LINEAR QUADRATIC GAUSSIAN (LQG) CONTROL

H_2/H_∞ CONTROL

Advanced and Multivariable Control

Linear Quadratic Gaussian (LQG) control

lots of implication of it

Riccardo Scattolini

main problem ... given the

System

syst described by em model + disturbances acting on state, output

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_x(t) \\ y(t) = Cx(t) + v_y(t) \end{cases}$$

$\Downarrow$ standard static feedback control law on estimated state + observer)

v_x, v_y, $x(0)$ satisfy all the assumption introduced for the Kalman Filter

(all assumptions on v_x, v_y, x) satisfyed same LQ control conditions

Goal of *LQG*: in the case of nonmeasurable state, minimize

↓ minimizing J..

(reformulate the problem)

↳ suitable controller design by classical index of merit:

$$J = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T (x'(t)Qx(t) + u'(t)Ru(t))\,dt\right]$$

to avoid continuous integration J→∞, we divide for T

required to have a finite cost function

because of v_x, v_y

required because *x* and *u* are stochastic processes, which do not tend to zero ($\tilde{Q} \geq 0$)

even if your system is A.S in closed loop → x, u are stochastic! noises acting on system, NO constant values (noises)

If the resulting closed-lop system is asymptotically stable, *x* and *u* are ***stationary processes*** and the cost function can be written as

easy formulation

$\Downarrow$

$$J = E\left[x'(t)Qx(t) + u'(t)Ru(t)\right]$$

Solution

solution by combining LQ CTRL PROB. refferd to system without noises (state measurable) + (K.F) providing $\hat{x}$ to use on control loop

The optimal control law is given by the combination of the solution to the corresponding deterministic LQ control problem and of the state estimated by the corresponding Kalman filter

solve K.F asymp. to compute $\bar{K}$ of observer

applyed on proper A, B, Q, R ↳ compute $\bar{K}$

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \bar{L}\left[y(t) - C\hat{x}(t)\right] \\
u(t) &= -\bar{K}\hat{x}(t)
\end{aligned}
$$

Solution of LQ_{inf} with Q, R

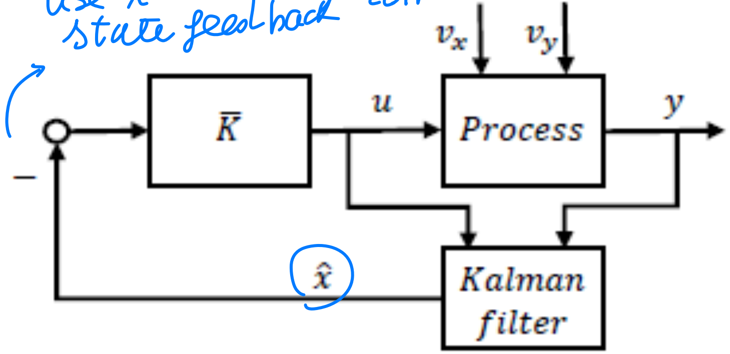
- pole placement for $\bar{K}$, $\bar{L}$
- OR • LQ + K.F to compute gains

} or mix the two procedures! to build overall system

Solution of KF with $\tilde{Q}, \tilde{R}$

↳ overall structure equal to pole placem. control

Use $\hat{x}$ on state feedback control



apply K.F

Comments

- The structure (state feedback + observer) is exactly equal to the one derived for pole placement control

- The ***separation principle*** holds as well, so that the closed-loop system has the eigenvalues of (***A-BK***) and (***A-LC***)

- From the equations of the state feedback (*LQ*) and of the observer (*KF*) it is easy to compute the equivalent regulator transfer function (same computations of the pole placement analysis)

taking $u(t)$, replace on $\hat{x}(t)$, go on laplace, compute T.F and trang o back...

$$U(s) = \underbrace{-\bar{K}\left(sI - \left(A - \bar{L}C - B\bar{K}\right)\right)^{-1}\bar{L}}_{R(s)}Y(s)$$

LQ_{inf} has robustness properties with respect to gain and phase variations at the plant input.

Does LQG inherit these properties?

the Robustness is lost! because:..

you loose All! ↳

NO

so we have some problems!

as seen on the overall scheme: IF you have uncertainty between B→A

↓

you design KF for nominal system → so our designed KF does not provide proper estimation

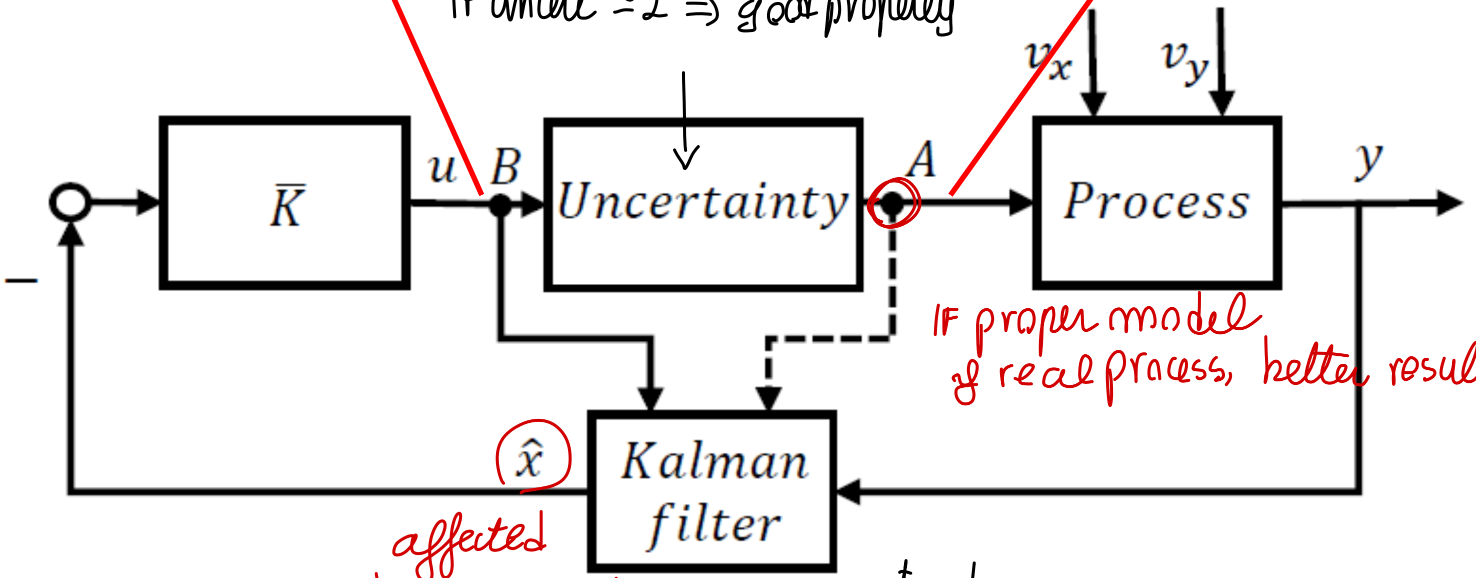
This is the signal available, the KF cannot provide the correct state estimation

during design of regulator you neglect it! because unknown

IF uncert ≃ 1 ⇒ good property

Differently!

If one could use this signal, the same robusness of LQ would be guaranteed



IF proper model of real process, better result!

affected by uncertainty! no guarantee!

Everything is lost? NO →

particular procedure that can be applied sometimes to recover ROBUSTNESS of LQ control ⇒

Loop Transfer Recovery procedure (*LTR*) - uncertainty at the plant input

Consider the **SISO** system (restrict on SISO case)

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

(consider syst. without noises)

characterized by: $G(s) = C(sI - A)^{-1}B = \dfrac{B_G(s)}{A_G(s)} = \dfrac{b_{n-1}s^{n-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$

and the state feedback ***LQ**$_{inf}$* control law $u(t) = -Kx(t)$, $K = \begin{bmatrix} k_0 & k_1 & \dots & k_{n-2} & k_{n-1} \end{bmatrix}$

SISO, K is a row vector!

The loop transfer function with robustness properties is

taking directly the state, no observer by LQ control → guarantee stability!

LQ control!

$$L_a^1(s) = K(sI - A)^{-1}B = \frac{\kappa(s)}{A_G(s)} \quad , \quad \kappa(s) = k_{n-1}s^{n-1} + \dots + k_0$$

using also (KF)

When the regulator $R(s) = K(sI - A + BK + LC)^{-1}L$ is used (with a generic ***L*** observer gain), the loop transfer function becomes

Guarantee phase/gain nominal properties of system → same robustness using a Reg including OBS.

$$L_a^2(s) = R(s)G(s) = K(sI - A + BK + LC)^{-1}LC(sI - A)^{-1}B$$

We want $L_a^2 \simeq L_a^1$ similar as much as possible!

Now, take the observer gain as

focus on that type of obs

$$\boxed{L = \rho B, \quad \rho > 0}$$

(on SISO case L has dimension n,m → 1×1)

(note that there are **no stability guarantees**). It can be shown (see the textbook) that ...

$$L_a^2(s) = \frac{\rho\kappa(s)}{(A_G(s) + \kappa(s) + \rho \cancel{B_G}(s))} \frac{\cancel{B_G}(s)}{A_G(s)}$$

for large ρ! $\rho B_G \gg A_G + K$

and

• so IF design an observer with Gain $L=\rho B$ and take ρ LARGE! the loop T.F of syst with observer ≃ loop T.F of static state feedback → ROBUSTNESS

$$L_a^2(s) \underset{\rho \to \infty}{\to} L_a^1(s) = \frac{\kappa(s)}{A_G(s)}$$

there is a cancellation!, meaning that ZEROS of O.L system are cancelled! needs to be A.S

The properties of the state feedback control law are recovered (LTR procedure)

check that zeros of O.L syst must be A.S

There is a cancellation of the open-loop zeros, that must be asymptotically stable

MIMO systems

$m \times p$ $\quad$ $m = p$! $\quad$ $m \times m$

The LTR procedure can be applied with $L = \rho B$, $\rho > 0$ provided that:

procedure only for square system without unst. invariant zeros

system $\quad$ $m = p$

1. it is square (same number of inputs and outputs),
2. its invariant zeros have negative real part. → on MIMO we need $\mathrm{Re}<0$ on invariant zeros! as before on normal zeros

So, one increases the value of ρ until the recovery of the loop transfer function with LQ_{inf} are obtained

↳(idea) provided (1)+(2) :⇒ take obs gain $L=\rho B$ with high ρ → does NOT guarantee stability of closed loop syst! (critical flow!)

But what about the stability of the observer?

to apply LTR, simply design LQ control, then design OBSERV. with K.F approach setting $\tilde{R}=I$, $\tilde{Q}=\alpha BB'$ α Big!

MIMO systems

this... provide A.S K.F design! ⇒ under suitable assumption

apply this procedure using KF gain properly

It can be proven that, under the same conditions previously introduced, *the LTR* is guaranteed also using a *KF* with

(FAST OBSERVER)

$$\tilde{Q} = \alpha BB', \tilde{R} = I \quad , \quad \alpha \longrightarrow \infty$$

for large α values

in practice... $\tilde{Q} \gg \tilde{R}$, so take $\alpha \to \infty$, try to make obs. FAST, so $\hat{x} \to x$, instead of $u=-K\hat{x} \simeq -Kx$ good state feedback control law

*and under suitable conditions, the **KF is asymptotically stable***

+ loop T.F tends to loop T.F static state feedback ⟶ so you obtain all Robustness of LQ ctrl!

+ problem solved!

I ORD system

noise acting

Exercise: given

$$\begin{cases}\dot{x} = -x + 2u + v_x & , v_x \sim WGN(0,\tilde{q}) \\ y = x + v_y & , v_y \sim (0,1)\end{cases}$$

initially compute state feedback control law..

(?)

A) Assuming v_x, v_y null, compute the LQ_∞ control law with $Q = 15/4$, $R = 1$ → JUST apply formula

B) Compute the gain margin of the system and compare it with the theoretical one **and also the phase margin**

C) Compute the KF with the specified characteristics of the noises

D) Compute the regulator transfer function $R(s)$ and use it to verify the LTR procedure $\Longrightarrow$

→ assumption satisfyed!

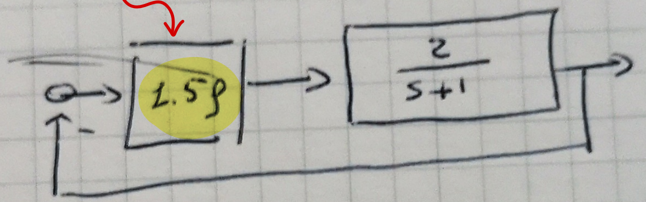
A) $A=-1$, $B=2$, $Q=15/4$, $R=1$ → given all values, solve Riccati equation

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P \rightarrow 0 = -2\bar{P} + 15/4 - 4\bar{P}^2$$

STATIONARY R.E

$\bar{P} = 0.75$, → corresponding $K = R^{-1}B'\bar{P} = 1.5$

B) ρ used to compute the gain margin



select ρ such that A.S system → gain margin

Characteristic equation $s+1+3\rho=0$ → $s=-(1+3\rho)$ eig value of the system I ORD

Stability for $\boxed{\rho > -\frac{1}{3}}$

(0.5, ∞)

Gain margin $(-1/3, \infty)$ larger than the theoretical one

↳ larger than theory!

compute K.F

C) $C=1$, $\tilde{R}=1$ → $\tilde{P} = L = -1+\sqrt{1+\tilde{Q}}$

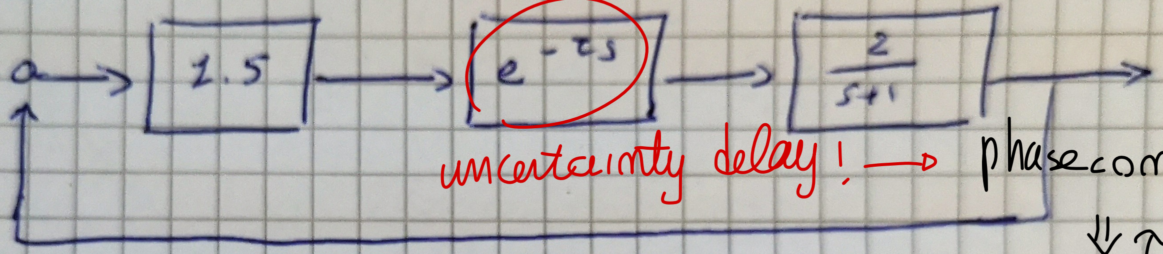
Riccati equation respect $\tilde{Q}$

$\boxed{\text{OBSERVER}}$

$$\dot{\hat{x}} = -\sqrt{1+\tilde{Q}}\,\hat{x} + 2u + \left(-1+\sqrt{1+\tilde{Q}}\right)y$$

phase margin

to compute φ_m, from feedback control law and $\bar{K}$

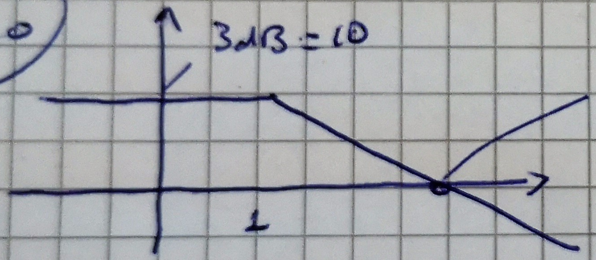


uncertainty delay! → phase contribution $-\omega\tau$

⇓ τ such that syst remains A.S

(Bode diagram)

$\tau = 0$



$\omega_c = 2.83$ rad/s

$\varphi_m = 108°$ ← larger φ_m than theory

tolerated phase variation

in nominal condition, you can introduce 108° delay, remain A.S

c) $C = 1, \tilde{R} = 1 \longrightarrow \tilde{P} = L = -1 + \sqrt{1 + \tilde{q}}$

$$\dot{\hat{x}} = -\sqrt{1+\tilde{q}}\,\hat{x} + 2u + \left(-1 + \sqrt{1+\tilde{q}}\right) y$$

check LTR procedure! from Regulator

D) $R(s) = K(sI - A + BK + LC)^{-1} L$ using K, L values still parametrized respect $\tilde{Q}$

$$= 1.5 \frac{-1+\sqrt{1+\tilde{Q}}}{s+3+\sqrt{1+\tilde{Q}}} = \frac{1.5}{\underbrace{\frac{1}{-1+\sqrt{1+\tilde{Q}}}}_{T} s + \underbrace{\frac{3+\sqrt{1+\tilde{Q}}}{-1+\sqrt{1+\tilde{Q}}}}_{\alpha}}$$

Same $m = p$ IN, OUT, and NO ZERO, so verified condition! LTR for simple SISO case, guarantee robustness

more sensitive to the noise!

in case of large $\tilde{Q}$ → Q big enlarge passing noises!

$\tilde{Q} \to \infty$ as LTR request, $T \to 0$, $\alpha \to 1$ .. → $R(s) \to 1.5$ as state feedback control law!

