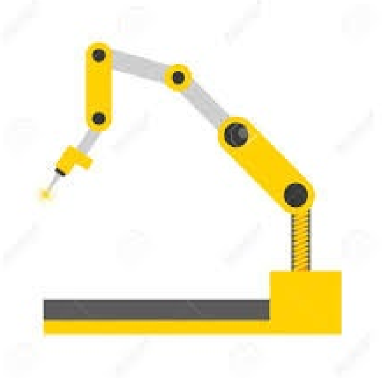
$$V(x) = 0.5(x_1^2 + x_2^2) \longrightarrow \dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -3x_1^2 + 2x_1^2x_2^2 + x_1u - x_2^4 - x_2^2$$

Control law

$$\boxed{u = -2x_1x_2^2} \longrightarrow \dot{V}(x) = -3x_1^2 - x_2^4 - x_2^2 < 0 \longrightarrow$$ origin *globally asympotically* stable

- not the only possible solution
- not a systematic approach

An industrial problem – control of a manipulator – Lyapunov based solution



Given the coordinates *q*, the goal is to drive the manipulator to the equilibrium defined by q_d

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v(q)\dot{q} + g(q) = \tau$$

torque to the joints
control variable

inertial *centrifugal & Coriolis* *friction* *gravitational*

Define $\tilde{q} = q_d - q$ and

$$V(\tilde{q},\dot{q}) = \frac{1}{2}\dot{q}'B(q)\dot{q} + \frac{1}{2}\tilde{q}'K_p\tilde{q}, \quad K_p > 0$$

kinetic energy *potential energy*

$$
\begin{aligned}
\dot{V} &= \dot{q}^{'} B(q)\ddot{q} + \frac{1}{2}\dot{q}^{'}\dot{B}(q)\dot{q} - \dot{q}' K_p \tilde{q} \\
&= \dot{q}^{'}\left[-C(q,\dot{q})\dot{q} - F_v(q)\dot{q} - g(q) + \tau\right] + \frac{1}{2}\dot{q}^{'}\dot{B}(q)\dot{q} - \dot{q}' K_p \tilde{q} \\
&= \frac{1}{2}\dot{q}^{'}\underbrace{\cancel{\left[\dot{B}(q) - 2C(q,\dot{q})\right]}}_{\text{structurally equal to zero}}\dot{q} - \dot{q}^{'} F_v(q)\dot{q} + \dot{q}^{'}\left[\tau - g(q) - K_p \tilde{q}\right]
\end{aligned}
$$

Consider the control law

$$
\boxed{\tau = g(q) + K_p \tilde{q} - K_d \dot{q}, \quad K_d > 0}
$$

$$
\dot{V} = -\dot{q}^{'} F_v(q)\dot{q} - \dot{q}^{'} K_d \dot{q} \leq 0 \longrightarrow \dot{V} < 0 \text{ when } \dot{q} \neq 0,\ \dot{V} = 0 \text{ for } \dot{q} = 0 \text{ and for any } \tilde{q}
$$

From the model equation with the selected control law

$$\cancel{B(q)\ddot{q}} + \cancel{C(q,\dot{q})\dot{q}} + \cancel{F_v(q)\dot{q}} = K_p\tilde{q} - \cancel{K_d\dot{q}} \qquad \text{for } \dot{q} = 0$$

defining $S = \left\{\xi = \begin{bmatrix} \tilde{q} \\ \dot{q} \end{bmatrix} : \dot{q} = 0\right\}$, one has $K_p\tilde{q} = 0$, $\forall \xi \in S$ and, since $K_p > 0$, $\tilde{q} = 0$. In conclusion, $q = q_d$ and $\dot{q} = 0$ is the only asymptotically stable equilibrium

Example of application of the Lyapunov theory – noise cancellation (from Astrom, Murray: «*Feedback systems*» WIKIBOOK)

