POLES / ZEROS IN MIMO SYST.

analysis for multivariable systems!

↓

Multivariable case ⟹ extend SISO concept to MIMO

Advanced and Multivariable Control

new concepts

Poles and zeros of MIMO systems

similar to SISO

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focus only on MIMO LINEAR

MIMO linear systems

different representations... STATE FORM

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$x \in R^n$, $u \in R^m$, $y \in R^p$

ORDER n

S.S to T.F easy

to move from S.S to T.F

T.F to S.S (NOT unique SISO, ∞ equivalent S.S representations)

SISO: canonical forms (easy)

the inverse trasformation NOT SO easy!

MIMO: more complex canonical forms

or nonminimal representations

exist a canonical form of A,B,C,D but complex made of BLOCKS, lots time to describe

as many rows as OUTPUTS (p)

columns number as inputs number (m)

$$Y(s) = G(s)U(s) \implies G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{p1}(s) & \dots & G_{pm}(s) \end{bmatrix}$$

T.F form

While on SISO G(s) is a RATIONAL ratio of polinomial on MIMO is a matrix [p × m]

⟹ ***G(s)*** **describes only the reachable and observable part of the system**

the T.F description (according to KALMAN DECOMPOSITION)

→ given G(s) I assume implicitly all syst fully reach/obs

If a system will be described by *G(s)*, it will be implicitly assumed that it is reachable and observable (or, at most, stabilizable and detectable) → eig values hidden are STABLES. goes to 0

We assume given G(s) well described syst.

$\boxed{\text{MIMO}}$: T.F $\xrightarrow{\text{TRANSFORM}}$ S.S you can describe the syst by S.S representation of each G_{ij} and combine it

finding NOT the minimal representation → { Large system redundant }

assume syst. with hidden eig values **stable** so NOT PROBLEMATIC on SISO we just look just denominator of G(s)

Poles

Reach / obs

For systems in minimal form, the poles, including their multiplicity, coincide with the eigenvalues of the matrix A, i.e. with the roots of the characteristic equation

from S.S form representation

$$\phi(s) = \det(sI - A) = 0$$

standard expression...

simply solve this equation

If the number of eigenvalues is larger than the number of poles, the system has unreachable and/or unobservable parts

↳ the eig values NOT PART of poles are unreach / unobs part !

NOT moving to S.S

Is it possible to compute the poles directly from the transfer function *G(s)*?

it is hard to obtain A from G(s) in MIMO syst... Poles from G(s) ?

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

by computing least common den → you compute poles of syst (trivial on SISO where you have just one minor) on MIMO

Example

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

looking single element you see the poles BUT NOT the multiplicity of it

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{3}{s+1} \end{bmatrix}$$

fictitious MIMO → is made by independent SISO of I ORD with pole in (-1)

A *fake* MIMO system! In practice, two first order systems

↓ independent syst toghether, set of poles is the union of the two

(single elements)

Minors of order 1

$$\frac{1}{s+1};\frac{3}{s+1}$$

Minor of order 2 = det (G(s))

$$\frac{1}{s+1}\frac{3}{s+1} = \frac{3}{(s+1)^2}$$

from definition of the poles... least common den of all non null min

Two poles in s=-1 *(as expected)*

for overall system

Example

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

looking $G(s)$ you see there are poles in $s=-1$ BUT anything about multiplicity!

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{s}{s+1} \end{bmatrix}$$

Seems to be of order 4, but …

by definition…

JUST the 4 elements of $G(s)$

Minors of order 1

$$\frac{1}{s+1};-\frac{1}{s+1};\frac{1}{s+1};\frac{s}{s+1}$$

same den with roots in $s=-1$

Minor of order 2

det$(G(s))$

$$\frac{1}{s+1}\frac{s}{s+1}+\frac{1}{s+1}\frac{1}{s+1}=\frac{s+1}{(s+1)^2}=\frac{1}{s+1}$$

Only one pole in s=-1

SYST. OF $\boxed{\text{ORDER } 1}$

$\phi(s) = (s+1)$ least common den of all minors of any order

It is clear that the poles of the system are poles of the single transfer functions of G(s), but their multiplicity cannot be immediately computed by looking at the single transfer functions

↳ you can find a 1 STATE S.S represent.

