POLE PLACEMENT IM S.S. DOMAIN

STATE OBSERVER

DISTURBANCE ESTIMATION

Advanced and Multivariable Control

Pole placement control

method for synthesis of stabilizing regulator on MIMO case

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BASIC

Problem statement

(from a linear system)

Consider the system

Assumption: ⇒ system such that the state $x(t)$ is measurable (available)

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in R^n, \quad u \in R^m$$

if $x(t)$ available you don't need output transform because the state info is greater than output info

and the control law

state feedback

$$u(t) = -Kx(t) + \gamma(t), \quad \underbrace{K \in R^{m,n}}_{m \times n \text{ MATRIX}}, \quad \gamma \in R^m$$

additional terms

GOAL

Problem: we want to design a feedback matrix ***K*** such that the closed loop system

$$\dot{x}(t) = (A - BK)\,x(t) + B\gamma(t)$$

additional input used for other reasons

(2nd step of design)

stated in this way $\gamma(t)$ seems useless, the specification depends on A - BK

has prescribed eigenvalues (of the matrix ***A-BK***)

has prescribed eig values

Remark: we are assuming that the state is measurable (not realistic in many cases). This hypothesis will be removed later on

NOT unique, it can take different meaning

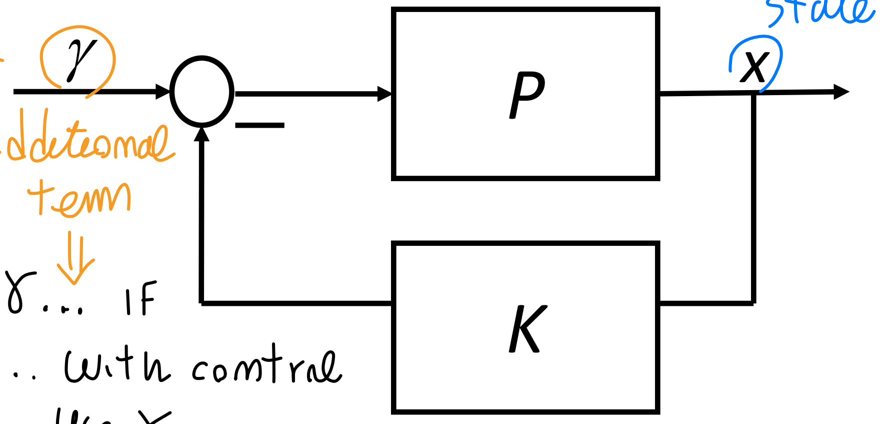
but sometimes this additional input can be useful for pole placement and to consider additional requirements

Remarks → overall scheme

x: as output of plant P:

produced by external regulator for move request

state



additional term

feedback

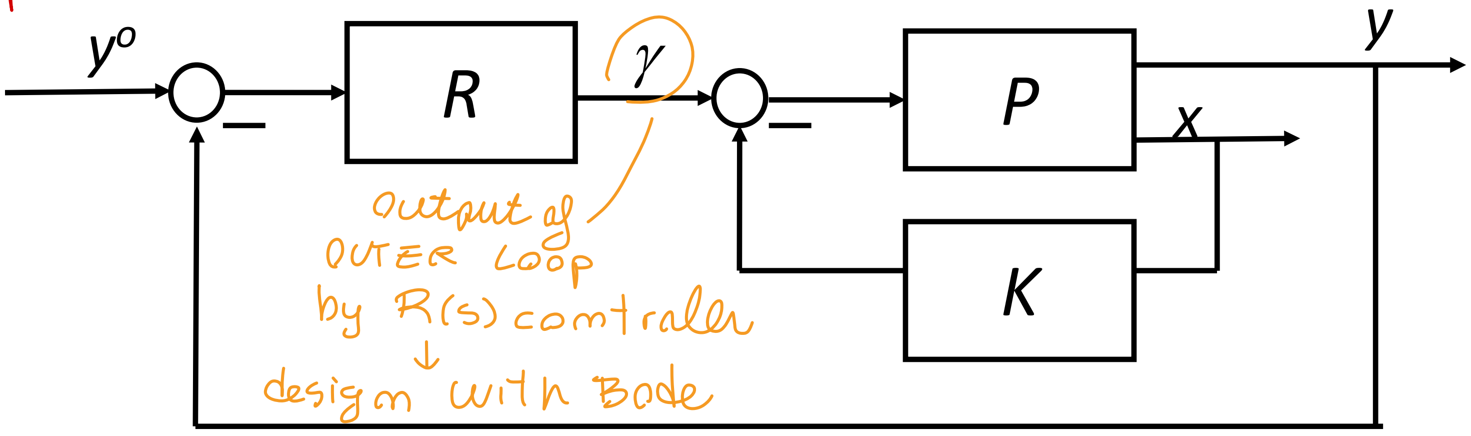
The additional input γ does not play any role, but it can be useful to design external loops, while *K* is used for stabilization

usefull γ... IF

for example we have an unstable system P... with control law stabilizing we stabilize the syst and than use γ to close an external loop (static performance or other)

IF P unstable → NO Bode → Nyquist

→ pole placement ⇒ by K MATRIX



Inner feedback used to stabilize, while *R(s)* used for performance (for example, one can use the Bode criterion)

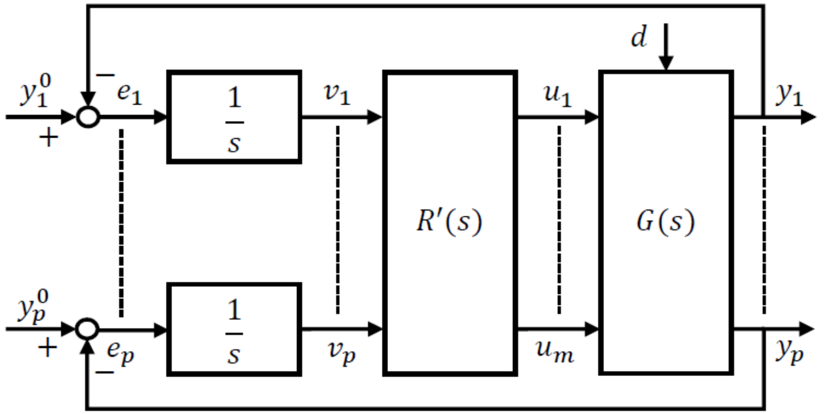
output of OUTER LOOP by R(s) controller

↓

design with Bode criterion, standard techniques on stable syst.

Scheme with integrators

If we want integrators, first we add integral action... than we design the pole placement controller for enlarged system made by plant state $x(t)$ and integrator state $v(t)$



$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} y^0$$

new matrix to consider for design

K designed for the new pair $\bar{A}, \bar{B}$ such that $\boxed{\bar{A} - \bar{B}K}$ has prescribed eigenvalues

Control law

$$u(t) = -K \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = -\begin{bmatrix} K_x & K_v \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

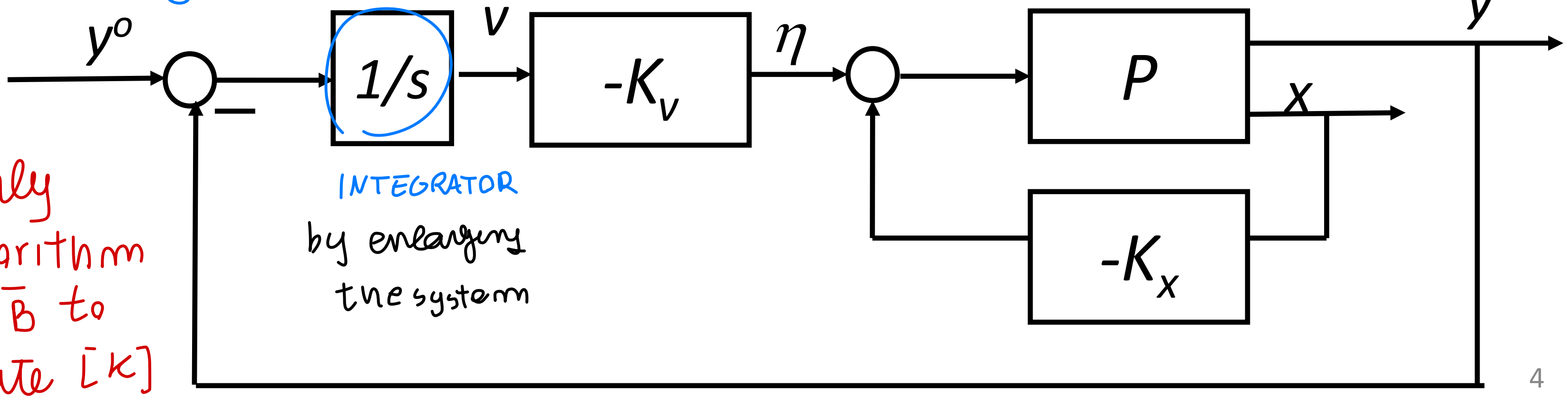
$u(t) = -K_x\, x(t) + K_v\, v(t)$

$\Downarrow$ new scheme!

PARTITION of [K] in 2 block $\begin{pmatrix} x \\ v \end{pmatrix}$

to use structure with integral action... new system enlarged described by $\bar{A}, \bar{B}$

apply algorithm to $\bar{A}, \bar{B}$ to compute [K]



INTEGRATOR by enlarging the system

Algorithm

Necessary and sufficient condition for the solution of the pole placement problem (with measurable state) is that the pair (A, B) is reachable

↑ for the control problem

↳ or of $(\bar{A}, \bar{B})$ of integrators

Sketch of the proof

⇓ given a generic lin. system you can partition A as triang. matrix

By means of a suitable state transformation, it is possible to write the system as

Reachable

$$\begin{bmatrix} \dot{\hat{x}}_r(t) \\ \dot{\hat{x}}_{nr}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_r & \hat{A}_x \\ 0 & \hat{A}_{nr} \end{bmatrix} \begin{bmatrix} \hat{x}_r(t) \\ \hat{x}_{nr}(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_r \\ 0 \end{bmatrix} u(t)$$

subset of state — unreachable

the pole assignment problem can be solved only if all reachable ...

If we take

$$u(t) = -\hat{K}\hat{x}(t) = -\begin{bmatrix} \hat{K}_r & \hat{K}_{nr} \end{bmatrix} \begin{bmatrix} \hat{x}_r(t) \\ \hat{x}_{nr}(t) \end{bmatrix}, \quad \hat{K}_r \in R^{m,\nu_r}$$

eig of $\hat{A}_r - \hat{B}_r\hat{K}_r$: can be moved by a proper choice of $\hat{K}_r$

we obtain

Close loop matrix

$$\boxed{\hat{A} - \hat{B}\hat{K}} = \begin{bmatrix} \hat{A}_r - \hat{B}_r\hat{K}_r & \hat{A}_x - \hat{B}_r\hat{K}_{nr} \\ 0 & \hat{A}_{nr} \end{bmatrix}$$

eig of $\hat{A}_{nr}$ → unreachable part

⇑

that cannot be modified !!

In case of stabilizable system ⇝ eig of unreach part are asymp. stable so their transient goes to 0 asymp.

in general we assum reachable system ⇐

and assign eig values of $A - B \cdot K$ (Pole placement)

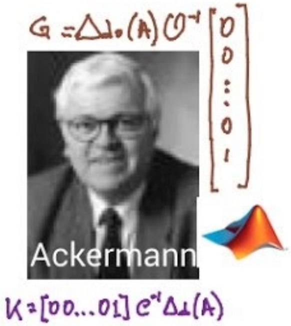
PROBLEM → at the end we have to find K such that $A-BK$ has prescribed eig values

Single input systems (*m=1*) - *The Ackermann's formula*

unique solution

ALGORITHM

[redacted]



[redacted]

[Acker.m] in Matlab!

Define the reachability matrix (square for *m=1*)

$$M_r = [\ B \quad AB \quad \dots \quad A^{n-1}B\ ]$$

for single input system is a square $n \times n$ system

reachable system ⇒ M_r NON SINGULAR

Assume you want to obtain with a closed-loop system with characteristic polynomial

$$P(s) = (s+\bar{p}_1)(s+\bar{p}_2)\dots(s+\bar{p}_n) = s^n + p_{n-1}s^{n-1} + \dots + p_1 s + p_0$$

You specify the closed loop eig values $\bar{p}_i$ desired → (obtaining)

Define

evaluate $P(s=A)$:

$$P(A) = A^n + p_{n-1}A^{n-1} + \dots + p_1 A + p_0 I$$

⇓

Then, the solution to the pole placement problem is given by the Ackermann's formula

desired polynomial @ $A=s$

on practical P.O.V there are other computation approaches to obtain K ⇒

$$\boxed{K = [\ 0 \quad 0 \quad \dots \quad 0 \quad 1\ ]\, M_r^{-1} P(A)}$$

unique matrix

M_r inverse, wich exist

In the textbook proofs and many more considerations

in open loop eig values on (-1)(+2) unstable!