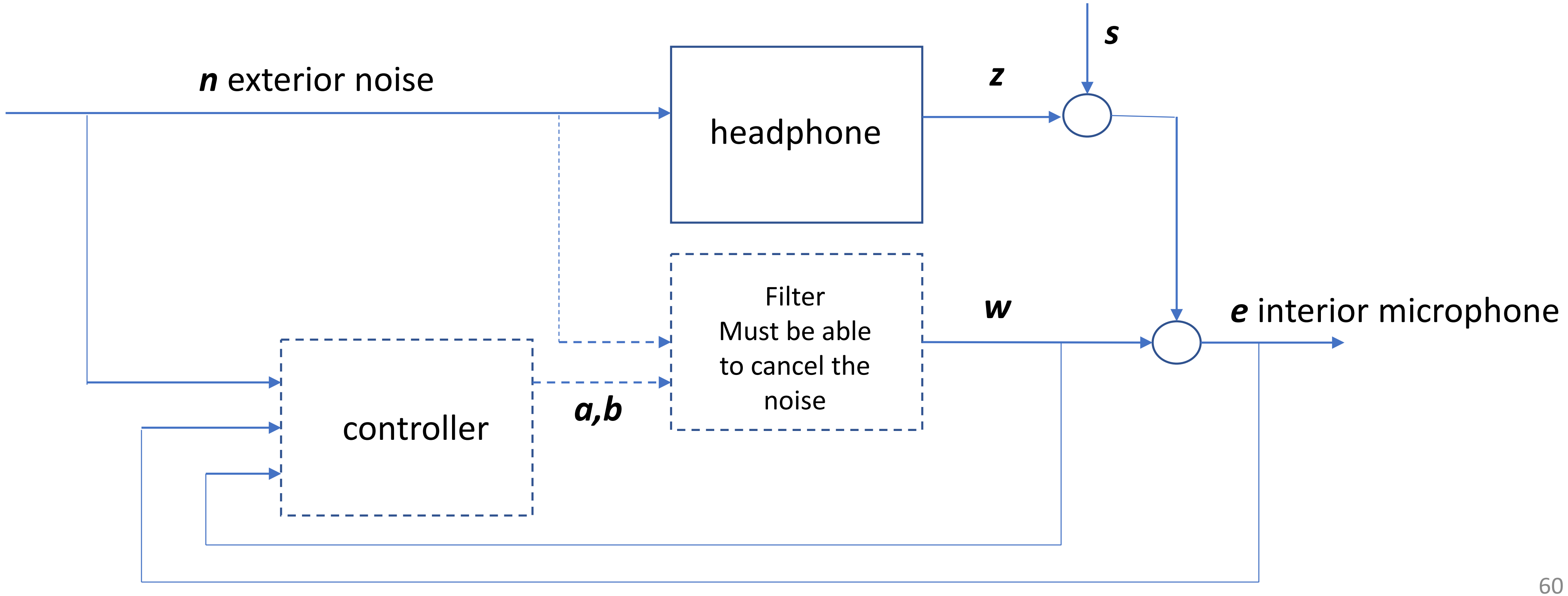
Internal microphone, picks up the signal ***e***, combination of the desired signal ***s*** and of the external noise

