Advanced and Multivariable Control

Pole placement control

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Problem statement

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in R^n, \quad u \in R^m$$

and the control law

$$u(t) = -Kx(t) + \gamma(t), \quad K \in R^{m,n}, \quad \gamma \in R^m$$

Problem: we want to design a feedback matrix ***K*** such that the closed loop system

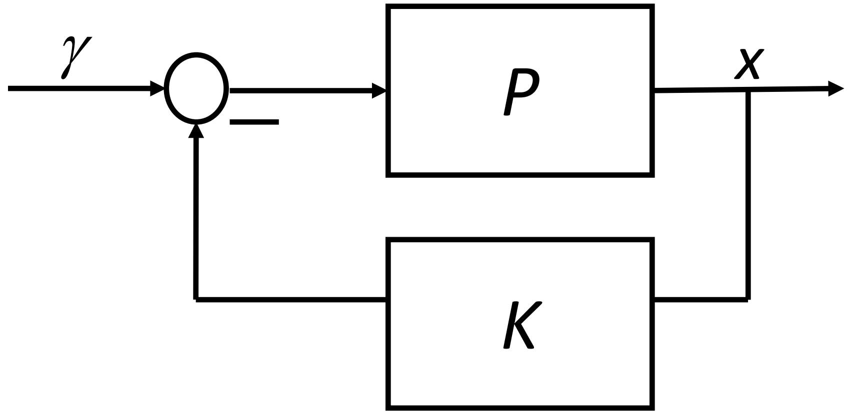
$$\dot{x}(t) = (A - BK)\, x(t) + B\gamma(t)$$

has prescribed eigenvalues (of the matrix ***A-BK***)

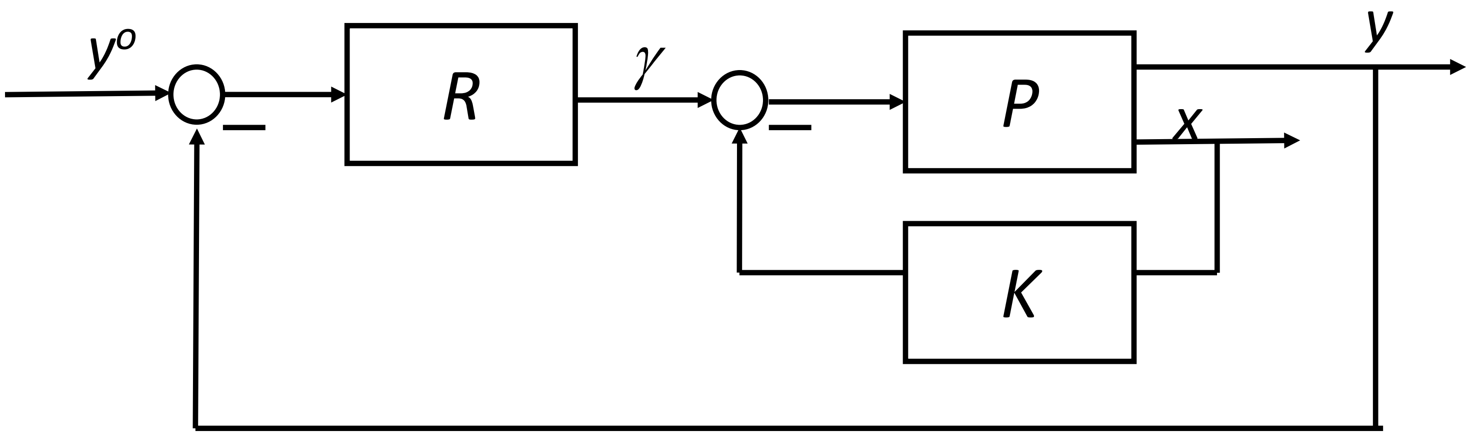
Remark: we are assuming that the state is measurable (not realistic in many cases). This hypothesis will be removed later on

Remarks

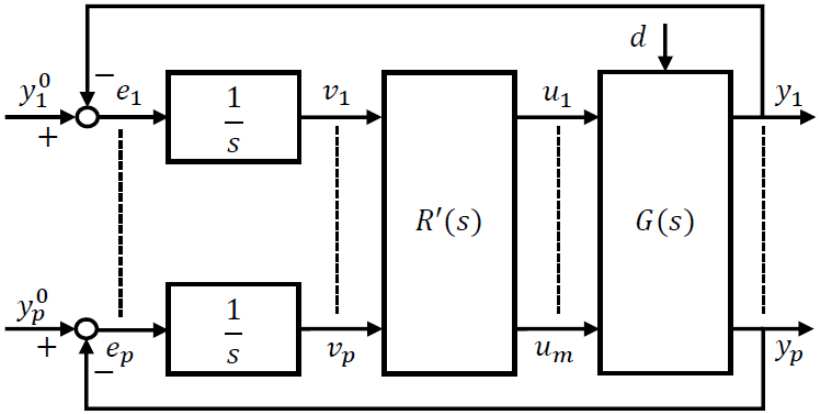
The additional input γ does not play any role, but it can be useful to design external loops, while *K* is used for stabilization



Inner feedback used to stabilize, while *R(s)* used for performance (for example, one can use the Bode criterion)



Scheme with integrators

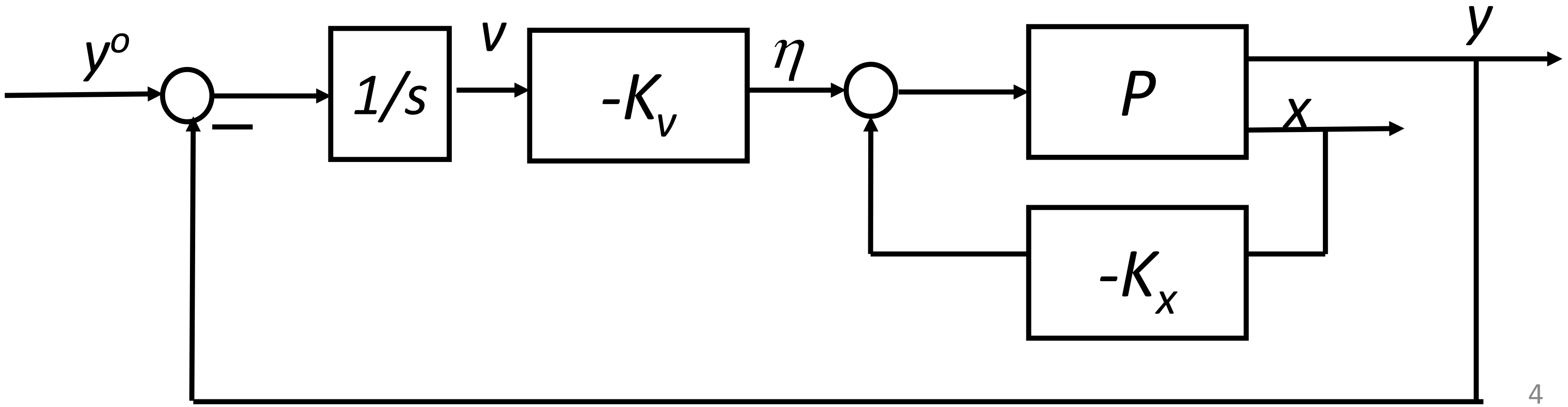


$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}^{\bar{A}} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{\bar{B}} u(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} y^0$$

Control law

K designed for the new pair $\bar{A}, \bar{B}$ ***such that*** $\bar{A} - \bar{B}K$ ***has prescribed eigenvalues***

$$u(t) = -K \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = -\begin{bmatrix} K_x & K_v \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$



Necessary and sufficient condition for the solution of the pole placement problem (with measurable state) is that the pair (A, B) is reachable

Sketch of the proof

By means of a suitable state transformation, it is possible to write the system as

$$\begin{bmatrix} \dot{\hat{x}}_r(t) \\ \dot{\hat{x}}_{nr}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_r & \hat{A}_x \\ 0 & \hat{A}_{nr} \end{bmatrix} \begin{bmatrix} \hat{x}_r(t) \\ \hat{x}_{nr}(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_r \\ 0 \end{bmatrix} u(t)$$

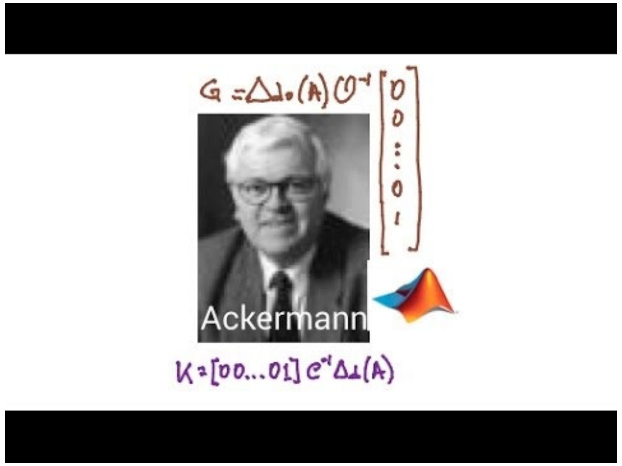
If we take

$$u(t) = -\hat{K}\hat{x}(t) = -\begin{bmatrix} \hat{K}_r & \hat{K}_{nr} \end{bmatrix} \begin{bmatrix} \hat{x}_r(t) \\ \hat{x}_{nr}(t) \end{bmatrix}, \quad \hat{K}_r \in R^{m,\nu_r}$$

we obtain

$$\hat{A} - \hat{B}\hat{K} = \begin{bmatrix} \hat{A}_r - \hat{B}_r\hat{K}_r & \hat{A}_x - \hat{B}_r\hat{K}_{nr} \\ 0 & \hat{A}_{nr} \end{bmatrix}$$

