Exercise session 1 - Structural Properties

Advanced and Multivariable Control

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Some preliminary information

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Teaching assistants: Lorenzo Nigro

Lab assistants: Lorenzo Nigro, Matteo Luigi De Pasali

Slide credits: Fabio Bonassi

- Anyone is invited to raise questions during the exercise sessions.
- If you wish to further discuss the topic, drop an e-mail asking for an appointment.
- No online classroom, registrations of last year available on WeBeep.
- **Attending exercise sessions and laboratories is highly recommended!**

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Some organizational stuff

- Exercise sessions are on Tuesday, exceptions will be notified in advance. One class only.
- Six laboratories of 3 hours each, held on Tuesdays, with me and Matteo Luigi De Pascali (S.0.1), One class only.
- A schedule of lessons, excercise Sessions and laboratories will be shared on Beep.

Concerning the laboratories, the following software will be necessary:

- MATLAB (my version is R2022a)
- pplane (see WeBeep)
- Control System Toolbox
- CasADI Toolbox (web.casadi.org/get)
  Download the .zip and add it to your Matlab path

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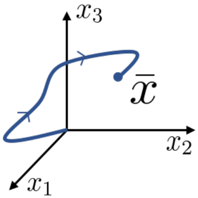
Definition - Reachability (continuous-time systems)

Given the continuous-time system $\dot{x}(t) = A\,x(t) + B\,u(t)$, a state $\bar{x}$ is said to be **reachable** if there exists an arbitrary finite time $\bar{t}$ and an input realization $\bar{u}(\tau)$, $\tau \in [0, \bar{t}\,)$, such that starting from the origin (i.e. $x(0) = 0$), $x(\bar{t}\,) = \bar{x}$.

In other words, a state is reachable if it is possible to design an input sequence to **bring the state** $x(t)$ **from the origin to the desired value**.



Definition

A system is said to be **fully reachable** if all its states are reachable.

Reachability - Example

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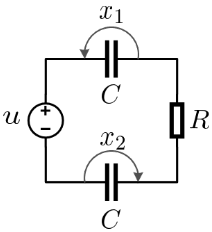
Consider a system made of two capacitors of same size, at equal initial charge.



$$\begin{cases} u = x_1 + R\,i + x_2 \\ C\dot{x}_1 = i \\ C\dot{x}_2 = i \end{cases} \rightarrow \begin{cases} i = -\frac{1}{R}\left[x_1 + x_2 - u\right] \\ \dot{x}_1 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \\ \dot{x}_2 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \end{cases}$$

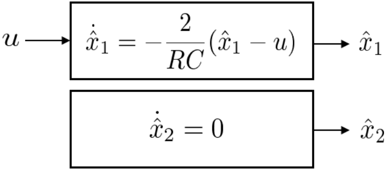
Taking $x = [x_1, x_2]^T$ as state vector, being u the input:

$$A = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{RC} \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{RC} \end{bmatrix}$$

Remark: The system is not fully reachable, since the two states x_1 and x_2 must be equal (capacitors with same size and initial states, fed by the same current i). The "target" states $\bar{x} = (\bar{x}_1, \bar{x}_2)$ are reachable if and only if $\bar{x}_1 = \bar{x}_2$.

To highlight this, let's make a change of variables: $\hat{x}_1 = x_1 + x_2$, $\hat{x}_2 = x_1 - x_2$. Then:

$$\begin{cases} \dot{\hat{x}}_1 = \dot{x}_1 + \dot{x}_2 = -\frac{2}{RC}\left[\hat{x}_1 - u\right] \\ \dot{\hat{x}}_2 = \dot{x}_1 - \dot{x}_2 = 0 \end{cases} \quad \rightarrow \quad \hat{A} = \begin{bmatrix} -\frac{2}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \frac{2}{RC} \\ 0 \end{bmatrix}$$



By means of this change of variables, the system is decomposed into:

- A reachable part, $\hat{x}_1$, function of the input u.
- An unreachable part, $\hat{x}_2$, onto which no control variable is acting.