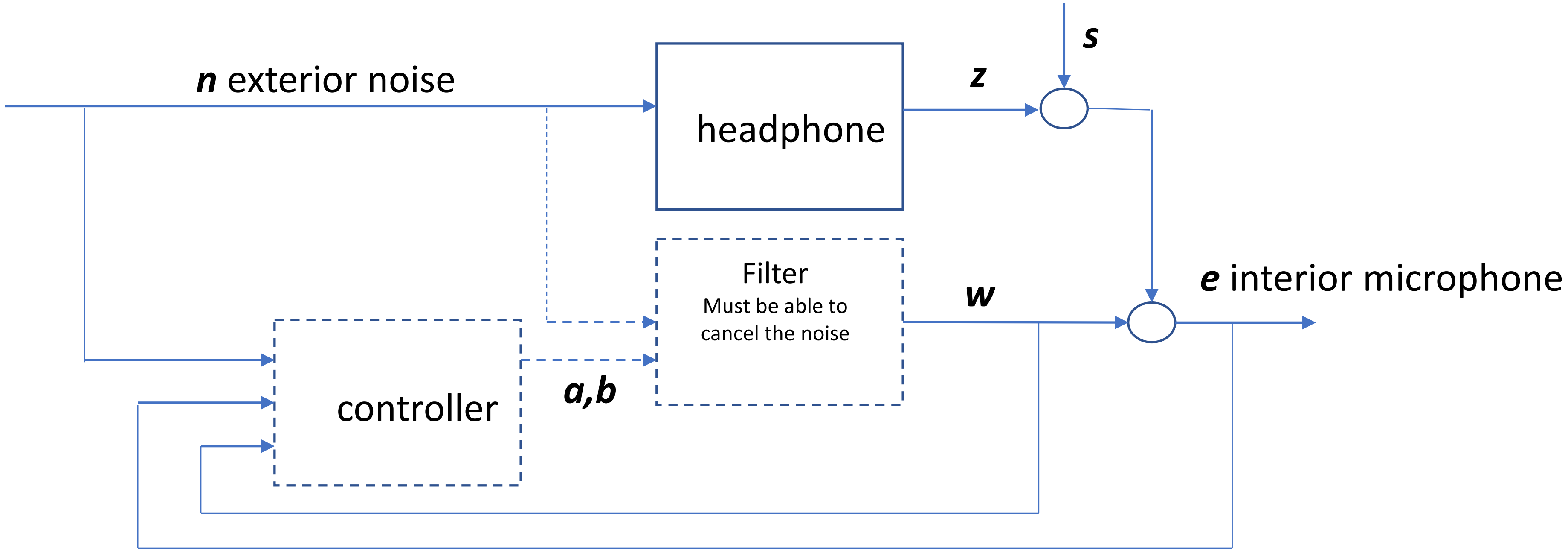
External microphone, picks up the external noise

Internal microphone, picks up the signal ***e***, combination of the desired signal ***s*** and of the external noise

External microphone, picks up the external noise

Control scheme





Assumption $\dot{z}(t) = a_o z(t) + b_o n(t),\ a_0 < 0,$ where a_o, b_o are unknown

Filter $\dot{w}(t) = aw(t) + bn(t)$

Consider $s = 0$ (we focus on noise)

Define (skip the dependence on time) $x_1 = w - z = e,\ x_2 = a - a_o,\ x_3 = b - b_o$

We would like to have (at least) ***x_1->0***

$$\dot{z}(t) = a_o z(t) + b_o n(t) \qquad \dot{w}(t) = a w(t) + b n(t) \qquad x_1 = e = w - z,\ x_2 = a - a_o,\ x_3 = b - b_o$$

$$\dot{x}_1 = \dot{w} - \dot{z} = aw + bn - a_o z - b_o n = (a - a_o)w + a_o(w - z) + (b - b_o)n =$$
$$x_2 w + a_o x_1 + x_3 n$$

Take the Lyapunov function $V(x) = 0.5(\alpha x_1^2 + x_2^2 + x_3^2), \quad \alpha > 0$

$$\dot{V}(x) = \alpha x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3 = \alpha a_o x_1^2 + x_2(\cancel{\alpha x_1 w + \dot{x}_2}) + x_3(\cancel{\alpha x_1 n + \dot{x}_3})$$

