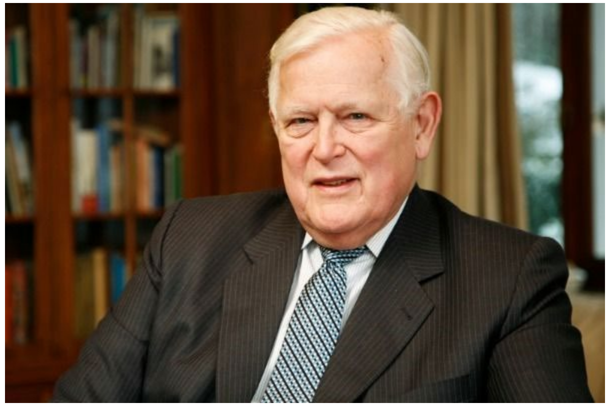

Advanced and Multivariable Control

Kalman Filter

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System

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + v_x(t) \\ y(t) &= Cx(t) + v_y(t) \end{aligned}$$

Disturbances

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \xrightarrow{\text{white gaussian noise}} \quad E\left[v(t)\right] = 0, \quad E\left[v(t_1)v(t_2)'\right] = V\delta(t_1 - t_2) \qquad \delta \text{ Kroenecker index}$$

$$V = \begin{bmatrix} \tilde{Q} & Z \\ Z' & \tilde{R} \end{bmatrix}, \quad \tilde{Q} \in R^{n,n}, \quad \tilde{R} \in R^{p,p}, \quad Z \in R^{n,p}$$

$\tilde{Q} \geq 0$

state noise

$\tilde{R} > 0$

measurement noise

usually
≠ 0
for instance some outputs and states can coincide

Initial assumption (to be removed) $\longrightarrow$ $Z = 0$

Initial state

$$x_0 = x(0) \xrightarrow{\text{Gaussian random variable}} E[x_0] = \bar{x}_0 \quad , \quad E\left[(x_0 - \bar{x}_0)\,(x_0 - \bar{x}_0)'\right] = \tilde{P}_0 \geq 0$$

Additional assumption

$$E\left[x_0 \begin{bmatrix} v_x' & v_y' \end{bmatrix}\right] = 0$$

Goal: to design a state observer «optimal» according to a given criterion which considers the *presence and characteristics* of the noises

otherwise a pole placement observer could be used

Some comments on the parameters

Proper knowledge of $\bar{x}_0, \tilde{P}_0$, not so critical

Knowledge of $\tilde{R} > 0$: possible, it represents the measurement noise, often provided by sensors' manufactures

Knowledge of $\tilde{Q} \geq 0$: often difficult. Can we use it as a design parameter to obtain faster or slower state estimation? See later

Structure of the filter/observer

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + v_x(t)\\ y(t) &= Cx(t) + v_y(t)\end{aligned}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(t)\,[y(t) - C\hat{x}(t)]$$

time varying gain to be selected according to an optimality criterion

State estimation error $\quad e(t) = x(t) - \hat{x}(t)$

$$\begin{aligned}\dot{e}(t) &= Ae(t) - L(t)\,[Ce(t) + v_y(t)] + v_x(t)\\ &= [A - L(t)C]\,e(t) + B_c(t)v(t)\\ &= A_c(t)e(t) + B_c(t)v(t)\end{aligned}$$

$$A_c(t) = [A - L(t)C]\,, \quad B_c(t) = \begin{bmatrix} I & -L(t) \end{bmatrix}$$

$$\dot{e}(t) = A_c(t)e(t) + B_c(t)v(t) \xrightarrow{\text{set } \bar{e}(t) = E\left[e(t)\right]} \dot{\bar{e}}(t) = A_c(t)\bar{e}(t)$$

$$\downarrow$$

Letting $\hat{x}(0) = \bar{x}_0$ $\longrightarrow$ $E\left[e(0)\right] = E\left[x(0) - \hat{x}(0)\right] = 0$ $\longrightarrow$ $\bar{e}(t) = 0, \forall t \geq 0$

$$\downarrow$$

Letting $\tilde{P}(t)$ be the covariance of $e(t)$ $\longrightarrow$ $\tilde{P}(t) = E\left[e(t)e(t)'\right], \quad \tilde{P}(0) = \tilde{P}_0$