Example Ackermann's formula

[mechanic approach]

DIFFICULT task: define specification

Where to place poles!?

II ORD system

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \quad , \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

m=1 {1 input on system}

desired eig position in (-3),(-4) in closed loop

1) WRITE P(s) $\Rightarrow$

$$P(s) = (s+3)(s+4) = s^2 + 7s + 12$$

EXAM description of how to apply this formula than using it

2) compute P(A) $\Rightarrow$

$$P(A) = A^2 + 7A + 12I = \begin{bmatrix} 6 & 24 \\ 0 & 30 \end{bmatrix}$$

compute inverse

3) compute reach. matrix

$$M_r = [\ B \quad AB\ ] = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \quad , \quad M_r^{-1} = \begin{bmatrix} -4/6 & 5/6 \\ 2/6 & -1/6 \end{bmatrix}$$

exist if reachable syst

$$K = [\ 0 \quad 1\ ]\, M_r^{-1} P(A) = [\ 2 \quad 3\ ] \quad , \quad \Rightarrow \boxed{A - BK} = \begin{bmatrix} -3 & 0 \\ -4 & -4 \end{bmatrix}$$

Ackermann's formula

overall closed loop

new eig values, positioned where desired!

Where to place the poles? ***This is the difficult task*** (NOT specific rules)

(Define specification thinking of I / II ORD systems!)

An idea is try to impose that the closed loop system (approximately) behaves as a first order or second order system

(specify that by phase margin is a good indicator of closed loop system characteristic for SISO)

φ_m small → II ORD with $\xi \simeq \varphi_m/100$ / φ_m LARGE → I ORD

In the case of first order systems, the «desired» transfer function and step response are

$$G(s) = \frac{\mu}{1+sT} \longleftrightarrow \boxed{y(t) = \mu(1 - e^{-t/T})}$$

step input response

you could find the required poles looking to step responce

respect our requirements you choose T to have desired behaviour

So, you can choose the time constant *T*, or (equivalently) the pole *-1/T* and fix one of the closed-loop poles at that value

(I imagine closed loop ≃ I ORD, select ideal pole position, the others @ higher freq)

Then, the remaining closed-loop poles can be chosen to be «$\underbrace{\text{much faster}}_{}$» than this pole, which is the «***dominant one***»

@ higher frequency!

Always remember that it is not advisable to choose a closed-loop system must faster than the open-loop one. This could lead to the saturation of the actuators, or to an excessive use of them

↑ too fast closed loop → SATURATION → NO more linear model

TECHNOLOGICAL problem of fast feedback!

~II ORD systems similar reasoning

Two dominant poles

I may request for little oscillation → so FASTER response

$$G(s) = \frac{\mu\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

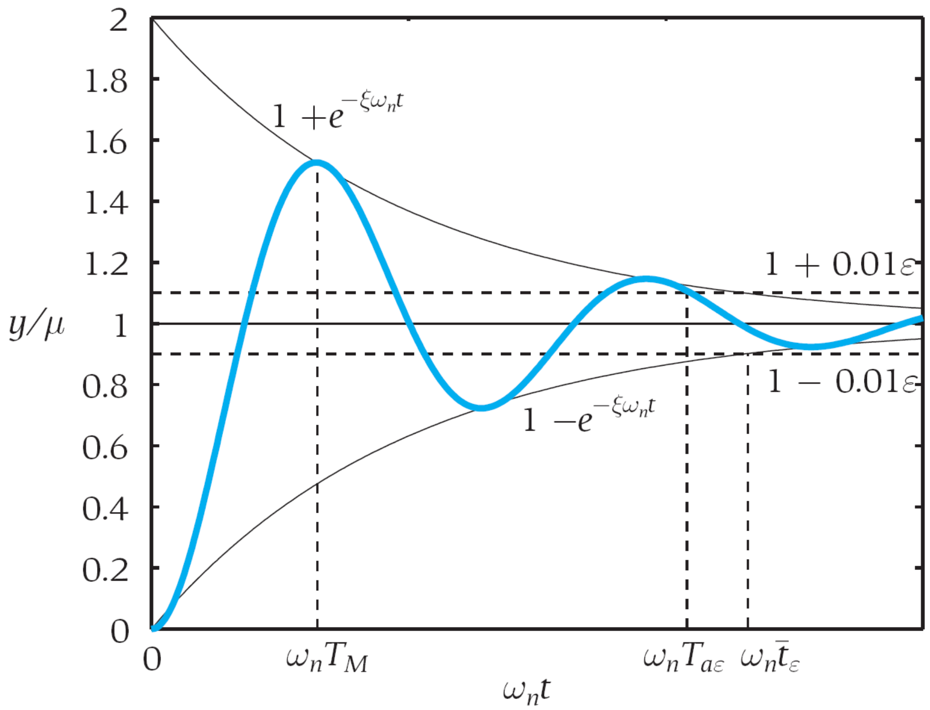
— settling time

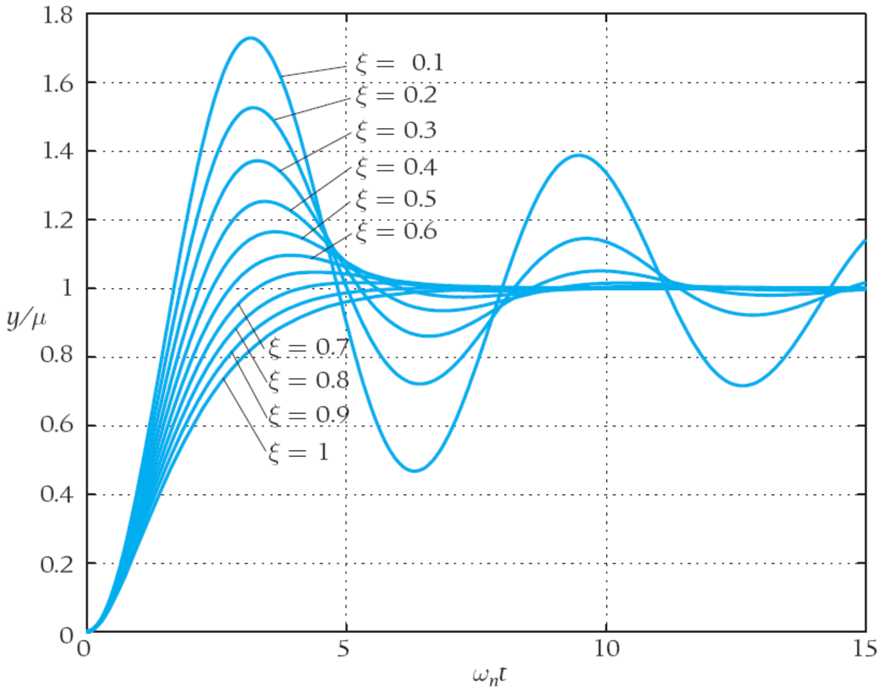
MAX overshoot

↓ what you want close loop

(II ORD resp.)

Characteristics of the step response





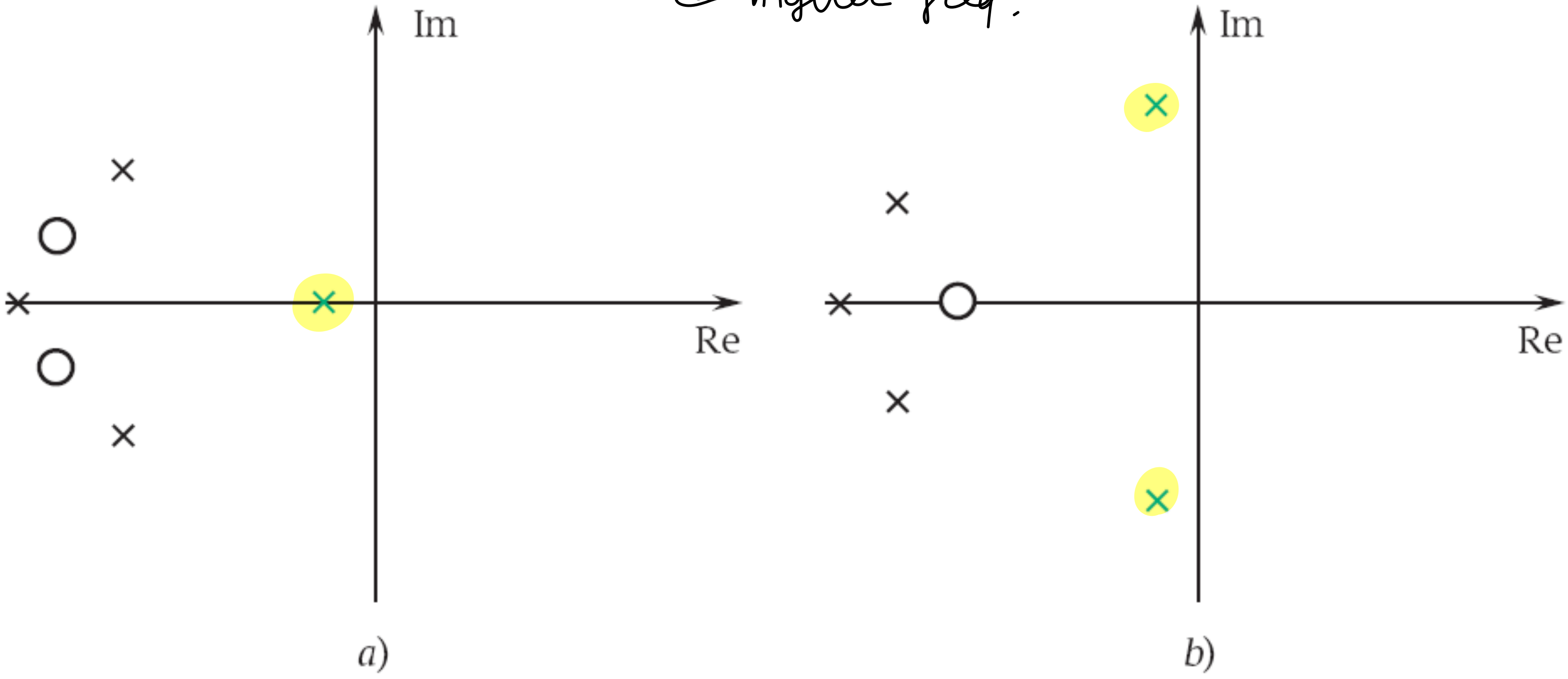
select 2 poles and the others @ higher freq suggestion idea ...

$\Uparrow$

other singularities @ higher freq.

one dominant pole

two dominant poles



where to place the poles!

Systems with many inputs (*m*>1)

MIMO case! for more inputs more DOF on the choice ⇒ on SIMO case you have just one choice! Solution (Ackermann)

The problem can have **many solutions**. In any case, the assumption of ***reachability is mandatory***

(difficult to understand BEST)

necessary condition

Example

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + b_1u_1(t) + b_2u_2(t)$$

{ 3 STATES
2 INPUT

{ canonical form }

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

2 INPUTS.. by 2 rows of B

because both reachable

Both (A, b_1) and (A, b_2) are reachable, so that we can use just one input to assign the closed-loop poles

verified reachability test!

⇓

Use one input for Ackermann formula to assign poles, while don't use the other input!

continued…

- **Solution 1**

use only $u_1(t)$ with Ackermann!

impose this eig

$$\begin{cases} u_1(t) = -K_1 x(t) \\ u_2(t) = 0 \end{cases} \longleftrightarrow \quad \dot{x}(t) = (\boxed{A - b_1 K_1})x(t)$$

don't use $u_2(t)$!

