Exercise for MA-INF 2201 Computer Vision WS19/20
18.11.2019
Submission on 23.11.2019

1. A function is *submodular* when it satisfies the equation:

$$P(\beta,\gamma) + P(\alpha,\delta) - P(\beta,\delta) - P(\alpha,\gamma) \geq 0$$

for all $\alpha, \beta, \gamma, \delta$ such that $\beta > \alpha$ and $\delta > \gamma$. Show that:

1.1. the *Quadratic Function* $P(\omega_m, \omega_n) = c(\omega_m - \omega_n)^2$ is *submodular*, ***(2 points)***

1.2. the *Potts model* $P(\omega_m, \omega_n) = c(1 - \delta(\omega_m - \omega_n))$ is not *submodular*, by providing a counter-example to the above criterion. ***(2 points)***

2. Provide a graph structure using the *alpha expansion* method that encodes the initial state of 6 nodes (a,b,c,d,e,f) with initial states $\beta\beta\gamma\alpha\alpha\gamma$ for the case where the label α is expanded. ***(4 points)***

3. Denoise the binary image using a Markov random field. For cut, use max-flow/min-cut algorithm. Read a noisy binary image *noise.png*

3.1. Create a graph for the image using all the pixels as nodes. Each pixel (node) are connected to the "source node" and the "sink node" with directed edges as well as the directed edges between its left, top, right and bottom neighboring pixel. ***(2 points)***

3.2. The Unaries are defined by

$$\begin{aligned} P(x_n|w_n = 0) &= Bern_{x_n}[\rho] \\ P(x_n|w_n = 1) &= Bern_{x_n}[1-\rho] \\ U_n(w_n) &= -log(P(x_n|w_n)) \end{aligned}$$

where, $\rho = 0.7$. Use the following, three combination of pairwise values and display all theres denoised outputs. ***(4 points)***

i. $P(w_m = 0, w_n = 0) = P(w_m = 1, w_n = 1) = 0.005$ and $P(w_m = 0, w_n = 1) = P(w_m = 1, w_n = 0) = 0.2$

ii. $P(w_m = 0, w_n = 0) = P(w_m = 1, w_n = 1) = 0.005$ and $P(w_m = 0, w_n = 1) = P(w_m = 1, w_n = 0) = 0.35$

iii. $P(w_m = 0, w_n = 0) = P(w_m = 1, w_n = 1) = 0.005$ and $P(w_m = 0, w_n = 1) = P(w_m = 1, w_n = 0) = 0.55$

Note: For min-cut/max-flow algorithm install "PyMaxflow"

4. Extend the algorithm in 3 for a grayscale image *noise2.png* using Alpha Expansion Algorithm. There are only three labels $[l_1, l_2, l_3]$ where l_1=1, l_2=2 and l_3=3 corresponding to gray values of (0,128,255) respectively. Unary costs are defined as:

$$\begin{aligned} P(x_n = l_i|w_n = l_i) &= \rho_{l_1} \\ P(x_n = l_j|w_n = l_i) &= \frac{(1-\rho_{l_1})}{2} \forall i \neq j \end{aligned}$$

where, $\rho_{l_i} = 0.8$. Define the pairwise cost using Potts Model: $P(\omega_m, \omega_n) = (1 - \delta(\omega_m - \omega_n))$. ***(6 points)***

$\boxed{1.1}$

Assuming $c \geq 0$, and given:

$$\boxed{\beta > \alpha} \quad \text{and} \quad \boxed{\delta > \gamma}$$

We formulate the following:

$$c\cdot(\beta-\gamma)^2 + c\cdot(\alpha-\delta)^2 - c\cdot(\beta-\delta)^2 - c\cdot(\alpha-\gamma)^2 \geq 0$$

$$\Rightarrow c\cdot(\beta^2 - 2\beta\gamma + \gamma^2) + c(\alpha^2 - 2\alpha\delta + \delta^2) - c\cdot(\beta^2 - 2\beta\delta + \delta^2) - c\cdot(\alpha^2 - 2\alpha\gamma + \gamma^2) \geq 0$$

$$\Rightarrow c\cdot\left[(\beta^2 - 2\beta\gamma + \gamma^2) + (\alpha^2 - 2\alpha\delta + \delta^2) - (\beta^2 - 2\beta\delta + \delta^2) - (\alpha^2 - 2\alpha\gamma + \gamma^2)\right] \geq 0$$

$$\Rightarrow c\cdot(\beta^2 - 2B\gamma + \gamma^2 + \alpha^2 - 2\alpha\delta + \delta^2 - B^2 + 2B\delta - \delta^2 - \alpha^2 + 2\alpha\gamma - \gamma^2) \geq 0$$

$$\Rightarrow c\cdot(-2B\gamma - 2\alpha\delta + 2B\delta + 2\alpha\gamma) \geq 0$$

$$\Rightarrow c\cdot(-2B\gamma + 2\alpha\gamma + 2B\delta - 2\alpha\delta) \geq 0$$

$$\Rightarrow 2c(-B\gamma + \alpha\gamma + B\delta - \alpha\delta) \geq 0$$

$$\Rightarrow \underbrace{B(\delta-\gamma)}_{\text{term 1}} + \underbrace{\alpha(\gamma-\delta)}_{\text{term 2}} \geq 0$$

Term 1 is bigger than term 2 because of the initial assumptions, yielding a value ≥ 0 for all $\alpha, \beta, \delta, \gamma$.

1.2

$$c(1-\delta(\beta-\gamma))+c(1-\delta(\alpha-\delta))-c(1-\delta(\beta-\delta))-c(1-\delta(\alpha-\gamma))\geq 0$$

The delta function is defined:

$$\delta(x-y)=\begin{cases}1 & \text{if } x=y\\ 0 & \text{otherwise}\end{cases}$$

We assume $c\geq 0$ and formulate the following:

$$\cancel{1}-\delta(\beta-\gamma)+\cancel{1}-\delta(\alpha-\delta)-\cancel{1}+\delta(\beta-\delta)-\cancel{1}+\delta(\alpha-\gamma)\geq 0$$

$$\Rightarrow -\delta(\beta-\gamma)-\delta(\alpha-\delta)+\delta(\beta-\delta)+\delta(\alpha-\gamma)\geq 0$$

Assuming $\beta=\gamma$, we obtain:

$$-1-0+0+0\leq 0$$

$-1 \leq 0$ which is not true

Therefore, Potts model is not submodular by definition.

2)

$\beta \beta \gamma \alpha \alpha \gamma$