***Goal**: minimize with respect to the gain **L(t)** the covariance of the estimation error*

$$\min_{L(t)} \gamma' \tilde{P}(t) \gamma$$

$\gamma \in R^{n,1}$ is a generic vector

The solution of the above problem is

$$L(t) = \tilde{P}(t)C'\tilde{R}^{-1}$$

where $\tilde{P}(t)$ is the solution of the Riccati equation

$$\dot{\tilde{P}}(t) = A\tilde{P}(t) + \tilde{P}(t)A' + \tilde{Q} - \tilde{P}(t)C'\tilde{R}^{-1}C\tilde{P}(t)$$

with initial condition

$$\tilde{P}(0) = \tilde{P}_0$$

Matrix differential Riccati equation to be solved

LQ - KF

$$\min_{K(t)} x'P(t)x$$

$$K(t) = R^{-1}B'P(t)$$

$$\dot{P}(t) + Q - P(t)BR^{-1}B'P'(t) + P(t)A + A'P'(t) = 0$$

$$P(T) = S$$

$$\xleftarrow{\quad t \quad}$$

$$\min_{L(t)} \gamma'\tilde{P}(t)\gamma$$

$$L(t) = \tilde{P}(t)C'\tilde{R}^{-1}$$

$$\dot{\tilde{P}}(t) = A\tilde{P}(t) + \tilde{P}(t)A' + \tilde{Q} - \tilde{P}(t)C'\tilde{R}^{-1}C\tilde{P}(t)$$

$$\tilde{P}(0) = \tilde{P}_0$$

$$\xrightarrow{\quad t \quad}$$

LQ	KF
A	A'
B	C'
Q	$\tilde{Q}$
R	$\tilde{R}$
P	$\tilde{P}$
t	$t^* - t$
K	L'

perfect duality previous results on LQ can be extended to KF

KF **with time varying gain** ***L(t)*****: not very useful and difficult to implement. Is it possible to compute a steady-state filter?**

In view of duality, we can extend the results of ***LQ$_{inf}$***

If

1. the pair (A, B_q) is reachable, where B_q is such that $\tilde{Q} = B_q B_q'$;
2. the pair (A, C) is observable;

then

A) the optimal estimator is

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \bar{L}\left[y(t) - C\hat{x}(t)\right] \\
&= \left(A - \bar{L}C\right)\hat{x}(t) + Bu(t) + \bar{L}y(t)
\end{aligned}
$$

with

$$\bar{L} = \bar{\tilde{P}}C'\tilde{R}^{-1}$$

where $\bar{\tilde{P}}$ is the unique positive definite solution of the stationary Riccati equation

$$0 = A\bar{\tilde{P}} + \bar{\tilde{P}}A' + \tilde{Q} - \bar{\tilde{P}}C'\tilde{R}^{-1}C\bar{\tilde{P}}$$

B) the observer is asymptotically stable, that is all the eigenvalues of $\left(A - \bar{L}C\right)$ have negative real part.

What to do if *Z≠0*?

Write the system as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + v_x(t) + Z\tilde{R}^{-1}(y(t) - y(t)) \\
&= Ax(t) + Bu(t) + v_x(t) + Z\tilde{R}^{-1}y(t) - Z\tilde{R}^{-1}Cx(t) \\
&\quad -Z\tilde{R}^{-1}v_y(t) \\
&= \underbrace{\left(A - Z\tilde{R}^{-1}C\right)}_{\bar{A}} x(t) + Bu(t) + Z\tilde{R}^{-1}y(t) + \bar{v}(t)
\end{aligned}
$$

$$\bar{v}(t) = v_x(t) - Z\tilde{R}^{-1}v_y(t)$$

$$
\begin{aligned}
E\left[\bar{v}(t)v_y(t)'\right] &= Z - Z\tilde{R}^{-1}\tilde{R} = 0 \\
E\left[\bar{v}(t)\bar{v}(t)'\right] &= \tilde{Q} - Z\tilde{R}^{-1}Z' = \tilde{\tilde{Q}} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}(t) &= \bar{A}x(t) + Bu(t) + Z\tilde{R}^{-1}y(t) + \bar{v}(t) \\
y(t) &= Cx(t) + v_y(t)
\end{aligned}
$$