K_1 can be computed with the Ackermann's formula (only one input!) to impose the eigenvalues of $(A\text{-}b_1K_1)$

2 SOLUTIONS! NOT unique!

- **Solution 2**

$$\begin{cases} u_1(t) = 0 \\ u_2(t) = -K_2 x(t) \end{cases} \longleftrightarrow \quad \dot{x}(t) = (A - b_2 K_2)x(t)$$

use only $u_2(t)$

same type of control law $\Rightarrow$

K_2 can be computed with the Ackermann's formula (only one input!) to impose the eigenvalues of $(A\text{-}b_2K_2)$

In both cases $u(t) = -Kx(t)$ where $K = \begin{bmatrix} K_1 \\ 0 \end{bmatrix}$ (first solution) or $K = \begin{bmatrix} 0 \\ K_2 \end{bmatrix}$ (second solution)

don't seems clever solution using just one input BUT both usable..

on MIMO case → we can have many different solutions

(STANDARD SOFTWARE) ⇐ NOT unique solution ↑

What to do in practice? (to compute [K] according to different criteria)

↳ algorithm such that from A, B nominal even for variation good solution

Use the function *place.m* of Matlab, it uses all the available inputs and guarantees some robustness of the solution with respect to small variations of the parameters of the matrices *A, B*.
The only warning is that you must specify closed-loop poles each one different from the others (not a big issue, there is no difference in placing two poles in -1 or one pole in -1 and the other one in -1.001)

take as input A, B matrix and location of eig values desired, by P

[K,PREC] = place(A,B,P)

returns PREC, an estimate of how closely the eigenvalues of A-B*K match the specified locations P (PREC measures the number of accurate decimal digits in the actual closed-loop poles). A warning is issued if some nonzero closed-loop pole is more than 10% off from the desired location.

multiple solutions! can compute [K] with more SWs ⇐ (LIMITATION) avoid to place the poles on same position of complex plane ⇒ use different pole position (in -1, -1.01 .. NOT 2 in -1)

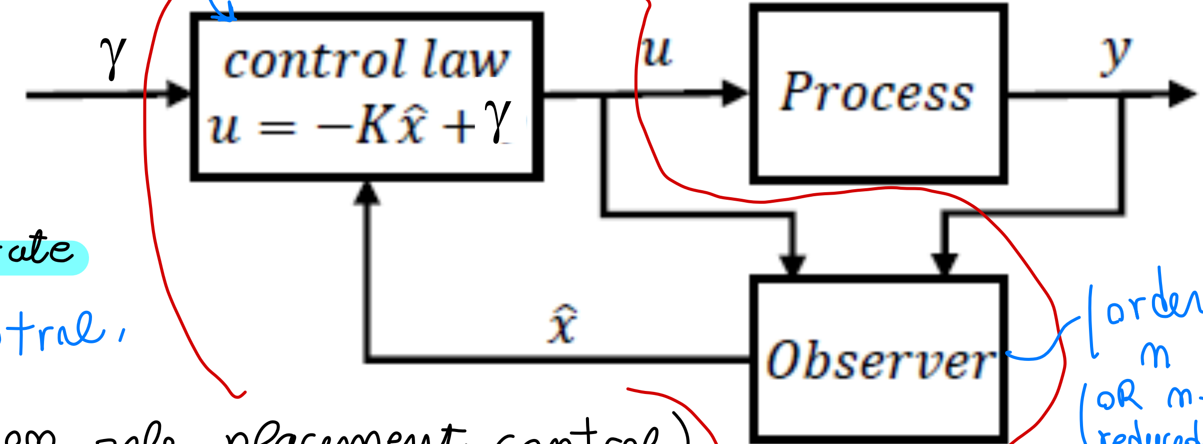
IF you don't have the STATE of syst..

Nonmeasurable state

↓ you need to use an element (new dyn system) which given the in/out of syst compute an estimation $\hat{x}$ of the state at any time

used for control,

(static) overall reg



(order n (OR n-p if reduced order)

(OBSERVER based on pole placement control)

(Khalman Filter)

↳ here simpler, using duality of control / observer design

Procedure

1. first a $\boxed{\textit{state observer}}$ is built. The observer is a system which, based on the input and output measurements, computes an estimate $\hat{x}$ of the system's state x. The observer introduced in the following is a linear time-invariant system with prescribed eigenvalues;

2. the estimated state $\hat{x}$ is used in the pole assignment control law in place of the real state x.

↳

State observers

System specified also by OUTPUT transform

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x \in R^n, \quad u \in R^m \\ y(t) = Cx(t) + Du(t), & y \in R^p \end{cases}$$

only measurable signal

We must consider also the output transformation now

Observer

from $u(t), y(t)$ we have to estimate $x(t)$

$$\dot{\hat{x}}(t) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{same system dyn. (OPEN LOOP)}} + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

we also use a feedback observer improving performance by a Gain (L) by difference of output and estimation

$L \in R^{n,p}$ is the gain of the observer (and its design parameter)

$L \in [n \times p]$

$e_y(t) = [y(t) - C\hat{x}(t) - Du(t)]$ is the output estimation error

IF $\hat{x} = x$: $y(t) - C\hat{x} - Du(t) = 0$

$e(t) = x(t) - \hat{x}(t)$ is the state estimation error

We have to compute L

we want to bring $e(t) \to 0$

from OBSERVER formula

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u(t) + Ly(t)$$

{on different notation}

on a PRACTICAL P.O.V

to implement it on MATLAB

to implement an observer

to understand how the systnem works...

Dynamics of the state estimation error

IF you know u(t) external signal existance you use it on state estim. dynamic so@ end it cancel out!

IF this is asymp. stable e(t) → 0 with time

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$
$$= Ax(t) + B\cancel{u}(t) - A\hat{x}(t) - B\cancel{u}(t) - L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$
$$= A(x(t) - \hat{x}(t)) - LC(x(t) - \hat{x}(t)) = (A - LC)e(t)$$

we have to ensure (A-LC) on stable region

no forcing input ← evolutation as the one of a linear system with only free motion

If the eigenvalues of (***A-LC***) are asymptotically stable, the state estimation error tends to zero. The idea is to use pole assignment to select the observer gain ***L*** to assign these eigenvalues

to design that regulator

FIND L such that (A-LC) has eig as desired

The eigenvalues of (***A-LC***) are the eigenvalues of (***A'-C'L'***)

└ TRANSPOSED

⇓

equivalent problem

The problem of assigning the eigenvalues of (***A'-C'L'***) is equivalent to assigning the eigenvalues of (***A-BK***)

solve equivalent problem as described before!

$$A \rightarrow A', \quad B \rightarrow C', \quad m \rightarrow p, \quad K \rightarrow L'$$

on pole placement control problem

as reachability means I'm able to have an input which modify system dyn. observability means that looking output I understand the state !!

A necessary and sufficient condition for the design of an asymptotic observer with arbitrarily specified eigenvalues is that the pair (***A,C***) is observable (CNS)

(detectability ! ⇒ non obs part has stable eig values, they cannot be well estimated BUT they don't affect the syst)

- For single output systems we can use the Ackermann's formula with the proper substitutions

[you can use and redefine ackerman formula from A', B' obtain L']

polynomial of desired eig on A'

$$L' = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} (M_o^{-1})' P(A') \text{ or } L = P(A) M_o^{-1} \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

inverse of observability, transposed

{ don't make syst too much faster or you will have problems phiscal } PREVIOUSLY

- For multi output systems the suggestion is to use *place.m* with the proper substitutions (***A->A',B->C'***)

for OBSERVER

faster than state feedback control law!

⟹ How to choose the eigenvalues of (***A-LC***)? Typically ***much faster*** than the ones of (***A-BK***).

The observer is an algorithm, it does not suffer from actuators' saturations

{ NO speed limitations } don't impact actuator (only problem for unbalanced problem) ↑ force the observer to reconstruct state faster

Matlab – Simulink implementation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

$$\dot{\hat{x}}(t) = \underbrace{(A - LC)}_{\tilde{A}}\hat{x}(t) + \underbrace{(B - LD)}_{\tilde{B}}u(t) + Ly(t)$$

$\tilde{B} = B - LDL$ because you have also $y(t)$

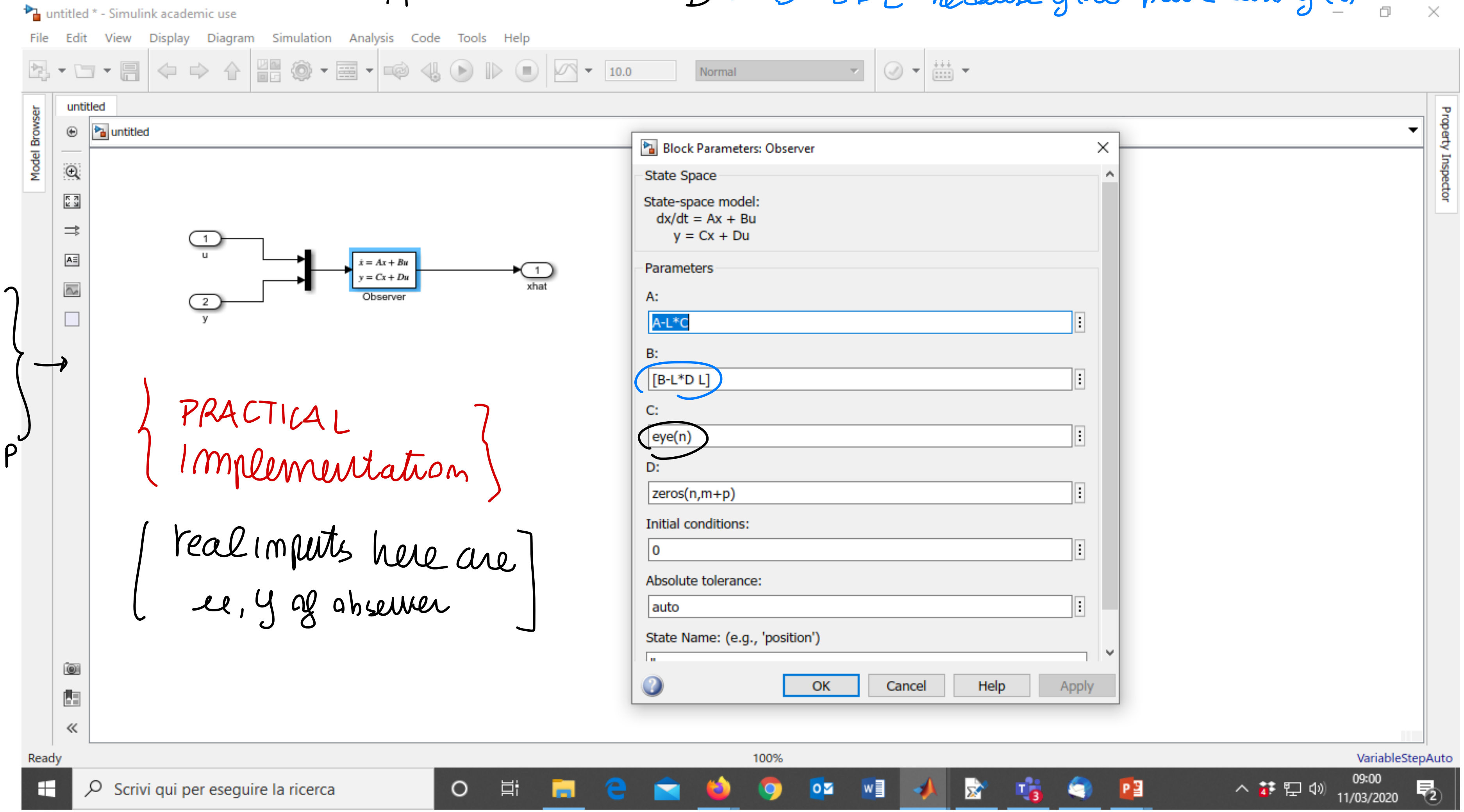
OBSERVER

$C = [I]$

because $y = \hat{x}$

and $D = [0]_{n,m+p}$

dyn system with proper I/O



PRACTICAL Implementation

real inputs here are u, y of observer

Systems with disturbances

disturbance additive on state evolution and output transform

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Md(t) \\ y(t) = Cx(t) + Du(t) + Nd(t) \end{cases}$$

(2 CASES)

1) ***measurable disturbance*** d(t) KNOWN → SIMPLE case, consider also d(t), include disturbance exactly as an INPUT u(t)

STATE influence

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + Md(t) + L\left[y(t) - C\hat{x}(t) - Du(t) - Nd(t)\right]$$

influence output error!

from computation $\dot{\hat{x}} - \dot{x} = \dot{e}(t)$

(we did cancel from this difference)

$$\dot{e}(t) = (A - LC)e(t)$$

same as before! same PROBLEM of state estimation

$$\hat{x} \rightarrow x$$

2) ***non measurable disturbance*** d(t) UNKNOWN

{more difficult!}

→ IF unknown d(t) you cannot use it for observer definition!