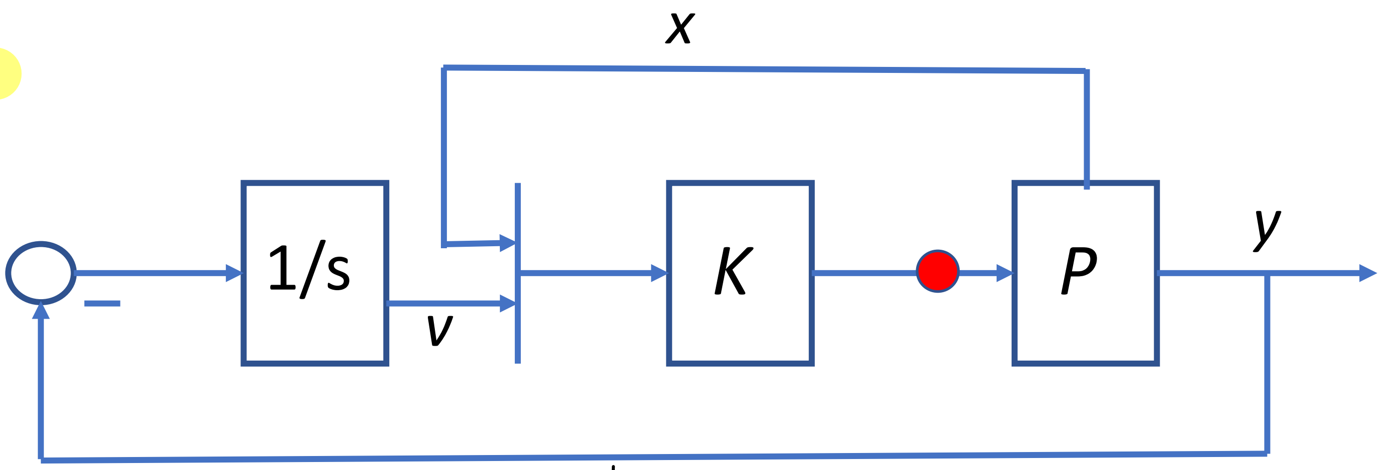
and the loop transfer function $L(s) \to \frac{1.5 \cdot 2}{s+1}$

exactly same of LQ

The one obtained with LQ

LTR - scheme with integrators – measurable state

to guarantee 0 error Reg @ steady state for const. Ref. → by including integrators acting on ERRORS inside loop := consider enlarged syst to design state feedback c.l



1) consider scheme
2) enlarge syst
3) obtain $\bar{A}$, $\bar{B}$, assumy meas. state
4) state feedback control law

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u(t)$$

$$u(t) = -K \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = -\begin{bmatrix} K_x & K_v \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

Loop transfer function at the plant input $L_a^1(s) = K(sI - \bar{A})^{-1}\bar{B}$

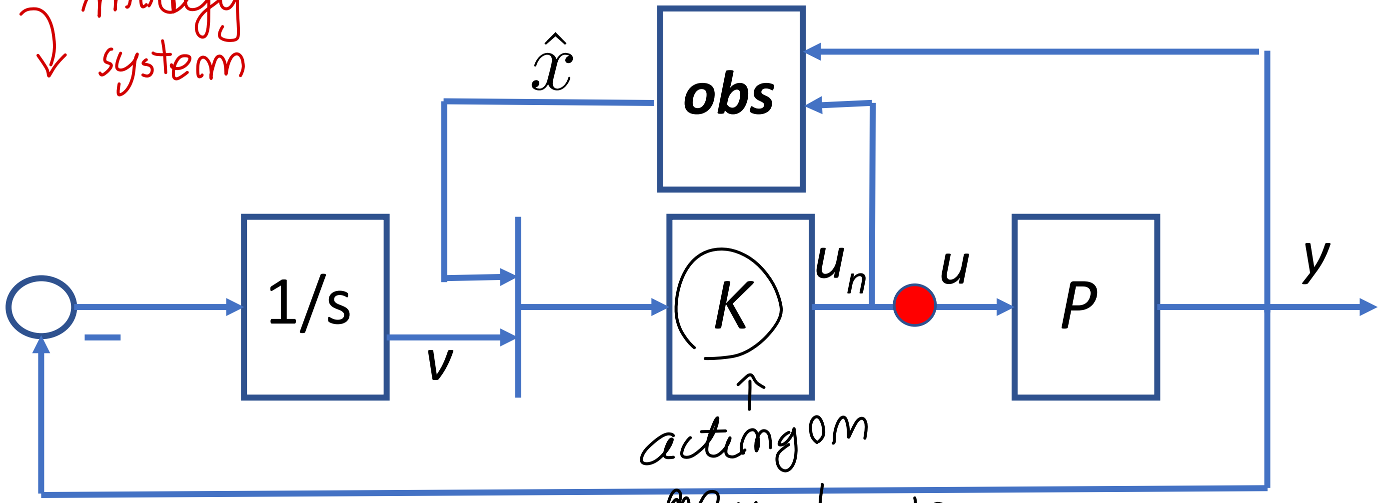
LTR - scheme with integrators – unmeasurable state

modefy system

estimate the state x introducing an estimator

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_n(t) + L[y(t) - C\hat{x}(t)]$$

properly designed



acting on enlarged system

$$u_n(t) = -\underbrace{\begin{bmatrix} 0 & K_v & K_x \end{bmatrix}}_{\tilde{K}} \begin{bmatrix} x(t) \\ v(t) \\ \hat{x}(t) \end{bmatrix}$$

{New control law} :=

enlarged state

overall system (enlarged + observer)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ -C & 0 & 0 \\ LC & -BK_v & A - BK_x - LC \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x(t) \\ v(t) \\ \hat{x}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{\tilde{B}} u(t)$$

Loop transfer function at the plant input $L_a^2(s) = \tilde{K}(sI - \tilde{A})^{-1}\tilde{B}$

I WANT $L_a^2 \simeq L_a^1$ ⇒ ... for proper OBS design

↳ system with integrators... compare known/unknown state

Example: linearized model of an aircraft

linear system made of 5 states, 3 inputs

Maciejowski, Jan Marian. "Multivariable feedback design." *Electronic Systems Engineering Series, Wokingham, England: Addison-Wesley,| c1989* (1989).

$x1$: altitude relative to some datum (m) $= Y1$

$x2$: forward speed ($\mathrm{m\,s^{-1}}$) $= Y2$

$x3$: pitch angle (degrees) $= Y3$

3 outputs

$x4$: pitch rate ($\mathrm{deg\,s^{-1}}$)

$x5$: vertical speed ($\mathrm{m\,s^{-1}}$)

$$A = \begin{bmatrix} 0 & 0 & 1.1320 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$u1$: spoiler angle (measured in tenths of a degree)

$u2$: forward acceleration ($\mathrm{m\,s^{-2}}$)

$u3$: elevator angle (degrees)

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1.0000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.665 \\ 1.5750 & 0 & -0.0732 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

D=0

NOT easy decoupling solution!
you need Multivar. approach

eigenvalues

s = 0 (integrators

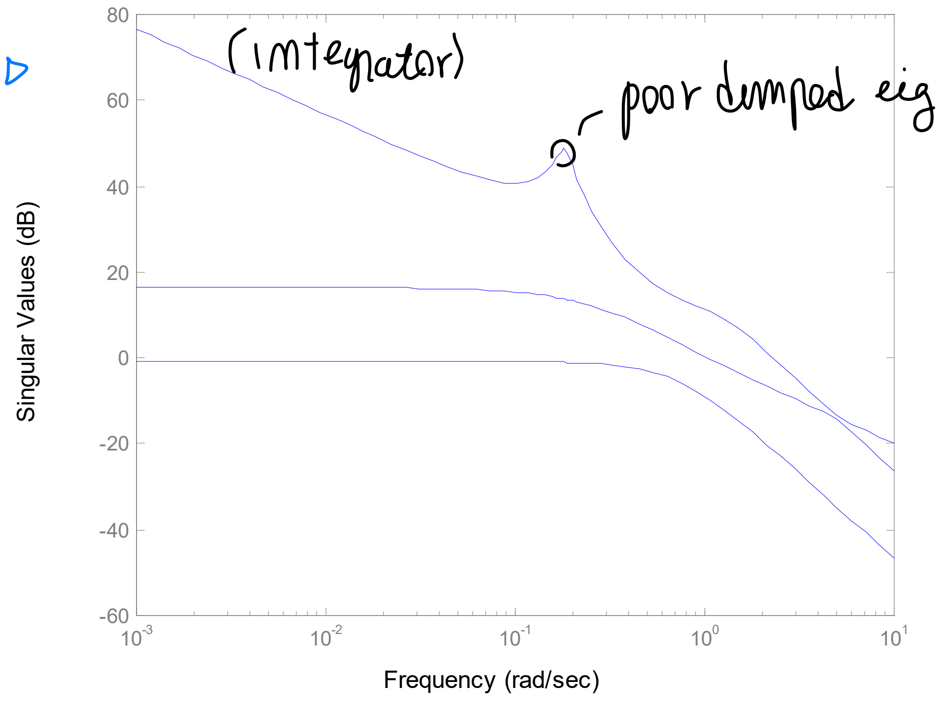
s = -0.7801 + 1.0296i
s = -0.7801 - 1.0296i
complex conj eig values
s = -0.0176 + 0.1826i
s = -0.0176 - 0.1826i

low damp

no zeros

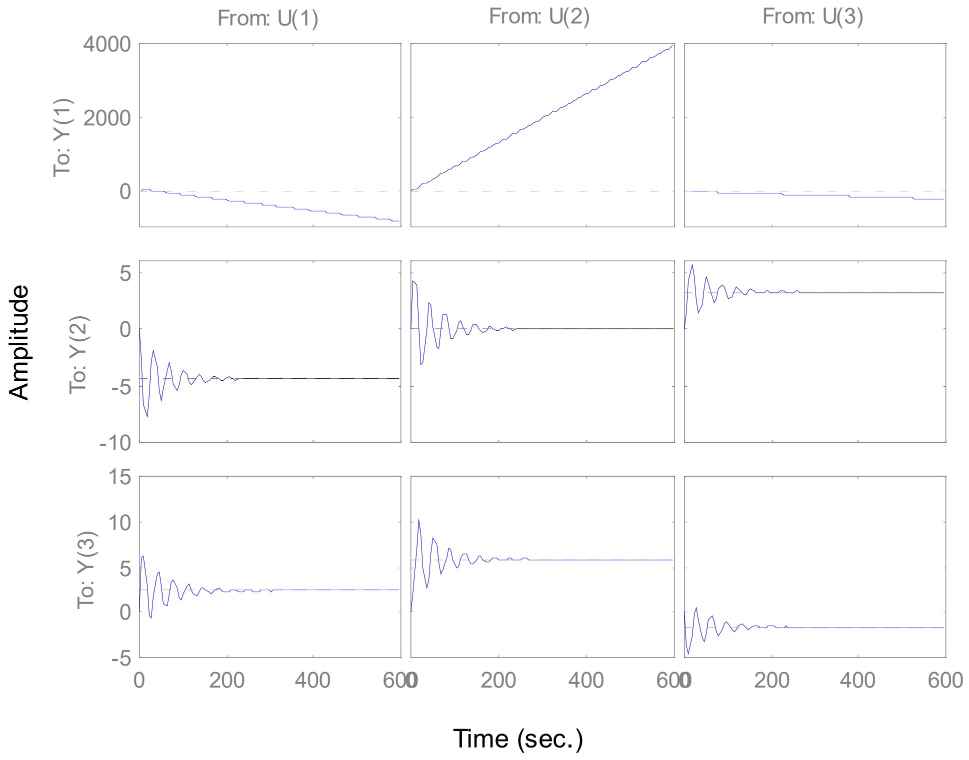
singular values

3 singular values of system

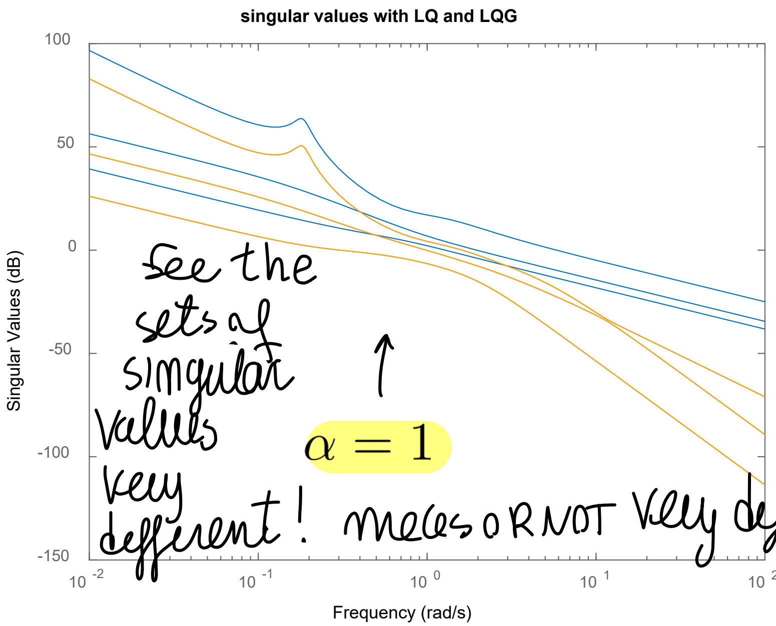


show integral action and oscillation

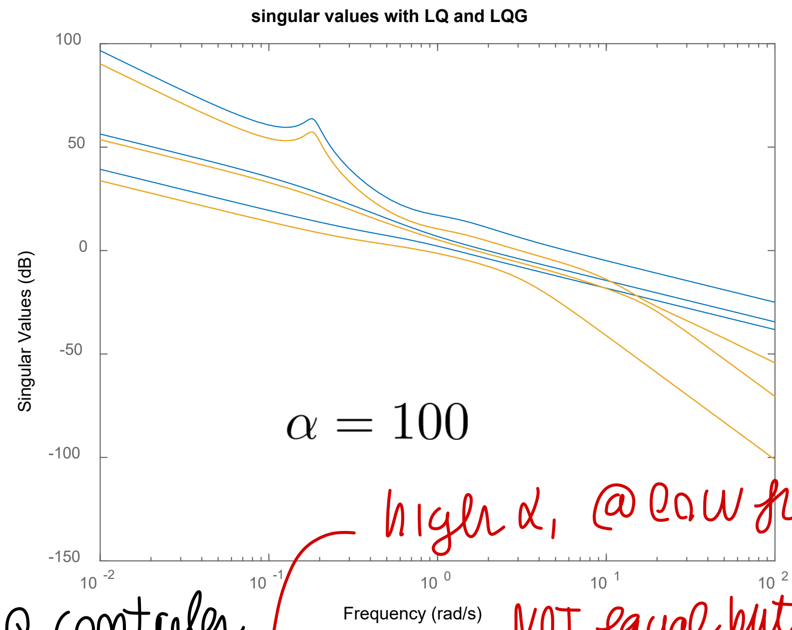
Open loop step responses



design LQG controller!
with matrix



See the sets of singular values very different! meas OR NOT very different



high α, @ low freq almost the same
NOT equal but similar!

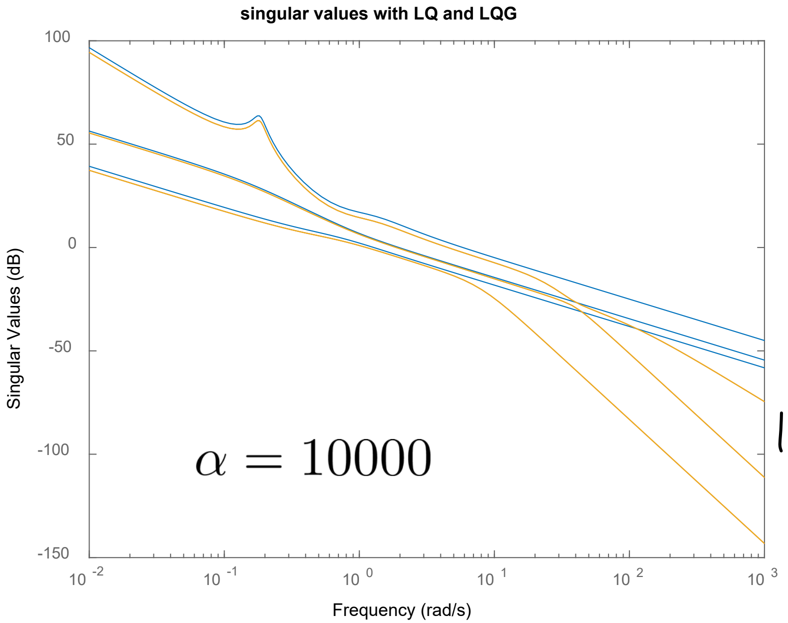
singular value of LQ control

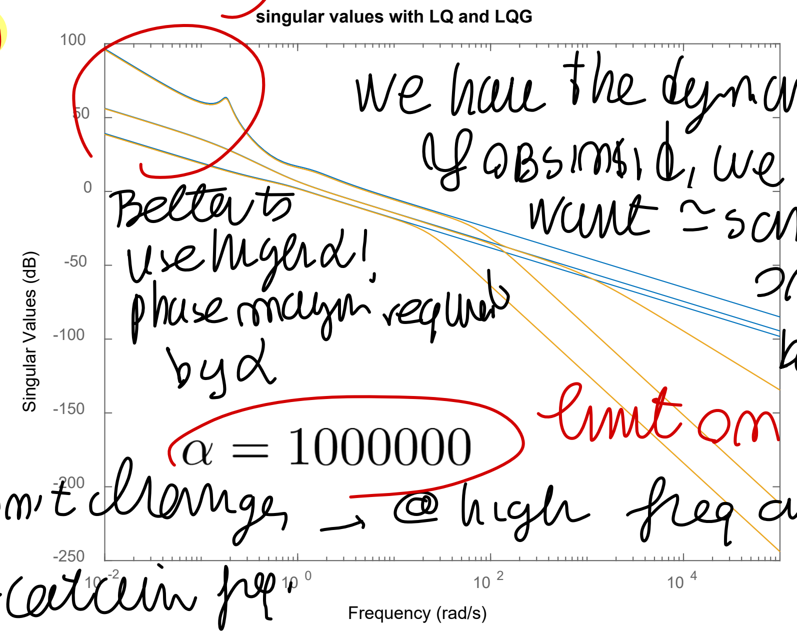
LQG control sing values

$(Q = I, R = I)$ LQ controller

$(\tilde{Q} = \alpha BB', \tilde{R} = I)$

OBS matrix according to LTR procedure, automatic satisfy zero condition





we have the dynamic of OBS inside, we want ≃ same behav on large band!