Satisfies the initial assumptions of KF

use it as a standard input

Duality with ***LQ*** allows one to understand how to choose the design parameters (provided that they are not obtained from physical considerations

LQ

LQ_{inf} | $KF_{statonary}$

$$Q \text{ "large"},\ R \text{ "small"} \rightarrow \text{"fast" feedback system}$$

$$\tilde{Q} \text{ "large"},\ \tilde{R} \text{ "small"} \rightarrow \text{"fast" observer}$$

and viceversa

Example: estimation of a constant from noisy measurements

$$
\begin{aligned}
\dot{x}(t) &= 0, \quad x(0) = \bar{x} \\
y(t) &= x(t) + v_y(t)
\end{aligned}
$$

$$A = 0,\ C = 1,\ \tilde{Q} = 0,\ \tilde{R} = r$$

$$\downarrow$$

$$\left(A, \tilde{Q}^{1/2}\right)$$ ***not reachable***

Riccati equation $$\frac{d\tilde{P}(t)}{dt} = -\frac{\tilde{P}^2(t)}{r} \longrightarrow \tilde{P}(t) = \frac{1}{r^{-1}t + \frac{1}{\tilde{P}_0}} \longrightarrow \tilde{P}(t) \to 0$$

$$L(t) = \tilde{P}(t)C\tilde{R}^{-1} = \frac{r^{-1}}{r^{-1}t + \frac{1}{\tilde{P}_0}}$$

The KF gain tends to 0, the estimate tends to be constant

$$\dot{\hat{x}}(t) = \frac{r^{-1}}{r^{-1}t + \frac{1}{\tilde{P}_0}}\left(y(t) - \hat{x}(t)\right)$$

Example

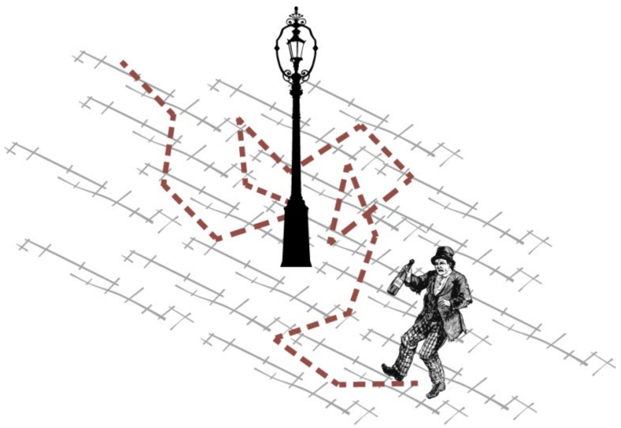
$$\begin{aligned} \dot{x}(t) &= Ax(t) + v_x(t) \\ y(t) &= v_y(t) \end{aligned} \longrightarrow C = 0 \longrightarrow L(t) = 0 \longrightarrow \dot{\hat{x}}(t) = A\hat{x}(t)$$

The output does not bring any information on the state
You can only run the estimator open-loop from the knowledge of the expected value of the initial state

Example

$$
\begin{aligned}
\dot{x}(t) &= v_x(t) \\
y(t) &= Cx(t) + v_y(t)
\end{aligned}
$$