Reachability in discrete-time

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Definition - Reachability (discrete-time systems)

Given the system $x(k+1) = A\,x(k) + B\,u(k)$, with $x(0) = 0$, a state $\bar{x}$ is said to be **reachable in $\bar{k}$ steps** if there exists an input sequence $u(0), \ldots, u(\bar{k})$ such that $x(\bar{k}) = \bar{x}$.

Definition

A system whose states are reachable in $\bar{k}$ steps is **fully reachable in $\bar{k}$ steps**.

Reachability Matrix

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How to assess the reachability? We start from discrete-time system and then extend the results to continuous-time ones. Starting from $x(0) = 0$, one has that:

$$
\begin{aligned}
x(1) &= Ax(0) + Bu(0) = Bu(0) \\
x(2) &= Ax(1) + Bu(1) = ABu(0) + Bu(1) \\
&\vdots \\
x(n) &= A^{n-1}Bu(0) + A^{n-2}Bu(1) + ... + ABu(n-2) + Bu(n-1)
\end{aligned}
$$

Definition - Reachability Matrix

$$
\mathcal{M}_R = \begin{bmatrix} B & AB & ...A^{n-1}B \end{bmatrix} \tag{1}
$$

Remark: $x(n) = \mathcal{M}_R \cdot \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$

Reachability Matrix (cont'd)

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Theorem 1 - Necessary and sufficient condition for reachability of linear systems

A linear system is **fully reachable** iff rank $(\mathcal{M}_R) = n$, where n is the system's order.

Remark: Why is $\mathcal{M}_R$ constructed using $x(n)$? Suppose to consider $x(n+1)$. Then $\mathcal{M}_R$ will contain $A^n B$ as well. But, in light of Cayley-Hamilton theorem, A^n can be written as a linear combination of $I, A^1, ..., A^{n-1}$, and thus this extra term does not affect the rank of $\mathcal{M}_R$.

Remark: In discrete-time linear systems, states are reachable **at most** in n steps.

Remark - Reachability of continuos-time systems

For **continuous-time** linear systems, $\mathcal{M}_R$ is computed as for discrete-time ones, i.e. by (1), and the same condition as Theorem 1 holds.

Reachability - Example 1 (cont'd)

Considering the previous example, let's compute $\mathcal{M}_R$ (in the original coordinates). Being $n = 2$:

$$\mathcal{M}_R = [B \quad AB] = \begin{bmatrix} \frac{1}{RC} & -\frac{2}{(RC)^2} \\ \frac{1}{RC} & -\frac{2}{(RC)^2} \end{bmatrix}$$

Since $\text{rank}(\mathcal{M}_R) = 1 < n$, the system is not fully reachable (as previously discussed).

We have also shown that, even if the system is not fully reachable, **by means of a suitable change of variables it is possible to decompose the system into its reachable and unreachable part**.

This leads to the following theorem, which is stated for continuous-time systems, but holds for discrete-time ones as well.

Reachable and unreachable parts

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Theorem - Reachability decomposition

Given the system $\dot{x} = Ax + Bu$, not fully reachable, there exists a non-unique change of variables $\hat{x} = T_R x$, where $\hat{x} = [\hat{x}_r^T, \hat{x}_{nr}^T]^T$, which allows to write the system as:

$$\begin{cases} \dot{\hat{x}}_r = \hat{A}_r \hat{x}_r + \hat{A}_x \hat{x}_{nr} + \hat{B}_r u \\ \dot{\hat{x}}_{nr} = \hat{A}_{nr} \hat{x}_{nr} \end{cases} \quad \Leftrightarrow \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$$

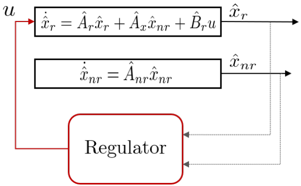
where $\hat{A} = \begin{bmatrix} \hat{A}_r & \hat{A}_x \\ \mathbf{0} & \hat{A}_{nr} \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} \hat{B}_r \\ \mathbf{0} \end{bmatrix}$.