Single input systems (*m=1*) - *The Ackermann's formula*



Define the reachability matrix (square for *m=1*)

$$M_r = \left[\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array} \right]$$

Assume you want to obtain with a closed-loop system with characteristic polynomial

$$P(s) = (s+\bar{p}_1)(s+\bar{p}_2)\dots(s+\bar{p}_n) = s^n + p_{n-1}s^{n-1} + \dots + p_1 s + p_0$$

Define

$$P(A) = A^n + p_{n-1}A^{n-1} + \dots + p_1 A + p_0 I$$

Then, the solution to the pole placement problem is given by the Ackermann's formula

$$\boxed{K = \left[\begin{array}{ccccc} 0 & 0 & \dots & 0 & 1 \end{array} \right] M_r^{-1} P(A)}$$

In the textbook proofs and many more considerations

Example Ackermann's formula

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \quad , \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$P(s) = (s+3)(s+4) = s^2 + 7s + 12$$

$$P(A) = A^2 + 7A + 12I = \begin{bmatrix} 6 & 24 \\ 0 & 30 \end{bmatrix}$$

$$M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \quad , \quad M_r^{-1} = \begin{bmatrix} -4/6 & 5/6 \\ 2/6 & -1/6 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} M_r^{-1} P(A) = \begin{bmatrix} 2 & 3 \end{bmatrix} \quad , \quad A - BK = \begin{bmatrix} -3 & 0 \\ -4 & -4 \end{bmatrix}$$

Where to place the poles? *This is the difficult task*

An idea is try to impose that the closed loop system (approximately) behaves as a first order or second order system

In the case of first order systems, the «desired» transfer function and step response are

$$G(s) = \frac{\mu}{1+sT} \longleftrightarrow y(t) = \mu(1 - e^{-t/T})$$

So, you can choose the time constant *T*, or (equivalently) the pole *-1/T* and fix one of the closed-loop poles at that value

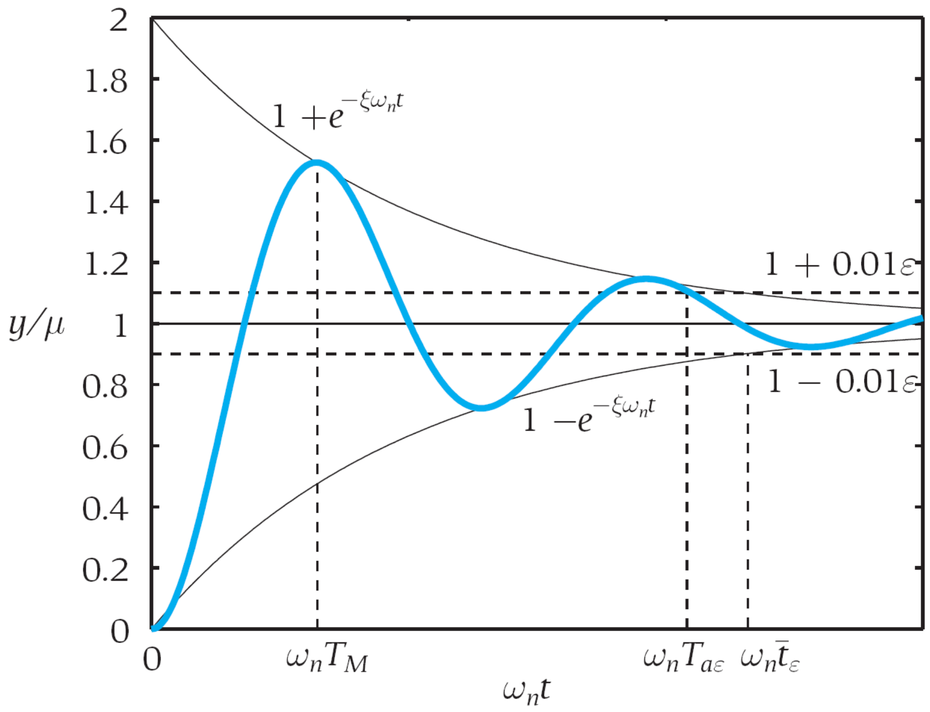
Then, the remaining closed-loop poles can be chosen to be «much faster» than this pole, which is the «***dominant one***»

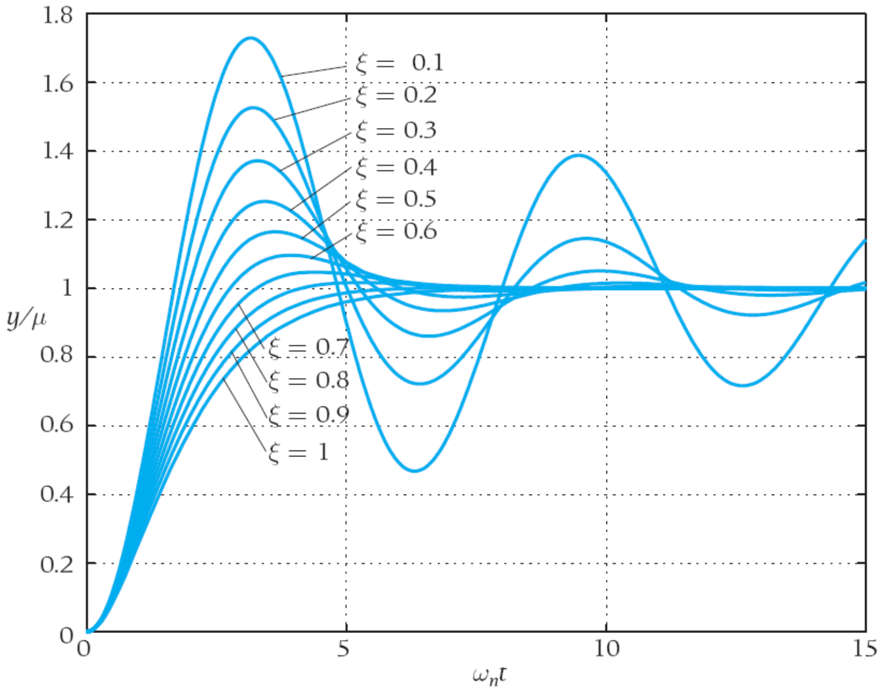
Always remember that it is not advisable to choose a closed-loop system must faster than the open-loop one.
This could lead to the saturation of the actuators, or to an excessive use of them

Two dominant poles

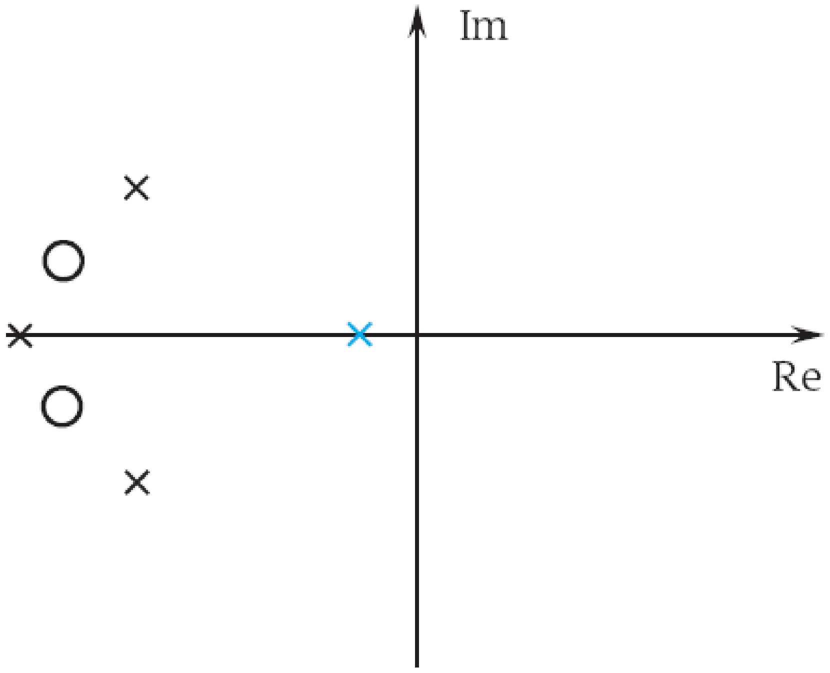
$$G(s) = \frac{\mu\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Characteristics of the step response



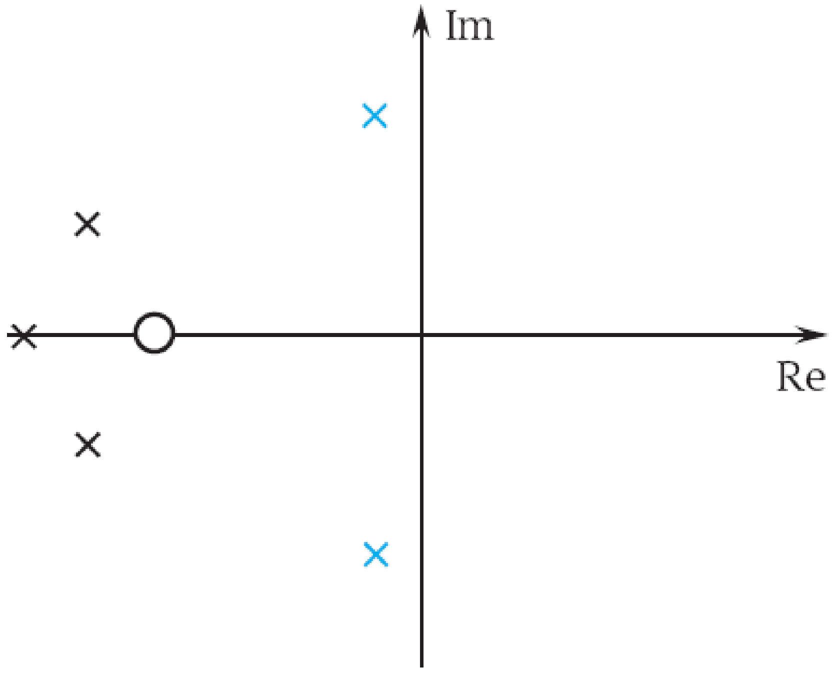


one dominant pole



a)

two dominant poles



b)

