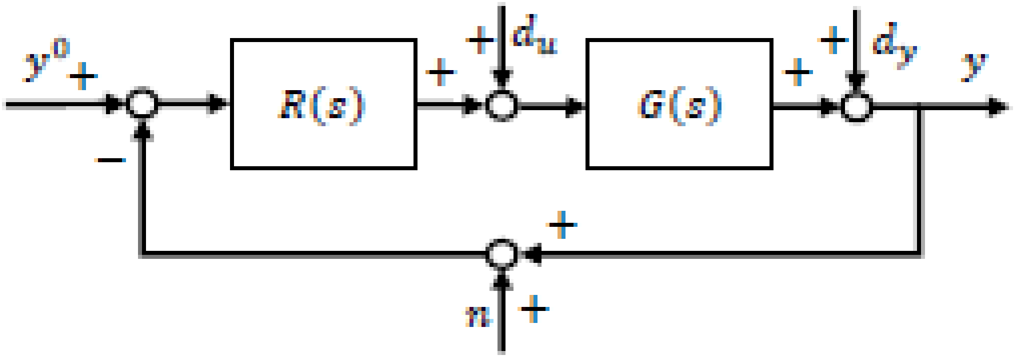
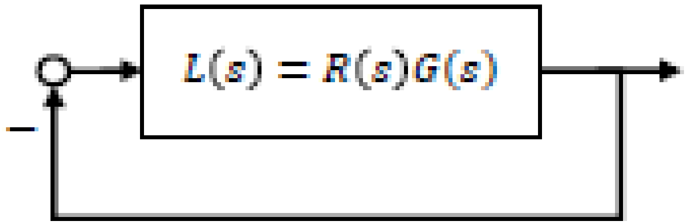
Advanced and Multivariable Control

Summary on control synthesis for SISO systems

Riccardo Scattolini

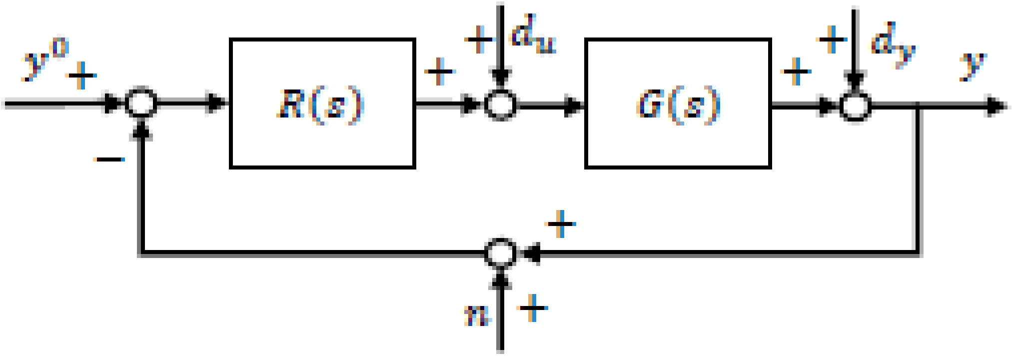




$$
\begin{aligned}
Y(s) &= T(s)(Y^o(s) - N(s)) + S(s)D_y(s) + S(s)G(s)D_u(s) \\
U(s) &= K(s)(Y^o(s) - N(s) - D_y(s)) + S(s)D_u(s)
\end{aligned}
$$

- *Loop transfer function* $L(s) = R(s)G(s)$
- *Sensitivity* $S(s) = \dfrac{1}{1 + R(s)G(s)}$
- *Complementary sensitivity* $T(s) = \dfrac{R(s)G(s)}{1 + R(s)G(s)}$
- *Control sensitivity* $K(s) = \dfrac{R(s)}{1 + R(s)G(s)} = S(s)R(s)$

Stability of the closed-loop system



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\begin{aligned}
Y(s) &= T(s)(Y^o(s) - N(s)) + S(s)D_y(s) + S(s)G(s)D_u(s) \\
U(s) &= K(s)(Y^o(s) - N(s) - D_y(s)) + S(s)D_u(s)
\end{aligned}
$$

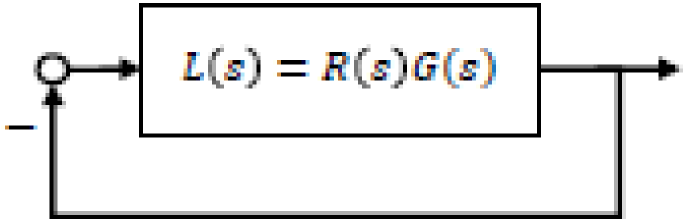
One could check the stability by looking at the poles of the functions *S(s), T(s), K(s), G(s)S(s)*

Note: all the four transfer functions must be studied to check the presence of hidden and forbidden cancellations

$$R(s) = \frac{s-1}{s} \quad , G(s) = \frac{1}{s-1} \rightarrow S(s) = \frac{s}{s+1} \quad , G(s)S(s) = \frac{s}{(s+1)(s-1)}$$

Not very practical and useful in the control design phase

Stability of the closed-loop system by looking at the open-loop transfer function _L(s)_



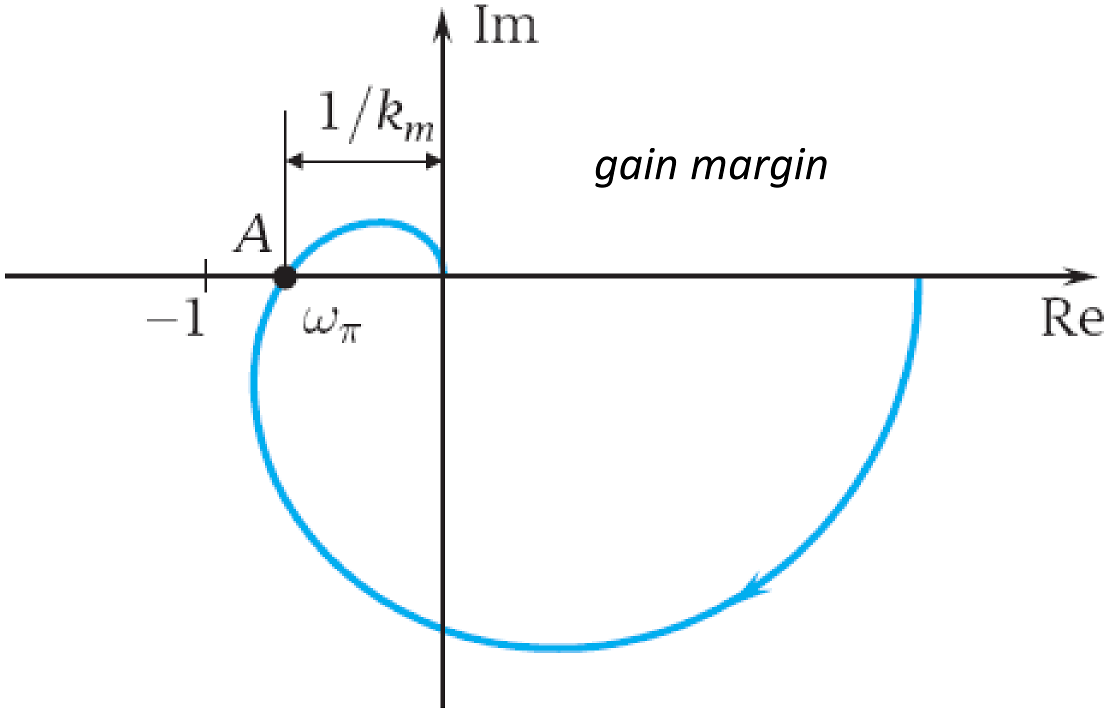
Nyquist criterion: the king of analysis methods (you must know it!), but not extremely useful for the synthesis of *R(S)*

Bode criterion: equivalent to the Nyquist criterion in many problems, more practical for the synthesis → *phase (and gain) margin* (see the next slide)