$\hat{x}_r$ is called **reachable part** of the system, while $\hat{x}_{nr}$ is called **unreachable part**.

In particular, if $\text{rank}(\mathcal{M}_R) = n_r$, then $\hat{A}_r$ is $n_r \times n_r$, $\hat{B}_r$ is $n_r \times m$, and

$$\text{rank}\left(\hat{\mathcal{M}}_R = \begin{bmatrix} \hat{B}_r & \hat{A}_r \hat{B}_r & \ldots & \hat{A}_r^{n_r - 1} \hat{B}_r \end{bmatrix}\right) = n_r$$

Stabilizability



Remark: The regulator can act **only** on the reachable part.

- If the unreachable part is asymptotically stable, $\hat{x}_{nr}$ goes to zero and its effect on $\hat{x}_r$ vanishes.
- If the unreachable part is unstable, **nothing can be done to stabilize the system**.

Definition - Stabilizability

A system whose unreachable part is asymptotically stable is said to be **stabilizable**.

Stabilizability is a milder condition than reachability. If a system is not fully reachable, we must check that the dynamics of the unreachable part are asymptotically stable, i.e.:

- All the eigenvalues of $\hat{A}_{nr}$ have negative real part (continuous-time systems)
- All the eigenvalues of $\hat{A}_{nr}$ lie within the unit circle (discrete-time systems)

PBH reachability test

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Checking stabilizability requires to find a change of variables that allows to decompose the system into its reachable and unreachable part, which is not straightforward.

PBH reachability test

The system is fully reachable if and only if

$$P_R(s) = \begin{bmatrix} sI - A & B \end{bmatrix}$$

has rank n for any complex value s.

Note that the only values of s that *could* decrease the rank of $P_R(s)$ are the eigenvalues of A, since for those values it holds that $\det(sI - A) = 0$.

Remark

To assess the reachability of a system, it is enough to check that for any s_*, eigenvalue of A:

$$\text{rank}(P_R(s_*)) = n$$

PBH stabilizability condition

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The PBH reachability test allows to derive a condition for the stabilizability of the system.

PBH stabilizability condition

If the rank of $P_R(s)$ is decreased only in correspondence of asymptotically stable eigenvalues, the system is stabilizable.

Example: Consider the system $\dot{x} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$. Its eigenvalues are $s_1 = -3, s_2 = 1$.

If you compute $\mathcal{M}_R$, rank$(\mathcal{M}_R) < n$. The system is not fully reachable. Is it at least stabilizable?

$$P_R(s) = [sI - A \qquad B] = \begin{bmatrix} s+1 & -2 & 1 \\ -2 & s+1 & 1 \end{bmatrix}$$

- rank $(P_R(s_1))$ = rank $\left(\begin{bmatrix} -2 & -2 & 1 \\ -2 & -2 & 1 \end{bmatrix}\right) = 1 < n$
- rank $(P_R(s_2))$ = rank $\left(\begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 1 \end{bmatrix}\right) = 2 = n$

$\Rightarrow$ The system is stabilizable, because the eigenvalue causing the loss of rank is asymptotically stable ($s_1 = -3$).

Controllability

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- Reachability: bring the state from the origin to any $\bar{x}$.
- **Controllability**: bring the states from any initial $\bar{x}_0$ to the origin.

Definition - Controllability (continuous-time systems)

Given a continuous-time system $\dot{x}(t) = Ax(t) + Bu(t)$, a state $x(0) = \bar{x}_0$ is said **controllable** if there exists a finite arbitrary $\bar{t} > 0$, and an input profile $u(\tau)$, $\tau \in [0, \bar{t})$, such that $x(\bar{t}) = 0$.

Definition - Controllability (discrete-time systems)

Given a discrete-time system $x(k+1) = Ax(k) + Bu(k)$, a state $x(0) = \bar{x}_0$ is said **controllable** in $\bar{k}$ steps if there exists an input sequence $u(0), \ldots, u(\bar{k})$ such that $x(\bar{k}) = 0$.

