Advanced and Multivariable Control

Linear Quadratic Gaussian (LQG) control

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System

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + v_x(t) \\ y(t) &= Cx(t) + v_y(t)\end{aligned}$$

$v_x,\ v_y,\ x(0)$ satisfy all the assumption introduced for the Kalman Filter

Goal of *LQG*: in the case of nonmeasurable state, minimize

$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \left(x'(t)Qx(t) + u'(t)Ru(t)\right) dt\right]$$

required to have a finite cost function

required because *x* and *u* are stochastic processes, which do not tend to zero ($\tilde{Q} \geq 0$)

If the resulting closed-lop system is asymptotically stable, *x* and *u* are ***stationary processes*** and the cost function can be written as

$$J = E\left[x'(t)Qx(t) + u'(t)Ru(t)\right]$$

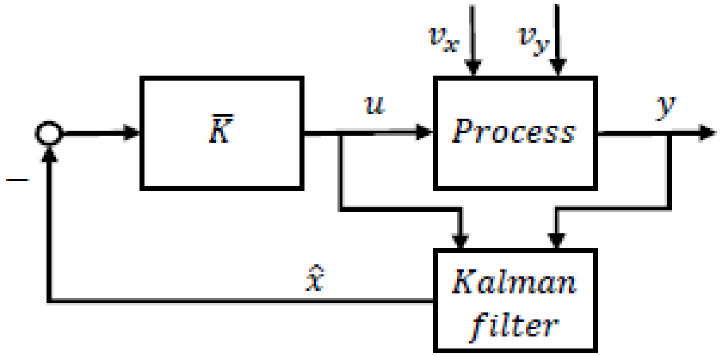
Solution

The optimal control law is given by the combination of the solution to the corresponding deterministic LQ control problem and of the state estimated by the corresponding Kalman filter

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \bar{L}[y(t) - C\hat{x}(t)] \\ u(t) &= -\bar{K}\hat{x}(t)\end{aligned}$$

Solution of *LQ*$_{inf}$ with *Q, R*

Solution of *KF* with $\tilde{Q}, \tilde{R}$



Comments

The structure (state feedback + observer) is exactly equal to the one derived for pole placement control

The ***separation principle*** holds as well, so that the closed-loop system has the eigenvalues of (***A-BK***) and (***A-LC***)

From the equations of the state feedback (*LQ*) and of the observer (*KF*) it is easy to compute the equivalent regulator transfer function (same computations of the pole placement analysis)

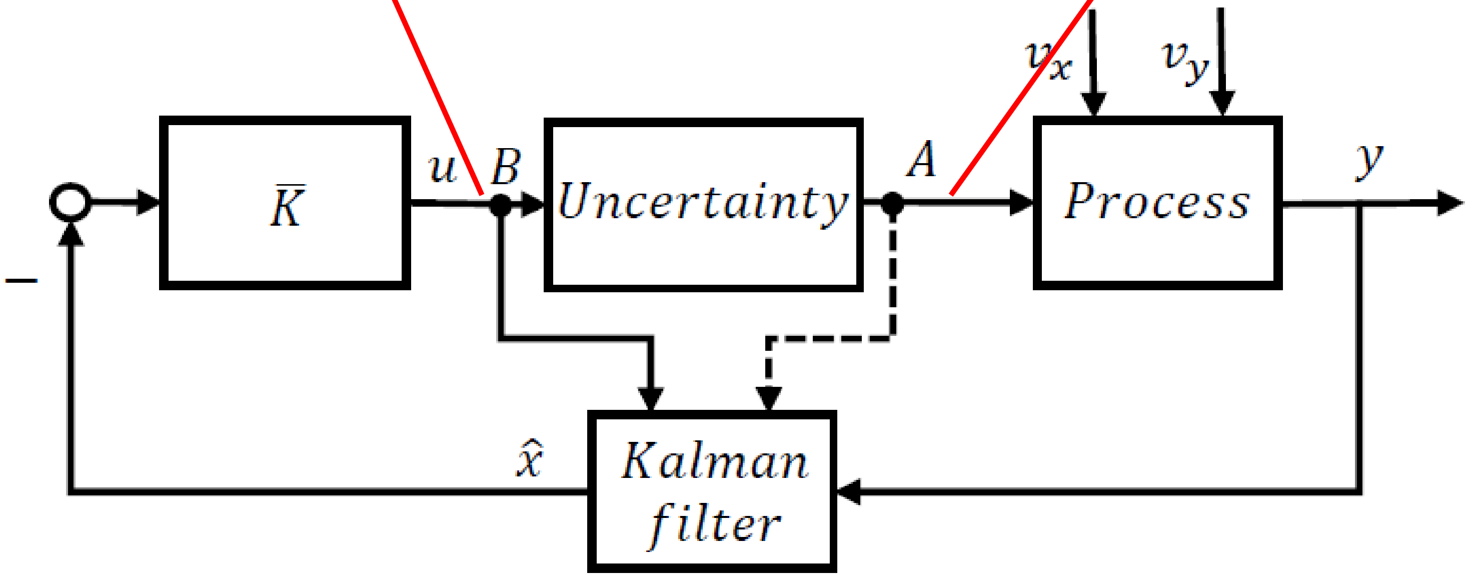
$$U(s) = -\underbrace{\bar{K}\left(sI - \left(A - \bar{L}C - B\bar{K}\right)\right)^{-1}\bar{L}}_{R(s)}Y(s)$$

LQ_{inf} has robustness properties with respect to gain and phase variations at the plant input. Does LQG inherit these properties?

NO

This is the signal available, the KF cannot provide the correct state estimation

If one could use this signal, the same robusness of LQ would be guaranteed



Everything is lost? *NO*

Loop Transfer Recovery procedure (*LTR*) - uncertainty at the plant input

Consider the **SISO** system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad G(s) = C(sI-A)^{-1}B = \frac{B_G(s)}{A_G(s)} = \frac{b_{n-1}s^{n-1} + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0}$$

and the state feedback ***LQ$_{inf}$*** control law $u(t) = -Kx(t) \quad , \quad K = \left[\begin{array}{ccccc} k_0 & k_1 & \dots & k_{n-2} & k_{n-1} \end{array} \right]$

The loop transfer function with robustness properties is

$$L_a^1(s) = K(sI-A)^{-1}B = \frac{\kappa(s)}{A_G(s)} \quad , \quad \kappa(s) = k_{n-1}s^{n-1} + ... + k_0$$

When the regulator $R(s) = K(sI - A + BK + LC)^{-1}L$ is used (with a generic ***L*** observer gain), the loop transfer function becomes

$$L_a^2(s) = R(s)G(s) = K(sI - A + BK + LC)^{-1}LC(sI-A)^{-1}B$$

Now, take the observer gain as

$$\boxed{L = \rho B, \quad \rho > 0}$$

(note that there are **no stability guarantees**). It can be shown (see the textbook) that

$$L_a^2(s) = \frac{\rho\kappa(s)}{(A_G(s) + \kappa(s) + \rho \cancel{B_G(s)})} \frac{\cancel{B_G(s)}}{A_G(s)}$$

and

$$\underset{\rho \to \infty}{L_a^2(s) \to L_a^1(s)} = \frac{\kappa(s)}{A_G(s)}$$

The properties of the state feedback control law are recovered (LTR procedure)

There is a cancellation of the open-loop zeros, that must be asymptotically stable

MIMO systems

The LTR procedure can be applied with $L = \rho B, \quad \rho > 0$ provided that:

1. it is square (same number of inputs and outputs),
2. its invariant zeros have negative real part.

So, one increases the value of ρ until the recovery of the loop transfer function with LQ_{inf} are obtained

But what about the stability of the observer?

MIMO systems

It can be proven that, under the same conditions previously introduced, *the LTR* is guaranteed also using a *KF* with

$$\tilde{Q} = \alpha BB', \, \tilde{R} = I \qquad , \qquad \alpha \longrightarrow \infty$$

and under suitable conditions, the ***KF is asymptotically stable***

Exercise: given

$$\begin{cases} \dot{x} = -x + 2u + v_x & , v_x \sim WGN(0, \tilde{q}) \\ y = x + v_y & , v_y \sim (0, 1) \end{cases}$$

A) Assuming v_x, v_y null, compute the LQ_∞ control law with $Q = 15/4$, $R = 1$

B) Compute the gain margin of the system and compare it with the theoretical one and also the phase margin

C) Compute the KF with the specified characteristics of the noises

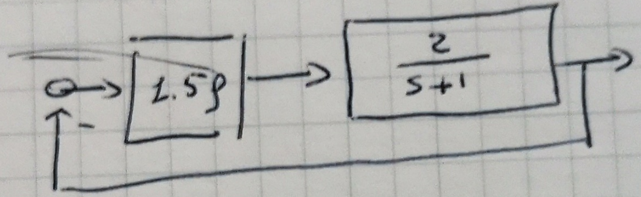
D) Compute the regulator transfer function $R(s)$ and use it to verify the LTR procedure

A) $A=-1,\ B=2,\ Q=15/4,\ R=1$

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P \rightarrow 0 = -2\bar{P} + 15/4 - 4\bar{P}^2$$

$$\bar{P} = 0.75, \quad K = R^{-1}B'\bar{P} = 1.5$$

B) ρ used to compute the gain margin



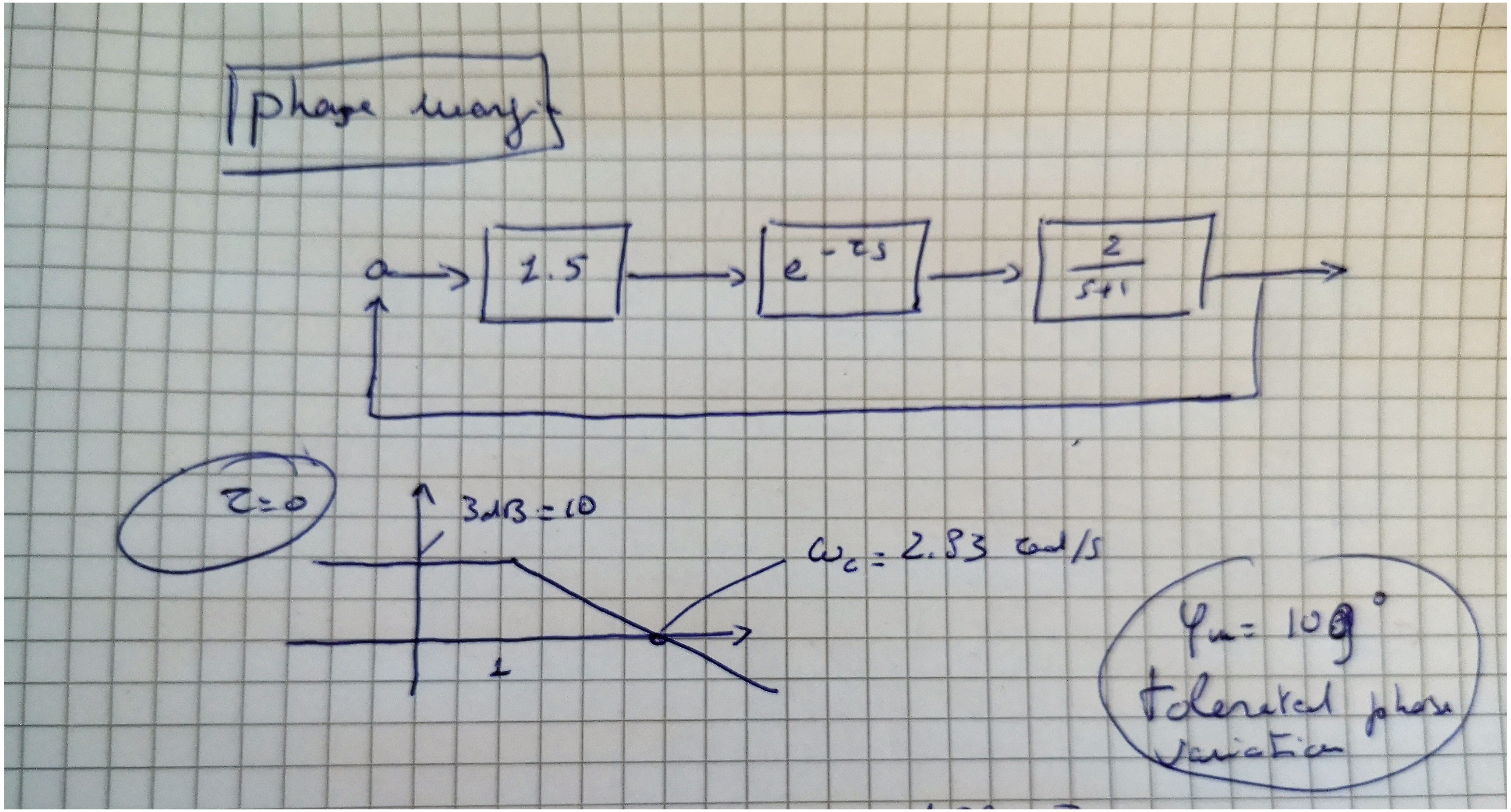
Characteristic equation $s+1+3\rho=0 \rightarrow s=-(1+3\rho)$

Stability for $\rho > -\frac{1}{3}$

Gain margin $(-1/3, \infty)$ larger than the theoretical one

C) $C=1,\ \tilde{R}=1 \rightarrow \tilde{P} = L = -1+\sqrt{1+\tilde{Q}}$

$$\dot{\hat{x}} = -\sqrt{1+\tilde{Q}}\,\hat{x} + 2u + \left(-1+\sqrt{1+\tilde{Q}}\right) y$$



c) $C=1,\ \tilde{R}=1 \longrightarrow \tilde{P} = L = -1+\sqrt{1+\tilde{\varphi}}$

$$\dot{\hat{x}} = -\sqrt{1+\tilde{\varphi}}\,\hat{x} + 2u + \left(-1+\sqrt{1+\tilde{\varphi}}\right)y$$

