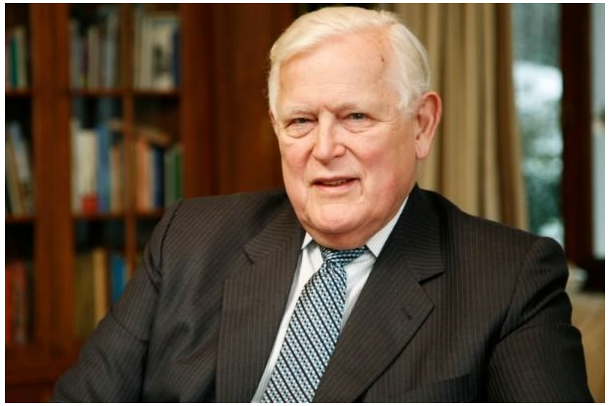
KALMAN FILTER

(continuous time)



Advanced and Multivariable Control

Kalman Filter

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Given the System → deterministic system, descibed by state equation, affected by stocastic noises

+ white Gaussian noise

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_x(t) \\ y(t) = Cx(t) + v_y(t) \end{cases}$$

← usual output transform with white noise affecting noise

v_x, v_y necessary, overall disturbance $d(t)$ due to v_x, v_y noise acting!

Disturbances

$$v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \xrightarrow{\text{white gaussian noise}} E\left[v(t)\right] = 0, \quad E\left[v(t_1)v(t_2)'\right] = V\delta(t_1 - t_2) \quad (\delta \text{ Kroenecker index})$$

1 when $t_1 = t_2$
0 when $t_1 \neq t_2$

(n+m) square matrix

$$V = \begin{bmatrix} \tilde{Q} & Z \\ Z' & \tilde{R} \end{bmatrix}, \quad \tilde{Q} \in R^{n,n}, \quad \tilde{R} \in R^{p,p}, \quad Z \in R^{n,p}$$

covariance of v_x

covariance of v_y

possible correlation between v_x, v_y

cross covariance

typically state-output are same variable! or anyway affected

$\tilde{Q} \geq 0$

state noise

state noise

$\tilde{R} > 0$

measurement noise

usually $\neq 0$

for instance some outputs and states can coincide

we expect dependence

→ $\tilde{Q}, \tilde{R}$ characteristic of the system

Initial assumption (to be removed) $\longrightarrow$ $Z = 0$

assumption!
v_x, v_y uncorrelated
Gaussian W.N

Initial state

x(0) is a
Gaussian random variable

$$x_0 = x(0) \longrightarrow E[x_0] = \bar{x}_0 \quad , \quad E\left[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'\right] = \bar{P}_0 \geq 0$$

as cov matrix, always spd!!

Covariance matrix

Additional assumption

$$E\left[x_0 \begin{bmatrix} v_x' & v_y' \end{bmatrix}\right] = 0$$

$\{x_0, v_x, v_y$ uncorrelated!$\}$ → most cases its real!

$\Downarrow$

K.F is a standard OBSERVER (like pole placement obs.) ↓ if v_x, v_y unkn. we use standard obs →

Goal: to design a state observer «optimal» according to a given criterion which considers the *presence and characteristics* of the noises

assuming previous properties!

otherwise a pole placement observer could be used

if data about noise are known we use it!!

includes stochastic terms for better result

Some comments on the parameters

Proper knowledge of $\bar{x}_0, \tilde{P}_0$, not so critical → you can have info about it (asymptotic considerations)

Knowledge of $\tilde{R} > 0$: possible, it represents the measurement noise, often provided by sensors' manufactures

covariance of noise acting on output → sensor measuring it!

Idea of noise acting!!

info of it from sensor manufacturer

Knowledge of $\tilde{Q} \geq 0$: often difficult. Can we use it as a design parameter to obtain faster or slower state estimation? See later

Noise of syst acting on state } hard to know! → difficult to assign

on K.F

for implementing

$\tilde{Q}$ is a design parameter → tooned for a given result!

lose knowledge about real noise ... tuning parameter

} $\tilde{R}, \tilde{Q}$ compared value depending on how much you use the data

Structure of the filter/observer

(as done on pole placem.)