Definition - Full controllability

A system is said to be fully controllable if all its states are controllable.

Controllability vs reachability

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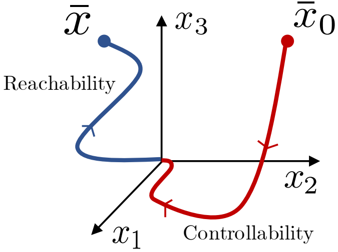


Remark

- For continuous systems, the set of reachable states matches the set of controllable states.
- For discrete systems, if a state $\bar{x}$ is reachable it is also controllable, **but not viceversa!**

Example: Given the system $x(k+1) = 0$:

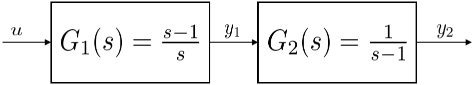
- All the states are controllable, since they are brough to 0 at $\bar{k} = 1$;
- Only the origin is reachable.

Cancellations and unreachability

Where does the unreachability of the system come from?

- From a problem of the model (e.g. the two-capacitors circuit)
- From a zero-pole cancellation

Example



$$G_1 : \begin{cases} \dot{x}_1 = -u \\ y_1 = x_1 + u \end{cases} \qquad G_2 : \begin{cases} \dot{x}_2 = x_2 + y_1 = x_1 + x_2 + u \\ y_2 = x_2 \end{cases}$$

The state-space equation of the system is hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

Cancellations and unreachability (cont'd)

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$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Let's now check the reachability of the system:

$$\mathcal{M}_R = [B \quad AB] = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\,(\mathcal{M}_R) = 1 < n$$

Due to the zero/pole cancellation, an unreachable part is created. Being such unreachable part associated to the unstable pole $(s = 1)$, the system is not stabilizable!

Final comments

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Comment

Reachability is a fundamental property describing systems' structure and the possibility to regulate them. However, **it does not "describe" how** the system reaches the target state $\bar{x}$, i.e. if it exhibits overshoots, oscillations, etc.

Comment

Typically, systems are characterized by the saturation of control variables. In these cases, all the previous results are **not valid**. In these cases, one shall resort to the concept of *constrained reachability*.

Observability in continuous-time

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Definition - Observability (continuous-time system)

Consider a continuous-time autonomous linear system:

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

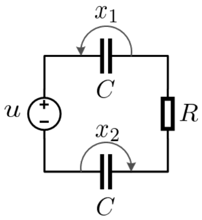
A non-null state $x(0) = \bar{x}_0$ is **non-observable** if, $\forall \bar{t} > 0$ finite, the corresponding free movement due to $\bar{x}_0$, denoted by $\bar{y}(t)$, is constantly zero, i.e. $\bar{y}(\tau) = 0 \quad \forall \tau \in [0, \bar{t}\,]$.

A system **without non observable** states is said to be **fully observable**.

- For observable systems, there do not exist non-null states $\bar{x}$ causing null outputs.
- If the system is not observable, there exist states $\bar{x}$ not "showing up" on the output. **Measuring the outputs is not sufficient to know what's going on with the system.**

Observability - Example

Consider the previous system made of two capacitors of same size, at equal initial charge.



$$\begin{cases} i = -\frac{1}{R}\left[x_1 + x_2 - u\right] \\ \dot{x}_1 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \\ \dot{x}_2 = -\frac{1}{RC}\left[x_1 + x_2 - u\right] \end{cases}$$

It is reminded that $x_1(t) = x_2(t)$.