(OBSERVER theory: to estimate unmeasurable quantity)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

computing $\dot{\hat{x}} - \dot{x}$ ...

$$\dot{e}(t) = (A - LC)e(t) + (M - LN)d(t)$$

as before Free motion $e_0 \rightarrow 0$

additional term! ⇒ cannot be cancelled, UNKNOWN terms!

$$\hat{x} \not\rightarrow x$$

also for constant *d*

for unknown disturbance, NOT possible to have $e(t) \rightarrow 0$! forcing terms

OBSERVER theory important in many application, also for monitoring process in safety application $\Rightarrow$ OBSERVER to consider VIRTUAL sensor

where you estimate variable

$\hat{x} \not\rightarrow x$

TROUBLE

$\hookrightarrow$ other consideration ...

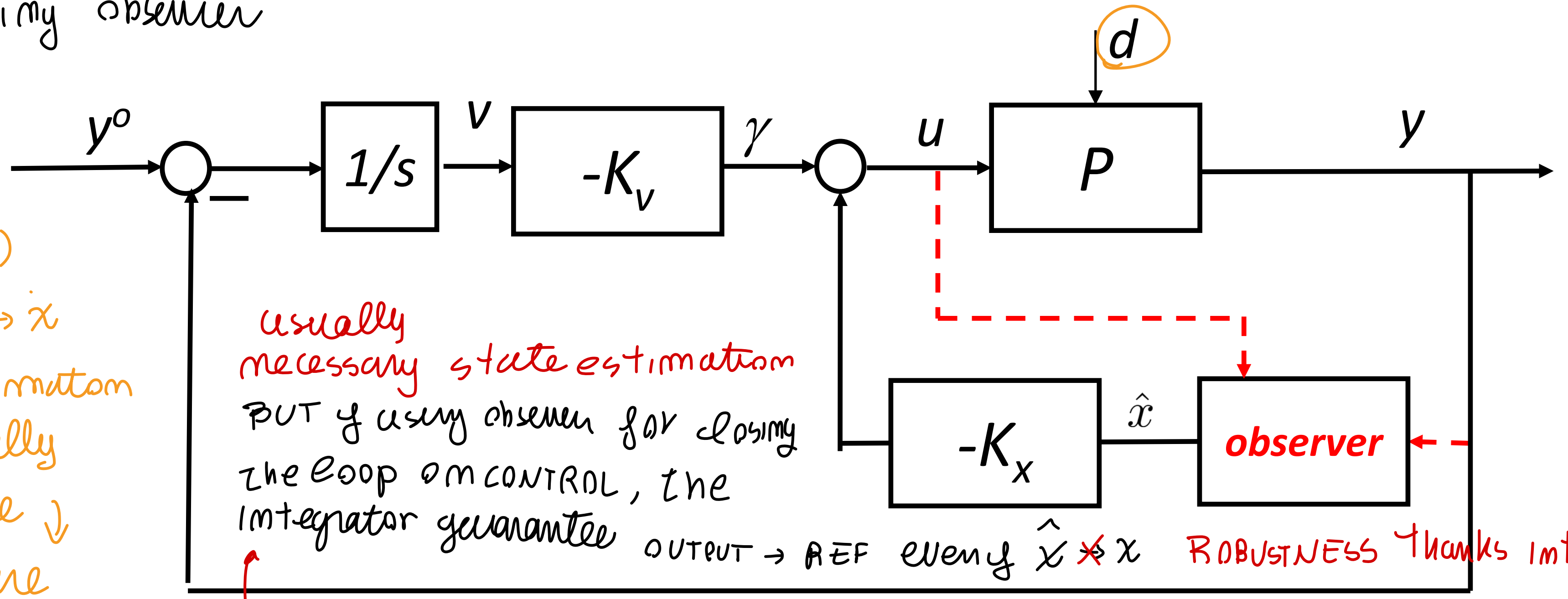
Is it really necessary to have the correct state estimation?

sometimes we need a state estimate to close the loop → instead of usual control law, we use $\hat{x}$, NOT x

Consider the system with integrators and observer (0 error for constant reference)

If the closed-loop system remains asymptotically stable, even when the state estimate is not correct, for constant references and disturbances the input of the integrators must be asymptotically zero

solve static optimiz by integrator ⇒ design K_x, K_v with pole placement, than stabilizing observer



IF act d(t) unlikely $\hat{x} \to x$
WRONG estimation
BUT if overally asymp. stable
if y^o, d are constant, all signal tends to constant value, so for same u goes to error e → 0 otherwise ramp!

usually necessary state estimation
BUT if using observer for closing the loop on CONTROL, the integrator guarantee OUTPUT → REF even if $\hat{x} \not\to x$
ROBUSTNESS thanks integrator!

IF $d(t)$ unknown impossible proper $x(t)$ reconstruct → on a PARTICULAR CASE

Estimation of constant disturbances

Assume to know that ***d*** is constant, or at least constant for long periods of time.
It is useful to estimate it for two reasons: → $d(t)$ can be estimated to have $e(t) \to 0$

1) It is possible to correctly estimate the state of the system

2) The estimate of the disturbance can be used in a control scheme with direct compensation

IF you have $d(t)$ estimation ⇒ compensator on CONTROL schem

Procedure

Assign a dynamics to the disturbance IF $d(t) = $ constant → (fictitious dynamic term)

$$\dot{d}(t) = 0, \quad d(0) = \bar{d} \in R^r$$

$r :=$ disturbances number

Enlarge the system

↳ well WORKING system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} A & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C & N \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + Du(t) \end{cases}$$

from Enlarged system... Basically provided some condition holds!
↳ $d(t)$ estimation properly
Algorithm to estimate $\hat{x}, \hat{d} \to x, d$!

Use an observer for the enlarged system. When is it possible?

apply observer theory on this new system,

Conditions I need conditions to guarantee OBSERVER existence

$\bar{A} \in R^{n+r,n+r}$

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} A & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C & N \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} + Du(t) \end{cases}$$

$\bar{C} \in R^{p,n+r}$

disturbances estimation

IF constant dynamic!

$\Downarrow$

we can use this approach (used also for d(t) ramp / sinusoid with proper generalization)

The pair $(\bar{A}, \bar{C})$ must be observable !

$\Downarrow$

Observable ***iff*** (CNS) for enlarged syst to be obs

1. (A,C) observable (ORIGINAL syst must be obs)

trivial! IF original non obs! impossible...

2. $rank \begin{bmatrix} A & M \\ C & N \end{bmatrix} = n + r$ $\longrightarrow$ $\boxed{r \leq p}$ $\leftarrow$ || *maximum numbr of disturbances that can be estimated* ||

you don't have invariant ZEROS...

you cannot reconstruct more disturbance than outputs!

$\Uparrow$

p as upper limit of d estimated

OBSERVABILITY TEST!

Exercise: use the $\boxed{\text{PBH test}}$ to prove this result

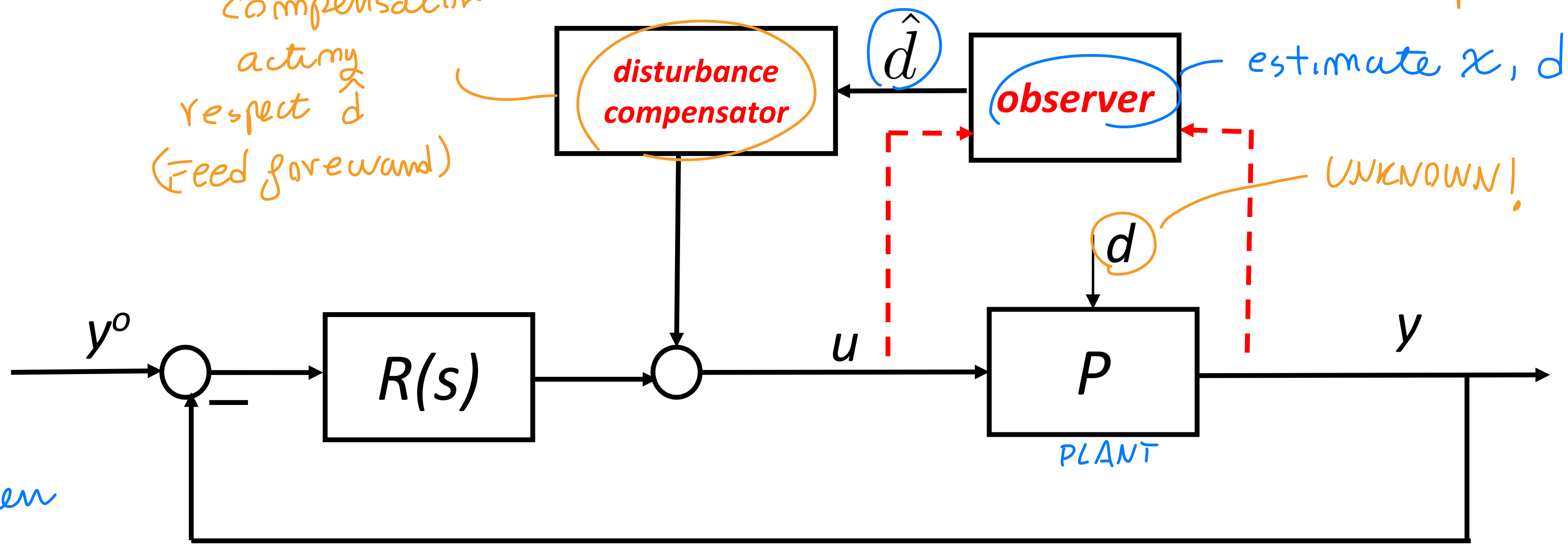
Scheme with $\boxed{\text{compensator}}$

usefull d(t) estimation also to design control scheme with compensation!

The idea is to design the compensator in such a way to force to be null the transfer function from the (estimated) disturbance to the output

state feedback control law + disturbance compensation

compensator acting respect $\hat{d}$ (feed foreward)



on long dynamic system, d(t) effect is seen after long Δt! ⇒ we wanna compensate d(t) to have GOOD result!

disturbance compensator: designed such that $H_{yd} = y/d \simeq 0$ overall → cancel dynamic!
and on practice → usage of compensator is foundamental to compensate external effect → BETTER PERFORMANCE !!

OPEN issues... after we saw the control law to assign feedback system poles! ⇒ BUT you can design observer A.S imposing A-LC

What are the eigenvalues of the closed-loop system?

linear system (S.S)

$$\begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{cases}$$

+

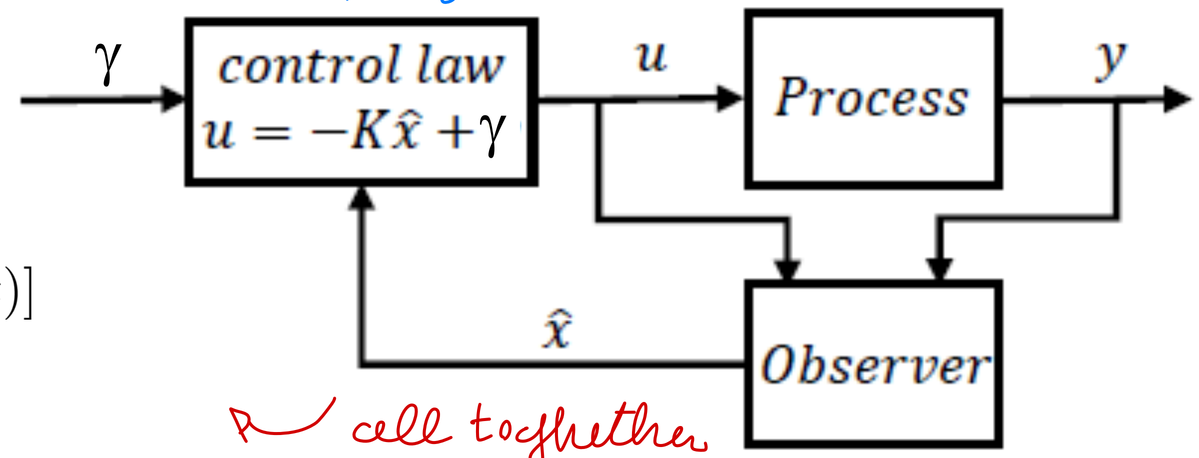
OBSERVER

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$

+

CONTROL LAW state feedback

$$u(t) = -K\hat{x}(t) + \gamma(t)$$



all together OBS + pole placement ⇒ all works well ??!