Brownian motion, or drunkard's Walk



Steady state Riccati equation
$$0 = \tilde{Q} - \frac{\bar{\tilde{P}}^2 C^2}{\tilde{R}} \longrightarrow \bar{\tilde{P}} = \sqrt{\frac{\tilde{Q}\tilde{R}}{C^2}} \longrightarrow \bar{L} = \bar{\tilde{P}} C' \tilde{R}^{-1} = \sqrt{\frac{\tilde{Q}\tilde{R}}{C^2}} \frac{C}{\tilde{R}} = \sqrt{\frac{\tilde{Q}}{\tilde{R}}}$$

$$\frac{d\hat{x}(t)}{dt} = \sqrt{\frac{\tilde{Q}}{\tilde{R}}} \left[y(t) - C\hat{x}(t)\right] \quad \left\{ \begin{array}{l} \frac{\tilde{Q}}{\tilde{R}} \text{ large: "fast observer"} \\ \\ \frac{\tilde{Q}}{\tilde{R}} \text{ small: "slow observer"} \end{array} \right.$$

$$\hat{X}(s) = \frac{1}{1 + \frac{1}{C}\sqrt{\frac{\tilde{R}}{\tilde{Q}}}\, s} Y(s)$$

Estimation of a Brownian motion

$$\begin{cases} \dot{x} = v_x \\ y_1 = x + v_y \end{cases} \qquad v_x \sim WGN(0,1), \; v_y \sim WGN(0,1)$$

$$A = 0, \quad C = 1$$

Riccati eq. $\rightarrow 1 - \tilde{P}^2 = 0 \longrightarrow \tilde{P} = 1$ asymptotic variance of the state estimation error

$$\tilde{P} = 1 \rightarrow L = \tilde{P} C' \tilde{R}^{-1} = 1$$

Observer $\dot{\hat{x}} = [y_1 - \hat{x}] \longrightarrow \xrightarrow{y_1} \boxed{\frac{1}{s+1}} \xrightarrow{\hat{x}}$

Now assume to have two measurements with the same measurement noise

$$\begin{cases} \dot{x} = v_x \\ y_1 = x + v_{y_1} \\ y_2 = x + v_{y_2} \end{cases} \qquad \tilde{Q} = 1, \ \tilde{R} = I_2, \ A = 0, \ C = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

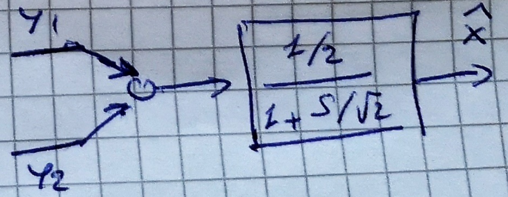
Riccati eq. $\quad 1 - \tilde{P} \begin{vmatrix} 1 & 1 \end{vmatrix} I \begin{vmatrix} 1 \\ 1 \end{vmatrix} \tilde{P} = 0 \longrightarrow 1 - 2\tilde{P}^2 = 0$

$\tilde{P} = \sqrt{1/2} \simeq 0.707$ new variance of the state estimation error

$$L = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \end{vmatrix}$$

$$\dot{\hat{x}} = \begin{vmatrix} 1 & 1 \end{vmatrix} \frac{1}{\sqrt{2}} \left\{ \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} - \begin{vmatrix} 1 \\ 1 \end{vmatrix} \hat{x} \right\}$$

$$= \frac{1}{\sqrt{2}} \left\{ y_1 + y_2 - 2\hat{x} \right\} \longrightarrow$$



Now assume to have 2 measurement with different measurement noise

$$\tilde{Q} = 1 \qquad \tilde{R} = \begin{vmatrix} 1 & 0 \\ 0 & r \end{vmatrix}$$

Riccati eq.

$$1 - \tilde{P} \underbrace{\begin{vmatrix} 1 & 1 \end{vmatrix}}_{C'} \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1/r \end{vmatrix}}_{R^{-1}} \underbrace{\begin{vmatrix} 1 \\ 1 \end{vmatrix}}_{C} \tilde{P} = 0$$

it is easy to compute $\tilde{P} = \sqrt{r/(r+1)} \longrightarrow L = \sqrt{\dfrac{r}{r+1}} \begin{vmatrix} 1 & \dfrac{1}{r} \end{vmatrix}$

$$\dot{\hat{x}} = \sqrt{\frac{r}{r+1}} \begin{vmatrix} 1 & 1/r \end{vmatrix} \begin{vmatrix} y_1 - \hat{x} \\ y_2 - \hat{x} \end{vmatrix}$$