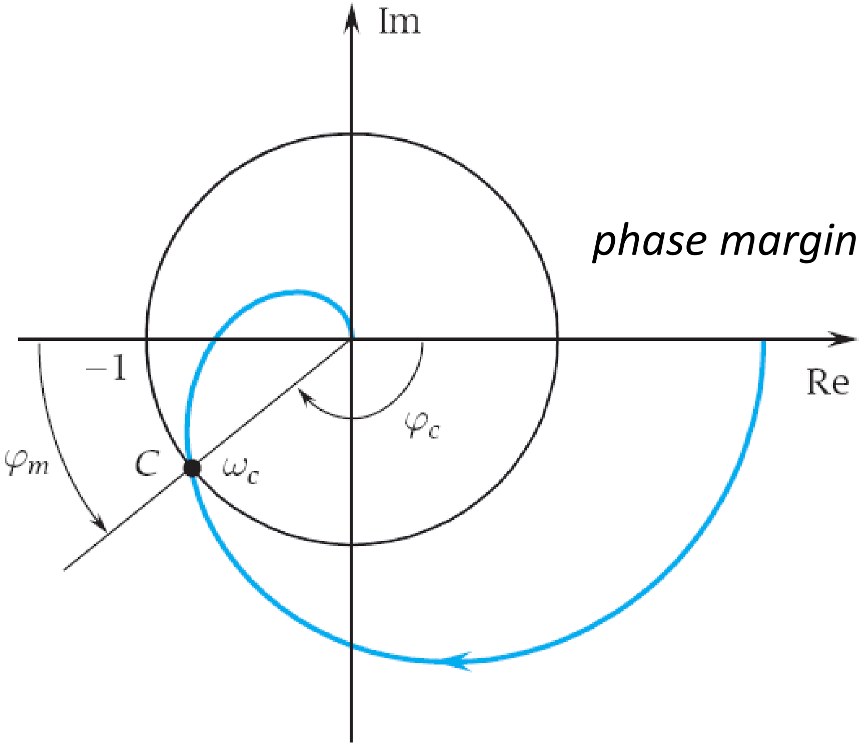
Root locus: we will use only the very basic rules

gain and phase margins

For many systems they represent a distance of the polar diagram of $G(j\omega)$ from point -1, in this sense a sufficiently high value is a good indicator of ***robustness*** with respect to modeling errors or variations



gain margin



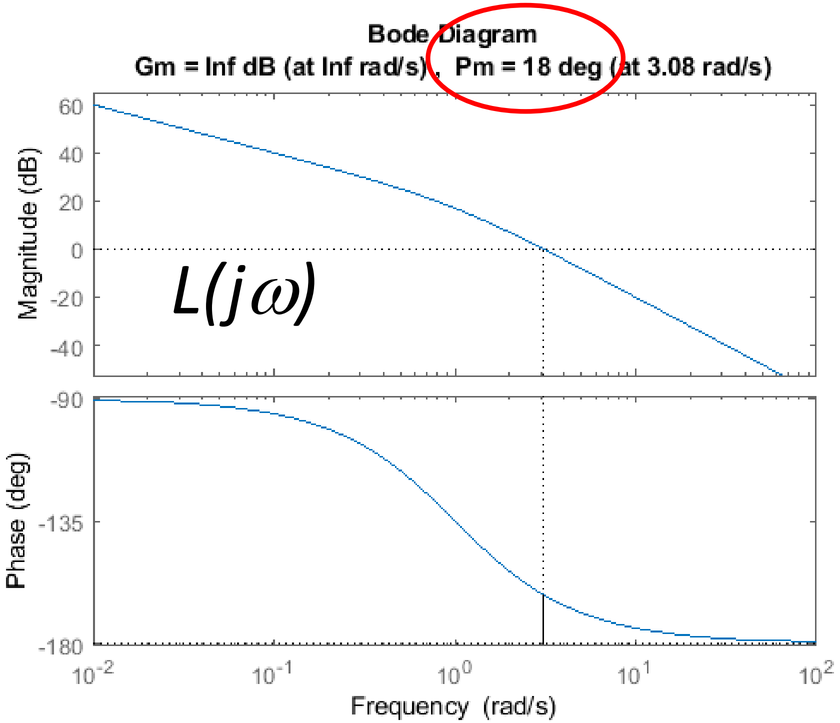
phase margin φ_m

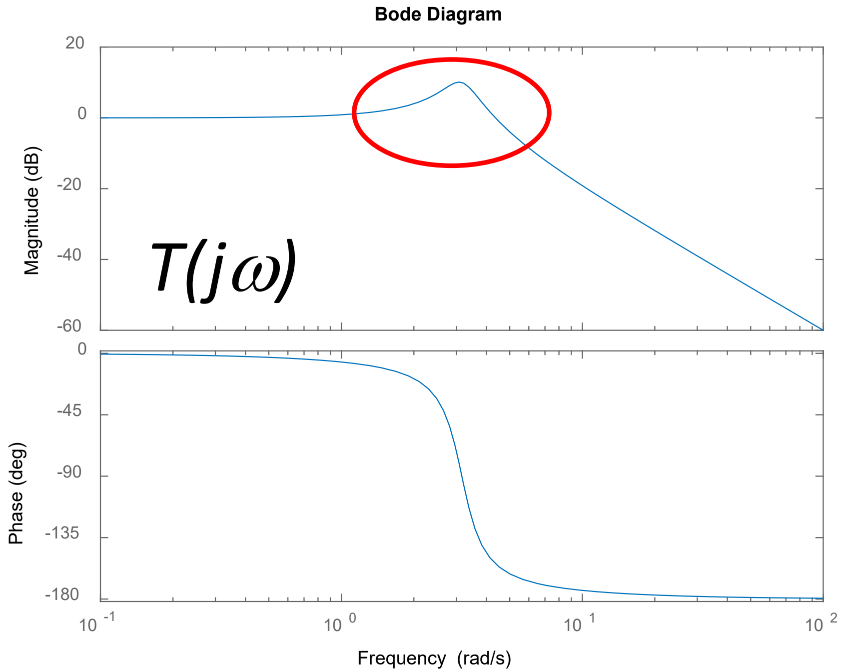
easier to use and required by the Bode criterion

Phase margin and closed-loop oscillations

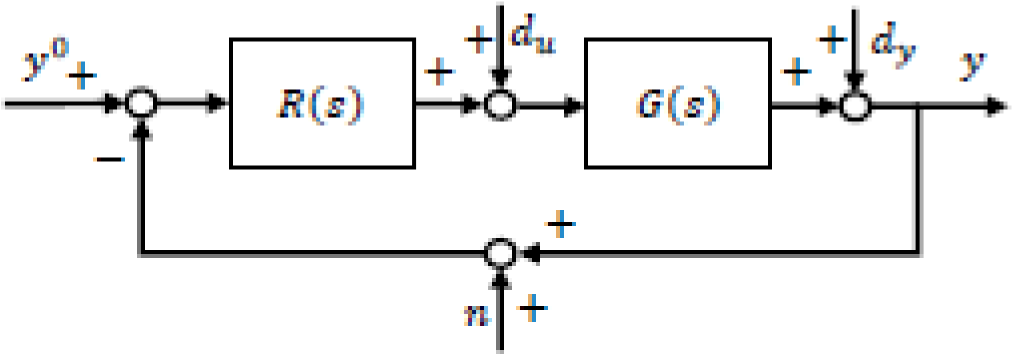
In many cases, the complementary sensitivity can be approximated by $T(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$ $\quad \xi \cong \frac{\varphi_m}{100}$

In order to reduce the peak of $T(j\omega)$ one must choose a sufficiently high φ_m