Better to use high α! phase margin required by α

limit on α_{MAX}!

Increasing α value, meas/unmeas & don't change → @ high freq are different!
L_a^1, L_a^2 valid until a certain freq.

Now, let's go back to Chapter 3

(regulator design with H2 – Hinf with shaping functions for SISO systems)

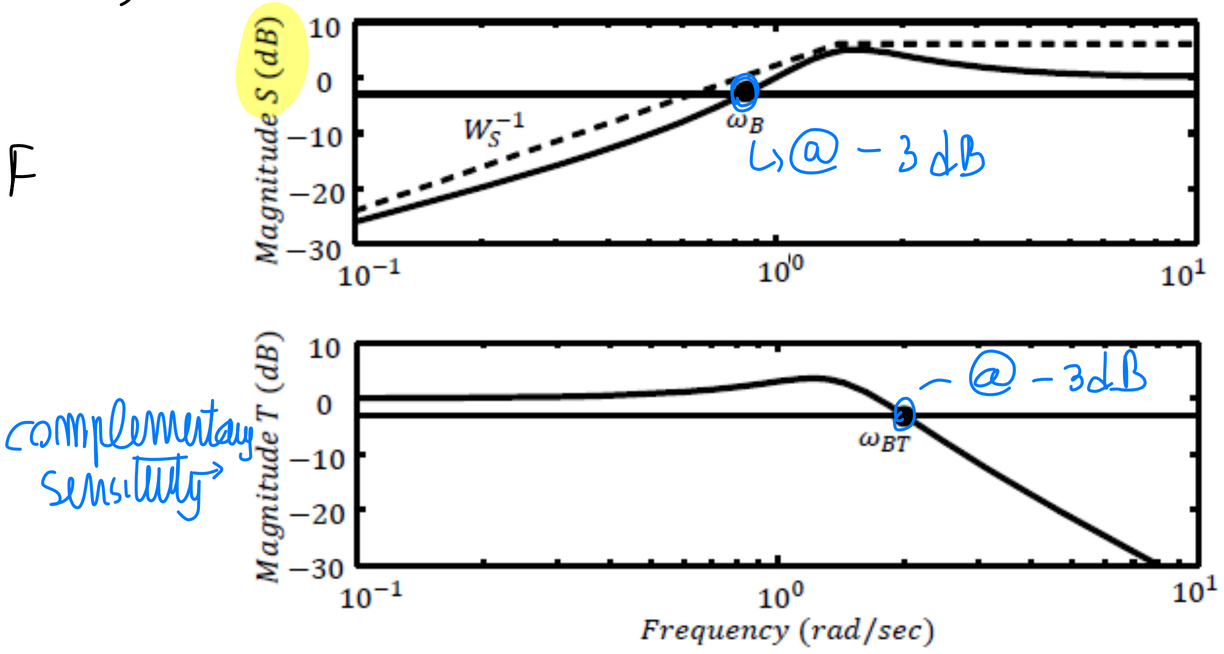
previously introduced/formulated problem, solved by a software → now we can extend for MIMO system!

- to show how LQG is a particular case of H_2
- and reformulate H_∞ such that solution can be given made by standard structure: state obs + state feedback control law on obs state

summarize what we see (on chapter 3)

Sensitivity functions and crossover frequency

on SISO case you can consider sensitivity function instead of loop T.F



Define by ω_B the frequency where $|S(j\omega)|$ crosses $1/\sqrt{2}$ (-3dB) from below and by ω_{BT} the frequency where $|T(j\omega)|$ crosses $1/\sqrt{2}$ (-3dB) from above

Then, if $\varphi_m < 90^o$, one has

$$\boxed{\omega_B < \omega_c < \omega_{BT}}$$

affecting system responce speed!

Also in this case, specifications can be given in terms of *S(s), T(s)*

Design specifications in terms of sensitivity functions

We could specify:

- shape of $S(s)$;
- minimum frequency ω_B;
- small or null asymptotic error for constant reference signals ($|S(j\omega)|$ small or Bode diagram of $|S(j\omega)|$ with shape +1 at low frequency);
- $M_S \leq \bar{M}_S$. ask for a max value of M_S (H_∞ gain limited)

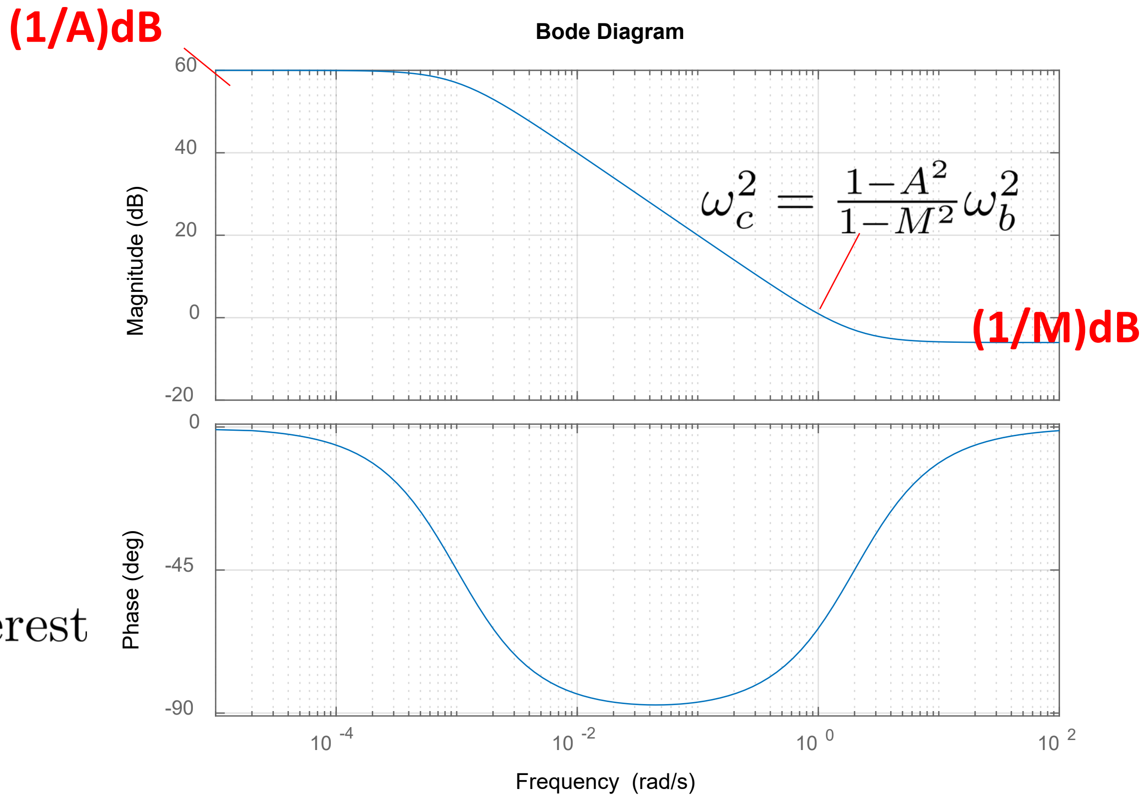
this defines a function $S_{desired}(s)$, and the function $W_S(s) = S^{-1}_{desired}(s)$ also named *(sensitivity) shaping function*

then, the regulator must be designed such that

$$|S(j\omega)| < \frac{1}{|W_S(j\omega)|}, \quad \forall\omega \quad \longleftrightarrow \quad \|W_S S\|_\infty < 1$$

Possible choice of $W_S(s)$

$$W_S(s) = \frac{s/M+\omega_B}{s+A\omega_B}$$



$A \ll 1$: desired attenuation of $S(s)$ in the band of interest

M required bound on the H_∞ norm of $S(s)$

How to synthetise the regulator? We'll see later in the course

Design specifications in terms of complementary sensitivity function

proper inverse!

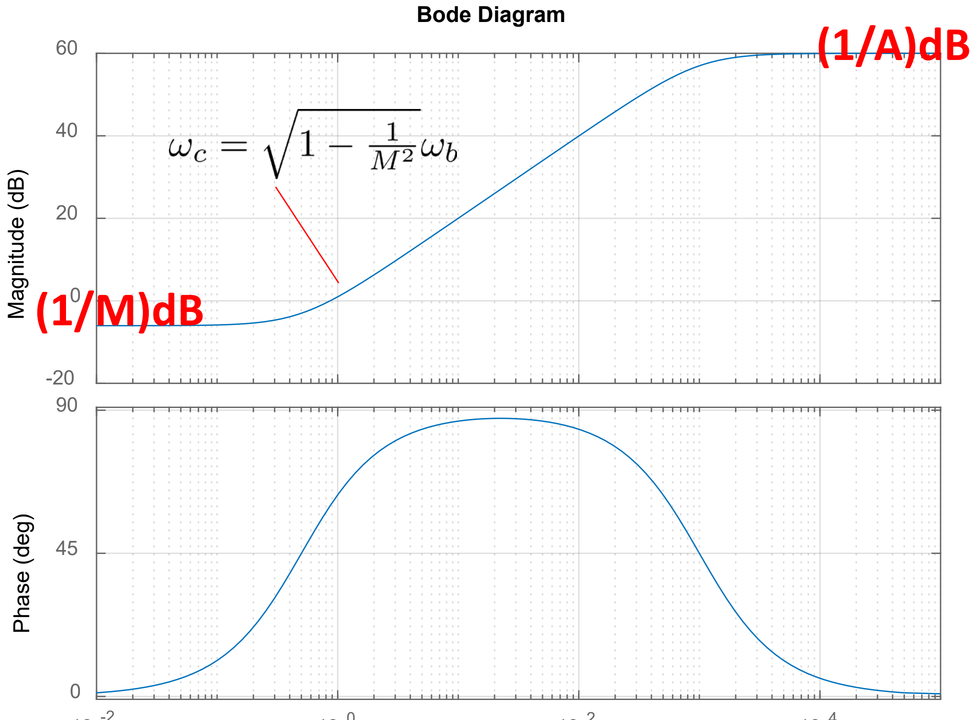
Also in this case, define a $T_{desired}(s)$, and its inverse $W_T(s) = T^{-1}_{desired}(s)$ also named *shaping function* $W_T(s)$

then, the regulator must be designed such that

$$|T(j\omega)| < \frac{1}{|W_T(j\omega)|}, \quad \forall\omega \quad \longleftrightarrow \quad \|W_T T\|_\infty < 1$$

Example

$$W_T(s) = \frac{s + \omega_{BT}/M}{As + \omega_{BT}},$$



Control sensitivity function

Same approach, define the control sensitivity function $W_K(s)$ and choose a regulator *R(s)* such that

$$|K(j\omega)| < \frac{1}{|W_K(j\omega)|}, \quad \forall \omega \Leftrightarrow \|W_K K\|_\infty < 1$$

The shaping function $W_S(s)$, $W_T(s)$, $W_K(s)$ must be selected as asymptotically stable systems

+

overall requirement!

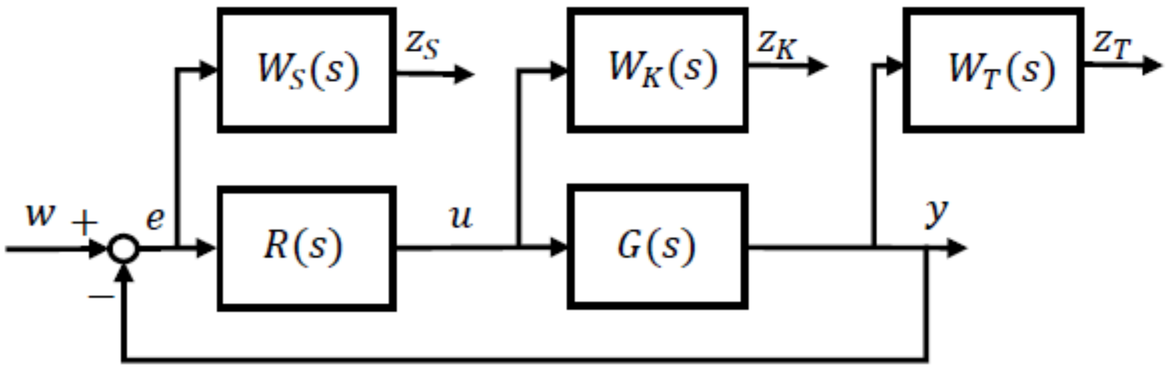
In summary, one must find a regulator such that

$$\|W_S S\|_\infty < 1, \quad \|W_T T\|_\infty < 1, \quad \|W_K K\|_\infty < 1$$

regulator such that this conditions are satisfyed

***Control problem formulation** (Chapter 3 SISO case - shaping functions at the output)*

Consider the enlarged system:



Define $$z=\begin{bmatrix} z_S \\ z_K \\ z_T \end{bmatrix}, \quad w = y^o$$ and note that $$z = G_{zw} w \quad , \quad G_{zw}(s) = \begin{bmatrix} W_S(s)S(s) \\ W_T(s)T(s) \\ W_K(s)K(s) \end{bmatrix}$$

containing shaping functions

shaping functions

The control problem consists in $$\min_R \quad \|G_{zw}\|_2 \text{ or } \|G_{zw}\|_\infty$$

Why do we need to use the shaping functions? Because otherwise the regulator would be null and because we want to specify the shape of the sensitivity functions

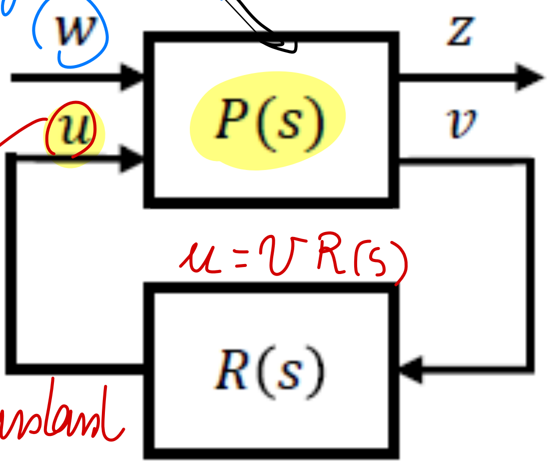
by define:

plant + shaping functions

exogenous input := set point / noise /...