D) $R(s) = K\left(sI - A + BK + LC\right)^{-1} L$

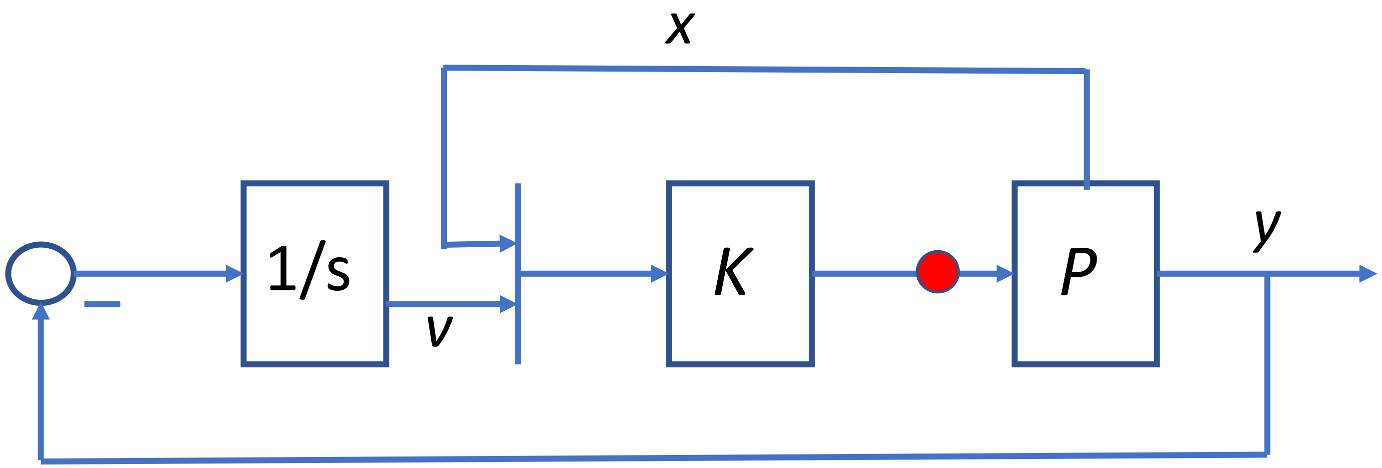
$$= 1.5\,\frac{-1+\sqrt{1+\tilde{Q}}}{s+3+\sqrt{1+\tilde{Q}}} = \frac{1.5}{\underbrace{\frac{1}{-1+\sqrt{1+\tilde{Q}}}}_{T}\,s + \underbrace{\frac{3+\sqrt{1+\tilde{Q}}}{-1+\sqrt{1+\tilde{Q}}}}_{\alpha}}$$

$\tilde{Q} \to \infty$, $T \to 0$, $\alpha \to 1$ $\to R(s) \to 1.5$

and the loop transfer function $L(s) \to \dfrac{1.5 \cdot 2}{s+1}$

The one obtained with LQ

LTR **- scheme with integrators – measurable state**

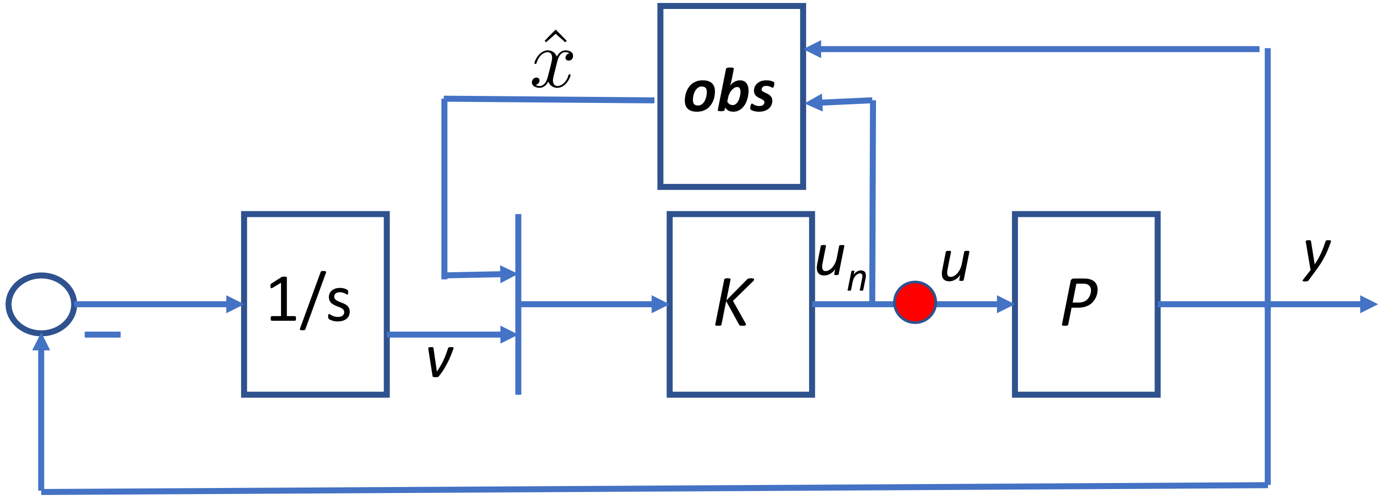


$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u(t)$$

$$u(t) = -K \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = -\begin{bmatrix} K_x & K_v \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Loop transfer function at the plant input $L_a^1(s) = K(sI - \bar{A})^{-1}\bar{B}$

LTR - scheme with integrators – unmeasurable state



$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_n(t) + L[y(t) - C\hat{x}(t)]$$

$$u_n(t) = -\underbrace{\begin{bmatrix} 0 & K_v & K_x \end{bmatrix}}_{\tilde{K}} \begin{bmatrix} x(t) \\ v(t) \\ \hat{x}(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ -C & 0 & 0 \\ LC & -BK_v & A - BK_x - LC \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x(t) \\ v(t) \\ \hat{x}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{\tilde{B}} u(t)$$

Loop transfer function at the plant input $L_a^2(s) = \tilde{K}(sI - \tilde{A})^{-1}\tilde{B}$

Example: linearized model of an aircraft

Maciejowski, Jan Marian. "Multivariable feedback design." *Electronic Systems Engineering Series, Wokingham, England: Addison-Wesley,| c1989* (1989).

$x1$: altitude relative to some datum (m)

$x2$: forward speed ($\mathrm{m\,s^{-1}}$)

$x3$: pitch angle (degrees)

$x4$: pitch rate ($\mathrm{deg\,s^{-1}}$)

$x5$: vertical speed ($\mathrm{m\,s^{-1}}$)

$$A = \begin{bmatrix} 0 & 0 & 1.1320 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$u1$: spoiler angle (measured in tenths of a degree)

$u2$: forward acceleration ($\mathrm{m\,s^{-2}}$)

$u3$: elevator angle (degrees)

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1.0000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.665 \\ 1.5750 & 0 & -0.0732 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