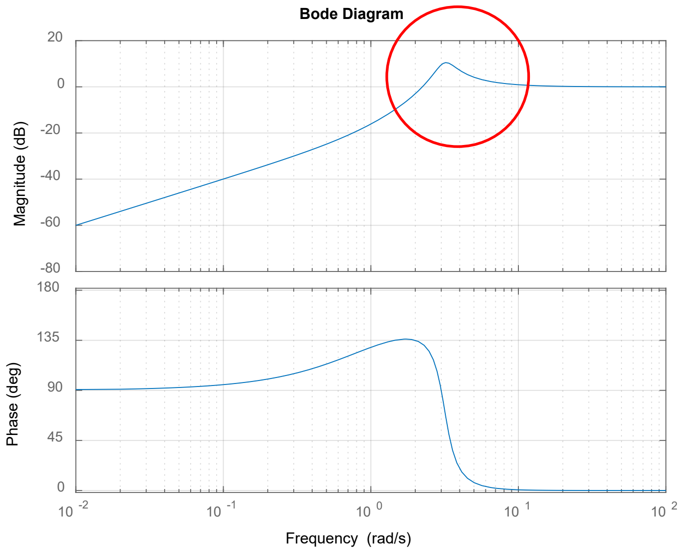


Sensitivity and performance



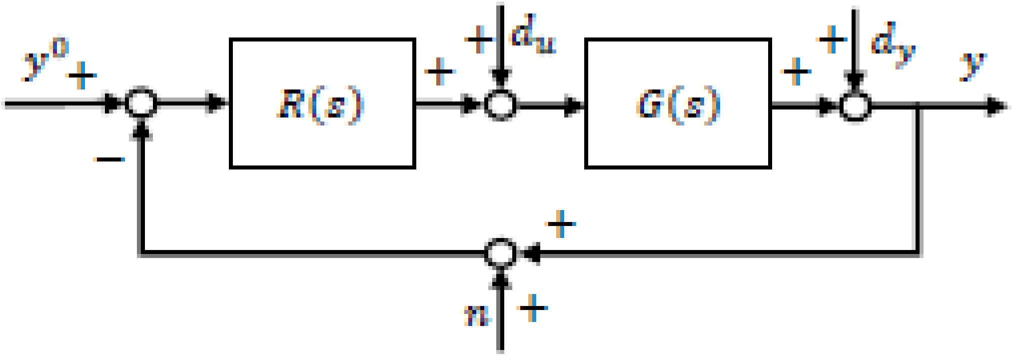
$$
\begin{aligned}
Y(s) &= T(s)(Y^o(s) - N(s)) + S(s)D_y(s) + S(s)G(s)D_u(s) \\
U(s) &= K(s)(Y^o(s) - N(s) - D_y(s)) + S(s)D_u(s)
\end{aligned}
$$

$$
|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \simeq \begin{cases} \dfrac{1}{|L(j\omega)|}, & |L(j\omega)| \gg 1, \quad \omega \ll \omega_c \\ 1, \quad |L(j\omega)| \ll 1, & \omega \gg \omega_c \end{cases}
$$



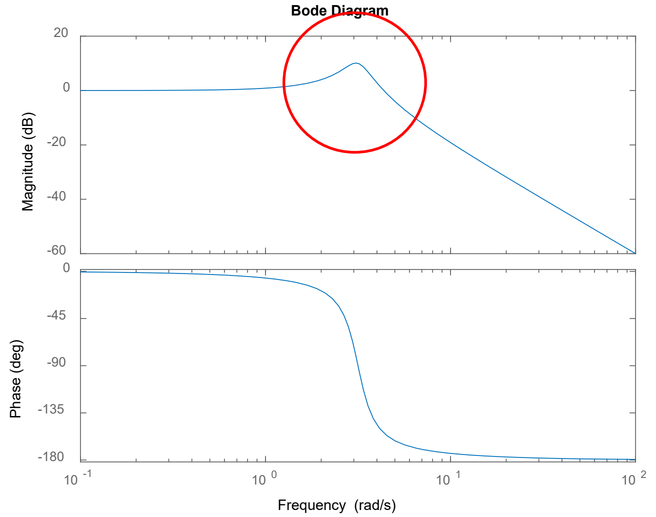
$|S|$ should be "small" ($|L|$ "big") where the spectrum d_y, d_u have significant harmonic components (usually at low frequencies)

Complementary sensitivity and performance



$$
\begin{aligned}
Y(s) &= T(s)(Y^o(s) - N(s)) + S(s)D_y(s) + S(s)G(s)D_u(s) \\
U(s) &= K(s)(Y^o(s) - N(s) - D_y(s)) + S(s)D_u(s)
\end{aligned}
$$

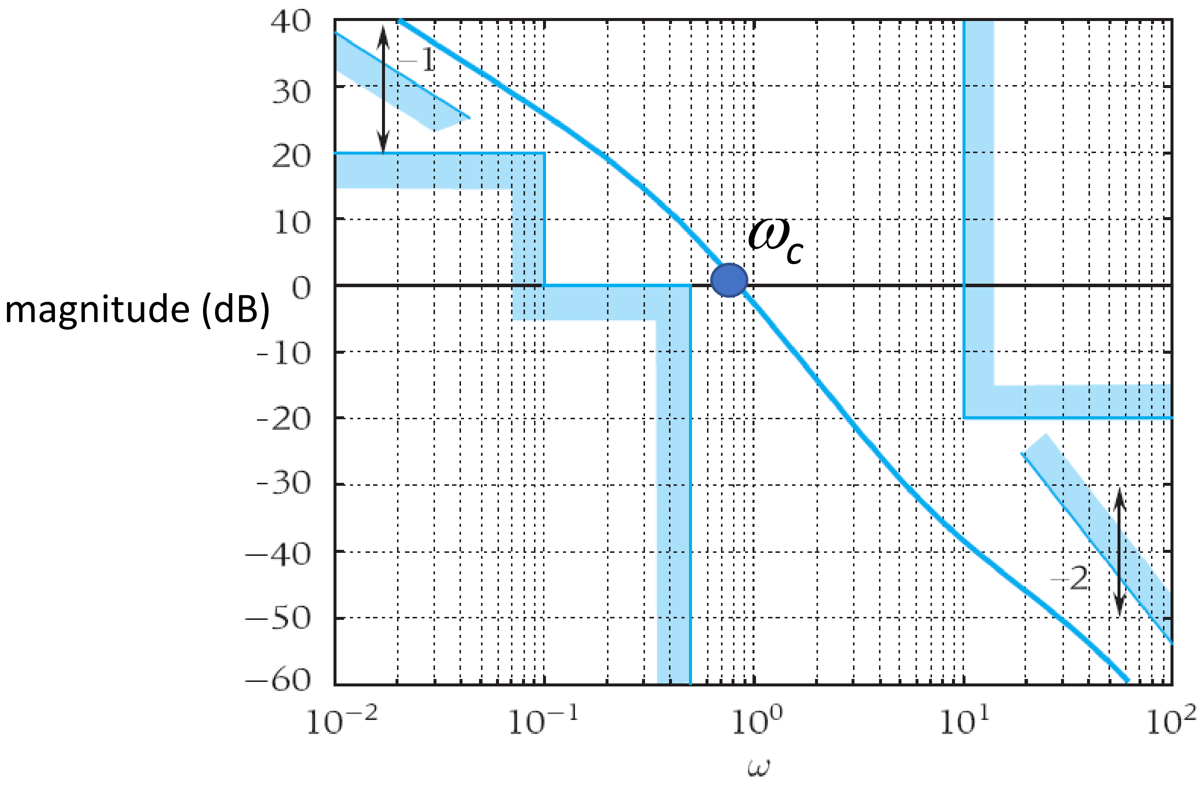
$$
|T(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \simeq \begin{cases} 1, & |L(j\omega)| \gg 1, \quad \omega \ll \omega_c \\ |L(j\omega)|, & |L(j\omega)| \ll 1, \quad \omega \gg \omega_c \end{cases}
$$



