# Advanced and Multivariable Control

***Poles and zeros of MIMO systems***

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## MIMO linear systems

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)\\ y(t) &= Cx(t)+Du(t)\end{aligned} \qquad x\in R^n, u\in R^m,\ y\in R^p$$

*easy* (↓)

↑ *SISO: canonical forms (easy)*

*MIMO: more complex canonical forms or nonminimal representations*

$$Y(s)=G(s)U(s) \qquad G(s)=C(sI-A)^{-1}B+D=\begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s)\\ \vdots & \ddots & \vdots\\ G_{p1}(s) & \dots & G_{pm}(s)\end{bmatrix}$$

***G(s)* describes only the reachable and observable part of the system**

**If a system will be described by *G(s)*, it will be implicitly assumed that it is reachable and observable (or, at most, stabilizable and detectable)**

## Poles

For systems in minimal form, the poles, including their multiplicity, coincide with the eigenvalues of the matrix $A$, i.e. with the roots of the characteristic equation

$$\phi(s) = \det(sI - A) = 0$$

If the number of eigenvalues is larger than the number of poles, the system has unreachable and/or unobservable parts

Is it possible to compute the poles directly from the transfer function *G(s)*?

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

**Example**

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{3}{s+1} \end{bmatrix}$$

**A *fake* MIMO system! In practice, two first order systems**

Minors of order 1

$$\frac{1}{s+1}; \frac{3}{s+1}$$

Minor of order 2

$$\frac{1}{s+1}\frac{3}{s+1} = \frac{3}{(s+1)^2}$$

***Two poles in s=-1*** *(as expected)*

**Example**

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{s}{s+1} \end{bmatrix}$$

***Seems to be of order 4, but …***

Minors of order 1

$$\frac{1}{s+1}; -\frac{1}{s+1}; \frac{1}{s+1}; \frac{s}{s+1}$$

Minor of order 2

$$\frac{1}{s+1}\frac{s}{s+1} + \frac{1}{s+1}\frac{1}{s+1} = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$$

***Only one pole in s=-1***

*It is clear that the poles of the system are poles of the single transfer functions of G(s), but their multiplicity cannot be immediately computed by looking at the single transfer functions*

**Example** *continued*

The characteristic polynomial $\phi(s)$ associated with a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all the non null minors of any order of $G(s)$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{s}{s+1} \end{bmatrix}$$

$$\downarrow$$

$$\begin{aligned} \dot{x}(t) &= -x(t) + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \end{aligned}$$

*Minimal first order realization*

**Example** $$G_3(s) = \left[\begin{array}{ccc} \frac{1}{(s+1)} & 0 & \frac{(s-1)}{(s+1)(s+2)} \\ \frac{-1}{(s-1)} & \frac{1}{(s+2)} & \frac{1}{(s+2)} \end{array}\right]$$

First order minors: the single transfer functions of *G(s)*

Minor of order 2 removing column 1 $$M_1 = \det\left[\begin{array}{cc} 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{(s+2)} \end{array}\right] = -\frac{(s-1)}{(s+1)(s+2)^2}$$

Minor of order 2 removing column 2 $$M_2 = \det\left[\begin{array}{cc} \frac{1}{s+1} & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{s-1} & \frac{1}{(s+2)} \end{array}\right] = \frac{1}{s+1}\frac{1}{s+2} + \frac{1}{(s+1)(s+2)} = \frac{2}{(s+1)(s+2)}$$

Minor of order 2 removing column 3 $$M_3 = \det\left[\begin{array}{cc} \frac{1}{s+1} & 0 \\ -\frac{1}{s-1} & \frac{1}{(s+2)} \end{array}\right] = \frac{1}{(s+1)(s+2)}$$

$$\phi(s) = (s+1)(s+2)^2(s-1) \quad \text{four poles in } -1, 1, -2-2$$

## Zeros of *SISO* systems – *remember that the zeros can limit the performance of closed-loop systems*

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}, \qquad G(s) = C(sI - A)^{-1}B + D$$

and the input $u(t) = u_0 e^{\lambda t}$,

If $\lambda$ is not an eigenvalue of $A$, there exists an initial state $x_0$ such that, given the input $u_0 e^{\lambda t}$, the output has the same exponential behavior:
$y(t) = G(\lambda) u_0 e^{\lambda t} = y_0 e^{\lambda t}$

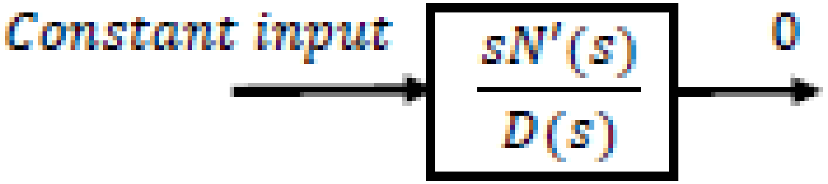
*Proof: see the textbook*

## Zeros of *SISO* systems – *the blocking property*

In view of the previous result

> If $\lambda$ is a zero of $G(s)$ $(G(\lambda) = 0)$, then there exists an initial state $x_0$ such that, given the exponential input $u_0 e^{\lambda t}$, the output is null at any $t$. The input is then blocked

If the system has a derivative action (*G(0)=0, $\lambda$=0),* then there exists $x_0$ such that, given the input $u_0$ (a step) the output is null at any $t \geq 0$