Systems with many inputs (*m>1*)

The problem can have **many solutions**. In any case, the assumption of ***reachability is mandatory***

Example

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + b_1 u_1(t) + b_2 u_2(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Both (A, b_1) and (A, b_2) are reachable, so that we can use just one input to assign the closed-loop poles

continued…

Solution 1

$$\begin{aligned} u_1(t) &= -K_1 x(t) \\ u_2(t) &= 0 \end{aligned} \quad \longleftrightarrow \quad \dot{x}(t) = (A - b_1 K_1) x(t)$$

K_1 can be computed with the Ackermann's formula (only one input!) to impose the eigenvalues of $(A\text{-}b_1K_1)$

Solution 2

$$\begin{aligned} u_1(t) &= 0 \\ u_2(t) &= -K_2 x(t) \end{aligned} \quad \longleftrightarrow \quad \dot{x}(t) = (A - b_2 K_2) x(t)$$

K_2 can be computed with the Ackermann's formula (only one input!) to impose the eigenvalues of $(A\text{-}b_2K_2)$

In both cases $u(t) = -Kx(t)$ where $K = \begin{bmatrix} K_1 \\ 0 \end{bmatrix}$ (first solution) or $K = \begin{bmatrix} 0 \\ K_2 \end{bmatrix}$ (second solution)

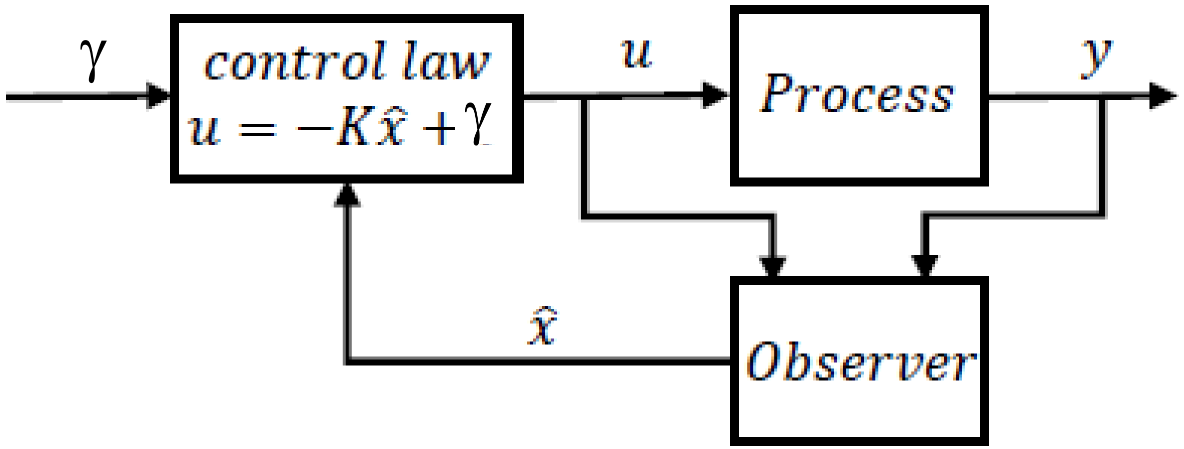
What to do in practice?

Use the function *place.m* of Matlab, it uses all the available inputs and guarantees some robustness of the solution with respect to small variations of the parameters of the matrices *A, B.*
The only warning is that you must specify closed-loop poles each one different from the others (not a big issue, there is no difference in placing two poles in -1 or one pole in -1 and the other one in -1.001)

```
[K,PREC] = place(A,B,P)
```

returns PREC, an estimate of how closely the eigenvalues of A-B*K match the specified locations P (PREC measures the number of accurate decimal digits in the actual closed-loop poles). A warning is issued if some nonzero closed-loop pole is more than 10% off from the desired location.

Nonmeasurable state



Procedure

1. first a *state observer* is built. The observer is a system which, based on the input and output measurements, computes an estimate $\hat{x}$ of the system's state x. The observer introduced in the following is a linear time-invariant system with prescribed eigenvalues;

2. the estimated state $\hat{x}$ is used in the pole assignment control law in place of the real state x.

State observers

System

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x \in R^n, \quad u \in R^m \\
y(t) &= Cx(t) + Du(t), \quad y \in R^p
\end{aligned}
$$

We must consider also the output transformation now

Observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

$L \in R^{n,p}$ is the gain of the observer (and its design parameter)

$e_y(t) = [y(t) - C\hat{x}(t) - Du(t)]$ is the output estimation error

$e(t) = x(t) - \hat{x}(t)$ is the state estimation error

Dynamics of the state estimation error

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= Ax(t) + Bu(t) - A\hat{x}(t) - Bu(t) - L\left[y(t) - C\hat{x}(t) - Du(t)\right] \\
&= A(x(t) - \hat{x}(t)) - LC(x(t) - \hat{x}(t)) = (A - LC)e(t)
\end{aligned}
$$

If the eigenvalues of (***A-LC***) are asymptotically stable, the state estimation error tends to zero. The idea is to use pole assignment to select the observer gain ***L*** to assign these eigenvalues

The eigenvalues of (***A-LC***) are the eigenvalues of (***A'-C'L'***)

The problem of assigning the eigenvalues of (***A'-C'L'***) is equivalent to assigning the eigenvalues of (***A-BK***)

$$A \to A', \quad B \to C', \quad m \to p, \quad K \to L'$$

A necessary and sufficient condition for the design of an asymptotic observer with arbitrarily specified eigenvalues is that the pair (***A,C***) is observable

For single output systems we can use the Ackermann's formula with the proper substitutions

$$L' = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} M_o^{-1'} P(A') \text{ or } L = P(A) M_o^{-1} \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

For multi output systems the suggestion is to use *place.m* with the proper substitutions (***A->A',B->C'***)

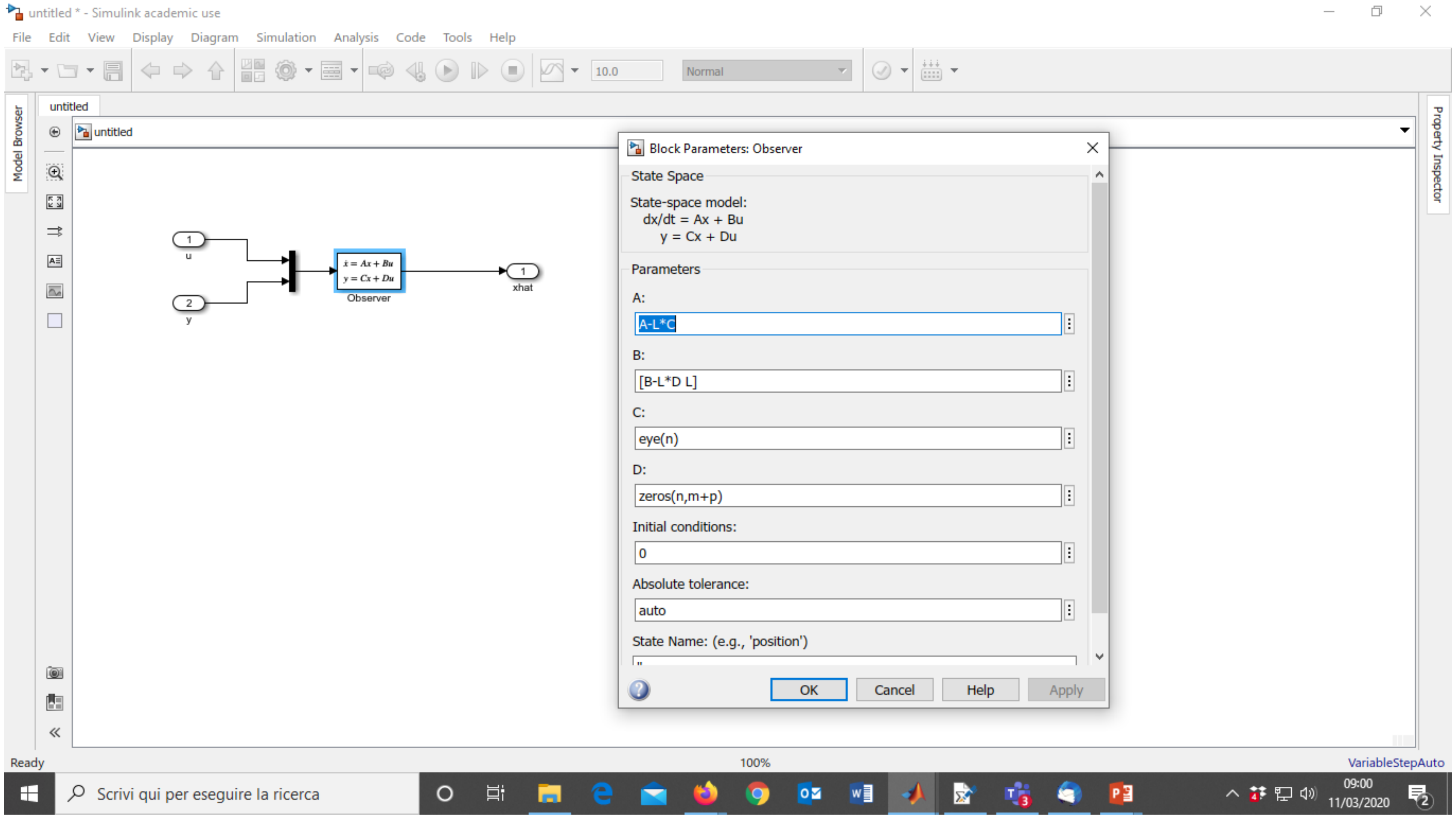
How to choose the eigenvalues of (***A-LC***)? Typically ***much faster*** than the ones of (***A-BK***).
The observer is an algorithm, it does not suffer from actuators' saturations