OVERALL

$$\begin{cases} \dot{x}(t) & = & Ax(t) - BK\hat{x}(t) + B\gamma(t) \pm BKx(t) \\ \dot{\hat{x}}(t) & = & A\hat{x}(t) - BK\hat{x}(t) + B\gamma(t) + L\left[Cx(t) - C\hat{x}(t)\right] \end{cases}$$

[NO u(t) exogenous variable effect!]

$e = x - \hat{x}$ STATE ESTIMATION error

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \gamma(t)$$

$(x - \hat{x})$

L dyn matrix of closed loop syst ⇒ block triangular

The eigenvalues are those of ***A-BK*** and ***A-LC***

Separation principle

You can independently design the state feedback control law and the observer, the eigenvalues are maintained when putting everything together

really important RESULT!

eig values overall are the ones of (A − BK) and (A − LC) OBSERVER

control law
+
observer

even overall mixing the two you can design K, L independently without issue, mantaining eig of design

⇓

SEPARATION PRINCIPLE

mantained eig values

↑ don't hold for non-lin systems! ⇒ more difficult study

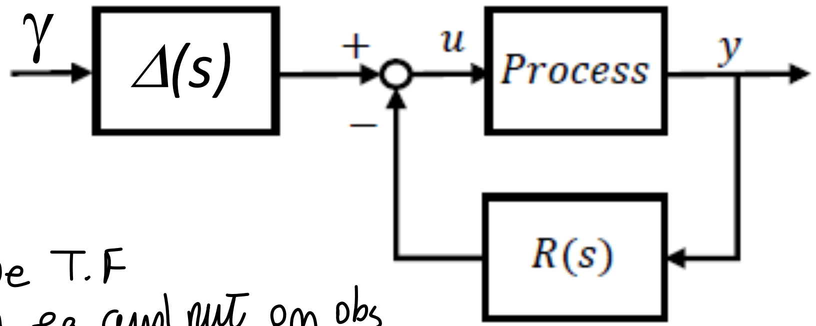
until now all studied reffered to system described on time domain → overall control scheme, CONTROLLER made by OBSERVER (L) + control law (K) in time domain

↳ how to go in T.F forms ⇒

T.F

Regulator transfer function

implementation in BLOCKS (on variable s)



$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y(t) - C\hat{x}(t) - Du(t)\right]$$ OBSERVER

+

$$u(t) = -K\hat{x}(t) + \gamma(t)$$ CONTROLLER

put controller on OBS! (1)

to obtain the T.F
1) take controller eq and put on obs
2) transform obs on s domain
3) go back to controller eq

$$\begin{cases} \dot{\hat{x}}(t) = (A - LC)\hat{x}(t) - (B - LD)K\hat{x}(t) + (B - LD)\gamma(t) + Ly(t) \\ u(t) = -K\hat{x}(t) + \gamma(t) \end{cases}$$

no more function of u(t)

obtain Regulator on time domain, but you can elaborate the equations from y, γ input

Redefine the matrix!

$$\bar{A} = A - LC - BK + LDK, \quad \bar{B} = (B - LD)$$

to write Regulator T.F

$$\begin{cases} \dot{\hat{x}}(t) = \bar{A}\hat{x}(t) + \bar{B}\gamma(t) + Ly(t) \\ u(t) = -K\hat{x}(t) + \gamma(t) \end{cases}$$

Retransform on Laplace domain

Y contribute

γ(t) contribution

$$U(s) = -K(sI - \bar{A})^{-1}LY(s) + \left[-K(sI - \bar{A})^{-1}\bar{B} + I\right]\Gamma(s)$$

$$R(s) = K(sI - \bar{A})^{-1}L$$

$$\Delta(s) = -K(sI - \bar{A})^{-1}\bar{B} + I$$

The overall regulator can be expressed in transfer function form

from time OR laplace representation of overall Regulator is a dynamic system

{ made by n state eq of observer (n ORD) + set of algebraic eq ($\emptyset$ ORD) }

$(R(s) + \Delta(s))$ are of order n

⟹ if $x(t)$ measurable you have only ALGEBRAIC equation of $R(s)$ (ORDER $\emptyset$)

all Reg dynamic comes from (state obs) of order n dyn syst

↓

When you have syst with $x(t)$ unmeasurable and a known OUTPUT transform $y(t) = C x(t)$

↓

we don't have $x(t)$ BUT we know $y(t)$ is a lin combin. of $y(t)$! ⟹ p variable (OUTPUT) as combination of the states

IDEA

{ (Basically) → make a STATE transformation (NOT unique)... such that the output of this transformed variables will coincide with some states !

⇓

so on transformed system → p outputs coincide with the states which are already available coinciding with output

↳ so p states known, I only have to enstimate remaining $n-p$ states !

(OBSERVER) estimating $x(t)$ → because $y(t) = C x(t)$ combination of states by C → by a proper state transform $x(t) = T\tilde{x}(t)$ corresponding to transformed state correspondig to the outputs $y(t)$ → I obtain state

REDUCED ORDER OBSERVER ⟹

sequence of operations to apply! (Focus on mainsteps)

Reduced order observers

OVERALL System

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

Idea: the output is a linear combination of the states

1. Apply a state transformation to the system so that the ***p*** outputs coincide with ***p*** new states
2. Estimate the remaining ***n-p*** *states*

1 transform the states by a suitable state transf Tx

*T_1 any matrix such that **T** is non singular*

(To find T)

$$\tilde{x} = Tx = \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ \tilde{x}_r \end{bmatrix}, \quad \tilde{x}_r \in R^{n-p}$$

T_1 easy to build such that T is not singular

(such that $\tilde{x}$ has first p values corresponding to y)

Transformed system

2 overall syst transform!

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) = \tilde{C}x(t) \end{cases}$$

decomposing matrix. (PARTITIONING of $\tilde{A}$)

KNOWN by measure

$$\dot{\tilde{x}} = \begin{cases} \dot{y}(t) = \tilde{A}_{11}y(t) + \tilde{A}_{12}\tilde{x}_r(t) + \tilde{B}_1 u(t) \\ \dot{\tilde{x}}_r(t) = \tilde{A}_{21}y(t) + \tilde{A}_{22}\tilde{x}_r(t) + \tilde{B}_2 u(t) \end{cases}$$

TO RECONSTRUCT

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}$$

PARTITION

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in R^{p,p}, \quad \tilde{A}_{22} \in R^{n-p,n-p}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{B}_1 \in R^{p,m}, \quad \tilde{B}_2 \in R^{n-p,m}$$

because $y = \tilde{C}\tilde{x}$ only

$y = Cx$

$$\tilde{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

Observer equation on new formulation KNOWN! measurable

$$\begin{cases} \dot{y}(t) = \tilde{A}_{11}y(t) + \tilde{A}_{12}\tilde{x}_r(t) + \tilde{B}_1u(t) \\ \dot{\tilde{x}}_r(t) = \tilde{A}_{21}y(t) + \tilde{A}_{22}\tilde{x}_r(t) + \tilde{B}_2u(t) \end{cases}$$

3 **define** redefine new variables!

$$\begin{cases} \eta(t) = \dot{y}(t) - \tilde{A}_{11}y(t) - \tilde{B}_1u(t) \\ \zeta(t) = \tilde{A}_{21}y(t) + \tilde{B}_2u(t) \end{cases}$$

to have on the right only KNOWN quantities!

4 Reduced order system

Now, design an observer for this system

s.s system form → STANDARD OBS FORM OBSERVER

dyn eq. of STATE KNOWN

$$\begin{cases} \dot{\tilde{x}}_r(t) = \tilde{A}_{22}\tilde{x}_r(t) + \zeta(t) \\ \eta(t) = \tilde{A}_{12}\tilde{x}_r(t) \end{cases}$$

OUTPUT TRANSF.

if (A, C) is observable, $(\tilde{A}_{22}, \tilde{A}_{12})$ is observable

if original syst obs.. same for new one

I can find L IF I can guarantee $(\tilde{A}_{22}, \tilde{A}_{12})$ is OBS, so of eig of OBS well selected

$$\boxed{\dot{\hat{x}}_r(t) = \tilde{A}_{22}\hat{x}_r(t) + \zeta(t) + L\left[\eta(t) - \tilde{A}_{12}\hat{x}_r(t)\right]}$$

or (equivalent) ||| equivalent

$$\boxed{\dot{\hat{x}}_r(t) = \left(\tilde{A}_{22} - L\tilde{A}_{12}\right)\hat{x}_r(t) + \tilde{A}_{21}y(t) + \tilde{B}_2u(t) + L\eta(t)}$$

IF this is asymp stable you will be able to solve your problem

Problem: η contains the derivative of **y**, not wise from a numerical point of view

I have an OBS of order $\hat{x}_r \to (n-p)$ obtained by my OBS design

ISSUE...

on $\dot{\hat{x}}_r(t)$ we treat $\underbrace{y(t)}$ as OUTPUT

BUT it is measurable depending on $\dot{y}(t)$

BAD dependency on numerical P.O.V

$\Downarrow$

We wanna remove the derivative dependency (Bad numerically)

remove $\dot{y}$!

from what I found before

$$\dot{\hat{x}}_r(t) = \left(\tilde{A}_{22} - L\tilde{A}_{12}\right)\hat{x}_r(t) + \tilde{A}_{21}y(t) + \tilde{B}_2u(t) + L\eta(t)$$

$$\eta(t) = \dot{y}(t) - \tilde{A}_{11}y(t) - \tilde{B}_1u(t)$$

put $\dot{y}$ on the left hand side of the equation

$$\dot{\hat{x}}_r(t) - L\dot{y}(t) = (\tilde{A}_{22} - L\tilde{A}_{12})\hat{x}_r(t) + (\tilde{A}_{21} - L\tilde{A}_{11})y(t) + (\tilde{B}_2 - L\tilde{B}_1)u(t)$$

sum and subtract $(\tilde{A}_{22} - L\tilde{A}_{12})Ly(t)$

define $\xi(t) = \hat{x}_r(t) - Ly(t), \quad \xi \in R^{n-p}$

Final reduced order observer

$$\boxed{\dot{\xi}(t) = (\tilde{A}_{22} - L\tilde{A}_{12})\xi(t) + (\tilde{A}_{21} - L\tilde{A}_{11} + \tilde{A}_{22}L - L\tilde{A}_{12}L)y(t) + (\tilde{B}_2 - L\tilde{B}_1)u(t)}$$

on state $\xi(t)$ and input $y(t), u(t)$ (STANDARD observer) to estimate ξ

then from ξ estimation I go back on $\hat{x}$ estimation!

$$\hat{x}_r(t) = \xi(t) + Ly(t)$$

from ξ definition!

$$\boxed{\hat{x}(t) = T^{-1}\begin{bmatrix} y(t) \\ \hat{x}_r(t) \end{bmatrix}}$$

(EXAM question about reduced observer and simple examples)

Example

Given the system

Design a reduced order observer for the system

using general steps:

1) we should tranform the state so that substate of output are states

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases} \qquad A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \quad C = \begin{vmatrix} 1 & 0 \end{vmatrix}$$

equation of II state dynamic

we have only to estimate x_2

One state already coincides with the output. There is no need of state transformation ($\tilde{x} = Tx$) $\underbrace{T = I_2}$

we don't need to modify A, B, C

According to the general Theory $\boxed{\eta = \dot{y} = x_2}$

$\dot{y} = x_2 = \eta$

output transformm

The observer can be written as $\Longrightarrow$

The observer can be written as

GAIN

$$\dot{\hat{x}}_2 = u + L(\dot{y} - \hat{x}_2) \mp L^2 y$$

estimation equation for STATE ESTIMATION

by sum / subtract

$$\underbrace{\dot{\hat{x}}_2 - L\dot{y}}_{\dot{\xi}} = -L\underbrace{(x_2 - Ly)}_{\xi} - L^2 y + u$$

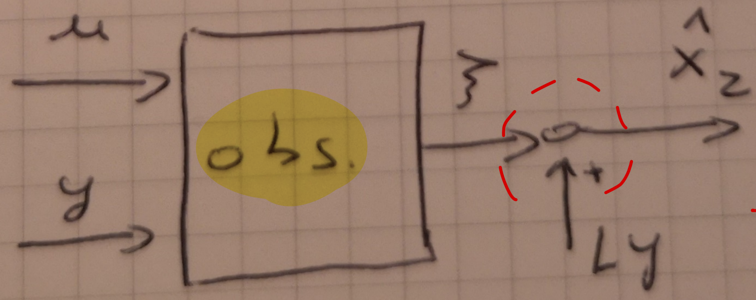
re-define, new variable ξ

$$\boxed{\dot{\xi} = -L\xi - L^2 y + u}$$

dyn. equation of the OBSERVER

following the theoretical steps

inputs u, y



$$\boxed{\hat{x}_2 = \xi + Ly}$$

to estimate $\hat{x}_2$

to come back

use of an OBSERVER to estimate a disturbance $d(t)$

Exercise

additive input acting on $u(t)$

Given the system $\xrightarrow{u} \circ \xrightarrow{} \boxed{\frac{1}{s+1}} \xrightarrow{}$, with d entering at $\circ$