D=0

eigenvalues

s = 0

s = -0.7801 + 1.0296i

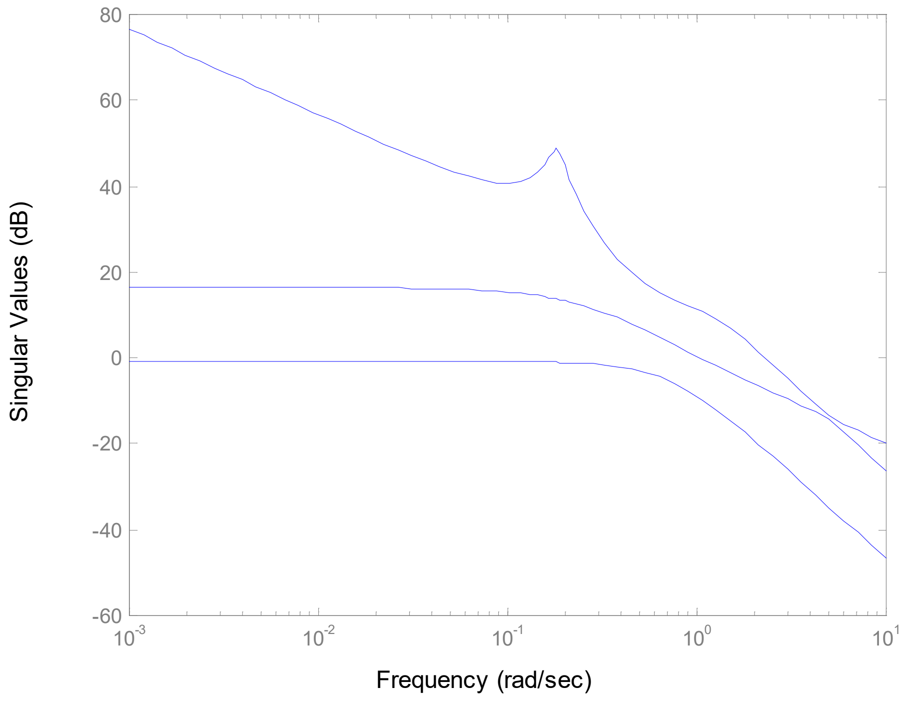
s = -0.7801 - 1.0296i

s = -0.0176 + 0.1826i

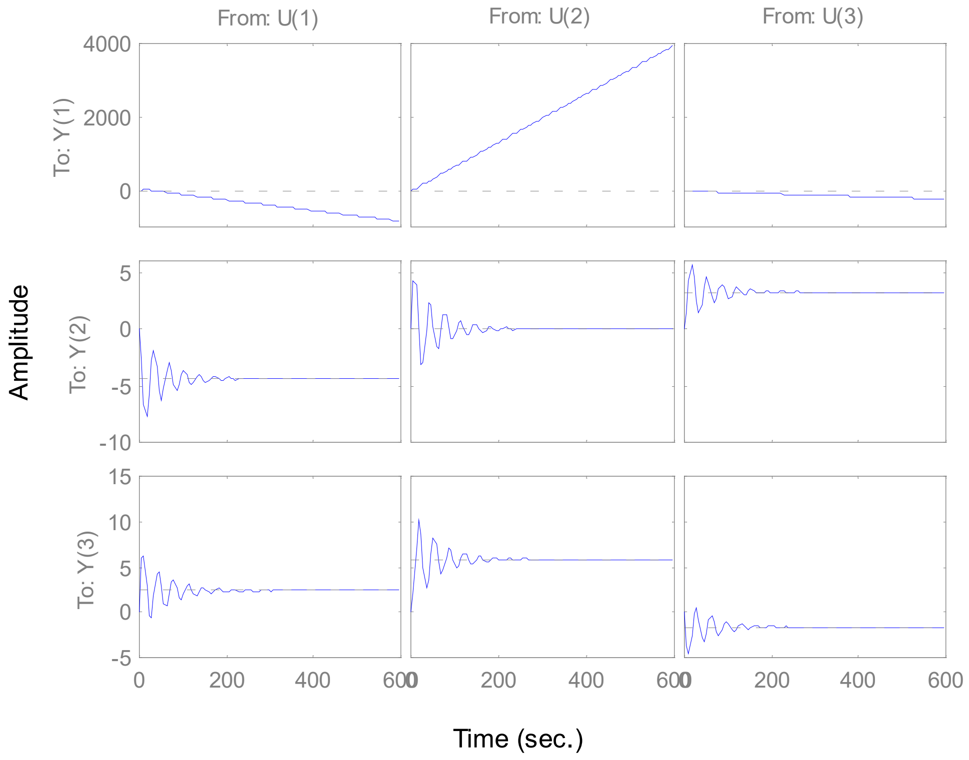
s = -0.0176 - 0.1826i

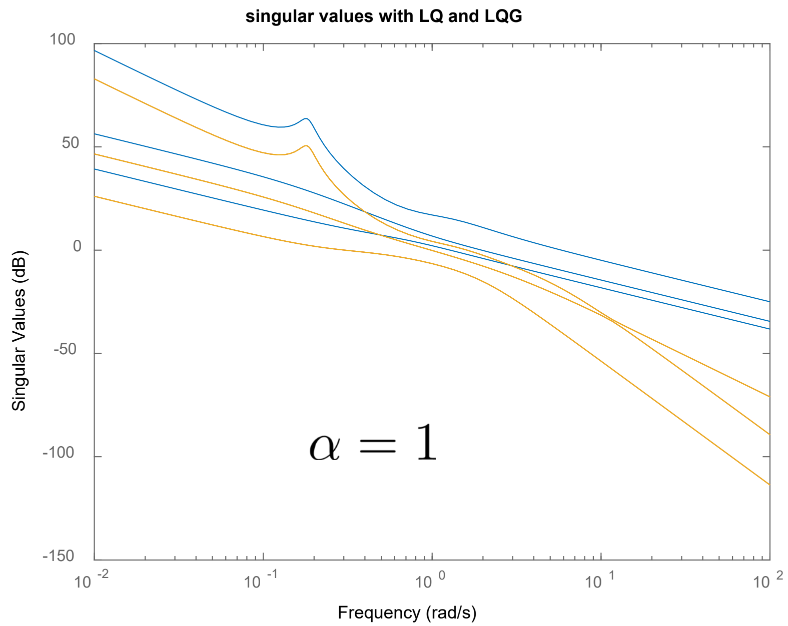
no zeros

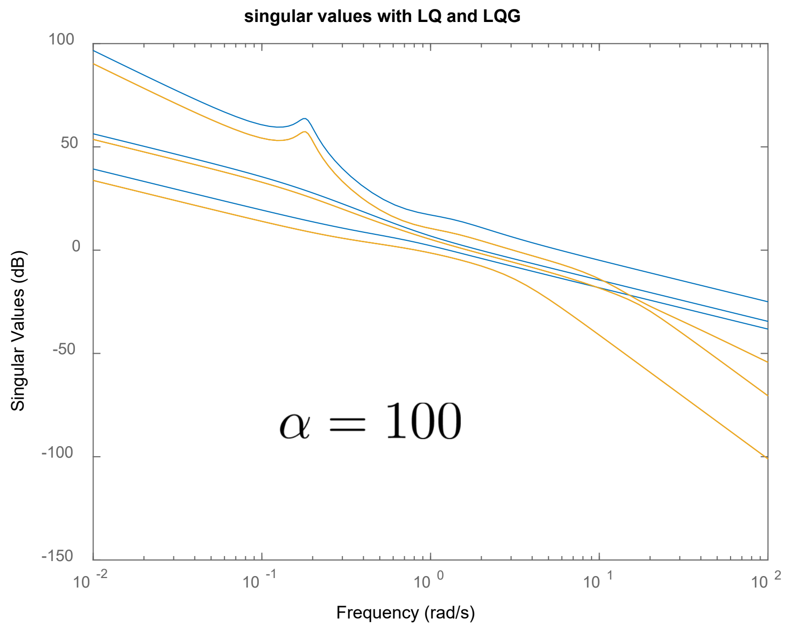
singular values



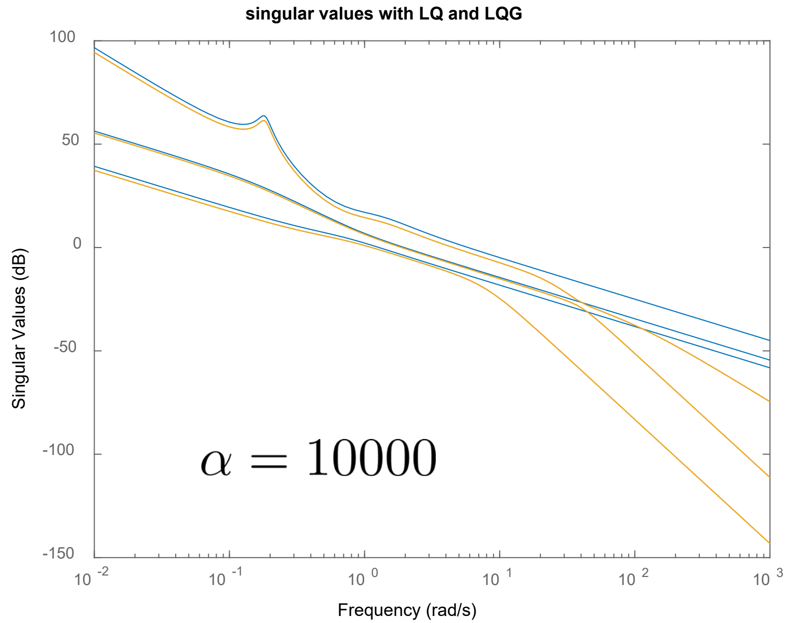
Open loop step responses

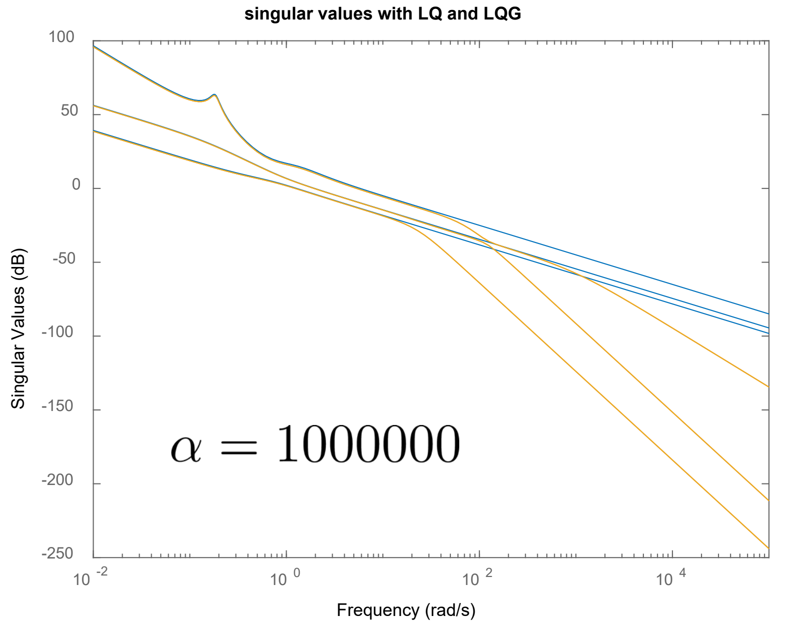






$$Q = I,\ R = I,$$
$$\tilde{Q} = \alpha BB',\ \tilde{R} = I$$

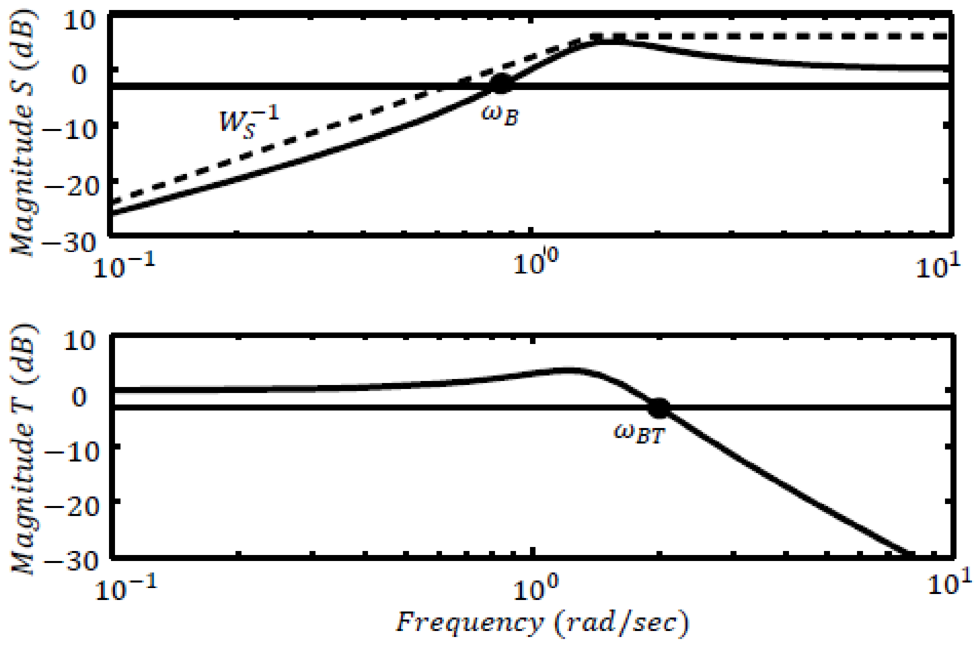




Now, let's go back to Chapter 3

(regulator design with H2 – Hinf with shaping functions for SISO systems)

Sensitivity functions and crossover frequency



Define by ω_B the frequency where $|S(j\omega)|$ crosses $1/\sqrt{2}$ (-3dB) from below and by ω_{BT} the frequency where $|T(j\omega)|$ crosses $1/\sqrt{2}$ (-3dB) from above

Then, if $\varphi_m < 90^o$, one has

$$\omega_B < \omega_c < \omega_{BT}$$

Also in this case, specifications can be given in terms of *S(s), T(s)*

Design specifications in terms of sensitivity functions

We could specify:

- shape of $S(s)$;
- minimum frequency ω_B;
- small or null asymptotic error for constant reference signals ($|S(j\omega)|$ small or Bode diagram of $|S(j\omega)|$ with shape $+1$ at low frequency);
- $M_S \leq \bar{M}_S$.

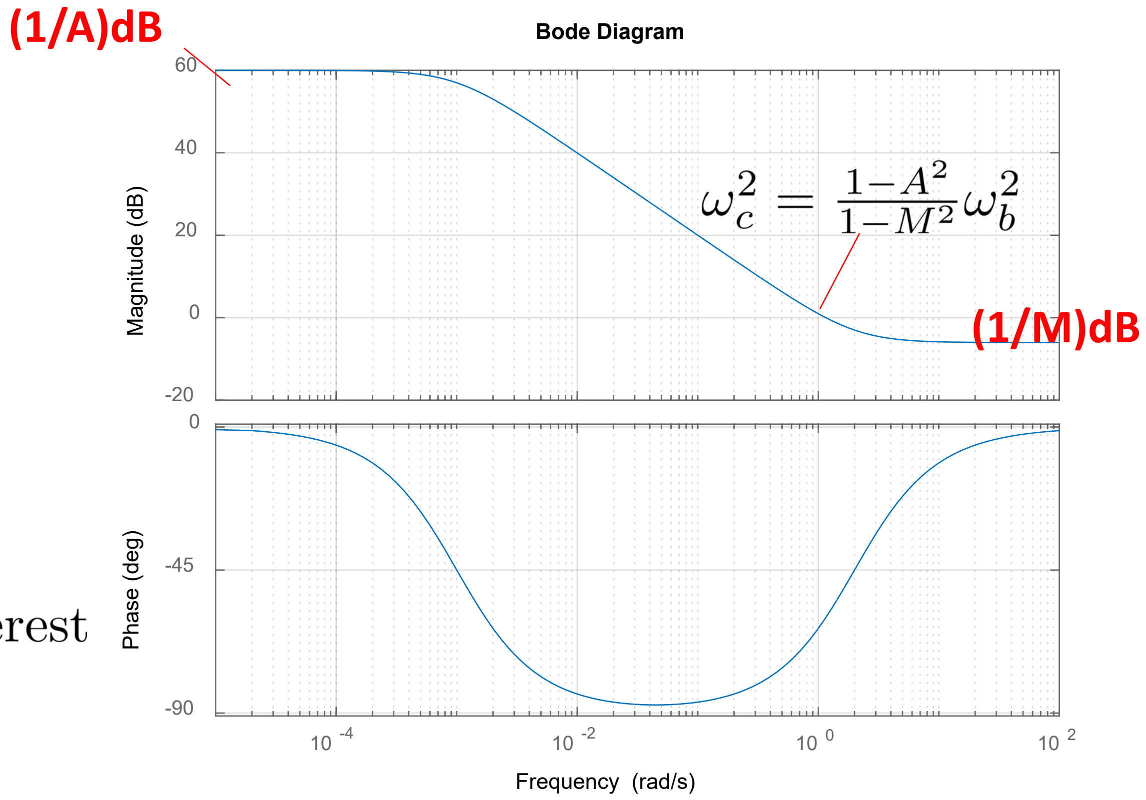
this defines a function $S_{desired}(s)$, and the function $W_S(s) = S^{-1}_{desired}(s)$ also named *(sensitivity) shaping function*

then, the regulator must be designed such that

$$|S(j\omega)| < \frac{1}{|W_S(j\omega)|}, \quad \forall \omega \qquad \longleftrightarrow \qquad \|W_S S\|_\infty < 1$$

Possible choice of $W_S(s)$

$$W_S(s) = \frac{s/M + \omega_B}{s + A\omega_B}$$



$A \ll 1$: desired attenuation of $S(s)$ in the band of interest

M required bound on the H_∞ norm of $S(s)$

How to synthetise the regulator? We'll see later in the course

Design specifications in terms of complementary sensitivity function

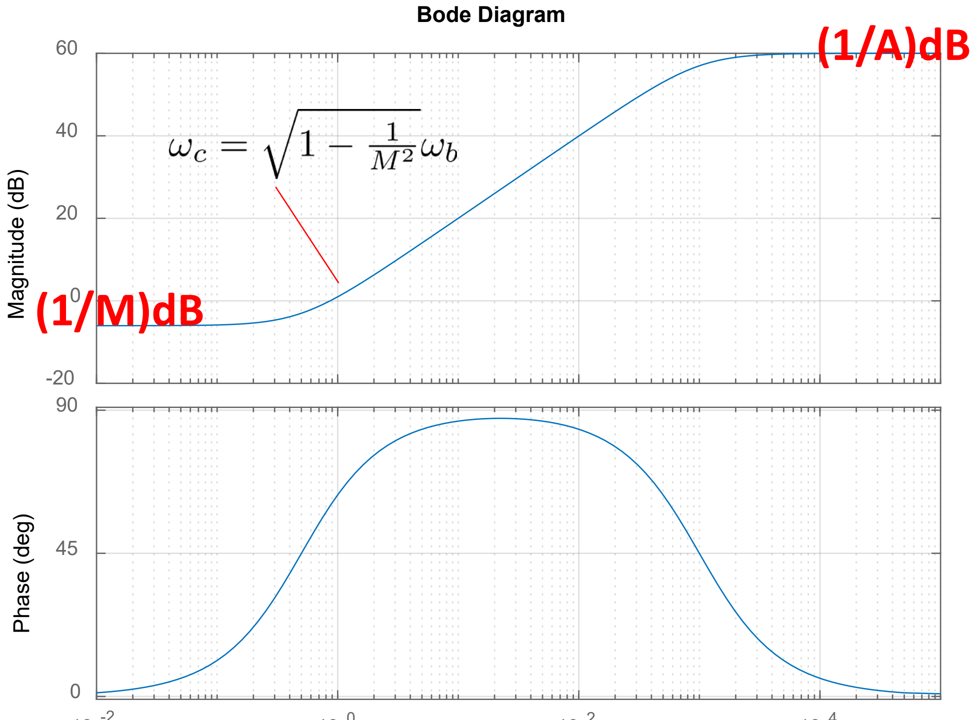
Also in this case, define a $T_{desired}(s)$, and its inverse $W_T(s) = T^{-1}_{desired}(s)$ also named *shaping function* $W_T(s)$

then, the regulator must be designed such that

$$|T(j\omega)| < \frac{1}{|W_T(j\omega)|}, \quad \forall\omega \quad \longleftrightarrow \quad \|W_T T\|_\infty < 1$$

Example

$$W_T(s) = \frac{s + \omega_{BT}/M}{As + \omega_{BT}},$$



Control sensitivity function

Same approach, define the control sensitivity function $W_K(s)$ and choose a regulator $R(s)$ such that

$$|K(j\omega)| < \frac{1}{|W_K(j\omega)|}, \quad \forall\omega \Leftrightarrow \|W_K K\|_\infty < 1$$

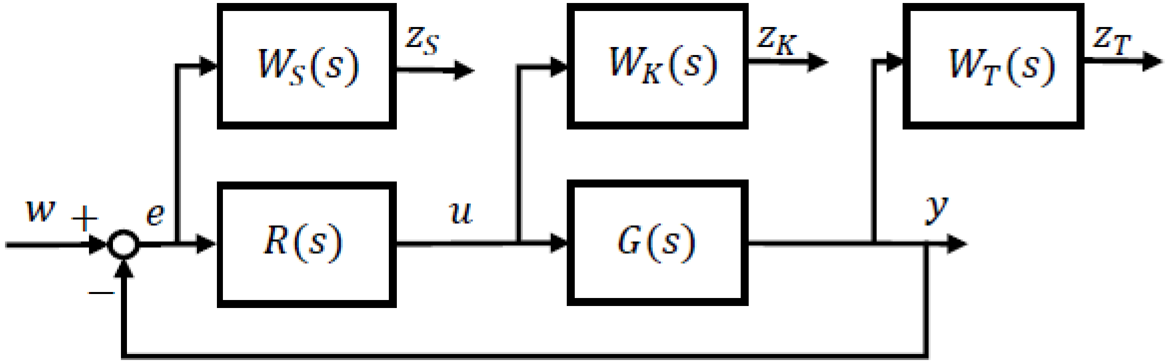
The shaping function $W_S(s)$, $W_T(s)$, $W_K(s)$ must be selected as asymptotically stable systems

In summary, one must find a regulator such that

$$\|W_S S\|_\infty < 1, \quad \|W_T T\|_\infty < 1, \quad \|W_K K\|_\infty < 1$$

Control problem formulation (Chapter 3 SISO case - shaping functions at the output)

Consider the enlarged system

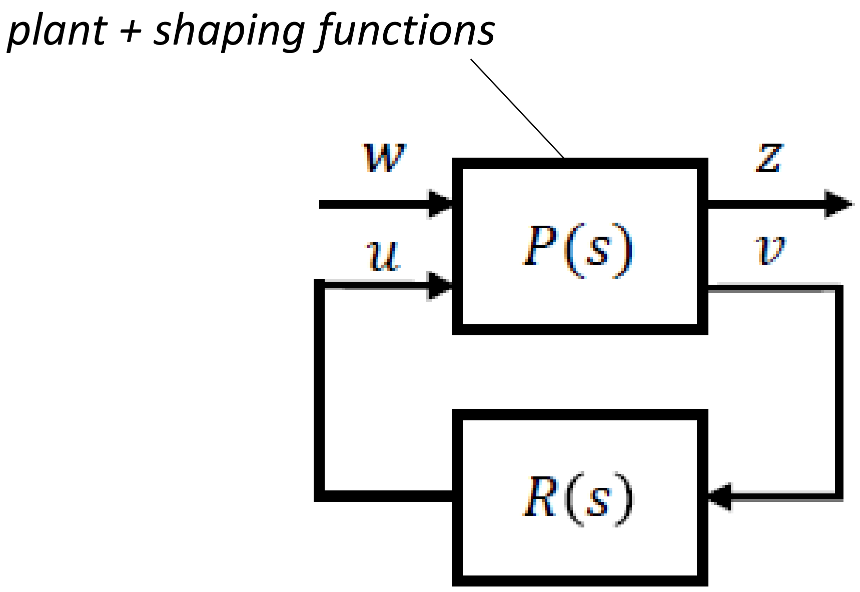


Define $z = \begin{bmatrix} z_S \\ z_K \\ z_T \end{bmatrix}$, $w = y^o$ and note that $z = G_{zw}w$, $G_{zw}(s) = \begin{bmatrix} W_S(s)S(s) \\ W_T(s)T(s) \\ W_K(s)K(s) \end{bmatrix}$

shaping functions

The control problem consists in $\min_R \quad \|G_{zw}\|_2$ or $\|G_{zw}\|_\infty$