Formal statement of *H2 – Hinf* control

state the problem as:



$u = v\,R(s)$

Inputs from standard scheme feedback evaluated from meas. output u through R(s)

Minimize the *2-norm* or *inf-norm* of $\boldsymbol{G_{zw}(s)}$ with respect to ***R(s)***

where

z are named *performance variables* z_S, z_K, z_T

v are named *measured variables*

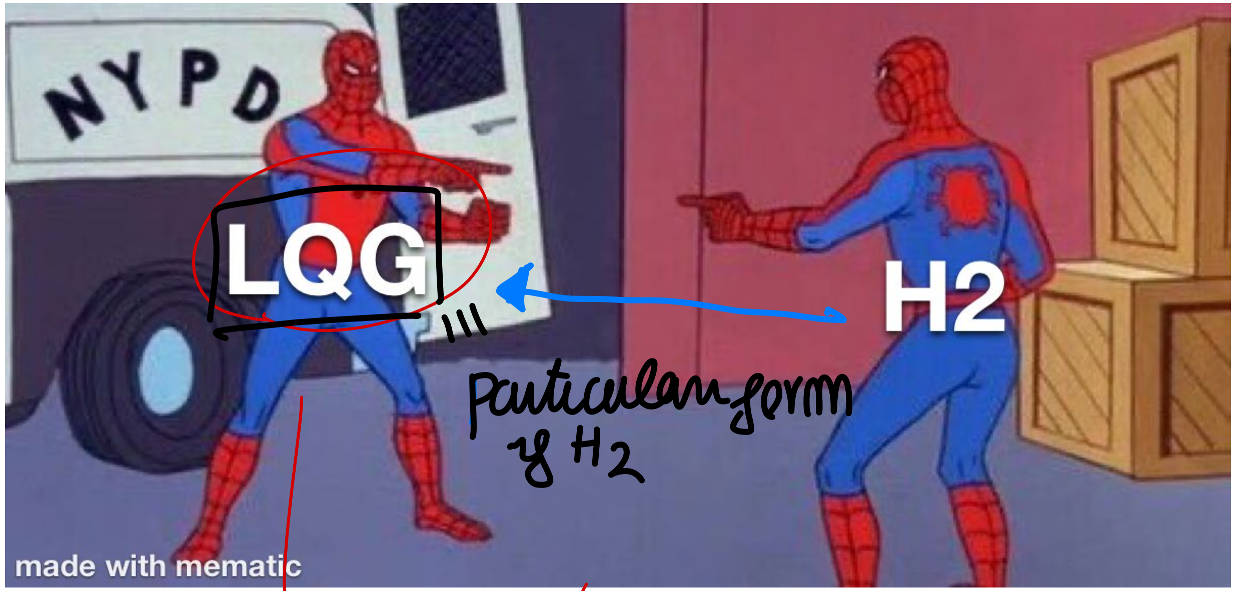
$w = \begin{bmatrix} d \\ y^o \\ n \end{bmatrix}$ are the *exogenous variables*

↳ min H_2, H_∞
chosing properly regulator $R(s)$!

this automatic regulator design is efficient!

→ on our case...

back to LQG — very usefull techniques to control systems with lin. model subject to noises on state/output
↳ with standard estimation problem statement
↓
related to H_2 / H_∞ control



$\equiv$ particular form of H_2

LQG is essentially a particular case of H_2 control!

→ reformulate LQG in terms of H_2

⇒

LQG and H2 control

(LQG equivalent cost funct. formulation)

The cost function $J = E\left[x'(t)Qx(t) + u'(t)Ru(t)\right]$ can be written as

$$J = E\left[x'(t)Qx(t) + u'(t)Ru(t)\right] = E\left[\underbrace{\begin{bmatrix} x' & u' \end{bmatrix}\begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}}_{z'}\underbrace{\begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}}\right]$$

$$z = \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix}$$

performance var := to min on J

define:

cov of v_x

w with unit cov matrix!

Let $\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \tilde{Q}^{1/2} & 0 \\ 0 & \tilde{R}^{1/2} \end{bmatrix} w$ where w is a white noise with zero mean and unit variance

cov of v_y

$w \sim WN(0,1)$

and define $v = y$. Then

↳ v as set of meas. variables → meas. var.

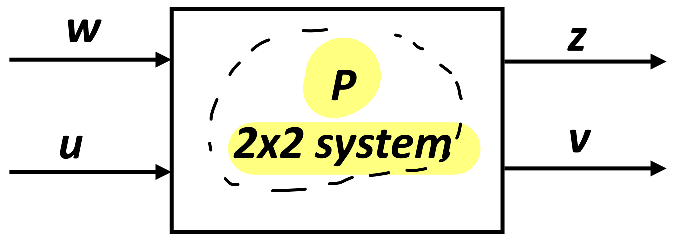
system description

H∞ control description

state equation $\dot{x} = Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu$

performance variable $z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u$

measured variable $v = Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w = y$

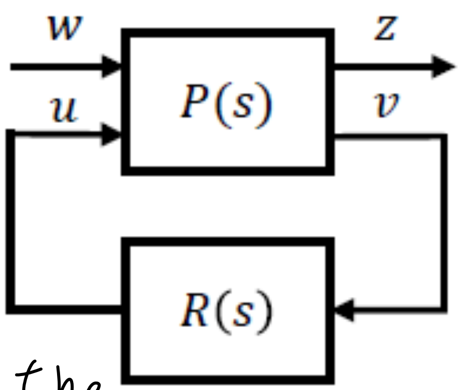


$[\min \{z^T z\}]$ ≡ LQG control!

(on the '60)

LQG and H2 control

↳ (in the '70/'80) → extused in mech field NOT on process control



The ***LQG*** problem consists in computing a regulator such that

$$min_R \quad J = E[z'z] \quad \longleftrightarrow \quad min_R \quad \|G_{zw}\|_2$$

equivalent to → minimize the 2 norm of (G_{zw}) from $w \to z$ of closed loop system

with

$$\begin{cases} \dot{x} = Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\ z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\ v = Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w \end{cases}$$

$\dot{x}$ depends on x, w, u

...

LQG ≃ H_2 control particular formulation!

This is a particular H2 control problem (go back to Chapters 2 and 3)

⇒ TO GENERALIZE the problem...

standard formulation:

More general formulation of H_2, H_{inf} control problems

system subject to external signals w(t)
+ performance variables (to impose behav.)
which can be IN/OUT...

Enlarged (2x2) plant

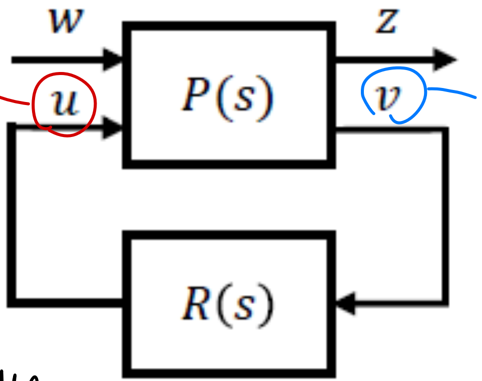
generic enlarged plant!

overall system description!

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ v(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{cases} \longleftrightarrow \quad P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

$B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22}$ are matrices to be suitably selected to meet specific design requirements

standard INPUT



measurable output

Goal → solve the problem of H_2, H_∞ control by minimize L_2, L_∞ Gain!

$$\min_R \; \underbrace{\|G_{zw}\|_2}_{} \text{ or } \underbrace{\|G_{zw}\|_\infty}_{}$$

Respect R(s) minimize

such that!
once you have
P(s) → you wanna close the loop through
R(s)

⇓

LQG as
particular case of this general case!

More general formulation of H_2, H_{inf} control problems

(SYSTEM MATRIX) contains all info

Enlarged (2x2) plant

2x2 syst description

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{aligned}
\quad \longleftrightarrow \quad
P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}
$$

$B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22}$ are matrices to be suitably selected to meet specific design requirements

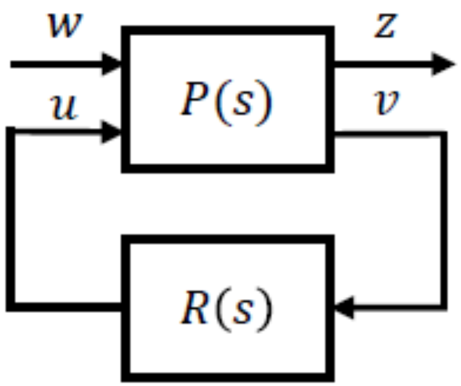
different design results! dependently on that MATRIX

from a 2x2 system described on that way

Goal

and min that norm

$$\min_R \quad \|G_{zw}\|_2 \text{ or } \|G_{zw}\|_\infty$$



in particular!

LQG as H_2 control problem

specifically design! syst represent in H_2/H_∞

In LQG control formulation

$$
\begin{cases}
\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{cases}
\quad \longleftrightarrow \quad
\begin{cases}
\dot{x} = Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\
z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
v = Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w
\end{cases}
$$

equivalence

using standard sw for this generic system simple apply algorithm on H_2/H_∞ specify that matrix, and solve LQG...

⇓ with this matrix definition

$$B_1 = \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} \quad , \quad B_2 = B \quad , \quad C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \quad , \quad C_2 = C \qquad D_{21} = \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} \quad , \quad D_{22} = 0$$

In many cases may be difficult to properly solve in $H_2 - H_\infty$ formulation!

→ to have specific performances desired:

How can we solve the problem with shaping functions?

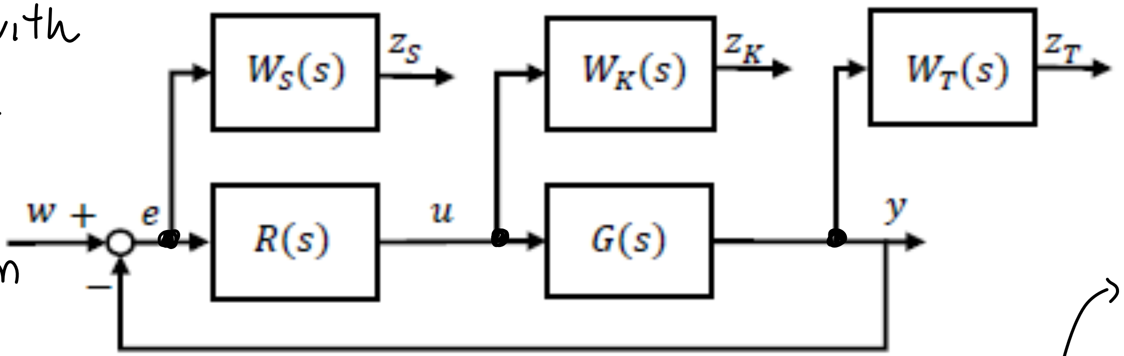
In general H control can specify the shaping functions...

We enlarge the system with the shaping functions at the process output (the same could be done at the process input, see the next slides)

to extend approach using shaping functions

with this design technique, we enlarge the system with W_S, W_K, W_T shaping functions!

Enlarge and then apply H_2-H_∞ theory to design $R(s)$ which min $\|G_{zw}\|_2$



(specified by us)
W_S, W_K, W_T square matrices!
typically selected diag!

How to do that?

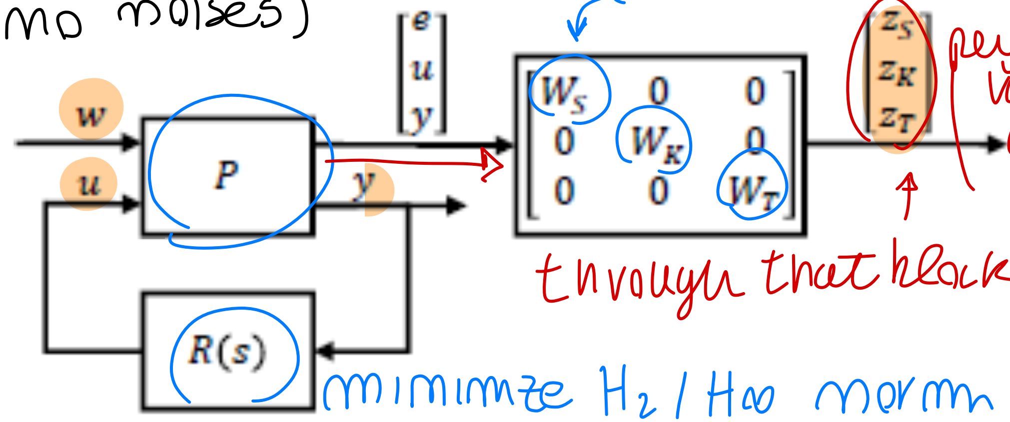
See the textbook pag. 160-162 and the Matlab example discussed in the following

Using shaping functions, assume made by (w, u) inputs on plant P

overall system (same represent) (multi dimensions)

($y = v$ now assume no noises)

enlarge your plant automatically



z

performance var of design phase

through that block you obt

enlarge system and use Matlab augmTF

(DOF given by shaping functions)

minimize H_2 / H_∞ norm by $R(s)$!

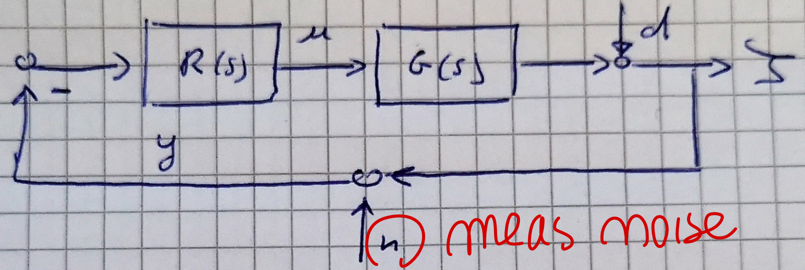
here we use shaping functions W_T, W_S, W_K @ system output
(by acascade)

↳ in general you can
change the vapresentation
using shaping function
@ input !

Example H_2 and "standard" synthesis problem

standard design problem!

Consider the feedback system (SISO system)



variable ξ

$$\begin{cases} \dot{x} = Ax + Bu & \text{(system dyn)} \\ \xi = Cx + d \\ y = Cx + n + d \end{cases}$$

y: output put into Reg R

performance var. $z = \begin{vmatrix} \xi \\ u \end{vmatrix}$, $w = \begin{vmatrix} d \\ n \end{vmatrix}$ exogenous signal, $v = y$

rewrite the problem as

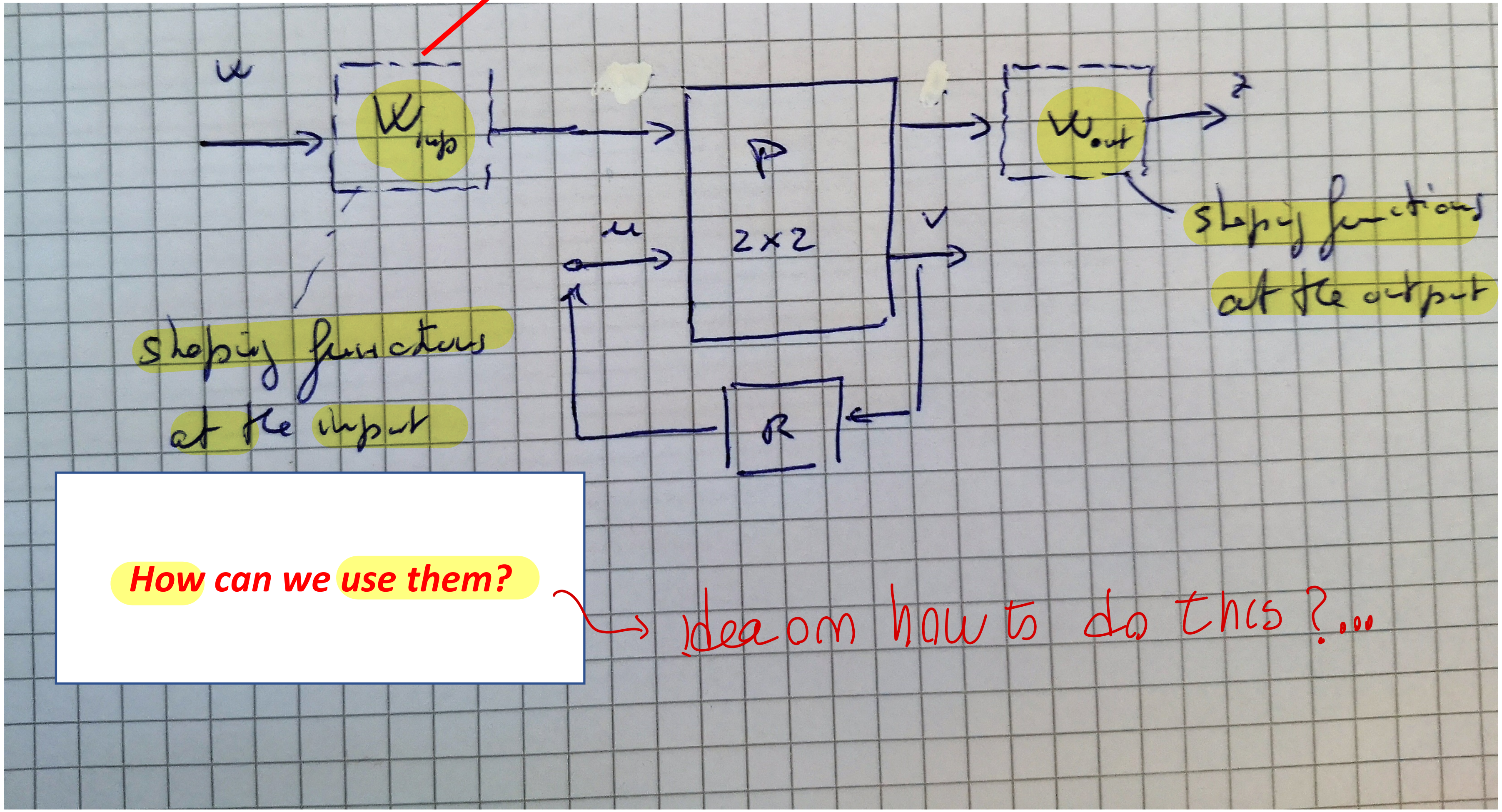
$$\begin{bmatrix} \dot{x} \\ z = \begin{vmatrix} \xi \\ u \end{vmatrix} \\ v = y \end{bmatrix} = \begin{bmatrix} A & B_1 = |1 \;\; 0| & B_2 = B \\ C_1 = \begin{vmatrix} C \\ 0 \end{vmatrix} & D_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & D_{12} = \begin{vmatrix} 0 \\ I \end{vmatrix} \\ C_2 = C & D_{21} = |I \;\; I| & D_{22} = 0 \end{bmatrix} \begin{vmatrix} x \\ w = \begin{vmatrix} d \\ n \end{vmatrix} \\ u \end{vmatrix}$$

$D_{22} = 0$ → not direct influence of u

Regulator design by H_2-H_∞ approach → from here minimize J

Alternatively, we could consider shaping functions at the input

Instead of using output shaping function you can use it at the input !