I ORDER system

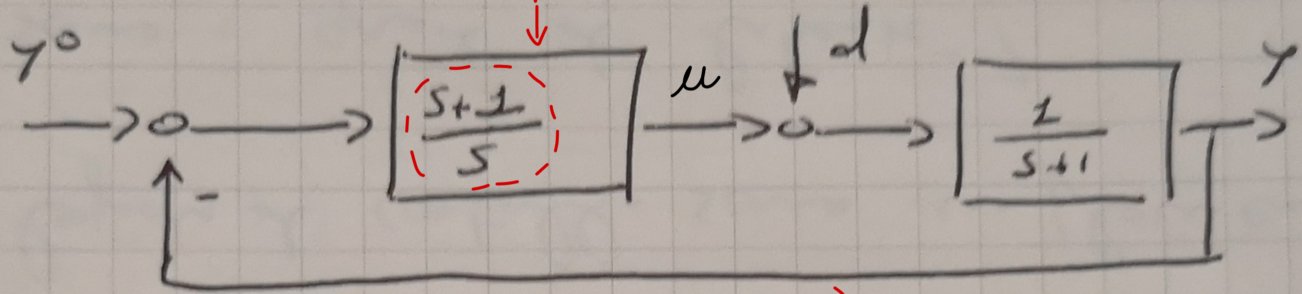
A) design a PI regulator

$d(t) \simeq \bar{d}$

B) Design an estimator of the disturbance, assumed to be constant and a compensator acting on the estimated value of d

Solution A) simple PI regulator design

zeros cancel the T.F pole



hidden pole in (-1)

$$y(s) = \frac{1}{s+1} y^o + \frac{s}{(s+1)^2} d$$

both poles visible

(note that due to the integrator in $R(s)$ constant disturbances are asymptotically rejected)

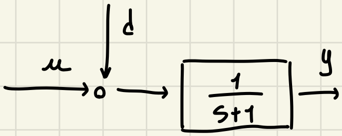
$L(s) = \frac{1}{s}$ so $\varphi_m = 90°$

with this regulator... nice T.F y/y^o

while y/d has two poles in (-1)

+ Derivative action from integrator $\Rightarrow$ steady state

$d(t) \to 0$ effect (derivative) error goes to 0 asymp



$$sY + Y = U + d$$

$$\dot{y} + y = u + d$$

Solution B

$d(t)$ can be estimated? and can we build a COMPENSATOR for $\hat{d}(t)$ (integrator already remove $\bar{d}$, but we want estimate $d(t)$)

The system is described in the state space by

$$\begin{cases} \dot{x} = -x + u + d \\ \dot{d} = 0 \\ y = x \end{cases}$$

model equation

disturbance equation → to estimate it, we assume a ficticious dyn. $\dot{d} = 0$ (d_0 unkn) ← fictitious dynamics

OUTPUT transform   output = state : We don't need $x(t)$ estimation

We can use a reduced order observer to estimate d

According to the theory, write the system as

$$\begin{cases} \dot d = 0 \\ \underbrace{\dot x + x - u}_{\gamma} = d \end{cases} \Longrightarrow \quad \text{ESTIMATOR} \quad \boxed{\dot{\hat d} = L\,[\underbrace{\dot y + y - u}_{\gamma} - \hat d\,]}$$

equation re-written

derivative of y on right → undesired!

$$\dot{\hat d} = L\dot y + Ly - Lu - L\hat d$$

⇓ new variable ξ

$$\underbrace{\dot{\hat d} - L\dot y}_{\dot \xi} = -L\hat d + Ly - Lu$$

$$\downarrow \; \mp L^2 y$$

$$\dot \xi = -L\,[\underbrace{\hat d - Ly}_{\xi}] - L^2 y + Ly - Lu$$

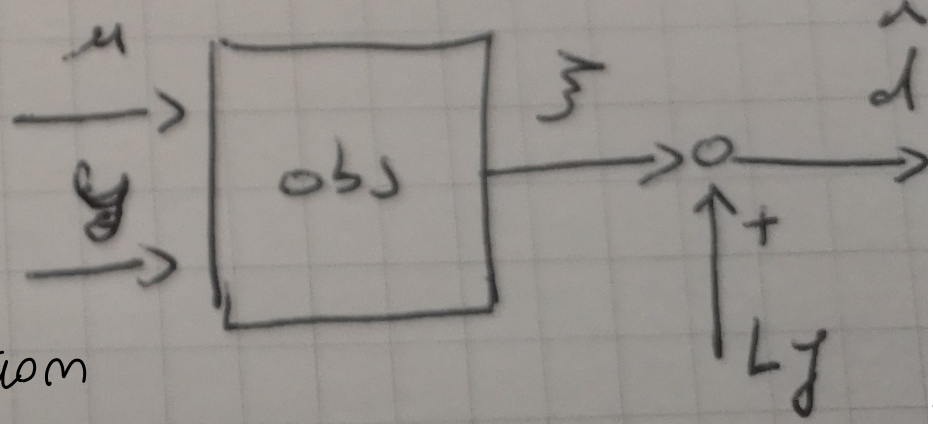
new OBSERV. of order I, to build for ξ ⇒

$$\underbrace{\dot{\hat{d}} - L\dot{y}}_{\dot{\xi}} = -L\hat{d} + Ly - Lu$$

$$\downarrow \mp L^2 y$$

$$\dot{\xi} = -L\underbrace{[\hat{d} - Ly]}_{\xi} - L^2 y + Ly - Lu$$

$$\boxed{\dot{\xi} = -L\xi - L^2 y + Ly - Lu}$$

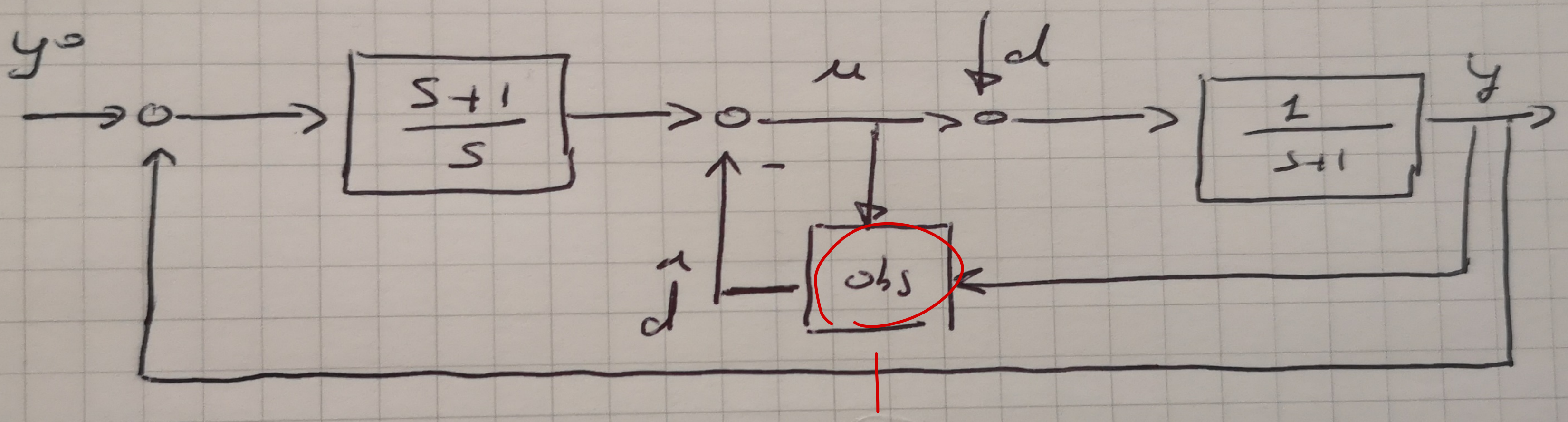


OBSERVER: given u, y compute ξ

by adding $Ly + \xi \rightarrow \hat{d}$ estimation by cancel the dynamic y/d using $\hat{d}$!

Scheme with disturbance compensation

LOT used for Better performance



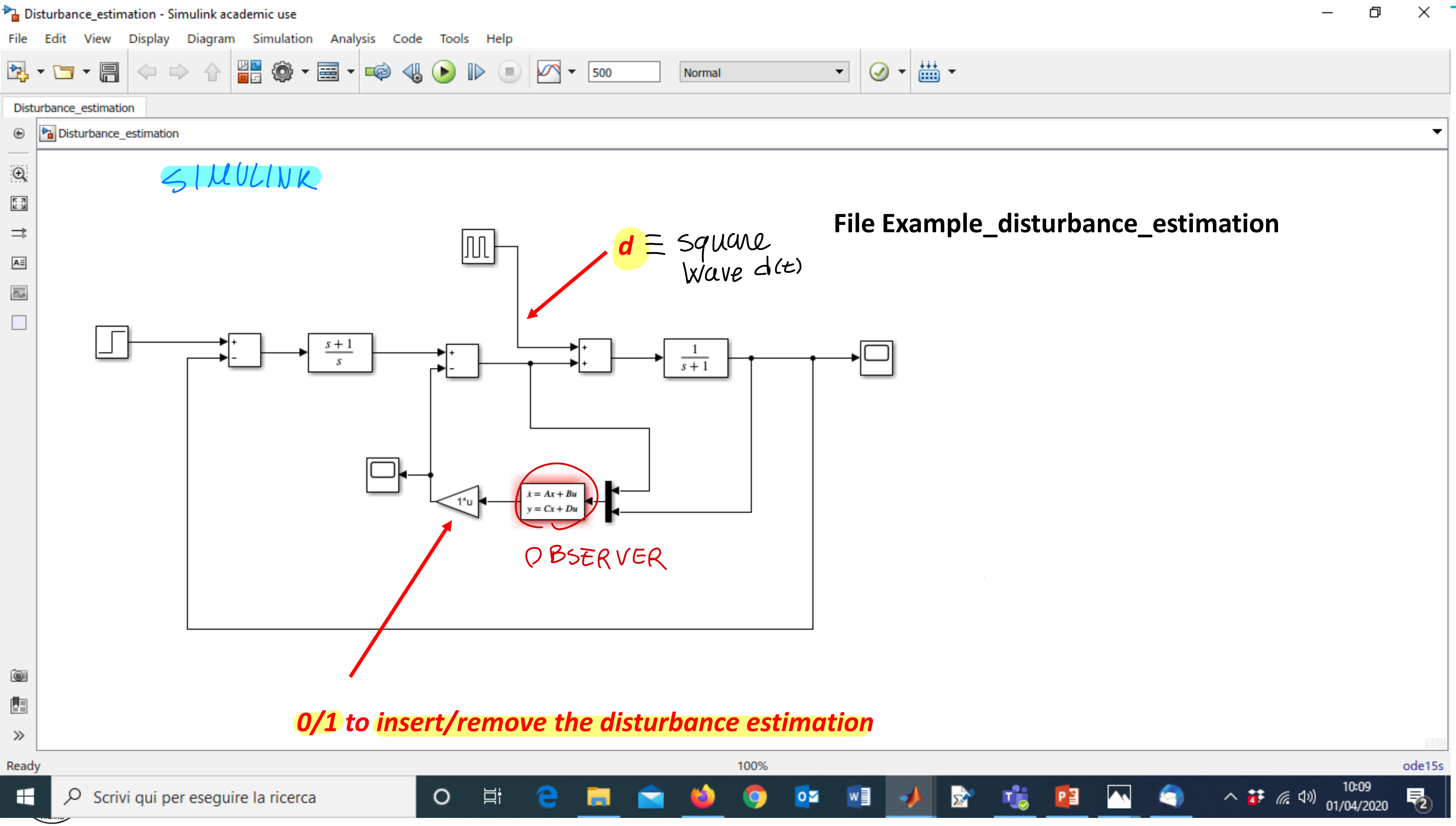
file Example_disturbance_estimation

{ Perfect compensation }

to cancel the effect of $d(t)$ on $y(t)$ through $\frac{1}{s+1}$, also $\hat{d}$ goes through $\frac{1}{s+1}$