OBSERVER

system $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_x(t) \\ y(t) = Cx(t) + v_y(t) \end{cases}$

$$\dot{\hat{x}}(t) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{system dynamic}} + L(t)[y(t) - C\hat{x}(t)]$$

Innovation respect estimation

time varying gain to be selected according to an optimality criterion

↳ L(t) by minimizing a suitable (KPI)

State estimation error $e(t) = x(t) - \hat{x}(t)$

↓ dynamics of it: $\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$

known external input to be included on $\dot{e}(t)$

$$\begin{aligned} \dot{e}(t) &= Ae(t) - L(t)\left[Ce(t) + v_y(t)\right] + v_x(t) \\ &= \left[A - L(t)C\right] e(t) + B_c(t)v(t) \\ &= \underbrace{A_c(t)e(t)}_{\text{time variant system...}} + \underbrace{B_c(t)v(t)}_{\text{stochastic term, force the system}} \end{aligned}$$

$\left\{ A_c(t) = [A - L(t)C], \quad B_c(t) = \begin{bmatrix} I & -L(t) \end{bmatrix} \right\}$

↳ hard to analyze!

how to get proper definition

$\to B_c(t)\,\mathbb{E}[\cancel{v}(t)]$ 0 mean noise! (Gaussian)

$$\dot{e}(t) = A_c(t)e(t) + B_c(t)v(t) \xrightarrow{\text{set } \bar{e}(t) = E[e(t)]} \dot{\bar{e}}(t) = A_c(t)\bar{e}(t)$$

$\mathbb{E}[v(t)] = 0$

No forcing

Linear unforced system for expected value!

free motion system with initial value null 0, so it remain

$\bar{e}(0) = 0$

Letting $\hat{x}(0) = \bar{x}_0 \longrightarrow E[e(0)] = E[x(0) - \hat{x}(0)] = 0 \longrightarrow \boxed{\bar{e}(t) = 0,}\ \forall t \geq 0$

initial state exp. value

on the origin always! No forcing effect

Letting $\tilde{P}(t)$ be the covariance of $e(t) \longrightarrow \tilde{P}(t) = E[e(t)e(t)'],\quad \tilde{P}(0) = \tilde{P}_0$

$(\bar{e} = 0)$ variance of error

to obtain an Algorithm!

Goal*: minimize with respect to the gain* ***L(t)*** *the covariance of the estimation error*

We wanna min the covariance of the error

so the error is almost null always

$$\min_{L(t)} \gamma' \tilde{P}(t) \gamma$$

"min a matrix", so min respect $L(t)$ the quadratic form respect the gain

$\gamma \in R^{n,1}$ is a generic vector

small $\tilde{P}$! so error $\simeq 0$: $x \simeq \hat{x}$

Gain depends on $\tilde{P}$, $\tilde{R}$, C

The solution of the above problem is

$$L(t) = \tilde{P}(t)C'\tilde{R}^{-1}$$

$\tilde{R} > 0$ p.d because
- measurement noise always pd
- to be invertable!

where $\tilde{P}(t)$ is the solution of the Riccati equation

$$\dot{\tilde{P}}(t) = A\tilde{P}(t) + \tilde{P}(t)A' + \tilde{Q} - \tilde{P}(t)C'\tilde{R}^{-1}C\tilde{P}(t)$$

differential! (nonstationary)

with initial condition

$$\tilde{P}(0) = \tilde{P}_0$$

Iteratively solve the (DRE), compute $\tilde{P}$ → evaluate $L(t)$ and observer!

Matrix differential Riccati equation to be solved

solve the Riccati Equation on finite control → ARE on discretized system

DIFFERENCES of LINEAR QUADRATIC / KALMAN FILTER

LQ - KF

LQ	KF
$$\min_{K(t)} x'P(t)x$$	min error covariance respect gain L(t) → $$\min_{L(t)} \gamma'\tilde{P}(t)\gamma$$
$$K(t)=R^{-1}B'P(t)$$	$$L(t)=\tilde{P}(t)C'\tilde{R}^{-1}$$
Require to solve a differential matrix equation	
$$\dot{P}(t)+Q-P(t)BR^{-1}B'P'(t)+P(t)A+A'P'(t)=0$$	$$\dot{\tilde{P}}(t)=A\tilde{P}(t)+\tilde{P}(t)A'+\tilde{Q}-\tilde{P}(t)C'\tilde{R}^{-1}C\tilde{P}(t)$$ ↳ leads to $\bar{\tilde{P}}$ static solution
$$P(T)=S$$	from initial value @ time 0 and go $$\tilde{P}(0)=\tilde{P}_0$$
$\overleftarrow{\ t\ }$ start from terminal point T and integrate backward	$\overrightarrow{\ t\ }$ forward