Why do we need to use the shaping functions? Because otherwise the regulator would be null and because we want to specify the shape of the sensitivity functions



Formal statement of *H2 – Hinf* control

Minimize the *2-norm* or *inf-norm* of $\boldsymbol{G_{zw}(s)}$ with respect to ***R(s)***

where

z are named *performance variables*

v are named *measured variables*

$$w = \begin{bmatrix} d \\ y^o \\ n \end{bmatrix}$$ are the *exogenous variables*

back to LQG

LQG and H2 control

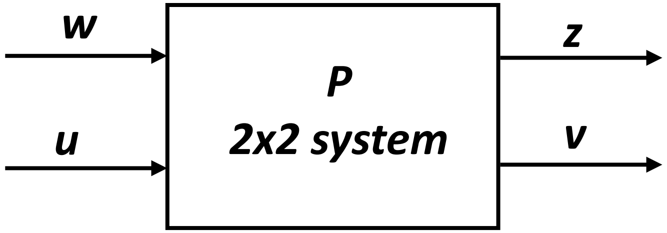
The cost function $J = E\left[x'(t)Qx(t) + u'(t)Ru(t)\right]$ can be written as

$$J = E\left[x'(t)Qx(t) + u'(t)Ru(t)\right] = E\left[\underbrace{\begin{bmatrix} x' & u' \end{bmatrix}\begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}}_{z'}\underbrace{\begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}}_{z = \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix}}\right]$$

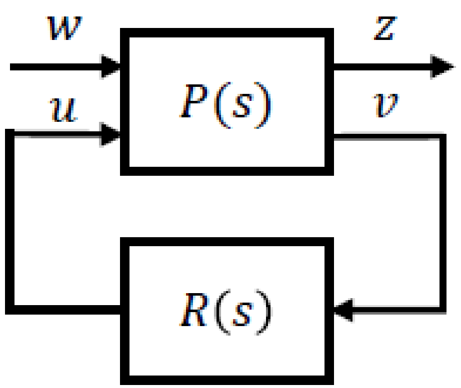
Let $\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \tilde{Q}^{1/2} & 0 \\ 0 & \tilde{R}^{1/2} \end{bmatrix} w$ where w is a white noise with zero mean and unit variance

and define $v = y$. Then

$$\begin{array}{rrcl} \textit{state equation} & \dot{x} & = & Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\ \textit{performance variable} & z & = & \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\ \textit{measured variable} & v & = & Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w \end{array}$$



LQG and H2 control



The ***LQG*** problem consists in computing a regulator such that

$$min_R \quad J = E[z'z] \quad \longleftrightarrow \quad min_R \quad \|G_{zw}\|_2$$

with

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\ z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\ v &= Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w \end{aligned}$$

This is a particular H2 control problem (go back to Chapters 2 and 3)

More general formulation of H_2, H_{inf} control problems

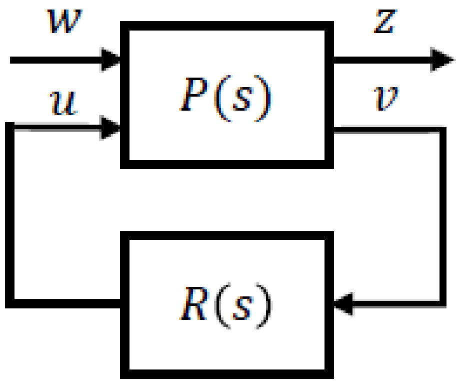
Enlarged (2x2) plant

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{aligned}
\quad \longleftrightarrow \quad
P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}
$$

B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22} are matrices to be suitably selected to meet specific design requirements

Goal

$$\min_R \quad \|G_{zw}\|_2 \text{ or } \|G_{zw}\|_\infty$$



LQG as *H_2* control problem

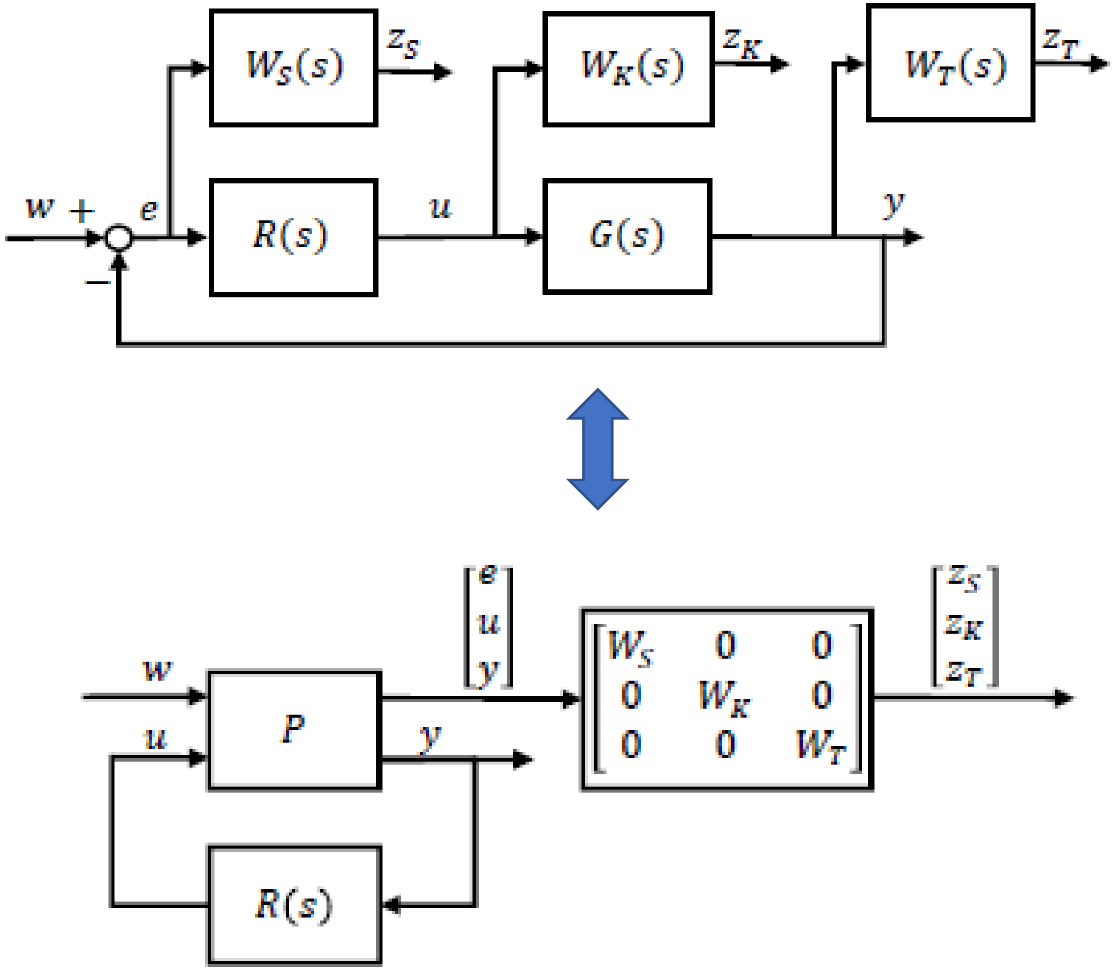
$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{aligned}
\quad \longleftrightarrow \quad
\begin{aligned}
\dot{x} &= Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\
z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
v &= Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w
\end{aligned}
$$

$$
B_1 = \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} \quad , \quad B_2 = B \quad , \quad C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}
$$

$$
D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \quad , \quad C_2 = C \qquad D_{21} = \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} \quad , \quad D_{22} = 0
$$

How can we solve the problem with shaping functions?

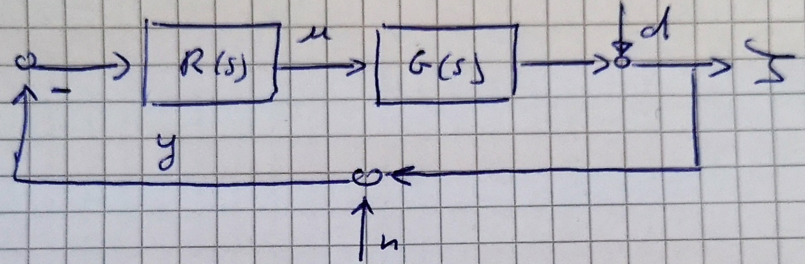
We enlarge the system with the shaping functions at the process output (the same could be done at the process input, see the next slides)



How to do that?
See the textbook pag. 160-162 and the Matlab example discussed in the following

Example H_2 and "standard" synthesis problem

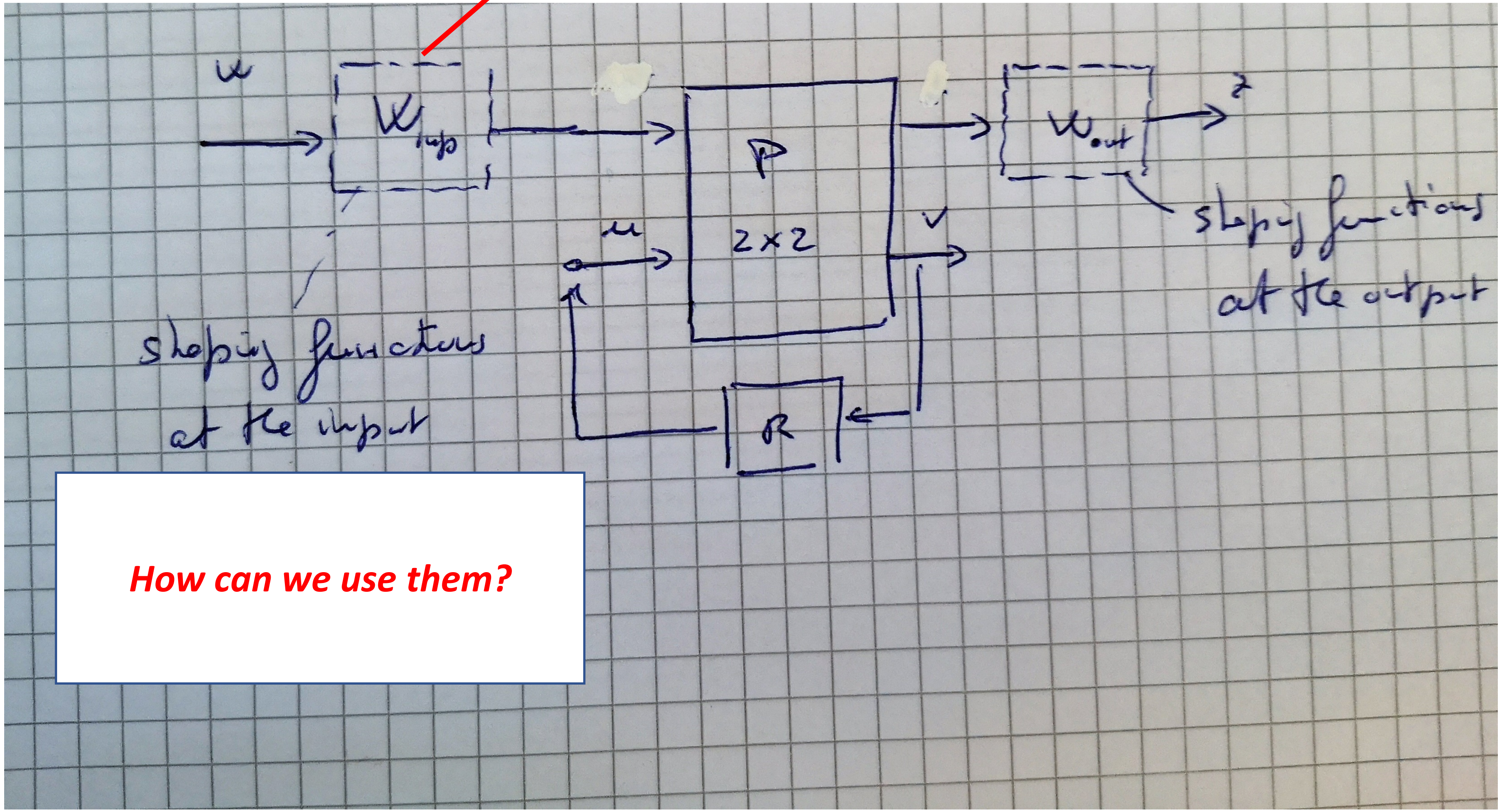
Consider the feedback system



$$\begin{cases} \dot{x} = Ax + Bu \\ \zeta = Cx + d \\ y = Cx + n + d \end{cases} \quad \longrightarrow \quad z = \begin{vmatrix} \zeta \\ u \end{vmatrix}, \quad w = \begin{vmatrix} d \\ n \end{vmatrix}, \quad v = y$$

$$\begin{bmatrix} \dot{x} \\ z = \begin{vmatrix} \zeta \\ u \end{vmatrix} \\ v = y \end{bmatrix} = \begin{bmatrix} A & B_1 = |0 \quad 0| & B_2 = B \\ C_1 = \begin{vmatrix} C \\ 0 \end{vmatrix} & D_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & D_{12} = \begin{vmatrix} 0 \\ I \end{vmatrix} \\ C_2 = C & D_{21} = |I \quad I| & D_{22} = 0 \end{bmatrix} \begin{vmatrix} x \\ w = \begin{vmatrix} d \\ n \end{vmatrix} \\ u \end{vmatrix}$$

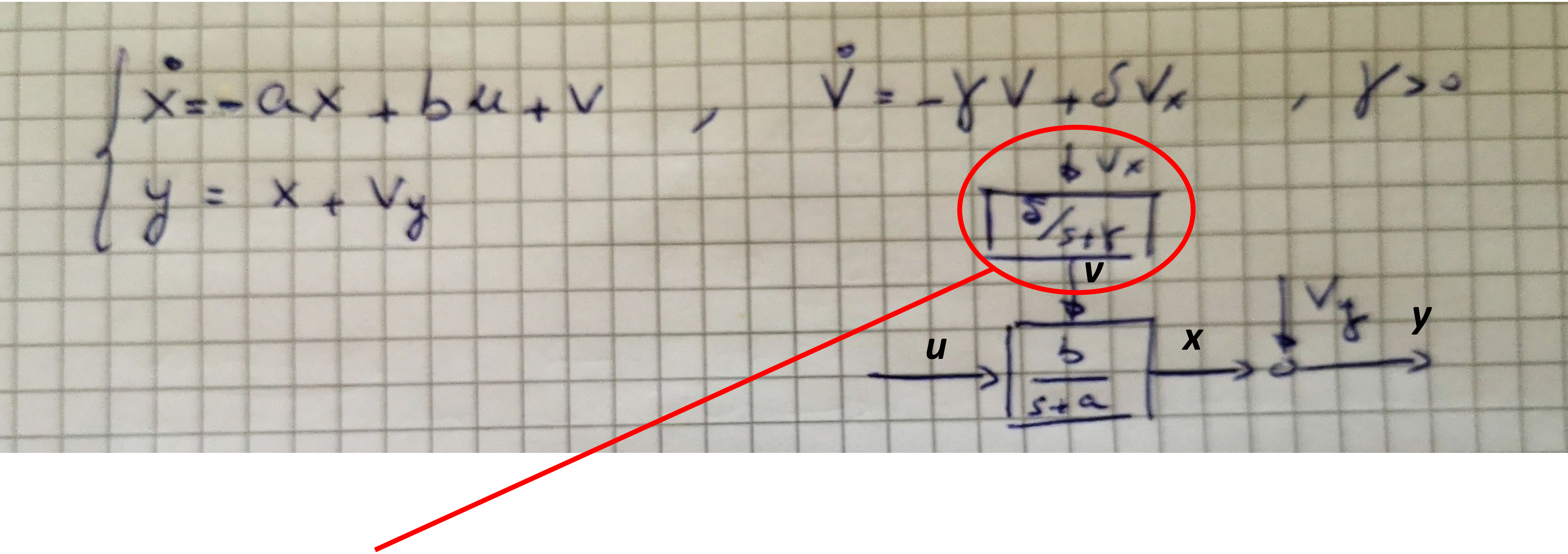
Alternatively, we could consider shaping functions at the input



How can we use them?