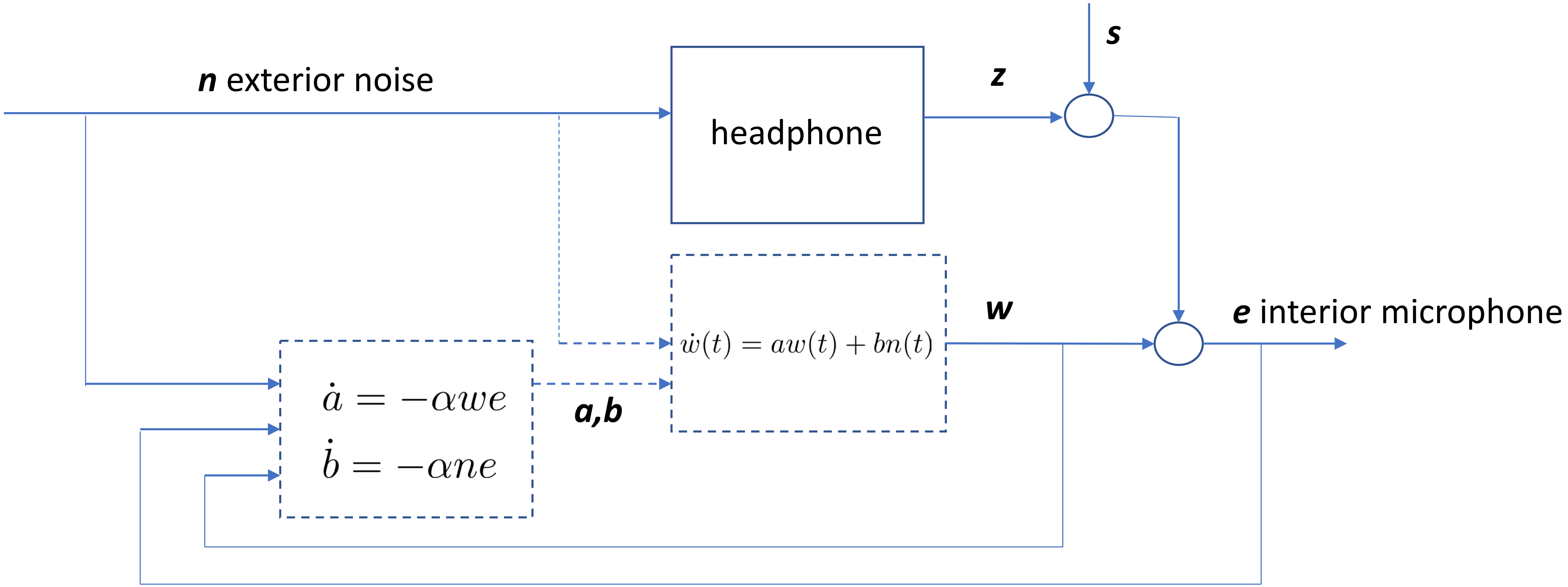
Set $\dot{x}_2 = -\alpha x_1 w = -\alpha w e \quad \dot{x}_3 = -\alpha x_1 n = -\alpha n e$

Then $\dot{V}(x) = \alpha a_o x_1^2 = \alpha a_o e^2 \leq 0 \longrightarrow e \to 0$

Note that $\dot{x}_2 = \dot{a} \quad \dot{x}_3 = \dot{b} \longrightarrow$

$$\dot{a} = -\alpha w e$$
$$\dot{b} = -\alpha n e$$

adaptive control law



n exterior noise
headphone
z
s
w
e interior microphone
$\dot{w}(t) = aw(t) + bn(t)$
$\dot{a} = -\alpha we$
$\dot{b} = -\alpha ne$
a,b

Overall signal: sinusoid (*s*) + broad band noise (*n*)

