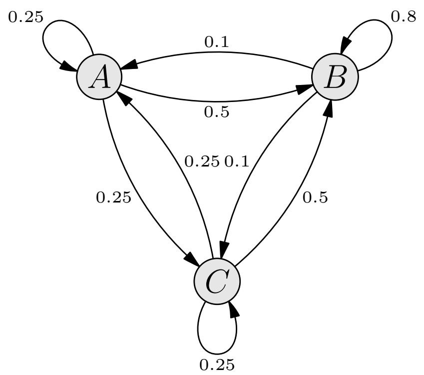
project 14: Markov chains

In this project, we are concerned with homogeneous discrete time Markov chains over discrete finite state spaces $S = \{s_1, s_2, \dots s_m\}$. That is, we are concerned with stochastic processes where

$$p\big(X_t = s_j \mid X_{t-1} = s_i\big) = p\big(s_j \mid s_i\big) = P_{ji}.$$

In the lecture we looked at examples such as this one (which models the behavior of “guarding the treasure in room B”)



$$\boldsymbol{P} = \begin{bmatrix} 0.25 & 0.10 & 0.25 \\ 0.50 & 0.80 & 0.50 \\ 0.25 & 0.10 & 0.25 \end{bmatrix}$$

Recall that the matrix of state transition probabilities of a homogeneous DTMC is column stochastic. In other words, if $|S| = m$, then $\boldsymbol{P} \in \mathbb{R}^{m \times m}$ and, importantly

$$P_{ji} \geq 0 \qquad \wedge \qquad \sum_j P_{ji} = 1.$$

Here is a puzzling statement: a Markov chain evolves over a state space and has states itself. But the states a Markov is in are not the states over which it evolves. In fact, states of a Markov chain are stochastic vectors $\boldsymbol{\pi} \in \mathbb{R}^m$ where

$$\pi_i \geq 0 \qquad \wedge \qquad \sum_i \pi_i = 1.$$

The significance of these state vectors is as follows: if we let $\boldsymbol{\pi}[t]$ denote the state vector at time t, we have

$$p\big(X_t = s_i\big) = \pi_i[t].$$

In other words, $\boldsymbol{\pi}[t]$ is a probability distribution over the set of states S and $\pi_i[t]$ indicates how likely the stochastic process is in state s_i after t time steps.
State vectors at time t are computed as follows: given an initial state distribution $\boldsymbol{\pi}[0]$, we have

$$
\begin{aligned}
\boldsymbol{\pi}[1] &= \boldsymbol{P}\boldsymbol{\pi}[0] \\
\boldsymbol{\pi}[2] &= \boldsymbol{P}\boldsymbol{\pi}[1] = \boldsymbol{P}\boldsymbol{P}\boldsymbol{\pi}[0] \\
\boldsymbol{\pi}[3] &= \boldsymbol{P}\boldsymbol{\pi}[2] = \boldsymbol{P}\boldsymbol{P}\boldsymbol{P}\boldsymbol{\pi}[0] \\
&\vdots \\
\boldsymbol{\pi}[t] &= \boldsymbol{P}\boldsymbol{\pi}[t-1] = \underbrace{\boldsymbol{P}\cdot\boldsymbol{P}\cdots\boldsymbol{P}}_{t \text{ times}}\boldsymbol{\pi}[0] = \boldsymbol{P}^t\boldsymbol{\pi}[0]
\end{aligned}
$$