Example *continued*

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{s}{s+1} \end{bmatrix}$$

$\downarrow$ possible state space form of syst (hard to transform)

$$\begin{aligned} \dot{x}(t) &= -x(t) + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \end{aligned}$$

Minimal first order realization

[2 × 3] system
$y \in \mathbb{R}^2$ $u \in \mathbb{R}^3$
2 outputs, 3 inputs

Example

$$G_3(s) = \begin{bmatrix} \dfrac{1}{(s+1)} & 0 & \dfrac{(s-1)}{(s+1)(s+2)} \\ \dfrac{-1}{(s-1)} & \dfrac{1}{(s+2)} & \dfrac{1}{(s+2)} \end{bmatrix}$$

↓ poles computation

anything about (multiplicity ?)

from minors of ORDER 1 → for sure $-1, +1, -2$ are poles of system

First order minors: the single transfer functions of *G(s)*

ORD 2 MINORS

Minor of order 2 removing column 1 $\quad M_1 = \det \begin{bmatrix} 0 & \dfrac{s-1}{(s+1)(s+2)} \\ \dfrac{1}{s+2} & \dfrac{1}{(s+2)} \end{bmatrix} = -\dfrac{(s-1)}{(s+1)(s+2)^2}$

$s=-2$ multiplicity 2 @ least

Minor of order 2 removing column 2 $\quad M_2 = \det \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s-1}{(s+1)(s+2)} \\ -\dfrac{1}{s-1} & \dfrac{1}{(s+2)} \end{bmatrix} = \dfrac{1}{s+1}\dfrac{1}{s+2} + \dfrac{1}{(s+1)(s+2)} = \dfrac{2}{(s+1)(s+2)}$

↓ poles $-2, -2, -1$

Minor of order 2 removing column 3 $\quad M_3 = \det \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ -\dfrac{1}{s-1} & \dfrac{1}{(s+2)} \end{bmatrix} = \dfrac{1}{(s+1)(s+2)}$

⇓

$$\phi(s) = (s+1)(s+2)^2\underbrace{(s-1)}_{\text{from minors of order 1 !}}$$

four poles in $-1, 1, -2 -2$ → S.S form with ORDER 4 state existing can be obtained

Poles for MIMO are trivial → easy to see from single T.F $G_{ij}(s)$ of system → only multiplicity hard to find

Zeros of *SISO* systems – *remember that the zeros can limit the performance of closed-loop systems*

↓ Important for Regulator design

a $\mathrm{Re}(s)>0$ zero cause limit in performance of ω_c...

LIMITS on ω_c max value so limit φ_m also

Consider the system

SISO

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad G(s) = C(sI - A)^{-1}B + D$$

↷ zeros.

routes of numerator of $G(s)$

and the input $u(t) = u_0 e^{\lambda t}$,

↷ on MIMO zeros don't coincide with the single zeros of the G_{ij}

If λ is not an eigenvalue of A, there exists an initial state x_0 such that, given the input $u_0 e^{\lambda t}$, the output has the same exponential behavior:
$y(t) = G(\lambda)u_0 e^{\lambda t} = y_0 e^{\lambda t}$ → this means that

Zeros important to check for limitations! also on MIMO

Proof: see the textbook

⇒ you can compute x_0 such that

$\forall t$ given the exponential function...

not JUST asymptotical but follow $\forall t$

↓

interesting performance...

IF $G(s)$ contains a ZERO in $\lambda = s$

⇒ means that $G(\lambda) = 0$

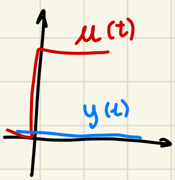
so $\exists x_0 : u(t) = u_0 e^{\lambda t}$ ⇒ $y(t) = 0 \ \forall t$ — 0 output @any time !

example

system: $s G'(s) = G(s)$

' you have a DERIVATIVE ACTION + $G(0) = 0$

in this case, with a STEP input: means that you can find an x_0 such that



$u(t) = u_0 e^{\lambda t}$ FOR $\lambda = 0$ (STEP u_0) $u(t) = u_0$

⇒ $y(t) = 0 \ \forall t$ $\lambda = 0 =$ ZERO

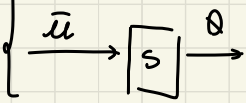
trivial asymptotically ⇒ for a constant input...

Syst with DERIVATIVE ACTION d/dt of const → 0

⇓

{ BUT this is valid $\forall t$ for some x_0 initial value }

asymp P.OV



asymptotically derivative goes to 0

↓ but is valid $\forall t$!

