

ADVANCED AND MULTIVARIABLE CONTROL

June 18, 2019

Surname and Name

University Id.....Signature.....

Exercise 1

Consider the system

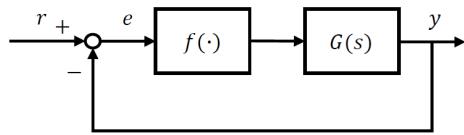
$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + 4x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - x_2^3(t)\end{aligned}$$

- a. Show that the origin is an asymptotically stable equilibrium using the quadratic Lyapunov function $V(x_1, x_2) = \frac{1}{2}(x_1^2 + ax_2^2)$ where $a > 0$ is a suitable design parameter.
- b. Discuss if the origin is a globally asymptotically stable equilibrium;
- c. Describe the form of the trajectories (corresponding to a node, a focus, a saddle point ...) in the neighbor of the origin by analyzing the linearized system.

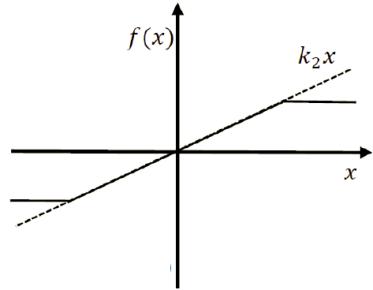
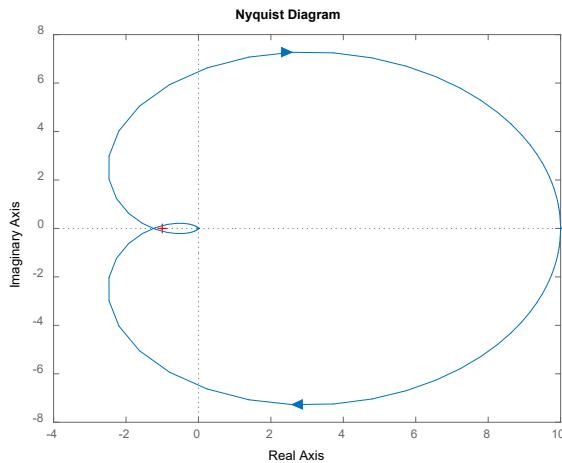
Exercise 2

A. Given the system $y(t) = S(u(t))$, define its gain and the condition for Input/Output stability.

B. Consider the feedback system



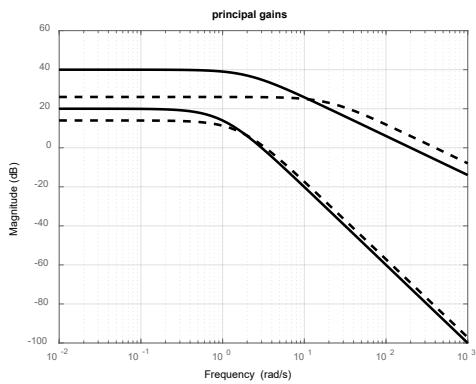
Where $G(s)$ is the transfer function of an asymptotically stable system with the Nyquist diagram reported below together with the form of the saturation $f(\cdot)$.



determine (qualitatively) the maximum value of k_2 guaranteeing the Input/Output stability of the system.

Exercise 3

Consider two MIMO systems with the following principal gains (system S1: continuous line, system S2: dotted line)



1. Which one guarantees more attenuation of low-frequency disturbances acting on the plant output?
2. Which one guarantees more attenuation of high-frequency disturbances acting on the measured output?

Exercise 4

Consider the system

$$\begin{aligned}\dot{x}(t) &= -x(t) + 2u(t) + v_x(t) \\ y(t) &= x(t) + v_y(t)\end{aligned}$$

Where $v_x \sim WN(0, \tilde{q})$, $v_y \sim WN(0, 1)$

- Assuming that the noises are null, compute the infinite horizon LQ control law with $Q=2$, $R=I$

Riccati equation

$$\dot{P}(t) = P(t)A + A'P(t) + Q - P(t)BR^{-1}B'P(t)$$

- Compute the gain margin of the resulting feedback control law and compare it with the one guaranteed by LQ control
- Assuming that the noises are not null, compute the Kalman Filter (KF) gain and write the expression of the KF
- Compute the regulator transfer function obtained by combining the LQ control law and the KF

Exercise 5

Describe one (of your choice) of the methods which can be used to include an integral action in the design of a Model Predictive Control algorithm.

Solution Exercise 1

The origin is the unique equilibrium point. In fact

$$0 = -\bar{x}_1 + 4\bar{x}_2$$

$$0 = -\bar{x}_1 - \bar{x}_2^3$$

has the unique real solution $\bar{x}_1 = \bar{x}_2 = 0$

Note that $V(x)$ is globally positive definite at the origin and radially unbounded. Moreover,

$$\dot{V}(x_1, x_2) = x_1 \dot{x}_1 + ax_2 \dot{x}_2 = -x_1^2 + 4x_1 x_2 - ax_1 x_2 - ax_2^4$$

Choose $a=4$, so that $\dot{V}(x_1, x_2) = -x_1^2 - 4x_2^4 < 0$ globally, and the global asymptotic stability of the origin is proven.

The linearized system at the origin is

$$\delta \dot{x}_1(t) = -\delta x_1 + 4\delta x_2$$

$$\delta \dot{x}_2(t) = -\delta x_1$$

With eigenvalues $s = \frac{-1 \pm \sqrt{-15}}{2}$, complex conjugate. Therefore, the trajectories in a neighbor of the system are those of a stable focus.

Solution Exercise 2

Question A: see the notes, pag. 32-34

Question B:

$$\frac{-1}{k_2} \simeq -3$$

Solution Exercise 3

The two questions are referred to the case where S1 and S2 are the loop transfer functions of two feedback systems with unit feedback. According to paragraph 5.2.5 of the notes (pag. 86-88), the solutions are:

Question 1: System S1 since the smaller singular value is larger than the one of S2

Question 2. System S1 since the larger singular value is smaller than the one of S2

Solution Exercise 4

1. $A = -1, B = 2$, Riccati eq. control $-2\bar{P} + 2 - 4P^2 = 0 \rightarrow \bar{P} = 1/2 \rightarrow K = 1$
Control law $u(t) = -x(t)$, closed-loop eigenvalue $A - BK = -3$

2. Process transfer function $G(s) = \frac{2}{s+1}$, regulator transfer function $R(s) = 1$, gain uncertainty term ρ , characteristic equation $s + 1 + 2\rho = 0 \rightarrow s = -1 - 2\rho$. Stability condition $\rho > -1/2$. Gain margin $(-1/2, \infty)$, guaranteed gain margin $(1/2, \infty)$.

3. $C = 1, \tilde{R} = 1$, Riccati eq. solution: $\tilde{P} = L = -1 + \sqrt{1 + \tilde{Q}}$. Kalman Filter:

$$\frac{d\hat{x}(t)}{dt} = -\sqrt{1+\tilde{Q}}\hat{x}(t) + 2u(t) + (-1 + \sqrt{1+\tilde{Q}})y(t)$$

4. Regulator transfer function $R(s) = K(sI - A + BK + LC)^{-1}L = \frac{-1 + \sqrt{1+\tilde{Q}}}{s + 2 + \sqrt{1+\tilde{Q}}} = \frac{\frac{-1}{2 + \sqrt{1+\tilde{Q}}} + 1}{\frac{-1}{2 + \sqrt{1+\tilde{Q}}}s + 1}$

Solution Exercise 5

See the notes, paragraph 12.2.3 (pag. 190-193), or alternatively Paragraph. 12.3.2 (pag. 199-200).