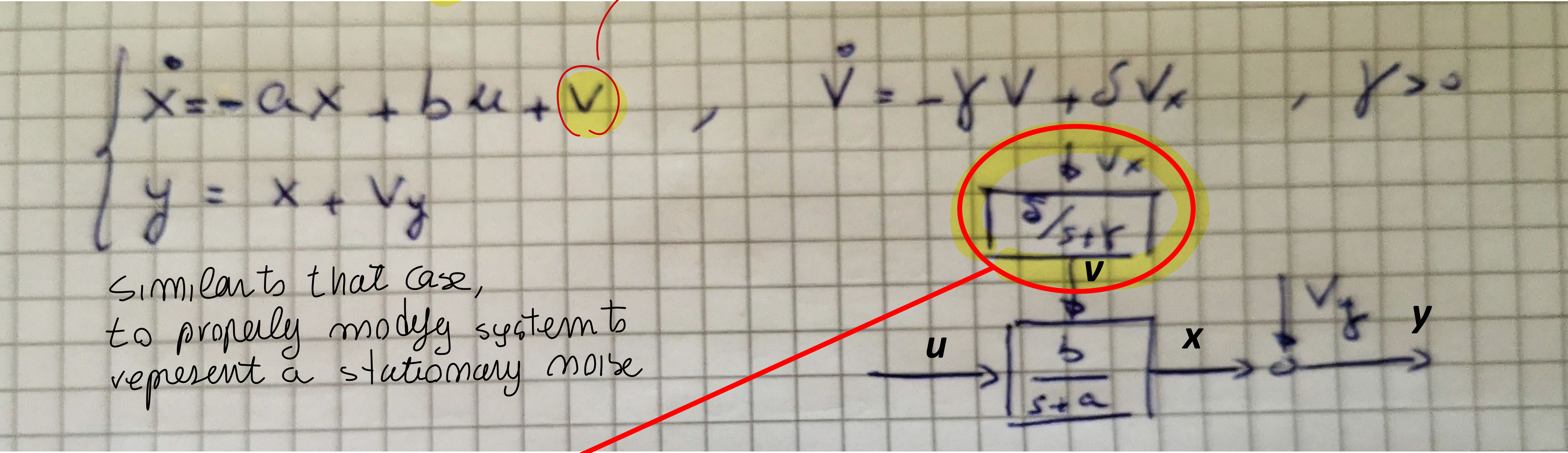
How can we use them? → idea on how to do this ?...

Shaping functions at the input

also stationary noise ≃ output of WN through some system

Remember the exercise on the KF

NOT WN, filter through T.F

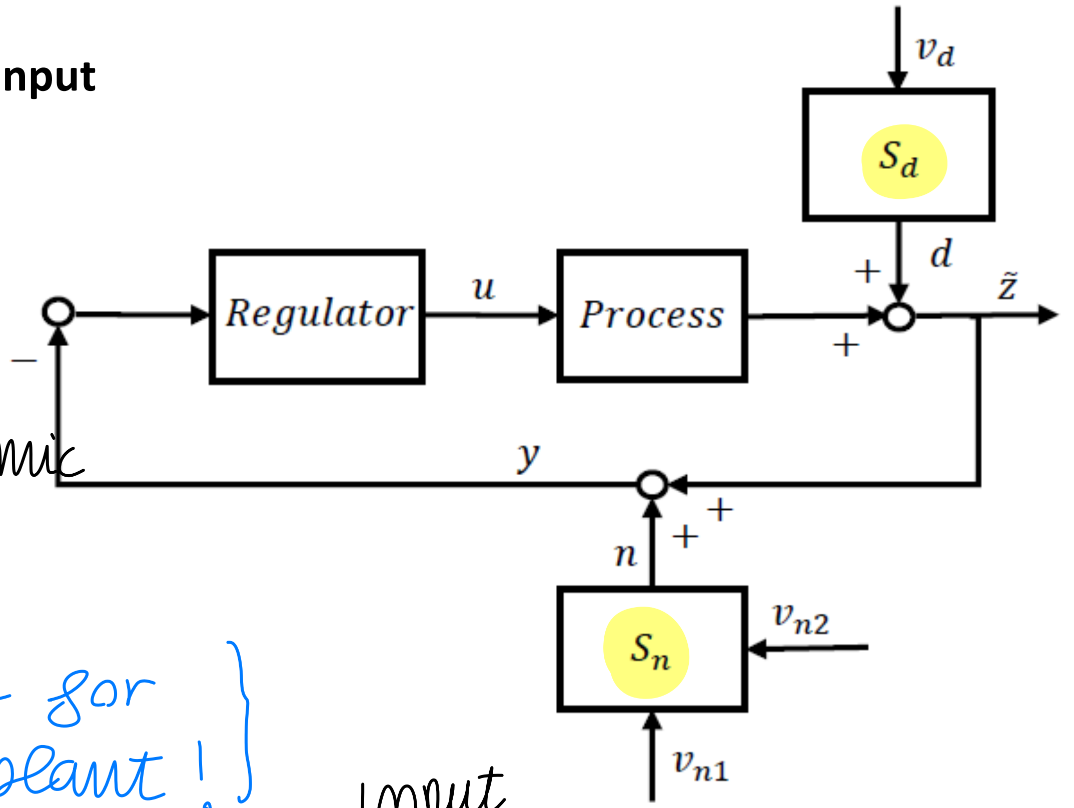


this can be seen !

This could be interpreted as a shaping function at the input, which modifies the characteristics of the KF

General idea

Basic ideas behind the use of shaping functions at the process input



Consider the system

v_d, v_{n1}, and v_{n2} are uncorrelated white noises

↓ let pass input through that dynamic

S_d, S_n are asymptotically stable dynamic systems

d and n are stationary noises

{ than solve for enlarged plant! }

input shaping function

the feedback system is asymptotically stable

↓

x, u, y, and $\tilde{z}$ are stationary stochastic processes

$$\begin{aligned} \tilde{Z}(s) &= -T(s)N(s) + S(s)D(s) \\ U(s) &= -R(s)S(s)\left[N(s) + D(s)\right] \end{aligned}$$

Recap of results on stationary stochastic signals

Consider a stationary stochastic signal *x(t)*

The autocorrelation function is $R_{xx}(t+\tau) = E[x(t)x(t+\tau)] = R_{xx}(\tau)$

The power spectral density is $\Phi_{xx}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau$

$$\begin{aligned}\tilde{Z}(s) &= -T(s)N(s) + S(s)D(s)\\ U(s) &= -R(s)S(s)\left[N(s)+D(s)\right]\end{aligned}$$

Recall the cost function to be minimized $J = E\left[\tilde{z}'(t)\tilde{z}(t) + u'(t)u(t)\right]$

$$E\left[\tilde{z}'(t)\tilde{z}(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Phi_{\tilde{z}\tilde{z}}(\omega)\right\} d\omega$$

$$E\left[u'(t)u(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Phi_{uu}(\omega)\right\} d\omega$$

$$\begin{aligned}tr\left\{\Phi_{\tilde{z}\tilde{z}}(\omega)\right\} &= tr\left\{T(j\omega)\Phi_{nn}(\omega)\, T'(-j\omega)\right\} + \\ &\quad + tr\left\{S(j\omega)\Phi_{dd}(\omega)\, S'(-j\omega)\right\}\end{aligned}$$

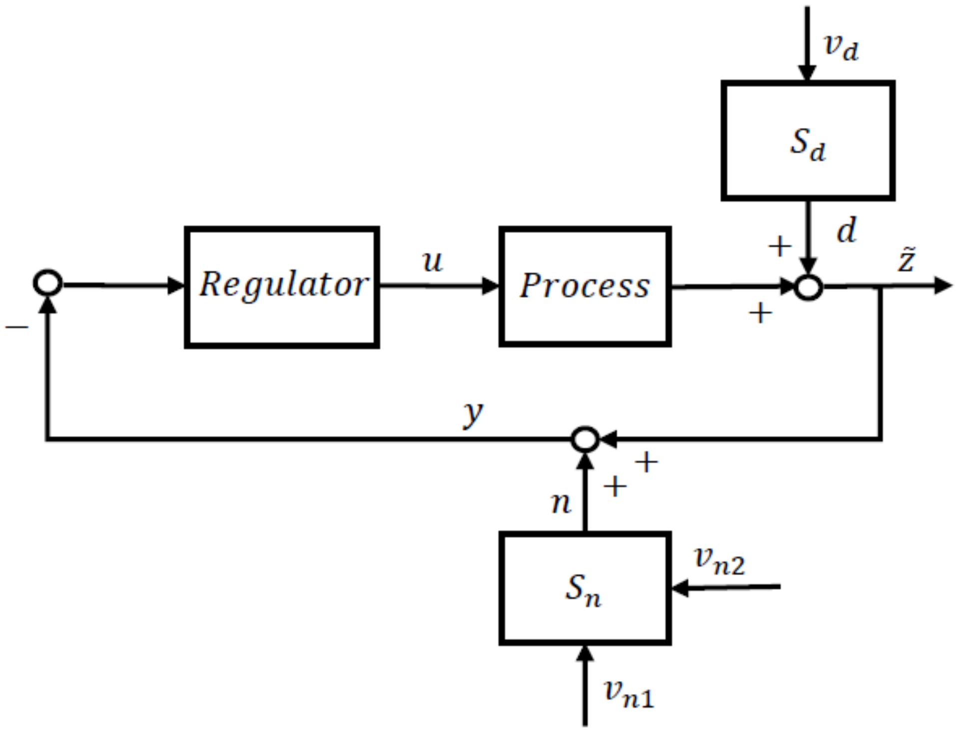
$$\begin{aligned}tr\left\{\Phi_{uu}(\omega)\right\} &= tr\{R(j\omega)S(j\omega)* \\ &\quad *\left(\Phi_{nn}(\omega)+\Phi_{dd}(\omega)\right) S'(-j\omega)R'(-j\omega)\}\end{aligned}$$

$$tr\left\{\Gamma(j\omega)\Gamma'(-j\omega)\right\} \qquad \Gamma(j\omega) = \begin{bmatrix} T(j\omega)\Phi_{nn}^{1/2}(\omega) \\ S(j\omega)\Phi_{dd}^{1/2}(\omega) \\ R(j\omega)S(j\omega)\left(\Phi_{nn}(\omega)+\Phi_{dd}(\omega)\right)^{1/2} \end{bmatrix}$$

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Gamma(j\omega)\Gamma'(-j\omega)\right\} d\omega = \left\|\Gamma(j\omega)\right\|_2^2$$

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\Gamma(j\omega)\Gamma'(-j\omega)\right\} d\omega = \|\Gamma(j\omega)\|_2^2 \qquad \Gamma(j\omega) = \begin{bmatrix} T(j\omega)\Phi_{nn}^{1/2}(\omega) \\ S(j\omega)\Phi_{dd}^{1/2}(\omega) \\ R(j\omega)S(j\omega)\left(\Phi_{nn}(\omega) + \Phi_{dd}(\omega)\right)^{1/2} \end{bmatrix}$$

shaping functions

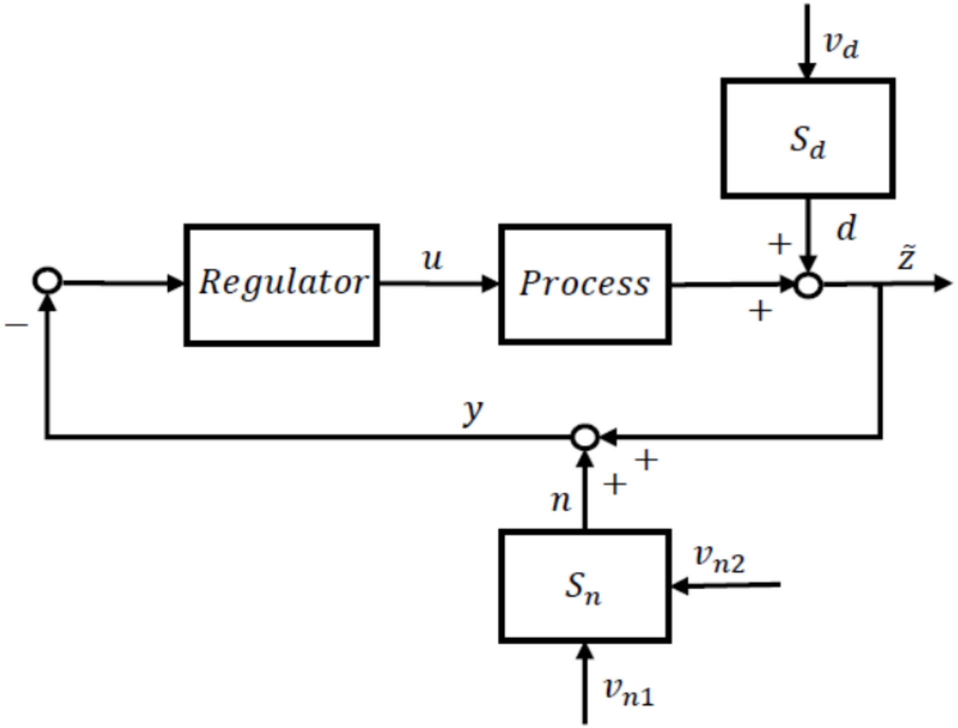


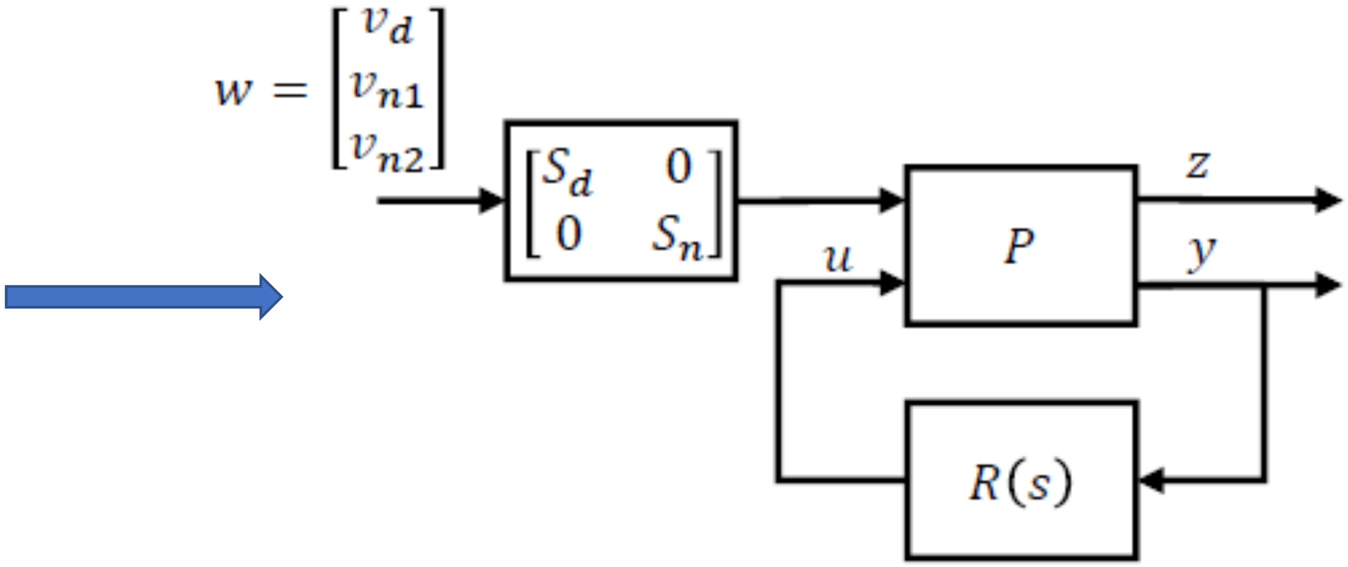
How to define S_d, S_n in order to obtain the required spectral densities Φ_{nn}, Φ_{dd} and the corresponding shaping functions is not trivial (see the notes)

properly weight sensitivity!
complementary ecc..