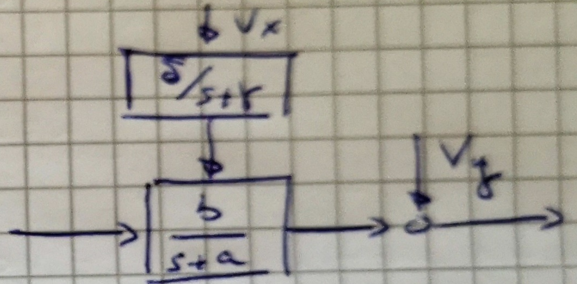
$$\overset{\circ}{\hat{x}} = \sqrt{\frac{r}{r+1}} \left\{ (y_1 - \hat{x}) + \frac{1}{r} (y_2 - \hat{x}) \right\}$$

If $r \to \infty$ $\quad \overset{\circ}{\hat{x}} = y_1 - \hat{x}$ The second measurement Y_2 is not reliable and not used

If $r \to 0$ $\quad$ only the second measurement is used because it is not affected by measurement noise

KF with non white noise

$$\begin{cases} \dot{x} = -ax + bu + v \\ y = x + v_y \end{cases} , \quad \dot{v} = -\gamma v + \delta v_x , \quad \gamma > 0$$



$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 1 \\ 0 & -\gamma \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} b \\ 0 \end{bmatrix}}_{B} u + \underbrace{\underbrace{\begin{bmatrix} 0 \\ \delta \end{bmatrix}}_{\Gamma} v_x}_{\tilde{v}_x} \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x \\ v \end{bmatrix} + v_y \end{cases}$$

$$E[v_x] = 0 \quad , \quad E[\tilde{v}_x] = 0$$

$$E[\tilde{v}_x \tilde{v}_x'] = M\, E[v_x v_x']\, M' = \underbrace{M \tilde{Q}_x M'}_{M \tilde{q}^2 M'} \qquad (\tilde{Q}_x = q^2 I)$$

$$= \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \tilde{q}^2 \delta^2 \end{bmatrix}}_{\tilde{Q} \geq 0}$$

$$B_q B_q' = \begin{vmatrix} 0 \\ \delta \tilde{q} \end{vmatrix} \begin{vmatrix} 0 & \delta \tilde{q} \end{vmatrix}$$

Observability $(A, C) \rightarrow M_o = \begin{vmatrix} C \\ CA \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -a & 1 \end{vmatrix}$ ✓

Reachability $(A, B_q) \rightarrow M_r = \begin{vmatrix} B_q & AB_q \end{vmatrix} = \begin{vmatrix} 0 & \delta\tilde{q} \\ \delta\tilde{q} & -\gamma\delta\tilde{q} \end{vmatrix} \rightarrow$ ok for $\delta \neq 0$, $\tilde{q} \neq 0$

Design of the filter for the enlarged state

$$\tilde{x} = \begin{vmatrix} x \\ v \end{vmatrix}$$

$$\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + Bu + L\left[y - C\hat{\tilde{x}}\right] \quad , L = \begin{vmatrix} l_1 \\ l_2 \end{vmatrix}$$

$$\begin{cases} \dot{\hat{x}} = -a\hat{x} + \hat{v} + bu + l_1\left(y - \hat{x}\right) \\ \dot{\hat{v}} = -\gamma\hat{v} + l_2\left[y - \hat{x}\right] \end{cases}$$