Matlab – Simulink implementation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

$$\downarrow$$

$$\dot{\hat{x}}(t) = (A - LC)\hat{x}(t) + (B - LD)u(t) + Ly(t)$$



Systems with disturbances

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Md(t)\\ y(t) &= Cx(t) + Du(t) + Nd(t)\end{aligned}$$

measurable disturbance

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + Md(t) + L\left[y(t) - C\hat{x}(t) - Du(t) - Nd(t)\right]$$

$$\downarrow$$

$$\dot{e}(t) = (A - LC)e(t) \qquad \hat{x} \to x$$

non measurable disturbance

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

$$\downarrow$$

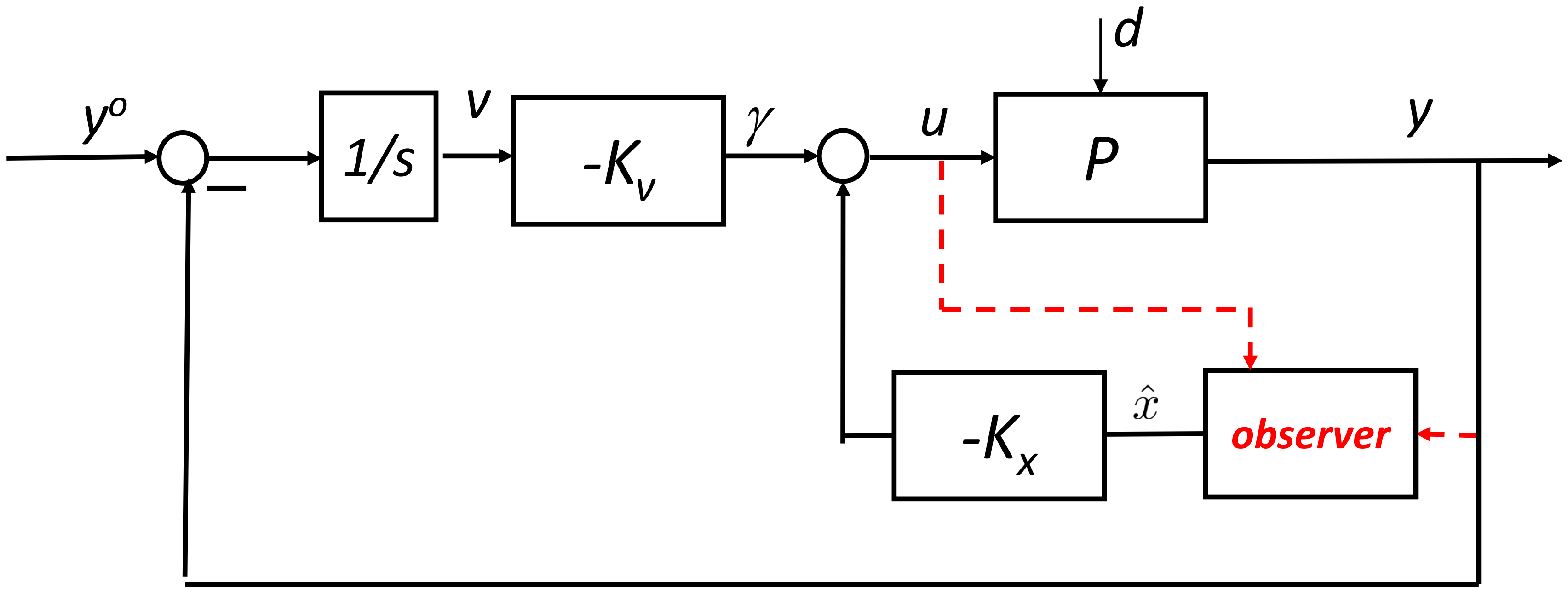
$$\dot{e}(t) = (A - LC)e(t) + (M - LN)d(t) \qquad \hat{x} \not\to x$$

also for constant *d*

Is it really necessary to have the correct state estimation?

Consider the system with integrators and observer

If the closed-loop system remains asymptotically stable, even when the state estimate is not correct, for constant references and disturbances the input of the integrators must be asymptotically zero



Estimation of constant disturbances

Assume to know that ***d*** is constant, or at least constant for long periods of time.
It is useful to estimate it for two reasons:

1) It is possible to correctly estimate the state of the system

2) The estimate of the disturbance can be used in a control scheme with direct compensation

Procedure

Assign a dynamics to the disturbance

$$\dot{d}(t) = 0, \quad d(0) = \bar{d} \in R^r$$

Enlarge the system

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{d}(t) \end{bmatrix} &= \begin{bmatrix} A & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & N \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + Du(t) \end{aligned}$$

Use an observer for the enlarged system. When is it possible?

Conditions

$$\bar{A} \in R^{n+r,n+r}$$

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{d}(t) \end{bmatrix} &= \begin{bmatrix} A & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} C & N \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + Du(t) \end{aligned}$$

$$\bar{C} \in R^{p,n+r}$$

The pair $(\bar{A}, \bar{C})$ must be observable

Observable ***iff***

1. (*A*,*C*) observable

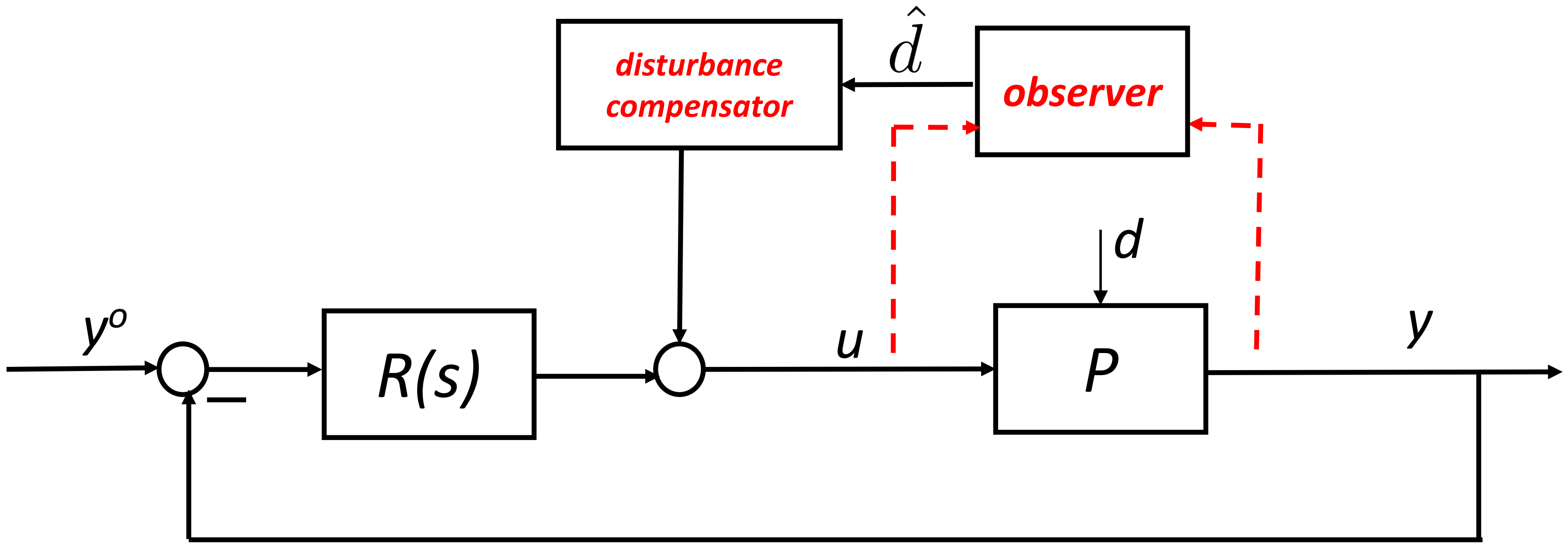
2. $rank \begin{bmatrix} A & M \\ C & N \end{bmatrix} = n + r \longrightarrow r \leq p$

maximum numbr of disturbances that can be estimated

Exercise: use the PBH test to prove this result

Scheme with compensator

The idea is to design the compensator in such a way to force to be null the transfer function from the (estimated) disturbance to the output



What are the eigenvalues of the closed-loop system?