- we can use some SW to solve, on Matlab you can use LQ function to solve KF
- We can extend to KF approach the result of LQ control about stability!

↳ next slide ..

$$\begin{array}{c|c}
LQ & KF \\
A & A' \\
B & C' \\
Q & \tilde{Q} \\
R & \tilde{R} \\
P & \tilde{P} \\
t & t^*-t \\
K & L'
\end{array}$$

perfect duality previous results on LQ can be extended to KF

same kind of problem and Algorithm

_KF_ with time varying gain _L(t)_: not very useful and difficult to implement. Is it possible to compute a steady-state filter?

In view of duality, we can extend the results of ***LQ***$_{inf}$

STABILITY TH. (like LQ control)

If

Bq is $\tilde{Q}$ partition

1. the pair (A, B_q) is reachable, where B_q is such that $\tilde{Q} = B_q B_q'$;
2. the pair (A, C) is observable;

then

$\Downarrow$ (A), (B) follows...

A) the optimal estimator is

$L(t) \to \bar{L}$ asymp. solution! Iterate forward Riccati equation using time invariant observer → time variant solution are bad for practical application

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \bar{L}[y(t) - C\hat{x}(t)] \\ &= (A - \bar{L}C)\,\hat{x}(t) + Bu(t) + \bar{L}y(t) \end{aligned}$$

with

$$\bar{L} = \bar{\tilde{P}}C'\tilde{R}^{-1}$$

where $\bar{\tilde{P}}$ is the unique positive definite solution of the stationary Riccati equation

no solvable by hand easly for syst of order ≥2

you can obtain more solutions → only one p.d $\bar{\tilde{P}} > 0$! To take!

$$\left(0 = A\bar{\tilde{P}} + \bar{\tilde{P}}A' + \tilde{Q} - \bar{\tilde{P}}C'\tilde{R}^{-1}C\bar{\tilde{P}} \right)$$

ARE solvable respect $\bar{\tilde{P}}$ instead of solving the DRE and iterate it

B) the observer is asymptotically stable, that is all the eigenvalues of $(A - \bar{L}C)$ have negative real part.

on **EXAM** can be given the RICCATI Eq. of LQ control problem

↓

and from here find the RICCATI Eq. of (K.F) observer!

↑

JUST consider proper sostitution duality (slide 8)

What to do if *Z≠0*?

removing Z=0 assumption! obviously y output and state coincide we have to deal with Z≠0!

Write the system as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + v_x(t) + Z\tilde{R}^{-1}(y(t) - y(t)) \\
&= Ax(t) + Bu(t) + v_x(t) + Z\tilde{R}^{-1}y(t) - Z\tilde{R}^{-1}Cx(t) \\
&\quad -Z\tilde{R}^{-1}v_y(t) \\
&= \underbrace{\left(A - Z\tilde{R}^{-1}C\right)}_{\bar{A}} x(t) + Bu(t) + Z\tilde{R}^{-1}y(t) + \bar{v}(t)
\end{aligned}
$$

new matrix $\bar{A}$

new state noise!

$$\bar{v}(t) = v_x(t) - Z\tilde{R}^{-1}v_y(t)$$

$$
\begin{aligned}
E\left[\bar{v}(t)v_y(t)'\right] &= Z - Z\tilde{R}^{-1}\tilde{R} = 0 \\
E\left[\bar{v}(t)\bar{v}(t)'\right] &= \tilde{Q} - Z\tilde{R}^{-1}Z' = \bar{\tilde{Q}} \geq 0
\end{aligned}
$$

(now uncorrelated noise!)

known term

$$
\begin{cases}
\dot{x}(t) = \bar{A}x(t) + Bu(t) + Z\tilde{R}^{-1}y(t) + \bar{v}(t) \\
y(t) = Cx(t) + v_y(t)
\end{cases}
$$

Satisfies the initial assumptions of KF

↑ it is a known input → standard theory

use it as a standard input

Duality with ***LQ*** allows one to understand how to choose the design parameters (provided that they are not obtained from physical considerations

similarity to understand
how to choice $\tilde{R}, \tilde{Q}$ parameter!