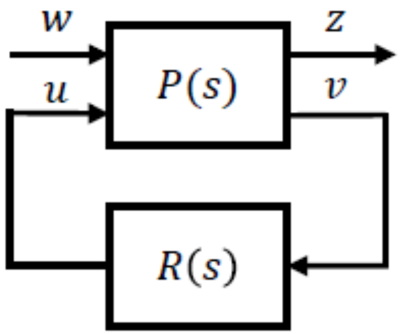
Interpretation as shaping functions at the process input

↓ equivalent to include input shaping functions (instead of output ones)





Partial summary of *H2* – *Hinf* discussion - I



1. LQG is a particular case of H2 control

$$
\begin{aligned}
\dot{x} &= Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\
z &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
v &= Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w
\end{aligned}
\quad \longleftrightarrow \quad
\begin{cases}
\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
v(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t)
\end{cases}
$$

Why to use this structure instead of classic LQG ?

Goal!

$$\min_R \quad \|G_{zw}\|_2$$

many way to design $\tilde{Q}, \tilde{R}$ ...

But sometimes it is difficult to tune LQG with a proper choice of the tuning parameters $Q, R, \tilde{Q}, \tilde{R}$

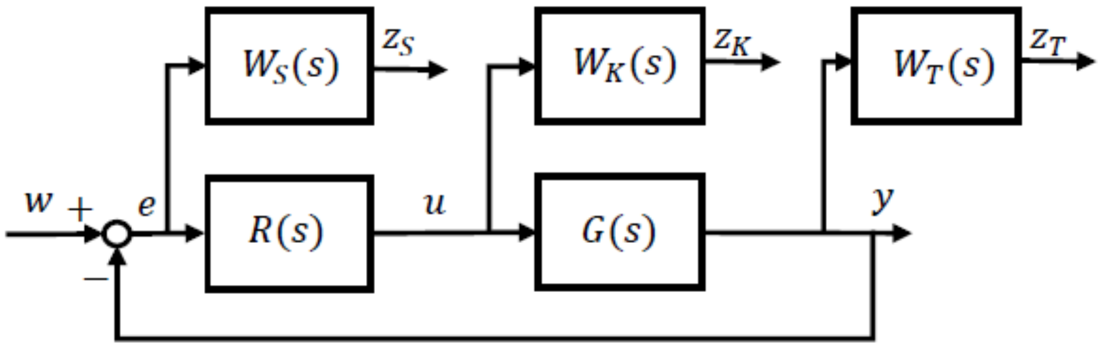
to obtain the desired characteristics (in frequency) of the closed-loop system

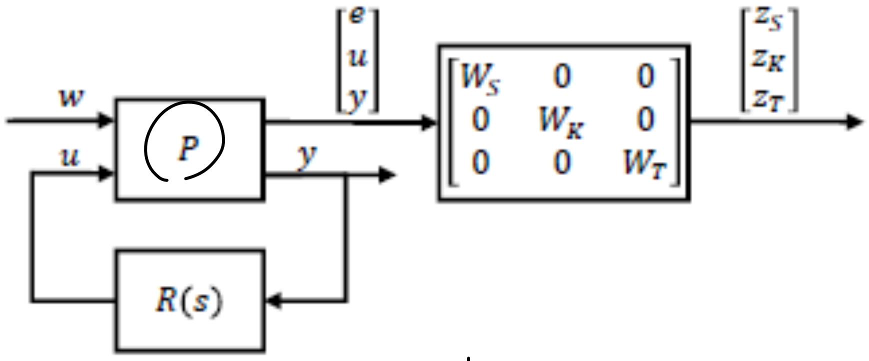
sometimes difficult to specify requirements in terms of LQG, easyer formulation by shaping

Partial summary of *H2 – Hinf* discussion - II

include shaping @ output

2. *We can use shaping functions at the process output*





enlarged system

order defined by P

order + shaping funct order

we enlarge the system!

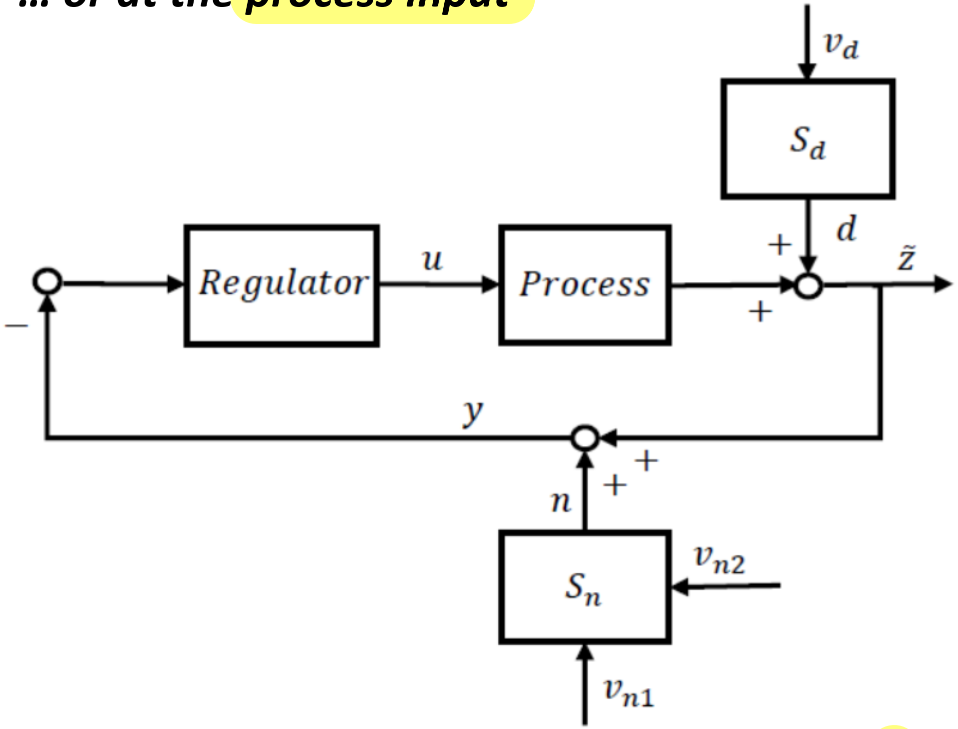
overall system increase order by W..

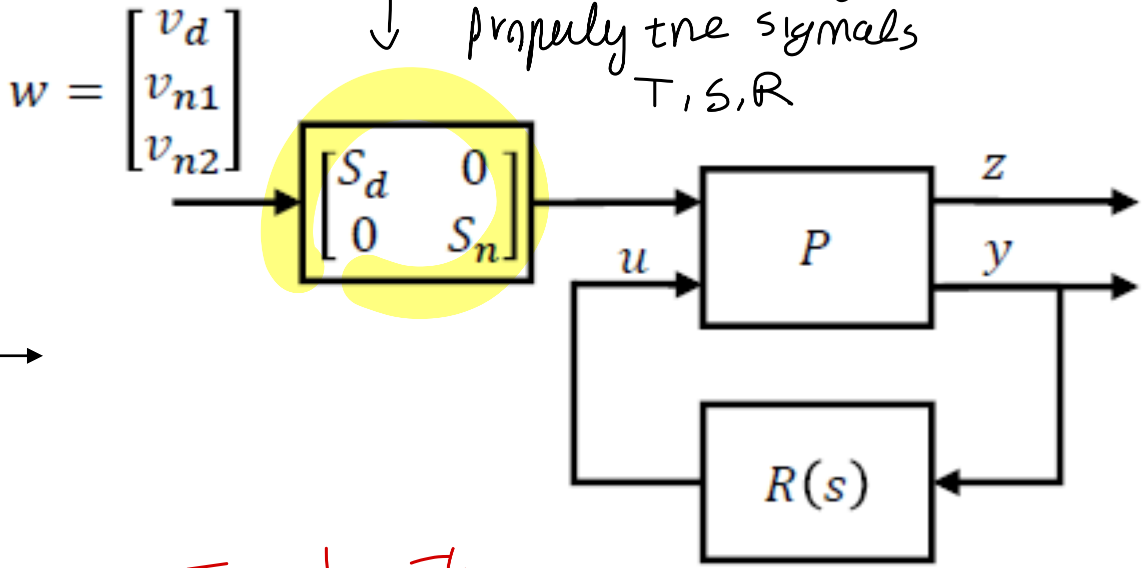
$$G_{zw}(s) = \begin{bmatrix} W_S(s)S(s) \\ W_T(s)T(s) \\ W_K(s)K(s) \end{bmatrix}$$

shaping functions → act on the system!

Partial summary of *H2 – Hinf* discussion - II

… or at the process input





Including shaping functions at the input, if we properly specify S_d, S_n you weight properly the signals T, S, R

output spectral density

from output shaping !

$$\neq \quad G_{zw}(j\omega) = \begin{bmatrix} T(j\omega)\Phi_{nn}^{1/2}(\omega) \\ S(j\omega)\Phi_{dd}^{1/2}(\omega) \\ R(j\omega)S(j\omega)\left(\Phi_{nn}(\omega) + \Phi_{dd}(\omega)\right)^{1/2} \end{bmatrix}$$

shaping functions

In any case, what is the structure of the solution of the $\boxed{H2}$ or $\boxed{Hinf}$ control problem?

$\Downarrow$

$$\min_R \quad \|G_{zw}\|_2 \text{ or } \|G_{zw}\|_\infty$$

once properly stated the problem and defined the shaping functions to obtain.. → than still open problem, ↓ (LQG solution)

we have to solve two Riccati equation to properly obtain observer and state feedback control

← (complete the design with studied tools ↓ in general.. ⇒)

 IN GENERAL...

Structure of the solution of the H_2 control problem

General idea of probeem...
typical
structure of control

on standard formul.
$B_2 \simeq B$

Under suitable assumptions (see the textbook, pag. 162)
(typical LQG assumption)

$$u(t) = -K\hat{x}(t) \quad , \quad K = B_2'P$$

(MATRIX)

control law as static feedback control law using an estimate of state instead of state itself

where ***P>0*** is the (unique positive definite) solution of

(RICCATI EQUATION)

$$A'P + PA - PB_2B_2'P + C_1'C_1 = 0$$

C
Q → linking state to performance vars

(OBSERVER)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_2u(t) + L(y(t) - C_2\hat{x}(t)) \qquad L = \tilde{P}C_2'$$

where $\tilde{P} > 0$ is the (unique positive definite) solution to

$$A\tilde{P} + \tilde{P}A' - \tilde{P}C_2'C_2\tilde{P} + B_1B_1' = 0$$

Basically the same structure of LQG control ! (standard structure)

small differences

Structure of the solution of the H_{inf} control problem

same with small modifications! (ROBUST CONTROL)

The problem is formulated as

$$\min \gamma \text{ such that } \boxed{\|G_{zw}\|_\infty < \gamma}$$

↳ solved by iterative Algorithm

not guarantee a priori! We try to minimize value of scalar γ such that $\|G_{zw}\|_\infty < \gamma$

Under suitable assumptions (see the textbook, pag. 162)

↓

GAIN (output estimation error)

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \gamma^{-2}B_1B_1'P\hat{x}(t) + B_2u(t) + ZL(y(t) - C_2\hat{x}(t)) \\ u(t) = -K\hat{x}(t) \end{cases}$$

↑ estimated state, obtained through standard filter!

state feedback + observer

where

$$Z = (I - \gamma^{-2}SP)^{-1}, \quad L = SC_2', \quad K = B_2'P$$

and ***S***, ***P*** are the solutions to two Riccati equations

↓

Iterative algorithms available

to minimize γ acting on Regulator R

Goal of H_2, H_∞ control... (Exam): ask R form, which answer is correct, {NOT observer expression OR RICCATI EQ} NO

Model reduction - motivations

describe How we can reduce model order! for the situation where you have large models reduced in state number without affecting its validity

Model reduction techniques are useful for many reasons, for example:

1. The original model of the system to be controlled can be in nonminimal form. This frequently happens when modeling large scale systems

   (• When you design complex system → you obtain huge # state model, redundant (NON MINIMAL form!)
   ↳ in modelling a pipe with discretization → you wanna reduce order)

2. The model contains pole/zero pairs that are not coincident, but very near each other. In this case removing these pairs does not reduce the validity of the model

   couple not cancellable, but close in complex plane → removed without affecting too much the model

3. The regulator computed with the ***H_2-H_{inf}*** synthesis with shaping functions is of order equal to the order of the plant + the order of W_S + the order of W_T + the order of W_K. Clearly, this number can become very large and removing poles and zeros almost equal significantly reduces its size

   enlarged plant when using shaping function → can lead to large system!
   so on H_2-H_∞ solution ⇒ Reg order is the same of enlarged plant (high) → difficult to implement!
   design $R(s)$ with H_2-H_∞ than try to reduce order without affecting system performance!

Model reduction

System (can be the regulator)

linear system standard structure

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

obtain a simpler system from the original?

Define

$$\begin{cases} P = \int_0^\infty e^{At}BB'e^{A't}dt \\ Q = \int_0^\infty e^{A't}C'Ce^{At}dt \end{cases}$$

controllability gramian depends on (A, B)

observability gramian depends on (A, C)

hard formula to compute

For asymptotically stable systems ***P*** and ***Q*** can be computed as the positive definite solutions of

$$\begin{cases} AP + PA' + BB' = 0 \\ A'Q + QA + C'C = 0 \end{cases}$$

2 lyap equations solutions

Balanced realization

when computing P, Q (positive def matrix)

In general ***P≠Q***, but with a proper state transformation it is possible to write the system in the *balanced realization* form with

decreasing order

$$P = Q = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad (\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0)$$

singular

Looking at the values of the ***σ***$_i$***'s***, the idea is to find a reduced state space representations with only ***k*** values $\sigma_1, \sigma_2, \ldots, \sigma_k$

checking of different order of magnitude, if $\sigma_{k+1} \gg \sigma_k$, we neglect, reducing syst order!

Accordingly, the balanced system can be partitioned as

partition system

$$A = \begin{array}{c c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \end{array}, \quad B = \begin{array}{c c} & m \\ & \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] & \begin{array}{c} k \\ n-k \end{array} \end{array}, \quad C = \begin{array}{c c} \begin{array}{cc} k & n-k \end{array} & \\ \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] & p \end{array}$$

where $A_{11} \in R^{k,k}$, $B_1 \in R^{k,m}$, $C_1 \in R^{p,k}$, and Σ is given by

$$\Sigma = \left[\begin{array}{cc} \underset{k \times k}{\Sigma_1} & 0 \\ 0 & \underset{(n-k)\times(n-k)}{\Sigma_2} \end{array}\right]$$

⟹ from here... realization form:

with $\Sigma_1 \in R^{k,k}$

Balanced truncation

(simply take only first part of the system)

1 The reduced model is described by (A_{11}, B_1, C_1, D)

(RESULT)

Let $G_a^k(s)$ be the transfer function of the reduced order model and $G(s)$ be the one of the original system

It is possible to prove that

$$\underbrace{\|G(s) - G_a^k(s)\|_\infty}_{\text{error when using reduced order model is}} \leq 2\overbrace{(\sigma_{k+1} + \sigma_{k+2} + \ldots}^{\text{bounded by the summ of neglected sing values}} + \sigma_n)$$

Bound

No state gain specification

Balanced residualization

2 Assume that the dynamics of the states to be neglected can be ignored, i.e. $\dot{x}_2 = 0$

(A_{21}, A_{22}) negligible $\dot{x}_2$ dynamic

set of algebraic equation $\dot{x}_2 = 0$

$A_{21}x_1 + A_{22}x_2 = 0$

$\hookrightarrow x_2 = -A_{22}^{-1}A_{21}x_1$

Assume that A_{22} is nonsingular

$\Downarrow$

The reduced order model is

$$\Sigma : \begin{cases} \dot{x}_1(t) = A_r x_1(t) + B_r u(t) \\ y(t) = C_r x_1(t) + D_r u(t) \end{cases}$$

$$A_r = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_r = B_1 - A_{12}A_{22}^{-1}B_2$$
$$C_r = C_1 - C_2A_{22}^{-1}A_{21}, \quad D_r = D - C_2A_{22}^{-1}B_2$$

∞ error norm of reduction is bounded by reduced sing values

Also in this case $\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_m)$

Moreover $\boxed{G_a^k(0) = G(0)}$, i.e. ***the static gain is maintained***

Guarantee at 0 frequency

Example: linearized model of an aircraft

Maciejowski, Jan Marian. "Multivariable feedback design." *Electronic Systems Engineering Series, Wokingham, England: Addison-Wesley, | c1989* (1989).

5 states

$x1$: altitude relative to some datum (m)

$x2$: forward speed ($\mathrm{m\,s^{-1}}$)

$x3$: pitch angle (degrees)

$x4$: pitch rate ($\mathrm{deg\,s^{-1}}$)

$x5$: vertical speed ($\mathrm{m\,s^{-1}}$)

3 outputs

3 inputs

$u1$: spoiler angle (measured in tenths of a degree)

$u2$: forward acceleration ($\mathrm{m\,s^{-2}}$)

$u3$: elevator angle (degrees)

$$A = \begin{bmatrix} 0 & 0 & 1.1320 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.013 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.120 & 1.0000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.665 \\ 1.5750 & 0 & -0.0732 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

design problem!

