Exercise 06 for MA-INF 2201 Computer Vision WS19/20
25.11.2019
Submission on 30.11.2019

1. **Theory Question**
   Prove the following property of k dimensional Gaussian distributions $\text{Norm}_x[\mu, \Sigma]$:

   $$\int \text{Norm}_x[a, A]\text{Norm}_x[b, B]dx = \text{Norm}_a[b, A+B] \int \text{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]dx$$

   where $\Sigma_* = (A^{-1} + B^{-1})^{-1}$.
   (5 points)

2. **Kalman Filtering**
   You need to implement the basic Kalman Filtering algorithm with fixed lag smoothing. You observe a set of 2D noisy observations (x_i, y_i) which are the coordinates of the 2D space. They are observed coordinated of a rotating object as shown in Figure 1.

   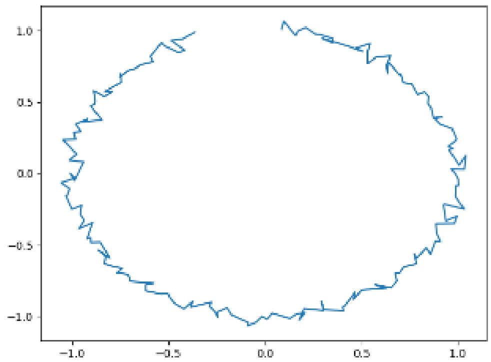

   Figure 1: Observations from location of a clockwise rotating object.

   State: The state of the object should be the 4D vector (x, y, v_x, v_y) which denote the location and the velocity in each axis.
   Initial State: You should consider the initial state of $(0, 1, 0, 0)$.
   Time Evolution Equation: What should be the time evolution equation?
   Measurement Equation: What should be the measurement equation?
   Code for reading observations is provided. You should write code for performing the filtering for different delays $\tau = 0$ and $\tau = 5$. You may use numpy for matrix operations. At the end visualize the filtered output. Use the template `kalman.py`.
   (5 points)

3. **Background Subtraction using Gaussian Mixture Models**
   In this exercise we want to perform background subtraction for the provided image. The image comes with a rectangular bounding box that contains some color pixels of a person (foreground). For this task you are required to implement a Gaussian Mixture Model and the EM algorithm for training. Assume that all covariance matrices are diagonal.

(a) Implement the function *fit_single_gaussian* which fits a single Gaussian to provided data.
(1 point)

(b) GMMs rely on a good initialization. One strategy is to start with a single Gaussian model, split it into two distributions (GMM with two mixtures) and train it using the EM algorithm. For a GMM with four mixtures, both of the previous distributions can be splitted again. Implement the *split* function that doubles the number of components in the current Gaussian mixture model. In particular, generate $2K$ components out of K components as follows:

- Duplicate the weights λ_k so you have $2K$ weights. Divide by two to ensure $\sum_k \lambda_k = 1$.
- For each mean μ_k, generate two new means $\mu_{k1} = \mu_k + \epsilon \cdot \sigma_k$ and $\mu_{k2} = \mu_k - \epsilon \cdot \sigma_k$.
- Duplicate the K diagonal covariance matrices so you have $2K$ diagonal covariance matrices.

(2 points)

(c) Implement the EM algorithm to train the GMM.
(5 points)

(d) **Background Subtraction** Train a GMM with 8 components (start with a single Gaussian and do 3 component splits) for the background pixels. Using the thresholding approach from the lecture, set every pixel in the image to zero which is above a threshold τ. Display the resulting image.
(2 points)