$|T| \simeq 1$ ($|L|$ "big") in the frequency band where the spectrum of the reference signal has significant harmonic components (usually at low-medium frequencies)

$|T|$ "small" ($|L|$ "small") in the frequency band where the spectrum of the measurement noise n has significant harmonic components (usually at high frequencies)

Design requirements on *L(s)*



- $\varphi_m \geq \bar{\varphi}_m$ (stability, robustness, limited peaks of $T(j\omega)$)
- $g_m \geq \bar{g}_m$ (robustness)
- $\omega_c \geq \bar{\omega}_c$ (speed of response, attenuation low frequency disturbances d_y)
- $\omega_c \leq \tilde{\omega}_c$ (limit control action, attenuation measurement noise n)

Limits to performance

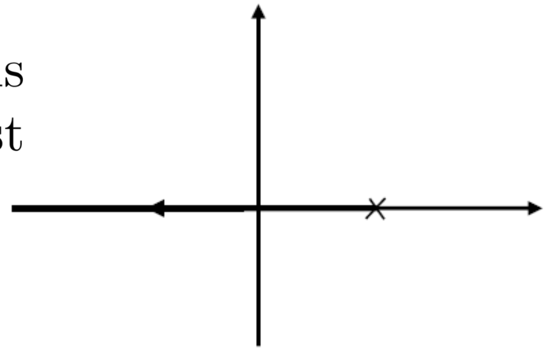
$$T(s) + S(s) = 1$$

Systems with delay: $e^{-\tau s}G(s)$, with $G(s)$ rational transfer functions. The delay introduces a negative phase contribution at the crossover frequency equal to $-\tau\omega_c$. The cutoff frequency ω_c must be limited to guarantee a reasonable phase margin

Systems with zeros with positive real part, such as: $(1-\tau s)G'(s)$, $\tau > 0$. The positive zero introduces a negative phase margin. In practice, it is impossible to have a crossover frequency greater than $1/\tau$ with an acceptable phase margin

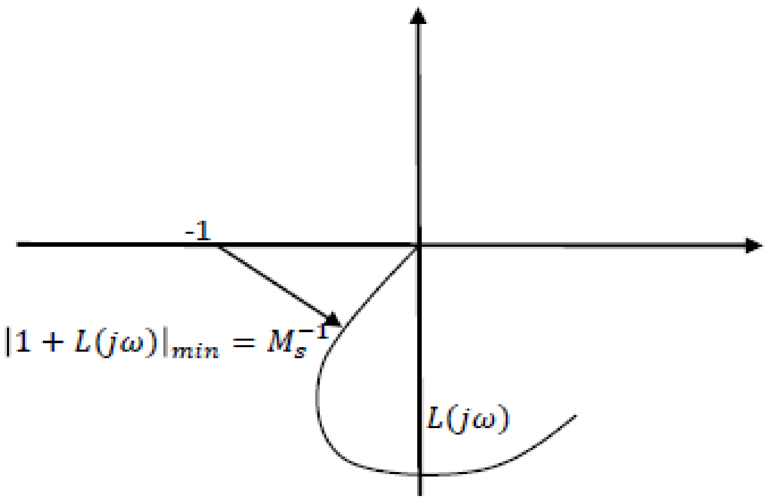
Examples are reported in the textbook

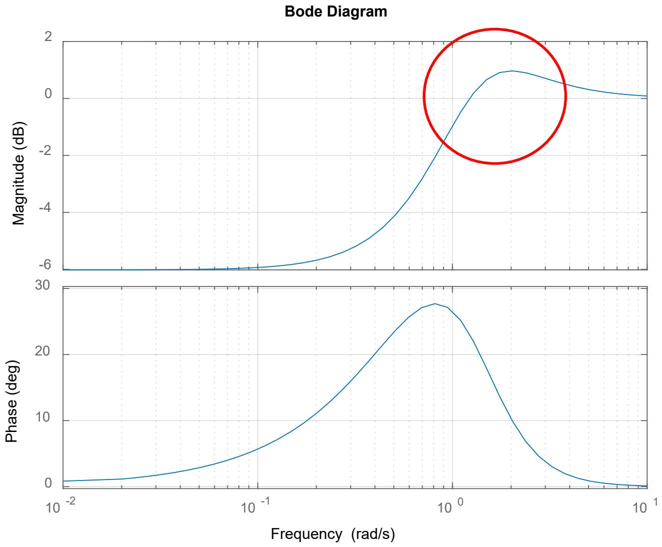
Systems with unstable poles: a regulator with *a sufficiently high gain* is required to stabilize the system. In terms of root locus, the unstable pole must be brought to the stability region

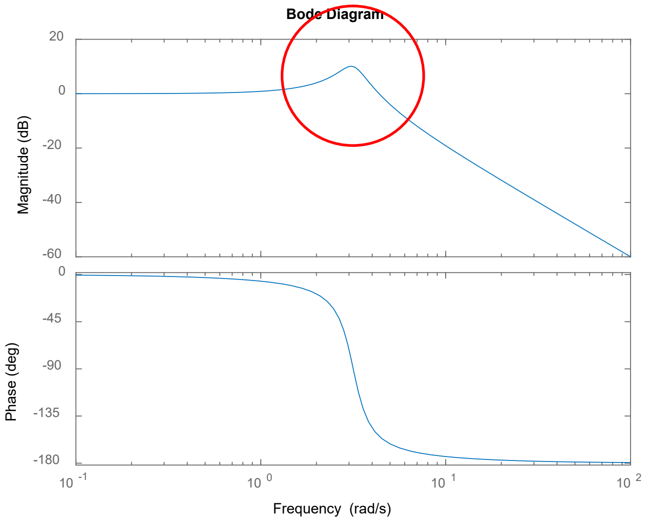


Is it truly impossible to provide specifications in terms of sensitivity functions?

Define







$$M_S = \sup_{\omega} |S(j\omega)| = \|S\|_\infty \quad , \quad M_T = \sup_{\omega} |T(j\omega)| = \|T\|_\infty$$

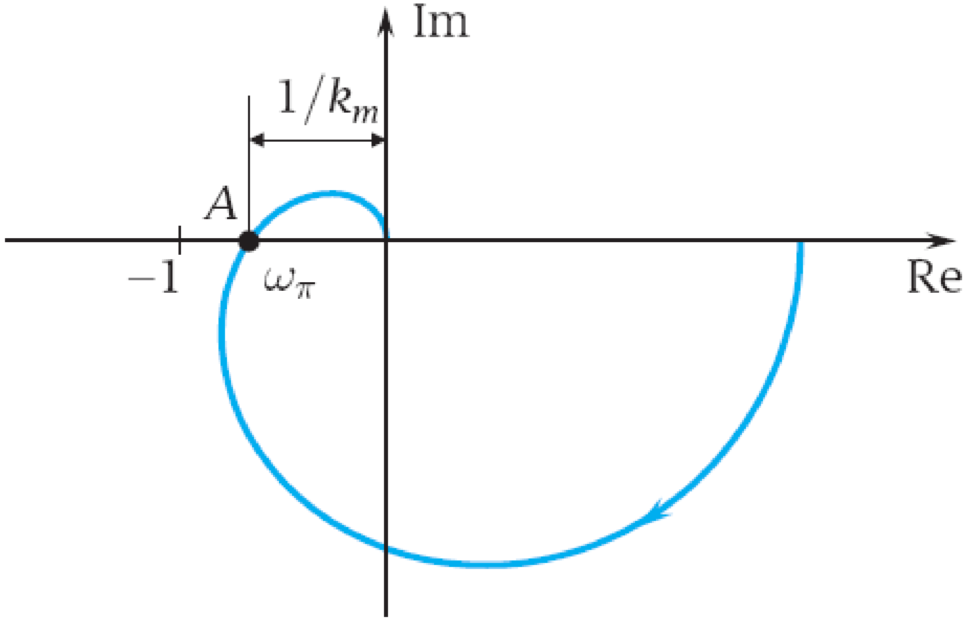
It is possible to prove that (see the textbook and following slides):

$$M_S \text{ and } M_T \text{ differ at most by } 1$$

$$\text{the gain margin } g_m \text{ is such that } g_m \geq 1 + \frac{1}{M_T} \ , \ g_m \geq \frac{M_S}{M_S - 1}$$

$$\varphi_m \geq 2 \arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T} \ , \ \varphi_m \geq 2 \arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$$