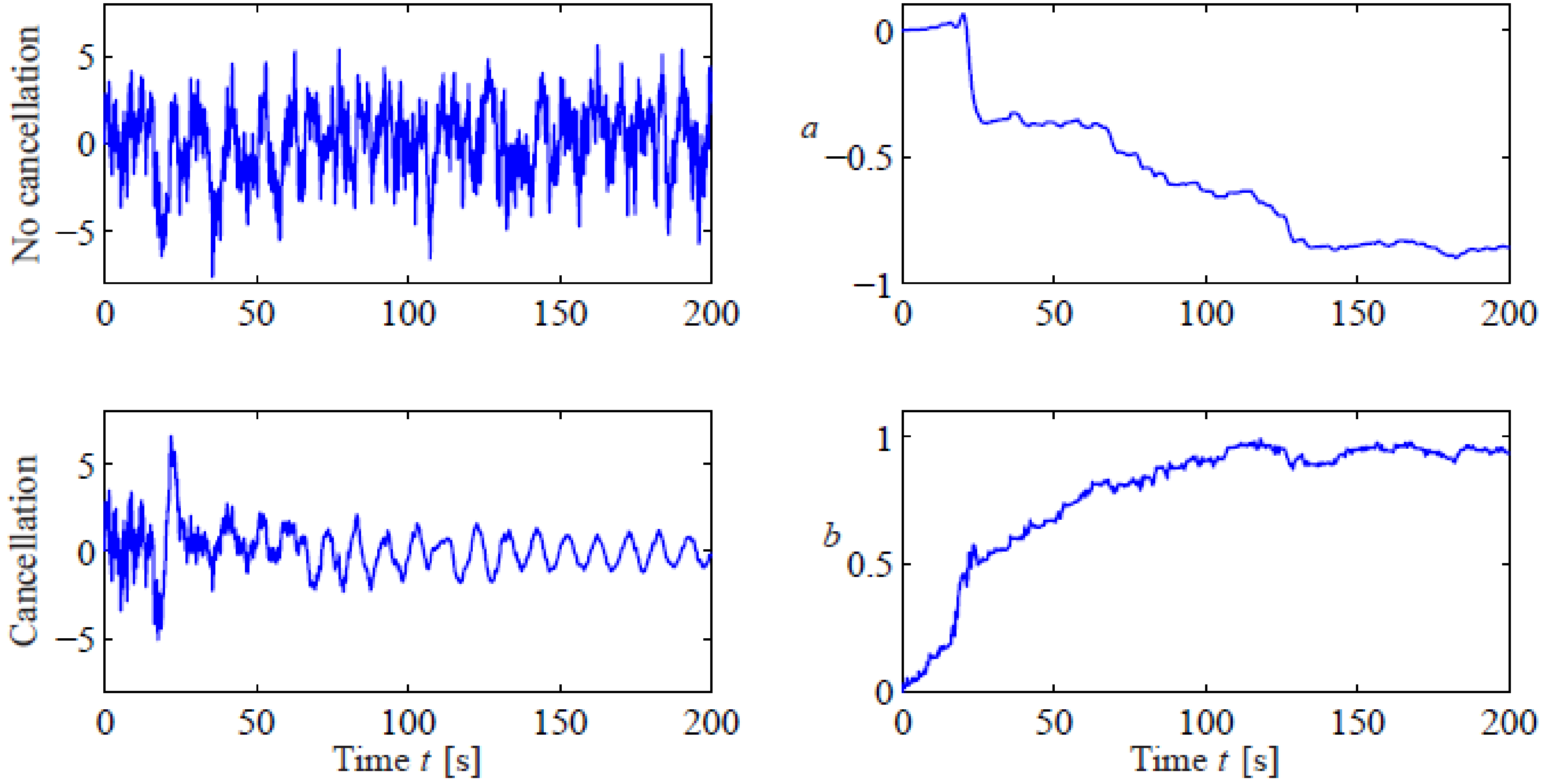


Figure 4.20: Simulation of noise cancellation. The top left figure shows the headphone signal without noise cancellation, and the bottom left figure shows the signal with noise cancellation. The right figures show the parameters a and b of the filter.

A systematic approach – the backstepping method (based on Lyapunov theory)

Consider the system

$$\begin{aligned} \dot{x}_1(t) &= f(x_1(t)) + g(x_1(t))x_2(t) \quad , \; x_1 \in R^n , \; x_2 \in R^1 \\ \dot{x}_2(t) &= u(t) \end{aligned}$$

where f and g are continuous and differentiable in a set $D \subset R^n$ and $f(0) = 0$

Assume to know a «*fictitious* control law»

$$x_2 = \phi_1(x_1) \, , \; \phi_1(0) = 0$$

Note: x_2 is a state, not a control variable

such that the origin of the fictitious closed-loop system

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1)$$

is an asymptotically stable equilibrium

Assume that for the fictitious closed-loop system

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1)$$

you know a Lyapunov function $V_1(x_1) > 0$ such that

$$\dot{V}_1(x_1) = \tfrac{dV_1}{dx_1}\left(f(x_1) + g(x_1)\phi_1(x_1)\right) < 0$$