General properties: Given initial state and exp input → output exp from $t \geq 0$ (NOT asymp.)

Zeros of *SISO* systems – *the blocking property*

and y exp coincide with one ZEROS you can get a $y(t) = 0 \ \forall t$

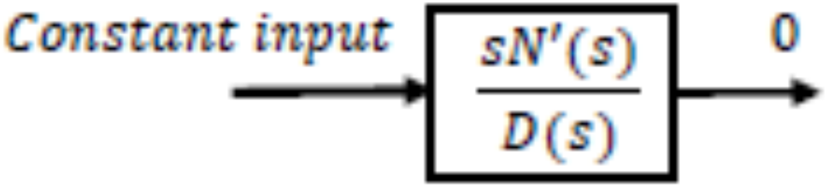
In view of the previous result (proof in the book notes)

IF $\lambda = 0$ → Derivative action such $y(t) = 0 \ \forall t$

If λ is a zero of $G(s)$ $(G(\lambda) = 0)$, then there exists an initial state x_0 such that, given the exponential input $u_0 e^{\lambda t}$, the output is null at any t. The input is then blocked

means that the syst can block the INPUT setting the output @ 0

If the system has a derivative action *(G(0)=0, λ=0),* then there exists x_0 such that, given the input u_0 (a step) the output is null at any $t \geq 0$



Not surprising, at least asymptotically

IF you must design a REGULATOR with derivative action...

$R(s) = s\,R'(s)$ You cannot moove the output

where you want ! → if Asymp stable syst

all signal tends to reach

constant value ⇒ so $u(t) \to 0$

(asymptotically 0 effect) ← (0 output)

ZEROS are foundamentals to look @,

if derivator → impossible

constant Ref syst

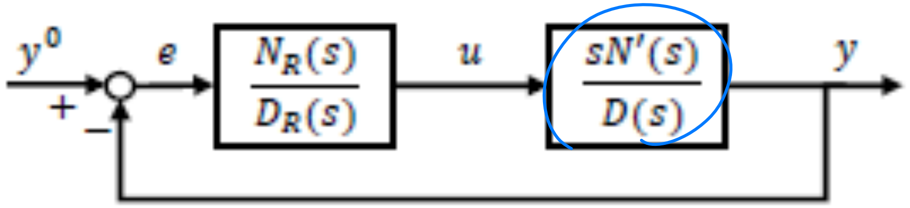
- unstable zeros → limit performance
- zeros @ 0 → limit the static performance (NO TRAKING)

Zeros of *SISO* systems – *limits to performance*

We have already noted that zeros with positive real part limit the achievable performance of closed-loop systems, in terms of crossover frequency

Derivative actions limit the static performance: it is not possible to force the output to reach a constant reference signal

G(s) → for any general Regulator R(s)



Y(s) as derivated of Y°(s)

$$Y(s) = \frac{R(s)G(s)}{1+R(s)G(s)}Y^0(s) = \frac{sN'(s)N_R(s)}{D_R(s)D(s)+N_R(s)sN'(s)}Y^0(s)$$

asymptotically s → 0 (t → ∞) produce output null

The open-loop zero remains as a closed-loop one (and cannot be canceled if it is *s=0*)

General result on closed loop → you cannot move the ZEROS, JUST cancel if NOT on 0 or unstable

How to extend these results to MIMO systems?

with same properties

how to treat for LTI MIMO ?

Zeros of *MIMO* systems – *system matrix*

here the ZEROS of the single $G_{ij}(s)$ are NOT ZEROS of overall system !!

to properly define ZEROS: on MIMO syst.

The system in the Laplace domain (with $x_0 = 0$) can be written as

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

{from s.s form in Laplace domain}

or

(matrix form)

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Y(s) \end{bmatrix}$$

$$\boxed{P(s)} = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

System matrix of our system

in general rectangular matrix! squared if same in/out number

The *normal rank* of $P(s)$ is its rank for any value of s, except for at most a finite number of singularities

normal rank by determinant expressed in s of same order of $P(s)$

IF for example n states, 3 IN, 3 OUT ⇒ P(S) square matrix

assume you have the system:
of I ORD → so A, B, C scalar

<u>normal rank</u> of $P(s)$: norm $\forall s$

det $\neq 0$ for lots s

$$P(s) = \begin{bmatrix} s-A & -B \\ C & D \end{bmatrix} \; ; \; \det(P(s)) = (s-A)D + BC =$$

↑ A, B, C scalar term

rank = 2 : [because det $\neq 0$ so rank 2 of $P(s)$]

↓ because det $\neq 0$

rank := rank $\forall s$ except for singularities

$$\left(\text{value } s = \frac{AD - BC}{D} \right) = A - BC$$

ZERO!