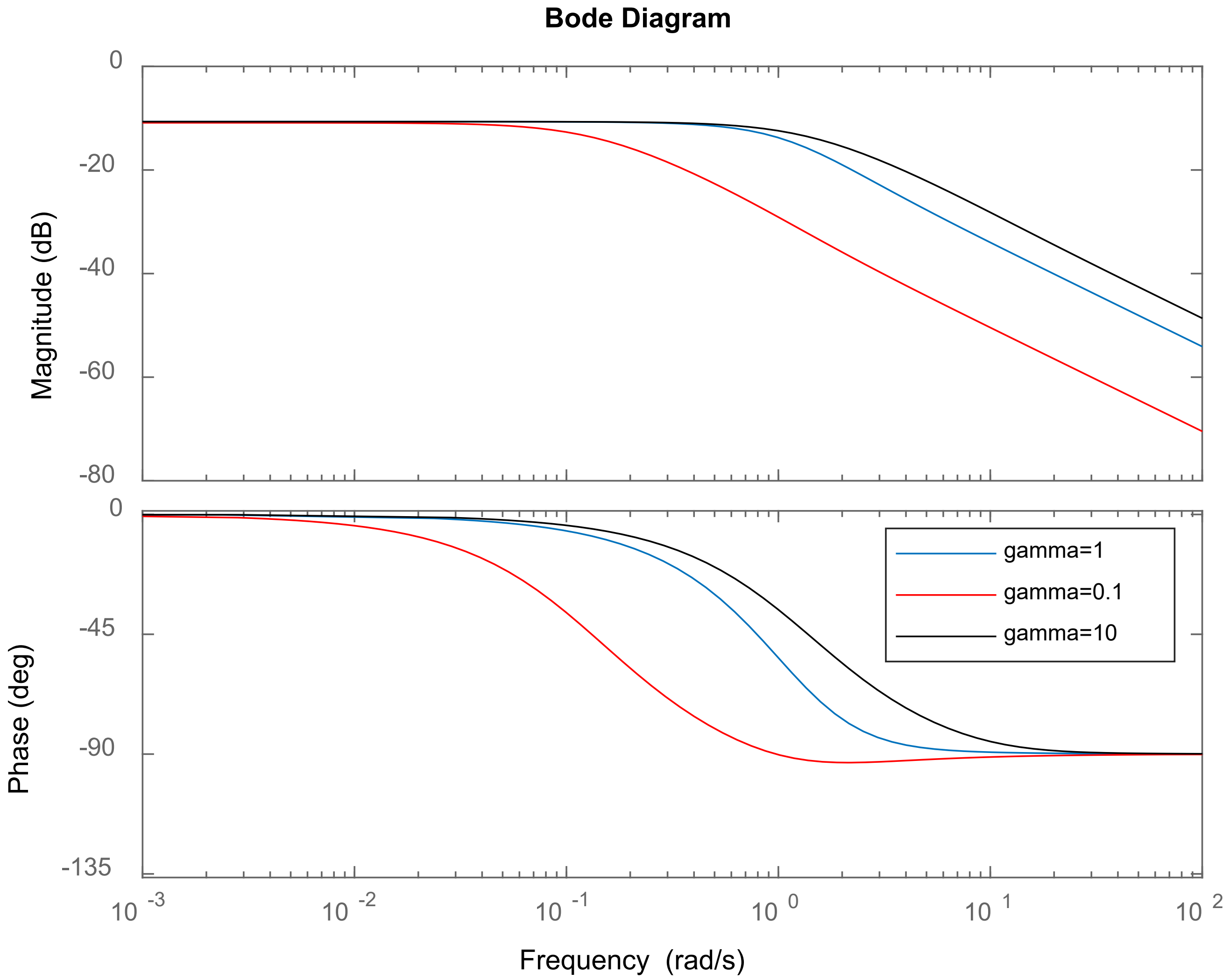
$$\Downarrow$$

$$\hat{x}(s) = \underbrace{C\,(sI-(A-LC))^{-1}L}_{\text{transfer function of the filter}}\,y(s)$$

Considering fixed values

$$a=1\;,\quad \tilde{R}=1\;,\quad \delta=\gamma\;,\quad \tilde{q}=1$$

we obtain different transfer functions of the filter



The slower the system of the noise, the slower the filter