Then, the control law

$$\boxed{u = -\frac{dV_1(x_1)}{dx_1}g(x_1) - k\left(x_2 - \phi_1(x_1)\right) + \frac{d\phi_1(x_1)}{dx_1}\left(f(x_1) + g(x_1)x_2\right)}$$

<u>*k>0* is a design parameter</u>

asymptotically stabilizes the original system with Lyapunov function

$$V_2(x_1, x_2) = V_1(x_1) + \frac{1}{2}\left(x_2 - \phi_1(x_1)\right)^2$$

Proof: nice exercise of the Lyapunov theory

Summary and example

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\
\dot{x}_2 &= u
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
&\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1) \quad , \quad \dot{x}_2 = u \\
&u = -\frac{dV_1(x_1)}{dx_1} g(x_1) - k\left(x_2 - \phi_1(x_1)\right) + \frac{d\phi_1(x_1)}{dx_1}\left(f(x_1) + g(x_1)x_2\right)
\end{aligned}
}
$$

Step 1: find a «fictitious control law» $x_2 = \phi_1(x_1) = -x_1^2 - x_1$

such that

$$
\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1) = -x_1^3 - x_1
$$

is asymptotically stable with $V_1(x_1) = 0.5x_1^2$

Step 2: apply the formula of the control law with

$$
\begin{aligned}
-\frac{dV_1(x_1)}{dx_1} g(x_1) &= -x_1 \\
-k\left(x_2 - \phi_1(x_1)\right) &= -(x_2 + x_1 + x_1^2) \\
\frac{d\phi_1(x_1)}{dx_1}\left(f(x_1) + g(x_1)x_2\right) &= -(2x_1 + 1)\left(x_1^2 - x_1^3 + x_2\right)
\end{aligned}
$$

$$
\boxed{u = -x_1 - (x_2 + x_1 + x_1^2) - (2x_1 + 1)\left(x_1^2 - x_1^3 + x_2\right)}
$$

A generalization

Consider the system

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t)) + g_1(x_1(t))x_2(t) \quad , \quad x_1 \in R^n, \;\; x_2 \in R^1 \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t))u(t)
\end{aligned}
$$

where $f_2 \in C^1$, $g_2 \in C^1$ and $g_2(x_1, x_2) \neq 0$ in the domain of interest

Set

$$
u = \frac{1}{g_2(x_1, x_2)} \{u_a - f_2(x_1, x_2)\}
$$

where u_a is a fictitious input to be properly selected. It then follows that

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1) + g_1(x_1)x_2(t) \\
\dot{x}_2(t) &= u_a
\end{aligned}
$$

This system has the standard form previously considered. Then, we can apply the corresponding theory

It turns out that

$$
\begin{aligned}
u &= \frac{1}{g_2(x_1,x_2)}\left\{\frac{d\phi_1(x_1)}{dx_1}\left(f_1(x_1)+g_1(x_1)x_2\right)-k\left(x_2-\phi_1(x_1)\right)\right. \\
&- \left.\frac{dV_1(x_1)}{dx_1}g_1(x_1)-f_2(x_1,x_2)\right\}
\end{aligned}
$$

and the Lyapunov function for the closed-loop system is

$$
V_2(x_1,x_2)=V_1(x_1)+\frac{1}{2}\left(x_2-\phi_1(x_1)\right)^2
$$

A systematic approach: *feedback linearization* (not strictly related to Lyapunov theory)

A simple idea is to algebraically transform the dynamics of the nonlinear system into a linear one, and then to apply linear control techniques to stabilize the transformed linear system

