1. **Theory Question**

Prove the following property of k dimensional Gaussian distributions $\text{Norm}_x[\mu, \Sigma]$:

$$\int \text{Norm}_x[a, A]\text{Norm}_x[b, B]dx = \text{Norm}_a[b, A+B] \int \text{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]dx$$

where $\Sigma_* = (A^{-1} + B^{-1})^{-1}$.

(5 points)

$\text{Norm}_x[\mu, \Sigma]$ is defined as:

$$\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \cdot \exp\left[-0.5(x-\mu)^T\Sigma^{-1}(x-\mu)\right]$$

Given two Normal distributions: $\mathcal{N}(a, A)$ and $\mathcal{N}(b, B)$ the product of the two is given by:

$$\mathcal{N}(a,A)\cdot\mathcal{N}(b,B) = \frac{1}{(2\pi)^{D/2}|A|^{1/2}} \cdot \exp\left[-0.5\cdot(x-a)^TA^{-1}(x-a)\right] \cdot \frac{1}{(2\pi)^{D/2}|B|^{1/2}} \cdot \exp\left[-0.5\cdot(x-b)^T\cdot B^{-1}(x-b)\right]$$

$$= \frac{1}{(2\pi)^D \cdot |A|^{1/2}|B|^{1/2}} \cdot \exp\left[-0.5(x-a)^TA^{-1}(x-a) - 0.5(x-b)^TB^{-1}(x-b)\right] \qquad k_1 = \frac{1}{(2\pi)^D \cdot |A|^{1/2} \cdot |B|^{1/2}}$$

$$= k_1 \cdot \exp\left\{-0.5\left[(x-a)^TA^{-1}(x-a) + (x-b)^TB^{-1}(x-b)\right]\right\}$$

$$= k_1 \cdot \exp\left\{-0.5\left[(x^TA^{-1} - a^TA^{-1})(x-a) + (x^TB^{-1} - b^TB^{-1})(x-b)\right]\right\}$$

$$= k_1 \cdot \exp\left\{-0.5\left[\underline{x^TA^{-1}x} - \underline{x^TA^{-1}a} - a^TA^{-1}x + \underline{a^TA^{-1}a} + \underline{x^TB^{-1}x} - \underline{x^TB^{-1}b} - b^TB^{-1}x + \underline{b^TB^{-1}b}\right]\right\}$$

$$= k_2 \cdot \exp\left\{-0.5\left[x^T(A^{-1}+B^{-1})x + 2(A^{-1}a + B^{-1}b)x\right]\right\}$$

$$\hookrightarrow k_1 \cdot \exp\left(-0.5 \cdot a^T A^{-1} a - 0.5\, b^T B^{-1} b\right)$$

$$= k3 \cdot \exp\left\{-0.5\left[x^T(A^{-1}+B^{-1})x + 2(A^{-1}a + B^{-1}b)x + (A^{-1}a + B^{-1}B)^T (A^{-1}+B^{-1})^{-1} \cdot (A^{-1}a + B^{-1}B)\right]\right\}$$

$$\hookrightarrow k3 = k2 \cdot \exp\left[0.5 \cdot (A^{-1}a + B^{-1}b)^T (A^{-1}+B^{-1})^{-1} (A^{-1}a + B^{-1}b)\right]$$

$$= k_3 \cdot \exp\left\{-0.5\left(x - (A^{-1}+B^{-1})^{-1}(A^{-1}a + B^{-1}b)\right)^T \cdot (A^{-1}+B^{-1}) \cdot \left(x - (A^{-1}+B^{-1})^{-1} \cdot (A^{-1}a + B^{-1}b)\right)\right\}$$

$$= \boxed{k \cdot \mathcal{N}_x\left[(A^{-1}+B^{-1})^{-1} \cdot (A^{-1}a + B^{-1}b), (A^{-1}+B^{-1})^{-1}\right]}$$

The constant k, is itself another Normal distribution which by unrolling the previous constant definitions and using calculus identities we obtain:

$$k = \frac{1}{(2\pi)^{\frac{D}{2}} |A+B|^{1/2}} \cdot \exp\left(-0.5 \cdot (a-b)^T (A+B)^{-1} (a-b)\right)$$

Therefore:

$$\boxed{k = \mathcal{N}(b, A+B)}$$

If we now apply the integral to the normal multiplication we obtain that:

$$\int \mathcal{N}(a, A) \cdot \mathcal{N}(b, B) =$$

$$= \mathcal{N}_x(b, A+B) \cdot \int \mathcal{N}_x\left((A^{-1}+B^{-1})^{-1} \cdot (A^{-1}a + B^{-1}b),\ (A^{-1}+B^{-1})^{-1}\right)$$

and where the above relation follows from the previous proof.