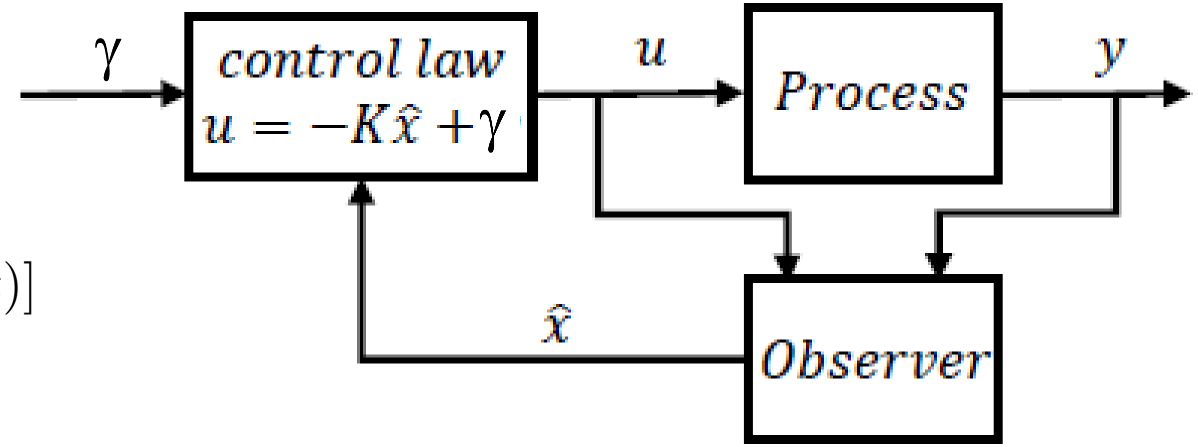
$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

+

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

+

$$u(t) = -K\hat{x}(t) + \gamma(t)$$



$$
\begin{aligned}
\dot{x}(t) &= Ax(t) - BK\hat{x}(t) + B\gamma(t) \\
\dot{\hat{x}}(t) &= A\hat{x}(t) - BK\hat{x}(t) + B\gamma(t) + L\left[Cx(t) - C\hat{x}(t)\right]
\end{aligned}
$$

$$e = x - \hat{x}$$

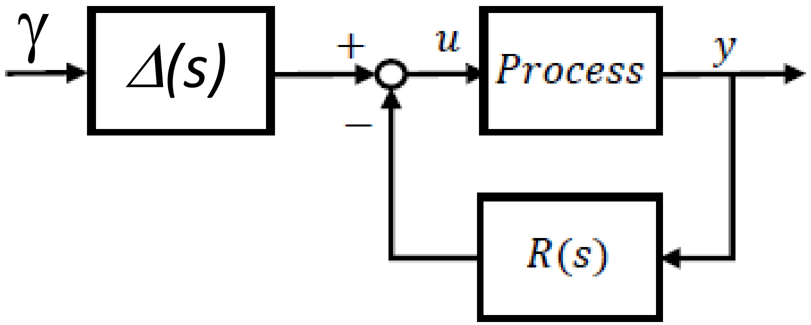
$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \gamma(t)$$

The eigenvalues are those of ***A-BK*** and ***A-LC***

Separation principle

You can independently design the state feedback control law and the observer, the eigenvalues are maintained when putting everything together

Regulator transfer function



$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

+

$$u(t) = -K\hat{x}(t) + \gamma(t)$$

$$\begin{aligned} \dot{\hat{x}}(t) &= (A - LC)\hat{x}(t) - (B - LD)K\hat{x}(t) + (B - LD)\gamma(t) + Ly(t) \\ u(t) &= -K\hat{x}(t) + \gamma(t) \end{aligned}$$

$$\bar{A} = A - LC - BK + LDK, \quad \bar{B} = (B - LD)$$

$$\begin{aligned} \dot{\hat{x}}(t) &= \bar{A}\hat{x}(t) + \bar{B}\gamma(t) + Ly(t) \\ u(t) &= -K\hat{x}(t) + \gamma(t) \end{aligned}$$

$$U(s) = -K(sI - \bar{A})^{-1}LY(s) + \left[-K(sI - \bar{A})^{-1}\bar{B} + I\right]\Gamma(s)$$

$$R(s) = K(sI - \bar{A})^{-1}L$$

$$\Delta(s) = -K(sI - \bar{A})^{-1}\bar{B} + I$$

The overall regulator can be expressed in transfer function form

Reduced order observers

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

Idea: the output is a linear combination of the states

1. Apply a state transformation to the system so that the ***p*** outputs coincide with ***p*** new states
2. Estimate the remaining ***n-p*** *states*

T_1 any matrix such that T is non singular

$$\tilde{x} = Tx = \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ \tilde{x}_r \end{bmatrix}, \quad \tilde{x}_r \in R^{n-p}$$

Transformed system

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t)+\tilde{B}u(t)\\ y(t) &= \tilde{C}x(t)\end{aligned} \quad \longleftrightarrow \quad \begin{aligned}\dot{y}(t) &= \tilde{A}_{11}y(t)+\tilde{A}_{12}\tilde{x}_r(t)+\tilde{B}_1u(t)\\ \dot{\tilde{x}}_r(t) &= \tilde{A}_{21}y(t)+\tilde{A}_{22}\tilde{x}_r(t)+\tilde{B}_2u(t)\end{aligned}$$

$$\tilde{A}=TAT^{-1},\ \tilde{B}=TB,\ \tilde{C}=CT^{-1}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in R^{p,p}, \quad \tilde{A}_{22} \in R^{n-p,n-p}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{B}_1 \in R^{p,m}, \quad \tilde{B}_2 \in R^{n-p,m}$$

$$\tilde{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

Observer

$$\begin{aligned}\dot{y}(t) &= \tilde{A}_{11}y(t) + \tilde{A}_{12}\tilde{x}_r(t) + \tilde{B}_1u(t)\\ \dot{\tilde{x}}_r(t) &= \tilde{A}_{21}y(t) + \tilde{A}_{22}\tilde{x}_r(t) + \tilde{B}_2u(t)\end{aligned}$$

define

$$\begin{aligned}\eta(t) &= \dot{y}(t) - \tilde{A}_{11}y(t) - \tilde{B}_1u(t)\\ \zeta(t) &= \tilde{A}_{21}y(t) + \tilde{B}_2u(t)\end{aligned}$$

Now, design an observer for this system

$$\begin{aligned}\dot{\tilde{x}}_r(t) &= \tilde{A}_{22}\tilde{x}_r(t) + \zeta(t)\\ \eta(t) &= \tilde{A}_{12}\tilde{x}_r(t)\end{aligned}$$

if (A, C) is observable, $(\tilde{A}_{22}, \tilde{A}_{12})$ is observable

$$\boxed{\dot{\hat{x}}_r(t) = \tilde{A}_{22}\hat{x}_r(t) + \zeta(t) + L\left[\eta(t) - \tilde{A}_{12}\hat{x}_r(t)\right]}$$

or (equivalent)

$$\boxed{\dot{\hat{x}}_r(t) = \left(\tilde{A}_{22} - L\tilde{A}_{12}\right)\hat{x}_r(t) + \tilde{A}_{21}y(t) + \tilde{B}_2u(t) + L\eta(t)}$$

Problem: $\boldsymbol{\eta}$ contains the derivative of ***y***, not wise from a numerical point of view

$$\dot{\hat{x}}_r(t) = \left(\tilde{A}_{22} - L\tilde{A}_{12}\right)\hat{x}_r(t) + \tilde{A}_{21}y(t) + \tilde{B}_2 u(t) + L\eta(t) \qquad \eta(t) = \dot{y}(t) - \tilde{A}_{11}y(t) - \tilde{B}_1 u(t)$$

$$\dot{\hat{x}}_r(t) - L\dot{y}(t) = (\tilde{A}_{22} - L\tilde{A}_{12})\hat{x}_r(t) + (\tilde{A}_{21} - L\tilde{A}_{11})y(t) + (\tilde{B}_2 - L\tilde{B}_1)u(t)$$

$\boxed{\text{sum and subtract } (\tilde{A}_{22} - L\tilde{A}_{12})Ly(t)}$

$\boxed{\text{define } \xi(t) = \hat{x}_r(t) - Ly(t), \quad \xi \in R^{n-p}}$

Final reduced order observer

$$\boxed{\dot{\xi}(t) = (\tilde{A}_{22} - L\tilde{A}_{12})\xi(t) + (\tilde{A}_{21} - L\tilde{A}_{11} + \tilde{A}_{22}L - L\tilde{A}_{12}L)y(t) + (\tilde{B}_2 - L\tilde{B}_1)u(t)}$$

$$\hat{x}_r(t) = \xi(t) + Ly(t)$$

$$\boxed{\hat{x}(t) = T^{-1}\begin{bmatrix} y(t) \\ \hat{x}_r(t) \end{bmatrix}}$$

Design a reduced order observer for the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases} \quad , \; A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \; B = \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \; C = \begin{vmatrix} 1 & 0 \end{vmatrix}$$

One state already coincides with the output. There is no need of state transformation ($\tilde{x} = Tx$)

According to the general Theory $\eta = \dot{y} = x_2$

The observer can be written as

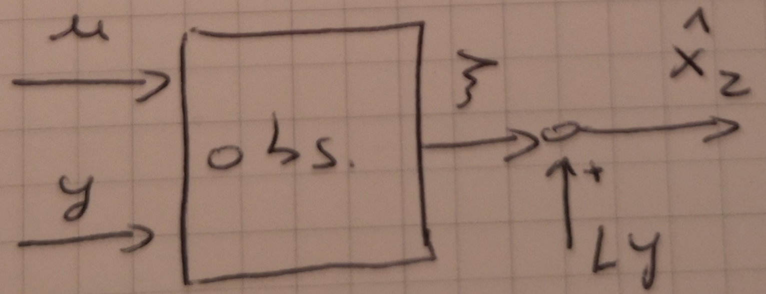
The observer can be written as

$$\dot{\hat{x}}_2 = u + L(\dot{y} - \hat{x}_2) \qquad \mp L^2 y$$

$$\underbrace{\dot{\hat{x}}_2 - L\dot{y}}_{\dot{\xi}} = -L\underbrace{(x_2 - Ly)}_{\xi} - L^2 y + u$$