Shaping functions at the input

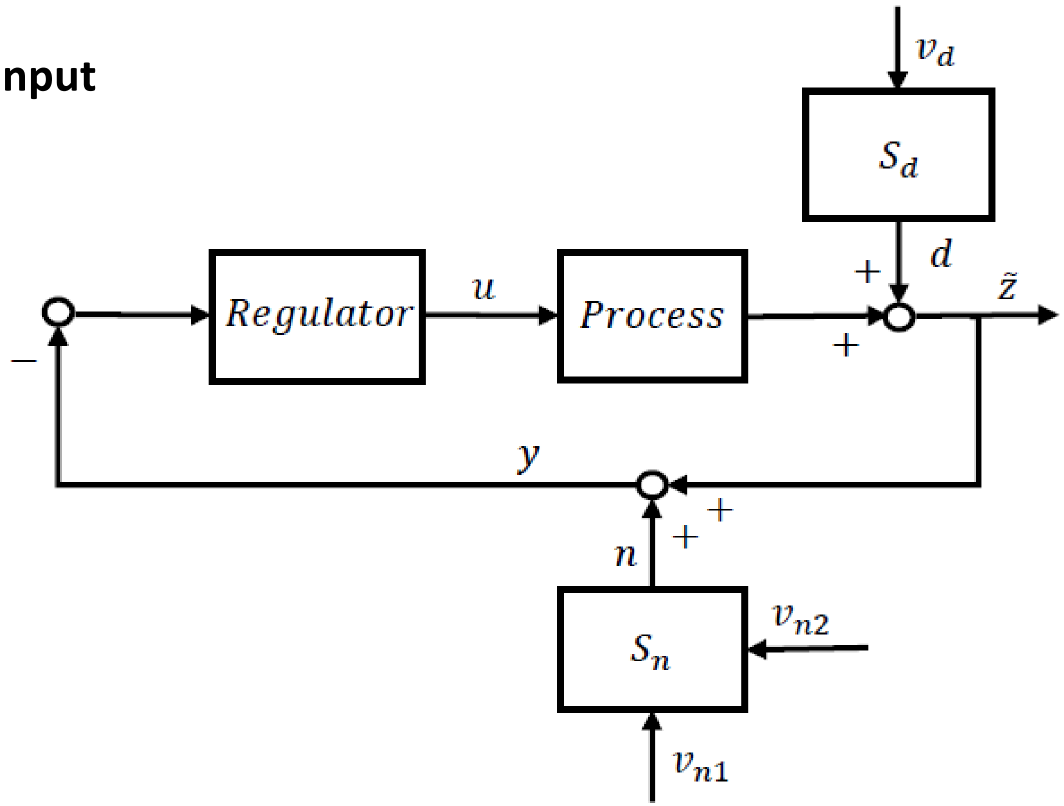
Remember the exercise on the KF



This could be interpreted as a shaping function at the input, which modifies the characteristics of the KF

Basic ideas behind the use of shaping functions at the process input

Consider the system



v_d, v_{n1}, and v_{n2} are uncorrelated white noises

S_d, S_n are asymptotically stable dynamic systems

d and n are stationary noises

the feedback system is asymptotically stable

$\downarrow$

x, u, y, and $\tilde{z}$ are stationary stochastic processes

$$\begin{aligned} \tilde{Z}(s) &= -T(s)N(s) + S(s)D(s) \\ U(s) &= -R(s)S(s)\left[N(s) + D(s)\right] \end{aligned}$$

Recap of results on stationary stochastic signals

Consider a stationary stochastic signal *x(t)*

The autocorrelation function is $R_{xx}(t+\tau) = E[x(t)x(t+\tau)] = R_{xx}(\tau)$

The power spectral density is $\Phi_{xx}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau$

$$\begin{aligned}\tilde{Z}(s) &= -T(s)N(s) + S(s)D(s) \\ U(s) &= -R(s)S(s)\left[N(s)+D(s)\right]\end{aligned}$$

Recall the cost function to be minimized $J = E\left[\tilde{z}'(t)\tilde{z}(t) + u'(t)u(t)\right]$

$$E\left[\tilde{z}'(t)\tilde{z}(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Phi_{\tilde{z}\tilde{z}}(\omega)\right\}d\omega$$

$$\begin{aligned}tr\left\{\Phi_{\tilde{z}\tilde{z}}(\omega)\right\} &= tr\left\{T(j\omega)\Phi_{nn}(\omega)\,T'(-j\omega)\right\} + \\ &\quad + tr\left\{S(j\omega)\Phi_{dd}(\omega)\,S'(-j\omega)\right\}\end{aligned}$$

$$E\left[u'(t)u(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Phi_{uu}(\omega)\right\}d\omega$$

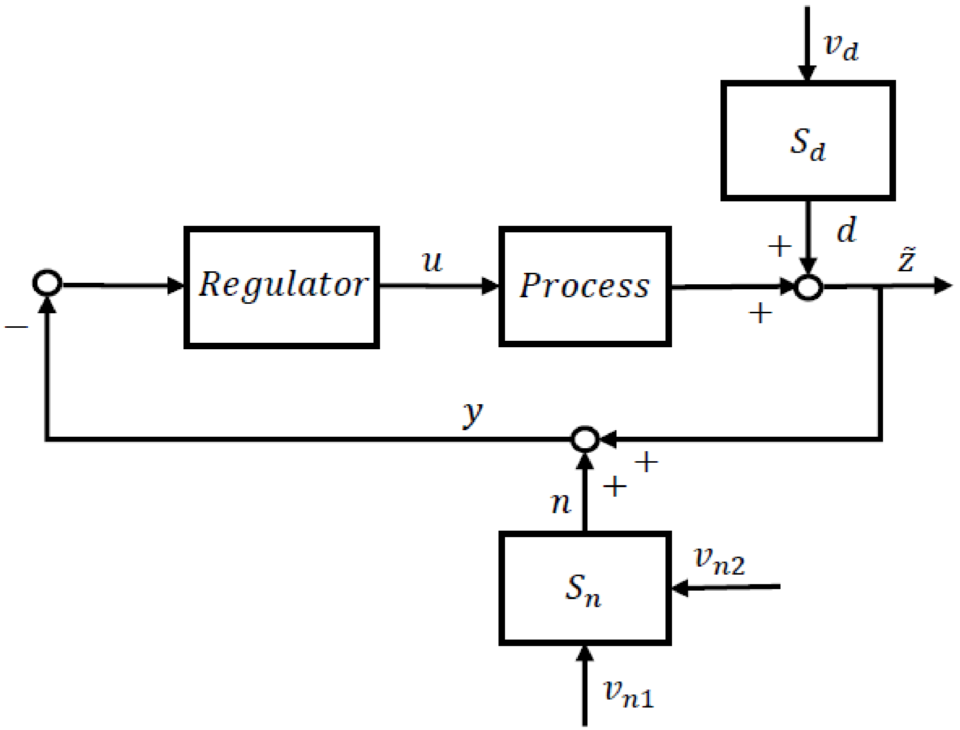
$$\begin{aligned}tr\left\{\Phi_{uu}(\omega)\right\} &= tr\{R(j\omega)S(j\omega)* \\ &\quad *\left(\Phi_{nn}(\omega)+\Phi_{dd}(\omega)\right)S'(-j\omega)R'(-j\omega)\}\end{aligned}$$

$$tr\left\{\Gamma(j\omega)\Gamma'(-j\omega)\right\} \qquad \Gamma(j\omega) = \begin{bmatrix} T(j\omega)\Phi_{nn}^{1/2}(\omega) \\ S(j\omega)\Phi_{dd}^{1/2}(\omega) \\ R(j\omega)S(j\omega)\left(\Phi_{nn}(\omega)+\Phi_{dd}(\omega)\right)^{1/2}\end{bmatrix}$$

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Gamma(j\omega)\Gamma'(-j\omega)\right\}d\omega = \left\|\Gamma(j\omega)\right\|_2^2$$

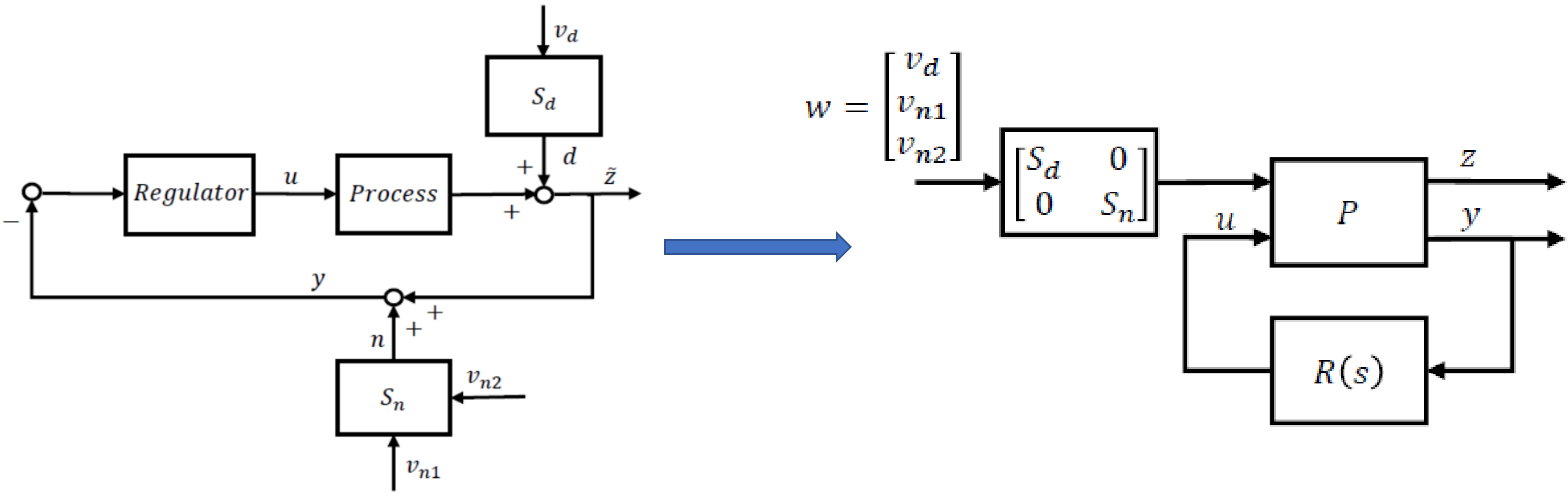
$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Gamma(j\omega)\Gamma'(-j\omega)\right\} d\omega = \|\Gamma(j\omega)\|_2^2 \qquad \Gamma(j\omega) = \begin{bmatrix} T(j\omega)\Phi_{nn}^{1/2}(\omega) \\ S(j\omega)\Phi_{dd}^{1/2}(\omega) \\ R(j\omega)S(j\omega)\left(\Phi_{nn}(\omega)+\Phi_{dd}(\omega)\right)^{1/2} \end{bmatrix}$$

shaping functions

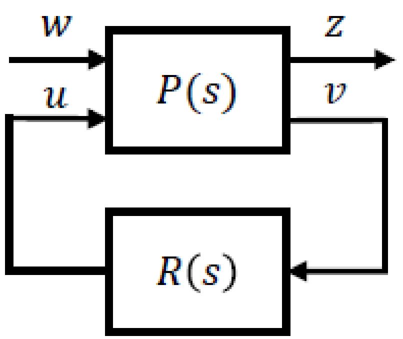


How to define S_d, S_n in order to obtain the required spectral densities Φ_{nn}, Φ_{dd} and the corresponding shaping functions is not trivial (see the notes)

Interpretation as shaping functions at the process input



Partial summary of *H2 – Hinf* discussion - I



1. LQG is a particular case of H2 control

$$
\begin{aligned}
\dot{x} &= Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\
z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
v &= Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w
\end{aligned}
\quad \longleftrightarrow \quad
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{aligned}
$$