Gain margin



Gain margin $\boldsymbol{k_m}$, or $\boldsymbol{g_m}$ in the notes (assumed to be >1)

$$L(j\omega_\pi) = -1/g_m \longrightarrow T(j\omega_\pi) = \frac{-1/g_m}{1-1/g_m} \longrightarrow M_T \geq |T(j\omega_\pi)| = \frac{1}{g_m - 1} \longrightarrow g_m \geq 1 + \frac{1}{M_T}$$

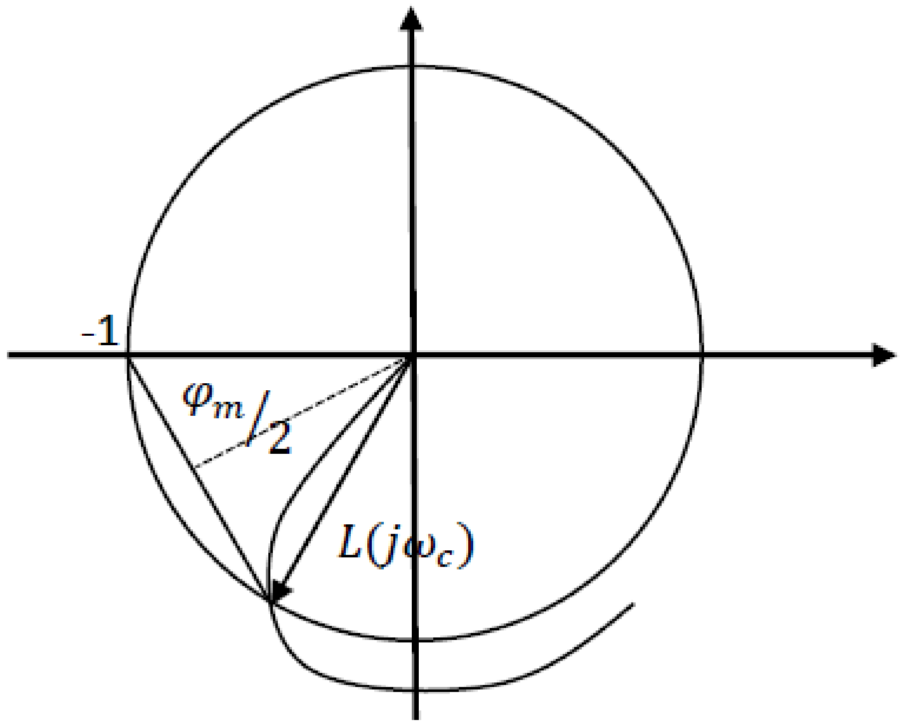
$$M_S \geq |S(j\omega_\pi)| = \frac{1}{1-1/g_m} \longrightarrow g_m \geq \frac{M_S}{M_S - 1}$$

Phase margin

$$\frac{1}{M_S} \leq |1 + L(j\omega_c)| = 2\sin\varphi_m/2$$

$$\downarrow$$

$$\varphi_m \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$$



Moreover

$$|T(j\omega_c)| = \frac{|L(j\omega_c)|}{|1+L(j\omega_c)|} = \frac{1}{|1+L(j\omega_c)|} = |S(j\omega_c)|$$

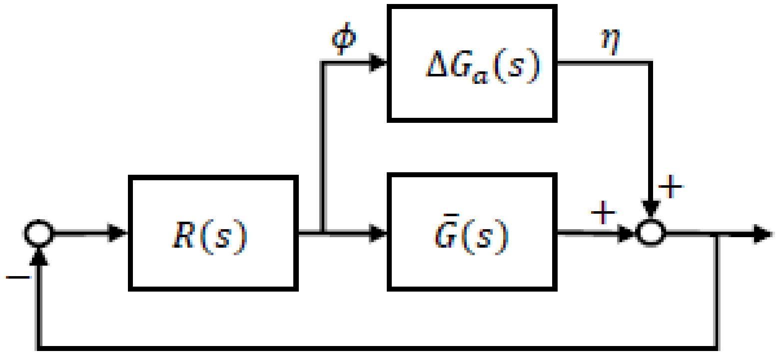
$$\downarrow$$

$$\varphi_m \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T}$$

M_S, M_T, and robustness – additive uncertainty

$$G(s) = \bar{G}(s) + \Delta G_a(s)$$

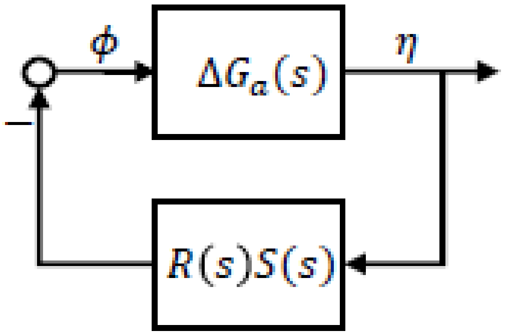
nominal *perturbation*



Assumptions

- closed-loop nominal system asymptotically stable;
- $\Delta G_a(s)$ asymptotically stable. Letting $P_{\bar{G}}$ and P_G be the number of poles of $\bar{G}(s)$ and $G(s)$ with positive real part, this means $P_{\bar{G}} = P_G$;
- no cancellations of poles and zeros of $R(s)$ and $G(s)$ with non negative real part.

$$\phi = -R(\eta + \bar{G}\phi) \rightarrow$$
$$(1 + R\bar{G})\phi = -R\eta \rightarrow$$
$$\phi = -\frac{1}{1+R\bar{G}} R\eta$$

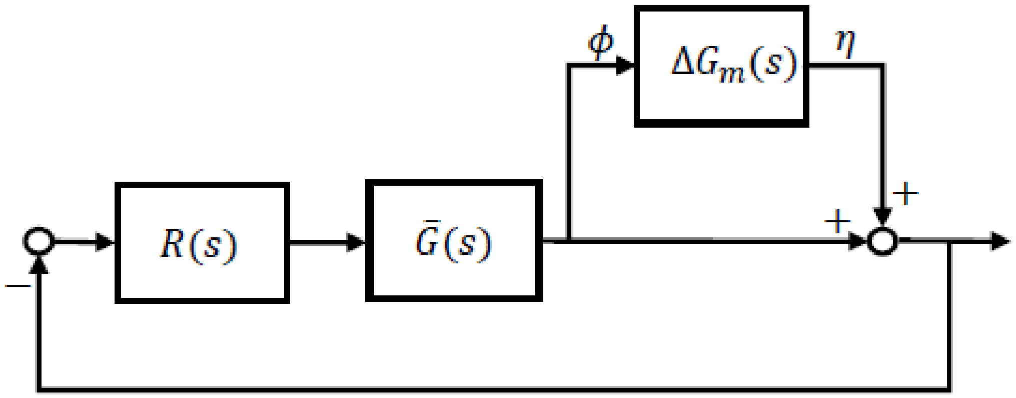


small gain theorem

$$\boxed{\|RS\Delta G_a\|_\infty < 1}$$

a small M_S helps

M_S, M_T, and robustness – multiplicative uncertainty



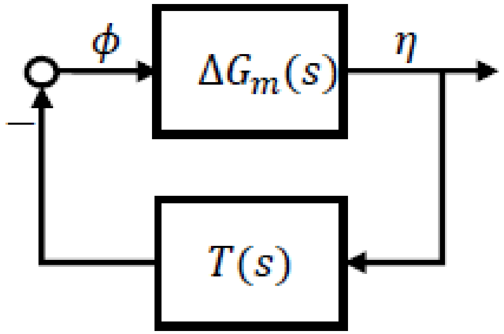
$$G(s) = \bar{G}(s)(1 + \Delta G_m(s))$$

nominal *perturbation*

$$\phi = -R\bar{G}(\eta + \phi) \rightarrow \phi = -\frac{R\bar{G}}{1+R\bar{G}}\eta$$

Assumptions

- closed-loop nominal system asymptotically stable;
- $\Delta G_m(s)$ asymptotically stable. Letting $P_{\bar{G}}$ and P_G the number of poles of $\bar{G}(s)$ and $G(s)$ with positive real part, this means that $P_{\bar{G}} = P_G$



small gain theorem $\longleftrightarrow$

$$\boxed{\|T\Delta G_m\|_\infty < 1}$$

a small M_T helps

How to model the uncertainty?

