Importance Sampling

Alena Shilova

Skoltech

June 05,2017

Normal distribution

For the normal distribution $\mathcal{N}(\mu, \Sigma)$ to look like the one from exponential family, let's make a change of variables: $S = \Sigma^{-1}$ and $m = \Sigma^{-1}\mu$. Then the density of distribution will look like the following:

$$f_{m,S}(x) = \frac{1}{2\pi} \exp\Big(m^T x - \frac{1}{2}\operatorname{tr}(Sxx^T)\Big) \exp\Big(-\frac{1}{2}\big(m^T S^{-1} m - \log|S|\big)\Big).$$

So in this case $A(m, S) = \frac{1}{2} m^T S^{-1} m - \frac{1}{2}\log|S|,\ T(x) = (x, -\frac{1}{2}xx^T)$ and $h(x) = \frac{1}{2\pi}$

Strong convexity

It is a well-known fact that e^x is a strongly convex function on some compact and if $x \in [-C, C]$ for some $C > 0$, then the coefficient of stong convexity is equal to e^{-C}. In our case if $\lambda(\Sigma) \in [\lambda_{min}, \lambda_{max}]$, then

$$\frac{1}{2}x^T S x - m^T x + \frac{1}{2}m^T S^{-1} m - \frac{1}{2}\log|S| \geq -\frac{1}{2}\log|S| \geq \frac{n}{2}\log\lambda_{min}$$

Therefore, $\alpha = \lambda_{min}^{\frac{n}{2}}$

Closing remark

If we have a proper $\alpha(x)$ (here $\alpha(x) = \lambda_{min}^{\frac{n}{2}}$) and a set $\mathbf{\Theta}$, then for all $\theta_1, \theta_2 \in \mathbf{\Theta}$ we will have:

$$\int \exp(A(t(\theta_1) + (1-t)\theta_2) - (t\theta_1 + (1-t)\theta_2)^T T(x)) f^2(x)\phi^2(x)h(x)\,dx \leq$$

$$\leq t \int \exp(A(\theta_1) - \theta_1^T T(x)) f^2(x)\phi^2(x)h(x)\,dx +$$

$$+(1-t)\int \exp(A(\theta_2) - \theta_2^T T(x)) f^2(x)\phi^2(x)h(x)\,dx -$$

$$-t(1-t)\frac{\lambda_{min}^{\frac{n}{2}}}{2\pi}\int f^2(x)\phi^2(x)\,dx \|\theta_1 - \theta_2\|_2^2$$

The coefficient will depend on the value of $\int\limits_{\mathbb{R}^n} f^2(x)\phi^2(x)\,dx$, but we can also take $\alpha(x) = \lambda_{min}^{\frac{n}{2}}\mathbb{I}(x \in X)$ for some X and then it is enough to know the value $\int\limits_{X} f^2(x)\phi^2(x)\,dx$