once build OBS, we want a control scheme with compensation

so just subtract $\hat{d}$ to u → ideally it cancel the d contribution


Disturbance_estimation - Simulink academic use
File Edit View Display Diagram Simulation Analysis Code Tools Help
500
Normal
Disturbance_estimation
SIMULINK
File Example_disturbance_estimation
d = square wave d(t)
$\frac{s+1}{s}$
$\frac{1}{s+1}$
1*u
$\dot{x} = Ax + Bu$
$y = Cx + Du$
OBSERVER
0/1 to insert/remove the disturbance estimation
Ready
100%
ode15s
Scrivi qui per eseguire la ricerca
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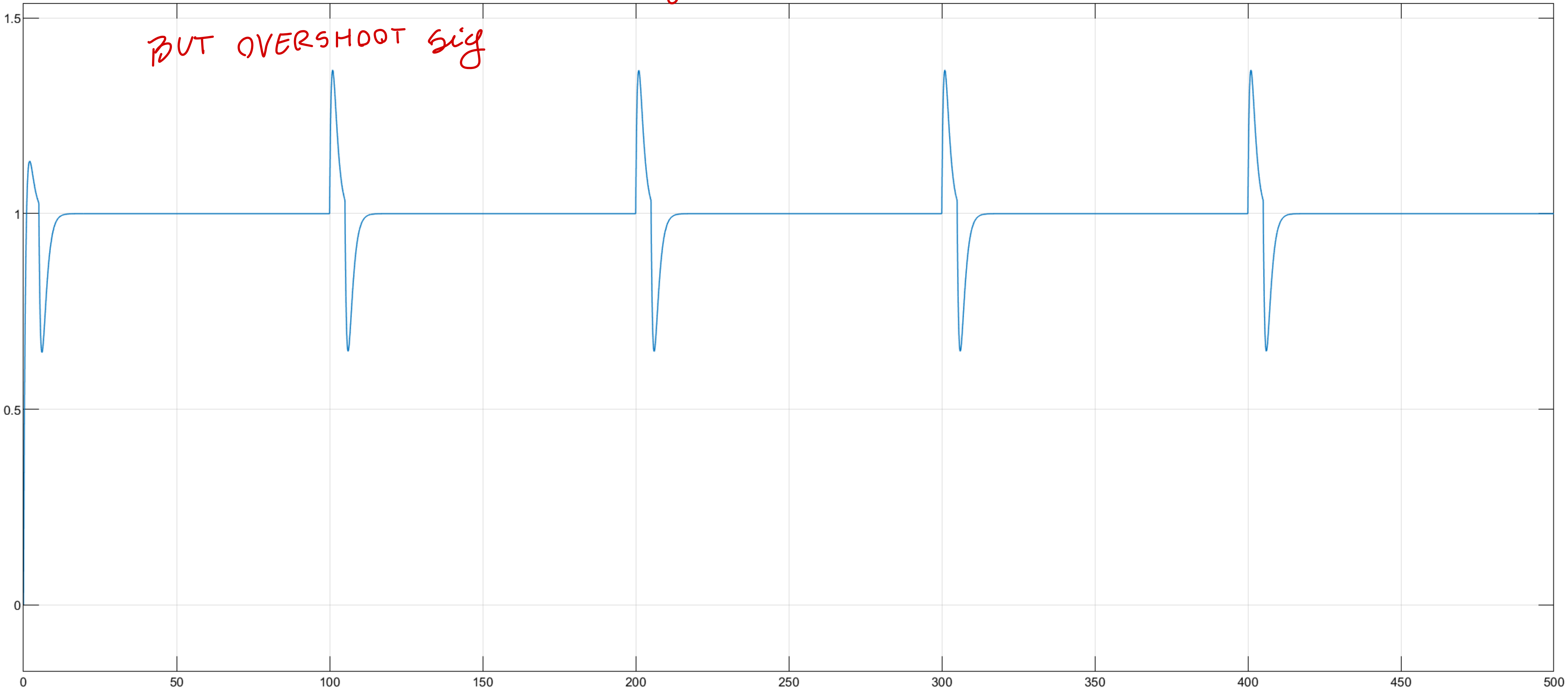
No disturbance compensation

effect of d(t) during t ! ⇒ perturbation

you reject asymp disturbance effect for each period

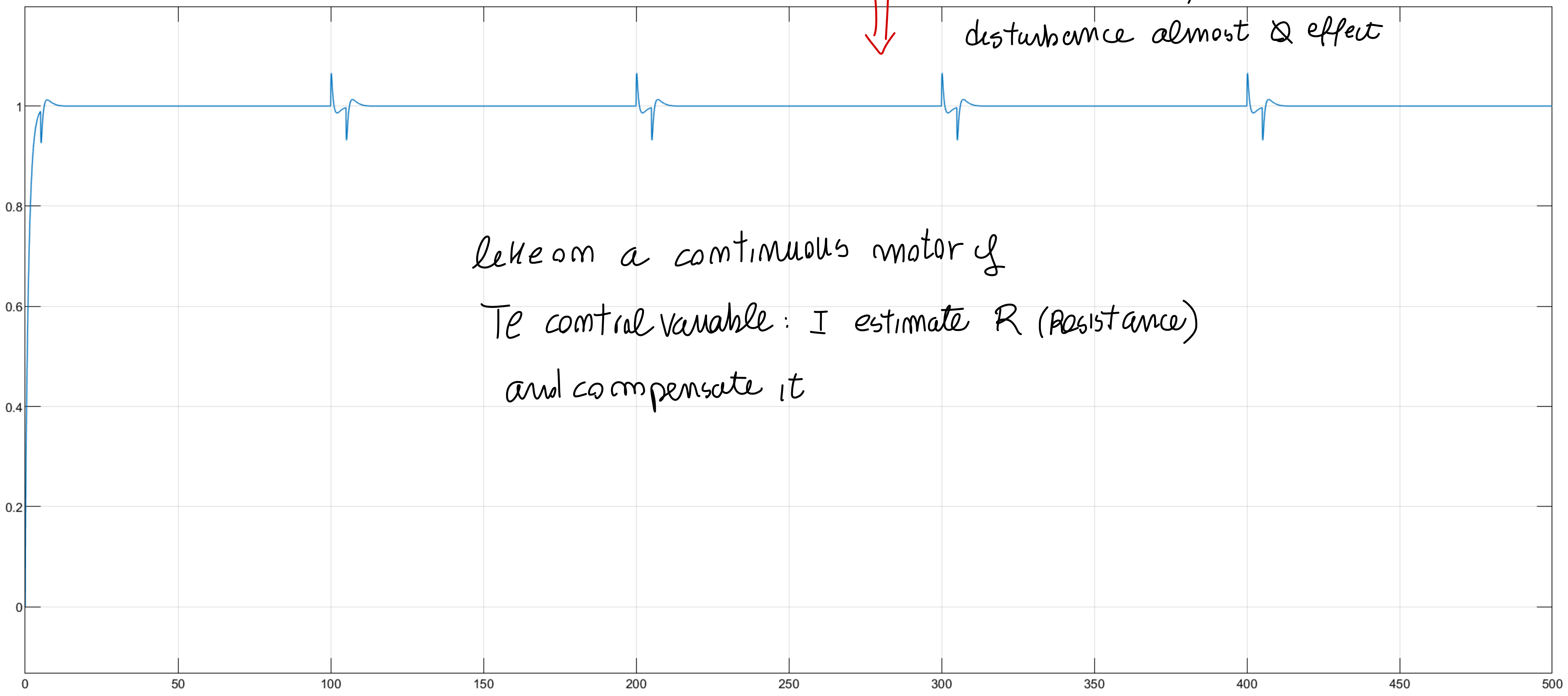
BUT OVERSHOOT sig



Disturbance estimation and compensation

With OBS + compensation

smaller overshoot,
disturbance almost ∅ effect



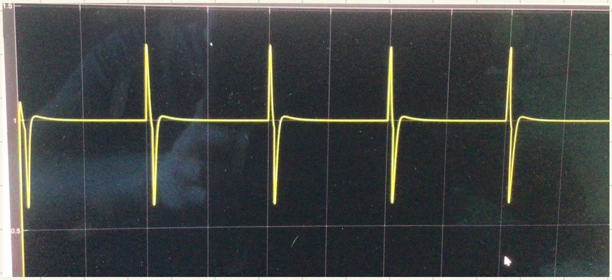
NOTICE on the estimator the eig are $(-L)$ values!

on simulation: for $L=10$: OBS has eig on -10...

- IF reducing $L \downarrow \Rightarrow$ eig becomes slow, so $\hat{d} \to d(t)$ in much longer time

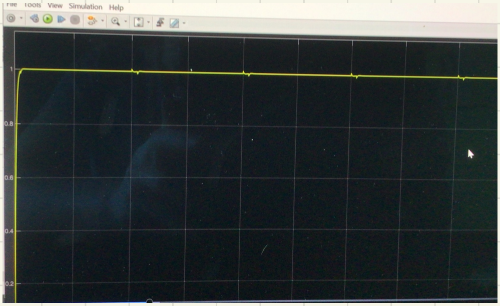
$\Downarrow$

for $L=0.1$ on estimator by SIMULINK we got a bad Result. ! too slow



SLOW OBSERVER $L=0.1$

for FASTER OBS. $L=100$



for FAST OBS you have $\hat{d}-d \to$ remove properly disturb

to select eig value $\Rightarrow$ OBS can be FAST,
but if you have OUTPUT noise,
the OBS make filtering action...

Without noise on $y(t)$ is better a fast OBS.

Extend the results for Discrete Time

easy estension, But take care of OBS. form!

Discrete time systems – measurable state

System (on S.S Form)

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), & x \in R^n, \quad u \in R^m \\ y(k) = Cx(k) + Du(k), & y \in R^p \end{cases}$$

Control law

state feedback control law to guarantee asymp. stable eig value of closed loop syst

$$u(k) = -Kx(k) + \gamma(k), \quad K \in R^{m,n}, \quad \gamma \in R^m$$

ALGEBRAIC control law (we know $x(k)$)

$\gamma(k)$ additional signal

Closed-loop

$$x(k+1) = (\boxed{A - BK})x(k) + B\gamma(k)$$

(same as continuous time) → $(A-BK)$ must have prescribed eig value!

The problem is exactly the same, the same algorithms can be used ($\underbrace{\textit{Ackermann's formula}}_{\text{single input systems}}$, $\underbrace{\textit{place.m}}_{\text{more inputs}}$,…)

same Algorithm

small difference with cont. time

Observers (already seen on MIDA)

↓

Two possibilities: (of design)

1. ***State predictor***: the estimated state at time *k* depends on the values of *u* and *y* up to time *k-1*
   @ $t = k$: you estimate $x(k)$ depending to u, y up to $k-1$

2. ***State filter***: the estimated state at time *k* depends on the values of *u* and *y* up to time *k*
   @ $t = k$: you estimate $x(k)$ using even the value of u, y up to k
   ⇒ most updated state estimation... But you can have implementation issue

1 **State predictor** (same as continuous time)

state estimate @ k+1, given data up to k (same as c.t observer!)

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L[y(k) - C\hat{x}(k|k-1) - Du(k)]$$

By simulating system dym. Gain * (difference of estimation)

state
Estimation error $\hat{e}(k|k-1) = x(k) - \hat{x}(k|k-1)$

dynamic of state estim error

$$\hat{e}(k+1|k) = (A - LC)\hat{e}(k|k-1)$$

you need L such that eig (A-LC) inside unit circle!

Also in this case the problem is to compute the gain ***L*** which assignes the eigenvalues of ***A-LC*** in prescribed positions (so $\hat{e} \to 0$, so $\hat{x} \to x$ @ regime)

observer such that

Deadbeat observers: all the eigenvalues of ***A-LC*** are at the origin. The state estimation error goes to zero in at most *n* steps (FASTEST possible observer in discrete time)

(sometimes NOT too fast or noise issue)

so you can design observer using pole placement algorithm to have (L)

2 **State filter** ↴ care about filter structure

$$\tilde{x}(k+1|k+1) = \overbrace{A\tilde{x}(k|k) + Bu(k)}^{\text{standard part: syst dyn}} \quad \text{state evolution of } \tilde{x}(k+1)$$
$$+ \underset{\text{GAIN}}{L}\underbrace{[y(k+1) - C(A\tilde{x}(k|k) + Bu(k)) - Du(k+1)]}_{\text{to use } y(k+1)}$$

given available measurable! you estimate $\tilde{x}(k+1)$

Estimation error $\tilde{e}(k|k) = x(k) - \tilde{x}(k|k)$

from system dyn eq. I obtain... → $\tilde{e}(k+1|k+1) = (\boxed{A - LCA})\tilde{e}(k|k)$ linear system without forcing

The gain ***L*** must be designed to assign the eigenvalues of (***A-LCA***). To this end, the pair (***A,CA***) must be observable

(so $\tilde{e} \to 0$) different condition!