- Assume to measure $y_1 = x_1 + x_2$:
  The states are **observable** (when x_1 and x_2 are non-null, y_1 is not-null)

- Assume instead to measure $y_2 = x_1 - x_2$:
  The states are **non-observable** (y_2 is null even when x_1 and x_2 are non-null)

Observability - Example

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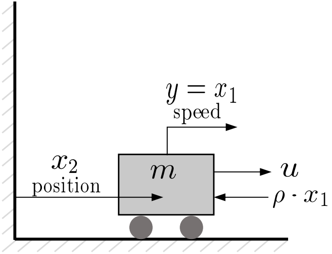

$$\begin{cases} \dot{x}_1 = \frac{1}{m}(u - \rho x_1) \\ \dot{x}_2 = x_1 \\ y(k) = x_1 \end{cases}$$

$$A = \begin{bmatrix} -\frac{\rho}{m} & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Since we measure the speed, we don't have any information on the position!

- If we initialize the system in $\bar{x}_0 = [0, 1]^T$, it stays still, since $y(t) = x_1(t) = 0$. The position does not affect the output.
- Measuring the output (the speed) we cannot reconstruct the position, unless we know the exact initial position).

The system is not fully observable.

Observability in discrete-time

Definition - Observability (discrete-time system)

Consider a discrete-time autonomous linear system:

$$\begin{cases} x(k+1) = Ax(k) \\ y(k) = Cx(k) \end{cases}$$

A non-null state $x(0) = \bar{x}_0$ is **non-observable** if, $\forall \bar{k} \in \mathbb{N}$ finite, the corresponding free movement due to $\bar{x}_0$, denoted by $\bar{y}(k)$, is constantly zero, i.e. $\bar{y}(k) = 0 \quad \forall k \in \{0, \bar{k}\}$.

A system **without non observable** states is said to be **fully observable**.

Observability Matrix

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How to assess the observability? We start from discrete-time systems, and then extend the results to continuous-time ones. Considering an initial state $x(0) = \bar{x}_0 \neq \mathbf{0}$, and no input:

$$\begin{aligned} \bar{y}(0) &= Cx(0) = C\bar{x}_0 \\ \bar{y}(1) &= C\left(Ax(0)\right) = CA\bar{x}_0 \\ &\vdots \\ \bar{y}(n-1) &= CA^{n-1}\bar{x}_0 \end{aligned}$$

Definition - Observability Matrix

$$\mathcal{M}_O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2}$$

Remark: $\bar{y} = \mathcal{M}_O \cdot \bar{x}_0$
Since $\bar{x}_0 \neq 0$, the only way in which $\bar{y}(k)$ can be always zero, is that $\bar{x}_0 \in \text{Ker}\left(\mathcal{M}_O\right)$.

Observability Matrix (cont'd)

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Theorem 2 - Necessary and sufficient condition for observability of linear systems

A linear system is fully observable iff $\text{rank}\left(\mathcal{M}_O\right) = n$, where n is system's order.

Remark - Observability of continuous-time systems

For continuous-time linear systems, $\mathcal{M}_O$ is computed as for discrete-time ones, i.e. by (2), and the same condition as Theorem 2 holds.

Example: In the cart example, the observability matrix is not full rank, therefore the system is not fully observable:

$$\mathcal{M}_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\rho}{m} & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\left(\mathcal{M}_O\right) = 1 < n$$

Observable and non-observable parts

Theorem - observability decomposition

If a system is not fully observable, there exists a non-unique change of variables, $\hat{x} = T_o\, x$, where $\hat{x} = \left[\hat{x}_o^T,\, \hat{x}_{no}^T\right]^T$, which allows to write the system as:

$$\begin{cases} \dot{\hat{x}}_o = \hat{A}_o \hat{x}_o + \hat{B}_o u \\ \dot{\hat{x}}_{no} = \hat{A}_x \hat{x}_o + \hat{A}_{no} \hat{x}_{no} + \hat{B}_{no} u \\ \hat{y} = \hat{C}_o \hat{x}_o \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{y} = \hat{C}\hat{x} \end{cases}$$

where $\hat{A} = \begin{bmatrix} \hat{A}_o & \mathbf{0} \\ \hat{A}_x & \hat{A}_{no} \end{bmatrix}$, $\hat{B} = \begin{bmatrix} \hat{B}_o \\ \hat{B}_{no} \end{bmatrix}$, and $\hat{C} = \begin{bmatrix} \hat{C}_o & \mathbf{0} \end{bmatrix}$.

