Importance Sampling. Minimization of Renyi Divergence. Strong convexity investigation

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Optimization problems

Optimization problem with $\alpha = 1$

$$\max_{\theta \in \Theta} \frac{1}{I} \int \varphi(x) f(x) \left[\theta^T T(x) - A(\theta)\right] \, dx$$

Optimization problem with $\alpha = 2$

$$\min_{\theta \in \Theta} \int \frac{\varphi^2(x) f^2(x)}{\exp(\theta^T T(x) - A(\theta)) h(x)} \, dx.$$

Normal distribution

For the normal distribution $\mathcal{N}(\mu, \Sigma)$ to look like the one from exponential family, let's make a change of variables: $S = \Sigma^{-1}$ and $m = \Sigma^{-1}\mu$. Then the density of distribution will look like the following:

$$g_{m,S}(x) = \frac{1}{2\pi}\exp\left(m^T x - \frac{1}{2}\operatorname{tr}(Sxx^T)\right)\exp\left(-\frac{1}{2}\left(m^T S^{-1} m - \log|S|\right)\right).$$

So in this case $A(m, S) = \frac{1}{2}m^T S^{-1} m - \frac{1}{2}\log|S|,\ T(x) = (x, -\frac{1}{2}xx^T)$ and $h(x) = \frac{1}{2\pi}$

Strong convexity of the first problem

Problem

$$\min_{m,S} \int \varphi(x) f(x) \left[\frac{1}{2} x^T S x - m^T x + \frac{1}{2} m^T S^{-1} m - \frac{1}{2} \log |S|\right] dx$$

As it was shown earlier and exploiting the equivalency of the norms, this objective function is strongly convex:

- with the coefficient $\frac{\lambda_{min}^2}{2}$ in the norm $\|\cdot\|_2$;
- with the coefficient $\frac{\lambda_{min}^2}{2}$ in the norm $\|\cdot\|_\infty$;
- with the coefficient $\frac{\lambda_{min}^2}{2\sqrt{n}}$ in the norm $\|\cdot\|_1$.

Strong convexity of the second problem

Problem

$$\min_{m,S} \int \varphi^2(x) f^2(x) \sqrt{(2\pi)^n} \exp\left[\frac{1}{2}x^T S x - m^T x + \frac{1}{2} m^T S^{-1} m - \frac{1}{2}\log|S|\right] dx$$

$e^{f(x)}$ is strongly convex $\Leftrightarrow$ f(x) is strongly convex and $f(X) \subseteq [-C, +\infty]$.
Indeed, for any x and y

$$\begin{aligned} &\langle e^{f(x)} \nabla f(x) - e^{f(y)} \nabla f(y), x - y \rangle \geq \\ \geq & e^{-C} \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \\ \geq & e^{-C} \beta \|x - y\|^2 \end{aligned}$$

Strong convexity of the second problem

Problem

$$\min_{m,S} \int \varphi^2(x) f^2(x) \sqrt{(2\pi)^n} \exp\left[\frac{1}{2}x^T S x - m^T x + \frac{1}{2} m^T S^{-1} m - \frac{1}{2}\log|S|\right] dx$$

From the things mentioned earlier and exploiting the equivalency of the norms, this objective function is strongly convex:

- with the coefficient $\frac{\lambda_{min}^{\frac{n}{2}+2}}{2}$ in the norm $\|\cdot\|_2$;
- with the coefficient $\frac{\lambda_{min}^{\frac{n}{2}+2}}{2}$ in the norm $\|\cdot\|_\infty$;
- with the coefficient $\frac{\lambda_{min}^{\frac{n}{2}+2}}{2\sqrt{n}}$ in the norm $\|\cdot\|_1$.