[in other case $\det(P(s)) = 0$] → lead to scales on rank number!

⇓

scalar

$$\left.\begin{aligned} A &= a \\ B &= b \\ C &= c \\ D &= 1 \end{aligned}\right\} \text{ we can compute } G(s) \text{ as}$$

$$G(s) = c \left(\frac{1}{s-a} \right) b + 1$$

similarly from T.F of system → numerator

$$= \frac{cb + s - a}{s - a} \rightarrow \text{ZERO } s = a - cb$$

SISO case!

the numerator is ZERO for

similarly from invariant zero definition, from $P(s)$ definition as rank 2 MATRIX ⇒ SISO case zeros / invariant zeros are same definitions!

- define $P(s)$
- normal rank of $P(s)$ ($\forall s$ rank) from $\det(P(s))$
  (so for SISO you can use invariant ZERO definition same as SISO)
  NOT same for MIMO, where looking on single G_{ij} T.F you don't conclude anything!

Invariant zeros of *MIMO* systems

General definitions

The invariant zeros of a system are the values of s such that the rank of the system matrix $P(s)$ is lower than the normal rank

In the SISO case they coincide with the zeros of $G(s)$ (see the example in the textbook) (on previous page proven!)

The *invariant zeros are not the zeros of the single elements of the transfer function matrix* $G(s)$, but they enjoy the following blocking property

!! IN General

defined on our way...

If λ is an invariant zero, there exist an initial state x_0 and a vector u_0 such that, given an input $u(t) = u_0 e^{\lambda t}$, the output is $y(t) = 0$, $t \geq 0$

same blocking property of SISO syst

slightly different formulation with respect to the SISO case

respect SISO where you take $u_0 e^{\lambda t}$ and find corresponding x_0 such $y = \exp$

here x_0, u_0 jointly defined

Computation of invariant zeros from *G(s)*

The polynomial $z(s)$ of the invariant zeros of $G(s)$ is the polynomial with roots coinciding with all and only the invariant zeros of $G(s)$

⇓ from s.s representation → compute syst matrix $P(s)$
→ compute zeros $z_1, z_2 \ldots$ $z(s) = (s-z_1)(s-z_2)\ldots$ (invariant zeros pol)

$$\boxed{\text{The polynomial } z(s) \text{ of the invariant zeros of } G(s) \text{ is the greatest common divisor of all the numerators of all the minors of order } r \text{ of } G(s)\text{, where } r \text{ is the normal rank of } G(s)\text{, assuming that these minors are written so that they have the polynomial } \phi(s) \text{ of the poles at the denominator}}$$

IF we have an I/O represent ⇒ hard to move to s.s
from $G(s)$ hard to get $P(s)$ of s.s !
is possible to compute z_i from $G(s)$? → Result!

Example

(2×2 system) study invariant ZEROS from G(s)

NO zeros

$$G(s)=\frac{1}{(0.2s+1)(1+s)}\begin{bmatrix}1 & 1\\ 1+2s & 2\end{bmatrix}$$

here the s = 0.5 ZERO IS NOT visible!

ZERO in −0.5 → NO conclusion

seems STABLE

characteristic polynomial $\phi(s)=(0.2s+1)^2(1+s)^2$

$\begin{cases} -1,-1 \\ -5,-5 \end{cases}$ poles

NOT!! UNSTABLE

From previous definition...

$$detG(s)=\frac{2-1-2s}{\phi(s)}=\boxed{\frac{1-2s}{\phi(s)}}$$

minor with denom $\phi(s)$

→ normal rank = 2 → **Invariant zero *s=0.5***

Routes of Num

the det (G(s)) can be found ... JUST one minor of order 2 (not other to study)

↳ write minor to have $\phi(s)$ @ denominator

looking G(s) elements you are NOT able to check zeros!