Consider the system

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned}$$

and assume that $g_2(x_1, x_2) \neq 0$. Then consider the control law

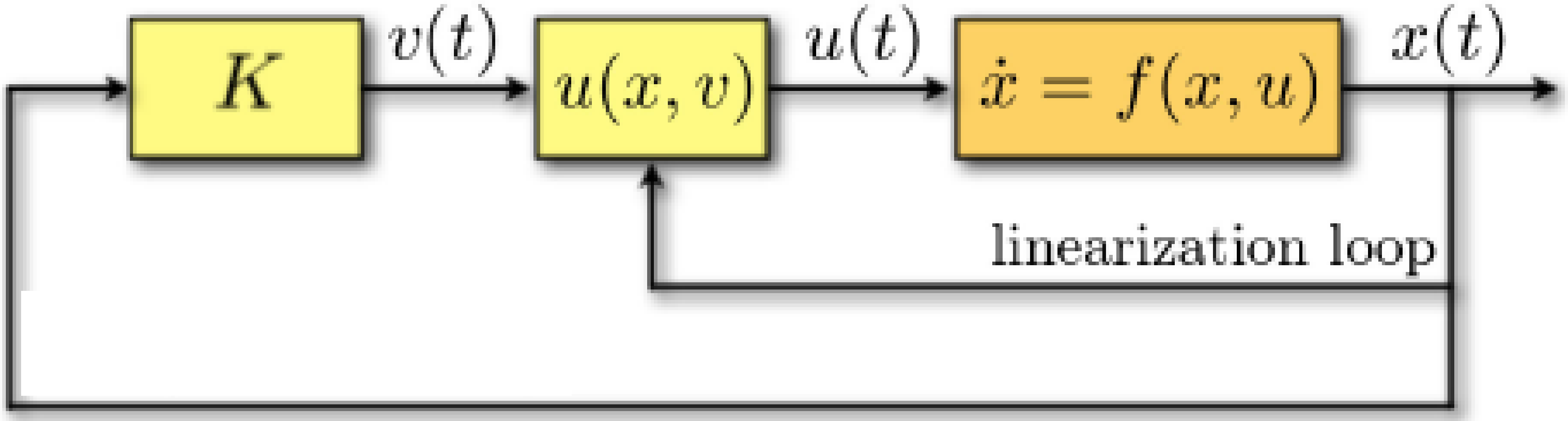
$$u = \frac{v - f_2(x_1, x_2)}{g_2(x_1, x_2)}$$

The system becomes linear

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= v \end{aligned}$$

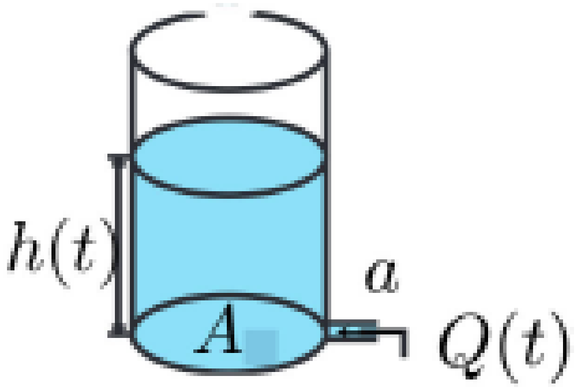
and with linear control theory one can seach for a stabilizing control law, for example $v = Kx$

A much more general theory has been developed. However, the basic idea is simply summarized in the following control scheme



Courtesy of Prof. A. Bemporad

An example



Model of the system $A\dot{h} = -a\sqrt{2gh} + u$

Reference level h^o

Feedback linearization law $u = -a\sqrt{2gh} + Av$

Resulting system $\dot{h} = v$ (integrator)

Proportional control law $v = ke, \quad e = h^o - h, \quad k > 0$

$$\dot{e} = -\dot{h} = -ke, \quad e \to 0, \quad h \to h^o$$

Exercise Lyapunov - Continuous time

$$\begin{cases} \dot{x}_1 = x_1 (x_1^2 + x_2^2 - 2) - 4 x_1 x_2^2 \\ \dot{x}_2 = 4 x_1^2 x_2 + x_2 (x_1^2 + x_2^2 - 2) \end{cases}$$

Study the stability of $\bar{x} = 0$

$$V(x) = \frac{1}{2} (x_1^2 + x_2^2)$$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1^2 (x_1^2 + x_2^2 - 2) - \cancel{4 x_1^2 x_2^2}$$

$$+ \cancel{4 x_1^2 x_2^2} + x_2^2 (x_1^2 + x_2^2 - 2)$$

$$= (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) < 0$$

< 0 in $x_1^2 + x_2^2 < 2$

Locally negative def.