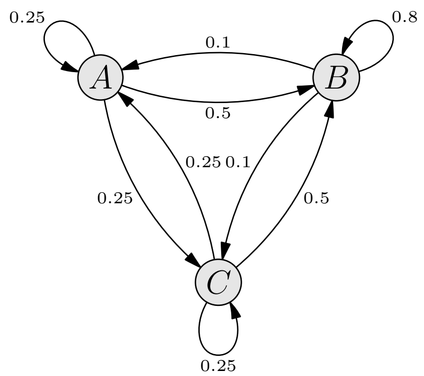
If this process converges for $t \to \infty$, it reaches a so called stationary distribution $\boldsymbol{\pi}$ where

$$\boldsymbol{\pi} = \boldsymbol{P}\boldsymbol{\pi}.$$

In other words, if a Markov chain has a stationary distribution $\boldsymbol{\pi}$, then $\boldsymbol{\pi}$ is an eigenvector of $\boldsymbol{P}$ and the corresponding eigenvalue is 1.

task 14.1: evolution of a Markov process

Consider again the Markov model for “guarding the treasure in room B”



$$\boldsymbol{P}_2 = \begin{bmatrix} 0.25 & 0.10 & 0.25 \\ 0.50 & 0.80 & 0.50 \\ 0.25 & 0.10 & 0.25 \end{bmatrix}$$

Given $\boldsymbol{P}_2$ as above, compute $\boldsymbol{\pi}[t]$ for

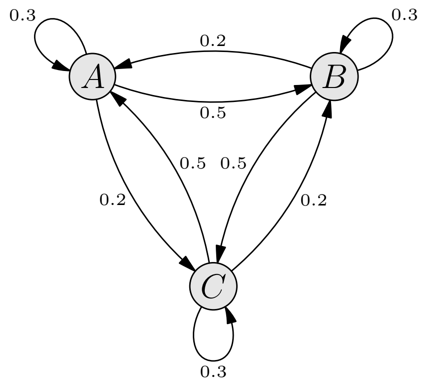
$$t \in \{1, 2, 4, 8, 16\}$$

and

$$\boldsymbol{\pi}[0] \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

What do you observe? Interpret your results. Answer the question: “what is the most likely state s_i for the stochastic process to be in after t steps?”

Next, consider our Markov model for “patrolling the map”



$$\boldsymbol{P}_1 = \begin{bmatrix} 0.30 & 0.20 & 0.50 \\ 0.50 & 0.30 & 0.20 \\ 0.20 & 0.50 & 0.30 \end{bmatrix}$$

and, given $\boldsymbol{P}_1$, compute the state distributions $\boldsymbol{\pi}[t]$ just as above. What do you observe in this case? Interpret your results.

Finally, compute the spectral decompositions

$$\boldsymbol{\Lambda}_1 = \boldsymbol{U}_1 \boldsymbol{P}_1 \boldsymbol{U}_1^\intercal$$
$$\boldsymbol{\Lambda}_2 = \boldsymbol{U}_2 \boldsymbol{P}_2 \boldsymbol{U}_2^\intercal$$

and have a look at the resulting eigenvector matrices $\boldsymbol{U}_1, \boldsymbol{U}_2$. What do you observe?

task 14.2: sampling a Markov process

Here is another puzzling statement: there is a difference between the most likely state of stochastic process to be in at time t and the state it is actually in at time t.
To see what this means, create a state sequence according to transition matrix $\boldsymbol{P}_2$.
In the supplementary material for lecture 19, we discussed that, given an appropriate `numpy` array `P2`, this can be accomplished as follows:

```
import numpy as np
import numpy.random as rnd

states  = ['A', 'B', 'C']
indices = range(len(states))

state2index = dict(zip(states, indices))
index2state = dict(zip(indices, states))

def generateStateSequence(X0, P, tau):
    sseq = [X0]

    iold = state2index[X0]

    for t in range(tau):
        inew = rnd.choice(indices, p=P[:,iold])
        sseq.append(index2state[inew])
        iold = inew

    return sseq
```

To generate and print a sequence of length 10 that starts with state `B`, use

```
>>> sequence = generateStateSequence('B', P2, 9)
>>> print (''.join(sequence))
```

In fact, generate 10.000 sequences of length 10 and have a look at their respective last element `sequence[-1]`. What do you observe? Given this sample of sequences, what is the empirically likeliest last element? How likely is it?