LQ

LQ_{inf} ***$KF_{statonary}$***

Q "large", R "small" $\rightarrow$ "fast" feedback system

$\tilde{Q}$ "large", $\tilde{R}$ "small" $\rightarrow$ "fast" observer

and viceversa

$\dot{x} = 0 := x = \bar{x}$ constant

Example: estimation of a constant from noisy measurements

$$\begin{cases} \dot{x}(t) = 0, \quad x(0) = \bar{x} \\ y(t) = x(t) + v_y(t) \end{cases}$$

$V_x = 0$ here!

noise

$V_x = 0$ const value r

$A = 0,\ C = 1,\ \tilde{Q} = 0,\ \tilde{R} = r$

BUT! $(A, \tilde{Q}^{1/2})$ **not reachable** since $\tilde{Q} = 0$

{No stationary equation}

looking Riccati equation

Riccati equation $\quad \boxed{\dfrac{d\tilde{P}(t)}{dt} = -\dfrac{\tilde{P}^2(t)}{r}} \longrightarrow \tilde{P}(t) = \dfrac{1}{r^{-1}t + \frac{1}{\tilde{P}_0}} \xrightarrow[\text{(for } t \to \infty)]{} \boxed{\tilde{P}(t) \to 0}$

meaning that cov. of $e(t) \to 0$ with time and data more!

$$L(t) = \tilde{P}(t)C\tilde{R}^{-1} = \frac{r^{-1}}{r^{-1}t + \frac{1}{\tilde{P}_0}}$$

gain of observer $L(t) \to 0$ for $t \to \infty$

The KF gain tends to 0, the estimate tends to be constant

$$\dot{\hat{x}}(t) = \frac{r^{-1}}{r^{-1}t + \frac{1}{\tilde{P}_0}}(y(t) - \hat{x}(t))$$

Example

no information about state

$$\begin{cases} \dot{x}(t) = Ax(t) + v_x(t) \\ y(t) = v_y(t) \end{cases} \longrightarrow C = 0 \longrightarrow L(t) = 0 \longrightarrow \boxed{\dot{\hat{x}}(t) = A\hat{x}(t)}$$

open loop system
because y(t) give
No informations!

The output does not bring any information on the state
You can only run the estimator open-loop from the knowledge of the expected value of the initial state

check observability (A, C) OK
+ reachability $(A, \tilde{Q})$ OK

Example

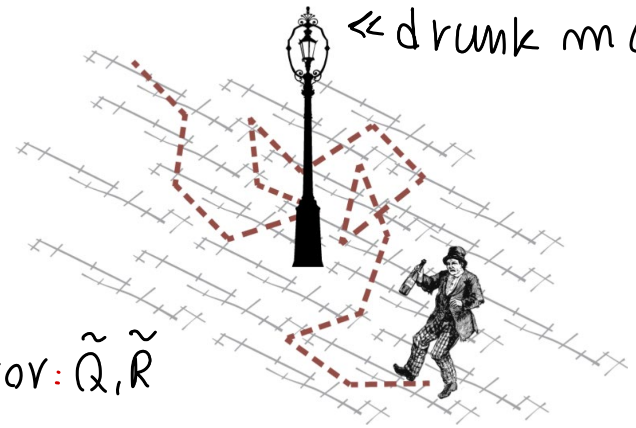
NO input syst or we have an additional input on observer!