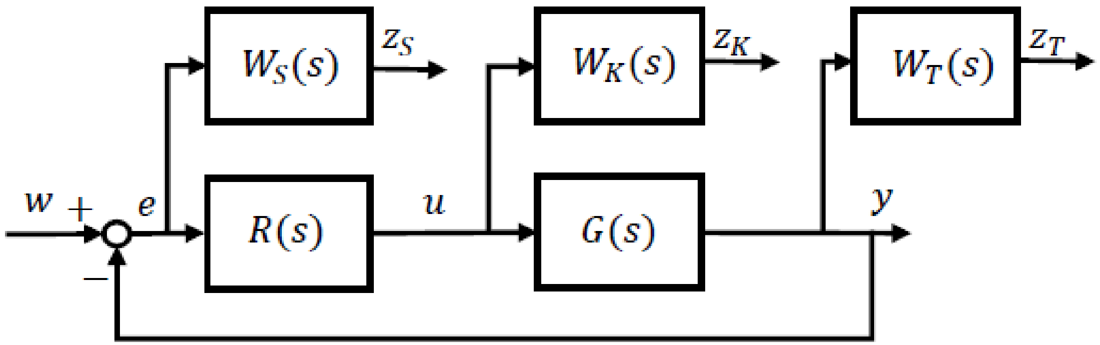
$$\min_R \quad \|G_{zw}\|_2$$

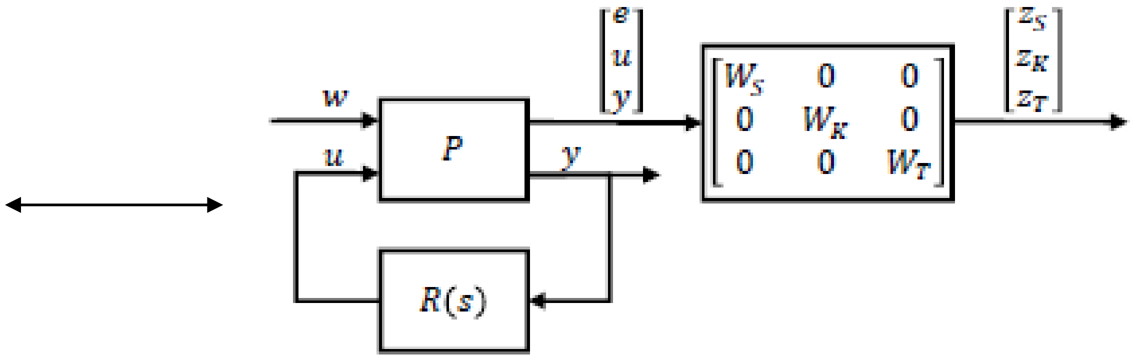
But sometimes it is difficult to tune LQG with a proper choice of the tuning parameters $Q, R, \tilde{Q}, \tilde{R}$

to obtain the desired characteristics (in frequency) of the closed-loop system

Partial summary of *H2 – Hinf* discussion - II

2. We can use shaping functions at the process output





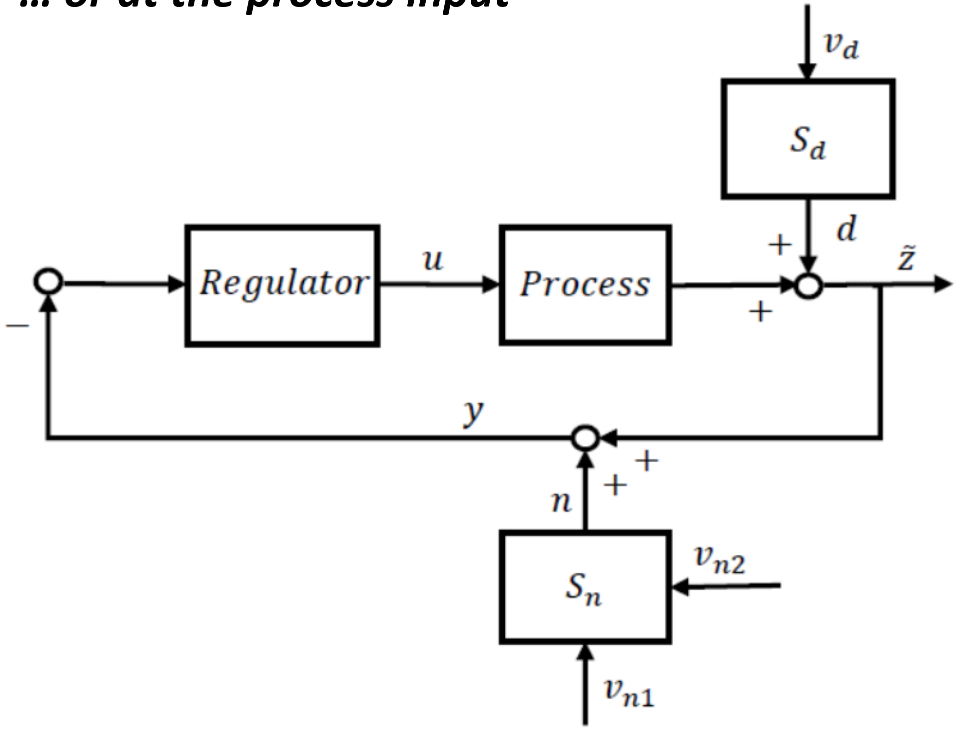
,

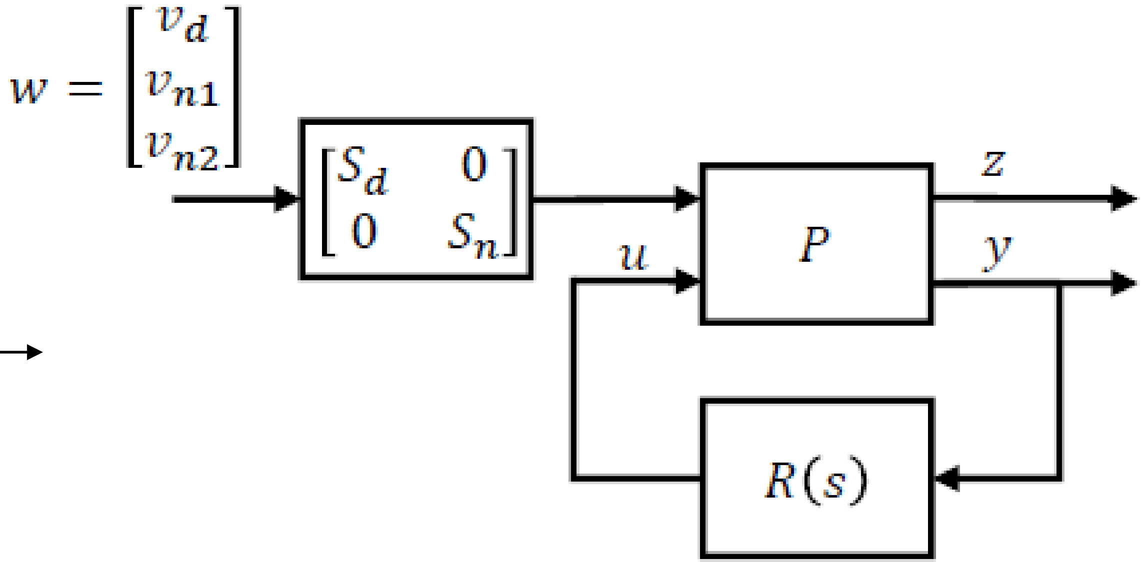
$$G_{zw}(s) = \begin{bmatrix} W_S(s)S(s) \\ W_T(s)T(s) \\ W_K(s)K(s) \end{bmatrix}$$

shaping functions

Partial summary of *H2 – Hinf* discussion - II

… or at the process input





$$G_{zw}(j\omega) = \begin{bmatrix} T(j\omega)\Phi_{nn}^{1/2}(\omega) \\ S(j\omega)\Phi_{dd}^{1/2}(\omega) \\ R(j\omega)S(j\omega)\left(\Phi_{nn}(\omega) + \Phi_{dd}(\omega)\right)^{1/2} \end{bmatrix}$$

shaping functions

In any case, what is the structure of the solution of the H2 or Hinf control problem?

$$\min_R \quad \|G_{zw}\|_2 \text{ or } \|G_{zw}\|_\infty$$

Structure of the solution of the H_2 control problem

Under suitable assumptions (see the textbook, pag. 162)

$$u(t) = -K\hat{x}(t) \quad , \quad K = B_2^{'}P$$

where ***P>0*** is the (unique positive definite) solution of

$$A'P + PA - PB_2B_2^{'}P + C_1^{'}C_1 = 0$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_2u(t) + L(y(t) - C_2\hat{x}(t)) \qquad L = \tilde{P}C_2^{'}$$

where $\tilde{P} > 0$ is the (unique positive definite) solution to

$$A\tilde{P} + \tilde{P}A' - \tilde{P}C_2'C_2\tilde{P} + B_1B_1^{'} = 0$$

Basically the same structure of LQG control

Structure of the solution of the H_{inf} control problem

The problem is formulated as

$$\min \gamma \text{ such that } \|G_{zw}\|_\infty < \gamma$$

Under suitable assumptions (see the textbook, pag. 162)

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \gamma^{-2}B_1B_1'P\hat{x}(t) + B_2u(t) + ZL\left(y(t) - C_2\hat{x}(t)\right) \\ u(t) &= -K\hat{x}(t) \end{aligned}$$

state feedback + observer

where

$$Z = \left(I - \gamma^{-2}SP\right)^{-1}, \;\; L = SC_2', \;\; K = B_2'P$$

and ***S***, ***P*** are the solutions to two Riccati equations

Iterative algorithms available

Model reduction - motivations

Model reduction techniques are useful for many reasons, for example:

1. The original model of the system to be controlled can be in nonminimal form. This frequently happens when modeling large scale systems

2. The model contains pole/zero pairs that are not coincident, but very near each other. In this case removing these pairs does not reduce the validity of the model

3. The regulator computed with the H_2-H_{inf} synthesis with shaping functions is of order equal to the order of the plant + the order of W_S + the order of W_T + the order of W_K. Clearly, this number can become very large and removing poles and zeros almost equal significantly reduces its size

Model reduction

System (can be the regulator)

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Define

$$P = \int_0^\infty e^{At} BB' e^{A't} dt \qquad \textit{controllability gramian}$$

$$Q = \int_0^\infty e^{A't} C'C e^{At} dt \qquad \textit{observability gramian}$$

For asymptotically stable systems ***P*** and ***Q*** can be computed as the positive definite solutions of

$$\begin{aligned}AP + PA' + BB' &= 0 \\ A'Q + QA + C'C &= 0\end{aligned}$$

Balanced realization

In general ***P≠Q***, but with a proper state transformation it is possible to write the system in the *balanced realization* form with

$$P = Q = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n),\ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$$

Looking at the values of the ***σ_i's***, the idea is to find a reduced state space representations with only ***k*** values $\sigma_1, \sigma_2, \ldots, \sigma_k$

Accordingly, the balanced system can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

where $A_{11} \in R^{k,k},\ B_1 \in R^{k,m},\ C_1 \in R^{p,k}$, and Σ is given by

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

with $\Sigma_1 \in R^{k,k}$

Balanced truncation

The reduced model is described by (A_{11}, B_1, C_1, D)

Let $G_a^k(s)$ be the transfer function of the reduced order model and $G(s)$ be the one of the original system

It is possible to prove that

$$\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_n)$$

Balanced residualization

Assume that the dynamics of the states to be neglected can be ignored, i.e. $\dot{x}_2 = 0$

Assume that A_{22} is nonsingular

The reduced order model is

$$\Sigma : \begin{cases} \dot{x}_1(t) = A_r x_1(t) + B_r u(t) \\ y(t) = C_r x_1(t) + D_r u(t) \end{cases}$$

$$A_r = A_{11} - A_{12} A_{22}^{-1} A_{21},\ B_r = B_1 - A_{12} A_{22}^{-1} B_2$$
$$C_r = C_1 - C_2 A_{22}^{-1} A_{21},\ D_r = D - C_2 A_{22}^{-1} B_2$$

Also in this case $\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_{k+n})$

Moreover $G_a^k(0) = G(0)$, i.e. ***the static gain is maintained***

Example: linearized model of an aircraft

Maciejowski, Jan Marian. "Multivariable feedback design." *Electronic Systems Engineering Series, Wokingham, England: Addison-Wesley,| c1989* (1989).

$x1$: altitude relative to some datum (m)

$x2$: forward speed ($\mathrm{m\,s^{-1}}$)

$x3$: pitch angle (degrees)

$x4$: pitch rate ($\mathrm{deg\,s^{-1}}$)

$x5$: vertical speed ($\mathrm{m\,s^{-1}}$)

$$A = \begin{bmatrix} 0 & 0 & 1.1320 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$u1$: spoiler angle (measured in tenths of a degree)

$u2$: forward acceleration ($\mathrm{m\,s^{-2}}$)

$u3$: elevator angle (degrees)

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1.0000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.665 \\ 1.5750 & 0 & -0.0732 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

D=0

FOLDER_EXAMPLE_LQG_H2_Hinf - file Synthesis_LQG_H2_Hinf.m

```
G=ss(A,B,C,D);

% weight LQ
 q=20;
Q=q*eye(n);
r=4;
R=r*eye(m);
W=blkdiag(Q,R);
```

LQ weights

```
% noises covariances
qt=100;
QT=qt*eye(n);
rt=2;
RT=rt*eye(p);
V=blkdiag(QT,RT);
```

Noise covariances for KF

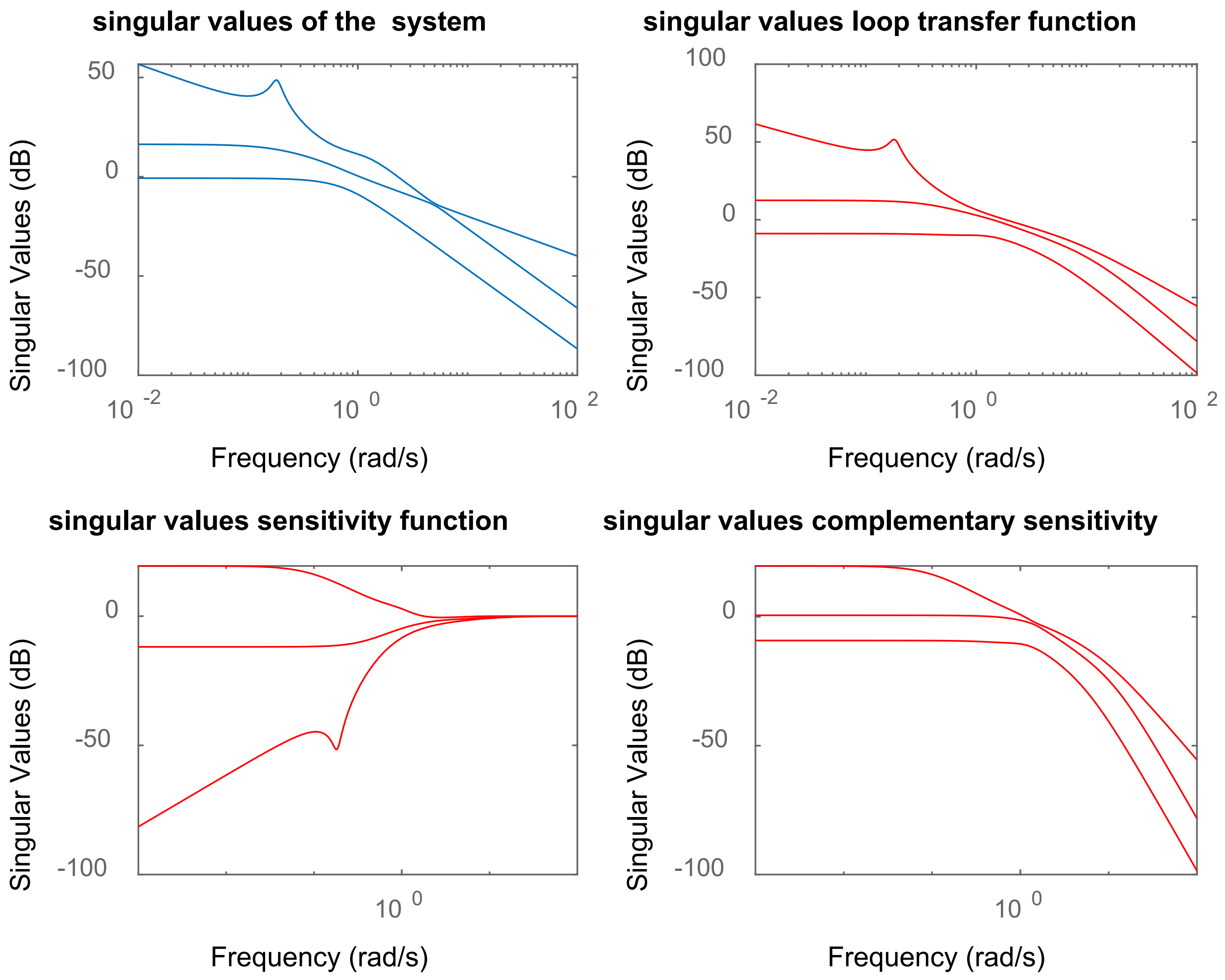
```
% synthesis LQG
[ALQG,BLQG,CLQG,DLQG] = lqg(A,B,C,D,W,V);
R=ss(ALQG,BLQG,CLQG,DLQG)
```