{ to assign for free the eig of that system you have to guarantee obs of the pair!

- The pair (A, CA) is observable iff the pair (A, C) is observable and A is nonsingular ← (CNS)

{ if $\tilde{C} = CA \to (A - L\tilde{C})$ same problem! so using standard Algorithm of pole placement }

(NOT practical PROBLEM)

Note however that if ***A*** is singular, the system is detectable, since the nonobservable eigenvalues are at the origin, i.e. the fastest possible ones. In other words, leave them where they are (algorithms in case of detectability are only slightly more complex)

{ ERROR to apply Ackerman... you should partition the system }

you cannot moove the eig values of A which are @ ORIGIN
{ NON OBS eig Values! ⤳ NOT a real problem, you cannot moove them but they have an evolution → 0

Estimation of constant disturbances

Assume to know that ***d*** is constant, or at least constant for long periods of time.
It is useful to estimate it for two reasons:

(otherwise $\tilde{e}(t) \not\to 0$ ERROR estimation)

1) It is possible to correctly estimate the state of the system (if you have d(t), you can estimate them for a proper overall state estimation)
you need for good estimate

2) The estimate of the disturbance can be used in a control scheme with direct compensation
estimate $\hat{d}$ so you build feed forward compensator

Procedure (similar to continuous time)

1. Assign a dynamics to the disturbance (fictitious dyn.) on discrete time $d(k+1)=d(k)$ (constant)

$$d(k+1)=d(k), \quad d(0)=\bar{d}\in R^r$$

2. Enlarge the system

provided that the pair $(\bar{A},\bar{C})$ is OBS. you can estimate $\begin{pmatrix} x \\ d \end{pmatrix}$ enlarged state!

$$\begin{cases} \begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} A & M \\ 0 & I \end{bmatrix}}^{\bar{A}} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) & \text{state eq. + Disturb} \\ y(k) = \underbrace{\begin{bmatrix} C & N \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + Du(k) & \text{output eq.} \end{cases}$$

by PBH test.. condition request (A,C) OBS, and if $r := \#$ disturb $d\in\mathbb{R}^r \Rightarrow r\le p$, $y\in\mathbb{R}^p$ and invariant zero of y/d are not in point (1)

Use an observer for the enlarged system. When is it possible? Try to find the $\boxed{\text{condition}}$

also in this case we can design reduced order for FILTER / PREDICTOR, they are of order n → we wanna find an obs of order (n-p) → usefull to have a FASTER OBSERV

Reduced order observer

Idea: the output is a linear combination of the states

1. Apply a state transformation to the system so that the ***p*** outputs coincide with ***p*** new states
2. Estimate the remaining ***n-p states***

$$\begin{cases} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k) \end{cases}$$

same trick of cont. time

1

T_1 any matrix such that T is non singular

$$\tilde{x} = Tx = \begin{bmatrix} C \\ T_1 \end{bmatrix} x = \begin{bmatrix} y \\ \tilde{x}_r \end{bmatrix}, \quad \tilde{x}_r \in R^{n-p}$$

so that outputs coincide with some x

available (measurable)

← to estimate

Transformed system

2

$$\begin{cases} y(k+1) &= \tilde{A}_{11}y(k) + \tilde{A}_{12}\tilde{x}_r(k) + \tilde{B}_1u(k) \\ \tilde{x}_r(k+1) &= \tilde{A}_{21}y(k) + \tilde{A}_{22}\tilde{x}_r(k) + \tilde{B}_2u(k) \end{cases}$$

output dyn

to estimate dyn

No derivative here!

$$\tilde{A} = TAT^{-1},\ \tilde{B} = TB,\ \tilde{C} = CT^{-1}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in R^{p,p}, \quad \tilde{A}_{22} \in R^{n-p,n-p}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \tilde{B}_1 \in R^{p,m}, \quad \tilde{B}_2 \in R^{n-p,m}$$

$$\tilde{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

3 Define

on right what we know, on left the unknown

$$\begin{cases} \eta(k+1) = y(k+1) - \tilde{A}_{11}y(k) - \tilde{B}_1u(k) \\ \zeta(k) = \tilde{A}_{21}y(k) + \tilde{B}_2u(k) \end{cases}$$

$$\begin{cases} y(k+1) = \tilde{A}_{11}y(k) + \tilde{A}_{12}\tilde{x}_r(k) + \tilde{B}_1u(k) \\ \tilde{x}_r(k+1) = \tilde{A}_{21}y(k) + \tilde{A}_{22}\tilde{x}_r(k) + \tilde{B}_2u(k) \end{cases}$$

$$\begin{cases} \tilde{x}_r(k+1) = \tilde{A}_{22}\tilde{x}_r(k) + \zeta(k) \\ \eta(k+1) = \tilde{A}_{12}\tilde{x}_r(k) \end{cases}$$

STATE EQ

OUTPUT transform

New system form

observer

4

$$\hat{x}_r(k+1|k+1) = \tilde{A}_{22}\hat{x}_r(k|k) + \zeta(k) + L\left[\eta(k+1) - \tilde{A}_{12}\hat{x}_r(k|k)\right]$$

by substituting η, ζ

or

$$\hat{x}_r(k+1|k+1) = \left(\tilde{A}_{22} - L\tilde{A}_{12}\right)\hat{x}_r(k|k) + \tilde{A}_{21}y(k) + \tilde{B}_2u(k)$$
$$+ L[y(k+1) - \tilde{A}_{11}y(k) - \tilde{B}_1u(k)]$$

{simple to implement}

this act like a sort of FILTER, the estimated @ k+1 uses data up to k+1 (NOT PREDICTOR!)

OBS eig values assigned by L, depending on request

L assign eig value of OBS by fixing L

|| **No problems with derivatives** || !

Regulator transfer function with state predictor

Which one to use? T.F with PREDICTOR or FILTER?!
IMPLEMENTATION aspect to choose between

System
$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases}$$

using the PREDICTOR

Control law
$$u(k) = -K\hat{x}(k|k-1)$$

Predictor
$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + L\left[y(k) - C\hat{x}(k|k-1)\right]$$

Regulator T.F ⟹

#ROWS = #INPUT

considered as SISO system!

$$U(z) = \boxed{-K(zI - A + BK + LC)^{-1}L}\,Y(z)$$

same form of cont time

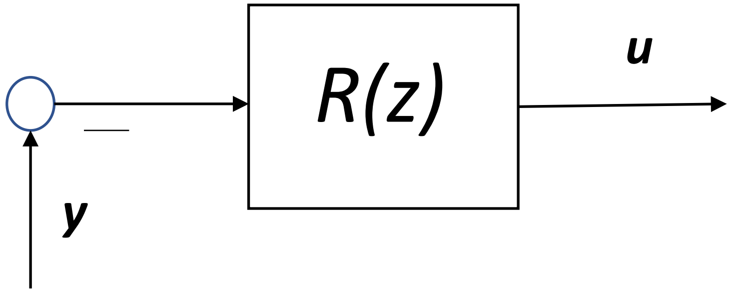
control law + Predictor equation

manipulation...

strictly proper transfer function R(z)

(more poles than zeros)

take $u(k)$ put on $\hat{x}(k+1|k)$ → transform on z domain, than reevaluate $U(z)$



→ R(z) classical SISO T.F

Regulator transfer function with state filter

System

$$\begin{cases} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k) \end{cases}$$

Control law

$$u(k) = -K\tilde{x}(k|k)$$

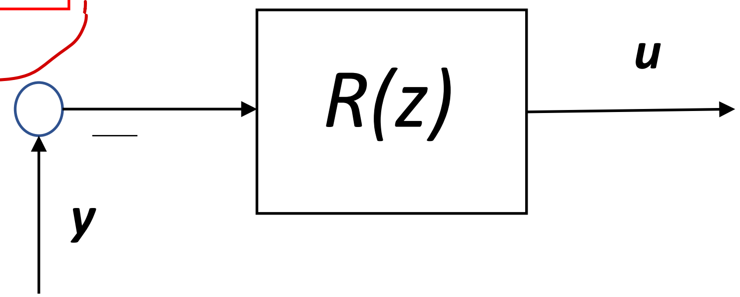
Filter

$$\begin{aligned} \tilde{x}(k+1|k+1) &= A\tilde{x}(k|k)+Bu(k) \\ &+ L\left[y(k+1) - C\left(A\tilde{x}(k|k)+Bu(k)\right)\right] \end{aligned}$$

Regulator T.F computed as before... you obtain a T.F matrix given by a proper T.F

$$U(z) = -\boxed{K(zI - A + BK + LCA + LCBK)^{-1}}\,L z Y(z)$$

proper transfer function R(z)

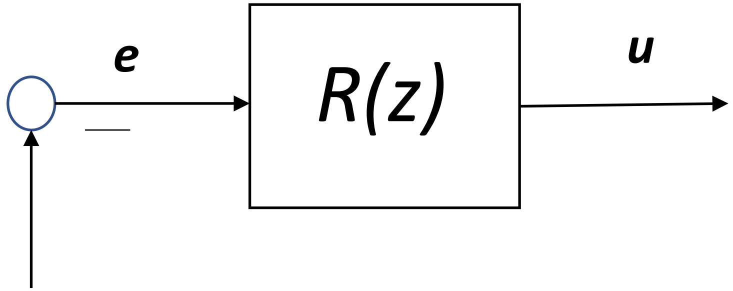


NOT strictly proper due to z! same poles-zeros number!

WHAT HAPPENS?

Strictly proper regulator transfer function

on SISO case (for simplicity)



$$R(z) = \frac{b_{n-1}z^{n-1}+\ldots+b_1z+b_0}{z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0}$$

to implement it by a suitable algorithm

$$\boxed{U(z) = R(z)E(z)}$$

$$\Downarrow$$

$$(z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0)U(z) = (b_{n-1}z^{n-1} + \ldots + b_1z + b_0)E(z))$$

z time shift transform

$$u(k) = -a_{n-1}u(k-1) - \ldots - a_1u(k-n+1) - a_0u(k-n) + b_{n-1}e(k-1) +$$
$$\ldots + b_1e(k-n+1) + b_0e(k-n)$$

to compute u(k) by strictly proper T.F.. u(k) depends on u, e up to (k-1)! ⇒ you have one sampling time Δt to compute u(k)

you can make comput. in one sampling time k=1 to estimate x(k) update and use it on feedback control law if you use an OBSERVER!

One sampling time to read the value of the error (A/D), make computations, write the input u (D/A)

… but less reactive

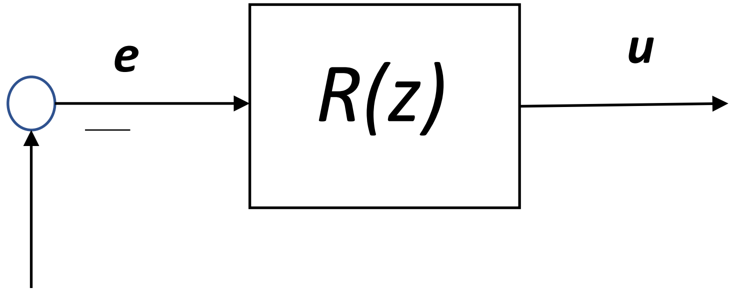
While!

Proper regulator transfer function

same number of poles/zeros

FILTER is more accurate, use updated value.. BUT if introducing too much delay → CRITICAL!

NOT relevant if high τ system, but very problematic for FAST systems!



$$R(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}$$

same computation

$$U(z) = R(z)E(z)$$

$$(z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0)U(z) = (b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0)E(z))$$

$$u(k) = -a_{n-1}u(k-1) - \ldots - a_1 u(k-n+1) - a_0 u(k-n) + b_n e(k) + b_{n-1} e(k-1) + \ldots + b_1 e(k-n+1) + b_0 e(k-n)$$

depends on e(k), so in k=0,

on 0 time I should read the error and compute u(k) → apply it!

More reactive, but introduces a delay due to A/D+computations+D/A

⇓

NOT possible, you need computation time... (issue) delay → phase lag critical for implementation