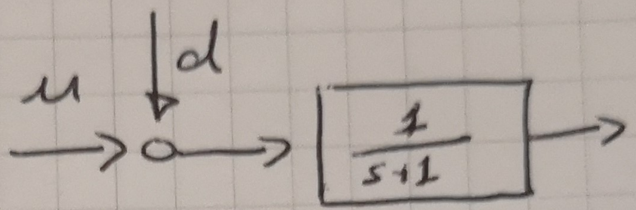
$$\Downarrow$$

$$\dot{\xi} = -L\xi - L^2 y + u$$



Exercise

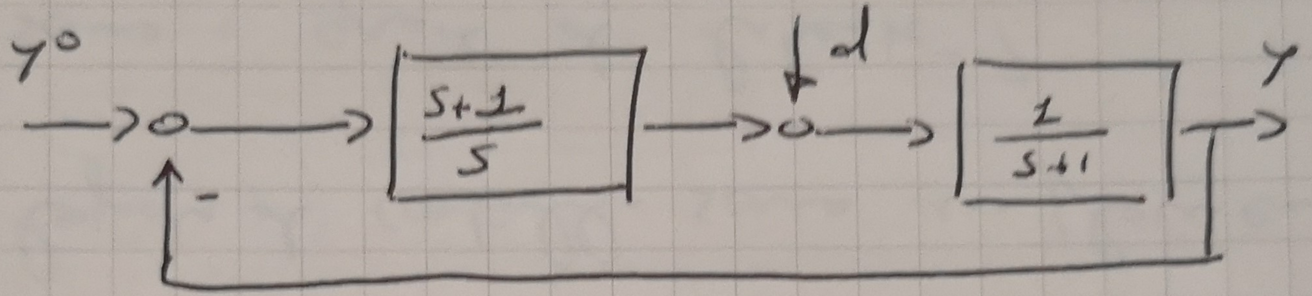
Given the system



A) design a PI regulator

B) Design an estimator of the disturbance, assumed to be constant and a compensator acting on the estimated value of d

Solution A)



$$y(s) = \frac{1}{s+1} y^o + \frac{s}{(s+1)^2} d$$

(note that due to the integrator in $R(s)$ constant disturbances are asymptotically rejected)

Solution B

The system is described in the state space by

$$\begin{cases} \dot{x} = -x + u + d \\ \dot{d} = 0 \quad \leftarrow \text{fictitious dynamics} \\ y = x \end{cases}$$

We can use a reduced order observer to estimate d

According to the theory, write the system as

$$\begin{cases} \dot d = 0 \\ \underbrace{\dot x + x - u}_{\gamma} = d \end{cases} \Longrightarrow \dot{\hat d} = L\,[\underbrace{\dot y + y - u}_{\gamma} - \hat d\,]$$

$$\dot{\hat d} = L\dot y + Ly - Lu - L\hat d$$

$$\underbrace{\dot{\hat d} - L\dot y}_{\dot\xi} = -L\hat d + Ly - Lu$$

$$\Big\downarrow \mp L^2 y$$

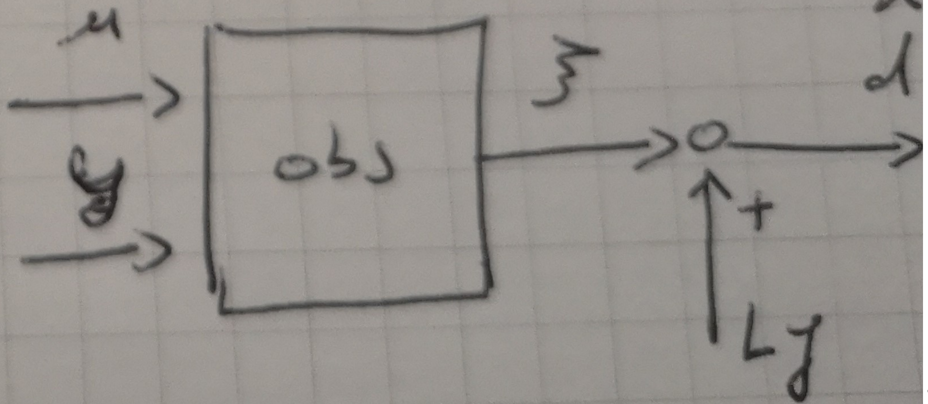
$$\dot\xi = -L\,[\underbrace{\hat d - Ly}_{\xi}\,] - L^2 y + Ly - Lu$$