LQG synthesis

```
% analysis of the transfer functions
 figure(1)
subplot(221)
sigma(G)
title('singular values of the  system')
hold on
 L=G*R;
subplot(222)
sigma(L)
title('singular values of the loop transfer function')
hold on
 S=inv(eye(p)+L); % Sensitivity
subplot(223)
sigma(S)
title('singular values of the sensitivity transfer function')
hold on
 T=eye(p)-S;     % complementary sensitivity
subplot(224)
sigma(T)
title('singular values of the complementary sensitivity ')
hold on
```

Sensitivity functions with LQG

LQG (LQ+KF) (blue), and H2 formulation (red)



```
% synthesis with H2, same choice of the matrices used for LQG control
% the controllers must be the same
```

```
B2=B;
C2=C;
D22=D;

B1=[sqrt(qt)*eye(n) zeros(n,p)];
C1=[sqrt(q)*eye(n);zeros(m,n)];
D11=[zeros(n,n) zeros(n,p);zeros(m,n) zeros(m,p)];
D12=[zeros(n,m);sqrt(r)*eye(m)];
D21=[zeros(p,n) sqrt(rt)*eye(p,p)];

AA=A;
BB=[B1 B2];
CC=[C1;C2];
DD=[D11 D12;D21 D22];
%
P=ss(AA,BB,CC,DD);
P1=mktito(P,p,m); % enlarged system for - H2 Hinf synthesis

[K2,CL2,GAM2,INFO2] = h2syn(P1); % synthesis H2 with the same characteristics LQG, the regulator is K2
```

General formulation 2x2 system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{aligned}
$$

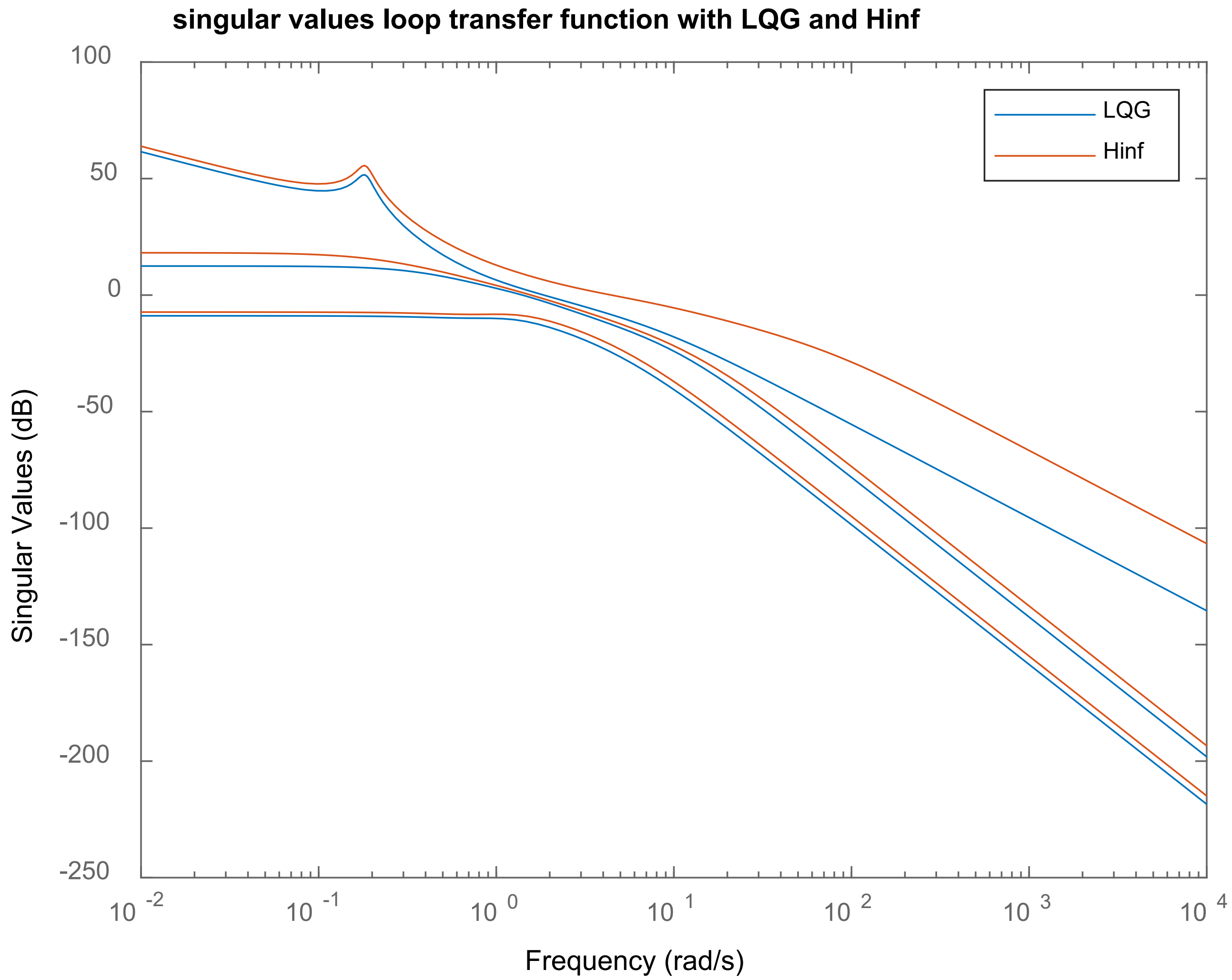
Specific choice - LQG

$$
\begin{aligned}
\dot{x} &= Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\
z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
v &= Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w
\end{aligned}
$$

```
K2;
LK=G*K2;
 subplot(222)
sigma(LK,'r')
title('singular values loop transfer function')
 SK=inv(eye(p)+LK); % Sensitivity
subplot(223)
sigma(SK,'r')
title('singular values sensitivity function')
 TK=eye(p)-SK;      % complementary sensitivity
subplot(224)
sigma(TK,'r')
title('singular values complementary sensitivity’)
```

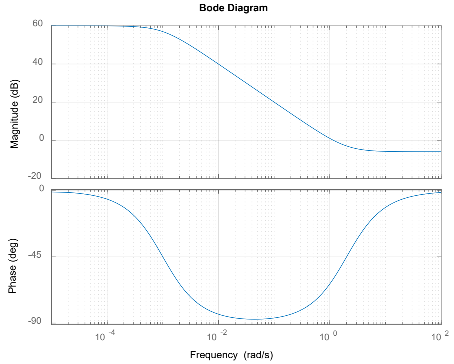
Sensitivity functions with H2 (same as LQG)

```
[Kinf,CLinf,GAMinf,INFOinf] = hinfsyn(P1); % synthesis Hinf with the same characteristics of LQG, the regulator is Kinf
 Kinf;
LKi=G*Kinf;
 figure(2)
sigma(L)
hold on
sigma(LKi)
title('singular values loop transfer function with LQG and Hinf')
legend('LQG','Hinf')
```


singular values loop transfer function with LQG and Hinf
100
50
0
-50
-100
-150
-200
-250
Singular Values (dB)
10 -2
10 -1
10 0
10 1
10 2
10 3
10 4
Frequency (rad/s)
LQG
Hinf

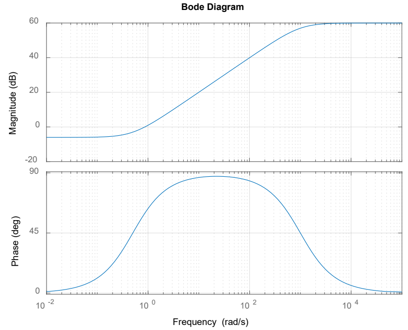
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% synthesis with shaping functions, omega = 1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% definition of the shaping functions
% and Bode diagrams
    wB=1;     % desired closed-loop bandwidth
    AA=1/1000;  % desired disturbance attenuation inside bandwidth
    M=2 ;      % desired bound on hinfnorm(S) & hinfnorm(T)
    s=tf('s'); % Laplace transform variable 's'
    WS=(s/M+wB)/(s+wB*AA); % Sensitivity weight
    WK=(0.001*s+1)/(0.01*s+1);  % Control weight can't be empty (d12).ne.0)
    WT=(s+wB/M)/(AA*s+wB); % Complementary sensitivity weight
    figure(3)
    bode(WS)
    hold on
    bode(WK)
    bode(WT)
    grid
    legend('WS','WK','WT')
```

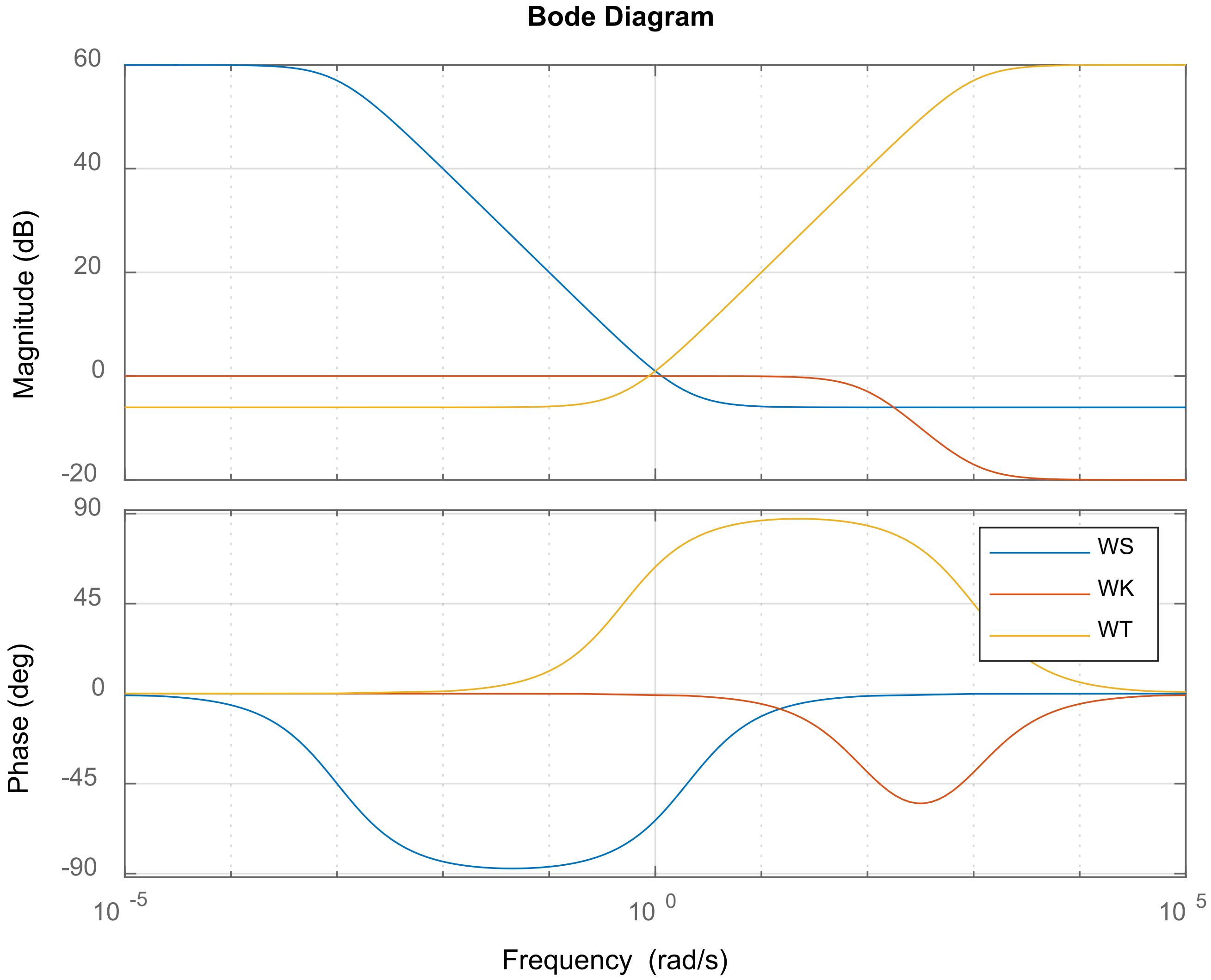
$$W_S(s) = \frac{s/M+\omega_B}{s+A\omega_B}$$



Scalar shaping functions

$$W_T(s) = \frac{s+\omega_{BT}/M}{As+\omega_{BT}} \ ,$$




Bode Diagram
Magnitude (dB)
60
40
20
0
-20
Phase (deg)
90
45
0
-45
-90
WS
WK
WT
10 -5
10 0
10 5
Frequency (rad/s)

% shaping functions written in block form

WWS=blkdiag(WS,WS,WS);
WWT=blkdiag(WT,WT,WT);
WWK=blkdiag(WK,WK,WK);

Ws, Wt, Wk
diagonal 3x3
matrices

% enlarged system withshaping functions

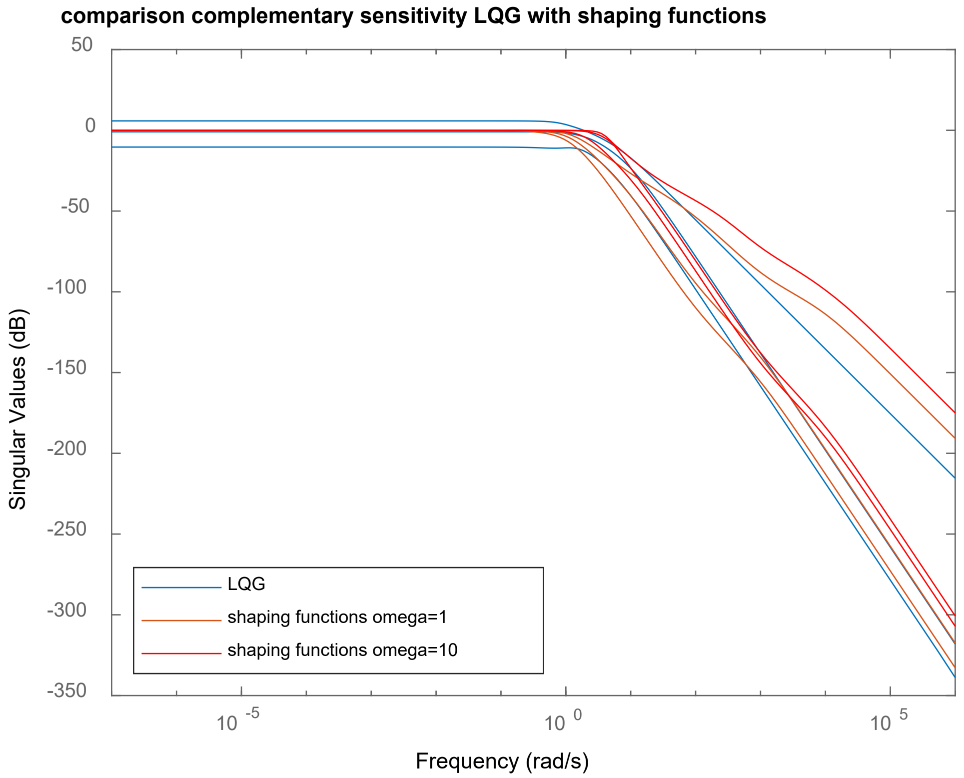
Plant + shaping functions at the output

SW=augw(G,WWS,WWK,WWT);
[KW,CLW,GAMW,INFOW]=hinfsyn(SW);

Hinf synthesis

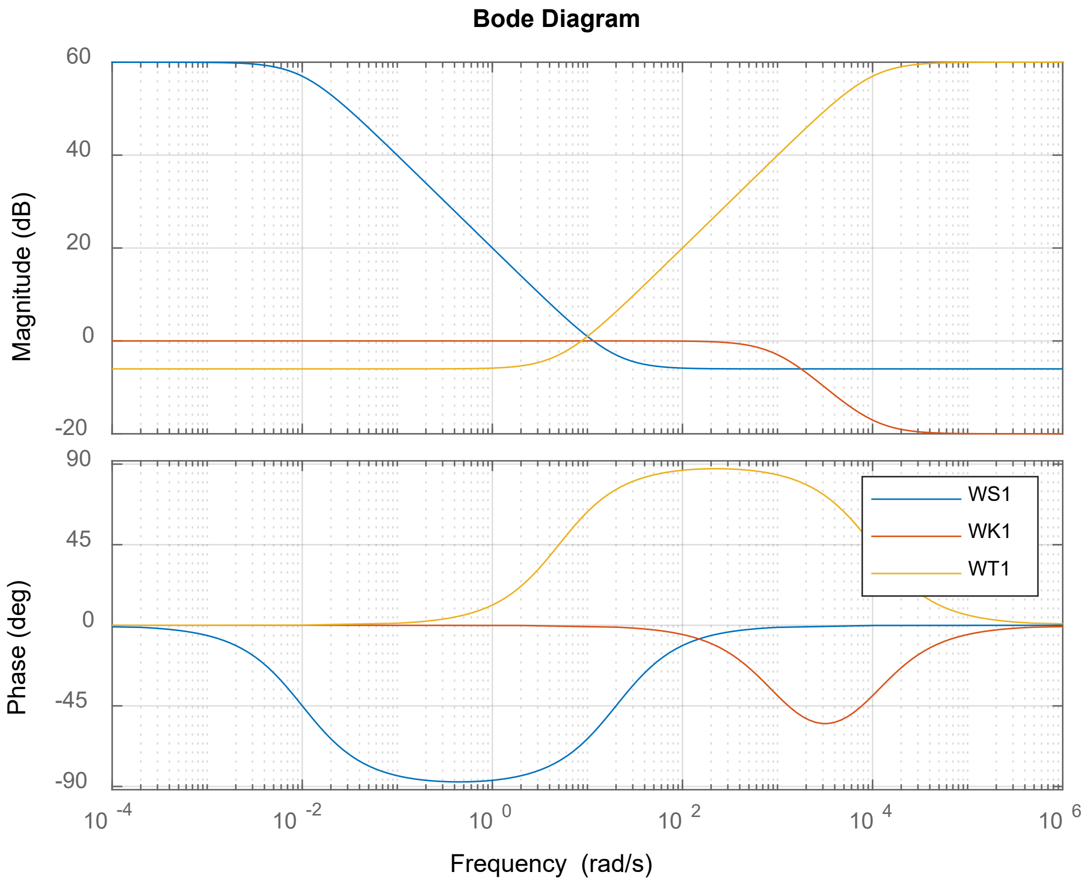
LKW=G*KW;
SKW=inv(eye(p)+LKW); % Sensitivity
TKW=eye(p)-SKW; % complementary sensitivity