integrator dynamic (particular case) leads moovement "random" « drunk man »

x(t) = integrated white noise

$$
\begin{cases}
\dot{x}(t) = v_x(t) \\
y(t) = Cx(t) + v_y(t)
\end{cases}
$$

Brownian motion or drunkard's Walk



apply KF theory asymp $A = 0$, C, $v_x, v_y \neq 0$ cov: $\tilde{Q}, \tilde{R}$

assumption Fullfilled → asymp. result

Steady state Riccati equation

$$0 = \tilde{Q} - \frac{\bar{\tilde{P}}^2 C^2}{\tilde{R}} \longrightarrow \bar{\tilde{P}} = \sqrt{\frac{\tilde{Q}\tilde{R}}{C^2}} \longrightarrow \bar{L} = \bar{\tilde{P}} C' \tilde{R}^{-1} = \sqrt{\frac{\tilde{Q}\tilde{R}}{C^2}} \frac{C}{\tilde{R}} = \sqrt{\frac{\tilde{Q}}{\tilde{R}}}$$

solve RE, positive solution

GAIN:

observer gain depends on $\tilde{Q}/\tilde{R}$ ration

estimator with gain depending on $\tilde{Q}, \tilde{R}$ to tune the system

(STATE ESTIMATOR) K.F form

$$\frac{d\hat{x}(t)}{dt} = \sqrt{\frac{\tilde{Q}}{\tilde{R}}}\,[y(t) - C\hat{x}(t)]$$

taking

- $\frac{\tilde{Q}}{\tilde{R}}$ large: "fast observer"
  ↳ eig value on left, FAST
- $\frac{\tilde{Q}}{\tilde{R}}$ small: "slow observer"
  ↳ eig value near origin, slow

we preffer fast, BUT IF noisy, hard to get too fast observer! Issue! you estimate too sensitive! + trade-off needed!

1st ORDER LINEAR FILTER with $\left(\tau = \frac{1}{C}\sqrt{\frac{\tilde{R}}{\tilde{Q}}}\right)$

$$\hat{X}(s) = \frac{1}{1 + \frac{1}{C}\sqrt{\frac{\tilde{R}}{\tilde{Q}}}\, s} Y(s)$$

$(Y \to \hat{X})$ dynamic.

T.F between output and state estimator!

τ: time constant! ~ $\tilde{Q}/\tilde{R}$ speed!

we study ROBUSTNESS of LQControl ⇒ can provide withouth trick a guarantee gain margin + phase margin

↓

Valid both for SISO, MIMO

(K.F) [duality between LQcontr. and K.F] → we can extend stability property to K.F interesting example brownian motion $\rightarrow$

Estimation of a Brownian motion

$$\begin{cases} \dot{x} = v_x \\ y_1 = x + v_y \end{cases}$$

Gaussian noise!

$\tilde{R} = 1$ $\qquad$ $\tilde{Q} = 1$

$v_x \sim WGN(0,1)$, $v_y \sim WGN(0,1)$

$\boxed{A=0}$, $\boxed{C=1}$

STEADY STATE $\longrightarrow$ cov of state estimat error

$\Downarrow$

Riccati eq. $\rightarrow 1 - \tilde{P}^2 = 0 \longrightarrow \bar{\tilde{P}} = 1$

asymptotic variance of the state estimation error

random process with covariance 1

$\tilde{P} = 1 \rightarrow L = \tilde{P} C' \tilde{R}^{-1} = 1$

FILTER

Observer $\quad \dot{\hat{x}} = [y_1 - \hat{x}] \longrightarrow \quad y_1 \rightarrow \boxed{\frac{1}{s+1}} \rightarrow \hat{x}$

Now assume to have two measurements with the same measurement noise

same output transform

IF 2 sensors y_1, y_2 →

$$\begin{cases} \mathring{x} = v_x \\ y_1 = x + v_{y_1} \\ y_2 = x + v_{y_2} \end{cases}$$

$\tilde{Q} = 1$, $\tilde{R} = I_2$ (2×2), $A = 0$, $C = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$\tilde{R} = I_2$: 2 sensors each one with covariance 1, independent!