D=0

FOLDER_EXAMPLE_LQG_H2_Hinf - file Synthesis_LQG_H2_Hinf.m

```
G=ss(A,B,C,D);
```
← s.s system description

(arbitrary choice)

```
% weight LQ
 q=20;
Q=q*eye(n);
r=4;
R=r*eye(m);
W=blkdiag(Q,R);
```
syst ORD.

define state/control var weights! to use on quadratic cost function

LQ weights

```
% noises covariances
qt=100;
QT=qt*eye(n);
rt=2;
RT=rt*eye(p);
V=blkdiag(QT,RT);
```

same to specify $\tilde{Q}, \tilde{R}$, cov. matrix of acting noises

Noise covariances for KF

s.s Regulator represent.

synthesize LQG controller

```
% synthesis LQG
[ALQG,BLQG,CLQG,DLQG] = lqg(A,B,C,D,W,V);
R=ss(ALQG,BLQG,CLQG,DLQG)
```

LQG synthesis

directly solved by Matlab, compact Algorithm solution

check system behaviours

```
% analysis of the transfer functions
 figure(1)
subplot(221)
sigma(G)
title('singular values of the  system')
hold on
 L=G*R;
subplot(222)
sigma(L)
title('singular values of the loop transfer function')
hold on
 S=inv(eye(p)+L); % Sensitivity
subplot(223)
sigma(S)
title('singular values of the sensitivity transfer function')
hold on
 T=eye(p)-S;     % complementary sensitivity
subplot(224)
sigma(T)
title('singular values of the complementary sensitivity ')
hold on
```

← original syst singular values

← loop T.F computation

sing values of L(s)

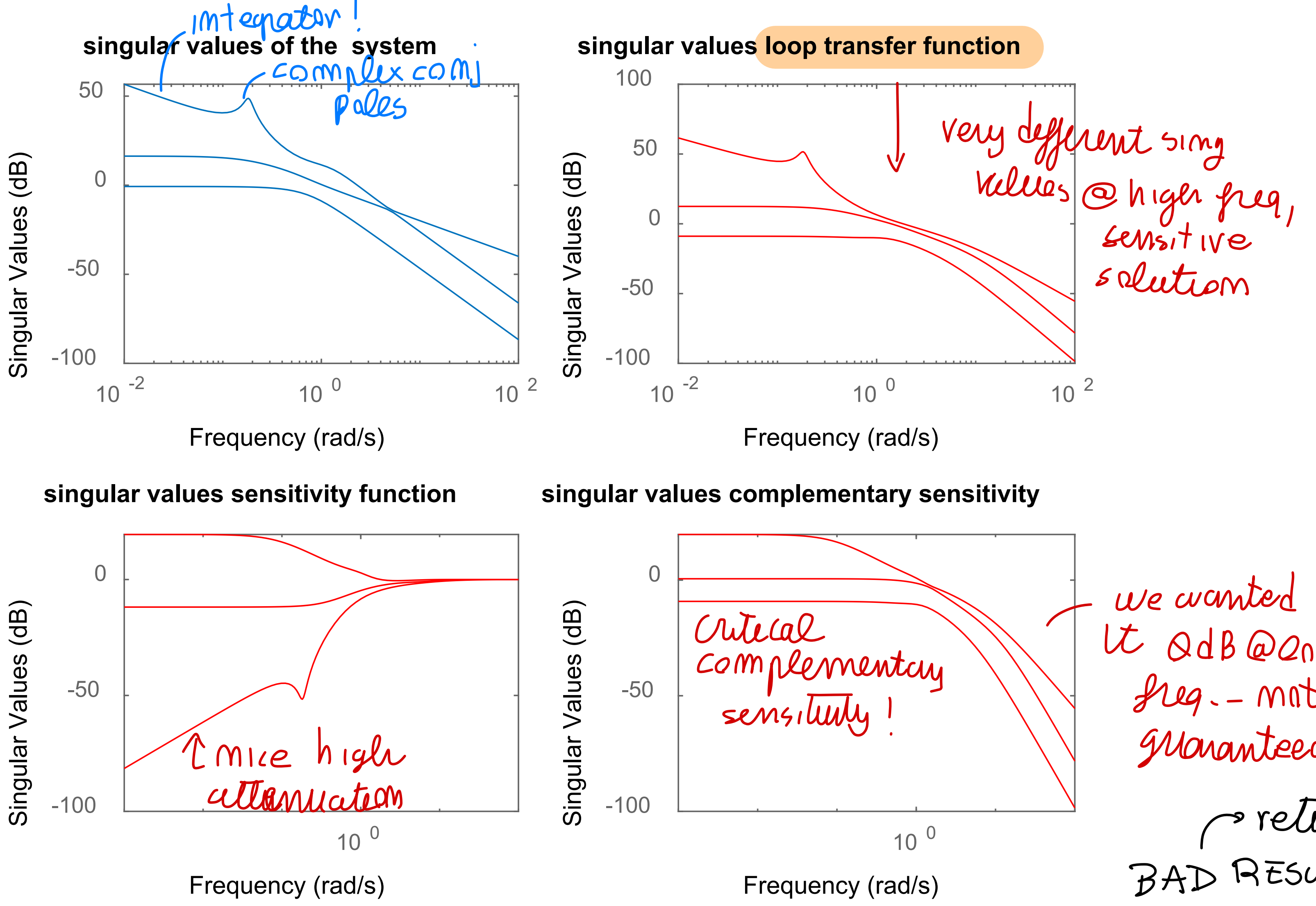
function

Sensitivity functions with LQG

OUTPUT

NOT so good results !

LQG (LQ+KF) (blue), and H2 formulation (red)



returning !

BAD RESULT

↳ from here we don't know how to modify the syst

```
% synthesis with H2, same choice of the matrices used for LQG control
% the controllers must be the same
```

{ H_2 as more general LQG control expression }

```
B2=B;
C2=C;
D22=D;
```

matrix partition of system H2

```
B1=[sqrt(qt)*eye(n) zeros(n,p)];
C1=[sqrt(q)*eye(n);zeros(m,n)];
D11=[zeros(n,n) zeros(n,p);zeros(m,n) zeros(m,p)];
D12=[zeros(n,m);sqrt(r)*eye(m)];
D21=[zeros(p,n) sqrt(rt)*eye(p,p)];
```

H_2 control matrices!

```
AA=A;
BB=[B1 B2];
CC=[C1;C2];
DD=[D11 D12;D21 D22];
%
P=ss(AA,BB,CC,DD);
P1=mktito(P,p,m); % enlarged system for - H2 Hinf synthesis
```

system

make 2x2 system (2 IN, 2 out "tito")

```
[K2,CL2,GAM2,INFO2] = h2syn(P1); % synthesis H2 with the same characteristics LQG, the regulator is K2
```

H2 control law computation!

K2 reg has to be similar to LQG one!

General formulation 2x2 system

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ v(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{cases}$$

Specific choice - LQG

$$\begin{cases} \dot{x} = Ax + \begin{bmatrix} \tilde{Q}^{1/2} & 0 \end{bmatrix} w + Bu \\ z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\ v = Cx + \begin{bmatrix} 0 & \tilde{R}^{1/2} \end{bmatrix} w \end{cases}$$

```
K2;
LK=G*K2;
 subplot(222)
sigma(LK,'r')
title('singular values loop transfer function')
 SK=inv(eye(p)+LK); % Sensitivity
subplot(223)
sigma(SK,'r')
title('singular values sensitivity function')
 TK=eye(p)-SK;      % complementary sensitivity
subplot(224)
sigma(TK,'r')
title('singular values complementary sensitivity')
```

all computations as before..

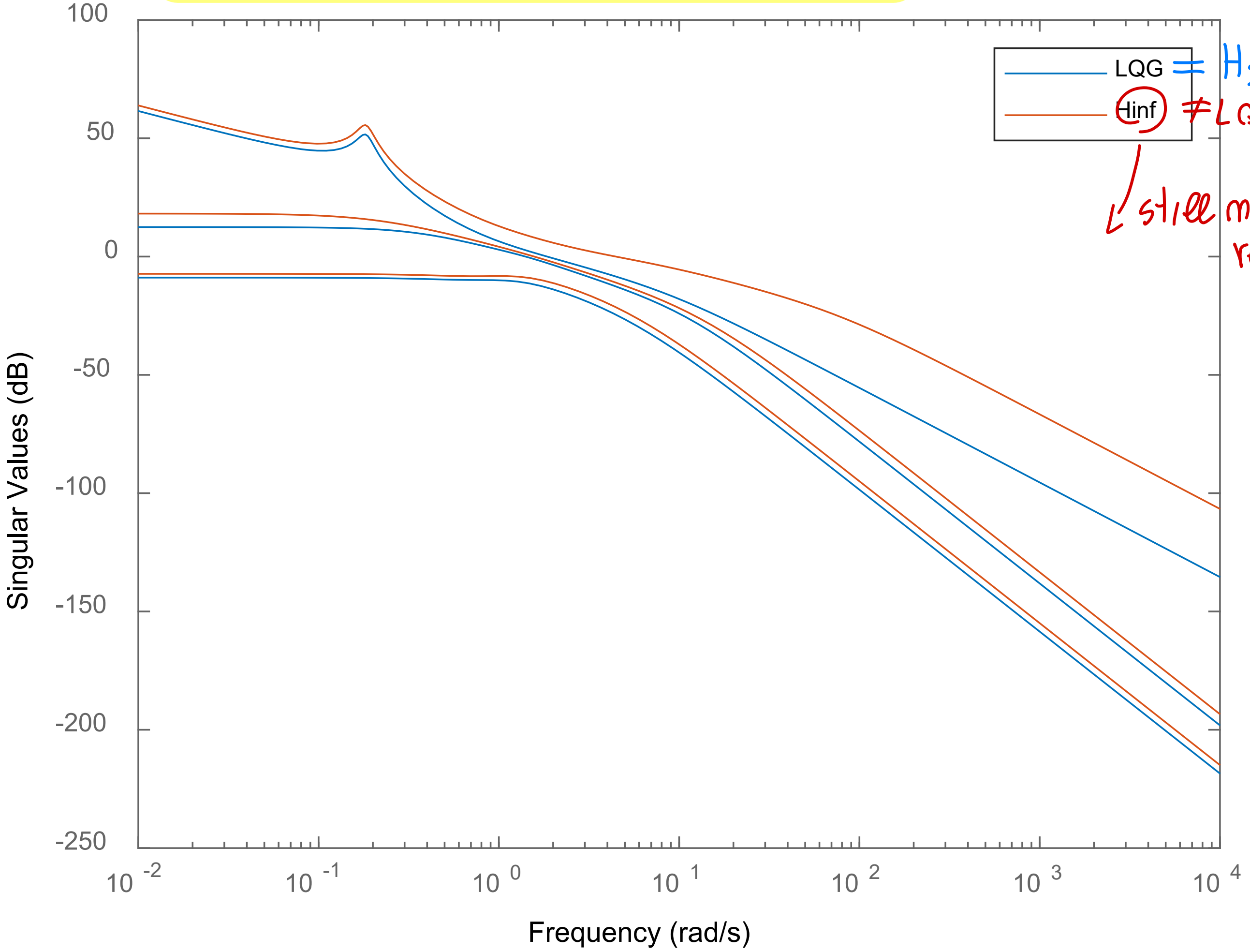
Sensitivity functions with H2 (same as LQG)

same results!

```
[Kinf,CLinf,GAMinf,INFOinf] = hinfsyn(P1); % synthesis Hinf with the same characteristics of LQG, the regulator is Kinf
 Kinf;
LKi=G*Kinf;
 figure(2)
sigma(L)
hold on
sigma(LKi)
title('singular values loop transfer function with LQG and Hinf')
legend('LQG','Hinf')
```

design by H∞ techniques
with same LQG characteristics design

singular values loop transfer function with LQG and Hinf



= H_2 (same)

$\neq$ LQG

still not so good! retune design parameter properly!

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% synthesis with shaping functions, omega = 1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% definition of the shaping functions
% and Bode diagrams
    wB=1;     % desired closed-loop bandwidth
    AA=1/1000;  % desired disturbance attenuation inside bandwidth
    M=2 ;      % desired bound on hinfnorm(S) & hinfnorm(T)
    s=tf('s'); % Laplace transform variable 's'
    WS=(s/M+wB)/(s+wB*AA); % Sensitivity weight
    WK=(0.001*s+1)/(0.01*s+1);  % Control weight can't be empty (d12).ne.0)
    WT=(s+wB/M)/(AA*s+wB); % Complementary sensitivity weight
    figure(3)
    bode(WS)
    hold on
    bode(WK)
    bode(WT)
    grid
    legend('WS','WK','WT')
```

we can moove from standard H∞ formulation to shaping funct to specify desired characteristics

← overshoot

$\rightarrow W_K = 1$ design this way

Bode diag of shaping functions

$$W_S(s) = \frac{s/M+\omega_B}{s+A\omega_B}$$

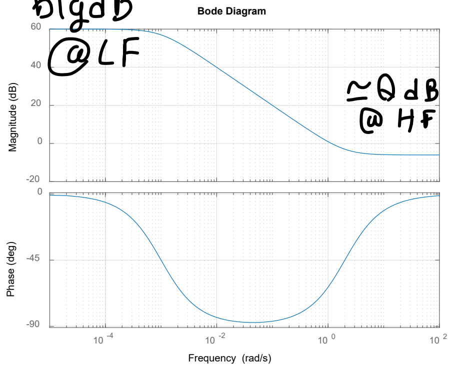
reasonable choise

to force for each A matrix

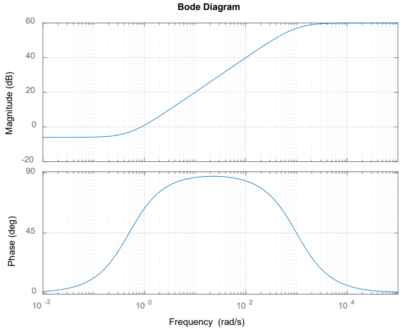
Scalar shaping functions

big dB @LF

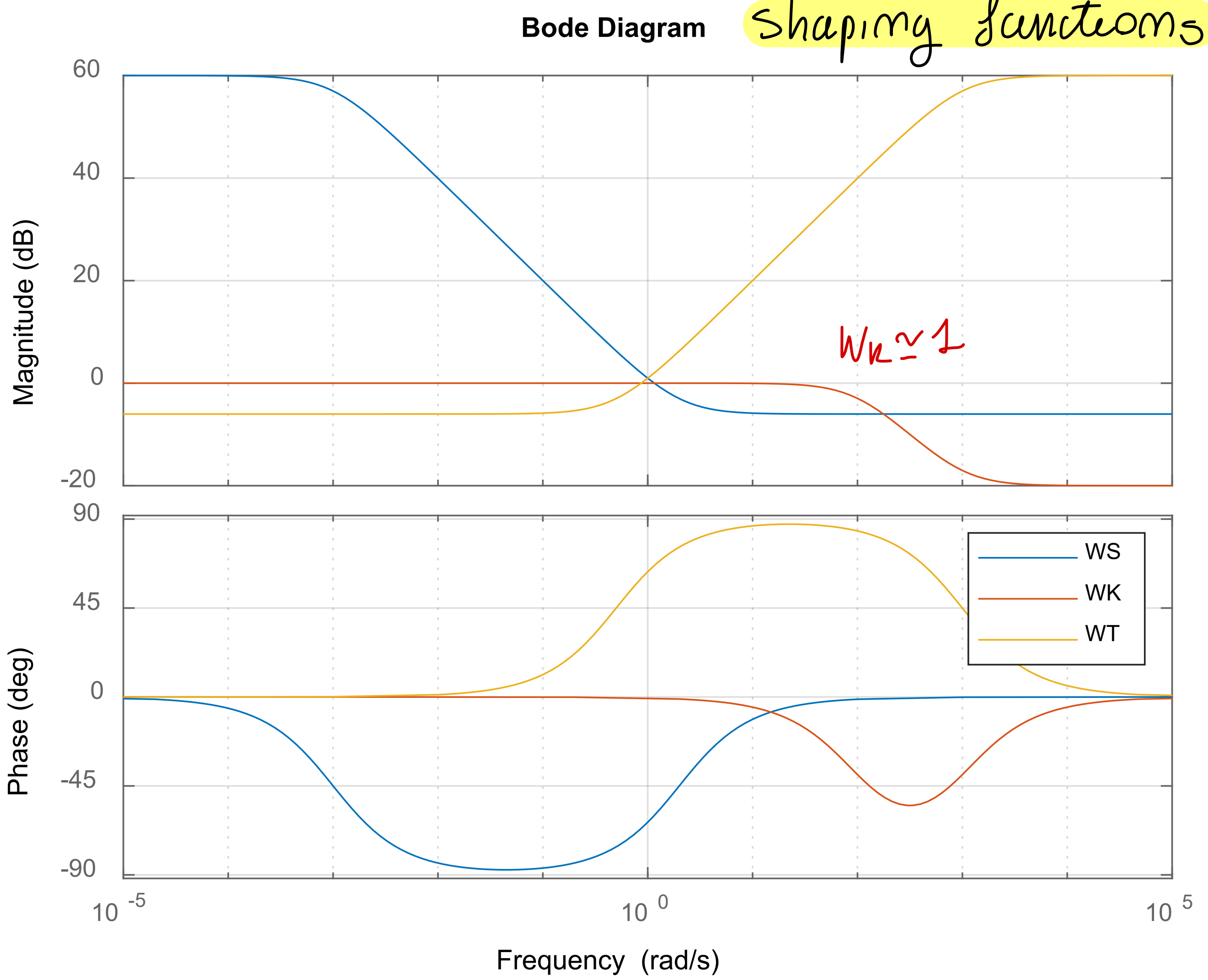
~0 dB @HF



$$W_T(s) = \frac{s+\omega_{BT}/M}{As+\omega_{BT}},$$



shaping functions



Build block diagram! (you can have multiple shaping functions)

% shaping functions written in block form

WWS=blkdiag(WS,WS,WS);
WWT=blkdiag(WT,WT,WT);
WWK=blkdiag(WK,WK,WK);

Ws, Wt, Wk diagonal 3x3 matrices

% enlarged system withshaping functions

augmented system

SW=augw(G,WWS,WWK,WWT);
[KW,CLW,GAMW,INFOW]=hinfsyn(SW);

Plant + shaping functions at the output

design H∞ control!

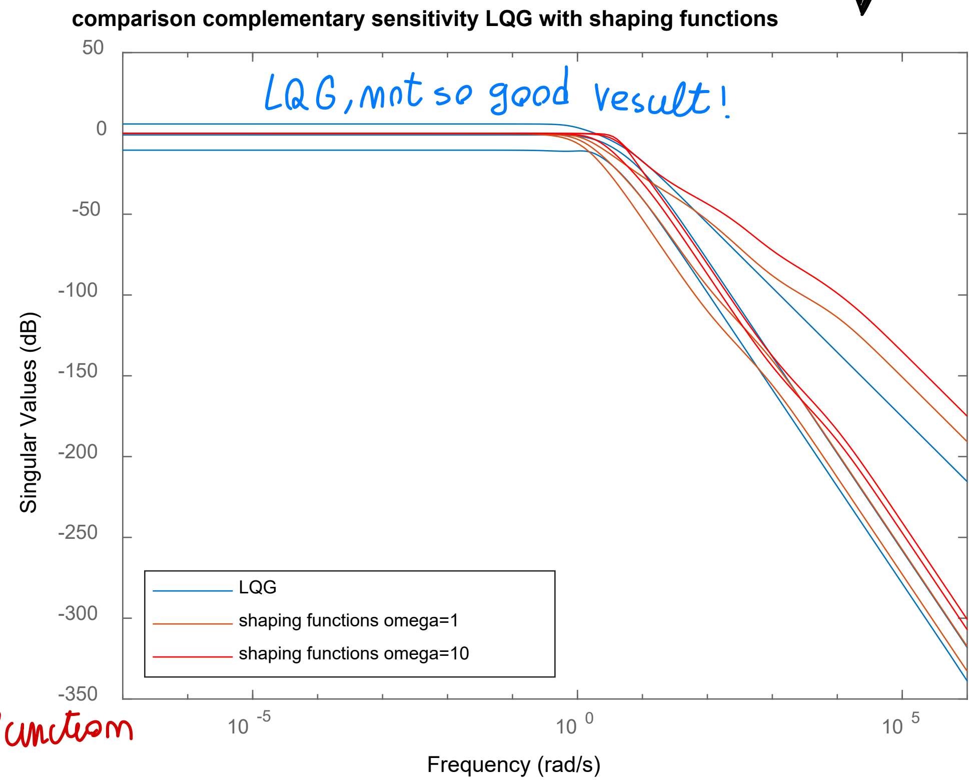
Hinf synthesis

Regulator

LKW=G*KW;
SKW=inv(eye(p)+LKW); % Sensitivity
TKW=eye(p)-SKW; % complementary sensitivity