Exercise Lyapunov discrete time

$$x(h+1) = \frac{x(h)}{\sqrt{x^2(h)+1}}$$

study the stability of the origin

Linearised system $\delta x(h+1) = \left. \frac{\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}}{x^2+1} \right|_{x=0} \delta x(h)$

$\delta x(h+1) = \delta x(h) \rightarrow$ eigenvalue $= 1$, no conclusion

With Lyapunov $V(x) = x^2$

$$\Delta V(x) = x^2(h+1) - x^2(h) = \left(\frac{x}{\sqrt{x^2+1}} \right)^2 - x^2 = \frac{-x^4}{x^2+1} < 0$$

asymptotically stable

EXERCISE

Consider the system

$$\dot{x}_1(t) = -x_1(t) + x_2(t)$$
$$\dot{x}_2(t) = -x_1^3(t) - x_2^3(t)$$

and the Lyapunov function $V(x) = bx_1^4 + ax_2^2(t)$. Select proper values of a, b such that $V(x)$ can be used to prove the stability of the origin.

a. *a=1, =b=1*
b. *a=2, b=1*
c. *a=0,b=1*

SOLUTION

Consider the system

$$\dot{x}_1(t) = -x_1(t) + x_2(t)$$
$$\dot{x}_2(t) = -x_1^3(t) - x_2^3(t)$$

and the Lyapunov function $V(x) = bx_1^4 + ax_2^2(t)$. Select proper values of a, b such that $V(x)$ can be used to prove the stability of the origin.

Vdot=$-4bx_1^4+4bx_1^3x_2-2ax_1^3x_2-2ax_2^4$

a. *a=1, =b=1 (Vdot=$-4x_1^4+2x_1^3x_2-2x_2^4$)*
b. *a=2, b=1 (Vdot=$-4x_1^2-4x_2^4<0$)*
c. *a=0,b=1 (V is not a Lyapunov function)*

SOLUTION

Consider the discrete time system

$$x_1(k+1) = 0.5x_1(k) + 1.5x_1^2(k)x_2(k)$$
$$x_2(k+1) = 0.5x_2(k)$$

By means of the quadratic Lyapunov function $V(x) = x_1^2 + x_2^2$, discuss the stability of the origin.

- ○ The origin is a locally asymptotically stable equilibrium
- ○ The origin is a globally asymptotically stable equilibrium
- ○ The origin is an unstable equilibrium
- ○ Nothing can be concluded with that Lyapunof function

EXERCISE

Consider the discrete time system

$$x_1(k+1) = 0.5x_1(k) + 1.5x_1^2(k)x_2(k)$$
$$x_2(k+1) = 0.5x_2(k)$$

By means of the quadratic Lyapunov function $V(x) = x_1^2 + x_2^2$, discuss the stability of the origin.

- The origin is a locally asymptotically stable equilibrium ←
- The origin is a globally asymptotically stable equilibrium
- The origin is an unstable equilibrium
- Nothing can be concluded with that Lyapunof function

$$\Delta V(x) = \underbrace{\left(0.5x_1 + 1.5x_1^2x_2\right)^2}_{x_1^2(k+1)} + \underbrace{(0.5x_2)^2}_{x_2^2(k+1)} - x_1^2 - x_2^2$$

$$= -0.75x_1^2 - 0.75x_2^2 + 1.5x_1^3x_2 + (1.5)^2x_1^4x_2^2$$

≤ 0 locally

Consider a second order system with dynamic matrix A, and a matrix Q equal to the 2x2 identity, the solution of the Lyapunov equation is $P = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$. Then, the system is

- asymptotically stable ← $P > 0$
- nothing can be concluded
- unstable
- simply stable