⇓

Riccati eq. $\quad 1 - \tilde{P} \begin{vmatrix} 1 & 1 \end{vmatrix} I \begin{vmatrix} 1 \\ 1 \end{vmatrix} \tilde{P} = 0 \quad \rightarrow \quad 1 - 2\tilde{P}^2 = 0$

(steady state) ⇒ $\tilde{P} = \sqrt{1/2} \simeq 0.707$ new variance of the state estimation error ← respect previous case, now filter provides smaller state est. error

↓

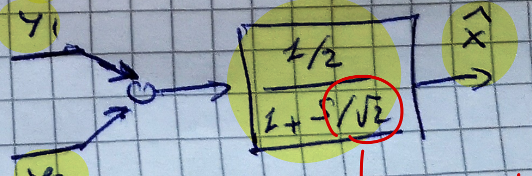
2 meas. independent ⇒ better result (more INFO)

$L = \dfrac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \end{vmatrix}$ (row vector) ∟ same dimension of m

K.F $\quad \hat{\mathring{x}} = \begin{vmatrix} 1 & 1 \end{vmatrix} \dfrac{1}{\sqrt{2}} \left\{ \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} - \begin{vmatrix} 1 \\ 1 \end{vmatrix} \hat{x} \right\}$, with $[C] = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$$= \frac{1}{\sqrt{2}} \left\{ y_1 + y_2 - 2\hat{x} \right\} \rightarrow$$

T.F $\quad y_1, y_2 \rightarrow \hat{x}$



∟ $\tau = 1/\sqrt{2}$ smaller τ, Faster system!

Now assume to have 2 measurement with different measurement noise

$$\tilde{Q} = 1 \qquad \tilde{R} = \begin{vmatrix} 1 & 0 \\ 0 & r \end{vmatrix}$$

cov of I sensor = 1

cov. of II sensor = r

Riccati eq.

$$1 - \tilde{P} \underbrace{\begin{vmatrix} 1 & 1 \end{vmatrix}}_{C'} \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1/r \end{vmatrix}}_{R^{-1}} \underbrace{\begin{vmatrix} 1 \\ 1 \end{vmatrix}}_{C} \tilde{P} = 0$$

it is easy to compute

$$\tilde{P} = \sqrt{r/(r+1)} \rightarrow L = \sqrt{\frac{r}{r+1}} \begin{vmatrix} 1 & \frac{1}{r} \end{vmatrix}$$

(overall state estimation)

$$\dot{\hat{x}} = \sqrt{\frac{r}{r+1}} \begin{vmatrix} 1 & 1/r \end{vmatrix} \begin{vmatrix} y_1 - \hat{x} \\ y_2 - \hat{x} \end{vmatrix}$$

|| performance ? ||

$$\mathring{\hat{x}} = \sqrt{\frac{r}{r+1}} \left\{ (y_1 - \hat{x}) + \frac{1}{r} (y_2 - \hat{x}) \right\}$$

(Large r value)

If $r \to \infty$ $\quad \mathring{\hat{x}} = y_1 - \hat{x}$ The second measurement y_2 is not reliable and not used

II meas has large cov. noise! so @ limit, you don't use II meas! NOT reliable!

If $r \to 0$ only the second measurement is used because it is not affected by measurement noise

NO noise on y_2, so you can use II meas!

We assume v_x, v_y WN, BUT in practice usually meas / noises are NOT White! → tracking a reference, you have noise to winds, waves ecc... high freq wave noise! → can be @ given spectrum ≈ stationary → Spectral theorem

KF with non white noise

to solve this..

I ORD case: syst.

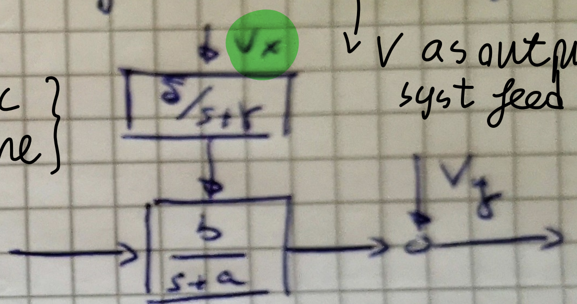
$$\begin{cases} \dot x = -ax + bu + v \\ y = x + v_y \end{cases} , \qquad \dot v = -\gamma v + \delta v_x , \quad \gamma > 0$$

shape of noise!

