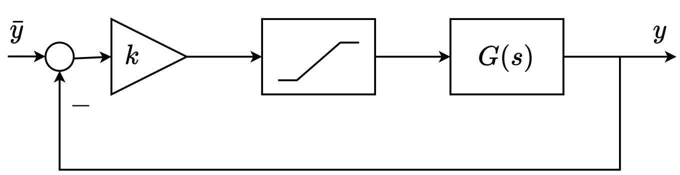
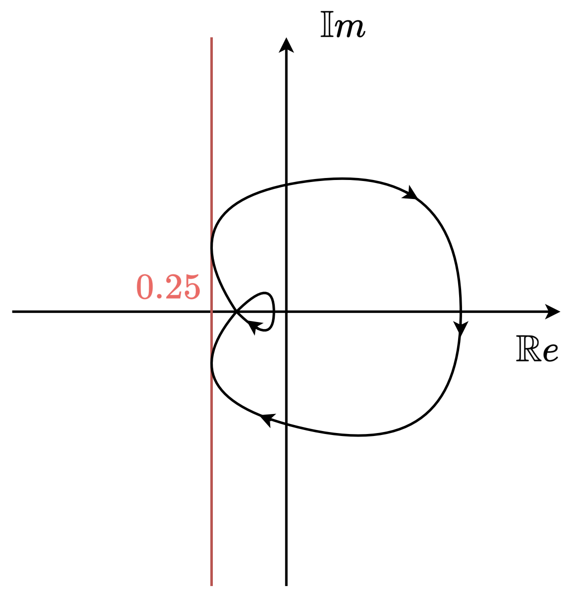
Exercises session 3: Circle Theorem, Uncertainty, MIMO Poles

Ex. 1: Consider the system depicted in Figure 1(a) where $G(s)$ is an asymptotically stable SISO system having the Nyquist diagram depicted in Figure 1(b)

1. Compute the maximum gain K that guarantees the closed-loop stability using the circle criterion.



(a) Block scheme



(b) Nyquist diagram

Figure 1

Ex. 2: Consider the nominal closed-loop system depicted in Figure 2, where

$$\bar{G}(s) \; = \; \frac{1}{1+sT}, \quad T \; > \; 0(A.S.), \tag{1}$$

while the real system is

$$G(s) \; = \; \frac{1}{(1+sT)(1+\alpha s)}, \quad \alpha \; > \; 0, \tag{2}$$

1. Model the uncertainty as both additive and multiplicative
2. Show how to design a controller which is robust to there uncertainties using the small gain theorem.

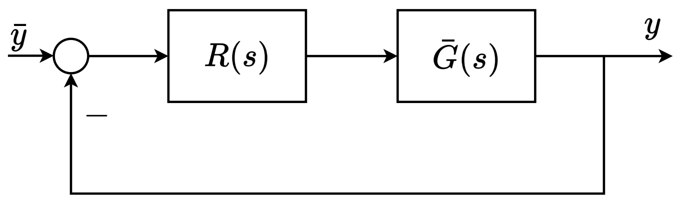


Figure 2: Ex. 2 Block diagram

Ex. 3: Given the MIMO transfer function

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{-1}{s+1} & \dfrac{10}{s+2} \\ \\ \dfrac{2}{s+1} & \dfrac{-0.5}{s+0.25} & \dfrac{10}{s+1} \end{bmatrix} \tag{3}$$

Compute

1. the poles of G(s)
2. and the zeros of G(s).

Ex. 4: Given the following continuos time system

$$\begin{cases} \dot{x} = A\ x + B\ u \\ y = C\ x + D\ u \end{cases} \tag{4}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

1. Compute the poles and zeros of the system
2. Check if the system is fully reachable and observable.
3. and the transfer function $G(s)$.
4. Evaluate the poles of $G(s)$.

Ex. 5: (Additional) Given the system

$$\begin{cases} \dot{x}_1 = -x_1^3 \ + \ x_2 \\ \dot{x}_2 = x_2^2 \ + \ u \end{cases} \tag{6}$$

Find the back stepping control law that stabilizes the origin given the formula

$$u = -\frac{dV_1(x_1)}{dx_1}g(x_1) - k(x_2 - \phi_1(x_1)) + \frac{d\phi_1(x_1)}{dx_1}(f(x_1) + g(x_1)x_2) \tag{7}$$

Hint: Use the extended formulation.

0.1 Additional Informations

Procedure for poles computation

:

1. Compute ALL the minors of any order of $G(s)$,
2. Find their least common denominator $\phi(s)$,
3. Find the roots of $\phi(s)$, well done!

Remark. *o A 'minor of order r' of A is the determinant of an rxr sub matrix of A.*

- *o The characteristic polynomial $\phi(s)$ of a MIMO system is the least common denominator of all the minors of any order of $G(s)$.*
- *o The poles of $G(s)$ are the roots of $G(s)$.*
- *o If a state space representation is available, it may be easier to directly compute the eigenvalues of A.*

Procedure for zeros computation

:

1. Compute the NORMAL RANK r_n of $G(s)$,
2. Compute all the minors of order r_n written to have $\phi(s)$ at the denominator,
3. Compute their maximum common divisor $z(s)$
4. Find the roots of $z(s)$, well done!

Remark. *o The 'Normal Rank' of a matrix $G(s)$ is the rank of $G(s)$ for all the values of s, except for a finite number of values.*

- *o The polynomial $z(s)$ is defined as the maximum common divisor of all the minors of order r_n (normal rank) of $G(s)$, written such that they have $\phi(s)$ as denominator.*
- *o The invariant zeros are all and only the roots of $z(s)$.*