



## Exercises session 6: LQ control, KF, LQG, loop transfer recovery

**Ex. 1:** Given the continuous time system

$$\begin{cases} \dot{x}(t) = -x(t) + u(t) \\ y(t) = x(t) \end{cases} \quad (1)$$

and the Riccati Differential Equation

$$\dot{P}(t) + A^T P(t) + Q - P(t)BR^{-1}B^T P(t) + P(t)A = 0 \quad (2)$$

1. Find the  $LQ_\infty$  control law with  $Q = 1$ ,  $R = 1$ .
2. Find the corresponding closed-loop poles, the closed loop T.F., the maximum gain variation and evaluate the phase margin.
3. Design a steady-state Kalman Filter with  $\tilde{Q} = \rho^2$ ,  $\tilde{R} = 1$ .
4. Compute the overall LQG regulator T.F.
5. Show how to apply the loop transfer recovery procedure (LTR).

**Ex. 2:** Given the system

$$\dot{x}(t) = 0.5x(t) + u(t) \quad (3)$$

1. Find the  $LQ_\infty$  control law with  $Q = 1$ ,  $R = 1$ .
2. Find the corresponding closed-loop poles.
3. Given  $u(t) = -\rho K_{LQ}x(t)$ , find the set of  $\rho$  for which the closed loop system is A.S.
4. Find the phase margin.
5. Which is the maximum time-delay that allows to maintain the asymptotic stability?
6. Enforce a closed loop pole faster than  $s = -2$

**Ex. 3:** Given discrete time system

$$\begin{cases} x(k+1) = -x(k) + u(k) + \nu_x(k) \\ y(k) = x(k) + v_y(k) \end{cases} \quad (4)$$

where  $\nu_x \sim WGN(0, \tilde{Q})$ ,  $\nu_y \sim WGN(0, \tilde{R})$

1. Find the  $LQ_\infty$  control law with  $Q = 1$ ,  $R = 2$ .
2. Compute the maximum gain variation allowed by the  $LQ_\infty$ .
3. Design a Kalman Filter using  $\tilde{Q} = 2$ ,  $\tilde{R} = 1.5$ .
4. Compute the regulator T.F.
5. Compute the closed loop poles.

The steady-state Riccati equation for discrete time systems is

$$\bar{P} = Q + A^T \bar{P} A - A^T \bar{P} B (R + B^T \bar{P} B)^{-1} B^T \bar{P} A \quad (5)$$

**Ex. 4:** Given the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ y = x_2 \end{cases} \quad (6)$$

1. Design a  $LQ_\infty$  control law with
- $$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (7)$$
- and  $R = 1$ .
2. Compute the closed loop poles.
  3. Design a Kalman Filter and apply the LTR procedure.

**Ex. 4:** Given the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ y = x_2 \end{cases}$$

1. Design a  $LQ_\infty$  control law with

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and  $R = 1$ .

2. Compute the closed loop poles.
3. Design a Kalman Filter and apply the LTR procedure.

... complete

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \quad u = -\sqrt{2} x_1 - x_2 = -K_{LQ} x$$

(2)

$$\dot{x} = Ax + Bu = Ax - BK_{LQ}x = \underbrace{(A - BK_{LQ})x}_{\downarrow}$$

$$A - BK_{LQ} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix}$$

$$\varphi(s) = \det(sI - (A - BK_{LQ})) = \det \begin{bmatrix} s + \sqrt{2} & 1 \\ -1 & s \end{bmatrix} = s^2 + \sqrt{2}s + 1$$

$$s_{1,2} = \frac{-\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$$

Re part neg! stable

solving this polynomial we find the poles...

### ③ K.F + LTR

$$\tilde{R} = 1 \quad \tilde{Q} = \rho B \tilde{R} B^T \quad \text{helps to recover stability of LQ control}$$

$$= \rho \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \text{ Diagonal} \Rightarrow \tilde{B}_q = \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & 0 \end{bmatrix}$$

to design a K.F we should check the conditions... I can take  $\tilde{B}_q$  from  $\tilde{Q}$

**CONDITIONS FOR K.F)**

- 1.  $(A, C)$  OBSERVABLE ✓
- 2.  $(A, \tilde{B}_q)$  REACHABLE ✓

( checking  
OBS, REACH. matrix )

$$M_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \det(M_0) \neq 0 \quad \text{clearly Full Rank} \quad (A, C) \text{ OBS}$$

$$M_R = \left[ \begin{array}{cc|cc} \tilde{B}_q & A \tilde{B}_q & 0 & 0 \\ \hline 0 & 0 & \sqrt{\rho} & 0 \end{array} \right] \quad \text{rank}(M_R) = 2 = \text{syst ORD}$$

OK REACHABLE

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & 0 \end{bmatrix}$$

OK → We can apply K.F

$$\dot{\hat{x}} = A \hat{x} + Bu - [L_{KF}] (y - \hat{y})$$

solved by K.F Riccati eq.

$$\text{R.E.(KF): } A\tilde{P} + \tilde{Q} - \tilde{P}C^T \tilde{R}^{-1} C \tilde{P} + \tilde{P}A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We need to define a parametric symm.  
SILVESTER CRITERION } to ensure pos. def.

$$\tilde{P} = \begin{bmatrix} \alpha & B \\ B & \gamma \end{bmatrix}$$

... Computation ...

$$\tilde{P} = \begin{bmatrix} \sqrt{2} \sqrt[4]{\rho^3} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2} \sqrt[4]{\rho} \end{bmatrix}$$
$$L_{KF} = \tilde{P} C^\top \tilde{R}^{-1} = \begin{bmatrix} \sqrt[4]{\rho^3} & " \\ " & \sqrt[4]{\rho} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} f^{-1} = \begin{bmatrix} \sqrt{\rho} \\ \sqrt{2} \sqrt[4]{\rho} \end{bmatrix}$$