output affect by WN

v as output of a syst feed by WN

{OVERALL scheme}



enlarged system! with states (x, v)

$$\begin{cases} \begin{vmatrix} \dot x \\ \dot v \end{vmatrix} = \underbrace{\begin{vmatrix} -a & 1 \\ 0 & -\gamma \end{vmatrix}}_{A} \begin{vmatrix} x \\ v \end{vmatrix} + \underbrace{\begin{vmatrix} b \\ 0 \end{vmatrix}}_{B} u + \underbrace{\underbrace{\begin{vmatrix} 0 \\ \delta \end{vmatrix}}_{M} v_x}_{\tilde v_x} \\ y = \underbrace{\begin{vmatrix} 1 & 0 \end{vmatrix}}_{C} \begin{vmatrix} x \\ v \end{vmatrix} + v_y \end{cases}$$

noise on enlarged state described by matrix M ⇒ $\tilde v_x$ new disturbance (WN)

this system, consider enlarged T.F with noise dynamic, we go back to standard K.F

Spectral theorem

from stationary noise → can be seen as output of an asymp. stable system with input V_x ($V_x \sim WN$)

↓

noise with a given spectrum as output of A.S system properly choosen feeded by WN! → key important when you have stationary noises (NOT WN!)

Tuning KF according to requirements

$$E[v_x] = 0 \,, \quad E[\tilde{v}_x] = 0$$ (by $\tilde{v}_x$ definition)

$$\mathrm{cov}(\tilde{v}_x) = E[\tilde{v}_x \tilde{v}_x'] = \Gamma E[v_x v_x'] \Gamma' = \underbrace{\Gamma \tilde{Q}_x \Gamma'}$$

deterministic := (outside E operator) $\Gamma \tilde{q}^2 \Gamma'$ $\quad (\tilde{Q}_x = q^2 I)$

$$= \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \tilde{q}^2 \delta^2 \end{bmatrix}}_{\tilde{Q} \geq 0}$$

asymp K.F conditions

$$B_q B_q' = \begin{vmatrix} 0 \\ \delta\tilde{q} \end{vmatrix} \begin{vmatrix} 0 & \delta\tilde{q} \end{vmatrix}$$

$$\Downarrow$$

Observability $(A, C) \rightarrow \Gamma_o = \begin{vmatrix} C \\ CA \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -a & 1 \end{vmatrix}$ ✓ verifyed always

Reachability $(A, B_q) \rightarrow \Gamma_r = \begin{vmatrix} B_q & AB_q \end{vmatrix} = \begin{vmatrix} 0 & \delta\tilde{q} \\ \delta\tilde{q} & -\gamma\delta\tilde{q} \end{vmatrix} \rightarrow$ ok for $\delta \neq 0$, $\tilde{q} \neq 0$

given $\tilde{q} \neq 0$, $\delta \neq 0$

Design of the filter for the enlarged state

$$\tilde{x} = \begin{vmatrix} x \\ v \end{vmatrix}$$

state x + the state of the filter noise dynamic

$\Downarrow$ standard asymp. K.F theory

$$\dot{\hat{\tilde{x}}} = A\hat{\tilde{x}} + Bu + L\left[y - C\hat{\tilde{x}}\right], \quad L = \begin{vmatrix} \ell_1 \\ \ell_2 \end{vmatrix}$$

$$\begin{cases} \dot{\hat{x}} = -a\hat{x} + \hat{v} + bu + \ell_1(y - \hat{x}) \\ \dot{\hat{v}} = -\gamma\hat{v} + \ell_2[y - \hat{x}] \end{cases}$$

$\Downarrow$ compute the T.F $\boxed{\tilde{X}(s)/Y(s)}$

$$\hat{x}(s) = \underbrace{C\,(sI-(A-LC))^{-1}L}_{\text{transfer function of the filter}}\,y(s)$$