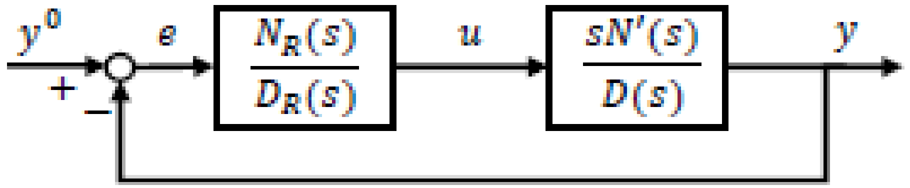



*Not surprising, at least asymptotically*

## Zeros of *SISO* systems – *limits to performance*

We have already noted that zeros with positive real part limit the achievable performance of closed-loop systems, in terms of crossover frequency

Derivative actions limit the static performance: it is not possible to force the output to reach a constant reference signal




$$Y(s) = \frac{R(s)G(s)}{1+R(s)G(s)}Y^0(s) = \frac{sN'(s)N_R(s)}{D_R(s)D(s)+N_R(s)sN'(s)}Y^0(s)$$

**The open-loop zero remains as a closed-loop one (and cannot be canceled if it is *s=0*)**

***How to extend these results to MIMO systems?***

## Zeros of *MIMO* systems – *system matrix*

The system in the Laplace domain (with $x_0 = 0$) can be written as

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

or

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Y(s) \end{bmatrix}$$

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$ ***System matrix***

The *normal rank* of $P(s)$ is its rank for any value of $s$, except for at most a finite number of singularities

# Invariant zeros of *MIMO* systems

The invariant zeros of a system are the values of $s$ such that the rank of the system matrix $P(s)$ is lower than the normal rank

In the SISO case they coincide with the zeros of $G(s)$ (see the example in the textbook)

The *invariant zeros are not the zeros of the single elements of the transfer function matrix* $G(s)$, but they enjoy the following blocking property

If $\lambda$ is an invariant zero, there exist an initial state $x_0$ and a vector $u_0$ such that, given an input $u(t) = u_0 e^{\lambda t}$, the output is $y(t) = 0$, $t \geq 0$

*slightly different formulation with respect to the SISO case*

## Computation of invariant zeros from *G(s)*

The polynomial $z(s)$ of the invariant zeros of $G(s)$ is the polynomial with roots coinciding with all and only the invariant zeros of $G(s)$

The polynomial $z(s)$ of the invariant zeros of $G(s)$ is the greatest common divisor of all the numerators of all the minors of order $r$ of $G(s)$, where $r$ is the normal rank of $G(s)$, assuming that these minors are written so that they have the polynomial $\phi(s)$ of the poles at the denominator

**Example**

$$G(s) = \frac{1}{(0.2s+1)\,(1+s)} \begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

characteristic polynomial $\quad \phi(s) = (0.2s+1)^2\,(1+s)^2$

$$detG(s) = \frac{2-1-2s}{\phi(s)} = \frac{1-2s}{\phi(s)}$$ ⟶ normal rank = 2 ⟶ **Invariant zero *s=0.5***

Positive zero, implies limits to performance

Nothing to do with the zeros of the single transfer functions

**Example**

$$G(s) = \left[ \begin{array}{cc} \frac{s+1}{(s+2)(s+3)} & \frac{4}{(s+2)(s+3)} \\ \frac{0.5}{(s+2)(s+3)} & \frac{2}{(s+2)(s+3)} \end{array} \right]$$

characteristic polynomial $\phi(s) = (s+2)^2\,(s+3)^2$

$$detG(s) = \frac{2s+2-2}{\phi(s)} = \frac{2s}{\phi(s)} \longrightarrow$$ normal rank = 2 $\longrightarrow$ **Invariant zero *s=0***
***Derivative action***

Interpretation: $G(0) = \left[ \begin{array}{cc} 1 & 4 \\ 0.5 & 2 \end{array} \right] \frac{1}{6} \quad , \quad detG(0) = 0$

impossible to find constant inputs $\bar{u}_1, \bar{u}_2$ such that, for arbitrary constant outputs $\bar{y}_1, \bar{y}_2$ one has

$$\left[ \begin{array}{c} \bar{y}_1 \\ \bar{y}_2 \end{array} \right] = G(0) \left[ \begin{array}{c} \bar{u}_1 \\ \bar{u}_2 \end{array} \right]$$

## Example - *continued*

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)} & 0 & \frac{(s-1)}{(s+1)(s+2)} \\ \frac{-1}{(s-1)} & \frac{1}{(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

characteristic polynomial $\phi(s) = (s+1)(s+2)^2(s-1)$

Second order minors

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2} = \frac{-(s-1)^2}{(s+1)(s+2)^2(s-1)}$$

$$M_2 = \frac{2}{(s+1)(s+2)} = \frac{2(s+2)(s-1)}{(s+1)(s+2)^2(s-1)} \longrightarrow z(s) = (s-1)$$

**zero *s=1***

$$M_3 = \frac{1}{(s+1)(s+2)} = \frac{(s+2)(s-1)}{(s+1)(s+2)^2(s-1)}$$

**Example**

$$G(s) = \frac{1}{(s+2)} \begin{bmatrix} (s-1) & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

characteristic polynomial $\phi(s) = s + 2$

Second order minor $\det(G(s)) = \dfrac{2(s-1)^2 - 18}{(s+2)^2} = \dfrac{2(s^2 - 2s - 8)}{(s+2)^2} = \dfrac{2(s-4)(s+2)}{(s+2)^2} = \dfrac{2(s-4)}{(s+2)}$

$$\text{pole } s = -2, \text{ zero } s = 4$$

## Poles and zeros of discrete time systems

No differences with respect to the definitions and algorithms provided for continuous time systems

System matrix

$$P(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}$$

Derivative action (zero in *z=1*). Same for invariant zeros

$$G(z) = \frac{(z-1)N'(z)}{D(z)}$$