- Positive zero, implies limits to performance
  important study
- Nothing to do with the zeros of the single transfer functions

Example

/ does NOT tells anything about INV. ZEROS

$$G(s) = \begin{bmatrix} \frac{s+1}{(s+2)(s+3)} & \frac{4}{(s+2)(s+3)} \\ \frac{0.5}{(s+2)(s+3)} & \frac{2}{(s+2)(s+3)} \end{bmatrix}$$

poles in −2, −3
+ study for MULTIPLICITY

characteristic polynomial $\phi(s) = (s+2)^2(s+3)^2$

$$det G(s) = \frac{2s+2-2}{\phi(s)} = \boxed{\frac{2s}{\phi(s)}} \longrightarrow \text{normal rank} = 2 \longrightarrow \text{Invariant zero } s=0$$

study the ZEROS

2×2 matrix det ≠ 0 → rank = 2

ZERO is the s for which the rank of G(s) decrease ⟹

Derivative action implicit on the system

hidden derivative action

⟹ Interpretation: $G(0) = \begin{bmatrix} 1 & 4 \\ 0.5 & 2 \end{bmatrix}\frac{1}{6}, \quad det G(0) = 0$

"DERIVATIVE ACTION hidden" on MIMO

(ISSUE)

impossible to find constant inputs $\bar{u}_1, \bar{u}_2$ such that, for arbitrary constant outputs $\bar{y}_1, \bar{y}_2$ one has

$$\hookrightarrow \quad \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = G(0) \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}$$

limits the STATIC performance of system

MIMO derivative

$$u(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

steady state consideration

$\downarrow$ we can reach set-point?

$$\bar{y} = G(0)\,\bar{u} \longrightarrow \begin{pmatrix} \bar{y}_1 \\ \bar{y}_1 \end{pmatrix} \text{ desired} \Rightarrow \bar{u}_1, \bar{u}_2 \text{ ?}$$

$\left\{ \begin{array}{l} \bar{y} \text{ desired} \\ \text{@ steady} \\ \text{state} \end{array} \right\}$

could be obtained computing

$$\bar{u} = (G(0))^{-1}\bar{y}$$

find $\bar{u}$ such that possible? but taking $G(0)$ has $\underbrace{\det(G(0)) = 0}$

$\nexists\, G^{-1}(0)$!

$\Downarrow$

unable to find inputs such that general output

DERIVATIVE ACTION $\rightarrow$ regime issues for $\bar{y}$ value!

Example - *continued*

we have seen the poles of it:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)} & 0 & \frac{(s-1)}{(s+1)(s+2)} \\ \frac{-1}{(s-1)} & \frac{1}{(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

characteristic polynomial $\phi(s) = (s+1)(s+2)^2(s-1)$

$\Downarrow$ to check the ZEROS..

to compute normal rank we should look to the minors of order 2 to check for which s goes to $\emptyset$

Second order minors

with den as poles $\phi(s)$

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2} = \frac{-(s-1)^2}{(s+1)(s+2)^2(s-1)}$$

$$M_2 = \frac{2}{(s+1)(s+2)} = \frac{2(s+2)(s-1)}{(s+1)(s+2)^2(s-1)}$$

$$M_3 = \frac{1}{(s+1)(s+2)} = \frac{(s+2)(s-1)}{(s+1)(s+2)^2(s-1)}$$

$\longrightarrow$ $z(s) = (s-1)$

zero *s=1*

seems normal rank = 2

but for $s = +1$

$M_1 = M_2 = M_3 = \emptyset$

NOTICE for big system you do it with MATLAB function!

Example

2x2 system

$$G(s) = \frac{1}{(s+2)} \begin{bmatrix} (s-1) & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

characteristic polynomial $\phi(s) = s + 2$

Second order minor $\det(G(s)) = \dfrac{2(s-1)^2 - 18}{(s+2)^2} = \dfrac{2(s^2 - 2s - 8)}{(s+2)^2} = \dfrac{2(s-4)(s+2)}{(s+2)^2} = \boxed{\dfrac{2(s-4)}{(s+2)}}$

pole $s = -2$, zero $s = 4$

↑ making the rank decreases!

from $G_{JK}(s)$ you don't see it!

many control syntesis implement. are on digital domain

Poles and zeros of discrete time systems

$\Downarrow$

all same propertes blocking ecc.. same Def / Algoritms of cont. time

No differences with respect to the definitions and algorithms provided for continuous time systems

System matrix

$$P(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}$$

Derivative action (zero in *z=1*). Same for invariant zeros on DISCRETE TIME

the origin coincide with $z = 1$

a zero on $z=1$ is a derivative action!

same on MIMO with invariant zero on 1!

$$G(z) = \frac{(z-1)N'(z)}{D(z)}$$