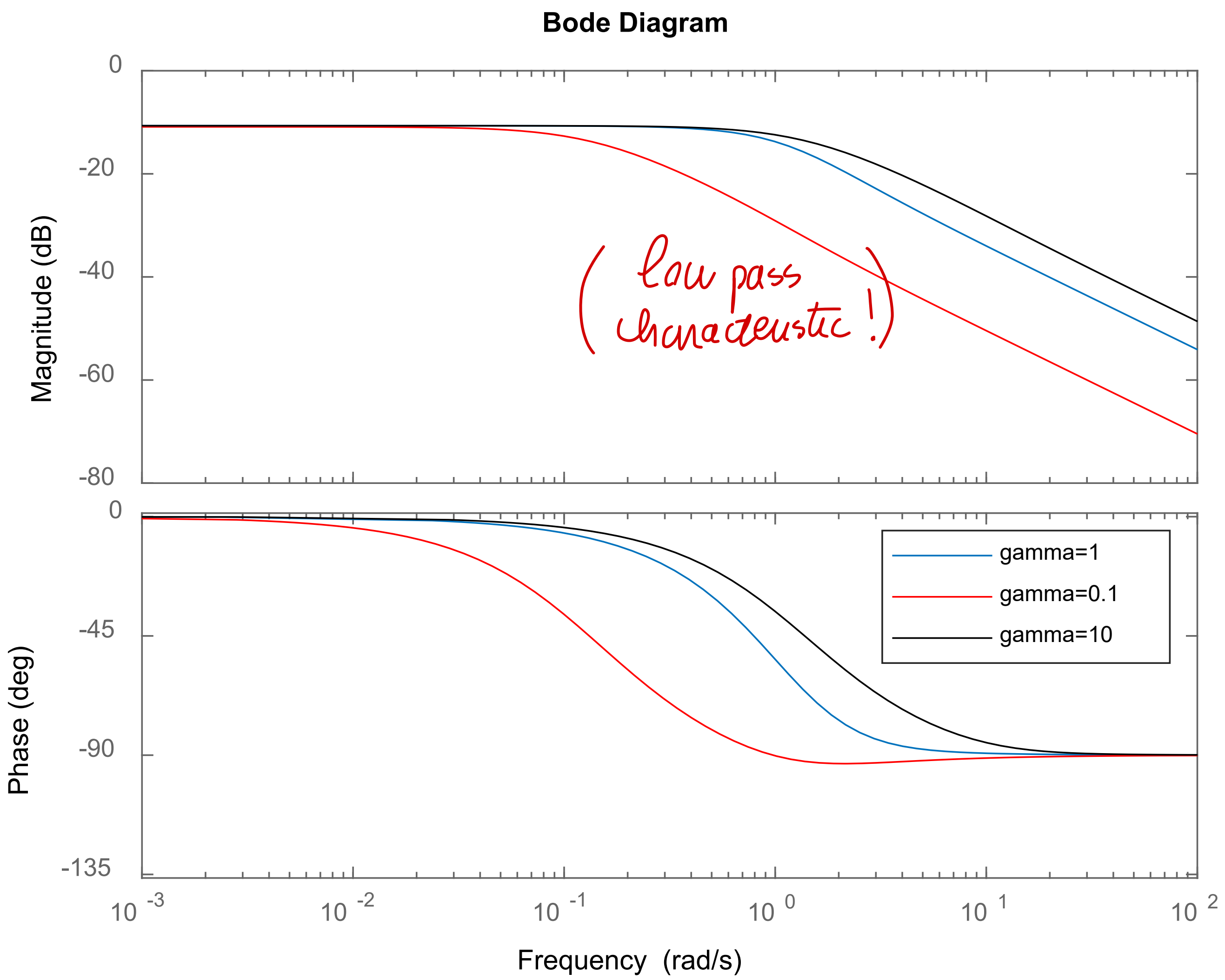
Considering fixed values

$a=1$ (original syst eig val), $\tilde{R}=1$ (meas cov), $\delta=\gamma$ (T.F of V/V_x shape), $\tilde{q}=1$ (cov V_x)

$\gamma > \delta$ Fast system

IF $\gamma < \delta$ slow system

we obtain different transfer functions of the filter



for low pass disturbance apply K.F.
IF you wanna tune K.F you moove γ, δ according to desire!

The slower the system of the noise, the slower the filter

FILTER Bode for different γ!

fast system $V_x \to V$, γ↑, you get a fast filter!
while slow V_x filtering lead to low K.F

(Relevant!)
you can consider this result as the theory application for known Data, BUT also consider to tune $V_x \to V$ dynamic to moove noise dyn... depend on the case