$$\underbrace{\dot{\hat{d}} - L\dot{y}}_{\dot{\xi}} = -L\hat{d} + Ly - Lu$$

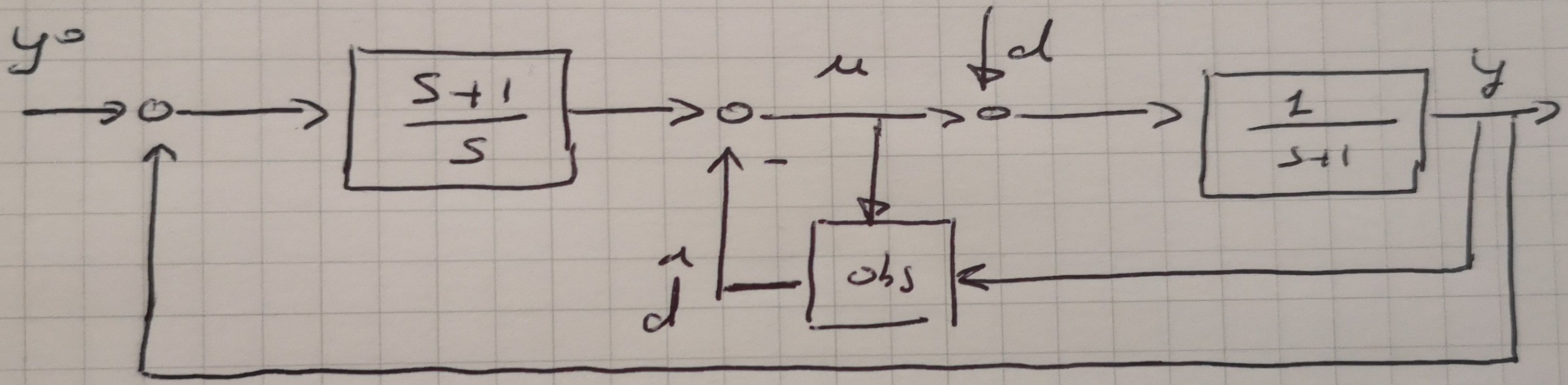
$$\downarrow \mp L^2 y$$

$$\dot{\xi} = -L\,[\underbrace{\hat{d} - Ly}_{\xi}] - L^2 y + Ly - Lu$$

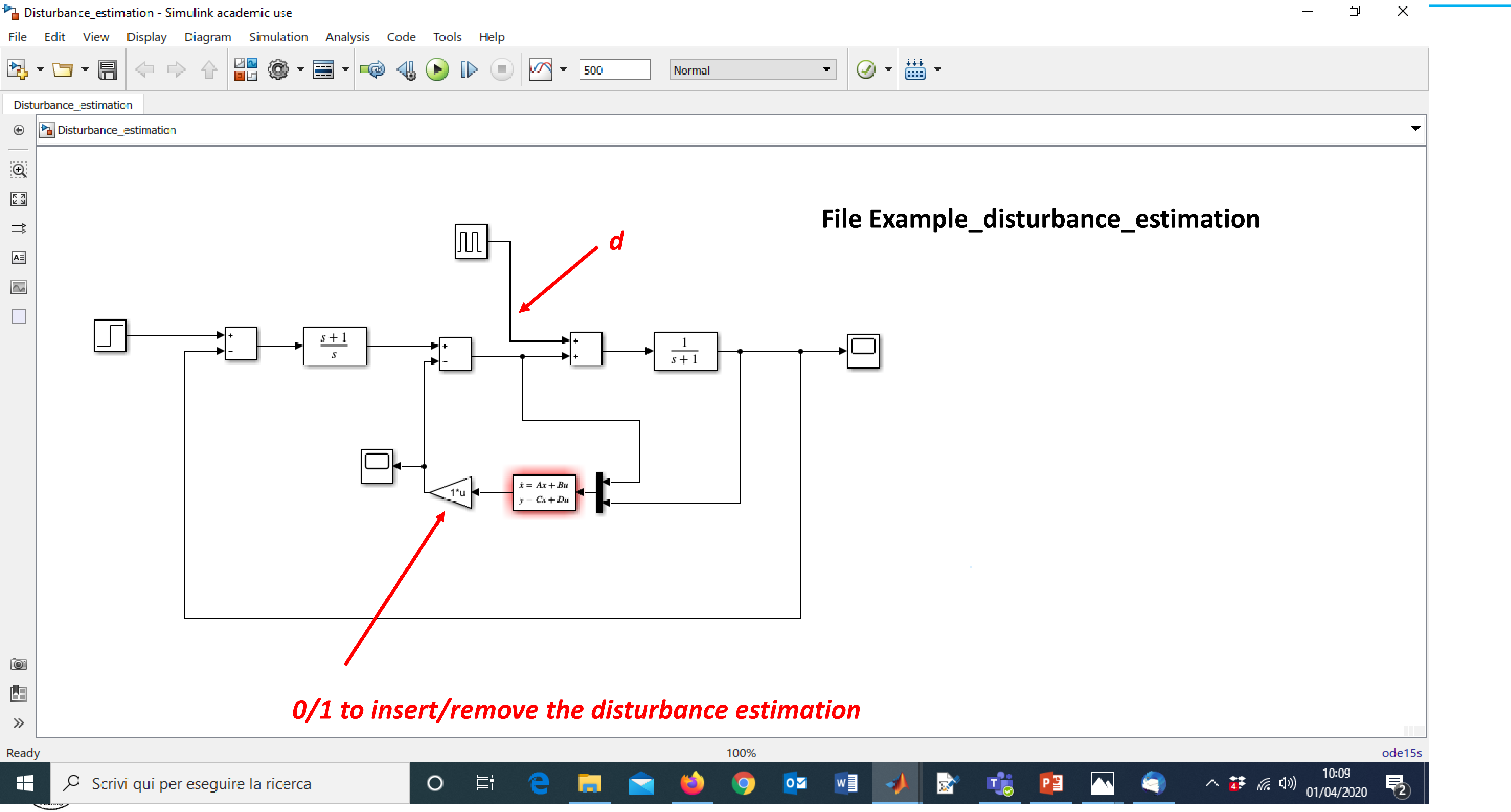
$$\dot{\xi} = -L\xi - L^2 y + Ly - Lu$$



Scheme with disturbance compensation



file Example_disturbance_estimation


Disturbance_estimation - Simulink academic use
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d
$\frac{s+1}{s}$
$\frac{1}{s+1}$
1*u
$\dot{x} = Ax + Bu$
$y = Cx + Du$
0/1 to insert/remove the disturbance estimation
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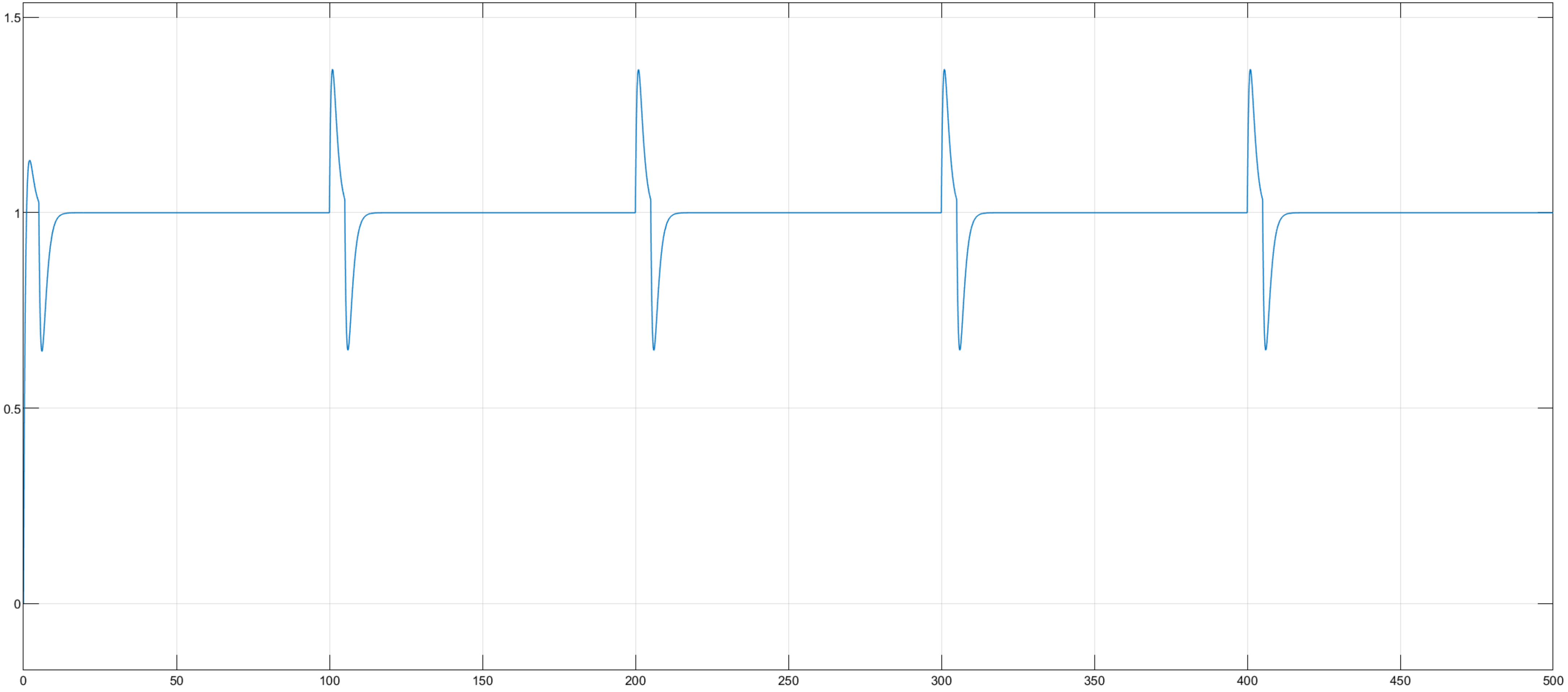
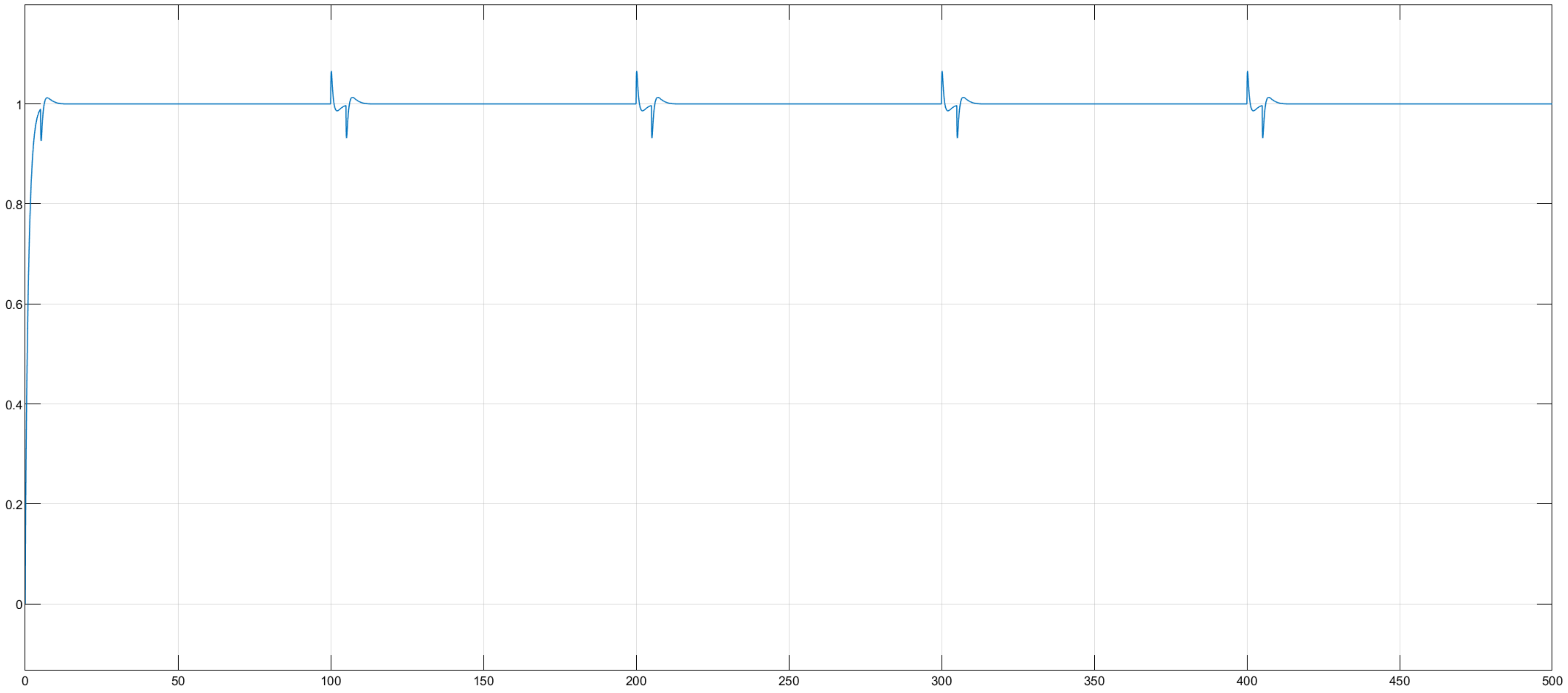
01/04/2020

No disturbance compensation



Disturbance estimation and compensation



Discrete time systems – measurable state

System

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x \in R^n, \quad u \in R^m \\ y(k) &= Cx(k) + Du(k), \quad y \in R^p \end{aligned}$$

Control law

$$u(k) = -Kx(k) + \gamma(k), \quad K \in R^{m,n}, \quad \gamma \in R^m$$

Closed-loop

$$x(k+1) = (A - BK)\, x(k) + B\gamma(k)$$

The problem is exactly the same, the same algorithms can be used (Ackermann's formula, place.m,…)

Observers

Two possibilities:

1. ***State predictor***: the estimated state at time *k* depends on the values of *u* and *y* up to time *k-1*

2. ***State filter***: the estimated state at time *k* depends on the values of *u* and *y* up to time *k*

State predictor

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L\left[y(k) - C\hat{x}(k|k-1) - Du(k)\right]$$

