Exercises session 2: Lyapunov functions

Lorenzo Nigro

Ex. 1: Given the continuous time system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2^2 \\ \dot{x}_2 = -x_2 \end{cases} \tag{1}$$

Show that

1. the $(\bar{x}_1,\ \bar{x}_2) = (0,\ 0)$ is an equilibrium point
2. and study its stability through the linearized system.

Given the function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, $\forall x_1,\ x_2 \neq 0$

3. Check that it is in fact a suitable Lyapunov function
4. and study the stability of the system with the given function

Ex. 2: Given the following parametric system

$$\begin{cases} \dot{x}_1 = x_1\ (k^2 - x_1^2 - x_2^2) + x_2\ (k^2 + x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1\ (k^2 + x_1^2 + x_2^2) + x_2\ (k^2 - x_1^2 - x_2^2) \end{cases} \tag{2}$$

and the function $V(x) = \frac{1}{2}(x_1^2 + x_2^2) \geq 0$,

1. Check that $V(x)$ is a Lyapunov function,
2. study the stability of the origin of the system for $k = 0$
3. and for $k \neq 0$

Ex. 3: Given the continuous time system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - \alpha x_1 - (x_1 + x_2)^2\ x_2 \end{cases} \tag{3}$$

with $\alpha > 0$ and the function $V(x) = \alpha x_1^2 + x_2^2$

1. Check that $V(x)$ is a Lyapunov function,
2. study the stability of the origin of the system
3. and characterize the trajectories of the state around the origin (linearized system's eigenvalues)

Ex. 4: Given the following discrete time system

$$\begin{cases} x_1(k+1) = x_2(k) \cdot \cos(x_1(k)) \\ x_2(k+1) = x_1(k) \cdot \cos(x_2(k)) \end{cases} \tag{4}$$

1. Study the stability of the origin of the linearized system,
2. Use the following Lyapunov function to study the stability of the system, $V(x) = x_1^2(k)+x_2^2(k)$.

Ex. 5: Given the following differential equation

$$\dot{x} = -x^2 + 3x - 2 \tag{5}$$

and the steady state control input $\bar{u} = 2$

1. find the equilibrium of the system for $\bar{u} = 2$.
2. Study the plane $\dot{x}-x$ to determine the stability of the equilibrium and the region of attraction.
3. Verify the results with the Lyapunov function $V(x - \hat{x}) = \dfrac{1}{2}(x - \hat{x})^2$

0.1 Additional exercises

Ex. 6: Consider the following differential equation

$$\dot{x} = -x^3 \tag{6}$$

1. Study the equilibrium points of the system
2. and analyse its stability with both the linearized system
3. and using the Lyapunov function $V(x) = x^2$.

Ex. 7: Given the following system

$$\begin{cases} \dot{x}_1 = x_1\ x_2^2\ -\ x_1 \\ \dot{x}_2 = -x_1^2\ x_2 \end{cases} \tag{7}$$

Study the stability of the origin using the Lyapunov function $V(x) = \dfrac{1}{2}(x_1^2 + x_2^2)$.

Ex. 8: Given the following system

$$\begin{cases} \dot{x}_1 = -x_1^3\ +\ x_2 \\ \dot{x}_2 = -x_2^2\ +\ u \end{cases} \tag{8}$$

and the Back-stepping formula (given at the exam)

$$u = -\frac{dV_1(x_1)}{dx_1} g(x_1) - k(x_2 - \phi_1(x_1)) + \frac{d\phi_1(x_1)}{dx_1}(f(x_1) + g(x_1)x_2) \tag{9}$$

Determine a control law stabilizing the origin using the Back-stepping method.

Ex. 9: Given the following system

$$\begin{cases} \dot{x}_1 = -x_1\ x_2^2\ -\ x_1\ -\ x_2^3\ +\ x_2 \\ \dot{x}_2 = x_1^4\ -\ x_1^2\ +\ x_1^2\ x_2\ -\ x_2 \end{cases} \tag{10}$$

and the phase plane in Figure 1

1. Study the equilibrium points of the system using the given phase plane,
2. verify the stability using the linearized system

and given the matrix

$$P = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \tag{11}$$

and using Lyapunov theorem for linearized systems, assessing its stability property.

Hint: For continuous time systems use $A^T P + PA = -Q$

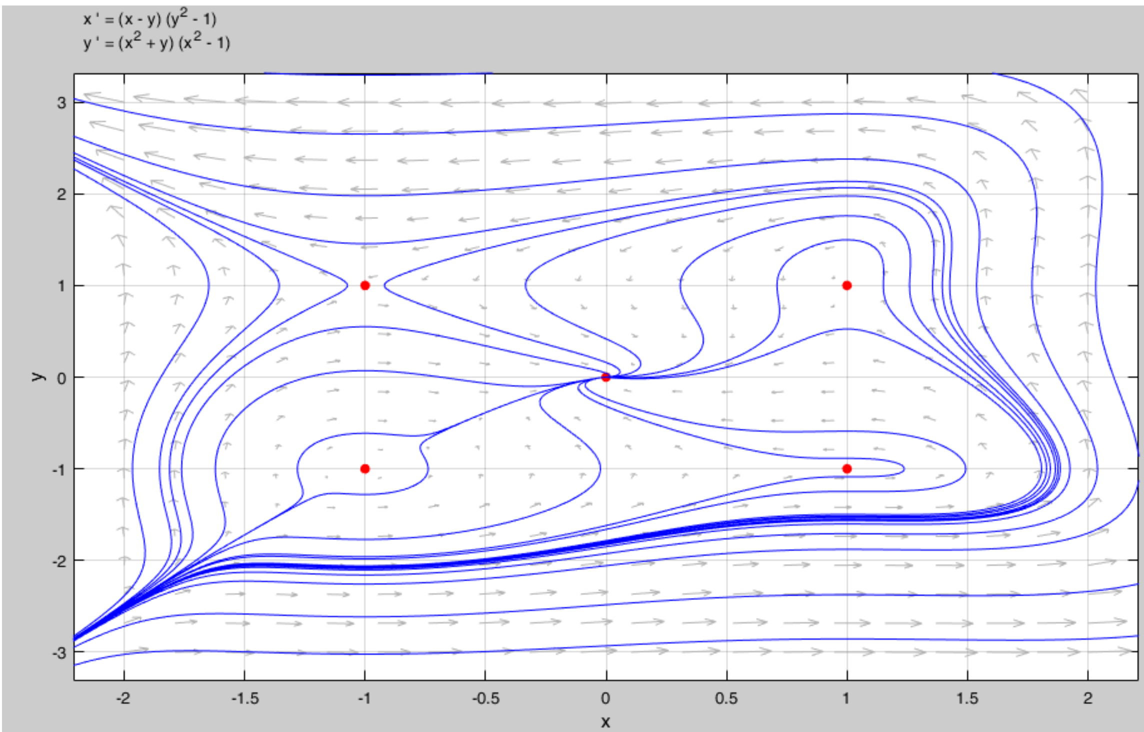

Figure 1: Phase plane of system given in exercise 9.