$\hat{x}_o$ is called **observable part** of the system, while $\hat{x}_{no}$ is called **non-observable part**.

Being rank$(\mathcal{M}_O)$ = rank$(\hat{\mathcal{M}}_O) = n_o$, then $\hat{A}_o$ is $n_o \times n_o$, $\hat{B}_o$ is $n_o \times m$, $\hat{C}_o$ is $p \times n_o$.

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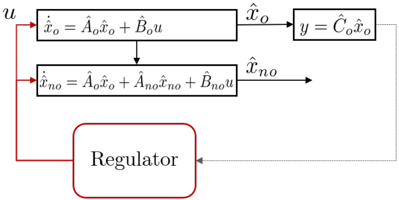

Remark: When we close the control loop, only the observable states are "accounted" by the regulator

- If the non-observable part is asymptotically stable, $\hat{x}_{no}$ vanishes.
- If the non-observable part is unstable, **diverging states cannot be detected from outputs**. The system cannot be controlled.

Definition - Detectability

A system whose non-observable part is asymptotically stable is said to be **detectable**.

Detectability is a milder condition than observability. If a system is not fully observable, we must check that the dynamics of the non-observable part are asymptotically stable.

PBH observability test

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PBH observability test

The system is fully observable if and only if

$$P_O(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix}$$

has rank n for any complex value s.

As for reachability, the only values of s that *could* decrease the rank of $P_O(s)$ are the eigenvalues of A.

Remark

To assess the observability of a system, it is enough to check that for any s_*, eigenvalue of A:

$$\text{rank}(P_O(s_*)) = n$$

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PBH detectability condition

If the rank of $P_O(s)$ is decreased only in correspondence of asymptotically stable eigenvalues, the system is detectable.

Example: Consider the system $\begin{cases} \dot{x} = \begin{bmatrix} -1 & -2 \\ -4 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & -1 \end{bmatrix} x \end{cases}$, with eigenvalues $s_1 = -3$, $s_2 = 3$.

If you compute $\mathcal{M}_O$, rank$(\mathcal{M}_O) < n$. The system is not fully observable. Is it at least detectable?

$$P_O(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} s+1 & 2 \\ 4 & s-1 \\ 1 & -1 \end{bmatrix}$$

- rank $(P_O(s_1))$ = rank $\left(\begin{bmatrix} -2 & 2 \\ 4 & -4 \\ 1 & -1 \end{bmatrix} \right) = 1 < n$
- rank $(P_O(s_2))$ = rank $\left(\begin{bmatrix} 4 & 2 \\ 4 & 2 \\ 1 & -1 \end{bmatrix} \right) = 2 = n$

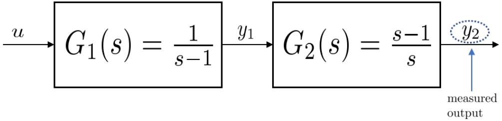
$\Rightarrow$ The system is detectable, because the eigenvalue causing the loss of rank is asymptotically stable ($s_1 = -3$).

Cancellations and non-observability

Where does the non-observability of the system come from?

- From a problem of the model (e.g. the cart with speed measurement)
- From a zero-pole cancellation

Example



$$G_1 : \begin{cases} \dot{x}_1 = x_1 + u \\ y_1 = x_1 \end{cases} \qquad G_2 : \begin{cases} \dot{x}_2 = -x_1 \\ y_2 = y_1 + x_2 = x_1 + x_2 \end{cases}$$

The state-space equation of the system is hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Cancellations and non-observability (cont'd)

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

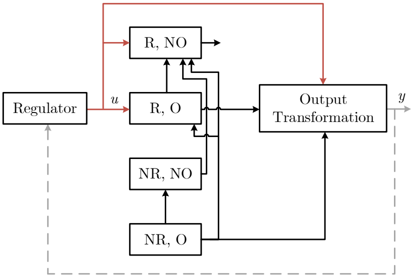
Let's now check both the reachability and the observability of the system:

$$\mathcal{M}_R = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\,(\mathcal{M}_R) = 2 = n$$

$$\mathcal{M}_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\,(\mathcal{M}_O) = 1 < n$$

Due to the zero/pole cancellation, a non-observable part is created. Being such non-observable part associated to the unstable pole ($s = 1$), the system is not detectable!