```
figure(4)
sigma(T)
hold on
sigma(TKW)
title('comparison complementary sensitivity LQG
with shaping functions')
```



```
% new shaping functions, the bandwith is multiplied by 10
 % definition of the shaping functions
     wB1=10;       % desired closed-loop bandwidth
    AA1=1/1000;   % desired disturbance attenuation inside bandwidth
    M1=2 ;       % desired bound on hinfnorm(S) & hinfnorm(T)
    WS1=(s/M1+wB1)/(s+wB1*AA1); % Sensitivity weight
    WK1=(0.0001*s+1)/(0.001*s+1);  % Control weight can't be empty (d12).ne.0)
    WT1=(s+wB1/M1)/(AA1*s+wB1); % Complementary sensitivity weight
    figure(5)
    bode(WS1)
    hold on
    bode(WK1)
    bode(WT1)
    grid
    legend('WS1','WK1','WT1')
% shaping functions written in block form
WWS1=blkdiag(WS1,WS1,WS1);
WWT1=blkdiag(WT1,WT1,WT1);
WWK1=blkdiag(WK1,WK1,WK1);
% enlarged system with shaping functions
SW1=augw(G,WWS1,WWK1,WWT1);
```

new enlarged system

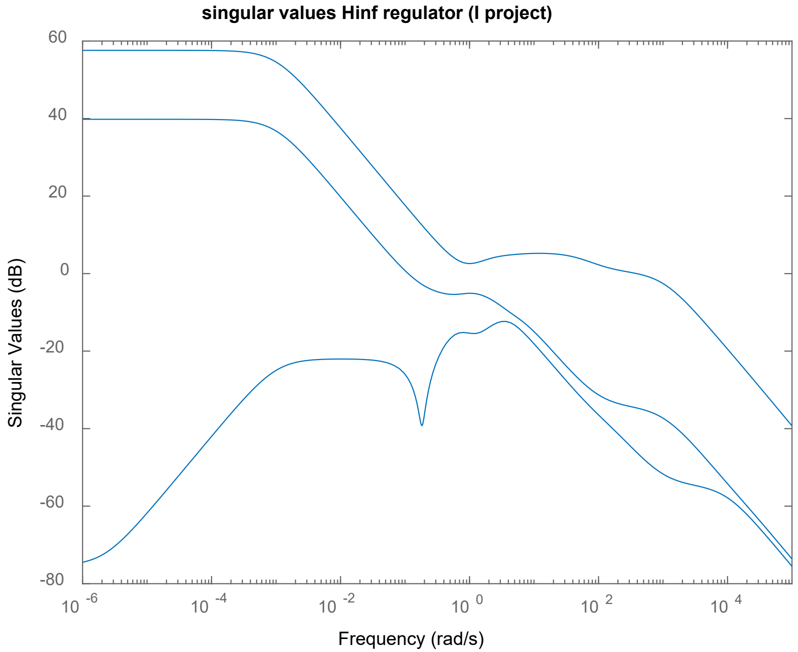

Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
60
40
20
0
-20
90
45
0
-45
-90
10 -4
10 -2
10 0
10 2
10 4
10 6
WS1
WK1
WT1

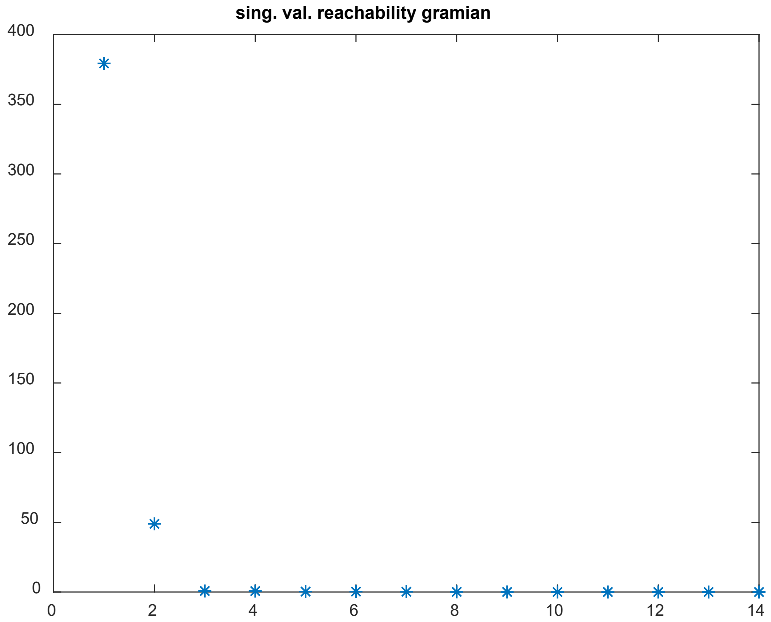
new Hinf synthesis

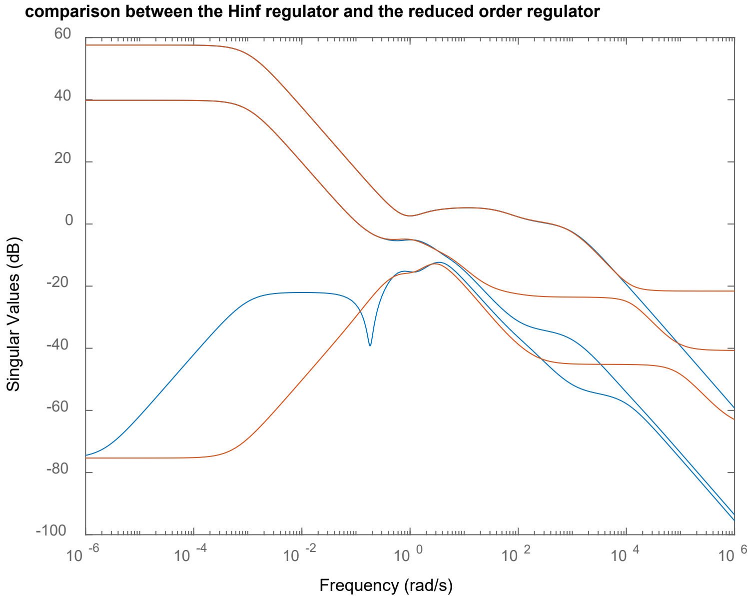
```
[KW1,CLW1,GAMW1,INFOW1]=hinfsyn(SW1);

LKW1=G*KW1;
SKW1=inv(eye(p)+LKW1);  % Sensitivity
TKW1=eye(p)-SKW1;       % complementary sensitivity

figure(4)
sigma(TKW1,'r')
legend('LQG','shaping functions omega=1','shaping functions omega=10','Location','SouthWest')
```

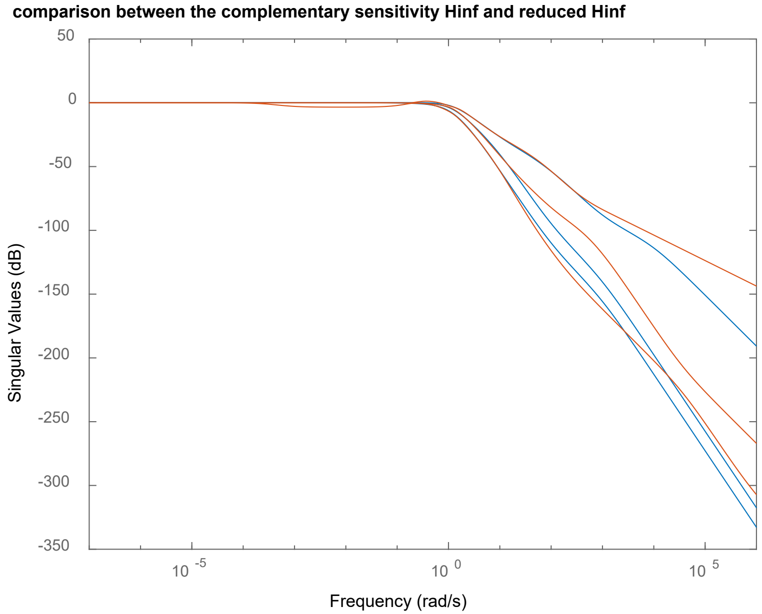
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% analysis of the singular values and order reduction of the Hinf regulator - first project
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure(6)
sigma(KW)
title('singular values Hinf regulator (I project)')
hold on
 KWBAL=balreal(KW); % compute the balanced realization of the Hinf regulator
GRAMr=gram(KWBAL,'c');
figure(7)
plot(diag(GRAMr),'*') % draws the singular values of the reachability gramian
title('sing. val. reachability gramian')
lim=0.05;            % limit of the singular values to be used in the order reduction
g=diag(GRAMr);
elim = (g<lim);             % removes the states corresponding to the singular values less than lim
KWRID = modred(KWBAL,elim);  % removes the states
 figure(8)
sigma(KW)
hold on
sigma(KWRID)
title('comparison between the Hinf regulator and the reduced order regulator')
```

```
figure(9)
LRID=G*KWRID;
SRID=inv(eye(p)+LRID);  % Sensitivity
TRID=eye(p)-SRID;       % complementary sensitivity
sigma(TKW)
hold on
sigma(TRID)
title('comparison between the complementary sensitivity Hinf and reduced Hinf')
```


singular values Hinf regulator (I project)
Singular Values (dB)
Frequency (rad/s)
60
40
20
0
-20
-40
-60
-80
10 -6
10 -4
10 -2
10 0
10 2
10 4


sing. val. reachability gramian
400
350
300
250
200
150
100
50
0
0
2
4
6
8
10
12
14


comparison between the Hinf regulator and the reduced order regulator
Singular Values (dB)
Frequency (rad/s)
60
40
20
0
-20
-40
-60
-80
-100
10 -6
10 -4
10 -2
10 0
10 2
10 4
10 6


comparison between the complementary sensitivity Hinf and reduced Hinf
Singular Values (dB)
Frequency (rad/s)
50
0
-50
-100
-150
-200
-250
-300
-350
10 -5
10 0
10 5