Additive or multiplicative? It depends on the problem

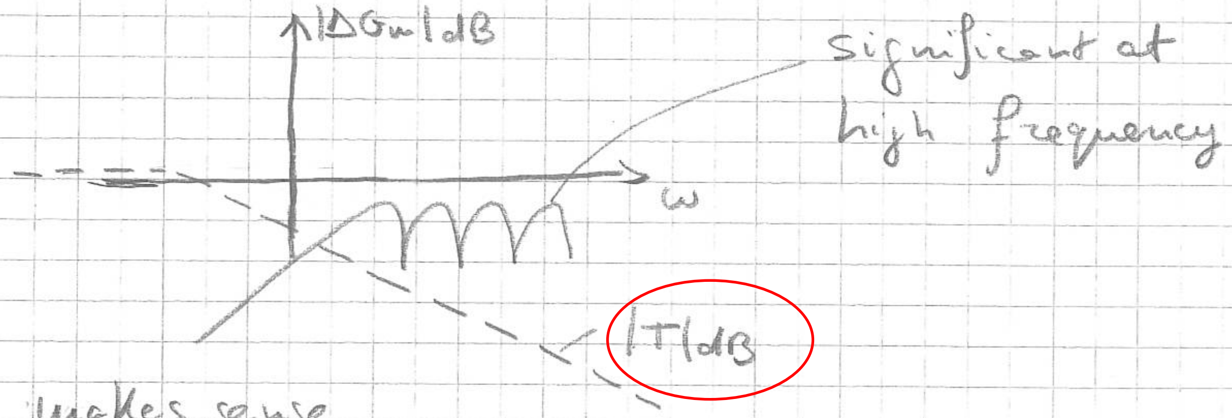
Example

$$G(s) = \frac{1}{s} e^{-\tau s} \quad , \quad \bar{G}(s) = \frac{1}{s}$$

Multiplicative uncertainty $G(s) = \bar{G}(s)\,\underbrace{(1 + \Delta G_m(s))}_{e^{-\tau s}}$

$\frac{1}{s} e^{-\tau s}$ $\quad$ $1/s$

so that $\Delta G_m(s) = e^{-\tau s} - 1$

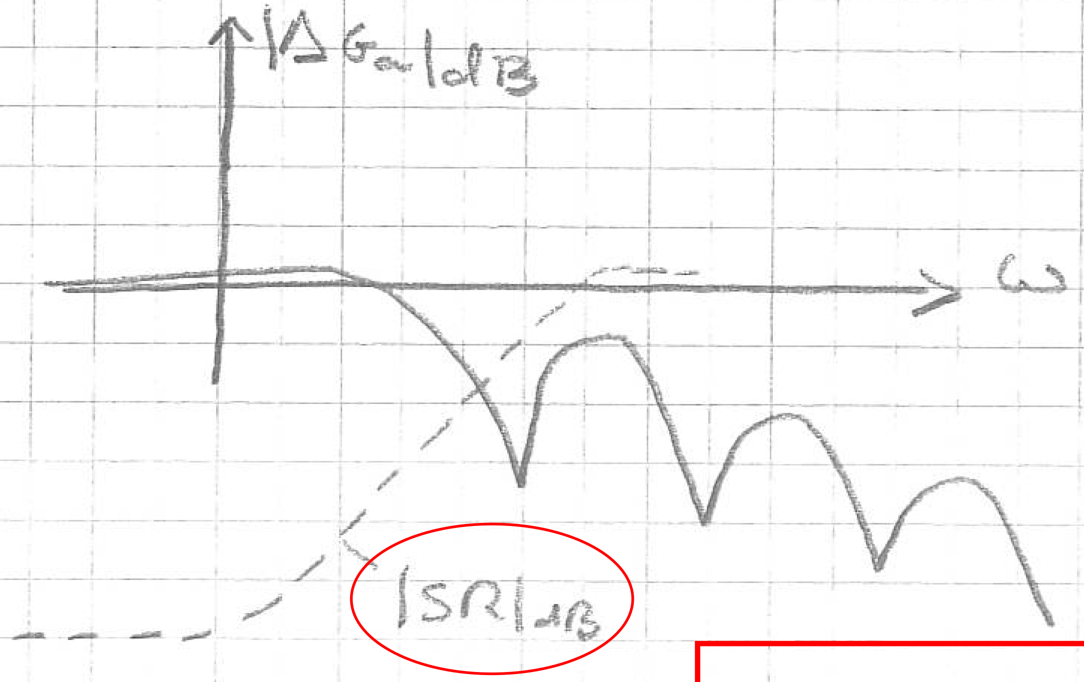


It makes sense to use the multiplicative uncertainty (small at high frequency)

$$\|T \Delta G_m\|_\infty < 1$$

Additive uncertainty

$$G(s) = \bar{G}(s) + \Delta G_a(s) \rightarrow \Delta G_a(s) = \frac{e^{-\tau s} - 1}{s}$$



One must force RS to be small at low frequency

$$\boxed{\|RS\Delta G_a\|_\infty < 1}$$

Example (gain uncertainty)

$$\bar{G}(s) = \frac{\bar{k}}{s+a} \quad , \quad G(s) = \frac{\bar{k}+\Delta k}{s+a}$$

Multiplicative uncertainty

$$\frac{\bar{k}+\Delta k}{s+a} = \frac{\bar{k}}{s+a}\left(1 + \Delta G_m(s)\right)$$

$$\Delta G_m(s) = \frac{\Delta k}{\bar{k}}$$

constant at all frequencies (previous condition on $T(s)$ not easy to satisfy)

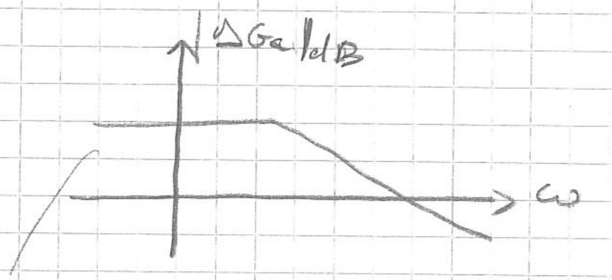
Example (gain uncertainty)

$$\bar{G}(s) = \frac{\bar{k}}{s+a}, \qquad G(s) = \frac{\bar{k}+\Delta k}{s+a}$$

Additive uncertainty

$$\frac{\bar{k}+\Delta k}{s+a} = \frac{\bar{k}}{s+a} + \Delta G_a(s)$$

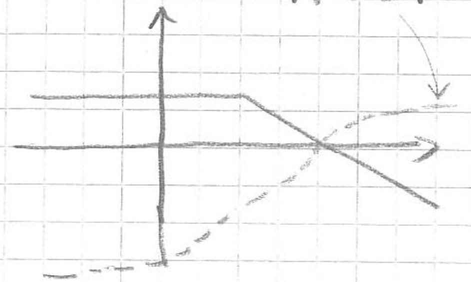
$$\Delta G_a(s) = \frac{\Delta k}{s+a}$$