```
figure(4)
sigma(T)
hold on
sigma(TKW)
title('comparison complementary sensitivity LQG with shaping functions')
```

draw loop T.F!



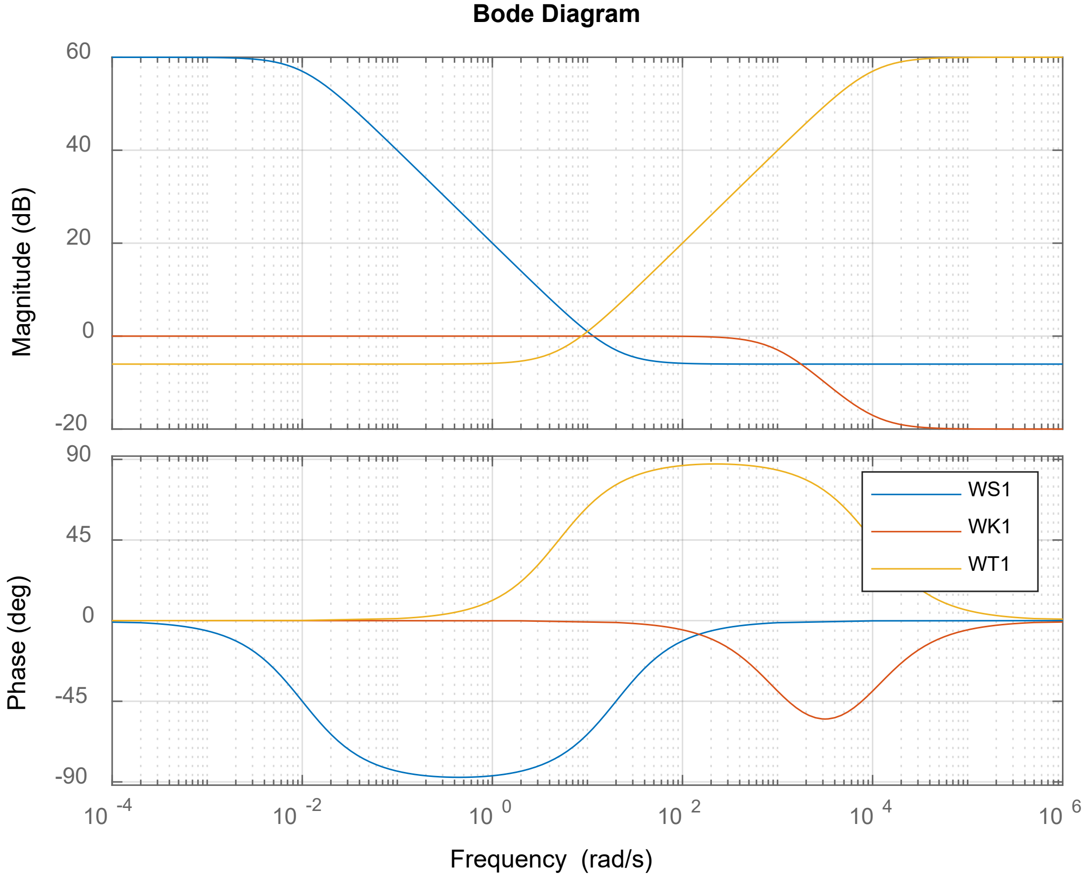
comparisons with LQG control...
on H∞ controller, for Ω=1, ...
shape of sing values changes properly! you repeat the design for Ω=10 you get larger bandwidth! → result behaviour through shaping function

```
% new shaping functions, the bandwith is multiplied by 10
 % definition of the shaping functions
    wB1=10;        % desired closed-loop bandwidth
    AA1=1/1000;   % desired disturbance attenuation inside bandwidth
    M1=2 ;       % desired bound on hinfnorm(S) & hinfnorm(T)
    WS1=(s/M1+wB1)/(s+wB1*AA1); % Sensitivity weight
    WK1=(0.0001*s+1)/(0.001*s+1);  % Control weight can't be empty (d12).ne.0)
    WT1=(s+wB1/M1)/(AA1*s+wB1); % Complementary sensitivity weight
    figure(5)
    bode(WS1)
    hold on
    bode(WK1)
    bode(WT1)
    grid
    legend('WS1','WK1','WT1')
% shaping functions written in block form
WWS1=blkdiag(WS1,WS1,WS1);
WWT1=blkdiag(WT1,WT1,WT1);
WWK1=blkdiag(WK1,WK1,WK1);
% enlarged system with shaping functions
SW1=augw(G,WWS1,WWK1,WWT1);
```

modify W_B, bandwidth!

new enlarged system

repeat computation with new shaping function



```
[KW1,CLW1,GAMW1,INFOW1]=hinfsyn(SW1);
```

new Hinf synthesis

```
LKW1=G*KW1;
SKW1=inv(eye(p)+LKW1);  % Sensitivity
TKW1=eye(p)-SKW1;       % complementary sensitivity

figure(4)
sigma(TKW1,'r')
legend('LQG','shaping functions omega=1','shaping functions omega=10','Location','SouthWest')
```

n = 5, high order

+ Ws order = 3

+ complementary = 3

+ control = 3

↳ Reg. of order 14 → try to reduce the Reg order! ⟹

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% analysis of the singular values and order reduction of the Hinf regulator - first project
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure(6)
sigma(KW)
title('singular values Hinf regulator (I project)')
hold on
```

gramiam

s.s form where P= Q diag

```
KWBAL=balreal(KW); % compute the balanced realization of the Hinf regulator
GRAMr=gram(KWBAL,'c');
```

compute gramiam

```
figure(7)
plot(diag(GRAMr),'*') % draws the singular values of the reachability gramian
title('sing. val. reachability gramian')
```

than plot sing values of gramian

```
lim=0.05;          % limit of the singular values to be used in the order reduction
g=diag(GRAMr);
elim = (g<lim);          % removes the states corresponding to the singular values less than lim
KWRID = modred(KWBAL,elim);  % removes the states
```

use model reduction technique to remove states of hig sing values!

```
 figure(8)
sigma(KW)
```

original reg sing value

```
hold on
sigma(KWRID)
```

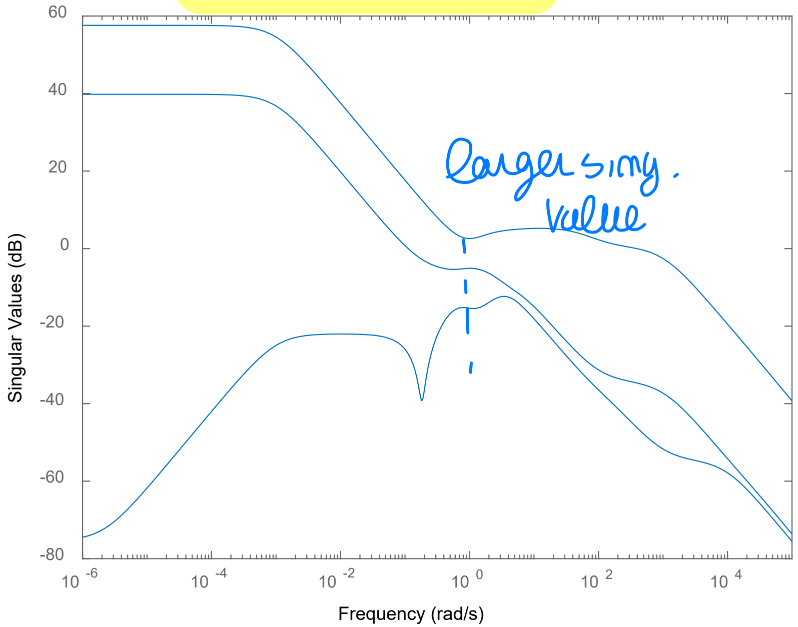
and reduced order

```
title('comparison between the Hinf regulator and the reduced order regulator')
```

allow for good result! specify requirements to have faster system ecc..

```
figure(9)
LRID=G*KWRID;
SRID=inv(eye(p)+LRID);  % Sensitivity
TRID=eye(p)-SRID;       % complementary sensitivity
sigma(TKW)
hold on
sigma(TRID)
title('comparison between the complementary sensitivity Hinf and reduced Hinf')
```

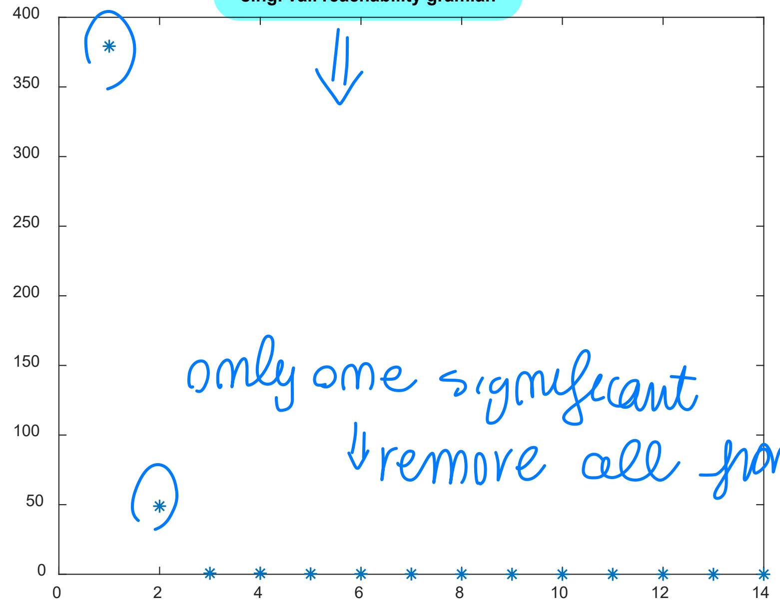
singular values Hinf regulator (I project)



larger sing. value

3 sing values (order 14 of Reg) → sing value as mimo order (3x3)

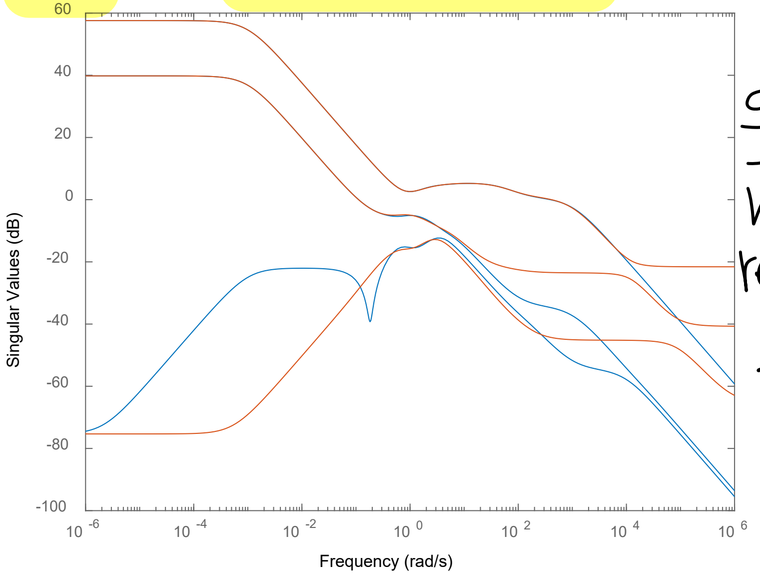
sing. val. reachability gramian



only one significant

↓ remove all from k>2!

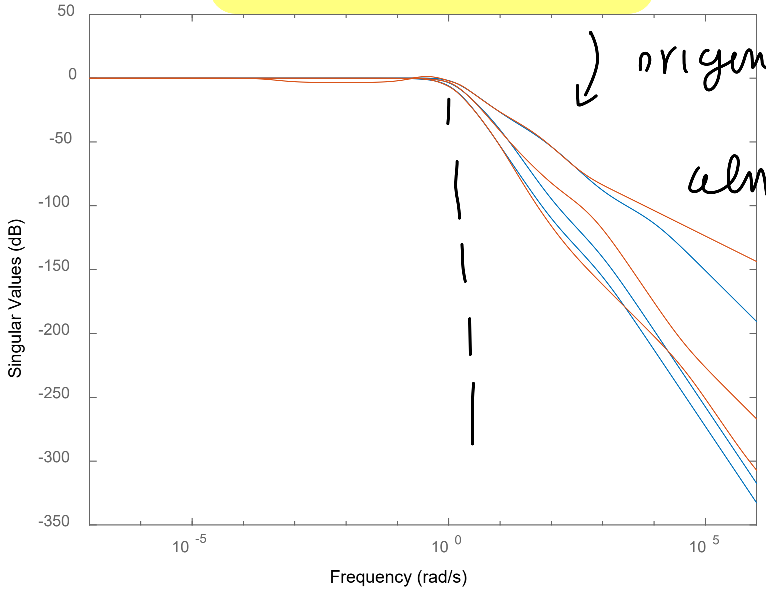
comparison between the Hinf regulator and the reduced order regulator



at the end I get sing values of reduced order syst.

– very similar largest sing val.

comparison between the complementary sensitivity Hinf and reduced Hinf



original vs reduced order

almost same behavior even @ high freq!

no penalize reduced order!