Estimation error $\hat{e}(k|k-1) = x(k) - \hat{x}(k|k-1)$

$$\hat{e}(k+1|k) = (A - LC)\hat{e}(k|k-1)$$

Also in this case the problem is to compute the gain ***L*** which assignes the eigenvalues of ***A-LC*** in prescribed positions

Deadbeat observers: all the eigenvalues of ***A-LC*** are at the origin. The state estimation error goes to zero in at most *n* steps

State filter

$$\begin{aligned}\tilde{x}(k+1|\,k+1) &= A\tilde{x}(k|\,k) + Bu(k) \\ &+ L\left[y(k+1) - C\left(A\tilde{x}(k|\,k) + Bu(k)\right) - Du(k+1)\right]\end{aligned}$$

Estimation error $\tilde{e}(k|\,k) = x(k) - \tilde{x}(k|\,k)$

$$\tilde{e}(k+1|\,k+1) = (A - LCA)\tilde{e}(k|\,k)$$

The gain ***L*** must be designed to assign the eigenvalues of (***A-LCA***). To this end, the pair (***A,CA***) must be observable

The pair (A, CA) is observable iff the pair (A, C) is observable and A is nonsingular

Note however that if ***A*** is singular, the system is detectable, since the nonobservable eigenvalues are at the origin, i.e. the fastest possible ones. In other words, leave them where they are (algorithms in case of detectability are only slightly more complex)

Estimation of constant disturbances

Assume to know that ***d*** is constant, or at least constant for long periods of time.
It is useful to estimate it for two reasons:

1) It is possible to correctly estimate the state of the system

2) The estimate of the disturbance can be used in a control scheme with direct compensation

Procedure

Assign a dynamics to the disturbance

$$d(k+1) = d(k), \quad d(0) = \bar{d} \in R^r$$

Enlarge the system

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} &= \begin{bmatrix} A & M \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} C & N \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + Du(k) \end{aligned}$$

Use an observer for the enlarged system. When is it possible? Try to find the condition

Reduced order observer

Idea: the output is a linear combination of the states

1. Apply a state transformation to the system so that the ***p*** outputs coincide with ***p*** new states
2. Estimate the remaining ***n-p*** *states*

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k)
\end{aligned}
$$

***T_1* any matrix such that *T* is non singular**

$$\tilde{x} = Tx = \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ \tilde{x}_r \end{bmatrix}, \quad \tilde{x}_r \in R^{n-p}$$

Transformed system

$$
\begin{aligned}
y(k+1) &= \tilde{A}_{11} y(k) + \tilde{A}_{12} \tilde{x}_r(k) + \tilde{B}_1 u(k) \\
\tilde{x}_r(k+1) &= \tilde{A}_{21} y(k) + \tilde{A}_{22} \tilde{x}_r(k) + \tilde{B}_2 u(k)
\end{aligned}
$$

$$\tilde{A} = TAT^{-1}, \, \tilde{B} = TB, \, \tilde{C} = CT^{-1}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in R^{p,p}, \quad \tilde{A}_{22} \in R^{n-p,n-p}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{B}_1 \in R^{p,m}, \quad \tilde{B}_2 \in R^{n-p,m}$$

$$\tilde{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

Define

$$\begin{aligned} \eta(k+1) &= y(k+1) - \tilde{A}_{11}y(k) - \tilde{B}_1u(k) \\ \zeta(k) &= \tilde{A}_{21}y(k) + \tilde{B}_2u(k) \end{aligned}$$

$$\begin{aligned} y(k+1) &= \tilde{A}_{11}y(k) + \tilde{A}_{12}\tilde{x}_r(k) + \tilde{B}_1u(k) \\ \tilde{x}_r(k+1) &= \tilde{A}_{21}y(k) + \tilde{A}_{22}\tilde{x}_r(k) + \tilde{B}_2u(k) \end{aligned}$$

$$\begin{aligned} \tilde{x}_r(k+1) &= \tilde{A}_{22}\tilde{x}_r(k) + \zeta(k) \\ \eta(k+1) &= \tilde{A}_{12}\tilde{x}_r(k) \end{aligned}$$

observer

$$\hat{x}_r(k+1\,|k+1\,) = \tilde{A}_{22}\hat{x}_r(k\,|k\,) + \zeta(k) + L\left[\eta(k+1) - \tilde{A}_{12}\hat{x}_r(k\,|k\,)\right]$$

or

$$\begin{aligned} \hat{x}_r(k+1\,|k+1\,) &= \left(\tilde{A}_{22} - L\tilde{A}_{12}\right)\hat{x}_r(k\,|k\,) + \tilde{A}_{21}y(k) + \tilde{B}_2u(k) \\ &+ L[y(k+1) - \tilde{A}_{11}y(k) - \tilde{B}_1u(k)] \end{aligned}$$

No problems with derivatives

Regulator transfer function with state predictor

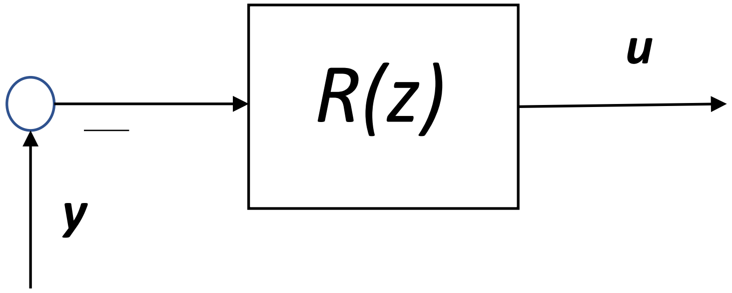
System
$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

Control law
$$u(k) = -K\hat{x}(k|k-1)$$

Predictor
$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L\left[y(k) - C\hat{x}(k|k-1)\right]$$

$$U(z) = -\boxed{K(zI - A + BK + LC)^{-1}L}Y(z)$$

strictly proper transfer function R(z)



Regulator transfer function with state filter

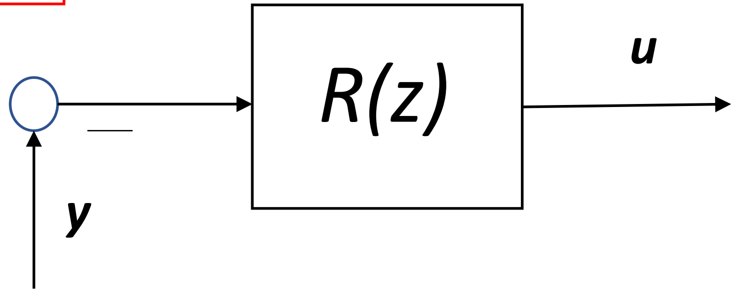
System
$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k)
\end{aligned}
$$

Control law
$$
u(k) = -K\tilde{x}(k|k)
$$

Filter
$$
\begin{aligned}
\tilde{x}(k+1|k+1) &= A\tilde{x}(k|k) + Bu(k) \\
&+ L\left[y(k+1) - C\left(A\tilde{x}(k|k) + Bu(k)\right)\right]
\end{aligned}
$$

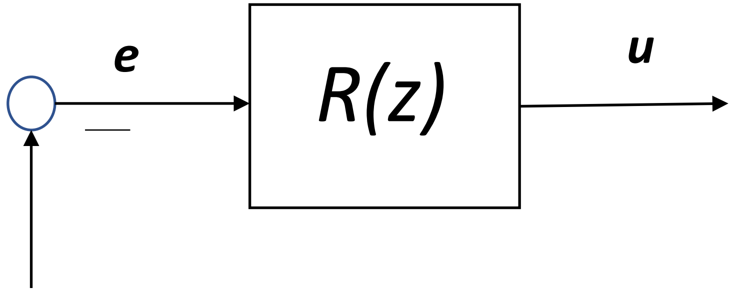
$$
U(z) = -\boxed{K(zI - A + BK + LCA + LCBK)^{-1}L}zY(z)
$$

proper transfer function R(z)



Strictly proper regulator transfer function

$$R(z) = \frac{b_{n-1}z^{n-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0}$$



$$U(z) = R(z)E(z)$$

$$(z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)U(z) = (b_{n-1}z^{n-1} + \ldots + b_1 z + b_0)E(z))$$

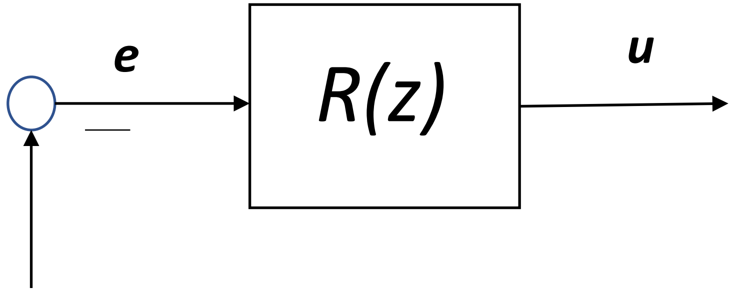
$$u(k) = -a_{n-1}u(k-1) - \ldots - a_1 u(k-n+1) - a_0 u(k-n) + b_{n-1}e(k-1) + \\ \ldots + b_1 e(k-n+1) + b_0 e(k-n)$$

One sampling time to read the value of the error (A/D), make computations, write the input u (D/A)

… but less reactive

Proper regulator transfer function

$$R(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}$$



$$U(z) = R(z)E(z)$$

$$(z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0)U(z) = (b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0)E(z))$$

$$u(k) = -a_{n-1}u(k-1) - \ldots - a_1 u(k-n+1) - a_0 u(k-n) + b_n e(k) + b_{n-1} e(k-1) + \ldots + b_1 e(k-n+1) + b_0 e(k-n)$$

More reactive, but introduces a delay due to A/D+computations+D/A