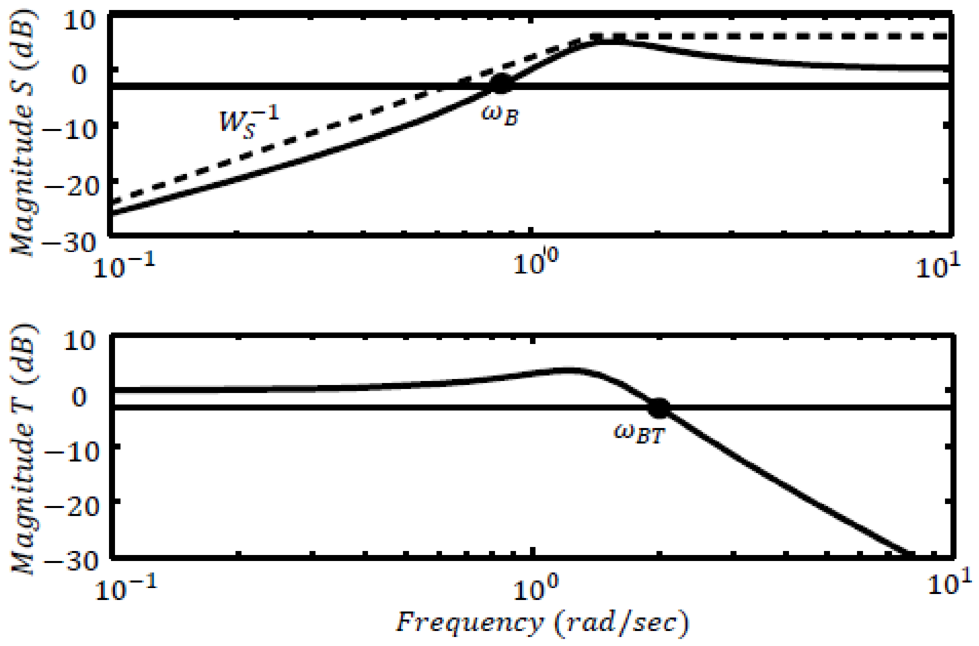
centered at low frequency

Easier to force the condition on

|R S|



Sensitivity functions and crossover frequency



Define by ω_B the frequency where $|S(j\omega)|$ crosses $1/\sqrt{2}$ (-3dB) from below and by ω_{BT} the frequency where $|T(j\omega)|$ crosses $1/\sqrt{2}$ (-3dB) from above

Then, if $\varphi_m < 90^o$, one has

$$\omega_B < \omega_c < \omega_{BT}$$

Also in this case, specifications can be given in terms of *S(s), T(s)*

Design specifications in terms of sensitivity functions

We could specify:

- shape of $S(s)$;
- minimum frequency ω_B;
- small or null asymptotic error for constant reference signals ($|S(j\omega)|$ small or Bode diagram of $|S(j\omega)|$ with shape +1 at low frequency);
- $M_S \leq \bar{M}_S$.

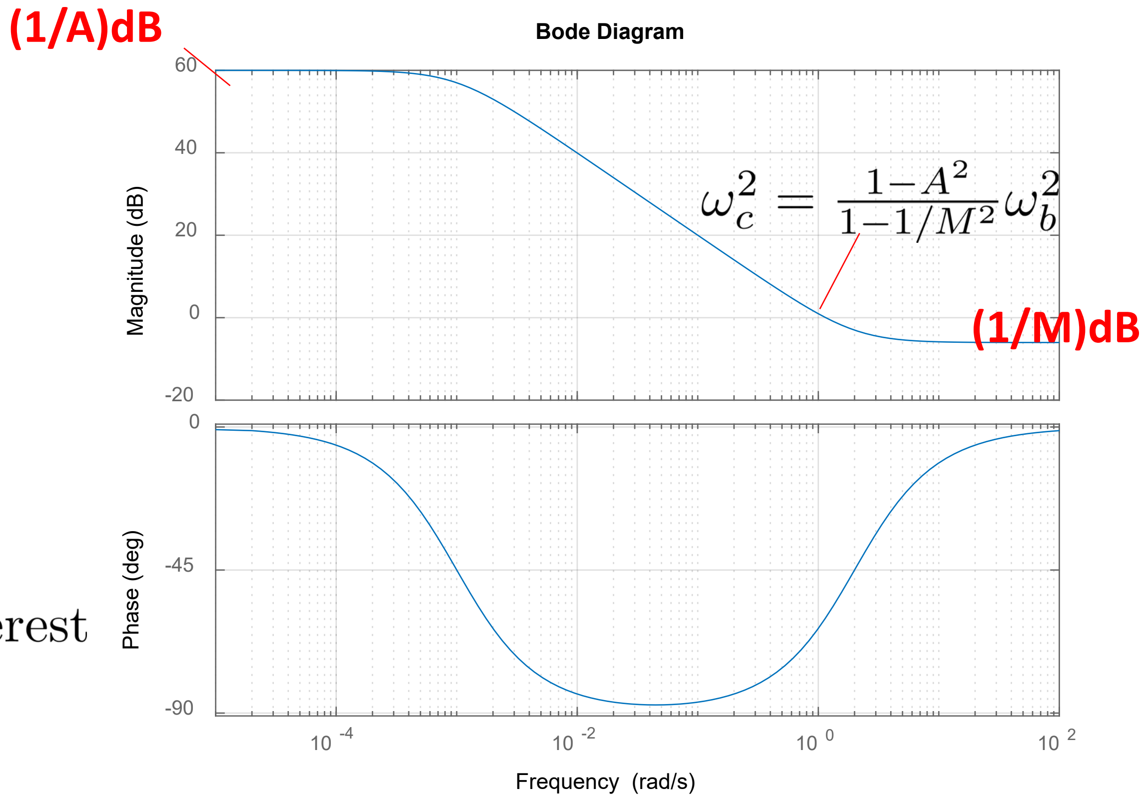
this defines a function $S_{desired}(s)$, and the function $W_S(s) = S^{-1}_{desired}(s)$ also named *(sensitivity) shaping function*

then, the regulator must be designed such that

$$|S(j\omega)| < \frac{1}{|W_S(j\omega)|}, \quad \forall \omega \qquad \longleftrightarrow \qquad \|W_S S\|_\infty < 1$$