Kalman Canonical Decomposition



Kalman Canonical Decomposition

For any linear system there exists a change of variable that allows to decompose the system into four parts:

- (R, O): Reachable and observable part
- (NR, NO): Unreachable and non-observable part
- (R, NO): Reachable and non-observable part
- (NR, O): Unreachable and observable part

Observations:

- **Any transfer function represents the reachable and observable (R, O) part only**.
- If the other parts are asymptotically stable, this is not a problem. Otherwise, it is not possible to regulate the system.
- **Remember that the cancellations of unstable poles and zeros are forbidden**, because they create unstable unreachable/non-observable parts.

Realization (SISO)

Given the transfer function of a SISO linear system, we want to **find the underlying state-space model** in its **minimal form**.

Consider a generic transfer function:

$$G(s) = \frac{\beta_n s^n + \beta_{n-1} s^{n-1} + ... + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + ... + \alpha_1 s + \alpha_0} = \hat{\beta}_n + \underbrace{\frac{\hat{\beta}_{n-1} s^{n-1} + ... + \hat{\beta}_1 s + \hat{\beta}_0}{s^n + \alpha_{n-1} s^{n-1} + ... + \alpha_1 s + \alpha_0}}_{\text{Strictly proper}}$$

where $\hat{\beta}_n = \beta_n$ and $\hat{\beta}_i = \beta_i - \alpha_i \beta_n$, for $i = 0, ..., n-1$.

Realization (SISO) (cont'd)

Definition - reachability canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} \hat{\beta}_0 & \hat{\beta}_1 & \dots & \hat{\beta}_{n-1} \end{bmatrix} \quad D = \hat{\beta}_n$$

Definition - observability canonical form

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & \dots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{n-1} \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad D = \hat{\beta}_n$$

Realization (MIMO)

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The realization problem is significantly more complex for MIMO systems.

Given $G(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s) \\ \vdots & \ddots & \vdots \\ G_{p1}(s) & \dots & G_{pm}(s) \end{bmatrix}$ we want to find the state-space model, such that

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

In case of MIMO system, an extensive use of canonical forms is required, which is out of scope.

Example

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Consider the system $\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$, which can be re-written as:

$$\begin{cases} y_1(s) = \tilde{y}_{11} + \tilde{y}_{12} = G_{11}(s)\, u_1(s) + G_{12}(s)\, u_2(s) \\ y_2(s) = \tilde{y}_{21} + \tilde{y}_{22} = G_{21}(s)\, u_1(s) + G_{22}(s)\, u_2(s) \end{cases}$$

To find a **non-minimal** realization of the system, we can find the state-space realization of each SISO element of the transfer matrix $G_{ij}(s)$ separately:

$$\begin{cases} \dot{\tilde{x}}_{ij} = A_{ij}\, \tilde{x}_{ij} + B_{ij}\, u_j \\ \tilde{y}_i = C_{ij}\, \tilde{x}_{ij} + D_{ij}\, u_j \end{cases}$$

Example (cont'd)

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The state-space model of the entire system is therefore:

$$\begin{bmatrix} \dot{\tilde{x}}_{11} \\ \dot{\tilde{x}}_{12} \\ \dot{\tilde{x}}_{21} \\ \dot{\tilde{x}}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & & & \\ & A_{12} & & \\ & & A_{21} & \\ & & & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{12} \\ \tilde{x}_{21} \\ \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & \\ & B_{12} \\ B_{21} & \\ & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & & \\ & & C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{12} \\ \tilde{x}_{21} \\ \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Remark

In general, this is a non-minimal realization.