Possible choice of $W_S(s)$

$$W_S(s) = \frac{s/M+\omega_B}{s+A\omega_B}$$



$A \ll 1$: desired attenuation of $S(s)$ in the band of interest

M required bound on the H_∞ norm of $S(s)$

How to synthetise the regulator? We'll see later in the course

Design specifications in terms of complementary sensitivity function

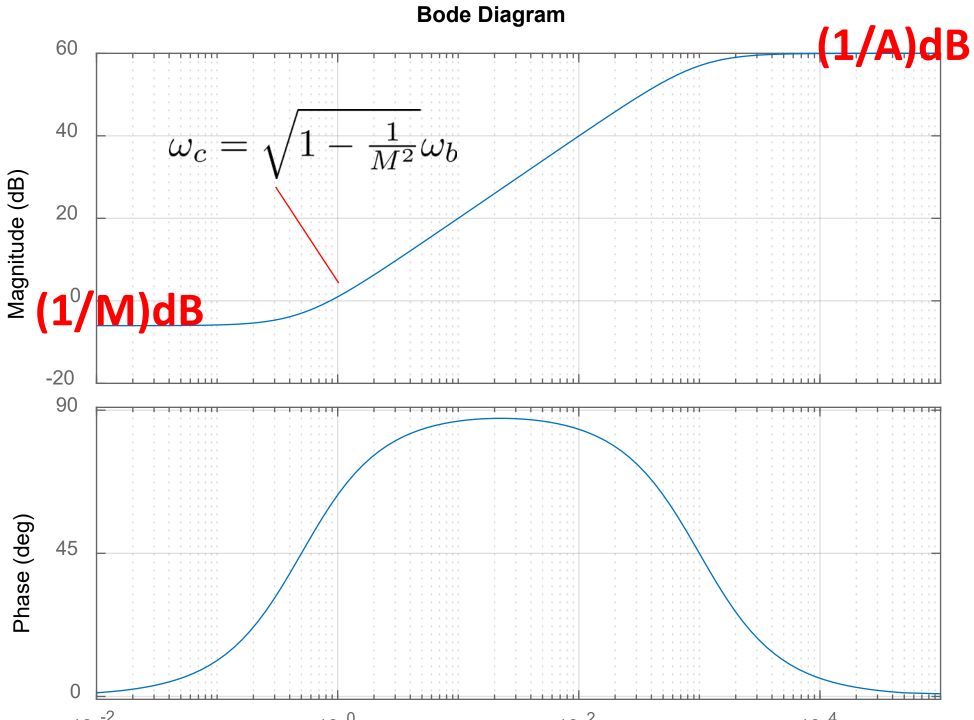
Also in this case, define a $T_{desired}(s)$, and its inverse $W_T(s) = T^{-1}_{desired}(s)$ also named *shaping function* $W_T(s)$

then, the regulator must be designed such that

$$|T(j\omega)| < \frac{1}{|W_T(j\omega)|}, \quad \forall\omega \qquad \longleftrightarrow \qquad \|W_T T\|_\infty < 1$$

Example

$$W_T(s) = \frac{s + \omega_{BT}/M}{As + \omega_{BT}},$$



Control sensitivity function

Same approach, define the control sensitivity function $W_K(s)$ and choose a regulator *R(s)* such that

$$|K(j\omega)| < \frac{1}{|W_K(j\omega)|}, \quad \forall \omega \Leftrightarrow \|W_K K\|_\infty < 1$$

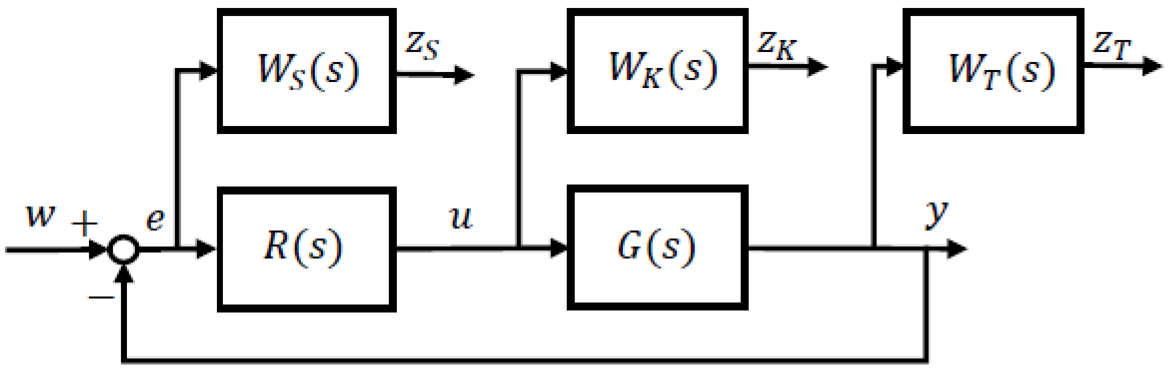
The shaping function $W_S(s)$, $W_T(s)$, $W_K(s)$ must be selected as asymptotically stable systems

In summary, one must find a regulator such that

$$\|W_S S\|_\infty < 1, \quad \|W_T T\|_\infty < 1, \quad \|W_K K\|_\infty < 1$$

An interpretation

Consider the enlarged system



,

Define $z = \begin{bmatrix} z_S \\ z_K \\ z_T \end{bmatrix}$, $w = y^o$ and note that $z = G_{zw} w$, $G_{zw}(s) = \begin{bmatrix} W_S(s)S(s) \\ W_T(s)T(s) \\ W_K(s)K(s) \end{bmatrix}$

the control synthesis can be completed by minimizing

$$\|G_{zw}\|_\infty = \sup_\omega \bar{\sigma}(G_{zw}(j\omega))$$

If the resulting regulator is such that $\|G_{zw}\|_\infty < \gamma$ one has

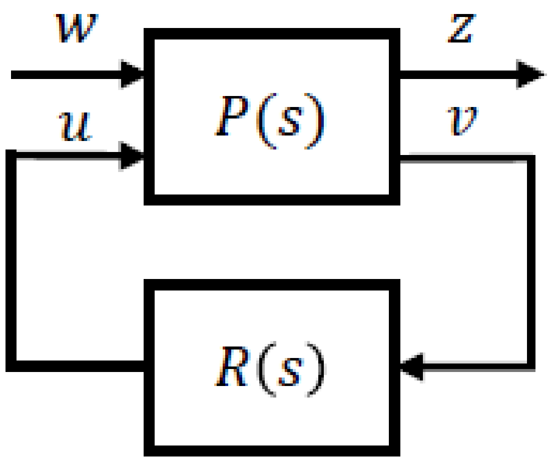
H_∞ control

$$\|W_S S\|_\infty < \gamma, \quad \|W_T T\|_\infty < \gamma, \quad \|W_K K\|_\infty < \gamma$$

H_2 control

$$
\begin{aligned}
J &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(|W_S(j\omega)S(j\omega)|^2 + |W_T(j\omega)T(j\omega)|^2 + |W_K(j\omega)K(j\omega)|^2\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}|G_{zw}(j\omega)|^2\,d\omega
\end{aligned}
$$

Formal statement of *H2 – Hinf* control



Minimize the *2-norm* or *inf-norm* of $\boldsymbol{G_{zw}(s)}$ with respect to ***R(s)***

where

z are named *performance variables*

v are named *measured variables*

$$w = \begin{bmatrix} d \\ y^o \\ n \end{bmatrix}$$ are the *exogenous variables*

Later in the course we'll study solution methods

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first_example (2018_10_19 17_00_40 UTC).m

```
wB=10;      %desired closed-loop bandwidth
A=1/10000;  %desired disturbance attenuation inside bandwidth
M=2 ;       %desired bound on hinfnorm(S) and hinfnorm(T)
s=tf('s'); %Laplace transform variable 's'
G=1/(s*(s+1));
numG=1;
denG=[1 1 0];
WS=(s/M+wB)/(s+wB*A); %Sensitivity weight
WK=[];                %Empty control weight
WT=(s+wB/M)/(A*s+wB); %Complementary sensitivity weight
[K,CL,GAM,INFO]=mixsyn(G,WS,WK,WT);

L=G*K;  %loop transfer function
S=inv(1+L); %Sensitivity
T=1-S;      %complementary sensitivity
figure(1)
sigma(GAM/WS,S);
title('GAM/WS and S')
figure(2)
sigma(GAM/WT,T);
title('GAM/WT and T')
figure(3)
margin(L)
% %regulator matrix A and eigenvalues
MatriceAreg=K.a
AutovReg=eig(K.a)
K.a(1,1)=0;
L1=G*K;
figure(4)
margin(L1)
[numR,denR]=ss2tf(K.a,K.b,K.c,K.d);
figure(5)
bode(numR,denR)
title('Bode diagram of the regulator')
```

Beep→folder→SW→first example

[K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3) or

[K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3,KEY1,VALUE1,KEY2,VALUE2,...)

mixsyn H-infinity mixed-sensitivity synthesis method for robust
control design. Controller K stabilizes plant G and minimizes
the H-infinity cost function

|| W1*S ||
|| W2*K*S ||
|| W3*T ||Hinf